Output Regulation and Internal Model

A Frequency Domain Approach

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OUTPUT REGULATION AND INTERNAL MODELS - A FREQUENCY DOMAIN APPROACH *

by

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ABSTRACT

The algebraic regulator problem is formulated and solved in a transfer matrix setting. It is shown that, provided the closed loop system disregarding disturbances is stable, a necessary and sufficient condition for output regulation to take place is that the open loop path consisting of the plant and compensator in cascade, contains a suitably defined internal model of the environment. The disturbance model is more general than the ones used before. The results also generalize earlier results on internal models since they are necessary and sufficient under weaker assumptions. The internal model property is used to construct a compensator which achieves output regulation and internal stability. It is shown that any such compensator can be obtained in two steps: (a) create an internal model of the environment in the forward path and (b) stabilize the system. Our concept of internal model generalize earlier definitions and, unlike earlier results, is valid even if structural stability (robustness) is not imposed.

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1. INTRODUCTION

A Basic requirement on a closed loop system must be its ability to regulate against disturbances and/or track reference signals, described by suitable dynamic model classes. One aspect of this problem is the so called algebraic regulator problem, which has attained much interest during the last few years, see e.g. [2, 3, 4, 6, 11, 14, 16]. With a few exceptions, see e.g. [14], the problem has been treated in a state space setting. In this paper we give a frequency analog and generalize some earlier results.

The concept of internal models plays a crucial role in regulator problems. The internal model principle can intuitively be expressed as: "Any good regulator must create a model of the dynamic structure of the environment in the closed loop system".

The necessity of internal models is discussed in [8] and more abstractly in [1]. In [4], and implicitly in [2], it is shown that a regulator which is to achieve steady state regulation despite of certain small perturbations in system data (structural stability) must contain a certain duplicated model of the environment in the feedback part. This feature is thus necessary under the perturbations considered.

To establish the internal model principle mathematically it is desirable to formalize the problem under as weak assumptions as possible both on the concept of regulation and the model classes considered. In this paper, the existence of internal models is established in a frequency domain setting. This is done under weaker assumptions (structural stability is not imposed) and using a more general disturbance model than before. We assume that the regulated variables coincide with the variables accessible for feedback. It is then both necessary and sufficient for output regulation to take place that the open loop path, consisting of the plant and the compensator in cascade, contains a suitably defined internal model of the environment. Our definition of the concept of internal model is different from that of [4], especially in that it is property of the open loop path rather than the regulator alone. The difference is, of course, due to the requirement of structural stability used in [4]. If this requirement is not imposed, there seems to be no reason to differ between the "plant" and the "regulator" in the feedback loop. Note also, that our definition of internal model is directly related to output regulation. The internal model criterion is stated in some different ways to show the relationships between different representations of a linear system and to illustrate the presence of an internal model in a signal flow graph.

An important feature of the internal model property is that it provides insight into the regulation problem without too much algebraic detail. Such considerations are of great importance in synthesis. Based on the internal model criterion, a compensator which achieves output regulation and internal stability is constructed. It is shown that any such compensator can be obtained as a cascade of two compensators, one which crates an internal model of the environment in the forward path and one which stabilizes the system. The first one will be of minimal order. This direct use of internal models for compensation is believed to be new. Also the results generalize earlier results since they are valid for a more general disturbance model.

The order of a rational matrix $T(s)$, written $\partial T(s)$, is defined as the sum of the degrees of the denominator polynomials in the Smith-McMillan form of $T(s)$. For proper rational matrices, the order of a rational matrix equals the order of a minimal state realization (and also the McMillan degree).

An arbitrary rational matrix $T(s)$ can uniquely be written as

$$T(s) = T(s)_c + T(s)_p \tag{2.1}$$

where $T(s)_c$ is strictly proper and $T(s)_p$ a polynomial matrix.

Many arguments from realization theory [5, 9] are used in the sequel.

<u>Polynomial Fraction Representations</u>.

The following results can be found in [5, 12, 13]. The exposition given here follows basically that of [5]. A (left) fraction representation of a rational matrix $T(s)$ is a pair of polynomial matrices $P(s)$ and $Q(s)$ such that

$$T(s) = Q(s)^{-1}P(s)$$

Such a representation is said to be <u>minimal</u> if $\deg\big(\det Q(s)\big)$ is the least possible. The <u>characteristic polynomial</u> for $T(s)$ is defined as $\det\big(Q(s)\big)$ where $Q(s)$, $P(s)$ is minimal.

The following result is taken from [5].

Lemma 1. The following statements are equivalent:

(i) $Q(s)^{-1}P(s)$ is a minimal fraction representation of $T(s)$.

(ii) $Q(s)$ and $P(s)$ are relatively left prime.

(iii) $\partial T(s) = \deg\left(\det Q(s)\right)$

(iv) There are polynomial matrices $X(s)$ and $Y(s)$ such that

$$Q(s)X(s) + P(s)Y(s) = I$$ □

Furthermore, if $Q_1(s)^{-1}P_1(s)$ and $Q_2(s)^{-1}P_2(s)$ are two minimal fraction representations of $T(s)$ then there exists a unimodular matrix $M(s)$ such that $Q_1(s) = M(s)Q_2(s)$ and $P_1(s) = M(s)P_2(s)$. The analogous results can directly be given for (right) fraction representations

$$T(s) = N(s)D(s)^{-1}$$

where $N(s)$ and $D(s)$ are polynomial matrices (consider the transpose).

3. FORMALIZATION

The Model.

The plant and its environment is described by one common linear, time invariant system. Unlike [4], where an internal (state space) model is used, the total system is in this case described by an external (transfer matrix) model:

$$y(s) = T(s)u(s) + D(s)w(s) \tag{3.1}$$

where $y(s)$ is the q-dimensional regulated output, $u(s)$ the m-dimensional control input and $w(s)$ an r-vector comprising all exogenous signals acting on the system. In (3.1), $D(s)$ is a proper rational matrix and $T(s)$ a strictly proper rational matrix. We assume that the regulated variables coincide with the variables accessible for feedback. The rational matrices $T(s)$ and $D(s)$ represent dynamic models of how the regulated variables are influenced by the control inputs and the exogenous inputs respectively. Therefore, we regard $T(s)$ as the <u>plant model</u> and $D(s)$ as the <u>disturbance model</u>.

To give physical interpretation to the concept of regulation below, the exogenous signal w can be either an impulse, the initial condition of a linear, time invariant system or the laplace transform of a bounded signal. It also makes sense to regard w as a white noise process.

A distinction should be made between the exogenous signal w and the actual disturbance or reference signal. The difference may be best illustrated by comparing with the state space formulation in [4]. In [4] the overall system is described by a state equation

$$\begin{aligned}
\dot{x}_1 &= A_1x_1 + A_3x_2 + B_1u \\
\dot{x}_2 &= A_2x_2 \qquad\qquad x_2(0) = x_{20} \\
y &= C_1x_1 + C_2x_2
\end{aligned} \tag{3.2}$$

In (3.2) x_1 is the state of the plant and x_2 the disturbance. A laplace transform of (3.2) yields the description (3.1) with

$$T(s) = C_1(s-A_1)^{-1}B_1$$
$$D(s) = \left(C_1(s-A_1)^{-1}A_3 + C_2\right)(s-A_2)^{-1} \tag{3.3}$$
$$w(s) = x_{20}$$

In this case the exogenous signal is the initial state x_{20} while the actual disturbance is x_2. Working with external descriptions, there is no need to identify the internal variable x_2 unless more specific properties such as e.g. internal stability is desired. Internal stability will be an issue first when the compensator design is discussed in Section 5. Here we follow basically the same approach as is done for state space descriptions in [16], i.e. identify the plant by a minimal realization of T(s) and regard the disturbance as described by the "remaining" dynamics.

