Filtering in Distributed Parameter Systems

A Survey Introduction

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Ruth Curtain

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1. INTRODUCTION.

In the early days of Kalman-type filtering, there appeared
in the journals much literature about the problem of esti-
mating a state vector $u(T)$ from an observation process $z(t)$
on $0 \leqslant t \leqslant T$, assuming a linear dynamical system for $u(t)$
with a "white noise" forcing term, i.e.

$$(1.1) \quad \begin{cases} \dot{u}(t) = A(t)u(t) + B(t)\xi(t) \\ u(0) = u_0 \\ z(t) = C(t)u(t) + \eta(t) \end{cases}$$

where A, B, C are continuous matrix functions and $\xi(t)$,
$\eta(t)$ are independent vector Gaussian white noises, i.e.

$$E\{\xi(t)\xi(s)^1\} = \delta(t-s)Q(t) \qquad\qquad (1.2)$$

$$E\{\eta(t)\eta(s)^1\} = \delta(t-s)R(t).$$

where Q,R are continuous in t, symmetric and positive mat-
rices and $R(t)$ is invertible.

Using this formal approach, filtering equations for a
best least squares estimate can be derived as for the
discrete time case. However, the definition of the dis-
turbance is not probabilistically well founded, because
$\xi(t)$ is equivalent to the derivative of a Wiener process
which simply does not exist. In other words, the state
vector $u(t)$ and observation $z(t)$ are not stochastic pro-
cesses in the probabilistic sense.

All of these problems are solved, if one uses the Itô in-
tegral and writes

$$\int_0^t B(t)dw(t)$$

for the "white noise" and defines the stochastic dyna-
mical system in integral form, i.e.

$$(1.3) \begin{cases} du(t) = A(t)u(t)dt + B(t)dw(t) & E\{w(t)w(s)^1\} = \int_0^{\min(t,s)} Q(\tau)d\tau \\ u(0) = u_0 \\ dz(t) = C(t)u(t)dt + dv(t) & E\{v(t)v(s)^1\} = \int_0^{\min(t,s)} R(\tau)d\tau \end{cases}$$

where w, v are independent vector Wiener processes. The
solutions u(t) and z(t) are now well defined vector sto-
chastic processes, and the best least squares estimate
of u(t) based on the observations z(s), $0 \leqslant s \leqslant t$ is $\hat{u}(t)$
and is given by the familiar Kalman-Bucy equations:

$$(1.4) \begin{cases} d\hat{u}(t) = A(t)\hat{u}(t)dt + P(t)C(t)^*R^{-1}(t)\left(dz(t) - C(t)\hat{u}(t)dt\right) \\ \hat{u}(0) = 0 \end{cases}$$

$$(1.5) \begin{cases} \dfrac{dP}{dt} - A(t)P(t) - P(t)A(t)^* + P(t)C(t)^*R^{-1}(t)C(t)P(t) = \\ \qquad = B(t)Q(t)B(t)^* \\ P(0) = P_0 = \mathrm{Cov}(x_0 x_0^1) \end{cases}$$

$\big(P(t)$ of course is the covariance of the error $u(t) - \hat{u}(t).\big)$

In many physical examples, the dynamical system one wishes
to estimate is described some partial-integro differential
equation; to mention a few of the often quoted ones from
the literature:

1. a heat exchanger

2. electrical, optical or acoustic waves which are random-
    ly excited

3. random variations on an electrical transmission line.

It seems to me that these motivations are only secondary, the primary one being a generalization of the Kalman-Bucy filtering theory to distributed parameter systems. As one would expect, the class of distributed parameter system which has been considered is small, namely parabolic partial differential equations with initial data and boundary conditions, so that the dynamical system always has a unique solution which depends continuously on the data. (There are some exceptions to this which will be mentioned later.) As for the finite dimensional case, the tricky mathematical problem is to suitably define the noise disturbances and as for the finite dimensional case, most authors have ignored the existence of this problem and have proceeded formally. So this survey will fall into two parts, namely the formal approaches and the mathematically rigorous ones.

