Suboptimal Linear Regulations for Linear Systems with Known Initial-State Statistics

Mårtensson, Krister

1970

Document Version:
Publisher's PDF, also known as Version of record

SUBOPTIMAL LINEAR REGULATORS FOR LINEAR SYSTEMS WITH KNOWN INITIAL-STATE STATISTICS.†

K. Mårtensson

ABSTRACT

For linear systems with quadratic loss the optimal control is a linear feedback from all the state variables of the system. If all the states are not measurable, it is sometimes possible to construct a suboptimal linear regulator which is almost as good as the full optimal. In this report an algorithm is deduced, which computes a suboptimal strategy when the statistics of the initial state of the system is known.

---

† This work was supported by the Swedish Board for Technical Development (Contract 69-631/U489).

1. INTRODUCTION.

When designing regulators for linear time-invariant systems, optimal control theory may often be very useful. The desired qualities of the regulator are then achieved from a suitable choice of a quadratic cost function. This formulation of the problem will result in a linear time-invariant feedback from the state of the system, and it is thus necessary to get information about all the state variables either through measurements or some kind of estimation.

In many situations, however, practical or economical reasons can make it impossible to implement the optimal regulator. For example, the instrumentation required to measure or estimate the state may be too expensive. In this case a suboptimal regulator could satisfy the demands on simplicity and low cost of the regulator.

The suboptimal strategy discussed in this paper is a linear time-invariant feedback from a reduced number of state variables. Since the optimization problem for such linear feedbacks generally will depend on the initial state of the system, some further information must be added. Here it is assumed that the initial state is a normally distributed random variable with known mean value and covariance. The problem is then reduced to determine a linear feedback matrix with a given structure, so that the mean value of the loss function is minimized.

A similar approach has recently been published by Dabke [3], where algebraic equations for the suboptimal strategy are deduced. However, these equations are possible to solve only for very simple problems unless a computer is used. The method proposed in this paper is based on an algorithm for numerical minimization, and thus requires a computer. In [1], [5], and [6] other approaches to the suboptimal regulator problem are presented.

2. STATEMENT OF THE PROBLEM.

Consider the linear time-invariant system

$$\frac{dx}{dt} = Ax + Bu \qquad x(0) = x_0 \tag{2.1}$$

and the quadratic loss function

$$V = \int_0^\infty \{x^T(s)Q_1x(s) + u^T(s)Q_2u(s)\}ds \tag{2.2}$$

x is the n dimensional state vector, u the r dimensional control vector, A an $n \times n$ matrix and B an $n \times r$ matrix. Q_1 is a nonnegative definite symmetric $n \times n$ matrix and Q_2 a positive definite symmetric $r \times r$ matrix. To guarantee the existence of an asymptotic stable optimal solution, it is assumed that the pair $[A,B]$ is stabilizable and the pair $[A,Q_1]$ detectable. The optimal solution is then given by the linear time-invariant feedback

$$u(t) = - L^*x(t) \tag{2.3}$$

where

$$L^* = Q_2^{-1}B^TS^* \tag{2.4}$$

S^* is the unique nonnegative definite symmetric solution of the stationary Riccati equation

$$A^TS^* + S^*A - S^*BQ_2^{-1}B^TS^* + Q_1 = 0 \tag{2.5}$$

The minimum of the loss function is

$$V = x^T(0)S^*x(0) \tag{2.6}$$

Notice that L^* is independent of the initial state of the system.

Now consider an arbitrary linear time-invariant feedback

$$u(t) = - Lx(t)$$

The closed loop system then is

$$\frac{dx}{dt} = (A - BL)x(t) \qquad\qquad x(0) = x_0$$

and the cost corresponding to L is

$$V(L) = \int_0^\infty \{x^T(s)Q_1x(s) + x^T(s)L^TQ_2Lx(s)\}ds \tag{2.7}$$

Introduce the fundamental matrix $\phi_L(t; t_0)$ associated with (A - BL).

$$\frac{d\phi_L(t; t_0)}{dt} = (A - BL)\phi_L(t; t_0) \qquad \phi_L(t_0; t_0) = I$$

