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HOMOGENIZATION OF THE MAXWELL EQUATIONS AT FIXED FREQUENCY*

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Abstract. The homogenization of the Maxwell equations at fixed frequency is addressed in this paper. The bulk (homogenized) electric and magnetic properties of a material with a periodic microstructure are found from the solution of a local problem on the unit cell by suitable averages. The material can be anisotropic and satisfies a coercivity condition. The exciting field is generated by an incident field from sources outside the material under investigation. A suitable sesquilinear form is defined for the interior problem, and the exterior Calderón operator is used to solve the exterior radiating fields. The concept of two-scale convergence is employed to solve the homogenization problem. A new a priori estimate is proved as well as a new result on the correctors.

Key words. Maxwell equations, homogenization, heterogeneous materials, periodic microstructure, effective properties, two-scale convergence, corrector results

AMS subject classifications. 35B27, 35Q60, 78A25, 78A40, 78A45, 78A48, 78M40

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1. Introduction.

The concept of two-scale convergence is a well-established tool in the theory of homogenization of elliptic equations with rapidly oscillating coefficients; see, e.g., [2, 3, 7, 10, 12, 14, 17, 19, 21, 26, 27, 28]. The results apply to several types of partial differential equations that are used in the engineering sciences, such as heat conduction, elastic deformation, porous media, and acoustics. The situation is, however, different with the Maxwell equations, and the few results that exist adopt boundary conditions that are of less importance in applications. Specifically, the boundary conditions employed in the literature (see, e.g., [4, 6, 15, 18, 20, 26, 27, 28]) are those of perfectly conducting walls. This situation applies to the case of a resonator filled with a heterogeneous material, but for other situations these boundary conditions are less applicable. Moreover, there is a need for a better understanding of how a microscopic structure alters the macroscopic electric and magnetic behavior of the material if the sources of the electromagnetic fields are located outside the heterogeneous material. In fact, most applications in the engineering sciences use external excitations, and to find the homogenized parameters of a heterogeneous material, other boundary conditions, such as the penetrable boundary conditions, must be used.

The engineering literature is dominated by the simple mixture formulae, which are derived using physical arguments. For an excellent overview and history of the mixture formulae, see [22].

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The object of this paper is to analyze thoroughly the homogenization of the Maxwell equations for a bounded object with penetrable boundary conditions. This homogenization problem seems not to have been published before in the literature. Moreover, the boundary condition implies that the excitation must be due to external sources. This situation is very important in many engineering applications, such as antenna applications. The two-scale convergence of the Maxwell equations depends on an a priori estimate of the fields. The external sources alter the traditional way of homogenization with two-scale convergence. In fact, in addition to the interior homogenization problem, there is an exterior scattering problem that couples via the boundary conditions to the interior problem. We solve this problem by introducing the Calderón operators, which map the tangential electric field to the tangential magnetic field on the bounding surface. In order to apply the boundary conditions and the Calderón operators, a new a priori estimate has been derived. The paper also includes new results on the correctors.

The paper is organized in the following way. Section 2 contains the prerequisites of the paper. The existence of solutions is proved in section 3, and the homogenization of the Maxwell equations is derived in section 4. We illustrate the exterior Calderón operator with two examples in section 5. The paper is concluded with a series of appendices that contain definitions of function spaces (Appendix A), and some important theorems (Appendix B). In Appendix C, the vector spherical waves used in section 5 are defined.

2. Formulation of the problem.

2.1. Domain and incident fields. Assume Ω is a bounded, open, simply connected set in $\mathbb{R}^3$ with $C^{1,1}$ boundary, $\partial\Omega$. The outward-pointing unit normal is $\hat{\boldsymbol{\nu}}$. The exterior of the volume Ω is denoted $\Omega_e = \mathbb{R}^3 \setminus \overline{\Omega}$, which is assumed vacuous. See Figure 2.1 for a typical geometry.

The incident fields, $\boldsymbol{E}_i$ and $\boldsymbol{H}_i$, are assumed to have their sources outside Ω in a bounded region Ω_i, i.e., $\Omega \cap \Omega_i = \emptyset$. It is assumed to be a fixed field throughout this paper. Outside this region the fields satisfy the time-harmonic Maxwell equations in vacuum time convention $e^{-i\omega t}$, i.e., they satisfy[1]

$$\begin{cases} \nabla \times \boldsymbol{E}_i(\boldsymbol{x}) = ik_0 \boldsymbol{H}_i(\boldsymbol{x}), \\ \nabla \times \boldsymbol{H}_i(\boldsymbol{x}) = -ik_0 \boldsymbol{E}_i(\boldsymbol{x}), \end{cases} \qquad \boldsymbol{x} \in \mathbb{R}^3.$$

The wave number in a vacuum is $k_0 = \omega/c_0$, where ω is the angular frequency of the fields, and c_0 is the speed of light in a vacuum. The incident fields $\boldsymbol{E}_i$ and $\boldsymbol{H}_i$ are assumed to have traces on $\partial\Omega$ belonging to $H^{-\frac{1}{2}}(\mathrm{div}, \partial\Omega)$, i.e., $(\hat{\boldsymbol{\nu}} \times \boldsymbol{E}_i, \hat{\boldsymbol{\nu}} \times \boldsymbol{H}_i) \in H^{-\frac{1}{2}}(\mathrm{div}, \partial\Omega) \times H^{-\frac{1}{2}}(\mathrm{div}, \partial\Omega)$; see Appendix A for definitions of the function spaces. Otherwise, the incident fields are arbitrary.

2.2. Interior problem. In Ω we assume there is a material modeled by the permittivity dyadic $\epsilon(\boldsymbol{x})$ and the permeability dyadic $\mu(\boldsymbol{x})$. The permittivity dyadic

[1] We use scaled electric and magnetic fields in this paper; i.e., the SI-unit fields $\boldsymbol{E}_{\mathrm{SI}}$ and $\boldsymbol{H}_{\mathrm{SI}}$ are related to the fields $\boldsymbol{E}$ and $\boldsymbol{H}$ used in this paper by

$$\boldsymbol{E}_{\mathrm{SI}}(\boldsymbol{x}) = \frac{\boldsymbol{E}(\boldsymbol{x})}{\sqrt{\epsilon_0}}, \qquad \boldsymbol{H}_{\mathrm{SI}}(\boldsymbol{x}) = \frac{\boldsymbol{H}(\boldsymbol{x})}{\sqrt{\mu_0}},$$

where the permittivity and permeability of vacuum are denoted ϵ_0 and μ_0, respectively.

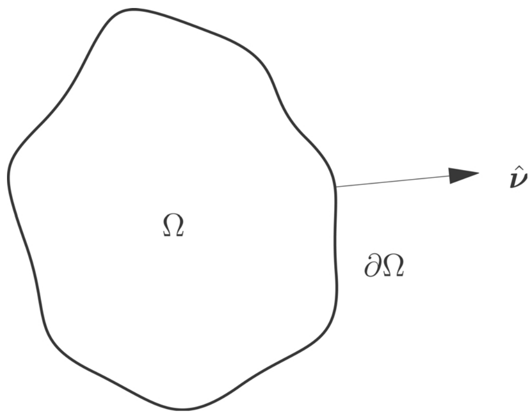

FIG. 2.1. *Typical geometry of the scattering problem in this paper.*

is assumed to satisfy

$$-ik_0\boldsymbol{\xi}\cdot\left(\epsilon(\boldsymbol{x})-\epsilon(\boldsymbol{x})^{\dagger}\right)\cdot\boldsymbol{\xi}^*\geq C_1|\boldsymbol{\xi}|^2\qquad\text{for all }\boldsymbol{\xi}\in\mathbb{C}^3\text{ and a.e. }\boldsymbol{x}\in\Omega \tag{2.1}$$

and

$$|\epsilon(\boldsymbol{x})\cdot\boldsymbol{\xi}|\leq C_2|\boldsymbol{\xi}|\qquad\text{for all }\boldsymbol{\xi}\in\mathbb{C}^3\text{ and a.e. }\boldsymbol{x}\in\Omega, \tag{2.2}$$

where † denotes the Hermitian of the dyadic ϵ and $C_i>0$, $i=1,2$. The condition in (2.1) corresponds physically to a passive material, i.e., a material that show dissipation. The entries of $\epsilon(\boldsymbol{x})$ are assumed to belong to $L^{\infty}(\Omega)$, which implies (2.2). Similar assumptions hold for the permeability μ. We note that it follows that ϵ and μ are invertible and that the inverses have the same kind of properties [9, p. 22].

In Ω the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{H}$ satisfy the Maxwell equations

$$\begin{cases}\nabla\times\boldsymbol{E}(\boldsymbol{x})=ik_0\mu(\boldsymbol{x})\cdot\boldsymbol{H}(\boldsymbol{x}),\\ \nabla\times\boldsymbol{H}(\boldsymbol{x})=-ik_0\epsilon(\boldsymbol{x})\cdot\boldsymbol{E}(\boldsymbol{x}),\end{cases}\qquad\boldsymbol{x}\in\Omega. \tag{2.3}$$

We are looking for solutions $\boldsymbol{E}$ and $\boldsymbol{H}$ of these equations in the space $H(\mathrm{rot},\Omega)$. A weak formulation of the solution to this problem is found in section 3.2.1.

2.3. Exterior problem. The presence of the material in the domain Ω distorts the incident fields $\boldsymbol{E}_i$ and $\boldsymbol{H}_i$. This distortion is denoted by the scattered fields, $\boldsymbol{E}_s$ and $\boldsymbol{H}_s$. They belong to $H_{\mathrm{loc}}(\mathrm{rot},\overline{\Omega}_e)$ and satisfy

$$\begin{cases}\nabla\times\boldsymbol{E}_s(\boldsymbol{x})=ik_0\boldsymbol{H}_s(\boldsymbol{x}),\\ \nabla\times\boldsymbol{H}_s(\boldsymbol{x})=-ik_0\boldsymbol{E}_s(\boldsymbol{x}),\end{cases}\qquad\boldsymbol{x}\in\Omega_e. \tag{2.4}$$

Moreover, the scattered fields satisfy the Silver–Müller radiation condition at infinity, i.e., one of the following conditions (see [11]):

$$\begin{cases}\hat{\boldsymbol{x}}\times\boldsymbol{E}_s(\boldsymbol{x})-\boldsymbol{H}_s(\boldsymbol{x})=o(1/x),\\ \hat{\boldsymbol{x}}\times\boldsymbol{H}_s(\boldsymbol{x})+\boldsymbol{E}_s(\boldsymbol{x})=o(1/x)\end{cases}\qquad\text{as }x\to\infty \tag{2.5}$$

uniformly in all directions $\hat{\boldsymbol{x}}$.

In Ω_e the sum of the incident and the scattered fields is defined as the total field, i.e.,

$$\begin{cases} \boldsymbol{E}_t(\boldsymbol{x}) = \boldsymbol{E}_i(\boldsymbol{x}) + \boldsymbol{E}_s(\boldsymbol{x}), \\ \boldsymbol{H}_t(\boldsymbol{x}) = \boldsymbol{H}_i(\boldsymbol{x}) + \boldsymbol{H}_s(\boldsymbol{x}), \end{cases} \quad \boldsymbol{x} \in \Omega_e.$$

The boundary conditions on $\partial\Omega$ are

$$\begin{cases} \hat{\boldsymbol{\nu}} \times \boldsymbol{E}_i|_{\partial\Omega} + \hat{\boldsymbol{\nu}} \times \boldsymbol{E}_s|_{\partial\Omega} = \hat{\boldsymbol{\nu}} \times \boldsymbol{E}|_{\partial\Omega}\,, \\ \hat{\boldsymbol{\nu}} \times \boldsymbol{H}_i|_{\partial\Omega} + \hat{\boldsymbol{\nu}} \times \boldsymbol{H}_s|_{\partial\Omega} = \hat{\boldsymbol{\nu}} \times \boldsymbol{H}|_{\partial\Omega}\,, \end{cases} \tag{2.6}$$

where the traces of the fields are taken from the outside (inside) in the left-hand (right-hand) side of the equations and belong to $H^{-\frac{1}{2}}(\mathrm{div}, \partial\Omega)$.

2.4. Calderón operators. The Calderón operator C^e utilizes the solution of a specific exterior problem. In fact, the following exterior problem, based upon (2.4) and (2.5) and given $\boldsymbol{m} \in H^{-\frac{1}{2}}(\mathrm{div}, \partial\Omega)$, is fundamental:

$$\begin{cases} (1) \quad (\boldsymbol{E}_s, \boldsymbol{H}_s) \in H_{\mathrm{loc}}(\mathrm{rot}, \overline{\Omega}_e) \times H_{\mathrm{loc}}(\mathrm{rot}, \overline{\Omega}_e), \\ (2) \quad \begin{cases} \nabla \times \boldsymbol{E}_s(\boldsymbol{x}) = ik_0 \boldsymbol{H}_s(\boldsymbol{x}), \\ \nabla \times \boldsymbol{H}_s(\boldsymbol{x}) = -ik_0 \boldsymbol{E}_s(\boldsymbol{x}), \end{cases} \quad \boldsymbol{x} \in \Omega_e, \\ (3) \quad \begin{cases} \hat{\boldsymbol{x}} \times \boldsymbol{E}_s(\boldsymbol{x}) - \boldsymbol{H}_s(\boldsymbol{x}) = o(1/x) \\ \text{or} \\ \hat{\boldsymbol{x}} \times \boldsymbol{H}_s(\boldsymbol{x}) + \boldsymbol{E}_s(\boldsymbol{x}) = o(1/x) \end{cases} \quad \text{as } x \to \infty, \\ (4) \quad \hat{\boldsymbol{\nu}} \times \boldsymbol{E}_s|_{\partial\Omega} = \boldsymbol{m} \in H^{-\frac{1}{2}}(\mathrm{div}, \partial\Omega). \end{cases} \quad \text{(Problem (R))} \tag{2.7}$$

This problem has a unique solution [4, 9]; see also section 3.1.

We have the following results proved in [9, p. 35].

THEOREM 2.1. *With the boundary $\partial\Omega$ of regularity $C^{1,1}$, the mapping*

$$\boldsymbol{\gamma}_\tau : \boldsymbol{u} \to \hat{\boldsymbol{\nu}} \times \boldsymbol{u}|_{\partial\Omega}$$

is a continuous mapping from $H_{loc}(\mathrm{rot}, \overline{\Omega}_e)$ onto $H^{-\frac{1}{2}}(\mathrm{div}, \partial\Omega)$.

The trace theorem is a local property of the field at the boundary, and the theorem shows that the field loses regularity on the boundary. We note that a similar result holds when the trace is taken from the inside of the boundary; see section 3.2.

The linear mapping of the electric field to the corresponding magnetic field on the boundary for a solution of the exterior problem is called the exterior Calderón operator. The following makes this definition precise.

DEFINITION 2.2. *The exterior Calderón operator C^e is defined as*

$$C^e : \boldsymbol{m} \to \hat{\boldsymbol{\nu}} \times \boldsymbol{H}_s|_{\partial\Omega}\,, \qquad H^{-\frac{1}{2}}(\mathrm{div}, \partial\Omega) \to H^{-\frac{1}{2}}(\mathrm{div}, \partial\Omega),$$

where $\boldsymbol{m} = \hat{\boldsymbol{\nu}} \times \boldsymbol{E}_s|_{\partial\Omega}$ and the fields $\boldsymbol{E}_s$ and $\boldsymbol{H}_s$ satisfy Problem (R) *in* (2.7).

Notice that the exterior Calderón operator C^e is uniquely defined for all $\boldsymbol{m} \in H^{-\frac{1}{2}}(\mathrm{div}, \partial\Omega)$, since Problem (R) has a unique solution. Two explicit examples of the exterior Calderón operator are given in section 5.

THEOREM 2.3. *The exterior Calderón operator defined in Definition* 2.2 *has the following properties:*

1. *The exterior Calderón operator satisfies the positivity condition*

$$\Re \iint_{\partial\Omega} C^e(\boldsymbol{m}) \cdot (\hat{\boldsymbol{\nu}} \times \boldsymbol{m}^*) \, dS \geq 0 \qquad \text{for all } \boldsymbol{m} \in H^{-\frac{1}{2}}(\mathrm{div}, \partial\Omega). \tag{2.8}$$

2. *The exterior Calderón operator satisfies*

$$(C^e)^2 = -\mathbf{I} \text{ on } H^{-\frac{1}{2}}(\mathrm{div}, \partial\Omega),$$

which implies that C^e is bounded on $H^{-\frac{1}{2}}(\mathrm{div}, \partial\Omega)$.

3. *The exterior Calderón operator is independent of the material properties inside Ω.*

Here dS denotes the surface measure of $\partial\Omega$.

Proof of Theorem 2.3. Property 1 is a simple consequence of the radiation condition and proved in, e.g., [9]. Specifically, the radiation conditions, (2.5), imply

$$\Re \iint_{\partial\Omega} \hat{\boldsymbol{\nu}} \cdot (\boldsymbol{E}_s \times \boldsymbol{H}_s^*) \, dS = \Re \iint_{|\boldsymbol{x}|=R} \hat{\boldsymbol{x}} \cdot (\boldsymbol{E}_s \times \boldsymbol{H}_s^*) \, dS = \iint_{|\boldsymbol{x}|=R} |\boldsymbol{E}_s|^2 \, dS + o(1)$$

as $R \to \infty$, which implies (2.8), since $\hat{\boldsymbol{\nu}} \cdot (\boldsymbol{E}_s^* \times \boldsymbol{H}_s) = -C^e(\hat{\boldsymbol{\nu}} \times \boldsymbol{E}_s) \cdot \boldsymbol{E}_s^*$.