2. FORMAL SOLUTIONS TO THE FILTERING PROBLEM.

2.1. Basic Linear Filtering Model.

Many authors [2], [14], [15], [16], [20], [21], [22], take as their basic system equations, the following mo-del

$$(2.1) \quad \begin{cases} \dfrac{\partial u(x,t)}{\partial t} = A(x,t)u(x,t) + B(x,t)\xi(x,t) \quad \text{on } D \subset R^n \\[2mm] \beta_x u(x,t) = 0 \quad \text{on } \partial D \\[2mm] u(x,0) = u_0(x) \\[2mm] z(x,t) = M(x,t)u(x,t) + \eta(x,t) \end{cases}$$

where the deterministic system with $B \equiv 0$ is a well-found problem of Hadamard, $A(x,t)$ is some spatial, linear, partial differential operator, β_x boundary conditions and $u_0(x)$ the initial data. $B(x,t)$ and $M(x,t)$ are matrix-valued with suitably smooth coefficients and $\xi(x,t)$, $\eta(x,t)$ are independent Gaussian distributed "stochastic processes" with zero expectation and covariances

$$(2.2) \quad \begin{aligned} E\{\xi(x,t)\xi(y,s)^1\} &= Q(x,y,t)\delta(t-s) \\ E\{\eta(x,t)\eta(y,s)^1\} &= R(x,y,t)\delta(t-s) \\ E\{\xi(x,t)\eta(y,s)^1\} &= S(x,y,t)\delta(t-s) \end{aligned}$$

$u_0(x)$ is a distributed Gaussian random variable with zero expectation and covariance matrix $P_0(x,y)$ and is independent of ξ and η. $R(x,y,t)$ is assumed to have an inverse.

Although several authors give less general examples, their ideas and techniques can be generalized to this case.

2.2. Best Least Squares Estimate.

A major contribution was made by Tzafestas and Nightingale [21] - [24] around 1968, 1969. In [21] he assumes that his best estimate $\hat{u}(x,t|t)$ of $u(x,t)$ based on $z(s,x)$ $0 \leqslant s \leqslant t$ has the form

$$(2.3) \qquad \hat{u}(x,t|t) = \int_0^t \int_D H(x,t;y,s)z(y,s)dDds$$

and minimizes the error in the sense of least squares. Then he is able to follow the finite dimensional proof to get analogous filtering equations to (1.4) and (1.5),

$$\frac{\partial \hat{u}}{\partial t} = A(x,t)\hat{u} + \int_D H(x,t;y,t)[z(y,t) - M(y,t)\hat{u}(y,t|t)]dD$$

and

$$H(x,t;y,t) = P(x,y,t)M^{*}(y,t)R^{-1}(x,y,t) \qquad\qquad (2.4)$$

where

$$\frac{\partial P}{\partial t}(x,y,t) = A(x,t)P(x,y,t) - P(x,y,t)A^{*}(y,t) +$$

$$+ B(x,t)Q(x,y,t)B^{*}(y,t) -$$

$$- \int_D P(y,\alpha,t)M^{*}(\alpha,t)R^{-1}(y,\alpha,t)M(\alpha,t)P(\alpha,y,t)dD$$

As these are partial differential equations, one needs boundary and initial conditions

$$(2.5) \qquad \begin{cases} \hat{u}(x,0|0) = 0 \\ \beta_x\hat{u}(x,t|t) = 0 \qquad x \in \partial D \\ P(x,y,0) = P_0(x,y) \\ \beta_x P(x,y,t|t) = 0 = \beta_y P(x,y,t|t) \qquad x,y \in \partial D \end{cases}$$

In [22] he gets the same result by introducing genera-
lized distributed characteristic functions and in [23]
he shows that this filter is also the best estimate in
the maximum likelihood sense.

Thau [20] in 1969 considered a special case of (2.1) for
the random heat equation, assuming his noise disturban-
ces were independent of x and taking a single point ob-
servation $z(t,x_1)$. He used the same least squares crite-
rion as Tzafestas and Nightingale and derived similar
filter equations. If nothing else, this paper demonstrates
the information gap between work in England and in Ameri-
ca.