The cost (2.7) then equals

$$V(L) = x^T(0)S(L)x(0) \tag{2.8}$$

where

$$S(L) = \int_0^\infty \phi_L^T(s; 0)\{Q_1 + L^TQ_2L\}\phi_L(s; 0)ds \tag{2.9}$$

The existence of S(L) is guaranteed if the closed loop system (A - BL) is asymptotic stable, and it is then easy to show that S(L) is a unique nonnegative definite symmetric solution of the algebraic equation

$$(A - BL)^TS(L) + S(L)(A - BL) = - Q_1 - L^TQ_2L \tag{2.10}$$

Subtracting (2.5) from (2.10) and rearranging the terms, equation (2.11) is obtained.

$$(A - BL)^T\left(S(L) - S^*\right) + \left(S(L) - S^*\right)(A - BL) = -(L - L^*)^T Q_2 (L - L^*) \tag{2.11}$$

Analoguous to (2.9) an explicit expression of the solution is

$$S(L) - S^* = \int_0^\infty \phi_L^T(s;0)(L - L^*)^T Q_2 (L - L^*)\phi_L(s;0)ds \tag{2.12}$$

and thus $S(L) \geq S^*$ when $L \neq L^*$.

Now assume that the feedback matrix $L = (\ell_{ij})$ should have a prescribed structure, i.e. $\ell_{ij} = 0$, when feedback is not allowed from state j to control variable i. With this structure, the feedback that minimizes V(L) is not the same for different initial states x(0) of the system, and thus the simplicity of the regulator is lost. To get a unique solution independent of the initial state, it is assumed that x(0) is a normally distributed n dimensional random variable, characterized by the mean value

$$E\{x(0)\} = m \tag{2.13}$$

and the covariance matrix

$$E\left\{\left(x(0) - m\right)\left(x(0) - m\right)^T\right\} = R \tag{2.14}$$

The feedback matrix L is then postulated to minimize the expected value of the loss function

$$\mu(L) = E\{x^T(0)S(L)x(0)\} \tag{2.15}$$

or the expected value of the deviation from the optimal strategy

$$\mu(L) = E\left\{x^T(0)\left(S(L) - S^*\right)x(0)\right\} \tag{2.16}$$

(2.16) is a better choice although it requires the solution of the optimal problem, since it will give some information about how efficient the suboptimal strategy is. As $x(0)$ is normally distributed, (2.16) is equivalent to

$$\mu(L) = m^T\left(S(L) - S^*\right)m + \operatorname{trace}\left[\left(S(L) - S^*\right)R\right] \tag{2.17}$$

or

$$\mu(L) = \operatorname{trace}\left\{\left(S(L) - S^*\right)(R + mm^T)\right\} \tag{2.18}$$

If the mean value does not equal zero, it follows from (2.18) that it can be included in the covariance matrix. In the following it will then be assumed that $m = 0$. The problem can now be stated as follows:

In the set of all stable linear time-invariant feedback matrices $L = (\ell_{ij})$ with a prescribed structure, find the one that minimizes the loss function $\mu(L)$.

3. NUMERICAL SOLUTION OF THE PROBLEM.

In this section an algorithm for numerical minimization of $\mu(L)$ is given. The method is straightforward, and is based on the Fletcher-Reeves conjugate gradient minimization method [4], but apply to any method which makes use both of the function values and the gradient. It is assumed that the optimal solution $S^* = (s^*_{ij})$ of (2.5) is known, and the loss function (2.18) is chosen. To simplify the notations, the argument L will be dropped from now on. The function value is thus given by

$$\mu(L) = \mathrm{trace}\big((S - S^*)R\big) = \sum_{i,j=1}^{n} (s_{ij} - s^*_{ij})r_{ji} \tag{3.1}$$

where $S = (s_{ij})$ is the unique nonnegative definite solution of

$$(A - BL)^T S + S(A - BL) = -(Q_1 + L^T Q_2 L) \tag{3.2}$$

There are many ways to solve this equation. If the order n of the system is not too large, Kronecker products can be used to rewrite (3.2) as a linear equation. Introduce $\tilde{s}$ as an $n(n+1)/2$ dimensional vector containing the upper triangular part of S.

$$\tilde{s}^T = (s_{11}, s_{12}, \ldots, s_{1n}, s_{22}, \ldots, s_{2n}, s_{33}, \ldots, s_{nn})$$

and $\tilde{q}$ as a vector of the same dimension containing the upper triangular part of $Q = Q_1 + L^T Q_2 L$.