Moreover, to prove property 2 we utilize the symmetry $\{\boldsymbol{E}_s, \boldsymbol{H}_s\} \to \{\boldsymbol{H}_s, -\boldsymbol{E}_s\}$ in (2.4) and the uniqueness of the exterior problem.

Property 3 is a consequence of the uniqueness of the exterior problem. □

An immediate consequence of the positivity property of C^e is that

$$-\Re \iint_{\partial\Omega} C^e(\hat{\boldsymbol{\nu}} \times \boldsymbol{E}_s) \cdot \boldsymbol{E}_s^* \, dS \geq 0 \qquad \text{for all } \hat{\boldsymbol{\nu}} \times \boldsymbol{E}_s \in H^{-\frac{1}{2}}(\mathrm{div}, \partial\Omega). \tag{2.9}$$

3. Existence of solutions. The existence of exterior and interior solutions is addressed in this section.

3.1. Exterior problem. The system (2.4) with the radiation condition (2.5) supplied with the boundary condition

$$\hat{\boldsymbol{\nu}} \times \boldsymbol{E}_s|_{\partial\Omega} = \boldsymbol{m} \ \in H^{-\frac{1}{2}}(\mathrm{div}, \partial\Omega),$$

i.e., Problem (R) in (2.7), has a unique solution in $(\boldsymbol{E}_s, \boldsymbol{H}_s) \in H_{\mathrm{loc}}(\mathrm{rot}, \overline{\Omega}_e) \times H_{\mathrm{loc}}(\mathrm{rot}, \overline{\Omega}_e)$ for any $\boldsymbol{m} \in H^{-\frac{1}{2}}(\mathrm{div}, \partial\Omega)$ [9, p. 107].

3.2. Interior problem. We have the interior trace result, similar to Theorem 2.1.

THEOREM 3.1. *With the boundary $\partial\Omega$ of regularity $C^{1,1}$, the mapping*

$$\boldsymbol{\gamma}_\tau : \boldsymbol{u} \to \hat{\boldsymbol{\nu}} \times \boldsymbol{u}|_{\partial\Omega}$$

is a continuous mapping from $H(\mathrm{rot}, \Omega)$ onto $H^{-\frac{1}{2}}(\mathrm{div}, \partial\Omega)$.

3.2.1. Sesquilinear form and weak solutions. Using Theorem 3.1, we can now define the sesquilinear form (see [9])

$$a(\boldsymbol{u}, \boldsymbol{v}) = -\iiint_{\Omega} \left\{ \frac{1}{ik_0} (\nabla \times \boldsymbol{v}^*) \cdot \mu^{-1} \cdot (\nabla \times \boldsymbol{u}) + ik_0 \boldsymbol{v}^* \cdot \epsilon \cdot \boldsymbol{u} \right\} dv$$
$$- \iint_{\partial\Omega} C^e(\hat{\boldsymbol{\nu}} \times \boldsymbol{u}) \cdot \boldsymbol{v}^* \, dS$$

for $\boldsymbol{u}$ and $\boldsymbol{v}$ in $H(\mathrm{rot}, \Omega)$. We denote the volume measure in $\mathbb{R}^3$ by dv in this paper.

A weak formulation of the original problem is then to find $\boldsymbol{E} \in H(\mathrm{rot}, \Omega)$ such that

$$a(\boldsymbol{E}, \boldsymbol{v}) = \iint_{\partial\Omega} (\hat{\boldsymbol{\nu}} \times \boldsymbol{H}_i - C^e(\hat{\boldsymbol{\nu}} \times \boldsymbol{E}_i)) \cdot \boldsymbol{v}^* \, dS \quad \text{for all } \boldsymbol{v} \in H(\mathrm{rot}, \Omega). \tag{3.1}$$

This solution satisfies the boundary conditions, (2.6), and couples to the exterior solution in (2.4)–(2.5). The corresponding magnetic field $\boldsymbol{H}$ is then constructed as[2]

$$\begin{cases} \boldsymbol{H}(\boldsymbol{x}) = -\frac{i}{k_0}\mu^{-1}(\boldsymbol{x}) \cdot (\nabla \times \boldsymbol{E}(\boldsymbol{x})), \\ \nabla \times \boldsymbol{H}(\boldsymbol{x}) = -ik_0\epsilon(\boldsymbol{x}) \cdot \boldsymbol{E}(\boldsymbol{x}), \end{cases} \qquad \boldsymbol{x} \in \Omega.$$

To see this, let $\boldsymbol{E}$ be a sufficiently regular solution to the Maxwell equations, (2.3). Then (3.1) is equivalent to the Maxwell equations with a coupling to an exterior solution since

$$\begin{aligned} a(\boldsymbol{E}, \boldsymbol{v}) &= -\iiint_{\Omega} \{(\nabla \times \boldsymbol{v}^*) \cdot \boldsymbol{H} - \boldsymbol{v}^* \cdot (\nabla \times \boldsymbol{H})\} \, dv - \iint_{\partial\Omega} C^e(\hat{\boldsymbol{\nu}} \times \boldsymbol{E}) \cdot \boldsymbol{v}^* \, dS \\ &= \iint_{\partial\Omega} \{(\hat{\boldsymbol{\nu}} \times \boldsymbol{H}) \cdot \boldsymbol{v}^* - C^e(\hat{\boldsymbol{\nu}} \times \boldsymbol{E}) \cdot \boldsymbol{v}^*\} \, dS, \end{aligned}$$

which is identical to (3.1) by the use of the boundary conditions on $\partial\Omega$ and by the definition

$$\iint_{\partial\Omega} C^e(\hat{\boldsymbol{\nu}} \times \boldsymbol{E}_s) \cdot \boldsymbol{v}^* \, dS = \iint_{\partial\Omega} (\hat{\boldsymbol{\nu}} \times \boldsymbol{H}_s) \cdot \boldsymbol{v}^* \, dS.$$

Moreover, the sesquilinear form a is coercive, i.e.,

$$\begin{aligned} \Re a(\boldsymbol{u}, \boldsymbol{u}) =& -\iiint_{\Omega} \frac{1}{ik_0} (\nabla \times \boldsymbol{u}^*) \cdot \left(\mu^{-1} - {\mu^{-1}}^{\dagger}\right) \cdot (\nabla \times \boldsymbol{u}) \, dv \\ & - \iiint_{\Omega} ik_0 \boldsymbol{u}^* \cdot \left(\epsilon - \epsilon^{\dagger}\right) \cdot \boldsymbol{u} \, dv \\ & - \Re \iint_{\partial\Omega} C^e(\hat{\boldsymbol{\nu}} \times \boldsymbol{u}) \cdot \boldsymbol{u}^* \, dS \geq C\|\boldsymbol{u}\|^2_{H(\mathrm{rot},\Omega)}, \end{aligned} \tag{3.2}$$

since from (2.1) we get[3]

$$\begin{cases} -ik_0\boldsymbol{\xi} \cdot \left(\epsilon(\boldsymbol{x}) - \epsilon(\boldsymbol{x})^{\dagger}\right) \cdot \boldsymbol{\xi}^* \geq C_1|\boldsymbol{\xi}|^2, \\ \frac{i}{k_0}\boldsymbol{\xi} \cdot \left(\mu^{-1}(\boldsymbol{x}) - {\mu^{-1}}^{\dagger}(\boldsymbol{x})\right) \cdot \boldsymbol{\xi}^* \geq C_2|\boldsymbol{\xi}|^2 \end{cases} \qquad \text{for all } \boldsymbol{\xi} \in \mathbb{C}^3 \text{ and a.e. } \boldsymbol{x} \in \Omega,$$

and we have also used (2.9).

---

[2]This construction is consistent since $-ik_0\epsilon(\boldsymbol{x}) \cdot \boldsymbol{E}(\boldsymbol{x})$ is the weak curl of $\boldsymbol{H}(\boldsymbol{x}) = -\frac{i}{k_0}\mu^{-1}(\boldsymbol{x}) \cdot (\nabla \times \boldsymbol{E}(\boldsymbol{x}))$. In fact, we have

$$(\boldsymbol{H}, \nabla \times \boldsymbol{\phi}) + ik_0 (\epsilon \cdot \boldsymbol{E}, \boldsymbol{\phi}) = 0 \qquad \text{for all } \boldsymbol{\phi} \in D(\Omega; \mathbb{C}^3)$$

since $a(\boldsymbol{E}, \boldsymbol{\phi}) = 0$ for all $\boldsymbol{\phi} \in D(\Omega; \mathbb{C}^3)$.

[3]With (2.1) we get

$$\frac{i}{k_0} (\mu^t \cdot \boldsymbol{\zeta}) \cdot \left(\mu^{-1} - {\mu^{-1}}^{\dagger}\right) \cdot (\mu^t \cdot \boldsymbol{\zeta})^* = \frac{i}{k_0} \boldsymbol{\zeta} \cdot \left(\mu^{\dagger} - \mu\right) \cdot \boldsymbol{\zeta}^* \geq \frac{C_2}{k_0^2}|\boldsymbol{\zeta}|^2.$$

Applying this result with $\boldsymbol{\zeta} = {\mu^t}^{-1} \cdot \boldsymbol{\xi}$, we get

$$\frac{i}{k_0}\boldsymbol{\xi} \cdot \left(\mu^{-1} - {\mu^{-1}}^{\dagger}\right) \cdot \boldsymbol{\xi}^* \geq \frac{C_2}{k_0^2}|{\mu^t}^{-1} \cdot \boldsymbol{\xi}|^2 \geq C|\boldsymbol{\xi}|^2$$

since μ is invertible.

3.2.2. Existence of a unique solution. Equation (3.1) has a unique solution in $H(\mathrm{rot},\Omega)$ due to the Lax–Milgram theorem (see Theorem A.1), since the sesquilinear form $a(\boldsymbol{u},\boldsymbol{v})$ is continuous, bounded, and coercive, and the right-hand side of (3.1) is continuous on $H(\mathrm{rot},\Omega)$. In fact,

$$\begin{aligned}
&\left|\iint_{\partial\Omega} (\hat{\boldsymbol{\nu}}\times\boldsymbol{H}_i - C^e(\hat{\boldsymbol{\nu}}\times\boldsymbol{E}_i))\cdot\boldsymbol{v}^*\,dS\right| \\
&\quad\le \left(\|\hat{\boldsymbol{\nu}}\times\boldsymbol{H}_i\|_{H^{-\frac{1}{2}}(\mathrm{div},\partial\Omega)} + \|C^e(\hat{\boldsymbol{\nu}}\times\boldsymbol{E}_i)\|_{H^{-\frac{1}{2}}(\mathrm{div},\partial\Omega)}\right)\|\boldsymbol{v}\|_{H^{-\frac{1}{2}}(\mathrm{rot},\partial\Omega)} \\
&\quad\le C'\left(\|\hat{\boldsymbol{\nu}}\times\boldsymbol{H}_i\|_{H^{-\frac{1}{2}}(\mathrm{div},\partial\Omega)} + \|(\hat{\boldsymbol{\nu}}\times\boldsymbol{E}_i)\|_{H^{-\frac{1}{2}}(\mathrm{div},\partial\Omega)}\right)\|\boldsymbol{v}\|_{H(\mathrm{rot},\Omega)}
\end{aligned}$$

by Minkowski's inequality, duality [9, p. 38], and the continuous dependence of the trace norm on the norm of the corresponding function space.

4. Homogenization. So far we have considered a general heterogeneous scattering problem with a unique solution in $H(\mathrm{rot},\Omega)$ for a given incident electromagnetic field. But if the heterogeneous material in Ω has a typical spatial scale which is much smaller than the size of the domain, then one runs into severe numerical problems if one tries to apply some standard numerical code, e.g., a finite element method (FEM). The principal obstacle is that the fine scale requires a very fine numerical mesh which generates a far too large linear system of equations for any computer to solve. However, if the wavelength of the incident field is much larger than the fine scale, then the field cannot resolve the fine scale and the solution of the Maxwell equations can be approximated by the solution of a scattering problem with constant coefficients; i.e., the heterogeneous material in Ω has been replaced by a homogeneous material with the same effective material properties. The procedure for finding these effective properties of the heterogeneous material is called homogenization.

4.1. Heterogeneous problem. Let us begin with the definition of a Y-cell which is the open unit cube in $\mathbb{R}^3$, i.e., $Y =]0,1[^3$. Further, from now on we assume that ϵ and μ are Y-periodic, which is defined as $\epsilon(\boldsymbol{x}+\hat{\boldsymbol{e}}_k) = \epsilon(\boldsymbol{x})$ for every $k = 1,2,3$, where $\hat{\boldsymbol{e}}_k$, $k = 1,2,3$, is the canonical basis in $\mathbb{R}^3$.

In the following, we assume that the material in the domain Ω is periodic with period ε in the three Cartesian coordinate directions, i.e., it is the union of a collection of disjoint, open identical cubes[4] with side length ε (Y^ε-cells); see Figure 4.1. It is easily verified that the scaled permeability and permittivity, $\epsilon(\boldsymbol{x}/\varepsilon)$ and $\mu(\boldsymbol{x}/\varepsilon)$, are periodic with period ε.

In Ω the fields satisfy the source-free Maxwell equations[5]

$$\begin{cases}
\nabla\times\boldsymbol{E}^\varepsilon(\boldsymbol{x}) = ik_0\boldsymbol{B}^\varepsilon(\boldsymbol{x}),\\
\nabla\times\boldsymbol{H}^\varepsilon(\boldsymbol{x}) = -ik_0\boldsymbol{D}^\varepsilon(\boldsymbol{x}),\\
\nabla\cdot\boldsymbol{B}^\varepsilon(\boldsymbol{x}) = 0,\\
\nabla\cdot\boldsymbol{D}^\varepsilon(\boldsymbol{x}) = 0,
\end{cases}\qquad \boldsymbol{x}\in\Omega,$$

[4]More generally, $Y = (0,a_1)\times(0,a_2)\times(0,a_3)$, where $a_i > 0$, $i = 1,2,3$, and $\epsilon(\boldsymbol{x}+a_k\hat{\boldsymbol{e}}_k) = \epsilon(\boldsymbol{x})$ for every $\boldsymbol{x}\in\mathbb{R}^3$ and for every $k = 1,2,3$. A similar result holds for the permeability μ.

[5]The electric and magnetic fields are scaled as above (see footnote 1), and the SI-unit flux densities $\boldsymbol{D}_{\mathrm{SI}}$ and $\boldsymbol{B}_{\mathrm{SI}}$ are related to the fields $\boldsymbol{D}$ and $\boldsymbol{B}$ used in this paper by

$$\boldsymbol{D}_{\mathrm{SI}}(\boldsymbol{x}) = \sqrt{\epsilon_0}\boldsymbol{D}(\boldsymbol{x}),\qquad \boldsymbol{B}_{\mathrm{SI}}(\boldsymbol{x}) = \sqrt{\mu_0}\boldsymbol{B}(\boldsymbol{x}).$$

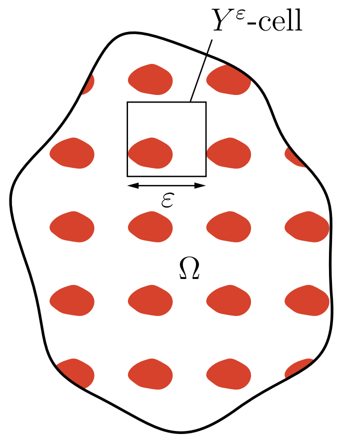

FIG. 4.1. *Typical periodic geometry of the material parameters.*

almost everywhere, with boundary conditions given by (2.6). By using the constitutive relations for the periodic material,

$$\begin{cases} \boldsymbol{D}^\varepsilon(\boldsymbol{x}) = \epsilon(\boldsymbol{x}/\varepsilon) \cdot \boldsymbol{E}^\varepsilon(\boldsymbol{x}), \\ \boldsymbol{B}^\varepsilon(\boldsymbol{x}) = \mu(\boldsymbol{x}/\varepsilon) \cdot \boldsymbol{H}^\varepsilon(\boldsymbol{x}), \end{cases} \qquad \boldsymbol{x} \in \Omega,$$

we eliminate $\boldsymbol{D}^\varepsilon$, $\boldsymbol{B}^\varepsilon$ and obtain

$$\begin{cases} \nabla \times \boldsymbol{E}^\varepsilon(\boldsymbol{x}) = ik_0\mu(\boldsymbol{x}/\varepsilon) \cdot \boldsymbol{H}^\varepsilon(\boldsymbol{x}), \\ \nabla \times \boldsymbol{H}^\varepsilon(\boldsymbol{x}) = -ik_0\epsilon(\boldsymbol{x}/\varepsilon) \cdot \boldsymbol{E}^\varepsilon(\boldsymbol{x}), \\ \nabla \cdot \{\epsilon(\boldsymbol{x}/\varepsilon) \cdot \boldsymbol{E}^\varepsilon(\boldsymbol{x})\} = 0, \\ \nabla \cdot \{\mu(\boldsymbol{x}/\varepsilon) \cdot \boldsymbol{H}^\varepsilon(\boldsymbol{x})\} = 0, \end{cases} \qquad \boldsymbol{x} \in \Omega, \tag{4.1}$$

where the solution $(\boldsymbol{E}^\varepsilon, \boldsymbol{H}^\varepsilon)$ is in $H(\mathrm{rot}, \Omega) \times H(\mathrm{rot}, \Omega)$ and belongs to a family of solutions, one for each ε. In the homogenization procedure we identify the limit of the fields $\boldsymbol{E}^\varepsilon, \boldsymbol{H}^\varepsilon$ when $\varepsilon \to 0$. This limit satisfies the homogenized system with constant coefficients, which is a model of a homogeneous material.