2.3. Minimization of Functional Approach.

Meditch in [14] 1971 considers (2.1) with $B \equiv 0$ and avoids
all specification of the random nature of $u_0(x)$ and $\eta(x,t)$;
they are simply "errors". He defines his best estimate
$\hat{u}(x,T|T)$ to be that which minimizes the following func-
tional

$$(2.6) \quad J = \frac{1}{2} \int_D \left(u(x,0) - u_0(x)\right)^1 A_{x,y} \left(u(y,0) - u_0(y)\right) dD +$$

$$+ \int_0^T \int_D \left(z(x,t) - M(x,t)u(x,t)\right)^1 R^{-1}(x,t) \cdot$$

$$\cdot \left(z(x,t) - M(x,t)u(x,t)\right) dD dt$$

where

$$A_{x,y}(\cdot) = \left(\int_D P_0(x,y)(\cdot)dD\right)^{-1}$$

and P_0, R are suitable weighting matrices, symmetric and positive.

This recasts the filtering problem as the deterministic optimal control problem of minimizing (2.6), which he solves by classical techniques, obtaining a two point boundary value problem (TPBVP) and then decouples to get the usual filtering equations for u and P. He notes that the case $B \neq 0$ can be included by adding an appropriate weighting term in J and he draws the analogy between P_0, R and the covariance matrices of the errors. Although in finite dimensions one can show that minimizing a quadratic functional is equivalent to finding the best least squares estimate, in infinite dimensions, this must be regarded as an "ad hoc" hypothesis. Both Phillipson and Mitter [15] 1967 and Balakrishnan and Lions [2] 1967 use this technique to solve the problem of estimating the control data $u_0(x)$ from noisy observations, nearly avoiding a probabilistic treatment. In fact, Bensoussan [4] does provide a probabilistic interpretation for minimizing a functional to find $\hat{u}$ (see §3).

2.4. Discrete Observations.

Meditch also suggests that in practice discrete observations are more desirable and provides a formal way of including these.

$$(2.7) \qquad z(t,x) = \sum_{i=1}^{k} M(t,s_i)\delta(s_i-x) + \sum_{i=1}^{k} n(t,s_i)\delta(t-s_i)$$

Other authors have since included these in their work.

2.5. Innovations Approach.

Atre and Lamba [1] 1971 borrow the innovations approach
from Kailath and use it to formally reproduce the fil-
tering equations. (This approach may be fruitful in a
rigorous treatment of linear distributed parameter sys-
tems.)

2.6. Noise on Boundary.

Sakawa [16] 1972 includes the case of noisy boundary da-
ta by assuming a boundary condition of the form

$$F(y)u(t,y) + \frac{\partial u(t,y)}{\partial \eta_A} = S(t,y)v(t,y) \qquad y \in \partial \Omega,$$

where F, S are smooth matrices, $\partial/\partial\eta_A$ is the normal spa-
tial derivative corresponding to A and $v(t,y)$ is Gaussian
white noise independent of ξ and η. He also assumes disc-
rete observations. However, he readily reduces the prob-
lem to one equivalent to (2.1) by adding the term $\delta(\psi(x)) \cdot$
$\cdot S(t,x)v(t,x)$ to the state equation, where $\delta(\psi(x))$ is a
generalized function concentrated on ∂D. Solutions of
(2.1) are taken in the generalized sense and the usual
formal analysis yields the generalized Kalman-Bucy fil-
ter equations.

2.7. Colored Noise.

Tzafestas in a recent paper [24], 1972, has generalized
his filtering results to include coloured noise in the
observation, i.e.

$$(2.8) \quad \begin{cases} z(x,t) = M(x,t)u(x,t) + N(x,t)\Omega(x,t) \\[2mm] \text{with} \\[2mm] \dfrac{\partial \Omega(x,t)}{\partial t} = B_x \Omega(x,t) + D(x,t)\eta(x,t) \\[2mm] \Omega(x,0) = \Omega_0(x) \\[2mm] \gamma_x \Omega(x,t) = 0 \quad \text{on } \partial D \end{cases}$$

where η is a distributed Gaussian white noise as before
and the coloured noise Ω is the output of the randomly
forced linear partial differential equation. In essence
his solution amounts to transforming this system into
one equivalent to (2.1).