Then $\tilde{s}$ is the solution of the linear equation

$$\mathcal{A}\tilde{s} = -\tilde{q} \tag{3.3}$$

where $\mathcal{A}$ is an $n(n+1)/2 \times n(n+1)/2$ matrix composed from the elements of $(A - BL)$ according to the rules given by the Kronecker product. The solution of (3.2) is then reduced to the solution of a system of linear equations in $n(n+1)/2$ variables.

To get the gradient, (3.1) is differentiated with respect to ℓ_{ij}.

$$\frac{\partial\mu}{\partial\ell_{ij}} = \sum_{p,q=1}^{n} \frac{\partial\mu}{\partial s_{pq}} \frac{\partial s_{pq}}{\partial\ell_{ij}} \tag{3.4}$$

which reduces to

$$\frac{\partial\mu}{\partial\ell_{ij}} = \sum_{p,q=1}^{n} r_{pq} \frac{\partial s_{pq}}{\partial\ell_{ij}} \tag{3.5}$$

The derivatives

$$\frac{\partial s_{pq}}{\partial\ell_{ij}}$$

are obtained from differentiating equation (3.2) with respect to ℓ_{ij}.

$$A^T \frac{\partial S}{\partial\ell_{ij}} - \left(\frac{\partial L}{\partial\ell_{ij}}\right)^T B^T S - L^T B^T \frac{\partial S}{\partial\ell_{ij}} + \frac{\partial S}{\partial\ell_{ij}} A - \frac{\partial S}{\partial\ell_{ij}} BL - SB \frac{\partial L}{\partial\ell_{ij}} =$$

$$= - \left(\frac{\partial L}{\partial\ell_{ij}}\right)^T Q_2 L - L^T Q_2 \frac{\partial L}{\partial\ell_{ij}} \tag{3.6}$$

Introduce

$$X_{ij} = \frac{\partial L}{\partial\ell_{ij}} \tag{3.7}$$

as an $r \times n$ matrix with zeroes in all entries except for $x_{ij} = 1$. (3.6) is then equivalent to

$$(A - BL)^T \frac{\partial S}{\partial\ell_{ij}} + \frac{\partial S}{\partial\ell_{ij}} (A - BL) = (SB - L^T Q_2)X_{ij} + X_{ij}^T (SB - L^T Q_2)^T \tag{3.8}$$

If feedback is not allowed from state i to control variable j, i.e. $\ell_{ij} \equiv 0$, it follows that X_{ij} is a null matrix, and then

$$\frac{\partial S}{\partial \ell_{ij}} = 0 \text{ and } \frac{\partial \mu}{\partial \ell_{ij}} = 0.$$

Notice, however, that the same composed matrix $\mathcal{A}$ is used to compute both S and all the derivatives

$$\frac{\partial S}{\partial \ell_{ij}} .$$

The computations are then reduced to the solution of a number of linear systems of equations with different right hand sides. Summarizing, the following algorithm is obtained:

1. Guess an initial value of L so that A - BL is stable.
2. Compose the matrix $\mathcal{A}$ and compute S and $\frac{\partial S}{\partial \ell_{ij}}$.
3. Compute μ and $\frac{\partial \mu}{\partial \ell_{ij}}$.
4. Use the function value and the gradient in the minimization algorithm to get a new estimation of the minimizing feedback matrix L.
5. Return to 2 if the minimum is not reached.

The algorithm has been implemented on a CDC 3600 and found to work well.

