Methods of Hopf and normal form in the analysis of non-linear and non-conservative systems

Arif, Ghazi Majid

1991

LUND UNIVERSITY

PO Box 117
221 00 Lund
+46 46-222 00 00

LUND UNIVERSITY · SWEDEN
INSTITUTE OF TECHNOLOGY
DEPARTMENT OF FIRE SAFETY ENGINEERING
CODEN: SE–LUTVDG/TVBB–3061
ISSN 0284–933X

GHAZI MAJID ARIF

METHODS OF HOPF AND NORMAL FORM IN THE ANALYSIS OF NON–LINEAR AND NON–CONSERVATIVE SYSTEMS

Lund, August, 1991

ABSTRACT

The method of normal form originating from the work of A.D. Brjuno has been used in the analysis of two–degrees–of–freedom discrete systems with eccentric follower force. Various types of external and internal damping has been taken into consideration. The program NORFOR2 is compared with program BIFOR2, which is based on Hopf Bifurcation, as to the capacity and speed in solving non–linear, non–conservative systems. The post–critical flutter of the systems has been classified into hard and soft types of flutter.

1. INTRODUCTION

A wide class of physical systems when analyzed for instability or local dynamical bifurcation is modelled by 4 dimensional differential equations with some varying controlling parameters. The equations being of mathematical as well as practical challenge have been the subject of extensive academic interest /1, 2/.

Reduction of the periodic differential equations systems to a simple form or a normal form originates from a thesis by Poincaré /3/ and the ideas of Birkoff /4/ and more recently of Brjuno /5, 6, 7/.

The systems we are concerned with are non–conservative and non–linear in which stability can be lost by divergence (saddle post instability) and Hopf bifurcation (flutter instability).

An example of non–conservative force is the follower or slave force which has no potential. The simplest example which then has been used by researchers is a double pendulum acted upon by a follower force.

Non–conservative forces appear in many problems such as fluid conveyed by flexible pipes and the motion of flexible missiles propelled by rockets /8, 9/. Many other important problems are associated with non–conservative forces in aerospace engineering, applied mechanics, astro–elasticity, electrical engineering as well as automatic control.

The stability of columns subjected to follower forces has witnessed a great surge of academic interest since the early 1950's, even though the first studies on the subject were published by E. Nikolai as early as 1928 /10, 11, 12/.

Stability of non–linear, non–conservative systems has been treated in several papers by Plaut /13, 14, 15/. Wiley solved the non–linear problem of a beam subjected to a partial follower force by the method of finite differences /16/. Burgess and Levinson investigated the postbuckling stability of discrete structural systems under non–conservative loading /17/. Burgess /18/ studied the non–linear and non–conservative systems using a perturbation method. Mandadi extended Huseyin's multiple parameter non–linear theory of stability to non–gradiant systems /19/.

Hagedorn used a procedure given by Salvadori to investigate the effect of non–linear damping on the stability of a double pendulum acted upon by a slave load /20/.

Hopf bifurcation of a double pendulum with a follower loading has been stu–died by Sethna and Shapiro using mathematical methods based on the method of integral manifolds and the method of averaging /21/.

Generalization of the Mandadi and Sethna studies /22/ has been achieved by Scheidl et al /23/ using a centre manifold theory and normal form theory of vector fields.

Normal form has been used by Hsu /24/ in the analysis of critical and post–critical buckling behaviour of non–linear systems acted upon by a slave load.

In practice, structural systems always contain influences of imperfections due to load eccentricity and/or influences of geometrical and material non–linearity. These influences must be taken into account in the post–buckling behaviour of structural systems. In this paper, the computer programs BIFOR2 and NORFOR2 have been used in the investigation of non–linear and non–conser–vative systems acted upon by eccentric slave forces. The effect of external, internal, linear, cubic and hysteresis damping have been included in the ana–lysis.

2. THE MECHANICAL MODEL

The double pendulum shown in Figure 1 will be considered. It has been used by many researchers as a discrete two–degree–of–freedom structural form of an elastic bar. It comprises two rigid links of lengths L_1 and L_2 connected by elastic springs C_1 and C_2. The configuration of the system is completely spe– cified by the angles ϕ_1 and ϕ_2 from the vertical and the angular misalign– ments (initial equilibrium angles) ϕ_{ko} $(k = 1,2)$.

Three types of internal damping have been introduced at the hinges, these are linear (B_1, B_2), cubic (B_{c1}, B_{c2}) and hysteresis (B_{h1}, B_{h2}). Linear external damping in the form of external damping force D and Coriolis force q have been included as well in the system.