2.8. Solution of the Filtering Equations.

Several of the authors included examples of solutions of
(2.4), (2.5) in their papers ([14], [20], [21] → [24])
and all were obtained by looking for a separation of va-
riables solution using eigenfunction expansions for $\hat{u}$
and P, where the eigenfunctions are those defined by the
deterministic version of (2.1). Sakawa [16] does a gene-
ral eigenfunction expansion, obtaining Kalman-Bucy equa-
tions for an infinite estimation vector $(\hat{u}_1(t), \hat{u}_2(t),$
$\dots)$ and an infinite covariance matrix $(P_{ij}(t))$, $i = 1,$
$\dots \infty$, $j = 1, \dots \infty$. Naturally these look just like the

finite dimensional Kalman-Bucy equations

However, these examples are purely illustrative and do
not represent any general theory. For a theory based on
approximation on finite dimensional subspaces, see Ben-
soussan [5] and a thesis to appear by J.C. Nedelec (I.R.
I.A).

2.9. Other Extensions.

Some authors have included partial-integro differential
operators in A(x,t), for example [22]. In fact all the
theories can be easily generalized to include integral
operators explicitly, as they are just bounded operators.
Using a formal approach it is possible to generalize most
results from lumped parameter to distributed parameter
systems. For example, Shukla and Srinath [19] give re-
sults for optimal filtering in a linear, distributed pa-
rameter system with time delays.

2.10. Smoothing and Estimation.

Although this report concerns the filtering problem, I
should note that similar techniques to the above ones
are applicable to the smoothing and estimation problem
and several authors have exploited these (see [1], [2],
[14], [15], [21], [23]. [24]).

2.11. Nonlinear Filtering.

The most recent papers in this field consider the filtering problem when the state equations are nonlinear. Tzafestas and Nightingale in [23] 1969 used a maximum likelihood approach to find a best filter for (2.1) where $A(x,t)$, β_x, $M(x,t)$ are allowed to be nonlinear differential or integro-differential operators. He uses a generalization of a likelihood functional to infinite dimensional Gaussian random variables, and shows that the best maximum likelihood estimate is one which minimizes a functional

$$(2.9) \quad J = \frac{1}{2} \int_{D_1} \int_{D_2} \left(u(x,0) - u_0\right)^1 P_0(x,y) \cdot$$

$$\cdot \left(u(y,0) - u_0(y)\right) dD_1 dD_2 +$$

$$+ \frac{1}{2} \int_0^t \int_{D_1} \int_{D_2} \left(z(t,x) - M(x,t)u(x,t)\right)^1 R^{-1}(x,y,t) \cdot$$

$$\cdot \left(z(t,y) - M(y,t)u(y,t)\right) dD_1 dD_2 dt +$$

$$+ \frac{1}{2} \int_0^t \int_{D_1} \int_{D_2} \left(\frac{\partial u(x,t)}{\partial t} - A(x,t,u)u\right];$$

$$Q^{-1}(x,y,t) \left(\frac{\partial u(y,t)}{\partial t} - A(y,t,u)u(t,y)\right) dD_1 dD_2 dt$$

subject to $\beta_x(u,x,t) = 0$ on ∂D and Q is actually the covariance matrix of $B(x,t)\xi(x,t)$

So this provides some justification for the functional

approach. Seinfeld [17], [18] 1969, 1971, Hwang, Sein-
feld and Gavalas [11] 1972 and Lamont and Kumar [12] 1972
all treat various nonlinear dynamical systems and add in
noise as desired and then write down some massive func-
tional which includes a term for every noise component
in the system. They define their best estimate to be the
solution of this deterministic minimization problem and
then proceed to solve it as a classical calculus of va-
riations problem. This yields a TPBVP which they embed
in a wider class of problems (i.e. "invariant embedding")
to yield a recursive Kalman-Bucy type filter. The mathe-
matics is quite formal and there is even some difference
in the choice of cost functionals by various authors.
Cf. [12] and [10].