We also see that our disturbance model is more general than the one used in (3.2) since D(s) is allowed to be an arbitrary proper rational matrix, cf. also (3.3).

Output Regulation.

The class of admissible controls in (3.1) is

$$u(s) = - F(s)y(s) \tag{3.4}$$

where F(s) is a proper rational matrix. The purpose of control is to regulate y(s) against the exogenous signal w(s). Before giving a rigorous definition of output regulation, consider the following simple example. Assume the disturbance model in (3.1) is a ramp, i.e.

$$D(s) = \frac{I_r}{s^2} ; \qquad w(s) = w_0 \in \mathbb{R}^r$$

There is no steady state error in y for any w_0 if and only if the transfer matrix from w to y in the closed loop system is stable, i.e. if and only if the poles of

$$\left(I + T(s)F(s)\right)^{-1} \frac{I_r}{s^2}$$

are all within the open left halfplane of the complex plane.

With this simple example in mind, let

$$\mathbb{C} = \mathbb{C}^+ \cup \mathbb{C}^- \tag{3.5}$$

be a disjoint partion of the complex plane, where $\mathbb{C}^-$ is symmetric with respect to the real axis and contains at least one real point. Here, $\mathbb{C}^-$ represents the "good" part and $\mathbb{C}^+$ the "bad" part of the complex plane as judged by the position of the poles of the transfer functions. Note that $\mathbb{C}^-$ is quite arbitrary and not just the open left halfplane, say.

A rational matrix T(s) can be written uniquely as the sum of two strictly proper rational matrices $T(s)_+$ and $T(s)_-$ and a polynomial matrix $T(s)_p$

as

$$T(s) = T(s)_+ + T(s)_- + T(s)_p \tag{3.6}$$

where the poles of $T(s)_+$ and $T(s)_-$ are all within the regions $\mathbb{C}^+$ and $\mathbb{C}^-$ respectively. Such factorizations are discussed in more detail in [7] and can easily be done e.g. using partial fraction expansions. A rational matrix $T(s)$ having all its poles within $\mathbb{C}^-$ can then be expressed as $T(s)_+ = 0$. A rational matrix with this property is said to be stable with respect to $\mathbb{C}^-$.

In analogy with the simple example above, it is required for output regulation that the closed loop transfer matrix from w to y obtained with feedback (3.1) is stable w.r.t. C^-, i.e.

$$\left((I + T_0(s))^{-1}D(s)\right)_+ = 0 \tag{3.7}$$

where

$$T_0(s) = T(s)F(s) \tag{3.8}$$

This formulation simply reflects the fact that the environment has an unsatisfactory dynamic behaviour, e.g. too slow, oscillative, unstable etc., in comparison with what is required from the closed loop system. Physically, (3.7) can be interpreted in different ways depending on the signal w and the choice of stable region $\mathbb{C}^-$. If w is an impulse and $\mathbb{C}^-$ the open left half plane, (3.7) is equivalent to that y tends to zero when time tends to infinity. The same is true if w represents an initial condition as in (3.3). Classical control problems such as steady state regulation against steps, ramps, sinusoids etc. are therefore included. Also (3.7) is equivalent to y being bounded for all bounded signals w. If w represents white noise, (3.7) implies that we "shape" the frequency spectrum for y by insisting that the poles of the closed loop transfer matrix are all within $\mathbb{C}^-$.

4. OUTPUT REGULATION AND INTERNAL MODELS.

The purpose of this section is to establish some principles for output regulation against modelled disturbances. It is shown that a model of the environment must be included in the feedback loop in a specified way. This is both necessary and sufficient for output regulation to take place, provided the closed loop system disregarding disturbances is stable.

Output Regulation.

Consider the total system (3.1) with feedback control (3.4). The signal flow for the closed loop system with exogenous signals is shown in fig. 1.

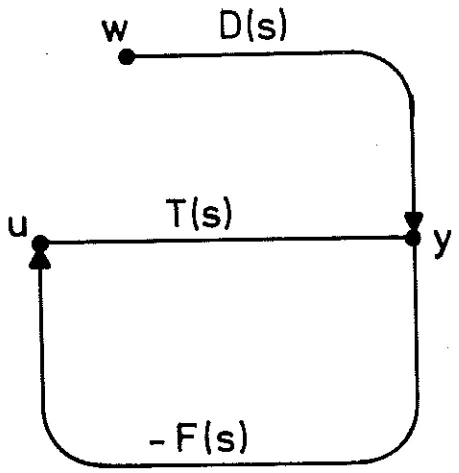


Fig. 1 - Signal flow for the closed loop system with exogenous signals.

In fig. 1 let $T_0(s) = T(s)F(s)$ be the open loop cascade of plant and compensator. Assume that $F(s)$ must be chosen so that the loop in fig. 1 disregarding disturbances, is stable for a signal injection at node y, i.e.

$$\left(I + T_0(s)\right)_+^{-1} = 0 \tag{4.1}$$

where $(\cdot)_+$ is defined with respect to (3.5), i.e. all the poles of (4.1) must be within $\mathbb{C}^-$. The property (4.1) is denoted <u>loop stability</u>. Note that loop stability is a weaker assumption than internal stability since unstable cancellations may occur in the cascade $T(s)F(s)$. Loop stability is, however, sufficient to establish the existence of an internal model as will be seen below. It is also required that <u>output regulation</u> takes place, i.e.

$$\left(\left(I + T_0(s)\right)^{-1} D(s)\right)_+ = 0 \tag{4.2}$$

It is now possible to establish the following result.

<u>Theorem 1</u>. <u>Assume that</u> $F(s)$ <u>is chosen so that loop stability holds. Output regulation takes place if and only if</u>

$$\partial T_0(s) = \partial\left[T_0(s) \quad D(s)_+\right]$$

<u>where</u> $\partial(\cdot)$ <u>is the order of a rational matrix and</u> $(\cdot)_+$ <u>is defined with respect to</u> (3.5).

To prove this theorem, the following lemma is used.

Lemma 2. *Let $R_1(s)$ and $R_2(s)$ be two arbitrary rational matrices with minimal fraction representations $Q_1(s)^{-1}P_1(s)$ and $Q_2(s)^{-1}P_2(s)$ respectively. Then $\partial R_1(s) = \partial[R_1(s) \;\; R_2(s)]$ if and only if $Q_2(s)$ is a right divisor of $Q_1(s)$.*

Proof. For brievity in exposition, we omit the argument s.

(if) There is a polynomial matrix D such that $Q_1 = DQ_2$. Hence,

$$[R_1 \;\; R_2] = Q_1^{-1}[P_1 \;\; DP_2]$$

is a fraction representation. Therefore, by Lemma 1 (iii), $\partial[R_1 \;\; R_2] \leq$ $\leq \deg\det Q_1 = \partial R_1$. Since the inequality trivially holds in the other direction, equality must be the case.