4.1.1. A priori estimate. We note that the heterogeneous system in (4.1) is of the same form as (2.3) and that the constitutive relations satisfy the same assumptions as in section 2.2. A weak formulation of the two first equations in (4.1) supplied with boundary conditions (2.6) reads

$$a^\varepsilon(\boldsymbol{E}^\varepsilon, \boldsymbol{v}) = \iint_{\partial\Omega} (\hat{\boldsymbol{\nu}} \times \boldsymbol{H}_i - C^e(\hat{\boldsymbol{\nu}} \times \boldsymbol{E}_i)) \cdot \boldsymbol{v}^* \, dS \quad \text{for all } \boldsymbol{v} \in H(\mathrm{rot}, \Omega), \tag{4.2}$$

where

$$\begin{aligned} a^\varepsilon(\boldsymbol{u}, \boldsymbol{v}) = -\iiint_\Omega \bigg\{ & \frac{1}{ik_0} (\nabla \times \boldsymbol{v}^*) \cdot \mu^{-1}(\boldsymbol{x}/\varepsilon) \cdot (\nabla \times \boldsymbol{u}) \\ & + ik_0 \boldsymbol{v}^* \cdot \epsilon(\boldsymbol{x}/\varepsilon) \cdot \boldsymbol{u} \bigg\} \, dv - \iint_{\partial\Omega} C^e(\hat{\boldsymbol{\nu}} \times \boldsymbol{u}) \cdot \boldsymbol{v}^* \, dS. \end{aligned} \tag{4.3}$$

We have the following a priori estimate.

THEOREM 4.1. *Let* $\boldsymbol{E}^\varepsilon, \boldsymbol{H}^\varepsilon$ *be a solution of* (4.2)*; then*

$$\|\boldsymbol{E}^\varepsilon\|_{H(\mathrm{rot},\Omega)} + \|\boldsymbol{H}^\varepsilon\|_{H(\mathrm{rot},\Omega)} \le C,$$

where the constant C *depends only on the domain* Ω*, the material parameters in* Ω*, and the strength of the incident field.*

Proof of Theorem 4.1. The sesquilinear form $a^\varepsilon(\boldsymbol{u},\boldsymbol{v})$ is coercive (cf. (3.2)), and the weak formulation (4.2) gives

$$\begin{aligned} C\|\boldsymbol{E}^\varepsilon\|^2_{H(\mathrm{rot},\Omega)} &\le \Re a^\varepsilon(\boldsymbol{E}^\varepsilon,\boldsymbol{E}^\varepsilon) \le |a^\varepsilon(\boldsymbol{E}^\varepsilon,\boldsymbol{E}^\varepsilon)| \\ &= \left| \iint_{\partial\Omega} (\hat{\boldsymbol{\nu}} \times \boldsymbol{H}_i - C^e(\hat{\boldsymbol{\nu}} \times \boldsymbol{E}_i)) \cdot (\boldsymbol{E}^\varepsilon)^* \, dS \right| \\ &\le \left(\|\hat{\boldsymbol{\nu}} \times \boldsymbol{H}_i\|_{H^{-\frac{1}{2}}(\mathrm{div},\partial\Omega)} + \|C^e(\hat{\boldsymbol{\nu}} \times \boldsymbol{E}_i)\|_{H^{-\frac{1}{2}}(\mathrm{div},\partial\Omega)} \right) \|\boldsymbol{E}^\varepsilon\|_{H^{-\frac{1}{2}}(\mathrm{rot},\partial\Omega)} \\ &\le C' \left(\|\hat{\boldsymbol{\nu}} \times \boldsymbol{H}_i\|_{H^{-\frac{1}{2}}(\mathrm{div},\partial\Omega)} + \|(\hat{\boldsymbol{\nu}} \times \boldsymbol{E}_i)\|_{H^{-\frac{1}{2}}(\mathrm{div},\partial\Omega)} \right) \|\boldsymbol{E}^\varepsilon\|_{H(\mathrm{rot},\Omega)} \end{aligned}$$

by Minkowski's inequality, duality [9, p. 38], and the continuous dependence of the trace norm on the norm of the corresponding function space. It follows now that

$$\|\boldsymbol{E}^\varepsilon\|_{H(\mathrm{rot},\Omega)} \le C' \left(\|\hat{\boldsymbol{\nu}} \times \boldsymbol{H}_i\|_{H^{-\frac{1}{2}}(\mathrm{div},\partial\Omega)} + \|(\hat{\boldsymbol{\nu}} \times \boldsymbol{E}_i)\|_{H^{-\frac{1}{2}}(\mathrm{div},\partial\Omega)} \right) \le C$$

by the assumption of the incident field. The bound of $\boldsymbol{E}^\varepsilon$ can now be used in (4.1) to get the estimate of $\boldsymbol{H}^\varepsilon$. ∎

4.2. Homogenized problem.

THEOREM 4.2. *The sequence of solutions* $(\boldsymbol{E}^\varepsilon, \boldsymbol{H}^\varepsilon)$ *of* (4.1) *converges weakly in* $H(\mathrm{rot},\Omega) \times H(\mathrm{rot},\Omega)$ *to* $(\boldsymbol{E},\boldsymbol{H}) \in H(\mathrm{rot},\Omega) \times H(\mathrm{rot},\Omega)$*, the unique solution of the homogenized Maxwell equations*

$$\left\{ \begin{array}{l} \nabla \times \boldsymbol{E}(\boldsymbol{x}) = ik_0 \mu^h \cdot \boldsymbol{H}(\boldsymbol{x}), \\ \nabla \times \boldsymbol{H}(\boldsymbol{x}) = -ik_0 \epsilon^h \cdot \boldsymbol{E}(\boldsymbol{x}), \\ \nabla \cdot \boldsymbol{B}(\boldsymbol{x}) = 0, \\ \nabla \cdot \boldsymbol{D}(\boldsymbol{x}) = 0, \end{array} \right. \tag{4.4}$$

which is coupled to the exterior problem (2.4)–(2.5) *via the boundary conditions* (2.6)*. The homogenized permeability and permittivity* ϵ^h *and* μ^h *are defined by*

$$\left\{ \begin{array}{l} \epsilon^h = \displaystyle\iiint_Y \epsilon(\boldsymbol{y}) \cdot (\mathbf{I}_3 - \nabla_{\boldsymbol{y}} \boldsymbol{\chi}_e(\boldsymbol{y})) \, dv_{\boldsymbol{y}}, \\ \mu^h = \displaystyle\iiint_Y \mu(\boldsymbol{y}) \cdot (\mathbf{I}_3 - \nabla_{\boldsymbol{y}} \boldsymbol{\chi}_h(\boldsymbol{y})) \, dv_{\boldsymbol{y}}, \end{array} \right.$$

$$\boldsymbol{\chi}_e(\boldsymbol{y}) = \sum_{i=1}^{3} \chi_e^i(\boldsymbol{y}) \hat{\boldsymbol{e}}_i, \qquad \boldsymbol{\chi}_h(\boldsymbol{y}) = \sum_{i=1}^{3} \chi_h^i(\boldsymbol{y}) \hat{\boldsymbol{e}}_i, \tag{4.5}$$

where $\chi_e^i(\boldsymbol{y})$ *and* $\chi_h^i(\boldsymbol{y})$*,* $i = 1, 2, 3$*, in* $H^1_\#(Y)/\mathbb{C}$ *solve the local elliptic problems*

$$\left\{ \begin{array}{l} \displaystyle\iiint_Y \nabla_{\boldsymbol{y}} w(\boldsymbol{y}) \cdot \epsilon(\boldsymbol{y}) \cdot \left(\hat{\boldsymbol{e}}_i - \nabla_{\boldsymbol{y}} \chi_e^i(\boldsymbol{y}) \right) dv_{\boldsymbol{y}} = 0, \\ \displaystyle\iiint_Y \nabla_{\boldsymbol{y}} w(\boldsymbol{y}) \cdot \mu(\boldsymbol{y}) \cdot \left(\hat{\boldsymbol{e}}_i - \nabla_{\boldsymbol{y}} \chi_h^i(\boldsymbol{y}) \right) dv_{\boldsymbol{y}} = 0 \end{array} \right. \tag{4.6}$$

for all $w \in H^1_\#(Y)$*.*

We note that the weak convergence is sharp in the sense that it never converges strongly in $H(\mathrm{rot}, \Omega)$ except in the electrostatic case (see the note after Theorem B.3). However, we can get strong convergence by the use of corrector functions; see section 4.2.2. These functions contain the fine-scale information in the problem and yield strong convergence when scaled and added to the homogenized solution.

Proof of Theorem 4.2. We use the concept of two-scale convergence; see Appendix B. Due to the a priori estimates there exists a subsequence which converges in the two-scale sense. We will keep the index ε for this subsequence. In the end we conclude that the whole original sequence converges due to the fact that the homogenized system has a unique solution. Let $\boldsymbol{\phi}(\boldsymbol{x}) = \varepsilon w(\boldsymbol{x}/\varepsilon)\boldsymbol{v}(\boldsymbol{x})$, where $w \in H^1_\#(Y)$ and $\boldsymbol{v} \in C_0^\infty(\Omega; \mathbb{C}^3)$. Then $\boldsymbol{\phi} \in H(\mathrm{rot}, \Omega)$ and is an admissible test function. We get in (4.1)

$$\begin{cases} \iiint_\Omega \boldsymbol{E}^\varepsilon(\boldsymbol{x}) \cdot \{\varepsilon w(\boldsymbol{x}/\varepsilon)\nabla_{\boldsymbol{x}} \times \boldsymbol{v}(\boldsymbol{x}) + \nabla_{\boldsymbol{y}} w(\boldsymbol{x}/\varepsilon) \times \boldsymbol{v}(\boldsymbol{x})\}\, dv \\ \qquad - ik_0 \iiint_\Omega \varepsilon w(\boldsymbol{x}/\varepsilon)\boldsymbol{v}(\boldsymbol{x}) \cdot \{\mu(\boldsymbol{x}/\varepsilon) \cdot \boldsymbol{H}^\varepsilon(\boldsymbol{x})\}\, dv = 0, \\ \iiint_\Omega \boldsymbol{H}^\varepsilon(\boldsymbol{x}) \cdot \{\varepsilon w(\boldsymbol{x}/\varepsilon)\nabla_{\boldsymbol{x}} \times \boldsymbol{v}(\boldsymbol{x}) + \nabla_{\boldsymbol{y}} w(\boldsymbol{x}/\varepsilon) \times \boldsymbol{v}(\boldsymbol{x})\}\, dv \\ \qquad + ik_0 \iiint_\Omega \varepsilon w(\boldsymbol{x}/\varepsilon)\boldsymbol{v}(\boldsymbol{x}) \cdot \{\epsilon(\boldsymbol{x}/\varepsilon) \cdot \boldsymbol{E}^\varepsilon(\boldsymbol{x})\}\, dv = 0. \end{cases}$$

In the limit $\varepsilon \searrow 0$ we get

$$\begin{cases} \iiint_\Omega \boldsymbol{E}^\varepsilon(\boldsymbol{x}) \cdot (\nabla_{\boldsymbol{y}} w(\boldsymbol{x}/\varepsilon) \times \boldsymbol{v}(\boldsymbol{x}))\, dv \to 0, \\ \iiint_\Omega \boldsymbol{H}^\varepsilon(\boldsymbol{x}) \cdot (\nabla_{\boldsymbol{y}} w(\boldsymbol{x}/\varepsilon) \times \boldsymbol{v}(\boldsymbol{x}))\, dv \to 0, \end{cases}$$

since $\boldsymbol{E}^\varepsilon$ and $\boldsymbol{H}^\varepsilon$ are uniformly bounded in ε in the $L^2(\Omega; \mathbb{C}^3)$-norm. By the use of Theorem B.6, we get

$$\begin{cases} \iiint_\Omega \iiint_Y \boldsymbol{E}_0(\boldsymbol{x}, \boldsymbol{y}) \cdot (\nabla_{\boldsymbol{y}} w(\boldsymbol{y}) \times \boldsymbol{v}(\boldsymbol{x}))\, dv_{\boldsymbol{y}}\, dv_{\boldsymbol{x}} = 0, \\ \iiint_\Omega \iiint_Y \boldsymbol{H}_0(\boldsymbol{x}, \boldsymbol{y}) \cdot (\nabla_{\boldsymbol{y}} w(\boldsymbol{y}) \times \boldsymbol{v}(\boldsymbol{x}))\, dv_{\boldsymbol{y}}\, dv_{\boldsymbol{x}} = 0, \end{cases}$$

which implies after cyclic permutation that

$$\begin{cases} \iiint_Y \boldsymbol{E}_0(\boldsymbol{x}, \boldsymbol{y}) \times \nabla_{\boldsymbol{y}} w(\boldsymbol{y})\, dv_{\boldsymbol{y}} = \boldsymbol{0}, \\ \iiint_Y \boldsymbol{H}_0(\boldsymbol{x}, \boldsymbol{y}) \times \nabla_{\boldsymbol{y}} w(\boldsymbol{y})\, dv_{\boldsymbol{y}} = \boldsymbol{0}, \end{cases} \qquad \boldsymbol{x} \in \Omega \text{ a.e.}$$

for all $w \in H^1_\#(Y)$. The functions $\boldsymbol{E}_0(\boldsymbol{x}, \boldsymbol{y})$ and $\boldsymbol{H}_0(\boldsymbol{x}, \boldsymbol{y})$ both belong to the space $L^2(\Omega; L^2_\#(Y; \mathbb{C}^3))$. From Lemma B.5 we conclude that the fields $\boldsymbol{E}_0(\boldsymbol{x}, \boldsymbol{y})$ and $\boldsymbol{H}_0(\boldsymbol{x}, \boldsymbol{y})$ can be decomposed as

$$\begin{cases} \boldsymbol{E}_0(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{E}(\boldsymbol{x}) + \nabla_{\boldsymbol{y}} \Phi_1(\boldsymbol{x}, \boldsymbol{y}), \\ \boldsymbol{H}_0(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{H}(\boldsymbol{x}) + \nabla_{\boldsymbol{y}} \Psi_1(\boldsymbol{x}, \boldsymbol{y}), \end{cases}$$

where

$$\boldsymbol{E}(\boldsymbol{x}) = \langle \boldsymbol{E}_0(\boldsymbol{x}, \boldsymbol{y}) \rangle = \iiint_Y \boldsymbol{E}_0(\boldsymbol{x}, \boldsymbol{y})\, dv_{\boldsymbol{y}}$$

and similarly for the field $\boldsymbol{H}_0(\boldsymbol{x},\boldsymbol{y})$. In summary,

$$\begin{cases} \boldsymbol{E}^\varepsilon(\boldsymbol{x}) \overset{\text{2-s}}{\rightharpoonup} \boldsymbol{E}(\boldsymbol{x}) + \nabla_{\boldsymbol{y}}\Phi_1(\boldsymbol{x},\boldsymbol{y}), \\ \boldsymbol{H}^\varepsilon(\boldsymbol{x}) \overset{\text{2-s}}{\rightharpoonup} \boldsymbol{H}(\boldsymbol{x}) + \nabla_{\boldsymbol{y}}\Psi_1(\boldsymbol{x},\boldsymbol{y}). \end{cases}$$

Multiplication of (4.1) by the admissible test functions $\boldsymbol{\phi} \in C_0^\infty(\Omega;\mathbb{C}^3)$ gives

$$\begin{cases} \iiint_\Omega \nabla_{\boldsymbol{x}} \times \boldsymbol{E}^\varepsilon(\boldsymbol{x}) \cdot \boldsymbol{\phi}(\boldsymbol{x})\, dv - ik_0 \iiint_\Omega \boldsymbol{\phi}(\boldsymbol{x}) \cdot \{\mu(\boldsymbol{x}/\varepsilon) \cdot \boldsymbol{H}^\varepsilon(\boldsymbol{x})\}\, dv = 0, \\ \iiint_\Omega \nabla_{\boldsymbol{x}} \times \boldsymbol{H}^\varepsilon(\boldsymbol{x}) \cdot \boldsymbol{\phi}(\boldsymbol{x})\, dv + ik_0 \iiint_\Omega \boldsymbol{\phi}(\boldsymbol{x}) \cdot \{\epsilon(\boldsymbol{x}/\varepsilon) \cdot \boldsymbol{E}^\varepsilon(\boldsymbol{x})\}\, dv = 0. \end{cases}$$

In the limit $\varepsilon \searrow 0$ we get

$$\begin{cases} \iiint_\Omega \nabla_{\boldsymbol{x}} \times \boldsymbol{E}(\boldsymbol{x}) \cdot \boldsymbol{\phi}(\boldsymbol{x})\, dv_{\boldsymbol{x}} \\ \qquad - ik_0 \iiint_\Omega \iiint_Y \boldsymbol{\phi}(\boldsymbol{x}) \cdot \mu(\boldsymbol{y}) \cdot (\boldsymbol{H}(\boldsymbol{x}) + \nabla_{\boldsymbol{y}}\Psi_1(\boldsymbol{x},\boldsymbol{y}))\, dv_{\boldsymbol{y}}\, dv_{\boldsymbol{x}} = 0, \\ \iiint_\Omega \nabla_{\boldsymbol{x}} \times \boldsymbol{H}(\boldsymbol{x}) \cdot \boldsymbol{\phi}(\boldsymbol{x})\, dv_{\boldsymbol{x}} \\ \qquad + ik_0 \iiint_\Omega \iiint_Y \boldsymbol{\phi}(\boldsymbol{x}) \cdot \epsilon(\boldsymbol{y}) \cdot (\boldsymbol{E}(\boldsymbol{x}) + \nabla_{\boldsymbol{y}}\Phi_1(\boldsymbol{x},\boldsymbol{y}))\, dv_{\boldsymbol{y}}\, dv_{\boldsymbol{x}} = 0. \end{cases} \tag{4.7}$$

Here we have used Theorem B.8, which states that

$$\nabla \times \boldsymbol{E}^\varepsilon \overset{\text{2-s}}{\rightharpoonup} \nabla_{\boldsymbol{x}} \times \boldsymbol{E}_0(\boldsymbol{x},\boldsymbol{y}) + \nabla_{\boldsymbol{y}} \times \boldsymbol{E}_1(\boldsymbol{x},\boldsymbol{y}),$$

which gives the weak limit $\nabla_{\boldsymbol{x}} \times \boldsymbol{E}(\boldsymbol{x})$ since the admissible test function $\boldsymbol{\phi}$ does not depend on $\boldsymbol{y}$.