3. MATHEMATICALLY RIGOROUS APPROACHES.

An early paper which posed an infinite dimensional fil-
tering problem was that by Falb [9] in 1967. The idea
was to generalize the Itô integral to a Hilbert space
and so to define a stochastic evolution equation, the
analogue of (1.3) in a Hilbert space context, and then
to proceed as for the finite dimensional case. The main
difficulty here is that for partial differential equa-
tions, the operator A(t) is a differential operator and
hence unbounded. In [9], Falb only considered A(t) boun-
ded which would cover integral equations, but in [7],
Curtain and Falb gave an existence theorem for a stochas-
tic evolution equation with unbounded A(t), which could
be applied fruitfully to the filtering problem.

Kushner [11] 1970 considers the filtering problem for
two classes of distributed systems. The first is a se-
cond order parabolic partial differential equation with
an integral operator in the state equation and Diriche-
let-type boundary conditions. The observation is a weigh-
ted average of the state vector over space and time and
the disturbance processes are Itô differentials of Wie-
ner processes in time only, i.e. he considers the dege-
nerate case where the noise is uniformly distributed in
space. Taking his best estimate to be the conditional
expectation of the state, given the observation process,
he derives filtering equations for this estimate and for
the covariance of the error, rigorously establishing
existence, uniqueness of the solutions. The second prob-
lem is a second order parabolic partial differential
equation with mixed boundary conditions. Again the dis-
turbance processes are only random in time, but now he
allows noise on the boundary as well as in the state
equation. Taking his observations averaged over the boun-
dary this time, he does a similar analysis to the first
problem.

Duncan [8] 1972 generalizes filtering results to Banach
valued stochastic processes, but as the operators are
bounded it does not apply to partial differential equa-
tions.

The rest of this survey will be concerned with the work
of Bensoussan and for a complete exposition, the reader
is referred to his recent book [4] 1971. He has solved
the problem of filtering for a very wide class of linear
distributed parameter systems, namely

$$(3.1) \quad \begin{cases} \dot{u}(t) + A(t)u = f(t) + B(t)\xi(t) & 0 \leqslant t \leqslant T \\ u(0) = u_0 + \rho \end{cases}$$
$$z(t) = C(t)u(t) + \eta(t)$$

where the state u(t) has values in a Hilbert space H,
the observation z(t) in another Hilbert space F, the
operators B(t), C(t) are bounded and linear, but A(t)
is linear and closed, i.e. a differential operator. If
we consider (3.1) as a deterministic system, then Lions
[13] has developed a very nice theory for the solution
of control problems of this form, and Bensoussan uses
this format. The random components are $\xi(t)$, ρ and $\eta(t)$
and have values in some Hilbert space, but instead of
treating them as random variables or stochastic proces-
ses in a Hilbert space, he introduces a more general
concept of a linear random functional or equivalently a
cylindrical probability. Really he treats them as "gene-
ralized random variables" in a manner analogous to the
use of generalized functions in partial differential
equations.

The class of second order linear random functionals (LRF)
which have uniquely defined expectations and covariance
operators are closed under affine continuous maps. (3.1)
defines u(t) and z(·) as affine, continuous maps from

$(\zeta, \xi(\cdot), \eta(\cdot))$ using appropriate spaces and so if we
assign a LRF associated with $(\zeta, \xi(\cdot), \eta(\cdot))$, this in-
duces LRF's associated with $u(T)$ and $z(\cdot)$. The estima-
tion problem is then defined as the best linear estimate
of the LRF associated with $u(T)$ by the LRF associated
with $z(\cdot)$. Using results from estimation theory in a
Hilbert space, Bensoussan reduces the problem to the
deterministic problem of minimizing the functional*

$$(3.2) \quad J(\zeta, \xi(\cdot)) = \langle P_0^{-1}\zeta, \zeta \rangle_H + \int_0^T \langle Q(t)^{-1}\xi(t), \xi(t) \rangle_E dt$$

$$+ \int_0^T \langle R^{-1}(t) z_d(t) - C(t)u(t;\zeta,\xi), z_d(t) - $$

$$- C(t) u(t; \zeta,\xi) \rangle_F dt$$

with respect to ζ, $\xi(\cdot)$, where P_0, $Q(t)$, $R(t)$ are the co-
variance operators for the LRF corresponding to ζ, $\xi(t)$ and
$\eta(t)$ respectively. P_0, $Q(t)$, $R(t)$ are self adjoint and posi-
tive and invertible.