4. COMPUTED EXAMPLES.

A. One-Dimensional Heat Diffusion Process.

The linear system

$$\frac{dx}{dt} = \begin{pmatrix} -32 & 16 & 0 \\ 16 & -32 & 16 \\ 0 & 16 & -32 \end{pmatrix} x + \begin{pmatrix} 16 & 0 \\ 0 & 0 \\ 0 & 16 \end{pmatrix} u \qquad (4.1)$$

is obtained from a difference approximation of a one-dimensional heat diffusion process (Fig. 1). The state variables describe the temperature at equally spaced points, and u_1 is the control variable.

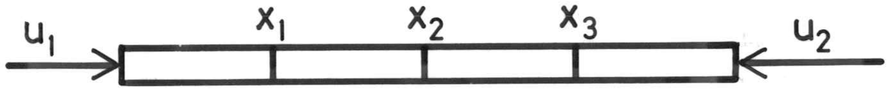

Fig. 1

u_2 is assumed to be an exponentially decreasing disturbance, $u_2(t) = e^{-1.6t}u_2(0)$, entering with a normally distributed initial value $u_2(0)$. The purpose of the regulator is to keep the temperature $x_2(t)$ close to zero.

Introducing the disturbance $u_2(t)$ as a new state variable x_4, the system equation is

$$\frac{dx}{dt} = \begin{pmatrix} -32 & 16 & 0 & 0 \\ 16 & -32 & 16 & 0 \\ 0 & 16 & -32 & 16 \\ 0 & 0 & 0 & -1.6 \end{pmatrix} x + \begin{pmatrix} 16 \\ 0 \\ 0 \\ 0 \end{pmatrix} u \qquad (4.2)$$

The loss function is chosen as

$$V = \int_0^\infty \{10x_2^2 + u^2\} ds \qquad (4.3)$$

which results in the optimal feedback matrix

$$L^* = (0.41,\ 0.91,\ 0.41,\ 0.43) \tag{4.4}$$

The initial disturbances are characterized by the mean value $m = 0$ and the covariance matrix

$$R = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.5}$$

Since the system equations are very well conditioned, it was thought that a linear feedback from x_2 or x_3 or both should be almost as good as an optimal feedback. The computed suboptimal strategies are

$$L_2 = (0,\ 4.55,\ 0,\ 0)$$

$$L_3 = (0,\ 0,\ 1.19,\ 0)$$

$$L_{23} = (0,\ -2.25,\ 1.80,\ 0)$$

The performance of the strategies are almost identical, and L_3 can then be considered as the best choice since it requires the least amplification gain. In Fig. 2 the time histories for $x_1(t)$ and $x_2(t)$ are shown for the three different strategies, L^*, L_3, and $\bar{L}_3 = (0, 0, 0.41, 0)$. In the latter the optimal amplification of x_3 is used. In Fig. 3 contour levels of μ are plotted versus the feedback matrix coefficients ℓ_{12} and ℓ_{13}.