The divergence equations are multiplied by $v(\boldsymbol{x}) = \varepsilon\psi(\boldsymbol{x})\phi(\boldsymbol{x}/\varepsilon)$, where $\psi \in C_0^\infty(\Omega)$, $\phi \in H^1_\#(Y)$. We note that $w_\epsilon(\boldsymbol{y}) = \hat{\boldsymbol{e}}_i \cdot \epsilon(\boldsymbol{y}) \cdot \hat{\boldsymbol{e}}_j \in L^\infty_\#(Y)$ and $w_\mu(\boldsymbol{y}) = \hat{\boldsymbol{e}}_i \cdot \mu(\boldsymbol{y}) \cdot \hat{\boldsymbol{e}}_j \in L^\infty_\#(Y)$, which implies that $w_\epsilon(\boldsymbol{y})\nabla_{\boldsymbol{y}}\phi$ and $w_\mu(\boldsymbol{y})\nabla_{\boldsymbol{y}}\phi \in L^2_\#(Y;\mathbb{C}^3)$. Theorem B.3 and an integration by parts give

$$\begin{aligned} &\lim_{\varepsilon\searrow 0} \iiint_\Omega \nabla \cdot \{\epsilon(\boldsymbol{x}/\varepsilon) \cdot \boldsymbol{E}^\varepsilon(\boldsymbol{x})\}\, \varepsilon\psi(\boldsymbol{x})\phi(\boldsymbol{x}/\varepsilon)\, dv_{\boldsymbol{x}} \\ &\quad = -\lim_{\varepsilon\searrow 0} \iiint_\Omega \{\varepsilon\nabla\psi(\boldsymbol{x})\phi(\boldsymbol{x}/\varepsilon) + \psi(\boldsymbol{x})\nabla_{\boldsymbol{y}}\phi(\boldsymbol{x}/\varepsilon)\} \cdot \{\epsilon(\boldsymbol{x}/\varepsilon) \cdot \boldsymbol{E}^\varepsilon(\boldsymbol{x})\}\, dv_{\boldsymbol{x}} \\ &\quad = -\iiint_\Omega \iiint_Y \psi(\boldsymbol{x})\nabla_{\boldsymbol{y}}\phi(\boldsymbol{y}) \cdot \epsilon(\boldsymbol{y}) \cdot \{\boldsymbol{E}(\boldsymbol{x}) + \nabla_{\boldsymbol{y}}\Phi_1(\boldsymbol{x},\boldsymbol{y})\}\, dv_{\boldsymbol{y}}\, dv_{\boldsymbol{x}} = 0 \end{aligned}$$

for all $\phi \in H^1_\#(Y)$ and all $\psi \in H^1_0(\Omega)$. Using similar arguments for the magnetic field, we get the local equations

$$\begin{cases} \iiint_Y \nabla_{\boldsymbol{y}}\phi(\boldsymbol{y}) \cdot \epsilon(\boldsymbol{y}) \cdot \{\boldsymbol{E}(\boldsymbol{x}) + \nabla_{\boldsymbol{y}}\Phi_1(\boldsymbol{x},\boldsymbol{y})\}\, dv_{\boldsymbol{y}} = 0, \\ \iiint_Y \nabla_{\boldsymbol{y}}\phi(\boldsymbol{y}) \cdot \mu(\boldsymbol{y}) \cdot \{\boldsymbol{H}(\boldsymbol{x}) + \nabla_{\boldsymbol{y}}\Psi_1(\boldsymbol{x},\boldsymbol{y})\}\, dv_{\boldsymbol{y}} = 0, \end{cases} \quad \boldsymbol{x} \in \Omega \text{ a.e.} \tag{4.8}$$

Define the vector fields

$$\boldsymbol{\chi}_e(\boldsymbol{y}) = \sum_{i=1}^3 \chi_e^i(\boldsymbol{y})\hat{\boldsymbol{e}}_i, \qquad \boldsymbol{\chi}_h(\boldsymbol{y}) = \sum_{i=1}^3 \chi_h^i(\boldsymbol{y})\hat{\boldsymbol{e}}_i.$$

The variables can be separated by using the ansatz

$$\begin{cases} \nabla_{\boldsymbol{y}}\Phi_1(\boldsymbol{x},\boldsymbol{y}) = -\nabla_{\boldsymbol{y}}\boldsymbol{\chi}_e(\boldsymbol{y})\cdot\boldsymbol{E}(\boldsymbol{x}), \\ \nabla_{\boldsymbol{y}}\Psi_1(\boldsymbol{x},\boldsymbol{y}) = -\nabla_{\boldsymbol{y}}\boldsymbol{\chi}_h(\boldsymbol{y})\cdot\boldsymbol{H}(\boldsymbol{x}) \end{cases}$$

inserted into (4.8), which gives

$$\begin{cases} \langle\nabla_{\boldsymbol{y}}\phi(\boldsymbol{y})\cdot(\epsilon(\boldsymbol{y})-\epsilon(\boldsymbol{y})\cdot\nabla_{\boldsymbol{y}}\boldsymbol{\chi}_e(\boldsymbol{y}))\rangle\cdot\boldsymbol{E}(\boldsymbol{x})=0, \\ \langle\nabla_{\boldsymbol{y}}\phi(\boldsymbol{y})\cdot(\mu(\boldsymbol{y})-\mu(\boldsymbol{y})\cdot\nabla_{\boldsymbol{y}}\boldsymbol{\chi}_h(\boldsymbol{y}))\rangle\cdot\boldsymbol{H}(\boldsymbol{x})=0 \end{cases}$$

for all $\phi \in H^1_{\#}(Y)$, i.e.,

$$\begin{cases} \nabla_{\boldsymbol{y}}\cdot(\epsilon(\boldsymbol{y})-\epsilon(\boldsymbol{y})\cdot\nabla_{\boldsymbol{y}}\boldsymbol{\chi}_e(\boldsymbol{y}))=0, \\ \nabla_{\boldsymbol{y}}\cdot(\mu(\boldsymbol{y})-\mu(\boldsymbol{y})\cdot\nabla_{\boldsymbol{y}}\boldsymbol{\chi}_h(\boldsymbol{y}))=0 \end{cases}$$

a.e. in $\Omega \times Y$. Inserting the solutions of the local equations into (4.7) yields the macroscopic homogenized equations

$$\begin{cases} \iiint_\Omega \nabla_{\boldsymbol{x}}\times\boldsymbol{E}(\boldsymbol{x})\cdot\boldsymbol{\phi}(\boldsymbol{x})\,dv_{\boldsymbol{x}} \\ \qquad - ik_0\iiint_\Omega\iiint_Y \boldsymbol{\phi}(\boldsymbol{x})\cdot(\mu(\boldsymbol{y})-\mu(\boldsymbol{y})\cdot\nabla_{\boldsymbol{y}}\boldsymbol{\chi}_h(\boldsymbol{y}))\;dv_{\boldsymbol{y}}\cdot\boldsymbol{H}(\boldsymbol{x})\,dv_{\boldsymbol{x}}=0, \\ \iiint_\Omega \nabla_{\boldsymbol{x}}\times\boldsymbol{H}(\boldsymbol{x})\cdot\boldsymbol{\phi}(\boldsymbol{x})\,dv_{\boldsymbol{x}} \\ \qquad + ik_0\iiint_\Omega\iiint_Y \boldsymbol{\phi}(\boldsymbol{x})\cdot(\epsilon(\boldsymbol{y})-\epsilon(\boldsymbol{y})\cdot\nabla_{\boldsymbol{y}}\boldsymbol{\chi}_e(\boldsymbol{y}))\;dv_{\boldsymbol{y}}\cdot\boldsymbol{E}(\boldsymbol{x})\,dv_{\boldsymbol{x}}=0 \end{cases}$$

and

$$\begin{cases} \nabla\cdot\boldsymbol{B}(\boldsymbol{x})=0, \\ \nabla\cdot\boldsymbol{D}(\boldsymbol{x})=0, \end{cases}$$

which defines the homogenized permeability and permittivity as

$$\begin{cases} \epsilon^h = \iiint_Y \epsilon(\boldsymbol{y})\cdot(\mathbf{I}_3-\nabla_{\boldsymbol{y}}\boldsymbol{\chi}_e(\boldsymbol{y}))\;dv_{\boldsymbol{y}}, \\ \mu^h = \iiint_Y \mu(\boldsymbol{y})\cdot(\mathbf{I}_3-\nabla_{\boldsymbol{y}}\boldsymbol{\chi}_h(\boldsymbol{y}))\;dv_{\boldsymbol{y}}, \end{cases}$$

i.e., $\boldsymbol{B} = \mu^h\cdot\boldsymbol{H}$ and $\boldsymbol{D} = \epsilon^h\cdot\boldsymbol{E}$. The existence of a unique solution of the homogenized system follows from the properties of the homogenized permeability and permittivity, μ^h and ϵ^h, respectively (see section 4.2.1), which satisfies the same assumptions as the material properties for the heterogeneous system. □

4.2.1. The properties of the homogenized parameters. An immediate consequence of Theorem 4.2 is that the homogenized parameters are independent of the properties of the domain Ω and of the properties of the incident field. Moreover, the homogenized material properties satisfy the same assumptions as the heterogeneous parameters do, i.e., they are coercive and bounded. Coercivity and boundedness follow from the fact that the homogenized parameters are bounded from below and above by the harmonic and arithmetic averages of the heterogeneous parameters; hence the homogenized parameters are bounded from below and above (e.g., see [5] or [24]). If the heterogeneous material parameters are symmetric (reciprocal material), then the homogenized parameters are also symmetric as proved below.

PROPOSITION 4.3. *The homogenized permeability and permittivity are symmetric, provided the heterogeneous parameters are symmetric.*

Proof of Proposition 4.3. We restrict ourselves to the electric parameters since the arguments for the permeability are the same. By assumption the material parameters are symmetrical, i.e., $\epsilon(\boldsymbol{y}) = \epsilon^t(\boldsymbol{y})$ and $\mu(\boldsymbol{y}) = \mu^t(\boldsymbol{y})$.

We define the average over the Y-cell by

$$\langle f \rangle = \iiint_Y f(\boldsymbol{y})\, dv_{\boldsymbol{y}}.$$

The local problem, (4.6), can be written as $(i = 1, 2, 3)$

$$\langle \nabla_{\boldsymbol{y}} w(\boldsymbol{y}) \cdot \epsilon(\boldsymbol{y}) \cdot \hat{\boldsymbol{e}}_i \rangle = \left\langle \nabla_{\boldsymbol{y}} w(\boldsymbol{y}) \cdot \epsilon(\boldsymbol{y}) \cdot \nabla_{\boldsymbol{y}} \chi_e^i(\boldsymbol{y}) \right\rangle$$

for all $w \in H^1_\#(Y)$. We rewrite these equations in one set of equations (see (4.5))

$$\langle \nabla_{\boldsymbol{y}} w(\boldsymbol{y}) \cdot \epsilon(\boldsymbol{y}) \rangle = \langle \nabla_{\boldsymbol{y}} w(\boldsymbol{y}) \cdot \epsilon(\boldsymbol{y}) \cdot \nabla_{\boldsymbol{y}} \boldsymbol{\chi}_e(\boldsymbol{y}) \rangle$$

for all $w \in H^1_\#(Y)$. Due to the symmetry in ϵ we get

$$\langle \epsilon(\boldsymbol{y}) \cdot \nabla_{\boldsymbol{y}} \boldsymbol{\chi}(\boldsymbol{y}) \rangle = \left\langle (\nabla_{\boldsymbol{y}} \boldsymbol{\chi}_e(\boldsymbol{y}))^t \cdot \epsilon(\boldsymbol{y}) \cdot \nabla_{\boldsymbol{y}} \boldsymbol{\chi}_e(\boldsymbol{y}) \right\rangle$$

if we choose $w = \chi_e^i$.

The homogenized parameters in (4.4) are

$$\begin{aligned} \epsilon^h &= \langle \epsilon(\boldsymbol{y}) \rangle - \langle \epsilon(\boldsymbol{y}) \cdot \nabla_{\boldsymbol{y}} \boldsymbol{\chi}_e(\boldsymbol{y}) \rangle \\ &= \langle \epsilon(\boldsymbol{y}) \rangle - \left\langle (\nabla_{\boldsymbol{y}} \boldsymbol{\chi}_e(\boldsymbol{y}))^t \cdot \epsilon(\boldsymbol{y}) \cdot \nabla_{\boldsymbol{y}} \boldsymbol{\chi}_e(\boldsymbol{y}) \right\rangle, \end{aligned}$$

which proves that ϵ^h is symmetric. □

4.2.2. Correctors. This section is concluded by the proof of a new result on correctors.

We begin with the two-scale limit of the heterogeneous system (4.1), which is given by

$$(4.9) \quad \begin{cases} \iiint_\Omega \iiint_Y (\nabla_{\boldsymbol{x}} \times \boldsymbol{E}_0(\boldsymbol{x},\boldsymbol{y}) + \nabla_{\boldsymbol{y}} \times \boldsymbol{E}_1(\boldsymbol{x},\boldsymbol{y})) \cdot \boldsymbol{\phi}(\boldsymbol{x},\boldsymbol{y})\, dv_{\boldsymbol{y}} dv_{\boldsymbol{x}} \\ = ik_0 \iiint_\Omega \iiint_Y \boldsymbol{\phi}(\boldsymbol{x},\boldsymbol{y}) \cdot \mu(\boldsymbol{y}) \cdot \boldsymbol{H}_0(\boldsymbol{x},\boldsymbol{y})\, dv_{\boldsymbol{y}} dv_{\boldsymbol{x}}, \\ \iiint_\Omega \iiint_Y (\nabla_{\boldsymbol{x}} \times \boldsymbol{H}_0(\boldsymbol{x},\boldsymbol{y}) + \nabla_{\boldsymbol{y}} \times \boldsymbol{H}_1(\boldsymbol{x},\boldsymbol{y})) \cdot \boldsymbol{\phi}(\boldsymbol{x},\boldsymbol{y})\, dv_{\boldsymbol{y}} dv_{\boldsymbol{x}} \\ = -ik_0 \iiint_\Omega \iiint_Y \boldsymbol{\phi}(\boldsymbol{x},\boldsymbol{y}) \cdot \epsilon(\boldsymbol{y}) \cdot \boldsymbol{E}_0(\boldsymbol{x},\boldsymbol{y})\, dv_{\boldsymbol{y}} dv_{\boldsymbol{x}} \end{cases}$$

for all $\boldsymbol{\phi} \in D(\Omega; C^\infty_\#(Y; \mathbb{C}^3))$. These equations follow from the fact that (see Appendix B)

$$\boldsymbol{E}^\varepsilon(\boldsymbol{x}) \overset{\text{2-s}}{\rightharpoonup} \boldsymbol{E}_0(\boldsymbol{x},\boldsymbol{y})$$

and

$$\nabla \times \boldsymbol{E}^{\varepsilon}(\boldsymbol{x}) \stackrel{\text{2-s}}{\rightharpoonup} \nabla_{\boldsymbol{x}} \times \boldsymbol{E}_0(\boldsymbol{x},\boldsymbol{y}) + \nabla_{\boldsymbol{y}} \times \boldsymbol{E}_1(\boldsymbol{x},\boldsymbol{y}),$$

where

$$\begin{cases} \boldsymbol{E}_0 \in L^2(\Omega; L^2_{\#}(Y;\mathbb{C}^3)), \\ \nabla_{\boldsymbol{x}} \times \boldsymbol{E}_0 \in L^2(\Omega; L^2_{\#}(Y;\mathbb{C}^3)), \\ \boldsymbol{E}_1 \in L^2(\Omega; H_{\#}(\mathrm{rot}, Y)/\mathbb{C}). \end{cases}$$

The system (4.9) contains macroscopic and microscopic information which gives the homogenized system when averaged over the local scale. The local equations and the two-scale limit system (4.9) provide us with the following correctors in the case when the solution of the homogenized system is smooth enough.