The upper end of the pendulum is acted upon by an eccentric follower force $P = \bar{P}\,(1 + k_1\dot{\phi}_2^{\,k_2})$, with $k_2 = 1$ or 2 in the case without eccentricity. In the eccentric case, a tangential load has been used, i.e. $k_1 = 0$. The dimen– sionless differential equations for the system are:

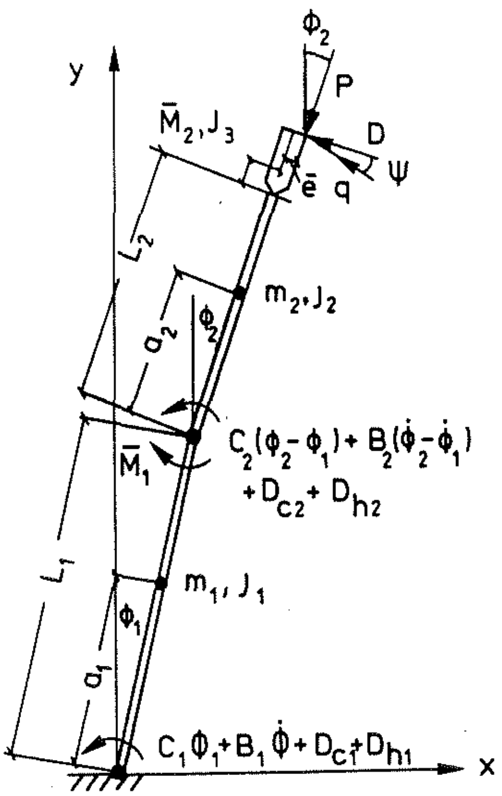

Fig. 1 A two–degree–of–freedom system:
D_{ci} = Moments at the hinges due to cubic damping;
D_{hi} = Moments at the hinges due to hysteresis

$$E_1\ddot{\phi}_1 + E_2\ddot{\phi}_2 \cos\phi = E_2\dot{\phi}_2^2 \sin\phi - c_3\phi_1 + c_2\phi_2 - b_3\dot{\phi}_1 +$$
$$+ b_2\dot{\phi}_2 - c_2\phi_{20} + c_3\phi_{10} - \beta\dot{\phi}_1 \cos\phi_1 \cos(\phi-\psi) -$$
$$- \tau\dot{\phi}_2 \cos\phi - \beta\alpha\,\dot{\phi}_2 \cos\phi_2 \cos(\phi-\psi) - b_{k1}\dot{\phi}_1^3 -$$
$$- b_{c2}(\dot{\phi}_2 - \dot{\phi}_1)^3 - b_{h1}\,\bar{\Phi}_1^2\,\dot{\phi}_1 - b_{h2}(\bar{\Phi}_2 - \bar{\Phi}_1)^2(\dot{\phi}_2 - \dot{\phi}_1) +$$
$$+ F\sin\phi$$
$$E_3\ddot{\phi}_2 + E_2\ddot{\phi}_1 \cos\phi = E_2\dot{\phi}_1^2 \sin\phi + c_2(\phi_1 - \phi_2) +$$
$$+ (\tau\alpha + b_2)\,\dot{\phi}_2 + b_2\dot{\phi}_1 - \beta\alpha^2\,\dot{\phi}_2 \cos\phi_1 \cos\psi +$$
$$+ c_2(\phi_{20} - \phi_{10}) - \beta\alpha\,\dot{\phi}_1 \cos\phi_1 \cos\psi +$$
$$+ b_{c2}(\dot{\phi}_1 - \dot{\phi}_2)^3 - b_{h2}(\bar{\Phi}_1 - \bar{\Phi}_2)^2(\dot{\phi}_1 - \dot{\phi}_2) + F\,e \qquad (1)$$

with

$$\phi = \phi_1 - \phi_2$$
$$\bar{\Phi}_1 = \phi_1 - \phi_{10},\ \bar{\Phi}_2 = \phi_2 - \phi_{20} \qquad (1.a)$$

where dots represent derivation with respect to the transformed time t. The following dimensionless quantities have been used

$$t = \sqrt{\frac{C_1}{m_1 L_1^2}}\,t',\ \mu = \frac{m_2}{m_1},\ F = \frac{P\ L_1}{C_1},$$
$$c_i = \frac{C_i}{C_1},\ \tau = f/\sqrt{m_1 C_1},\ \alpha = \frac{L_2}{L_1} \qquad (2)$$
$$M_1 = \frac{\bar{M}_1}{m_1},\ M_2 = \frac{\bar{M}_2}{m_1},\ \beta = \frac{bL_1}{\sqrt{m_1 C_1}}$$
$$\alpha_i = \frac{a_i}{L_1},\ b_i = \frac{B_i}{L_1\sqrt{m_1 C_1}},$$
$$\gamma = \bar{a}/L_1,\ I_i = \frac{J_i}{m_1 L_1{}^2},$$
$$b_{ci} = B_{ci}\sqrt{\frac{C_1}{(m_1 L_1^2)^3}};\ b_{hi} = \frac{B_{hi}}{\sqrt{C_1 m_1 L_1^2}}$$
$$e = \bar{e}/L_1$$