(only if) Let $Q^{-1}P = Q_1Q_2^{-1}P_2$ for a minimal fraction representation $Q^{-1}P$. Then

$$[R_1 \;\; R_2] = (QQ_1)^{-1}[QP_1 \;\; P] \tag{4.4}$$

is also a fraction representation. By Lemma 1, there are polynomial matrices X_i, i = 1, 2, 3, 4, such that $QX_1 + PX_2 = I$ and $Q_1X_3 + P_1X_4 = I$. Multiply the second expression from right by X_1 and from left by Q and substitute QX_1 from the first expression. This yields

$$QQ_1(X_3X_1) + [QP_1 \;\; P]\begin{pmatrix} X_4X_1 \\ X_2 \end{pmatrix} = I$$

i.e. (4.4) is a minimal fraction representation according to Lemma 1. Therefore, by Lemma 1 (iii)

$$\partial[R_1 \quad R_2] = \deg\det(QQ_1) = \deg\det Q_1 + \deg\det Q$$
$$\partial R_1 = \deg\det Q_1$$

Since equality holds, $\deg(\det Q) = 0$, i.e. $\det Q$ must be a nonzero real number since Q^{-1} exists. Hence, Q is unimodular and therefore $Q_1Q_2^{-1}P_2$ is a polynomial matrix, and since Q_2 and P_2 are relatively left prime, $Q_1Q_2^{-1}$ is a polynomial matrix. □

<u>Proof of Theorem 1</u>. Write $D = D_+ + D_- + D_p$ analogous to (3.6). Then

$$(I+T_0)^{-1}D = (I+T_0)^{-1}D_+ + (I+T_0)^{-1}(D_-+D_p)$$

Since $(I+T_0)^{-1}$ is stable w.r.t. $\mathbb{C}^-$ by assumption, the second term must always be stable. Therefore, (4.2) is equivalent to

$$\left((I+T_0)^{-1}D_+\right)_+ = 0$$

Now, let $T_0 = Q_0^{-1}P_0$ and $D_+ = Q_1^{-1}P_1$ be minimal fraction representations. Then

$$(I+T_0)^{-1} = (Q_0+P_0)^{-1}Q_0$$

which shows that $\det(Q_0+P_0)$ has all its zeros within $\mathbb{C}^-$. Furthermore,

$$T = (I+T_0)^{-1}D_+ = (Q_0+P_0)^{-1}Q_0Q_1^{-1}P_1$$

Since $(Q_0+P_0)^{-1}$ is stable and Q_1^{-1} completely unstable, T^* is stable if $Q_0Q_1^{-1}P_1$ is a polynomial matrix. Since Q_1, P_1 is relatively left prime, T^* is stable if $Q_0Q_1^{-1}$ is a polynomial matrix, which by Lemma 2 is equivalent to the condition in the theorem. □

The relationship between the result of Theorem 1 and internal models is discussed in more detail below.

Internal Models.

Condition (4.3), which thus is necessary and sufficient for output regulation to take place under loop stability, can be viewed as a property of the open loop paths of the signal flow shown in fig. 1, i.e.

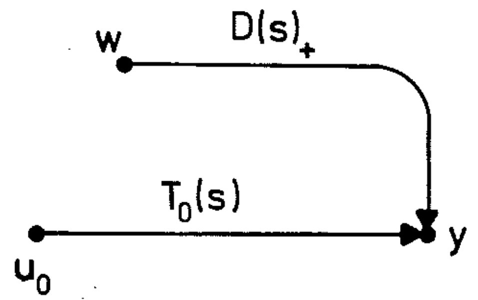


Fig. 2 - Open loop paths from the signal flow in fig. 1 where $T_0(s) =$
$= T(s)F(s)$.

The rational matrix $T_0(s)$ is here the open loop cascade of the plant and the compensator and $D(s)_+$ represents the "unstable" part of the disturbance model as determined by the expression (3.6). Condition (4.3) says that a minimal state realization of the total signal flow in fig. 2 yields the same dynamic order as a minimal realization of $T_0(s)$ alone. In other words, $T_0(s)$ contains a model of $D(s)_+$ in the sense that, given a minimal state realization of $T_0(s)$ in fig. 2, no extra state variables have to be introduced to realize the total signal flow including $D(s)_+$.

Let us therefore take the following definition.

Definition 1. Let $T(s)$ and $D(s)$ be arbitrary rational matrices. Then $T(s)$ is said to contain an internal model of $D(s)$ if

$$\partial T(s) = \partial[T(s) \quad R(s)] \qquad \square$$

To illustrate the internal model criterion for different representations of a linear system and to illustrate the presence of an internal model in a signal flow graph, the following theorem is given.

Theorem 2. Let $R(s)$ and $H(s)$ be arbitrary proper rational matrices with the same number of rows. The following statements are equivalent.

(i) $R(s)$ contains an internal model of $H(s)$.

(ii) A minimal (state) realization of

$$y = [H(s) \quad R(s)]\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

is controllable from u_1 alone.

(iii) Let $R(s) = Q_1(s)^{-1}P_1(s)$ and $H(s) = Q_2(s)^{-1}P_2(s)$ be minimal fraction representations. Then $Q_2(s)$ is a right divisor of $Q_1(s)$.

(iv) Let (A_R, B_R, C_R, D_R) and (A_H, B_H, C_H, D_H) be minimal state realizations of $R(s)$ and $H(s)$ respectively. Let their state dimensions be n_R and n_H. There is a monomorphism P such that the following diagram commutes

$$\begin{tikzcd}
\mathbb{R}^{n_R} \arrow[r, "A_R"] & \mathbb{R}^{n_R} \arrow[dr, "C_R"] & \\
 & & \mathbb{R}^{q} \\
\mathbb{R}^{n_R} \arrow[uu, "P"] \arrow[r, "A_H"'] & \mathbb{R}^{n_H} \arrow[uu, "P"] \arrow[ur, "C_H"'] &
\end{tikzcd}$$

(v) There are rational matrices $R_i(s)$, $i = 1, 2, 3$, such that

$$R(s) = \left(R_1(s) + H(s)R_2(s)\right)R_3(s)$$

$$\partial R(s) = \sum_{i=1}^{3} \partial R_i(s) + \partial H(s)$$

(vi) There is a real matrices E and proper rational matrices $R_1(s)$ and $R_2(s)$ such that

$$R(s) = R_1(s)R_2(s); \qquad \partial R(s) = \partial R_1(s) + \partial R_2(s)$$

$$H(s) = R_1(s)E$$

Proof. The equivalence between (i) and (iii) is proven in Lemma 2.

((i) ⇔ (ii)) Let

$$(A, [B_1 \;\; B_2], C, [D_1 \;\; D_2]) \tag{4.6}$$

be a minimal state realization of $[R(s) \;\; H(s)]$, i.e.

$$R(s) = C(s-A)^{-1}B_1 + D_1$$
$$H(s) = C(s-A)^{-1}B_2 + D_2$$

Then (i) implies that (A, B_1, C, D_1) is a minimal realization of $R(s)$, i.e. (A, B_1) is a controllable pair. Conversely, if (4.6) is a minimal realization with (A, B_1) controllable, then (A, B_1, C, D_1) is a minimal realization of $R(s)$ since (A, C) is an observable pair. This yields the same order of the minimal realization and therefore (i) holds.