THEOREM 4.4. *Let $\boldsymbol{E}^{\varepsilon}, \boldsymbol{H}^{\varepsilon}$ be the solution of* (4.1)*, let $\boldsymbol{E}, \boldsymbol{H}$ be the solution of the homogenized Maxwell equations* (4.4)*, and let $\boldsymbol{E}_1, \boldsymbol{H}_1$ solve the two-scale limit system* (4.9)*. If $\boldsymbol{E}_0$, $\boldsymbol{H}_0$, $\boldsymbol{E}_1$, $\boldsymbol{H}_1$, $\nabla_{\boldsymbol{x}} \times \boldsymbol{E}_0$, $\nabla_{\boldsymbol{x}} \times \boldsymbol{H}_0$, $\nabla_{\boldsymbol{x}} \times \boldsymbol{E}_1$, $\nabla_{\boldsymbol{x}} \times \boldsymbol{H}_1$, $\nabla_{\boldsymbol{y}} \times \boldsymbol{E}_1$, and $\nabla_{\boldsymbol{y}} \times \boldsymbol{H}_1$ are admissible test functions, then*

$$\begin{cases} \lim_{\varepsilon\to 0} \|\boldsymbol{E}^{\varepsilon}(\boldsymbol{x}) - \boldsymbol{E}_0(\boldsymbol{x},\boldsymbol{x}/\varepsilon) - \varepsilon \boldsymbol{E}_1(\boldsymbol{x},\boldsymbol{x}/\varepsilon)\|_{H(\mathrm{rot},\Omega)} = 0, \\ \lim_{\varepsilon\to 0} \|\boldsymbol{H}^{\varepsilon}(\boldsymbol{x}) - \boldsymbol{H}_0(\boldsymbol{x},\boldsymbol{x}/\varepsilon) - \varepsilon \boldsymbol{H}_1(\boldsymbol{x},\boldsymbol{x}/\varepsilon)\|_{H(\mathrm{rot},\Omega)} = 0, \end{cases}$$

where

$$\begin{cases} \boldsymbol{E}_0(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{E}(\boldsymbol{x}) - \nabla_{\boldsymbol{y}} \boldsymbol{\chi}_e(\boldsymbol{y}) \cdot \boldsymbol{E}(\boldsymbol{x}), \\ \boldsymbol{H}_0(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{H}(\boldsymbol{x}) - \nabla_{\boldsymbol{y}} \boldsymbol{\chi}_h(\boldsymbol{y}) \cdot \boldsymbol{H}(\boldsymbol{x}), \end{cases}$$

$$\boldsymbol{\chi}_e(\boldsymbol{y}) = \sum_{i=1}^{3} \chi_e^i(\boldsymbol{y}) \hat{\boldsymbol{e}}_i, \qquad \boldsymbol{\chi}_h(\boldsymbol{y}) = \sum_{i=1}^{3} \chi_h^i(\boldsymbol{y}) \hat{\boldsymbol{e}}_i,$$

and where $\chi_e^i(\boldsymbol{y})$ and $\chi_h^i(\boldsymbol{y})$, $i = 1, 2, 3$, in $H^1_{\#}(Y)$ solve the local problems (4.6).

Proof. The assumptions imply that (see Theorem B.8)

$$\begin{cases} \boldsymbol{E}^{\varepsilon} \stackrel{\text{2-s}}{\rightharpoonup} \boldsymbol{E}_0(\boldsymbol{x},\boldsymbol{y}), \\ \nabla \times \boldsymbol{E}^{\varepsilon} \stackrel{\text{2-s}}{\rightharpoonup} \nabla_{\boldsymbol{x}} \times \boldsymbol{E}_0(\boldsymbol{x},\boldsymbol{y}) + \nabla_{\boldsymbol{y}} \times \boldsymbol{E}_1(\boldsymbol{x},\boldsymbol{y}) \end{cases}$$

and $\nabla_{\boldsymbol{y}} \times \boldsymbol{E}_0(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{0}$.

The proof is carried out using the sesquilinear form

$$Q^{\varepsilon}(\boldsymbol{u},\boldsymbol{v}) = -\iiint_{\Omega} \left\{ \frac{1}{ik_0} (\nabla \times \boldsymbol{v}^*) \cdot \mu^{-1}(\boldsymbol{x}/\varepsilon) \cdot (\nabla \times \boldsymbol{u}) + ik_0 \boldsymbol{v}^* \cdot \epsilon(\boldsymbol{x}/\varepsilon) \cdot \boldsymbol{u} \right\} dv,$$

which is identical to (4.3) but without the surface integral term.

The coercivity assumption, (2.1), implies

$$C\|\boldsymbol{u}(\boldsymbol{x})\|^2_{H(\mathrm{rot},\Omega)} \le \Re Q^{\varepsilon}(\boldsymbol{u},\boldsymbol{u}).$$

We get

$$C\|\boldsymbol{E}^{\varepsilon}(\boldsymbol{x}) - \boldsymbol{E}_0(\boldsymbol{x},\boldsymbol{x}/\varepsilon) - \varepsilon \boldsymbol{E}_1(\boldsymbol{x},\boldsymbol{x}/\varepsilon)\|^2_{H(\mathrm{rot},\Omega)} \le I_1^{\varepsilon} + I_2^{\varepsilon},$$

where

$$\begin{cases} I_1^\varepsilon = \Re Q^\varepsilon\left(\boldsymbol{E}^\varepsilon(\boldsymbol{x}), \boldsymbol{A}_\varepsilon(\boldsymbol{x})\right), \\ I_2^\varepsilon = -\Re Q^\varepsilon\left(\boldsymbol{E}_0(\boldsymbol{x}, \boldsymbol{x}/\varepsilon) + \varepsilon \boldsymbol{E}_1(\boldsymbol{x}, \boldsymbol{x}/\varepsilon), \boldsymbol{A}_\varepsilon(\boldsymbol{x})\right), \end{cases}$$

where, for short, we denote $\boldsymbol{A}_\varepsilon(\boldsymbol{x}) = \boldsymbol{E}^\varepsilon(\boldsymbol{x}) - \boldsymbol{E}_0(\boldsymbol{x}, \boldsymbol{x}/\varepsilon) - \varepsilon \boldsymbol{E}_1(\boldsymbol{x}, \boldsymbol{x}/\varepsilon)$. Due to the assumptions of the fields in $\boldsymbol{A}_\varepsilon(\boldsymbol{x})$, we have

$$\begin{cases} \boldsymbol{A}_\varepsilon \overset{\text{2-s}}{\rightharpoonup} \boldsymbol{0}, \\ \nabla \times \boldsymbol{A}_\varepsilon \overset{\text{2-s}}{\rightharpoonup} \boldsymbol{0}, \end{cases}$$

since

$$\begin{cases} \boldsymbol{E}^\varepsilon \overset{\text{2-s}}{\rightharpoonup} \boldsymbol{E}_0(\boldsymbol{x}, \boldsymbol{y}), \\ \nabla \times \boldsymbol{E}^\varepsilon \overset{\text{2-s}}{\rightharpoonup} \nabla_{\boldsymbol{x}} \times \boldsymbol{E}_0(\boldsymbol{x}, \boldsymbol{y}) + \nabla_{\boldsymbol{y}} \times \boldsymbol{E}_1(\boldsymbol{x}, \boldsymbol{y}) \end{cases}$$

and $\nabla_{\boldsymbol{y}} \times \boldsymbol{E}_0(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{0}$.

We start by analyzing the first integral I_1^ε. Since $\boldsymbol{E}^\varepsilon$ satisfies the Maxwell equations, (4.1), we get

$$I_1^\varepsilon = \Re \iiint_\Omega (\nabla \times \boldsymbol{H}^\varepsilon(\boldsymbol{x})) \cdot \boldsymbol{A}_\varepsilon(\boldsymbol{x})^*\, dv - \Re \iiint_\Omega \boldsymbol{H}^\varepsilon(\boldsymbol{x}) \cdot (\nabla \times \boldsymbol{A}_\varepsilon(\boldsymbol{x}))^*\, dv.$$

We now use $\nabla \cdot (\nabla \times \boldsymbol{H}^\varepsilon) = 0$ and $\nabla \cdot (\nabla \times \boldsymbol{A}_\varepsilon) = 0$ and, moreover, the fact that $\nabla \times \boldsymbol{H}^\varepsilon \in L^2(\Omega; \mathbb{C}^3)$ and $\nabla \times \boldsymbol{A}_\varepsilon \in L^2(\Omega; \mathbb{C}^3)$. The div-curl lemma (see [24, 25]) can be used and the limit is zero, since

$$\boldsymbol{A}_\varepsilon(\boldsymbol{x}) \rightharpoonup \boldsymbol{0} \text{ and } \nabla \times \boldsymbol{A}_\varepsilon(\boldsymbol{x}) \rightharpoonup \boldsymbol{0}$$

weakly in $L^2(\Omega; \mathbb{C}^3)$.

The second integral is now analyzed:

$$\begin{aligned} I_2^\varepsilon = -\Re \iiint_\Omega \Big\{ & \frac{1}{ik_0} (\nabla \times \boldsymbol{A}_\varepsilon(\boldsymbol{x}))^* \cdot \mu^{-1}(\boldsymbol{x}/\varepsilon) \\ & \cdot\ (\nabla_{\boldsymbol{x}} \times \boldsymbol{E}_0(\boldsymbol{x}, \boldsymbol{x}/\varepsilon) + \varepsilon \nabla_{\boldsymbol{x}} \times \boldsymbol{E}_1(\boldsymbol{x}, \boldsymbol{x}/\varepsilon) + \nabla_{\boldsymbol{y}} \times \boldsymbol{E}_1(\boldsymbol{x}, \boldsymbol{x}/\varepsilon)) \\ & + ik_0 \boldsymbol{A}_\varepsilon(\boldsymbol{x})^* \cdot \epsilon(\boldsymbol{x}/\varepsilon) \cdot (\boldsymbol{E}_0(\boldsymbol{x}, \boldsymbol{x}/\varepsilon) + \varepsilon \boldsymbol{E}_1(\boldsymbol{x}, \boldsymbol{x}/\varepsilon)) \Big\}\, dv_{\boldsymbol{x}}. \end{aligned}$$

We pass to the limit, $\varepsilon \searrow 0$, and use that $\mu^{-1}(\boldsymbol{x}/\varepsilon) \cdot (\nabla_{\boldsymbol{x}} \times \boldsymbol{E}_0(\boldsymbol{x}, \boldsymbol{x}/\varepsilon) + \varepsilon \nabla_{\boldsymbol{x}} \times \boldsymbol{E}_1(\boldsymbol{x}, \boldsymbol{x}/\varepsilon) + \nabla_{\boldsymbol{y}} \times \boldsymbol{E}_1(\boldsymbol{x}, \boldsymbol{x}/\varepsilon))$ and $\epsilon(\boldsymbol{x}/\varepsilon) \cdot (\boldsymbol{E}_0(\boldsymbol{x}, \boldsymbol{x}/\varepsilon) + \varepsilon \boldsymbol{E}_1(\boldsymbol{x}, \boldsymbol{x}/\varepsilon))$ are admissible test functions and obtain

$$\lim_{\varepsilon \searrow 0} I_2^\varepsilon = 0;$$

the theorem is proved. ∎

Remark 4.1. It is still an open question how irregular a function can be and still be an admissible test function. However, if the homogenized solution $\boldsymbol{E} \in C(\overline{\Omega}; \mathbb{C}^3)$, then $\boldsymbol{E}_0 \in L^2_\#(Y; C(\overline{\Omega}; \mathbb{C}^3))$ is admissible (see Appendix B). Further, if $\boldsymbol{E} \in C(\overline{\Omega}; \mathbb{C}^3)$, then $\boldsymbol{H} \in C(\overline{\Omega}; \mathbb{C}^3)$ by symmetry, and via (4.9) we find that $\nabla_{\boldsymbol{x}} \times \boldsymbol{E}_0 + \nabla_{\boldsymbol{y}} \times \boldsymbol{E}_1$ is smooth in x, and for sufficient smoothness $\nabla_{\boldsymbol{x}} \times \boldsymbol{E}_1$ is also an admissible test function. To the knowledge of the authors there exist no results about regularity of the solutions of the Maxwell equations in the anisotropic, constant coefficient case. However, we believe that for sufficient regular boundary and incident fields, the solutions are admissible test functions.

5. Examples. In this section, we give two explicit examples of the exterior Calderón operator.

5.1. Plane boundary. The general representation of the solution to Problem (R) in (2.7) in a region $x_3 > c$ (plane interface Ω, $x_3 = c$) is found by a Fourier transform in the lateral coordinates x_1 and x_2.

The Fourier transform $\boldsymbol{E}(\boldsymbol{\xi}, x_3)$ of the electric field $\boldsymbol{E}(\boldsymbol{x})$, $\boldsymbol{x} = \hat{\boldsymbol{e}}_1 x_1 + \hat{\boldsymbol{e}}_2 x_2 + \hat{\boldsymbol{e}}_3 x_3$, with respect to the lateral position vector $\boldsymbol{\rho} = \hat{\boldsymbol{e}}_1 x_1 + \hat{\boldsymbol{e}}_2 x_2$ is defined by

$$\boldsymbol{E}(\boldsymbol{\xi}, x_3) = \iint_{\mathbb{R}^2} \boldsymbol{E}(\boldsymbol{x}) e^{-i\boldsymbol{\xi}\cdot\boldsymbol{\rho}}\, d\boldsymbol{\rho},$$

where the Fourier variable $\boldsymbol{\xi}$ is

$$\boldsymbol{\xi} = \hat{\boldsymbol{e}}_1 \xi_1 + \hat{\boldsymbol{e}}_2 \xi_2$$

and $d\boldsymbol{\rho} = dx_1\, dx_2$. The modulus of this vector is denoted ξ, i.e.,

$$\xi = \sqrt{\xi_1^2 + \xi_2^2}.$$

By the Fourier inversion formula,

$$\boldsymbol{E}(\boldsymbol{x}) = \frac{1}{4\pi^2} \iint_{\mathbb{R}^2} \boldsymbol{E}(\boldsymbol{\xi}, x_3) e^{i\boldsymbol{\xi}\cdot\boldsymbol{\rho}}\, d\boldsymbol{\xi},$$

where $d\boldsymbol{\xi} = d\xi_1\, d\xi_2$. Specifically, the tangential electric field on the surface $\partial\Omega$ is

$$-\hat{\boldsymbol{e}}_3 \times (\hat{\boldsymbol{e}}_3 \times \boldsymbol{E}(\boldsymbol{x}))|_{\partial\Omega} = \frac{1}{4\pi^2} \iint_{\mathbb{R}^2} \boldsymbol{A}(\boldsymbol{\xi}) e^{i\boldsymbol{\xi}\cdot\boldsymbol{\rho}}\, d\boldsymbol{\xi},$$

where $\boldsymbol{A}(\boldsymbol{\xi})$ is the Fourier transform of the trace of the tangential electric field.

The general form of the solution to Problem (R) in (2.7) in a region $x_3 > c$ is (see [16])

$$\begin{cases} \boldsymbol{E}(\boldsymbol{x}) = \frac{1}{4\pi^2} \iint_{\mathbb{R}^2} \left(\mathbf{I}_2 - \dfrac{\xi}{\xi_3} \hat{\boldsymbol{e}}_3 \hat{\boldsymbol{e}}_\parallel\right) \cdot \boldsymbol{A}(\boldsymbol{\xi}) e^{i\boldsymbol{\xi}\cdot\boldsymbol{\rho} + i\xi_3(x_3 - c)}\, d\boldsymbol{\xi}, \\ \boldsymbol{H}(\boldsymbol{x}) = \frac{1}{4\pi^2} \iint_{\mathbb{R}^2} \left(\dfrac{\boldsymbol{\xi}}{k_0} + \dfrac{\xi_3}{k_0} \hat{\boldsymbol{e}}_3\right) \times \left(\mathbf{I}_2 - \dfrac{\xi}{\xi_3} \hat{\boldsymbol{e}}_3 \hat{\boldsymbol{e}}_\parallel\right) \cdot \boldsymbol{A}(\boldsymbol{\xi}) e^{i\boldsymbol{\xi}\cdot\boldsymbol{\rho} + i\xi_3(x_3 - c)}\, d\boldsymbol{\xi}, \end{cases}$$

where $\mathbf{I}_2$ is the identity dyadic in $\mathbb{R}^2$, and a pertinent orthogonal basis in $\mathbb{R}^2$ is $\{\hat{\boldsymbol{e}}_\parallel, \hat{\boldsymbol{e}}_\perp\}$, defined by

$$\hat{\boldsymbol{e}}_\parallel = \boldsymbol{\xi}/\xi, \qquad \hat{\boldsymbol{e}}_\perp = \hat{\boldsymbol{e}}_3 \times \hat{\boldsymbol{e}}_\parallel$$

and where

$$\xi_3 = \left(k_0^2 - \xi^2\right)^{1/2} = \begin{cases} \sqrt{k_0^2 - \xi^2} & \text{for } \xi < k_0, \\ i\sqrt{\xi^2 - k_0^2} & \text{for } \xi > k_0 \end{cases}$$

and the standard convention of the square root of a nonnegative argument is intended.