The best estimate is $\hat{\omega}_T = u(T; \hat{\zeta}, \hat{\xi}(\cdot))$, where $(\hat{\zeta}, \hat{\xi}(\cdot))$ is
the minimizing pair.*

He also derives the Kalman - Bucy equations for $\hat{\omega}_r$.

$$(3.3) \quad \frac{dr}{dt} + A(t) r(t) + P(t)C^*(t)R^{-1}(t) C(t) r(t) =$$

$$g(t) + P(t) C^*(t)R^{-1}(t)z_d(t)$$
$$r(0) = u_0$$

---

* This gives some insight into the justification for defining the best
estimate a priori in terms of minimizing a functional. Cf. 2.3.

$$(3.4) \begin{cases} \dfrac{dP(t)}{dt} + P(t)A^{*}(t) + A(t)P(t) + \\ \qquad + P(t)C(t)^{*}R^{-1}(t)C(t)P(t) = \\ \qquad = B(t)\phi(t)B^{*}(t)\phi(t) \\ P(0) = P_{0} \end{cases}$$

(actually only a weaker version of the $P(t)$ equation holds.)

and $\hat{u}_{T} = r(T)$.

Finally he shows that (3.3), (3.4) define the best LRF estimate for the general case where P_{0} and $Q(t)$ are not invertible. Applying this result to the special equation (2.1), one finds agreement between both filtering equations. If one applies these results to the finite dimensional case, then (3.3), (3.4) correspond to the filtering equations one obtains using the formal approach of (1.1). So this LRF approach gives new insight into the finite dimensional filtering problem as well.

Note that (3.1) is not a stochastic differential equation in the strict sense and that $u(t)$ is not a well-defined stochastic process, but has some linear random functional identified with it. For a stochastic evolution equation approach, Bensoussan defines a Hilbert space-valued Wiener process and stochastic integration in a Hilbert space along the lines of Falb [9]. Then he considers the following system model

$$\begin{cases} \dot{u}(t,\omega) + \int_0^t A(\tau)u(\tau,\omega)d\tau = \\ \\ \qquad = u_0(\omega) + \int_0^t f(\tau)d\tau + \int_0^t B(t)d\xi(t) \\ \\ z(t,\omega) = \int_0^t C(\tau)u(\tau,\omega)d\tau + \eta(t,\omega) \end{cases}$$

(3.5)

where $\xi(t)$, $\eta(t)$ are Hilbert space valued independent Wiener processes with covariance operators $Q(t)$, $R(t)$ respectively and $u_0(\omega)$ is a Gaussian stochastic process independent of $\xi(t)$ and $\eta(t)$ with zero expectation and covariance P_0.

This theory requires that P_0, $Q(t)$, $R(t)$ be nuclear operators (as well as positive and self adjoint). For filtering, one requires $R(t)$ to be invertible and so F must be finite dimensional (i.e. a finite dimensional observation space).

Under these conditions, Bensoussan proves that the best global estimate for the random variable $u(T,\omega)$ based on the stochastic process $z(t,\omega)$; $0 \leqslant t \leqslant T$ is given by the solution of the stochastic differential equation

(3.6) $\quad \hat{u}(T,\omega) + \int_0^T A(t)\hat{u}(t)dt + \int_0^T P(t)C^{\varkappa}(t)R^{-1}(t)C(t)\hat{u}(t)dt =$

$$= \int_0^T f(t)dt + \int_0^T P(t)C^{\varkappa}(t)R^{-1}(t)dz(t) + u_0(\omega)$$

where $P(t)$ is given by (3.4).

This is the required generalization of the Kalman-Bucy
filter equations (1.4), (1.5) to infinite dimensions.

Bensoussan has also entended these results to systems
with second order in t to obtain similar results. In [6]
he has looked at the problem of optimization of location
of sensors when you have discrete observations and he
has proved a separation principle for distributed para-
meter systems in [5]. The reader is referred to [3] for
his treatment of the estimation problem.

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