$\left((ii) \Leftrightarrow (iv)\right)$ First, (iv) implies that (A_R, E, C_R, D_H) is a realization of $H(s)$ with $E = PB_H$. Therefore, $(A_R, [B_R \;\; E], C_R, [D_R \;\; D_H])$ is a minimal realization of $[R(s) \;\; H(s)]$ with (A_R, B_R) being controllable. Conversely, let $(A, [B_R \;\; E], C, [D_R \;\; D])$ be a realization of $[R(s) \;\; H(s)]$ with (A, B_R) being controllable. Then (A, B_R, C, D_R) is a minimal realization of $R(s)$. Moreover, let $\mathcal{R}$ be the controllable subspace for the pair (A, E) and let P be the inclusion of $\mathcal{R}$ into $\mathbb{R}^{n_R}$. Define (A_H, B_H, C_H, D_H) by

$$AP = PA_H; \qquad E = PB_H; \qquad C_H = CP \tag{4.7}$$
$$D_H = D$$

It is easily verified that (A_H, B_H, C_H, D_H) is a realization of $H(s)$. Also, (A_H, B_H) is controllable by construction. Since (A, C) is observable, so is (A_H, C_H). Therefore, the realization is minimal. The diagram commutes by (4.7). Since all minimal realization are isomorphic, there obviously exists a P such that the diagram commutes for arbitrary minimal realizations.

$\left((iii) \Rightarrow (v)\right)$ Let D be a polynomial matrix such that $Q_1 = DQ_2$. There are polynomial matrices X and Y such that $Q_2X + P_1Y = I$. Multiply from left by Q_2^{-1} and from right by $D^{-1}P_1$. Then

$$R = Q_1^{-1}P_1 = (X+HY)D^{-1}P_1$$

and

$$\partial X + \partial Y + \partial D^{-1}P_1 + \partial H = 0 + 0 + \deg\det D + \deg\det Q_2 =$$
$$= \deg\det(DQ_2) = \partial R$$

$\big((vi) \Rightarrow (i)\big)$ The conditions in (i) imply that $[R\ \ H] = R_1[R_2\ \ E]$. Let (A_i, B_i, C_i, D_i) be minimal realizations of $R_i(s)$, $i = 1, 2$, with state dimensions n_i. A realization (A, B, C, D) of $[R\ \ H]$ is then given by

$$A = \begin{pmatrix} A_1 & B_1C_2 \\ 0 & A_2 \end{pmatrix} \qquad B = \begin{pmatrix} B_1D_2 & B_1E \\ B_2 & 0 \end{pmatrix}$$

$$C = (C_1 \quad D_1C_2) \qquad D = (D_1D_2 \quad D_1E)$$

with dimension $n = n_1 + n_2$. Hence, $\partial R = \partial R_1 + \partial R_2 = n_1 + n_2 \geq \partial[R\ \ H]$ and equality must hold since the inequality always holds in the other direction.

$\big((ii) \Rightarrow (vi)\big)$ Let $(A, [B_1\ \ B_2], C, [D_1\ \ D_2])$ be a minimal realization with (A, B_1) controllable. Let

$$R_1(s) = [C(s-A)^{-1} \quad I]$$

$$R_2(s) = \begin{vmatrix} B_1 \\ D_1 \end{vmatrix} \qquad E = \begin{vmatrix} B_2 \\ D_2 \end{vmatrix}$$

By this choice, all the conditions in (vi) are satisfied. □

Let us briefly discuss the implications of this theorem by illustrating the condition (4.3), i.e. put $R(s) = T_0(s)$ and $H(s) = D(s)_+$ in Theorem 2. Condition (v) implies that $T_0(s)$ can be represented by the following signal flow graph.

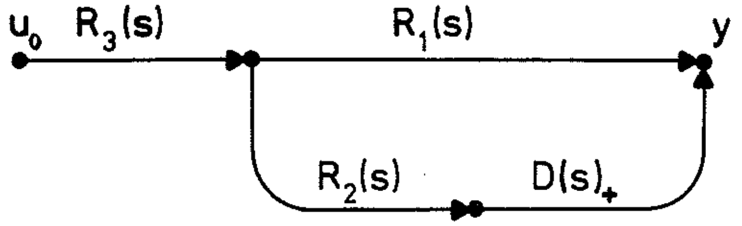


Fig. 3 - Signal flow for $T_0(s)$ with internal model $D(s)_+$.

In fig. 3 the order of $T_0(s)$ equals the sum of the orders of the component transfer matrices. The latter condition guarantees that no cancellation occurs. The presence of a model of $D(s)_+$ in $T_0(s)$ is apparent. Note, however, that $R_i(s)$, $i = 1, 2, 3$, are not necessarily proper.

To have a description with proper rational matrices, condition (vi) can be used. This condition implies that the signal flow in fig. 2 can be given the following alternative form.

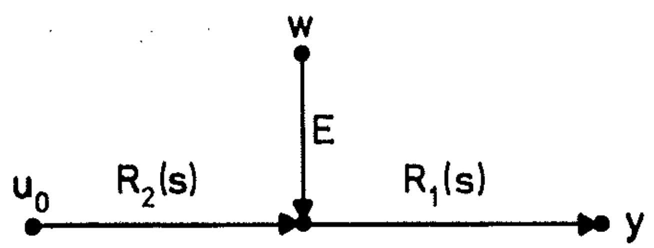


Fig. 4 - Alternative signal flow for the system in fig. 2 if $T_0(s)$ contains internal model of $D(s)_+$.

In Fig.4, $T_0(s) = R_1(s)R_2(s)$ and $D(s)_+ = R_1(s)E$ and no cancellation occurs in the cascade $R_1(s)R_2(s)$. In this case $R_1(s)$ contains an internal model of $D(s)_+$.

Finally, (iv) represents the state space analog and implies that there exists a nonsingular matrix T such that

$$TA_0T^{-1} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_+ \end{pmatrix}$$

$$C_0T^{-1} = (C_1 \quad C_+)$$

for some real matrices A_{11}, A_{12} and C_1.

Some immediate <u>necessary</u> conditions for output regulation to take place can be derived using Theorem 2. If $d_0(s)$ and $d_+(s)$ are the characteristic polynomials for $T_0(s)$ and $D(s)_+$ respectively, a necessary condition for output regulation is obtained directly from Theorem 2 (ii) as

$$d_+(s) | d_0(s) \tag{4.8}$$

This is also sufficient if $m = q = 1$. If in the state space model (3.2) the eigenvalues of A_2 are all within $\mathbb{C}^+$, a necessary condition for output regulation is also

$$\begin{aligned} &d_2(s) | d_0(s) \\ &d_2(s) = \deg(s-A_2) \end{aligned} \tag{4.9}$$

These results can be strengthened using the following observation, which follows from Theorem 2 (iii).