where E_1, E_2, E_3 c_3 and b_3 are given by

$$E_1 = M_1 + M_2 + \alpha_1^{\ 2} + \mu + I_1$$

$$E_2 = M_2(\alpha+\gamma) + \mu\ \alpha_2$$
$$E_3 = M_2(\alpha+\gamma)^2 + \mu\ \alpha_2^2 + I_2 + I_3$$

$$c_3 = c_1 + c_2$$

$$b_3 = b_1 + b_2$$

$$q = f\ \dot{\phi}_2 \tag{3}$$

Changing the system of equation (1) to first degree differential equations of the form $\dot{Z} = F(z,\eta)$ is achieved by denoting the variables as

$$\overline{Z}_1 = \phi_1,\ Z_2 = \dot{\phi}_1,\ \overline{Z}_3 = \phi_2 \text{ and } Z_4 = \dot{\phi}_2 \tag{4}$$

In order to be able to use programs NORFOR2 and BIFOR2 a change of coordinates must be made in the following manner.

The equations of the double pendulum acted upon by a static load are

$$\phi_1 - \phi_2 = G$$
$$c_3\ \phi_1 - c_2\ \phi_2 = F \sin G + R \tag{5}$$

where

$$G = K_4 - K_3$$
$$R = c_2(\phi_{10} - \phi_{20}) + c_1\ \phi_{10} \tag{6}$$

and

$$K_3 = \frac{Fe}{c_2}$$

$$K_4 = \phi_{10} - \phi_{20}$$

thus the new variables are given by

$$Z_1 = \phi_1 - \overset{*}{Z}_1$$
$$Z_3 = \phi_2 - \overset{*}{Z}_3 \tag{7}$$

where

$$\overset{*}{Z}_1 = \frac{F(\sin\ G + e)}{c_1} + \phi_{10}$$
$$\overset{*}{Z}_3 = \frac{F(ec_3 + c_2 \sin G)}{c_1 c_2} + \phi_{20} \tag{7.1}$$

and

$$\phi_1 - \phi_2 = Z_1 - Z_3 + G \tag{7.2}$$

The characteristic equation of the linear part of the changed system of equa– tions (1) is:

$$Q(F) = \begin{vmatrix} 0 & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & 0 & 0 & 1 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} \tag{8}$$

in which

$$a_{21} = A_1[E_3(F \cos G - c_3) - E_2 c_2 \cos G]$$
$$a_{22} = A_1[E_3(-b_3 - \beta \cos K_5 \cos G) -$$
$$- E_2(b_2 - \beta \alpha \cos K_5) \cos G]$$
$$A_{23} = A_1(E_3 c_2 + E_2 c_2 \cos G - E_3 F \cos G)$$
$$a_{24} = A_1[E_3(b_2 - \tau \cos G - \beta \alpha \cos K_6) +$$
$$+ E_2(\tau\alpha + b_2 + \beta \alpha^2 \cos K_6) \cos G]$$
$$a_{41} = A_1(E_1 c_2 - E_2 F \cos^2 G + E_2 c_3 \cos G)$$
$$a_{42} = A_1[E_1(b_2 - \beta \alpha \cos K_6) +$$
$$+ E_2(b_3 + \beta \cos K_5) \cos G]$$

$$a_{43} = A_1(-E_1\ c_2 - E_2\ c_2 \cos G + E_2\ F \cos^2 G)$$
$$a_{44} = A_1[E_1(-\tau\alpha - b_2 - \beta\ \alpha^2 \cos K_6) -$$
$$- E_2\ (b_2 - \tau \cos G - \beta\ \alpha \cos K_6\ ) \cos G] \qquad (8.1)$$

where

$$A_1 = \frac{1}{P_1{}^2}\ (P_1 - E_2^2 \sin^2 G)$$

and

$$P_1 = E_1\ E_3 - E_2^2\ ,\ K_5 = \frac{F(\sin G + e)}{c_1}$$
$$K_6 = \frac{F(c_3 e + c_2 \sin G)}{c_1 c_2} \qquad (8.2)$$

The critical values for dimensionless load F_c for the damped case are found from application of the Routh–Hurwitz criterion.

The critical load values and the critical eigenvalues are found by subroutines FAZA2 and CRIT.

The critical eigenvalues are

$$\lambda_1 = -\lambda_2 = i\ \Omega_o \qquad (9)$$

where

$$\Omega_o = (n_3/n_1)^{1/2}$$
$$n_3 = a_{43}\ a_{22} - a_{23}\ a_{42} + a_{44}\ a_{21} - a_{24}\ a_{41}$$
$$n_1 = -\ a_{22} - a_{44} \qquad (9.1)$$

The influence of eccentricity and the initial angular misalignments on the critical loading depending on damping are shown in Figures 2–6. The calculations were performed for Pettersson's model ($\mu = 1.0$, $\alpha_1 = \alpha_2 = 0.5$) and for Herrmann's model ($M_1 = 2.0$, $M_2 = 1.0$, $\alpha = 1.0$). Equation system (1) was integrated by fourth order Runge–Kutta method ($b_{h1} = b_{h2} = b_{c1} =$

$= b_{c2} = \tau = \beta = 0$) and the solution trajectories obtained were projected from (Z_1, Z_2, Z_3, Z_4) space on the (Z_2, Z_1) space.