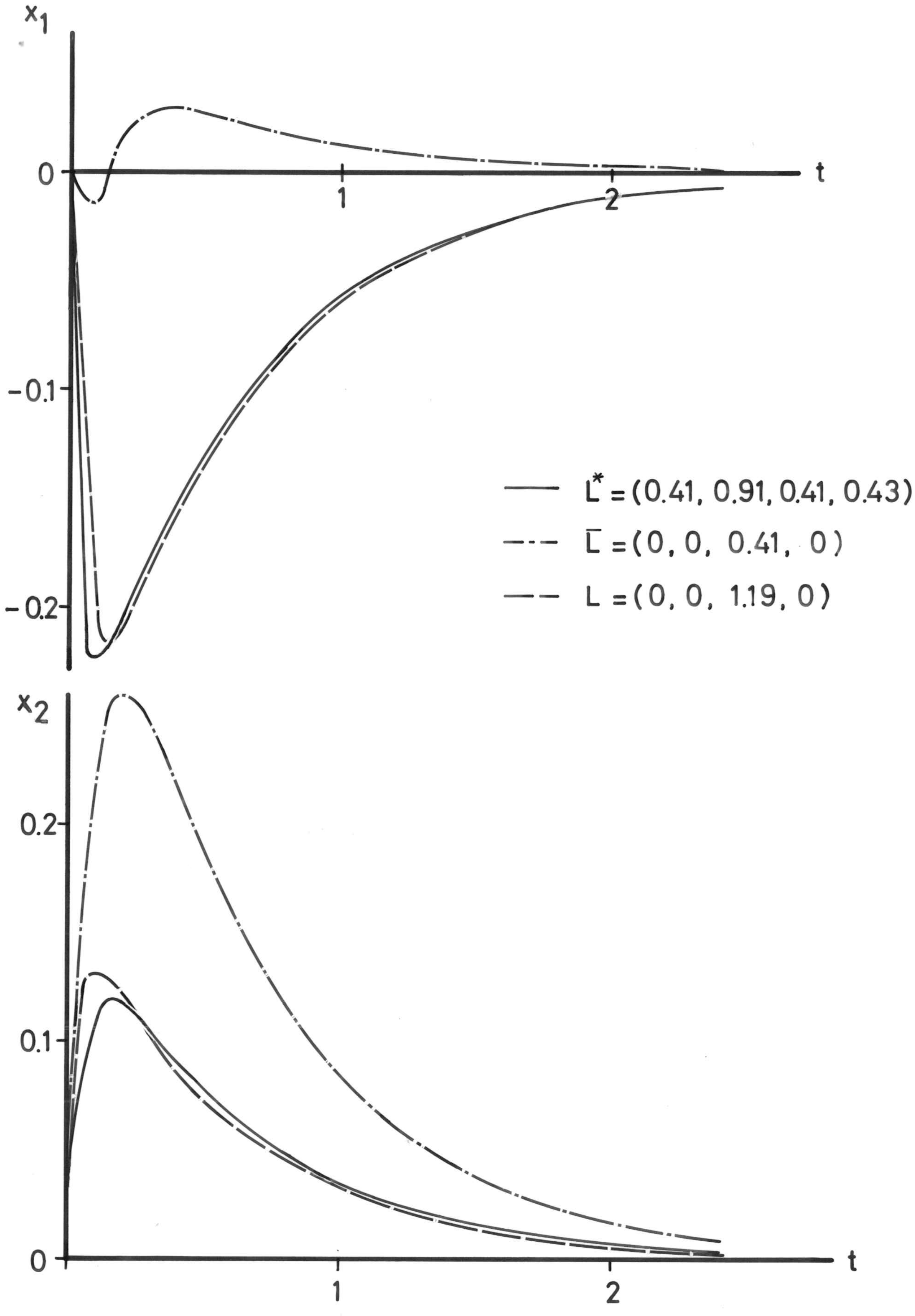

Fig. 2 - Closed loop response for different feedback matrices.