Proposition 1. *If* R(s) *contains an internal model of* H(s) *then also* M(s)R(s) *contains an internal model of* M(s)H(s) *for any polynomial matrix* M(s) *such that the matrix products are defined.*

Proof. Let $R = Q^{-1}P$ and $H = D^{-1}N$ be minimal fraction representations. Then by Theorem 2(iii), $QD^{-1} = S$ is a polynomial matrix.

Consider

$$R_1 = MQ^{-1}P$$
$$H_1 = MD^{-1}N$$

Let $MQ^{-1} = Q_1^{-1}M_1$ and $MD^{-1} = Q_2^{-1}M_2$ where the right hand sides are minimal fraction representations. Then

$$R_1 = Q_1^{-1}M_1P$$
$$H_1 = Q_2^{-1}M_2N$$

are also minimal fraction representations. Since, Q_2 and M_2 are relatively left prime, there are polynomial matrices X and Y such that $Q_2X + M_2Y = I$. By some straightforward calculations

$$X + Q_2^{-1}M_2Y = Q_2^{-1}$$
$$Q_1X + Q_1Q_2^{-1}M_2Y = Q_1Q_2^{-1}$$

$$Q_1X + Q_1MD^{-1}Y = Q_1Q_2^{-1}$$
$$Q_1X + M_1QD^{-1}Y = Q_1Q_2^{-1}$$

which shows that $Q_1Q_2^{-1}$ is a polynomial matrix since QD^{-1} is a polynomial matrix. The result then follows by Theorem 2 (iii). □

This proposition shows that if in the composite system shown in fig. 2, $T_0(s)$ contains an internal model of $D(s)_+$, all subsystems of the form

$$\hat{y}(s) = M(s)y(s) = M(s)T_0(s)u_0 + M(s)D(s)_+w$$

where M(s) is a polynomial matrix, are such that $M(s)T_0(s)$ contains an internal model of $M(s)D(s)_+$. Especially, we can take $M(s) = e_i^T$, where e_i is the unit vector with a nonzero element in the i:th position only. Combining this with (4.8) shows that

$$d_{i+}(s) | d_{i0}(s)$$

where $d_{i+}(s)$ and $d_{i0}(s)$ are least common denominators of the i:th rows of $D(s)_+$ and $T_0(s)$ respectively.

For synthesis, it is necessary to find a compensator F(s) which <u>creates</u> an internal model of $D(s)_+$ in the forward path. This is the topic of the next section.

<u>Remark</u>. Theorem 1 holds even if D(s) and T(s) are not proper rational matrices. The properness assumptions were introduced merely to restrict the attention to causal systems. Also Theorem 2 holds (except condition (vi))

if R(s) and H(s) are nonproper if we by a minimal realization of a nonproper rational matrix R(s) mean $(A, B, C, R(s)_p)$ where (A, B, C) is a minimal realization of $R(s)_c$, cf. (2.1).

5. COMPENSATOR DESIGN

In this section it is shown how a compensator F(s) can be designed using the internal model property. First, the following concept of <u>internal stability</u> is introduced. In the composite system shown in fig. 1, let

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{5.1}$$

be a minimal realization of the plant T(s). Also let

$$\begin{aligned} \dot{x}_c &= A_c x_c + B_c y \\ u &= C_c x_c + D_c y \end{aligned} \tag{5.2}$$

be a minimal realization of the compensator F(s). The closed loop system is said to be <u>internally stable</u> with respect to $\mathbb{C}^-$ if the composite system (5.1) and (5.2) is stable w.r.t. $\mathbb{C}^-$, i.e. if the following matrix

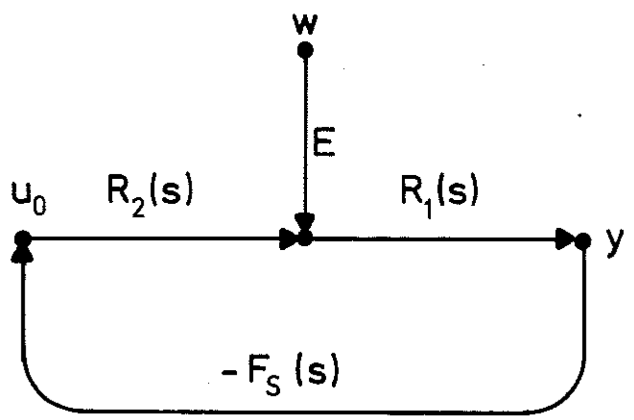
$$\begin{pmatrix} A-BD_cC & -B_c \\ B_cC & A_c \end{pmatrix} \tag{5.3}$$

has all its eigenvalues within $\mathbb{C}^-$. A compensator F(s) with this property is said to be a <u>stabilizator</u> for $T(s)$. It is well known from observer and pole assignement theory, see e.g. [10, 17], that stabilizators always exist and can be fairly easily constructed.

The effect of closing a system with a stabilizator is explained in the following proposition.

<u>Proposition 2</u>. <u>Assume that</u> $T_0(s)$ <u>is strictly proper and contains an internal model of</u> $D(s)_+$ <u>and let</u> $F_s(s)$ <u>be a stabilizator for</u> $T_0(s)$. <u>The</u> $T_0(s)F_s(s)$ <u>contains an internal model of</u> $D(s)_+$.

<u>Proof</u>. Using Theorem 2 (vi), the composite system shown in fig. 2, closed by F_s, can be represented by the following signal flow graph.



<u>Fig. 5</u> - Signal flow for composite system shown in fig. 2 with a stabilizator $F_s(s)$.

Here, R_1 and R_2 are proper and $\partial(R_1R_2) = \partial R_1 + \partial R_2$. Since F_s is a stabilizator for $T_0 = R_1R_2$, the transfer matrix from w to y must be stable. Moreover, loop stability holds. By Theorem 1, T_0F_s contains an internal model of $R_1E = D_+$. □

The purpose of design is to find a proper compensator $F(s)$, connected as in fig. 1, such that (a) output regulation and (b) internal stability hold. Since internal stability implies loop stability, there follows by Theorem 1 that $F(s)$ has the desired properties if and only if

(a) $F(s)$ creates an internal model of $D(s)_+$ in the cascade $T(s)F(s)$ (5.4)

(b) $F(s)$ is stabilizator for $T(s)$

To proceed, note that any $F(s)$ which is a candidate for (b) must avoid unstable cancellation in the cascade $T(s)F(s)$, or more precisely

$$\partial\big(T(s)F(s)\big)_+ = \partial T(s)_+ + \partial F(s)_+ \tag{5.5}$$

Otherwise, there appear eigenvalues in (5.3) which are within $\mathbb{C}^+$. Conversely, if there is no unstable cancellation in $T(s)F_1(s)$ and $T(s)F_1(s)$ contains an internal model of $D(s)_+$, there follows from Proposition 2 that we can take an arbitrary stabilizator $F_s(s)$ for $T(s)F_1(s)$ and

$$F(s) = F_1(s)F_s(s)$$

satisfies (5.4). Hence, the problem of finding a proper $F(s)$ satisfying (5.4a) and (5.4b) can be solved if and only if there is a proper $F(s)$ satisfying (5.4a) and (5.5).

Finally, the following proposition shows that if we can find <u>any</u> (including nonproper) compensator $F(s)$ satisfying (5.4a) and((5.5) we can also find a <u>proper</u> compensator of the same order satisfying (5.4a) and (5.5).