The representation of the fields can be simplified using dyadic calculus:

$$\begin{cases} \boldsymbol{E}(\boldsymbol{x}) = \dfrac{1}{4\pi^2} \iint_{\mathbb{R}^2} \left(\mathbf{I}_2 - \dfrac{\xi}{\xi_3} \hat{\boldsymbol{e}}_3 \hat{\boldsymbol{e}}_\parallel\right) \cdot \boldsymbol{A}(\boldsymbol{\xi}) e^{i\boldsymbol{\xi}\cdot\boldsymbol{\rho} + i\xi_3(x_3 - c)}\, d\boldsymbol{\xi}, \\ \boldsymbol{H}(\boldsymbol{x}) = \dfrac{1}{4\pi^2} \iint_{\mathbb{R}^2} \left(\dfrac{\xi}{k_0} \hat{\boldsymbol{e}}_3 \hat{\boldsymbol{e}}_\perp + \dfrac{k_0}{\xi_3} \hat{\boldsymbol{e}}_\perp \hat{\boldsymbol{e}}_\parallel - \dfrac{\xi_3}{k_0} \hat{\boldsymbol{e}}_\parallel \hat{\boldsymbol{e}}_\perp\right) \cdot \boldsymbol{A}(\boldsymbol{\xi}) e^{i\boldsymbol{\xi}\cdot\boldsymbol{\rho} + i\xi_3(x_3 - c)}\, d\boldsymbol{\xi}. \end{cases}$$

From these relations the exterior Calderón operator is the transformation from

$$\hat{\boldsymbol{e}}_3 \times \boldsymbol{E}(\boldsymbol{x})|_{\partial\Omega} = \frac{1}{4\pi^2} \iint_{\mathbb{R}^2} \hat{\boldsymbol{e}}_3 \times \boldsymbol{A}(\boldsymbol{\xi}) e^{i\boldsymbol{\xi}\cdot\boldsymbol{\rho}}\, d\boldsymbol{\xi}$$

to

$$\hat{\boldsymbol{e}}_3 \times \boldsymbol{H}(\boldsymbol{x})|_{\partial\Omega} = -\frac{1}{4\pi^2} \iint_{\mathbb{R}^2} \left(\frac{k_0}{\xi_3} \hat{\boldsymbol{e}}_\parallel \hat{\boldsymbol{e}}_\parallel - \frac{\xi_3}{k_0} \hat{\boldsymbol{e}}_\perp \hat{\boldsymbol{e}}_\perp \right) \cdot \boldsymbol{A}(\boldsymbol{\xi}) e^{i\boldsymbol{\xi}\cdot\boldsymbol{\rho}}\, d\boldsymbol{\xi},$$

where the vector field $\boldsymbol{A}(\boldsymbol{\xi})$ is determined from $\hat{\boldsymbol{e}}_3 \times \boldsymbol{E}(\boldsymbol{x})|_{\partial\Omega}$ by

$$\boldsymbol{A}(\boldsymbol{\xi}) = -\iint_{\mathbb{R}^2} \hat{\boldsymbol{e}}_3 \times (\hat{\boldsymbol{e}}_3 \times \boldsymbol{E}(\boldsymbol{x}))|_{\partial\Omega}\, e^{-i\boldsymbol{\xi}\cdot\boldsymbol{\rho}}\, d\boldsymbol{\rho}.$$

We note that in this example the domain and the boundary are unbounded, which yields other function spaces for the traces. We refer to [9] for the details.

5.2. Spherical boundary. For a spherical boundary, $x = a$, the exterior Calderón operator can be represented in a series of vector spherical waves; see Appendix C.

The general form of the solution to Problem (R) in (2.7) in a region $x > a$ is (see (C.2) and (C.3))

$$\begin{cases} \boldsymbol{E}(\boldsymbol{x}) = \sum_{\tau n} a_{\tau n} \boldsymbol{u}_{\tau n}(k_0 \boldsymbol{x}), \\ \boldsymbol{H}(\boldsymbol{x}) = -i \sum_{\tau n} a_{\tau n} \boldsymbol{u}_{\bar{\tau} n}(k_0 \boldsymbol{x}), \end{cases}$$

where the index $\bar{\tau}$ is the dual index of τ, defined by $\bar{1} = 2$ and $\bar{2} = 1$.

The traces of the electric and the magnetic fields are ($\kappa = k_0 a$)

$$\begin{cases} \hat{\boldsymbol{x}} \times \boldsymbol{E}(\boldsymbol{x})|_{\partial\Omega} = \sum_n \left(a_{1n} h_l^{(1)}(\kappa) \boldsymbol{A}_{2n}(\hat{\boldsymbol{x}}) - a_{2n} \frac{(\kappa h_l^{(1)}(\kappa))'}{\kappa} \boldsymbol{A}_{1n}(\hat{\boldsymbol{x}}) \right), \\ \hat{\boldsymbol{x}} \times \boldsymbol{H}(\boldsymbol{x})|_{\partial\Omega} = -i \sum_n \left(a_{2n} h_l^{(1)}(\kappa) \boldsymbol{A}_{2n}(\hat{\boldsymbol{x}}) - a_{1n} \frac{(\kappa h_l^{(1)}(\kappa))'}{\kappa} \boldsymbol{A}_{1n}(\hat{\boldsymbol{x}}) \right). \end{cases}$$

For a given tangential field $\hat{\boldsymbol{x}} \times \boldsymbol{E}(\boldsymbol{x})|_{\partial\Omega}$, the expansion coefficients $a_{\tau n}$ are found by the orthogonality relation (see (C.1))

$$\begin{cases} a_{1n} = \dfrac{1}{h_l^{(1)}(\kappa)} \displaystyle\iint_\gamma \boldsymbol{A}_{2n}(\hat{\boldsymbol{x}}) \cdot (\hat{\boldsymbol{x}} \times \boldsymbol{E}(\boldsymbol{x})|_{\partial\Omega}), \\ a_{2n} = -\dfrac{\kappa}{(\kappa h_l^{(1)}(\kappa))'} \displaystyle\iint_\gamma \boldsymbol{A}_{1n}(\hat{\boldsymbol{x}}) \cdot (\hat{\boldsymbol{x}} \times \boldsymbol{E}(\boldsymbol{x})|_{\partial\Omega}). \end{cases}$$

The exterior Calderón mapping is the mapping from $\hat{\boldsymbol{x}} \times \boldsymbol{E}(\boldsymbol{x})|_{\partial\Omega}$ (which determines the expansion coefficients $a_{\tau n}$ uniquely) to $\hat{\boldsymbol{x}} \times \boldsymbol{H}(\boldsymbol{x})|_{\partial\Omega}$.

6. Conclusions. This paper analyzes the homogenization of the Maxwell equations for a material with periodic microscale. The material can be anisotropic and satisfies a coercivity condition (passive material), and the sources of the excitation are located in the region outside the heterogeneous material in Ω. We utilize the concept of two-scale convergence. A new a priori estimate is established, and a proof of strong convergence of the corrector fields is presented. The homogenized parameters are shown to be independent of the properties of the domain Ω and of the properties of the incident field.

Appendix A. Function spaces. In this appendix, we list the various function spaces used in this paper. Let Ω be a bounded, open, simply connected set in $\mathbb{R}^3$ with Lipschitz boundary $\partial\Omega$. A Y-periodic function, f, is defined as $f(\boldsymbol{x}+\hat{\boldsymbol{e}}_k) = f(\boldsymbol{x})$ for every $k = 1, 2, 3$, where $\hat{\boldsymbol{e}}_k$, $k = 1, 2, 3$, is the canonical basis in $\mathbb{R}^3$.

The space $C(\Omega)$ is the space of continuous functions in Ω. We also use $C_0(\overline{\Omega})$, which consists of all uniformly continuous functions which are zero at the boundary. The space $C^\infty(\Omega)$ is the space of infinitely continuously differentiable functions in Ω, and $C_0^\infty(\Omega)$ are the functions in this space with compact support in Ω, which we also denote $D(\Omega)$. Moreover,

$$C_{\#}^\infty(Y) = \left\{\phi \in C^\infty(\mathbb{R}^3), \phi \ Y\text{-periodic}\right\}.$$

Several function spaces with square integrable functions are used in this paper. The basic space is

$$L^2(\Omega) \stackrel{\text{def}}{=} \left\{u(\boldsymbol{x}) : u \text{ Lebesgue integrable}, \iiint_\Omega |u(\boldsymbol{x})|^2 \, dv_{\boldsymbol{x}} < \infty\right\}$$

with norm

$$\|u\|_{L^2(\Omega)} = \left\{\iiint_\Omega |u(\boldsymbol{x})|^2 \, dv_{\boldsymbol{x}}\right\}^{1/2}.$$

Similarly for vector-valued spaces we have the norm

$$\|\boldsymbol{u}\|_{L^2(\Omega;\mathbb{C}^3)} = \left\{\iiint_\Omega |\boldsymbol{u}(\boldsymbol{x})|^2 \, dv_{\boldsymbol{x}}\right\}^{1/2}.$$

We also define two function spaces of periodic functions:

$$L_{\#}^2(Y) \stackrel{\text{def}}{=} \left\{\text{the completion of } C_{\#}^\infty(Y) \text{ in the } L^2(Y)\text{-norm}\right\}$$

and

$$L_{\#}^\infty(Y) \stackrel{\text{def}}{=} \left\{\phi \in L^\infty(\mathbb{R}^3), \phi \ Y\text{-periodic}\right\},$$

$$\begin{cases} H(\text{div}, \Omega) \stackrel{\text{def}}{=} \left\{\boldsymbol{u} \in L^2(\Omega; \mathbb{C}^3) : \nabla \cdot \boldsymbol{u} \in L^2(\Omega)\right\}, \\ H(\text{rot}, \Omega) \stackrel{\text{def}}{=} \left\{\boldsymbol{u} \in L^2(\Omega; \mathbb{C}^3) : \nabla \times \boldsymbol{u} \in L^2(\Omega; \mathbb{C}^3)\right\}, \end{cases}$$

which are Hilbert spaces with norms

$$\begin{cases} \|\boldsymbol{u}\|_{H(\text{div},\Omega)} = \left(\|\boldsymbol{u}\|_{L^2(\Omega;\mathbb{C}^3)}^2 + \|\nabla \cdot \boldsymbol{u}\|_{L^2(\Omega)}^2\right)^{1/2}, \\ \|\boldsymbol{u}\|_{H(\text{rot},\Omega)} = \left(\|\boldsymbol{u}\|_{L^2(\Omega;\mathbb{C}^3)}^2 + \|\nabla \times \boldsymbol{u}\|_{L^2(\Omega;\mathbb{C}^3)}^2\right)^{1/2}. \end{cases}$$

The curl and the divergence are defined in the weak sense as

$$\begin{cases} (\nabla \times \boldsymbol{u}, \boldsymbol{\phi}) = (\boldsymbol{u}, \nabla \times \boldsymbol{\phi}) \qquad \text{for all } \boldsymbol{\phi} \in D(\Omega; \mathbb{C}^3), \\ (\nabla \cdot \boldsymbol{u}, \phi) = -(\boldsymbol{u}, \nabla\phi) \qquad \text{for all } \phi \in D(\Omega). \end{cases}$$

In the exterior region, we define spaces of locally integrable functions as

$$\begin{cases} H_{\mathrm{loc}}(\mathrm{div},\overline{\Omega}_e) \stackrel{\mathrm{def}}{=} \left\{\boldsymbol{u}\in D'(\Omega_e;\mathbb{C}^3) : \xi\boldsymbol{u}\in H(\mathrm{div},\Omega_e) \text{ for all } \xi\in D(\mathbb{R}^3)\right\}, \\ H_{\mathrm{loc}}(\mathrm{rot},\overline{\Omega}_e) \stackrel{\mathrm{def}}{=} \left\{\boldsymbol{u}\in D'(\Omega_e;\mathbb{C}^3) : \xi\nabla\times\boldsymbol{u}\in H(\mathrm{rot},\Omega_e) \text{ for all } \xi\in D(\mathbb{R}^3)\right\}, \end{cases}$$

where $\Omega_e = \mathbb{R}^3\setminus\overline{\Omega}$ and $D'(\Omega_e)$ is the space of distributions. The appropriate trace spaces used in this paper are $H^{-\frac{1}{2}}(\mathrm{div},\partial\Omega)$ and $H^{-\frac{1}{2}}(\mathrm{rot},\partial\Omega)$ defined by

$$\begin{cases} H^{-\frac{1}{2}}(\mathrm{div},\partial\Omega) \stackrel{\mathrm{def}}{=} \left\{\boldsymbol{u}\in H^{-\frac{1}{2}}(\partial\Omega;\mathbb{C}^3),\ \hat{\boldsymbol{\nu}}\cdot\boldsymbol{u}=0,\ \mathrm{div}_{\partial\Omega}\boldsymbol{u}\in H^{-\frac{1}{2}}(\partial\Omega)\right\}, \\ H^{-\frac{1}{2}}(\mathrm{rot},\partial\Omega) \stackrel{\mathrm{def}}{=} \left\{\boldsymbol{u}\in H^{-\frac{1}{2}}(\partial\Omega;\mathbb{C}^3),\ \hat{\boldsymbol{\nu}}\cdot\boldsymbol{u}=0,\ \mathrm{rot}_{\partial\Omega}\boldsymbol{u}\in H^{-\frac{1}{2}}(\partial\Omega)\right\}, \end{cases}$$

where the surface divergence, $\mathrm{div}_{\partial\Omega}$, and the surface rotation, $\mathrm{rot}_{\partial\Omega}$, are defined by duality and restriction,

$$\begin{cases} (\mathrm{div}_{\partial\Omega}\boldsymbol{u},\phi) = -(\boldsymbol{u},\mathrm{grad}_{\partial\Omega}\phi) \qquad \text{for all } \phi\in D(\partial\Omega), \\ \mathrm{rot}_{\partial\Omega}\boldsymbol{u} = \hat{\boldsymbol{\nu}}\cdot(\nabla\times\boldsymbol{u})|_{\partial\Omega}, \end{cases}$$

and the surface gradient, $\mathrm{grad}_{\partial\Omega}$, is defined by the orthogonal projection of ∇ on the surface $\partial\Omega$.

We also define the function spaces

$$\begin{cases} H_{\#}(\mathrm{div},Y) \stackrel{\mathrm{def}}{=} \{\boldsymbol{u}\in H(\mathrm{div},Y), \boldsymbol{u}\ Y\text{-periodic}\}, \\ H_{\#}(\mathrm{rot},Y) \stackrel{\mathrm{def}}{=} \{\boldsymbol{u}\in H(\mathrm{rot},Y), \boldsymbol{u}\ Y\text{-periodic}\} \end{cases}$$

and

$$\begin{cases} H^1_{\#}(Y) \stackrel{\mathrm{def}}{=} \left\{\text{the completion of } C^\infty_{\#}(Y) \text{ in the } H^1(Y)\text{-norm}\right\}, \\ H^1_{\#}(Y)/\mathbb{C} \stackrel{\mathrm{def}}{=} \left\{\phi\in H^1_{\#}(Y),\ \text{equivalent up to a complex constant}\right\}. \end{cases}$$

If γ denotes the unit sphere in $\mathbb{R}^3$, the following norms are used in the paper:

$$\begin{cases} \|\boldsymbol{u}\|_\gamma = \left\{\iint_\gamma |\boldsymbol{u}(\hat{\boldsymbol{x}})|^2\, d\gamma\right\}^{1/2}, \\ \|\boldsymbol{u}\|_\infty = \sup_{|\hat{\boldsymbol{x}}|=1} |\boldsymbol{u}(\hat{\boldsymbol{x}})|, \end{cases}$$

and $d\gamma$ denotes the surface measure on the unit sphere in $\mathbb{R}^3$.

We conclude this appendix by stating the Lax–Milgram theorem [13].

THEOREM A.1 (Lax–Milgram). *Assume that H is a Hilbert space with norm $\|\cdot\|$. Moreover, assume that*

$$B : H\times H\to\mathbb{C}$$

is a sesquilinear functional on H, for which there exists constants $a, b>0$ such that

$$|B[u,v]| \le a\|u\|\|v\| \qquad \text{for all } u,v\in H$$

and

$$b\|u\|^2 \le |B[u,u]| \qquad \text{for all } u\in H.$$

Finally, let $f : H\to\mathbb{C}$ be a bounded linear functional on H.

Then there exists a unique $u \in H$ *such that*

$$B[u, v] = f(v) \quad \text{for all } v \in H.$$

Appendix B. Two-scale convergence.

DEFINITION B.1. *A sequence* $\{\boldsymbol{u}^\varepsilon\}$ *in* $L^2(\Omega; \mathbb{C}^3)$ *two-scale converges to* $\boldsymbol{u}_0 \in L^2(\Omega \times Y; \mathbb{C}^3)$ *if*

$$\lim_{\varepsilon \searrow 0} \iiint_\Omega \boldsymbol{u}^\varepsilon(\boldsymbol{x}) \cdot \boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{x}/\varepsilon)\, dv_{\boldsymbol{x}} = \iiint_\Omega \iiint_Y \boldsymbol{u}_0(\boldsymbol{x}, \boldsymbol{y}) \cdot \boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{y})\, dv_{\boldsymbol{y}}\, dv_{\boldsymbol{x}}$$

for every $\boldsymbol{\phi} \in D(\Omega; C^\infty_\#(Y; \mathbb{C}^3))$. *We denote this by* $\boldsymbol{u}^\varepsilon \overset{2\text{-}s}{\rightharpoonup} \boldsymbol{u}_0$.