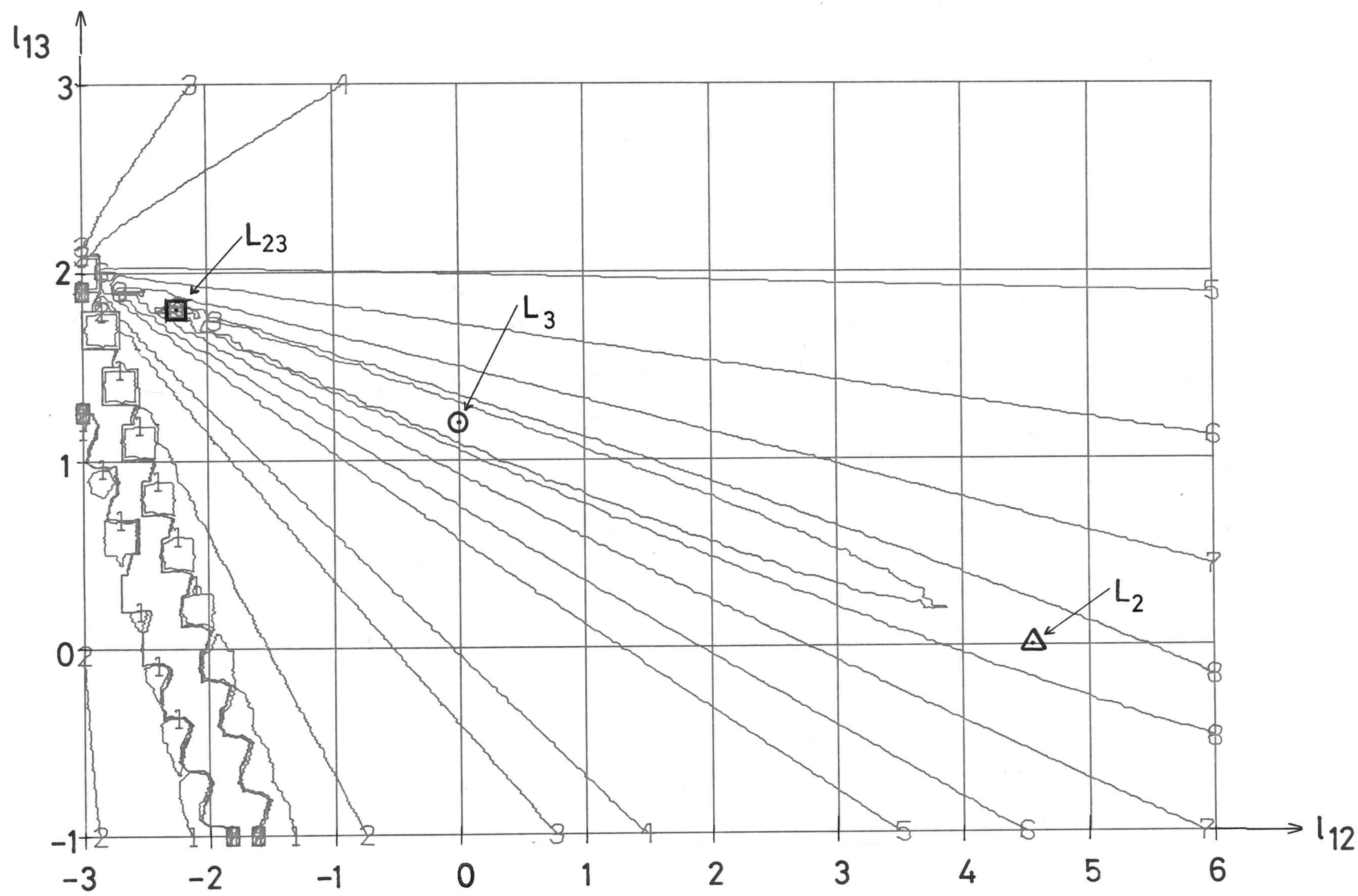

Fig. 3 - Contours with constant value of $\mu(L)$. The different suboptimal strategies are indicated. □ - L = (0, -2.25, 1.80, 0), 0 - L = (0, 0, 1.19, 0), Δ - L = (0, 4.55, 0, 0).

B. A Multivariable System.

The following example originates from Athans [2], and has also been used in [6] to design minimax controllers.

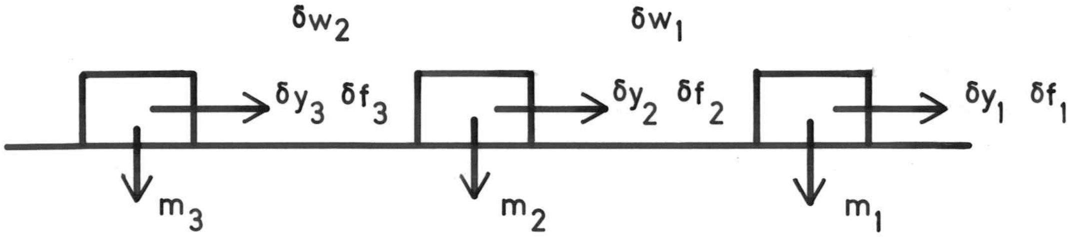

Fig. 4 - A string of three vehicles.