In Figures (6, 10) time histories for Z_1, Z_3 as well as phase portraits for various values of (b_1, b_2) and 3 different values of eccentricity are shown.

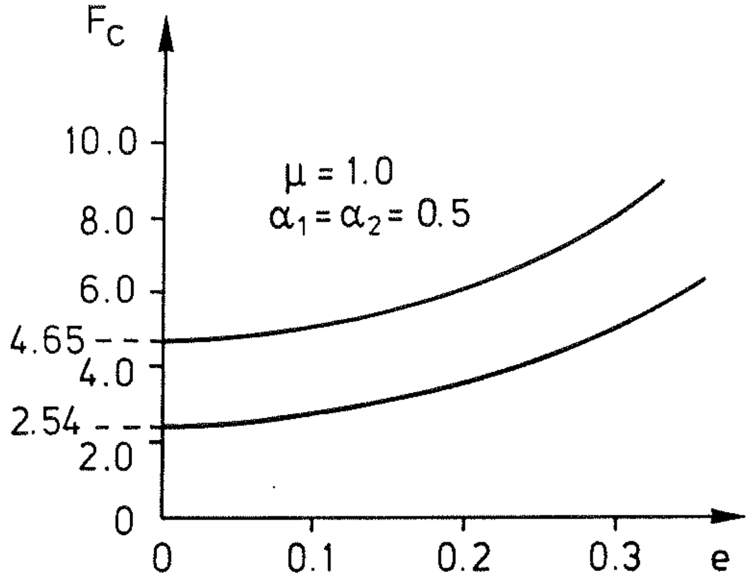

Fig. 2 Critical buckling load F_c versus eccentricity for Pettersson's model

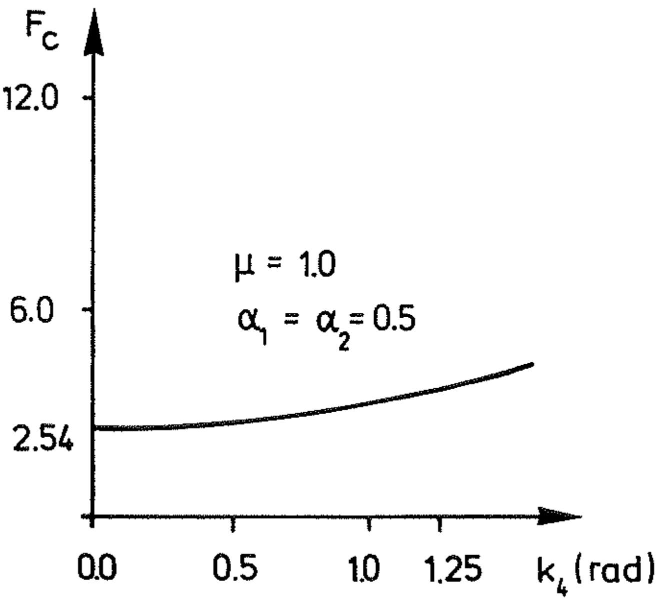

Fig. 3 Critical buckling load F_c versus initial angular misalignments for Pettersson's model

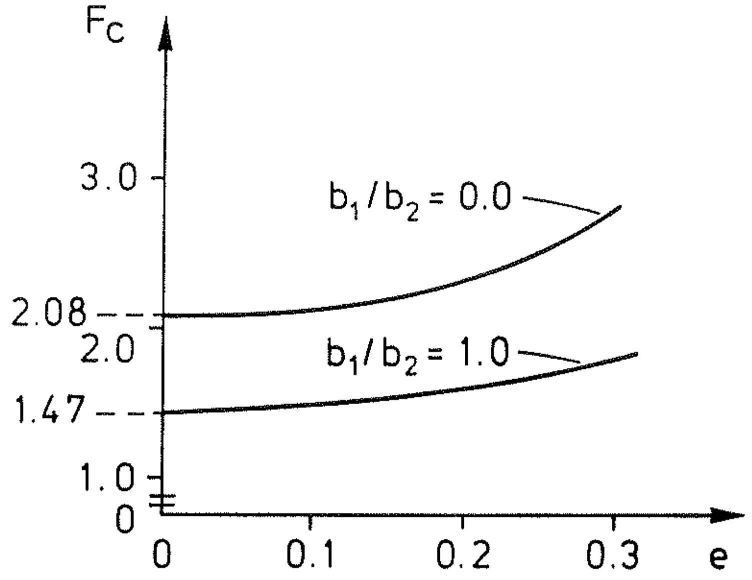

Fig. 4 Critical buckling load F_c versus eccentricity for Herrmann's model with weak damping at both hinges