Proposition 3. Assume there exists a rational matrix $F(s)$, not necessarily proper, such that (5.4a) and (5.5) are satisfied. Also let

$$F(s)_c = C_c(s-A_c)^{-1}B_c \tag{5.6}$$

where $F(s)_c$ is defined as in (2.1) and (A_c, B_c, C_c) is a minimal state realization of $F(s)_c$. Then the following proper compensators also satisfy (5.4a) and (5.5)

$$\begin{aligned} F_1(s) &= [I \quad F_c(s)] \\ F_2(s) &= [I \quad C_c(s-A_c)^{-1}] \end{aligned} \tag{5.7}$$

Lemma 3. Assume that $\partial R(s) = \partial[R(s) \quad H(s)]$. Then also $\partial[R(s) \quad S(s)] = \partial[R(s) \quad S(s) \quad H(s)]$ for an arbitrary rational matrix $S(s)$.

Proof. Let $R = Q_1^{-1}P_1$, $D = Q_2^{-1}P_2$ be minimal fraction representations. By Lemma 2, $Q_1Q_2^{-1} = M$ is a polynomial matrix. Also, let $Q_1S = Q_3^{-1}P_3$ be a minimal fraction representation. Then $[R \quad S] = (Q_3Q_1)^{-1}[Q_3P_1 \quad P_3]$ is a minimal fraction representation. Moreover, $Q_3Q_1Q_2^{-1} = Q_3M$ and the lemma follows by Lemma 2. □

Proof of Proposition 3. By Lemma 3, it follows directly that $\partial[T[I \quad F]] = \partial[T[I \quad F] \quad D_+]$ so $T[I \quad F]$ contains an internal model of D_+. Write $F = F_c + F_p$. Generally, if R_1 contains an internal model of R_2, then also R_1M contains an internal model of R_2 if M is a unimodular matrix. This follows e.g. from Lemma 2. Now

$$T[I \quad F]\begin{pmatrix} I & -F_p \\ 0 & I \end{pmatrix} = T[I \quad F_c] \tag{5.8}$$

Since we have multiplied with a unimodular matrix, $T[I \;\; F_c]$ contains an internal model of D_+. Again, using Lemma 3, $T[I \;\; C_c(s-A_c)^{-1}]$ contains an internal model of D_+, and therefore F_1 and F_2 both satisfy (5.4a). Next, it is obvious that if TF does not contain any unstable cancellation, neither does $T[I \;\; F]$. Furthermore, by (5.8) there follows that $T[I \;\; F_c]$ does not contain any unstable cancellation and the same is true for $T[I \;\; C_c(s-A_c)^{-1}]$. □

The original design problem has now been converted to a pure mathematical problem: <u>find a rational matrix</u> $F(s)$ <u>such that</u>

$$\begin{array}{lll} \text{(a)} & T(s)F(s) \text{ \underline{contains an internal model of} } D(s)_+ & \quad (5.9) \\ \text{(b)} & \partial\big(T(s)F(s)\big)_+ = \partial T(s)_+ + \partial F(s)_+ & \end{array}$$

Once such a compensator has been found, a proper compensator of the same order is directly constructed using Proposition 3 and a stabilizator using Proposition 2.

Represent $T(s)$ and $D(s)_+$ by minimal fraction representations:

$$\begin{aligned} T(s) &= Q(s)^{-1}P(s) \\ D(s)_+ &= Q_1(s)^{-1}P_1(s) \end{aligned} \qquad (5.10)$$

Also, let

$$Q(s)Q_1(s)^{-1} = Q_d(s)^{-1}P_d(s) \qquad (5.11)$$

where the right hand side is a minimal fraction representation.

Theorem 3. There is a compensator $F(s)$ such that (a) $T(s)F(s)$ contains an internal model of $D(s)_+$ and (b) $T(s)F(s)$ contains no unstable cancellation, if and only if there is a polynomial solution $X(s)$, $Y(s)$ to the linear equation

$$P(s)X(s) + Y(s)Q_d(s) = I \tag{5.12}$$

Furthermore, if a solution $X(s)$, $Y(s)$ to (5.12) exists, a compensator of least possible order satisfying (a) and (b) is given by

$$F(s) = [I_m \;\; X(s)Q_d(s)^{-1}] \tag{5.13}$$

A corresponding proper compensator is directly obtained using Proposition 3.

To prove this theorem, the following lemma is used.

Lemma 3. If $T(s)H(s)^{-1}$ contains an internal model of $D(s)_+$ and $H(s)$ is a polynomial matrix such that det $H(s)$ is nonzero and has all its zeros within $\mathbb{C}^-$, then also $T(s)$ contains an internal model of $D(s)_+$.

Proof. Let $T = Q^{-1}P$ and $D_+ = Q_1^{-1}P_1$ be minimal fraction representations. Now

$$TH^{-1} = Q^{-1}PH^{-1} = Q^{-1}\tilde{D}^{-1}\tilde{P} \tag{5.14}$$

where $\tilde{D}^{-1}\tilde{P} = PH^{-1}$ and the left hand side is a minimal fraction representation. Then, (5.14) is also minimal. By Theorem 2 (ii), $\tilde{D}QQ_1^{-1} = M$, where M is a polynomial matrix. Hence, $\tilde{D}^{-1}M = QQ_1^{-1}$. Now det $\tilde{D}$ divides det H which implies that the left hand side is stable. Also, the righthand side is completely unstable. Therefore, both $\tilde{D}^{-1}M$ and QQ_1^{-1} must be polynomial matrices, i.e. Q_1 is a right divisor of Q which proves the result by Theorem 2 (ii). □

Proof of Theorem 3. (If) First consider

$$TF = Q^{-1}P[I \quad XQ_d^{-1}]$$
$$= Q^{-1}[P \quad PX]\begin{pmatrix} I & 0 \\ 0 & Q_d \end{pmatrix}^{-1} = Q^{-1}\hat{P}\hat{Q}^{-1}$$

Using (5.12) we have

$$\begin{pmatrix} 0 \\ I \end{pmatrix}[P \quad PX] + \begin{pmatrix} I & 0 \\ -P & Y \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & Q_d \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

and therefore $\hat{P}\hat{Q}^{-1}$ is minimal by Lemma 1. Since $Q^{-1}P$ is minimal, so is $Q^{-1}\hat{P}$. Hence

$$\partial(TF) = \deg(\det Q) + \deg(\det \hat{Q})$$
$$= \deg(\det Q) + \deg(\det Q_d) = \partial T + \partial F \qquad (5.15)$$

which shows that no cancellation occurs (and therefore no unstable one). Rewrite (5.12) as

$$PXQ_d^{-1} = Q_d^{-1}(I-Q_dY)$$

Then

$$TF = Q^{-1}Q_d^{-1}[Q_dP \quad (I-Q_dY)]$$

This fraction representation is minimal by (5.15) and Lemma 1. Using (5.11), we have $Q_dQQ_1^{-1} = P_d$ and therefore TF contains an internal model of $D(s)_+$, cf. Theorem 2 (ii).