Consider the three vehicles shown in Fig. 4. If all the masses and friction coefficients equal one, the system is described by the linear equations

$$\frac{d}{dt}\begin{pmatrix}\delta y_1\\ \delta w_1\\ \delta y_2\\ \delta w_2\\ \delta y_3\end{pmatrix} = \begin{pmatrix}-1 & 0 & 0 & 0 & 0\\ 1 & 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0 & -1\\ 0 & 0 & 0 & 0 & -1\end{pmatrix}\begin{pmatrix}\delta y_1\\ \delta w_1\\ \delta y_2\\ \delta w_2\\ \delta y_3\end{pmatrix} + \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\delta f_1\\ \delta f_2\\ \delta f_3\end{pmatrix} \tag{4.6}$$

where δy_i are the velocity deviations from the desired string velocity, δw_i the deviations from the desired spacing and δf_i the incremental forces acting on the vehicles. The purpose of the regulator is to keep the spacing between the vehicles constant. A suitable choice of the loss function is

$$V = \int_0^\infty \{10\delta w_1^2 + 10\delta w_2^2 + \delta f_1^2 + \delta f_2^2 + \delta f_3^2\} ds \tag{4.7}$$

which results in the optimal feedback matrix

$$L^* = \begin{pmatrix} 1.26 & 2.49 & -0.82 & 0.67 & -0.44 \\ -0.82 & -1.83 & 1.64 & 1.83 & -0.82 \\ -0.44 & -0.67 & -0.82 & -2.49 & 1.26 \end{pmatrix} \quad (4.8)$$

The magnitude of the entries in L^* indicate that the incremental forces δf_i are mainly influenced by the velocity deviation δy_i of the same vehicle, and of the spacing deviation of the vehicle next to it. A suboptimal feedback thus should have the structure

$$L = \begin{pmatrix} x & x & 0 & 0 & 0 \\ 0 & x & x & x & 0 \\ 0 & 0 & 0 & x & x \end{pmatrix} \quad (4.9)$$

The disturbances on the system are assumed to appear as deviations from the desired positions of the vehicles. If these are assumed to be independent and normally distributed with mean value 0 and variance 1, the two-dimensional random variable $(\delta w_1(0), \delta w_2(0))$ is normally distributed with mean value

$$E\{(\delta w_1(0), \delta w_2(0))\} = (0,0) \quad (4.10)$$

and covariance matrix

$$E\{(\delta w_1(0), \delta w_2(0))^T(\delta w_1(0), \delta w_2(0))\} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad (4.11)$$

The covariance of the initial state of the system then is

$$R = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.12)$$

and the suboptimal strategy

$$L = \begin{pmatrix} 1.49 & 2.29 & 0 & 0 & 0 \\ 0 & -1.84 & 3.37 & 1.84 & 0 \\ 0 & 0 & 0 & -2.29 & 1.49 \end{pmatrix} \quad (4.13)$$