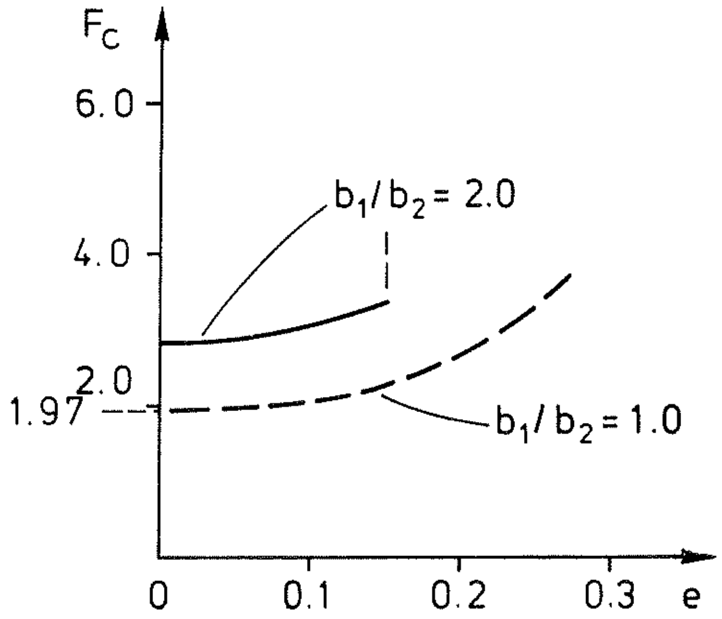

Fig. 5 Critical buckling F_c load versus eccentricity for Herrmann's model with moderate damping

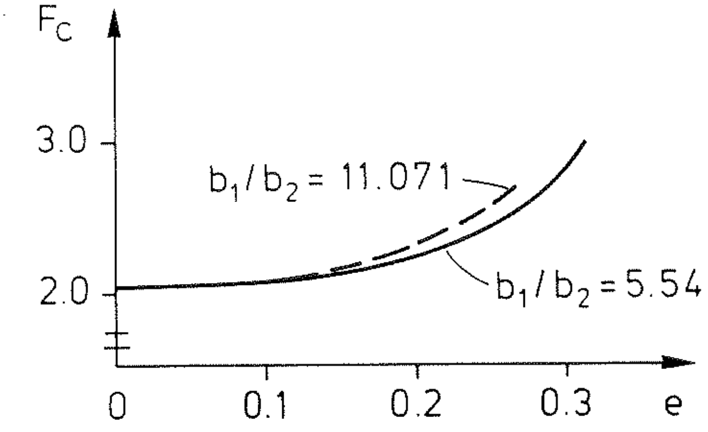

Fig. 6 Critical buckling load F_c versus eccentricity for Herrmann's model for small damping $b_i \leq 1.0$