The class of test functions can be enlarged to all admissible test functions defined below [2].

DEFINITION B.2. *We say that* $\boldsymbol{\phi} \in L^2(\Omega; L^2_\#(Y; \mathbb{C}^3))$ *is an admissible test function if* $\boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{x}/\varepsilon)$ *is measurable and*

$$\lim_{\varepsilon \searrow 0} \|\boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{x}/\varepsilon)\|_{L^2(\Omega; \mathbb{C}^3)} = \|\boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{y})\|_{L^2(\Omega \times Y; \mathbb{C}^3)}.$$

Remark B.1. Some examples of admissible test functions are $L^2(\Omega; C_\#(Y; \mathbb{C}^3))$ and for Ω bounded $L^2_\#(Y; C(\overline{\Omega}; \mathbb{C}^3))$.

We cite two important theorems by Nguetseng [19].

THEOREM B.3 (Nguetseng [19]). *Let* $u^\varepsilon \in L^2(\Omega)$. *Suppose that there exists a constant* $C > 0$ *such that*

$$\|u^\varepsilon\|_{L^2(\Omega)} \leq C \text{ for all } \varepsilon.$$

Then a subsequence (still denoted by ε*) can be extracted from* ε *such that, letting* $\varepsilon \searrow 0$,

$$\iiint_\Omega u^\varepsilon(\boldsymbol{x}) \Psi(\boldsymbol{x}, \boldsymbol{x}/\varepsilon)\, dv_{\boldsymbol{x}} \to \iiint_\Omega \iiint_Y u_0(\boldsymbol{x}, \boldsymbol{y}) \Psi(\boldsymbol{x}, \boldsymbol{y})\, dv_{\boldsymbol{y}}\, dv_{\boldsymbol{x}}$$

for all $\Psi \in C_0(\overline{\Omega}; C_\#(Y))$, *where* $u_0 \in L^2(\Omega; L^2_\#(Y))$. *Moreover,*

$$\iiint_\Omega u^\varepsilon(\boldsymbol{x}) v(\boldsymbol{x}) w(\boldsymbol{x}/\varepsilon)\, dv_{\boldsymbol{x}} \to \iiint_\Omega \iiint_Y u_0(\boldsymbol{x}, \boldsymbol{y}) v(\boldsymbol{x}) w(\boldsymbol{y})\, dv_{\boldsymbol{y}}\, dv_{\boldsymbol{x}}$$

for all $v \in C_0(\overline{\Omega})$ *and all* $w \in L^2_\#(Y)$.

We note that if u^ε is a sequence in $L^2(\Omega)$, which two-scale converges to the limit $u_0 \in L^2(\Omega \times Y)$, then u^ε also converges to $u(\boldsymbol{x}) = \iiint_Y u_0(\boldsymbol{x}, \boldsymbol{y})\, dv_{\boldsymbol{y}}$ in $L^2(\Omega)$ weakly [2]. Moreover, if u^ε converges strongly to $u(\boldsymbol{x})$ in $L^2(\Omega)$, then u^ε two-scale converges to the same limit $u(\boldsymbol{x})$. The second theorem is the following.

THEOREM B.4 (Nguetseng). *Let* $u^\varepsilon \in H^1(\Omega)$. *Suppose that there exists a constant* $C > 0$ *such that*

$$\|u^\varepsilon\|_{H^1(\Omega)} \leq C \text{ for all } \varepsilon.$$

Then a subsequence (still denoted by ε*) can be extracted from* ε *such that, letting* $\varepsilon \searrow 0$,

$$u^\varepsilon \to u \text{ in } H^1(\Omega)\text{-weak}$$

and

$$\iiint_\Omega \frac{\partial u^\varepsilon(\boldsymbol{x})}{\partial x_j} v(\boldsymbol{x}) w(\boldsymbol{x}/\varepsilon)\, dv_{\boldsymbol{x}} \to \iiint_\Omega \iiint_Y \left\{ \frac{\partial u(\boldsymbol{x})}{\partial x_j} + \frac{\partial u_1(\boldsymbol{x},\boldsymbol{y})}{\partial y_j} \right\} v(\boldsymbol{x}) w(\boldsymbol{y})\, dv_{\boldsymbol{y}}\, dv_{\boldsymbol{x}},$$

$j = 1,2,3$, *for all* $v \in C_0(\overline{\Omega})$ *and all* $w \in L^2_\#(Y)$, *where* $u_1 \in L^2(\Omega; H^1_\#(Y)/\mathbb{C})$.

In addition to these two theorems, we observe that, taking $w = 1$, we get from Theorem B.3

$$\iiint_\Omega u^\varepsilon(\boldsymbol{x}) v(\boldsymbol{x})\, dv_{\boldsymbol{x}} \to \iiint_\Omega u(\boldsymbol{x}) v(\boldsymbol{x})\, dv_{\boldsymbol{x}}$$

for all $v \in C_0(\overline{\Omega})$, where

$$u(\boldsymbol{x}) = \iiint_Y u_0(\boldsymbol{x},\boldsymbol{y})\, dv_{\boldsymbol{y}}$$

is the usual weak $L^2(\Omega)$-limit of $u^\varepsilon(\boldsymbol{x})$. It follows that u_0 is uniquely expressed in the form

$$u_0(\boldsymbol{x},\boldsymbol{y}) = u(\boldsymbol{x}) + \tilde{u}_0(\boldsymbol{x},\boldsymbol{y}),$$

where

$$\iiint_Y \tilde{u}_0(\boldsymbol{x},\boldsymbol{y})\, dv_{\boldsymbol{y}} = 0.$$

LEMMA B.5. *Let* $\boldsymbol{f} \in H^1_\#(Y;\mathbb{C}^3)$ *and assume that* $\nabla_{\boldsymbol{y}} \times \boldsymbol{f}(\boldsymbol{y}) = \boldsymbol{0}$. *Moreover, assume that* $\langle \boldsymbol{f} \rangle = 0$. *Then there exists a unique function* $q \in H^1_\#(Y)/\mathbb{C}$ *such that*

$$\boldsymbol{f}(\boldsymbol{y}) = \nabla_{\boldsymbol{y}} q(\boldsymbol{y}).$$

Proof of Lemma B.5. The periodicity of the function $\boldsymbol{f} \in H^1_\#(Y;\mathbb{C}^3)$ implies that $\boldsymbol{f}$ has a Fourier expansion

$$\boldsymbol{f}(\boldsymbol{y}) = \sum_n \boldsymbol{f}_n e^{i\boldsymbol{k}_n\cdot\boldsymbol{y}},$$

where the vector $\boldsymbol{k}_n$ is defined as

$$\boldsymbol{k}_n = 2\pi n_1 \hat{\boldsymbol{e}}_1 + 2\pi n_2 \hat{\boldsymbol{e}}_2 + 2\pi n_3 \hat{\boldsymbol{e}}_3$$

and where n_1, n_2, n_3 are integers and $n = (n_1, n_2, n_3)$. The sequence $\boldsymbol{f}_n$ belongs to $\left(\ell^2_1\right)^3$. The assumption that $\langle \boldsymbol{f} \rangle = 0$ implies that $\boldsymbol{f}_{(0,0,0)} = 0$. Moreover, the coefficients $\boldsymbol{f}_n$ satisfy

$$\boldsymbol{k}_n \times \boldsymbol{f}_n = \boldsymbol{0} \quad \text{for all } n.$$

Therefore $\boldsymbol{f}_n$ has the form

$$\boldsymbol{f}_n = \hat{\boldsymbol{k}}_n \left(\hat{\boldsymbol{k}}_n \cdot \boldsymbol{f}_n \right).$$

Define q_n as

$$\begin{cases} q_n = -i(\hat{\boldsymbol{k}}_n \cdot \boldsymbol{f}_n)/k_n \text{ for } n \neq (0,0,0), \\ q_{(0,0,0)} \text{ arbitrary,} \end{cases}$$

where $k_n = |\boldsymbol{k}_n|$. The coefficients $q_n \in \left(\ell_1^2\right)^3$,

$$\boldsymbol{f}_n = i\boldsymbol{k}_n q_n \quad \text{for all } n,$$

and

$$\boldsymbol{f}(\boldsymbol{y}) = \sum_n i\boldsymbol{k}_n q_n e^{i\boldsymbol{k}_n \cdot \boldsymbol{y}} = \nabla_{\boldsymbol{y}} q(\boldsymbol{y}),$$

where

$$q(\boldsymbol{y}) = \sum_n q_n e^{i\boldsymbol{k}_n \cdot \boldsymbol{y}} \in H^1_\#(Y)/\mathbb{C},$$

since $q_{(0,0,0)}$ is arbitrary and the lemma is proved. $\square$

The obvious vector analogous theorems follow.

THEOREM B.6. *Let $\boldsymbol{u}^\varepsilon \in L^2(\Omega;\mathbb{C}^3)$. Suppose that there exists a constant $C > 0$ such that*

$$\|\boldsymbol{u}^\varepsilon\|_{L^2(\Omega;\mathbb{C}^3)} \leq C \text{ for all } \varepsilon.$$

Then a subsequence (still denoted by ε) can be extracted from ε such that, letting $\varepsilon \searrow 0$,

$$\iiint_\Omega \boldsymbol{u}^\varepsilon(\boldsymbol{x}) \cdot \boldsymbol{\Psi}(\boldsymbol{x}, \boldsymbol{x}/\varepsilon)\, dv_{\boldsymbol{x}} \to \iiint_\Omega \iiint_Y \boldsymbol{u}_0(\boldsymbol{x}, \boldsymbol{y}) \cdot \boldsymbol{\Psi}(\boldsymbol{x}, \boldsymbol{y})\, dv_{\boldsymbol{y}}\, dv_{\boldsymbol{x}}$$

for all $\boldsymbol{\Psi} \in C_0(\overline{\Omega}; C_\#(Y;\mathbb{C}^3))$, where $\boldsymbol{u}_0 \in L^2(\Omega; L^2_\#(Y;\mathbb{C}^3))$. Moreover,

$$\iiint_\Omega \boldsymbol{u}^\varepsilon(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x}) w(\boldsymbol{x}/\varepsilon)\, dv_{\boldsymbol{x}} \to \iiint_\Omega \iiint_Y \boldsymbol{u}_0(\boldsymbol{x}, \boldsymbol{y}) \cdot \boldsymbol{v}(\boldsymbol{x}) w(\boldsymbol{y})\, dv_{\boldsymbol{y}}\, dv_{\boldsymbol{x}}$$

for all $\boldsymbol{v} \in C_0(\overline{\Omega};\mathbb{C}^3)$ and all $w \in L^2_\#(Y)$.

The field $\boldsymbol{u}_0$ is uniquely expressed in the form

$$\boldsymbol{u}_0(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{u}(\boldsymbol{x}) + \tilde{\boldsymbol{u}}_0(\boldsymbol{x}, \boldsymbol{y}),$$

where

$$\iiint_Y \tilde{\boldsymbol{u}}_0(\boldsymbol{x}, \boldsymbol{y})\, dv_{\boldsymbol{y}} = \boldsymbol{0}.$$

We have the following results proved in [23].

THEOREM B.7. *Let $\boldsymbol{u}^\varepsilon \in H(\mathrm{div}, \Omega)$. Suppose that there exists a constant $C > 0$ such that*

$$\|\boldsymbol{u}^\varepsilon\|_{H(\mathrm{div},\Omega)} \leq C \quad \text{for all } \varepsilon.$$

Then a subsequence (still denoted by ε) can be extracted from ε such that, letting $\varepsilon \searrow 0$,

$$\boldsymbol{u}^\varepsilon \to \boldsymbol{u} \text{ in } L^2(\Omega;\mathbb{C}^3)\text{-weak}$$

and

$$\iiint_\Omega \nabla_{\boldsymbol{x}} \cdot \boldsymbol{u}^\varepsilon(\boldsymbol{x}) v(\boldsymbol{x}) w(\boldsymbol{x}/\varepsilon)\, dv_{\boldsymbol{x}}$$

$$\to \iiint_\Omega \iiint_Y \{\nabla_{\boldsymbol{x}} \cdot \boldsymbol{u}(\boldsymbol{x}) + \nabla_{\boldsymbol{y}} \cdot \boldsymbol{u}_1(\boldsymbol{x},\boldsymbol{y})\}\, v(\boldsymbol{x}) w(\boldsymbol{y})\, dv_{\boldsymbol{y}}\, dv_{\boldsymbol{x}}$$

for all $v \in C_0(\overline{\Omega})$ *and all* $w \in L^2_\#(Y)$, *where* $\boldsymbol{u}(\boldsymbol{x}) = \iiint_Y \boldsymbol{u}_0(\boldsymbol{x},\boldsymbol{y})\, dv_{\boldsymbol{y}}$, $\boldsymbol{u}_0$ *is the two-scale limit of* $\boldsymbol{u}^\varepsilon$, *and* $\boldsymbol{u}_1 \in L^2(\Omega; H_\#(\mathrm{div}, Y))$.

THEOREM B.8. *Let* $\boldsymbol{u}^\varepsilon \in H(\mathrm{rot}, \Omega)$. *Suppose that there exists a constant* $C > 0$ *such that*

$$\|\boldsymbol{u}^\varepsilon\|_{H(\mathrm{rot},\Omega)} \le C \quad \textit{for all } \varepsilon.$$

Then a subsequence (still denoted by ε*) can be extracted from* ε *such that, letting* $\varepsilon \searrow 0$,

$$\boldsymbol{u}^\varepsilon \to \boldsymbol{u}_0 \ \textit{in } L^2(\Omega; \mathbb{C}^3)\textit{-weak}$$

and

$$\iiint_\Omega \nabla \times \boldsymbol{u}^\varepsilon(\boldsymbol{x}) \cdot v(\boldsymbol{x}) \boldsymbol{w}(\boldsymbol{x}/\varepsilon)\, dv_{\boldsymbol{x}}$$

$$\to \iiint_\Omega \iiint_Y \{\nabla_{\boldsymbol{x}} \times \boldsymbol{u}_0(\boldsymbol{x},\boldsymbol{y}) + \nabla_{\boldsymbol{y}} \times \boldsymbol{u}_1(\boldsymbol{x},\boldsymbol{y})\} \cdot v(\boldsymbol{x}) \boldsymbol{w}(\boldsymbol{y})\, dv_{\boldsymbol{y}}\, dv_{\boldsymbol{x}}$$

for all $v \in C_0(\overline{\Omega})$ *and all* $\boldsymbol{w} \in L^2_\#(Y; \mathbb{C}^3)$, *where* $\boldsymbol{u}_1 \in L^2(\Omega; H_\#(\mathrm{rot}, Y))$.