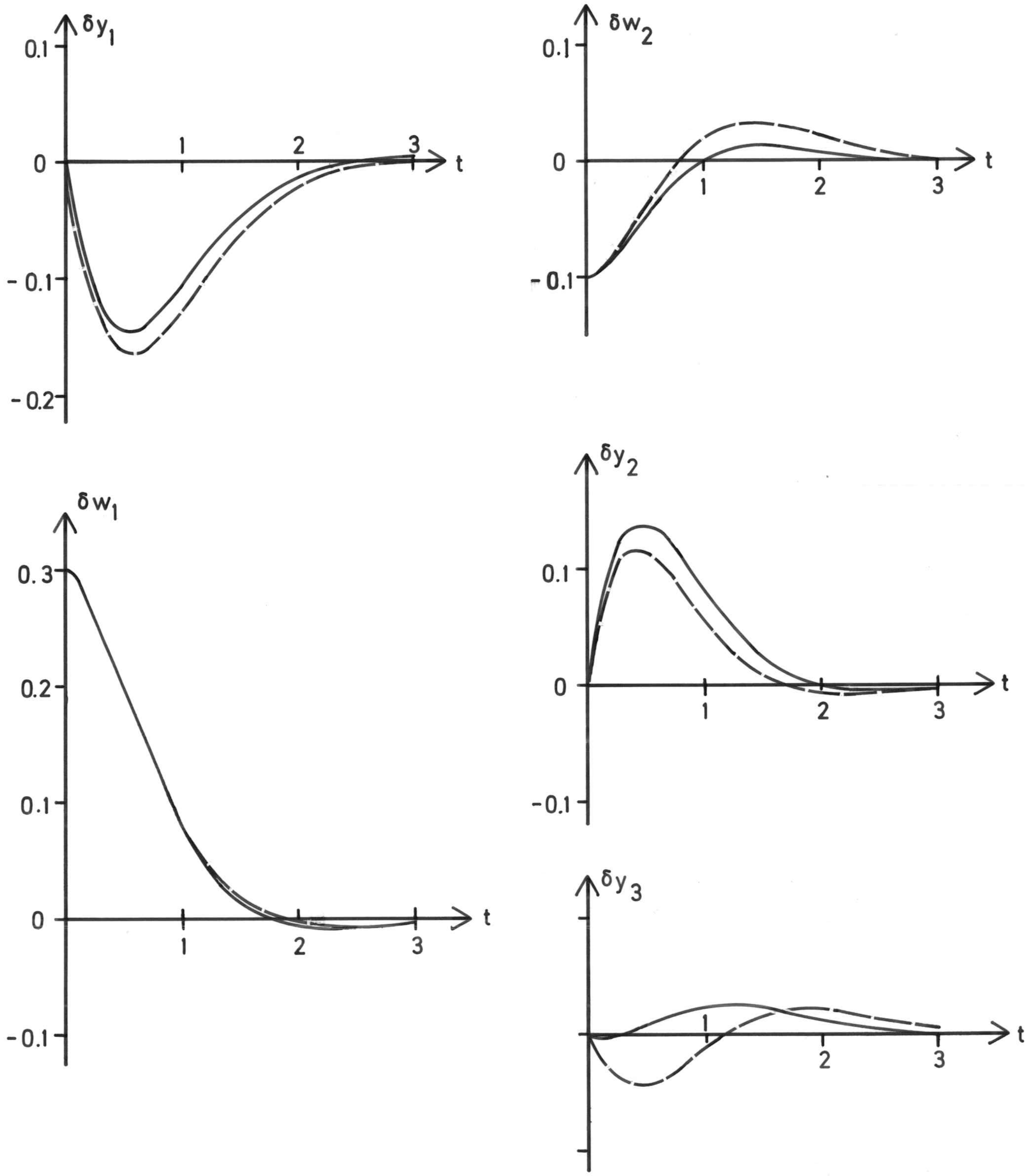

Fig. 5 - Response of the optimal (——) and suboptimal (----) regulator after the initial disturbance $\delta w_1 = 0.3$ $\delta w_2 = -0.1$.

The behaviour of the closed loop system A - BL and of the optimal system is shown in Fig. 5. Initial disturbances are $\delta w_1(0) = 0.3$ and $\delta w_2(0) = -0.1$. Except for the state variable δy_3, the performance of the suboptimal regulator is close to the optimal one. The deviation in δy_3 is due to the lack of information from vehicle 1 to vehicle 3. In the first instant, number 3 tries to increase the distance to vehicle 2, and it has no information about the distance between the other two, which in this case is too large. The overall performance of the suboptimal regulator can, however, be considered as quite satisfactory.

REFERENCES.

[1] Athans, Kleinman, Fortman: On the Design of Linear Systems with Piecewise Constant Feedback Gains, IEEE Trans. on Automatic Control, vol. AC-13, August, 1968.

[2] Athans, Levine: On the Optimal Error Regulation of a String of Moving Vehicles, IEEE Trans. on Automatic Control, vol. AC-11, July, 1966.

[3] Dabke: Linear Control with Incomplete State Feedback and Known Initial-State Statistics, Int. J. Control, vol. 11, No. 1, 1970.

[4] Fletcher, Reeves: Function Minimization by Conjugate Gradients, Computer Journal, vol. 7, 1964.

[5] Rekasius: Optimal Linear Regulators with Incomplete State Feedback, IEEE Trans. on Automatic Control, vol. AC-12, June, 1967.

[6] Salmon: Minimax Controller Design, IEEE Trans. on Automatic Control, vol. AC-13, August, 1968.