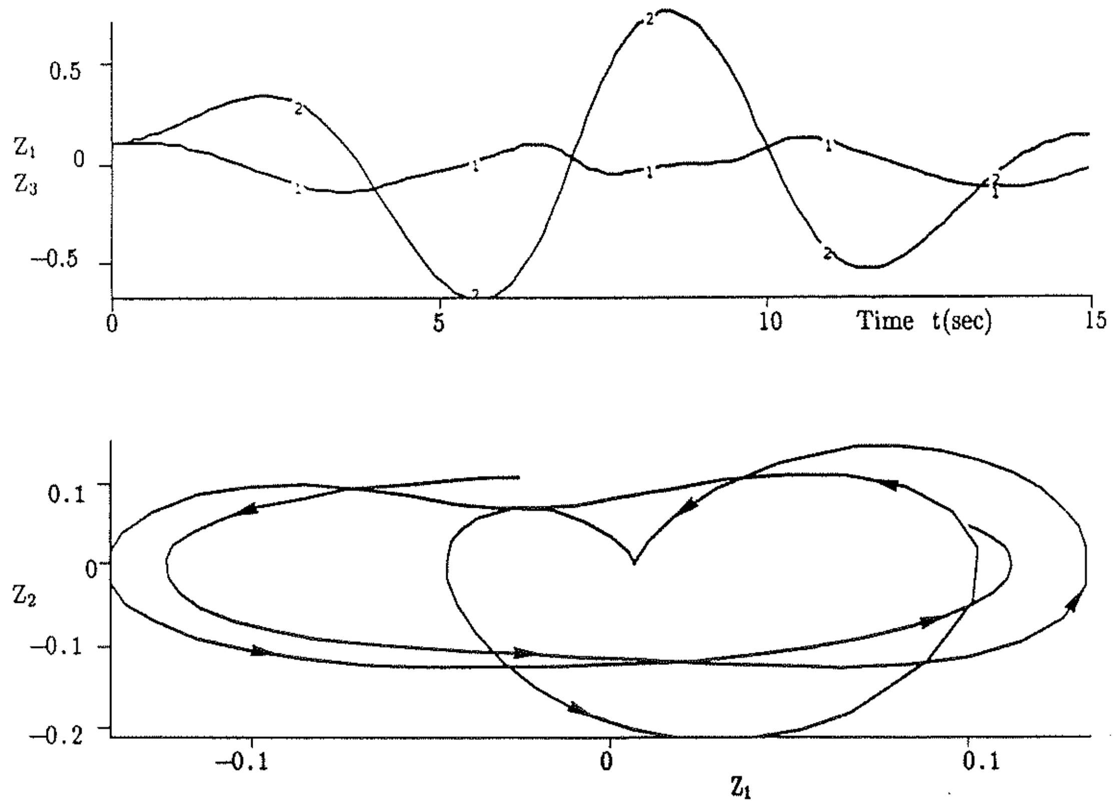

Fig. 7 Angular displacements versus time t (sec) and phase portrait of undamped nonlinear Herrmann's model with e = 0 and F = 2.086

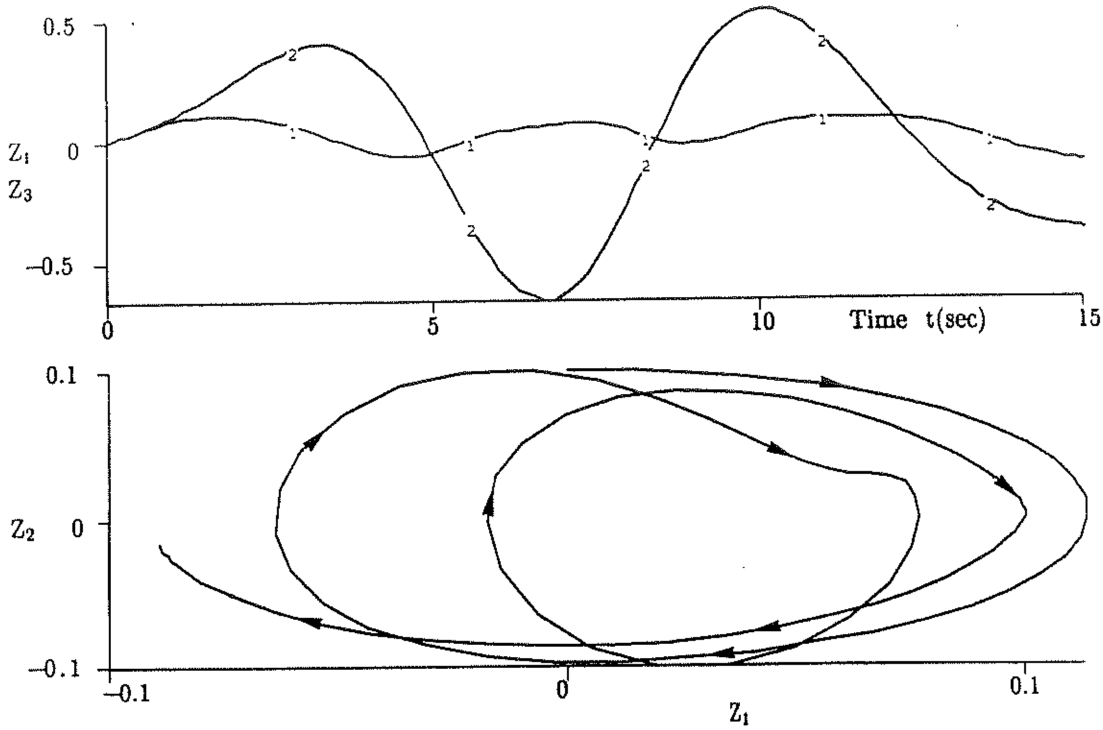

Fig. 8 Angular displacements versus time t (sec) and phase portrait of nonlinear Herrmann's model with $e = 0.1$, $F = 2.019$ and $b_1 = b_2 = 0.001$