(Only if) Let F be a compensator such that (a) and (b) are satisfied and let $F = NR^{-1}$ be a minimal fraction representation. Also represent T by (5.10). Factorize R and Q as $R = R_2R_1$ and $Q = Q_1Q_2$, where the zeros of det Q_1 and det R_1 are within $\mathbb{C}^+$ and the zeros of det Q_2 and det R_2 within $\mathbb{C}^-$. Now

$$TF = Q_2^{-1}Q_1^{-1}PNR_1^{-1}R_2^{-1} \tag{5.16}$$

Since TF contains an internal model of D_+, there follows by Proposition 1 and Lemma 3 that

$$T_1 = Q_1^{-1}PNR_1^{-1} \tag{5.17}$$

contains an internal model of Q_2D_+, i.e. of $Q_2Q_d^{-1}P_1$. By (5.16), Q_1, PN are relatively left prime and PN, R_1 are relatively right prime, since otherwise an unstable cancellation occurs. Now, write $\tilde{R}^{-1}\tilde{P} = PNR_1^{-1}$ where $\tilde{R}^{-1}\tilde{P}$ is a minimal fraction representation. A substitution into (5.17) yields $T_1 = Q_1^{-1}\tilde{R}^{-1}\tilde{P}$ which is also a minimal fraction representation. Since T_1 contains an internal model of $Q_2Q_1^{-1}P_1$, Theorem 2 (ii) implies that $\tilde{R}Q_1Q_2Q_1^{-1} = \tilde{R}QQ_1^{-1} = \tilde{R}Q_d^{-1}P_d$ is a polynomial matrix. This can only be true if Q_d is a right divisor of $\tilde{R}$ i.e.

$$\tilde{R} = \hat{R}Q_d \tag{5.18}$$

for some polynomial matrix $\hat{R}$. Since $\tilde{R}$ and $\tilde{P}$ are relatively left prime, there are polynomial matrices Z and W such that $\tilde{P}Z + \tilde{R}W = I$. Some straightforward manipulations give

$$\tilde{R}^{-1}\tilde{P}Z + W = \tilde{R}^{-1}$$

$$PNR_1^{-1}Z + W = \tilde{R}^{-1}$$

$$PNR_1^{-1}Z\tilde{R} + W\tilde{R} = I$$

This shows that $PNR_1^{-1}Z\tilde{R}$ is a polynomial matrix. Since PN and R_1 are relatively right prime, $R_1^{-1}Z\tilde{R} = M$ is a polynomial matrix. This together with (5.18) yields

$$PNM + W\hat{R}Q_d = I$$

i.e. $X = NM$ and $Y = W\hat{R}$ is a solution to (5.12).

Finally, to see that the compensator (5.13) is minimal, we note that for any compensator (5.18) holds, i.e.

$$\begin{aligned} \partial F = \deg(\det R) &\geq \deg(\det R_1) = \deg \det \tilde{R} \\ &\geq \deg \det Q_d \end{aligned}$$

However, $\deg(\det Q_d)$ is the order of the compensator in (5.13). □

The compensator design is now complete. The synthesis is summarized in fig. 6 which shows the final compensator.

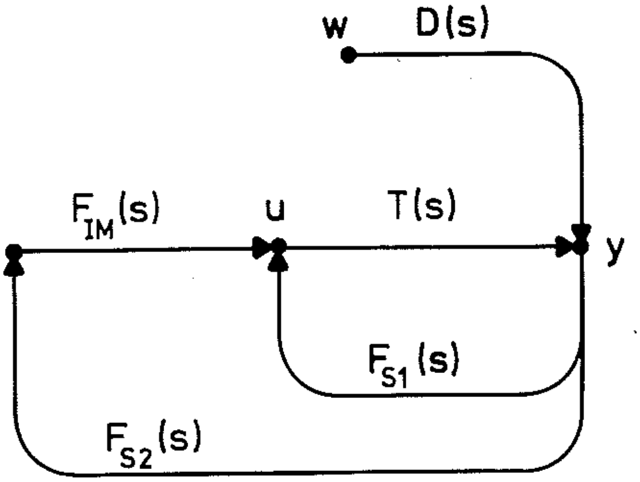


Fig. 6 - Signal flow for the final compensator.

In this compensator structure, the components are obtained as follows.

(1) By solving the linear matrix equation (5.12) we directly obtain a compensator $F(s) = [I_m \;\; X(s)Q_d(s)^{-1}]$ of least possible order which creates an internal model of $D(s)_+$ in $T(s)F(s)$ without unstable cancellations. Minimal proper compensators with the same properties are obtained via Proposition 3 as

$$F_1(s) = [I_m \;\; R(s)_c]; \qquad R(s) = X(s)Q_d(s)^{-1} \tag{5.19}$$

or

$$F_2(s) = [I_m \;\; C_c(s-A_c)^{-1}] \tag{5.20}$$

where $R(s)_c$ is the strictly proper part as in (2.1) and (A_c, B_c, C_c) is a minimal state realization of $R(s)_c$. Hence

$$F_{IM}(s) = R(s)_c \tag{5.21}$$

or

$$F_{IM}(s) = C_c(s-A_c)^{-1} \tag{5.22}$$

The sole purpose of F_{IM} is thus to create an internal model without unstable cancellation with the least possible dynamics.

(2) With F_{IM} defined as in (1), find a stabilizator

$$F_s(s) = \begin{bmatrix} F_{S1}(s) \\ F_{S2}(s) \end{bmatrix} \tag{5.23}$$

for $T(s)[I \quad F_{IM}]$. This stabilization can be done using any standard technique, e.g. pole assignement and observers or generalized Nyquist criteria. To have as much freedom as possible available in the stabilization step, it is advantageous to select the second form (5.22) for F_{IM}. If we use a minimal dual observer[10], the order of the final compensator becomes $n + n_c - m$ where $n = \partial T$, $n_c = \partial F_{IM}$ and m is the number of control inputs to the plant.

The main computational step is to solve the algebraic equation (5.12). Since it is linear this is in principle simple. For instance, by identifying the coefficient matrices for different powers in s on both sides of (5.12), we get a number of linear equations for the coefficients in X(s) and Y(s) which can be solved by standard techniques, see also Appendix. Also, (5.12) can be solved by hand in many cases. Note that once (5.12) is solved, we obtain the minimal compensator almost directly from the solution, cf. (5.13).

The section is concluded by a simple example.