Proof of Theorem B.8. From Theorem B.6 we get

$$\iiint_\Omega \boldsymbol{u}^\varepsilon(\boldsymbol{x}) \cdot \boldsymbol{\Psi}(\boldsymbol{x}, \boldsymbol{x}/\varepsilon)\, dv_{\boldsymbol{x}} \to \iiint_\Omega \iiint_Y \boldsymbol{u}_0(\boldsymbol{x},\boldsymbol{y}) \cdot \boldsymbol{\Psi}(\boldsymbol{x},\boldsymbol{y})\, dv_{\boldsymbol{y}}\, dv_{\boldsymbol{x}}$$

and

$$\iiint_\Omega \nabla \times \boldsymbol{u}^\varepsilon(\boldsymbol{x}) \cdot \boldsymbol{\Psi}(\boldsymbol{x}, \boldsymbol{x}/\varepsilon)\, dv_{\boldsymbol{x}} \to \iiint_\Omega \iiint_Y \boldsymbol{\chi}_0(\boldsymbol{x},\boldsymbol{y}) \cdot \boldsymbol{\Psi}(\boldsymbol{x},\boldsymbol{y})\, dv_{\boldsymbol{y}}\, dv_{\boldsymbol{x}}$$

for all $\boldsymbol{\Psi} \in C_0(\overline{\Omega}; C_\#(Y; \mathbb{C}^3))$, where $\boldsymbol{u}_0, \boldsymbol{\chi}_0 \in L^2(\Omega; L^2_\#(Y; \mathbb{C}^3))$. Choose test functions $\boldsymbol{\Psi} \in C_0(\overline{\Omega}; C_\#(Y; \mathbb{C}^3))$ such that $\nabla_{\boldsymbol{y}} \times \boldsymbol{\Psi} = 0$. We get by integration by parts

$$\begin{aligned}
\iiint_\Omega \nabla \times \boldsymbol{u}^\varepsilon(\boldsymbol{x}) \cdot \boldsymbol{\Psi}(\boldsymbol{x}, \boldsymbol{x}/\varepsilon)\, dv_{\boldsymbol{x}} &= \iiint_\Omega \boldsymbol{u}^\varepsilon(\boldsymbol{x}) \cdot \nabla \times \boldsymbol{\Psi}(\boldsymbol{x}, \boldsymbol{x}/\varepsilon)\, dv_{\boldsymbol{x}} \\
= \quad & \iiint_\Omega \boldsymbol{u}^\varepsilon(\boldsymbol{x}) \cdot \nabla_{\boldsymbol{x}} \times \boldsymbol{\Psi}(\boldsymbol{x}, \boldsymbol{x}/\varepsilon)\, dv_{\boldsymbol{x}} \\
\to \quad & \iiint_\Omega \iiint_Y \boldsymbol{u}_0(\boldsymbol{x},\boldsymbol{y}) \cdot \nabla_{\boldsymbol{x}} \times \boldsymbol{\Psi}(\boldsymbol{x},\boldsymbol{y})\, dv_{\boldsymbol{y}}\, dv_{\boldsymbol{x}} \\
= \quad & \iiint_\Omega \iiint_Y \nabla_{\boldsymbol{x}} \times \boldsymbol{u}_0(\boldsymbol{x},\boldsymbol{y}) \cdot \boldsymbol{\Psi}(\boldsymbol{x},\boldsymbol{y})\, dv_{\boldsymbol{y}}\, dv_{\boldsymbol{x}}.
\end{aligned}$$

This means that

$$\iiint_\Omega \iiint_Y (\boldsymbol{\chi}_0(\boldsymbol{x},\boldsymbol{y}) - \nabla_{\boldsymbol{x}} \times \boldsymbol{u}_0(\boldsymbol{x},\boldsymbol{y})) \cdot \boldsymbol{\Psi}(\boldsymbol{x},\boldsymbol{y})\, dv_{\boldsymbol{y}}\, dv_{\boldsymbol{x}} = 0$$

for all $\boldsymbol{\Psi} \in C_0(\overline{\Omega}; C_\#(Y;\mathbb{C}^3))$ such that $\nabla_{\boldsymbol{y}} \times \boldsymbol{\Psi} = 0$. By the decomposition of $L^2(\Omega;\mathbb{C}^3)$ (e.g., see [9]) there exists a function $\boldsymbol{u}_1 \in L^2(\Omega; H_\#(\mathrm{rot}, Y))$ such that

$$\nabla_{\boldsymbol{y}} \times \boldsymbol{u}_1 = \boldsymbol{\chi}_0(\boldsymbol{x},\boldsymbol{y}) - \nabla_{\boldsymbol{x}} \times \boldsymbol{u}_0(\boldsymbol{x},\boldsymbol{y}). \qquad \square$$

THEOREM B.9 (see Wellander [26] or [27]). *Let* $\boldsymbol{u}^\varepsilon \in H(\mathrm{rot},\Omega)$. *Suppose that there exists a constant* $C > 0$ *such that*

$$\|\boldsymbol{u}^\varepsilon\|_{H(\mathrm{rot},\Omega)} \leq C \quad \textit{for all } \varepsilon.$$

Then a subsequence (still denoted by ε*) can be extracted from* ε *such that, letting* $\varepsilon \searrow 0$,

$$\boldsymbol{u}^\varepsilon \overset{2\text{-}s}{\rightharpoonup} \boldsymbol{u}(\boldsymbol{x}) + \nabla_{\boldsymbol{y}}\phi(\boldsymbol{x},\boldsymbol{y}),$$

where $\phi \in L^2(\Omega; H^1_\#(Y))$ *is a scalar-valued function satisfying*

$$\iiint_Y \nabla_{\boldsymbol{y}}\phi(\boldsymbol{x},\boldsymbol{y})\, dv_{\boldsymbol{y}} = \mathbf{0}.$$

Moreover,

$$\nabla \times \boldsymbol{u}^\varepsilon \rightharpoonup \nabla \times \boldsymbol{u}(\boldsymbol{x}) \textit{ in } L^2(\Omega;\mathbb{C}^3).$$

THEOREM B.10 (see Wellander [26] or [27]). *Let* $\boldsymbol{u}^\varepsilon \in H(\mathrm{div},\Omega)$. *Suppose that there exists a constant* $C > 0$ *such that*

$$\|\boldsymbol{u}^\varepsilon\|_{H(\mathrm{div},\Omega)} \leq C \quad \textit{for all } \varepsilon.$$

Then a subsequence (still denoted by ε*) can be extracted from* ε *such that, letting* $\varepsilon \searrow 0$,

$$\boldsymbol{u}^\varepsilon \overset{2\text{-}s}{\rightharpoonup} \boldsymbol{u}_0(\boldsymbol{x},\boldsymbol{y})$$

and

$$\varepsilon\nabla \cdot \boldsymbol{u}^\varepsilon \overset{2\text{-}s}{\rightharpoonup} \nabla_{\boldsymbol{y}} \cdot \boldsymbol{u}_0(\boldsymbol{x},\boldsymbol{y}).$$

Proof of Theorem B.10. From Theorem B.6 we get

$$\iiint_\Omega \boldsymbol{u}^\varepsilon(\boldsymbol{x}) \cdot \boldsymbol{\Psi}(\boldsymbol{x},\boldsymbol{x}/\varepsilon)\, dv_{\boldsymbol{x}} \to \iiint_\Omega \iiint_Y \boldsymbol{u}_0(\boldsymbol{x},\boldsymbol{y}) \cdot \boldsymbol{\Psi}(\boldsymbol{x},\boldsymbol{y})\, dv_{\boldsymbol{y}}\, dv_{\boldsymbol{x}}$$

and

$$\iiint_\Omega \nabla \cdot \boldsymbol{u}^\varepsilon(\boldsymbol{x}) \Psi(\boldsymbol{x},\boldsymbol{x}/\varepsilon)\, dv_{\boldsymbol{x}} \to \iiint_\Omega \iiint_Y \chi_0(\boldsymbol{x},\boldsymbol{y}) \Psi(\boldsymbol{x},\boldsymbol{y})\, dv_{\boldsymbol{y}}\, dv_{\boldsymbol{x}}$$

for all $\boldsymbol{\Psi} \in C_0(\overline{\Omega}; C_\#(Y;\mathbb{C}^3))$ and $\Psi \in C_0(\overline{\Omega}; C_\#(Y))$, where $\boldsymbol{u}_0 \in L^2(\Omega; L^2_\#(Y;\mathbb{C}^3))$ and $\chi_0 \in L^2(\Omega; L^2_\#(Y))$.

We get by integration by parts

$$\begin{aligned}
&\iiint_\Omega \varepsilon\nabla \cdot \boldsymbol{u}^\varepsilon(\boldsymbol{x}) \Psi(\boldsymbol{x},\boldsymbol{x}/\varepsilon)\, dv_{\boldsymbol{x}} = -\iiint_\Omega \varepsilon\boldsymbol{u}^\varepsilon(\boldsymbol{x}) \cdot \nabla\Psi(\boldsymbol{x},\boldsymbol{x}/\varepsilon)\, dv_{\boldsymbol{x}} \\
&\quad = -\iiint_\Omega \varepsilon\boldsymbol{u}^\varepsilon(\boldsymbol{x}) \cdot \nabla_{\boldsymbol{x}}\Psi(\boldsymbol{x},\boldsymbol{x}/\varepsilon)\, dv_{\boldsymbol{x}} - \iiint_\Omega \boldsymbol{u}^\varepsilon(\boldsymbol{x}) \cdot \nabla_{\boldsymbol{y}}\Psi(\boldsymbol{x},\boldsymbol{x}/\varepsilon)\, dv_{\boldsymbol{x}} \\
&\quad \to -\iiint_\Omega \iiint_Y \boldsymbol{u}_0(\boldsymbol{x},\boldsymbol{y}) \cdot \nabla_{\boldsymbol{y}}\Psi(\boldsymbol{x},\boldsymbol{y})\, dv_{\boldsymbol{y}}\, dv_{\boldsymbol{x}} \\
&\quad = \iiint_\Omega \iiint_Y \nabla_{\boldsymbol{y}} \cdot \boldsymbol{u}_0(\boldsymbol{x},\boldsymbol{y}) \Psi(\boldsymbol{x},\boldsymbol{y})\, dv_{\boldsymbol{y}}\, dv_{\boldsymbol{x}}. \qquad \square
\end{aligned}$$

Appendix C. Vector spherical harmonics. The vector spherical harmonics are defined as (see [8])

$$\begin{cases} \boldsymbol{A}_{1n}(\hat{\boldsymbol{x}}) = \frac{1}{\sqrt{l(l+1)}}\nabla\times(\boldsymbol{x}Y_n(\hat{\boldsymbol{x}})) = \frac{1}{\sqrt{l(l+1)}}\nabla Y_n(\hat{\boldsymbol{x}})\times\boldsymbol{x}, \\ \boldsymbol{A}_{2n}(\hat{\boldsymbol{x}}) = \frac{1}{\sqrt{l(l+1)}}x\nabla Y_n(\hat{\boldsymbol{x}}), \\ \boldsymbol{A}_{3n}(\hat{\boldsymbol{x}}) = \hat{\boldsymbol{x}}Y_n(\hat{\boldsymbol{x}}), \end{cases}$$

where the spherical harmonics are denoted $Y_n(\hat{\boldsymbol{x}})$. The index n is a multi-index for the integer indices $l = 0, 1, 2, 3, \dots$, $m = 0, 1, \dots, l$, and $\sigma =$ e,o (even and odd in the azimuthal angle). From these definitions we see that the first two vector spherical harmonics, $\boldsymbol{A}_{1n}(\hat{\boldsymbol{x}})$ and $\boldsymbol{A}_{2n}(\hat{\boldsymbol{x}})$, are tangential to the unit sphere γ in $\mathbb{R}^3$ and they are related by

$$\begin{cases} \hat{\boldsymbol{x}}\times\boldsymbol{A}_{1n}(\hat{\boldsymbol{x}}) = \boldsymbol{A}_{2n}(\hat{\boldsymbol{x}}), \\ \hat{\boldsymbol{x}}\times\boldsymbol{A}_{2n}(\hat{\boldsymbol{x}}) = -\boldsymbol{A}_{1n}(\hat{\boldsymbol{x}}). \end{cases}$$

The vector spherical harmonics form an orthonormal set over the unit sphere γ in $\mathbb{R}^3$, i.e.,

$$\iint_\gamma \boldsymbol{A}_{\tau n}(\hat{\boldsymbol{x}})\cdot\boldsymbol{A}_{\tau' n'}(\hat{\boldsymbol{x}})\,d\gamma = \delta_{nn'}\delta_{\tau\tau'}. \tag{C.1}$$

The radiating solutions to the Maxwell equations in a vacuum are defined as

$$\begin{cases} \boldsymbol{u}_{1n}(k_0\boldsymbol{x}) = h_l^{(1)}(k_0x)\boldsymbol{A}_{1n}(\hat{\boldsymbol{x}}), \\ \boldsymbol{u}_{2n}(k_0\boldsymbol{x}) = \frac{1}{k_0}\nabla\times\left(h_l^{(1)}(k_0x)\boldsymbol{A}_{1n}(\hat{\boldsymbol{x}})\right), \end{cases}$$

where $h_l^{(1)}(k_0x)$ is the spherical Hankel function of the first kind [1]. These vector waves satisfy

$$\nabla\times(\nabla\times\boldsymbol{u}_{\tau n}(k_0\boldsymbol{x})) - k_0^2\boldsymbol{u}_{\tau n}(k_0\boldsymbol{x}) = \boldsymbol{0}, \qquad \tau = 1, 2, \tag{C.2}$$

and they also satisfy the radiation condition in (2.5). Another representation of the definition of the vector waves is

$$\begin{cases} \boldsymbol{u}_{1n}(k_0\boldsymbol{x}) = h_l^{(1)}(k_0x)\boldsymbol{A}_{1n}(\hat{\boldsymbol{x}}), \\ \boldsymbol{u}_{2n}(k_0\boldsymbol{x}) = \frac{(k_0xh_l^{(1)}(k_0x))'}{k_0x}\boldsymbol{A}_{2n}(\hat{\boldsymbol{x}}) + \sqrt{l(l+1)}\frac{h_l^{(1)}(k_0x)}{k_0x}\boldsymbol{A}_{3n}(\hat{\boldsymbol{x}}), \end{cases}$$

where $'$ denotes differentiation with respect to the argument of the spherical Hankel function. A simple consequence of these definitions is

$$\begin{cases} \boldsymbol{u}_{1n}(k_0\boldsymbol{x}) = \frac{1}{k_0}\nabla\times\boldsymbol{u}_{2n}(k_0\boldsymbol{x}), \\ \boldsymbol{u}_{2n}(k_0\boldsymbol{x}) = \frac{1}{k_0}\nabla\times\boldsymbol{u}_{1n}(k_0\boldsymbol{x}). \end{cases} \tag{C.3}$$

REFERENCES

[1] M. Abramowitz and I. A. Stegun, eds., *Handbook of Mathematical Functions*, Applied Mathematics Series 55, National Bureau of Standards, Washington, DC, 1970.

[2] G. Allaire, *Homogenization and two-scale convergence*, SIAM J. Math. Anal., 23 (1992), pp. 1482–1518.

[3] Y. Amirat, K. Hamdache, and A. Ziani, *Homogenization of degenerate wave equations with periodic coefficients*, SIAM J. Math. Anal., 24 (1993), pp. 1226–1253.
[4] M. Artola, *Homogenization and electromagnetic wave propagation in composite media with high conductivity inclusions*, in Proceedings of the Second Workshop on Composite Media and Homogenization Theory, G. Dal Maso and G. Dell'Antonio, eds., World Scientific, Singapore, 1995, pp. 1–15.
[5] H. Attouch, *Variational Convergence of Functions and Operators*, Pitman, London, 1984.
[6] A. Bensoussan, J. L. Lions, and G. Papanicolaou, *Asymptotic Analysis for Periodic Structures*, Stud. Math. Appl. 5, North–Holland, Amsterdam, 1978.
[7] A. Bossavit, *On the homogenization of Maxwell equations*, COMPEL, 14 (1995), pp. 23–26.
[8] A. Boström, G. Kristensson, and S. Ström, *Transformation properties of plane, spherical and cylindrical scalar and vector wave functions*, in Field Representations and Introduction to Scattering, V. V. Varadan, A. Lakhtakia, and V. K. Varadan, eds., Acoustic, Electromagnetic and Elastic Wave Scattering 1, North–Holland, Amsterdam, 1991, pp. 165–210.
[9] M. Cessenat, *Mathematical Methods in Electromagnetism*, Ser. Adv. Math. Appl. Sci. 41, World Scientific, Singapore, 1996.
[10] D. Cioranescu and P. Donato, *An Introduction to Homogenization*, Oxford University Press, Oxford, 1999.
[11] D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, Springer-Verlag, Berlin, 1992.
[12] C. Conca and M. Vanninathan, *Homogenization of periodic structures via Bloch decomposition*, SIAM J. Appl. Math., 57 (1997), pp. 1639–1659.
[13] L. C. Evans, *Partial Differential Equations*, AMS, Providence, RI, 1998.
[14] A. Holmbom, *Homogenization of parabolic equations: An alternative approach and some corrector-type results*, Appl. Math., 42 (1997), pp. 321–343.
[15] V. V. Jikov, S. M. Kozlov, and O. A. Oleinik, *Homogenization of Differential Operators and Integral Functionals*, Springer-Verlag, Berlin, 1994.
[16] G. Kristensson, S. Poulsen, and S. Rikte, *Propagators and Scattering of Electromagnetic waves in Planar Bianisotropic Slabs—An Application to Frequency Selective Structures*, Technical report LUTEDX/(TEAT-7099)/1–32/(2001), Lund Institute of Technology, Department of Electroscience, Lund, Sweden, 2001.
[17] D. Lukkassen, G. Nguetseng, and P. Wall, *Two scale convergence*, J. Pure Appl. Math., 2 (2002), pp. 35–86.
[18] P. A. Markowich and F. Poupaud, *The Maxwell equation in a periodic medium: Homogenization of the energy density*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 23 (1996), pp. 301–324.
[19] G. Nguetseng, *A general convergence result for a functional related to the theory of homogenization*, SIAM J. Math. Anal., 20 (1989), pp. 608–623.
[20] E. Sanchez-Palencia, *Non-Homogeneous Media and Vibration Theory*, Lecture Notes in Phys. 127, Springer-Verlag, Berlin, 1980.
[21] S. Shkoller and G. Hegemier, *Homogenization of plane weave composites using two-scale convergence*, Internat. J. Solids Structures, 32 (1995), pp. 783–794.
[22] A. Sihvola, *Electromagnetic Mixing Formulae and Applications*, IEE Electromagnet. Waves Ser. 47, IEE, London, 1999.
[23] N. Svanstedt and N. Wellander, *A Note on Two-Scale Limits of Differential Operators*, Technical report 19, Department of Mathematics, Chalmers University of Technology, Göteborg, Sweden, 2001.
[24] L. Tartar, *Cours peccot au Collège de France*, manuscript, 1977.
[25] L. Tartar, *Compensated compactness and applications to partial differential equations*, in Nonlinear Analysis and Mechanics: Heriot-Watt Symposium, Vol. 4, Res. Notes in Math. 39, Pitman, London, 1979, pp. 136–212.
[26] N. Wellander, *Homogenization of Some Linear and Nonlinear Partial Differential Equations*, Ph.D. thesis, Luleå University of Technology, Luleå, Sweden, 1998.
[27] N. Wellander, *Homogenization of the Maxwell equations: Case* I. *Linear theory*, Appl. Math., 46 (2001), pp. 29–51.
[28] N. Wellander, *Homogenization of the Maxwell equations: Case* II. *Nonlinear conductivity*, Appl. Math., 47 (2002), pp. 255–283.