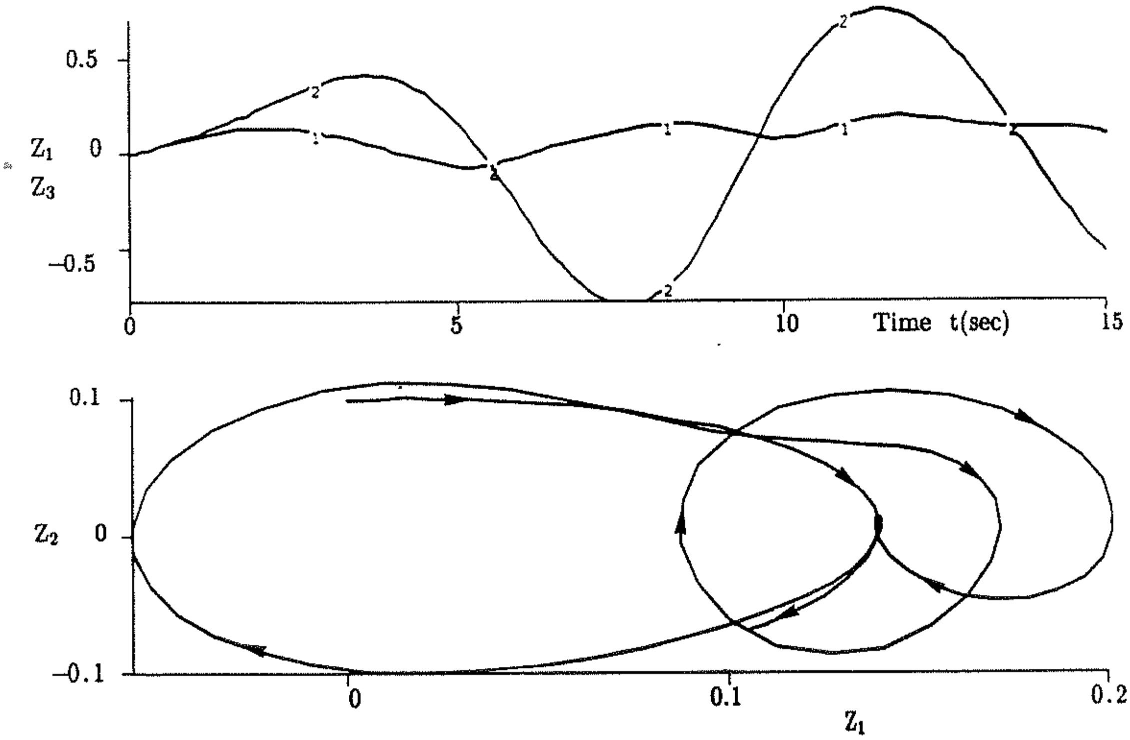

Fig. 9 Angular displacements versus time t (sec) and phase portrait of a damped nonlinear Herrmann's model with $e = 0.2$, $F = 2.0842$ and $b_1 = b_2 = 0.001$

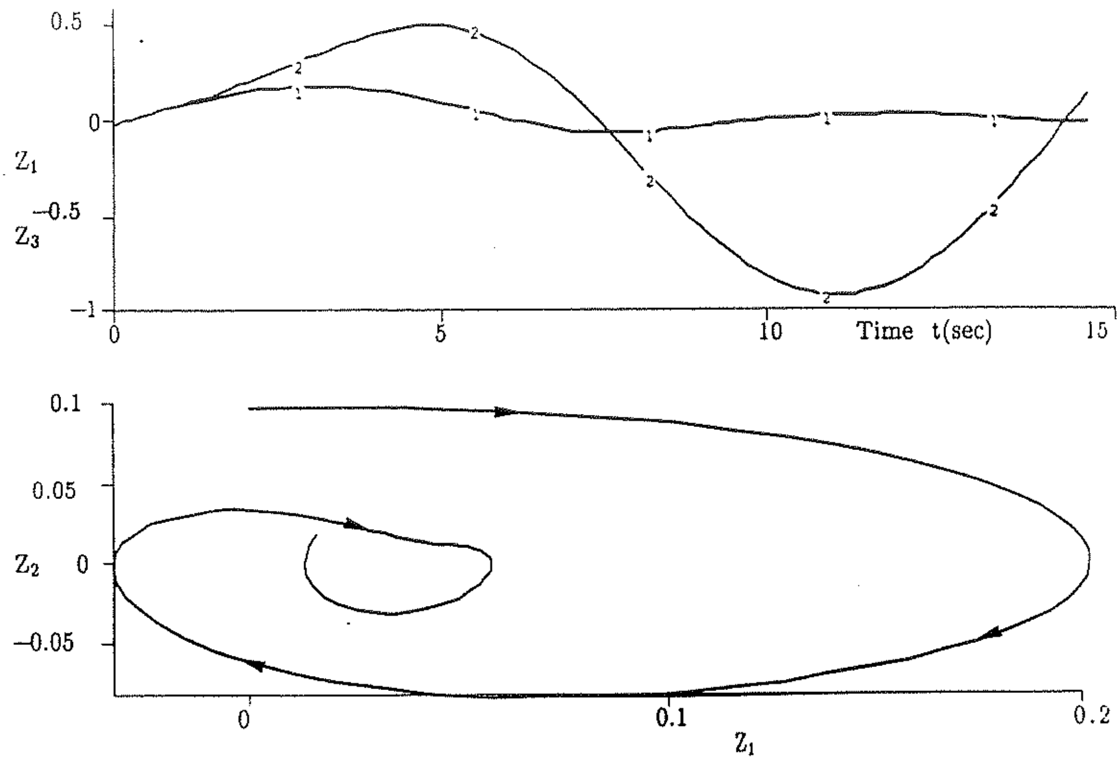

Fig. 10 Angular displacements versus time t (sec) and phase portrait of a damped nonlinear Herrmann's model with e = 0.3, F = 2.0842 and $b_1 = b_2 = 0.001$

3. HOPF'S METHOD

Method of Hopf has been used in this section to distinguish between the hard and the soft flutter of the system of equations(1).