<u>Example</u>. Consider the system

$$y(s) = \begin{pmatrix} \frac{s-1}{s^2} & \frac{2}{s} \\ \frac{2s-1}{s^2} & \frac{2}{s} \end{pmatrix} u(s) + \begin{pmatrix} \frac{1}{s+1} \\ \frac{2}{s+1} \end{pmatrix} w_1(s)$$

Find a compensator such that there is no steady state error in y for ramp disturbances $w_1(s) = w/s^2$. This is a problem of type (4.1) and (4.2) with $\mathbb{C}^-$ being the open left halfplane. In this case

$$T(s) = \begin{pmatrix} \frac{s-1}{s^2} & \frac{2}{s} \\ \frac{2s-1}{s^2} & \frac{2}{s} \end{pmatrix} \qquad D(s) = \begin{pmatrix} \frac{1}{s^2(s+1)} \\ \frac{2}{s^2(s+1)} \end{pmatrix}$$

A partial fraction expansion of D(s) yields

$$D(s)_+ = \begin{pmatrix} \frac{1-s}{s^2} \\ \frac{2(1-s)}{s^2} \end{pmatrix}.$$

The polynomial fraction representations become

$$T(s) = \begin{pmatrix} s-1 & 1 \\ -s & s \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$$

$$D(s)_+ = \begin{pmatrix} s^2 & 0 \\ -2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 - s \\ 0 \end{pmatrix}$$

Moreover, (5.11) becomes

$$Q(s)Q_1(s)^{-1} = \begin{pmatrix} \frac{s + 1}{s^2} & 1 \\ \frac{1}{s} & s \end{pmatrix} = \begin{pmatrix} s & -1 \\ 0 & s \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 1 & s^2 \end{pmatrix}$$

To obtain a compensator F(s) which creates an internal model in T(s)F(s) with no unstable cancellation, solve (5.12) i.e.

$$\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} X(s) + Y(s) \begin{pmatrix} s & -1 \\ 0 & s \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

One solution is

$$X(s) = \frac{1}{2}\begin{pmatrix} 0 & 2(1-s) \\ (1-s) & s \end{pmatrix} \qquad Y(s) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Then from (5.13)

$$X(s)Q_d(s)^{-1} = \frac{1}{2}\begin{pmatrix} 0 & \frac{2(1-s)}{s} \\ \frac{1 - s}{s} & \frac{1 - s}{s^2} + 1 \end{pmatrix}$$

Taking the strictly causal part yields

$$F_{IM} = \left(X(s)Q_s(s)\right)_c^{-1} = \frac{1}{2}\begin{bmatrix} 0 & \frac{2}{s} \\ \frac{1}{s} & \frac{1-s}{s^2} \end{bmatrix}$$

Here F_{IM} is a minimal compensator such that $T[I \quad F_{IM}]$ contains an internal model of $D(s)_+$, cf. fig. 6. The problem is the solved by taking any stabilizator

$$\begin{pmatrix} F_{S1}(s) \\ F_{S2}(s) \end{pmatrix}$$

in fig. 6. Since this is a standard problem, it is omitted here.

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Appendix A - On the linear equation (5.12)

Consider the linear equation (5.12), i.e.

$$P(s)\ X(s) + Y(s)\ Q_d(s) = I \tag{A.1}$$

which shall be solved for some polynomial matrices X(s) and Y(s). Here, $Q_d(s)$ is obtained via the minimal fraction representation (5.11). Given one $Q_d(s)$, the set of all is generated by $N(s)Q_d(s)$, N(s) unimodular, i.e. all polynomial matrices that are row equivalent with $Q_d(s)$. Since Theorem 3 is independent of the choice of $Q_d(s)$ within the equivalent class, we may assume that $Q_d(s)$ is such that the rational matrix

$$T(s) = Q_d(s)^{-1} \tag{A.2}$$

is strictly proper. If we are given one $Q_d(s)$ which does not satisfy this condition, we may transform $Q_d(s)$ to row proper forms ([13], Th. 2.5.11) in which case (A.2) is strictly proper.

To proceed we need a different version of the division algorithm.

Lemma A.1

Let P(s), $m \times r$, *and* Q(s), $m \times m$, *be polynomial matrices such that* $\det Q(s) \neq 0$. *There are polynomial matrices* H(s) *and* R(s) *such that* $P(s) = H(s)Q(s) + R(s)$ *where* $\deg R(s) < \deg Q(s)$ *and* $R(s)Q(s)^{-1}$ *is strictly proper*.

Proof

Consider the rational matrix $T = PQ^{-1}$. Using conventional realization theory, there are real matrices A, B and C and a polynomial matrix

$D(s)$ such that $T(s) = C(s-A)^{-1}B + D(s)$. Let $P_1Q_1^{-1} = (s-A)^{-1}B$ be a minimal fraction representation; then also $T = (CP_1 + DQ_1)Q_1^{-1}$ is a minimal fraction representation. Therefore, there is a nonsingular pol. matrix N such that $Q = Q_1N$ and $P = CP_1N + DQ_1N = CP_1N + DQ$. Let $R = CP_1N$ and $H = D$. Then $P = HQ + R$. By the equality $(s-A)P_1N = BQ$, there follows that $\deg(P_1N) < \deg Q$, and since C is real, $\deg R < \deg Q$. Also, $RQ^{-1} = CP_1N(Q_1N)^{-1} = C(s-A)^{-1}B$ which is strictly proper.

□

The following bounds on the degree of $X(s)$ and $Y(s)$ in (A.1) can now be given.

Proposition A.1

Assume that $Q_d(s)$ has been chosen so that $Q_d(s)^{-1}$ is strictly proper. If there exists any solution to (A.1), there exists one with $\deg X(s) < \deg Q(s)$ and $\deg Y(s) < \deg P(s)$.

Proof

Let $X_0(s)$ and $Y_0(s)$ be any solution to (A.1). Applying Lemma A.1, there exists polynomial matrices X and D such that $X = X_0 - DQ_d$, where $\deg X < \deg Q_d$ and XQ_d^{-1} is strictly proper. Then $X = X_0 - DQ_d$ and $Y = Y_0 + PD$ also satisfies (A.1) and we have satisfied the bound on $\deg X$. Next, by (A.1)

$$Y = Q_d^{-1} - P\,X\,Q_d^{-1} = \frac{\operatorname{Adj} Q_d - P\,X\operatorname{Adj} Q_d}{\det Q_d}$$

Now, Q_d^{-1} and XQ_d^{-1} are both strictly proper and therefore $\deg(\operatorname{Adj} Q_d) < \deg\det Q_d$ and $\deg(X\operatorname{Adj} Q_d) < \deg\det Q_d$ Since Y is a polynomial matrix, $\det Q_d$ must divide each element in $\operatorname{Adj} Q_d - PX\operatorname{Adj} Q_d$

Hence,

$$\begin{aligned}\deg Y &= \deg(\operatorname{Adj} Q_d\, PX \operatorname{Adj} Q_d) - \deg\det Q_d \\ &\leq \deg P + \deg X \operatorname{Adj} Q_d - \deg\det Q_d \\ &< \deg P\end{aligned}$$

□

Since we can set a priori bounds on $X(s)$ and $Y(s)$ in (A.1) we can now transform (A.1) to a linear equation with real numbers by identifying the coefficients in (A.1). On substituting

$$P(s) = \sum_{i=0}^{p} P_i\, s^i$$

$$Q_d(s) = \sum_{i=0}^{q} Q_0\, s^i$$

and (using Prop. A.1)

$$X(s) = \sum_{i=0}^{q-1} X_i\, s^i$$

$$Y(s) = \sum_{i=0}^{p-1} Y_i\, s^i$$

into (A.1), i.e.

$$\left(\sum_{i=0}^{p} P_i\, s^i\right)\left(\sum_{i=0}^{q-1} X_i\, s^i\right) + \left(\sum_{i=0}^{p-1} Y_i\, s^i\right)\left(\sum_{i=0}^{q} Q_i\, s^i\right) = I \qquad \text{(A.3)}$$

we get a set of linear equations in the coefficients X_i and Y_i by identifying the coefficients for different powers in s. Hence, by arranging the data X_i, Y_i into a vector x, we can rewrite as

$$A\,x = b \qquad \text{(A.4)}$$