Hopf treated the bifurcation of periodic orbits at a simple complex eigenvalue of a real n dimensional ($n \geq 2$) first order system of autonomous ordinary differential equations.

In order to explain Hopf's work briefly: consider an autonomous system characterized by

$$\frac{dz}{dt} = F(z,\eta) \tag{10}$$

$z \in \mathbb{R}^n \qquad \eta \in \mathbb{R}$

where η is a real parameter. It is assumed that the function F is real and smooth at least in the region of $(z,\eta) = (a^o,0)$ and that the Jacobian matrix $F_z(a^o,0)$ has exactly two, nonzero, purely imaginary eigenvalues $(\lambda_{1,2}(\eta) = \pm i\omega_o)$.

Suppose that η has been increased gradually, then at a critical value $(\eta=\eta_c)$ a pair of complex conjugate eigenvalues $(\lambda_{1,2}(\eta) = \xi(\eta) + i\omega(\eta))$ crosses the imaginary axis with non–zero velocity such that

$$\dot{\xi} = \frac{d\xi}{d\eta} \neq 0$$

Hopf's bifurcation theorem provides sufficient conditions for the existance and uniqueness as well as information regarding stability of time periodic solutions of a system of ordinary differential equations.

The three theorems will not be given here. The interested reader is referred to, for instance, Hassard et al /25/.

The program BIFOR2 was developed by Hassard for CDC Machines and it was implemented on a Sperry Univac 1100/80 Machine at Lund University. All the calculations have been done in double precision.

Three subroutines CRIT, PC and PCFUN have been added to the program BIFOR2.

The bifurcation formulae, primarily applied in present study from /25/, are

$$F(z,\eta) = F_*(\eta_c) + \left[\frac{\eta-\eta_c}{\mu_2}\right]^{1/2} \mathrm{Re}(e^{2\pi iz/T}v_1) + o(\eta-\eta_c)$$

$$T(\eta) = \frac{2\pi}{\omega_o}\left[1 + \tau_2\left[\frac{\eta-\eta_c}{\mu_2}\right] + o(\eta-\eta_c)^2\right]$$

$$\beta(\eta) = \beta_2\left[\frac{\eta-\eta_c}{\mu_2}\right] + o(\eta-\eta_c)^2 \qquad (11)$$

where

$$\mu_2 = -\text{Re } c_1(o)/\alpha'(o)$$
$$\tau_2 = -(\text{Im } c_1(o) + \mu_2 \ \omega'(o)/\omega_o) \qquad (11.1)$$
$$\beta_2 = 2\text{Re } c_1(o)$$

and

$$\alpha'(o) = \text{Re } \lambda_1'(\eta_c)$$
$$\omega'(o) = \text{Im } \lambda_1'(\eta_c) \qquad (11.2)$$

$c_1(o)$ is given in Ref /25/ pp 86–90. $F_*(\eta_c)$ is the stationary point satisfying the hypotheses of Hopf's theorem and v_1 is the right eigenvector.

In the above formulae, $T(\eta)$ is the period of oscillation of the periodic solutions and $\beta(\eta)$ is the characteristic exponent which determines their orbital stability. For classification of the type of instability, the following rules are employed:

1. If $\mu_2 > 0$ and $\beta_2 < 0$ the bifurcation is supercritical
2. If $\mu_2 < 0$ and $\beta_2 > 0$ the bifurcation is subcritical
3. If $\mu_2 < 0$ and $\beta_2 < 0$ the system has stable small amplitude oscillations.

3.1 RESULTS AND DISCUSSION

In BIFOR2 program the values of μ_2, τ_2 and β_2 are found numerically for a selected bifurcation parameter η ($\eta_c = F_c$). The recipe is summarized in Hassard /25/ pp 86–91.

The effect of various parameters on the stability of the system (1) has been shown in the tables 1–9.

As an explanation of the results of BIFOR2 program in table (1) (see program output in Appendix 1), the calculations have been carried out for Herrmann's model ($M_1 = 2.0$, $M_2 = 1.0$ and $\alpha = 1.0$) with $b_1/b_2 = 1.0$ unless otherwise stated. The critical load value is $F_c = 1.469$ (ANU in the program) and the critical pair of purely imaginary eigenvalues are $\lambda_1 = \lambda_2 = i\omega_{cr}$ (EV1 in the program), where $\omega_{cr} = 0.5345$. The values of $\mu_2' > 0$ and $\beta_2 < 0$ from table (1) provide the stability criterion and the bifurcation is un–stable (supercritical).

In the case of cubic damping, table (3), the stability of the system depends on the ratio of the damping coefficients if F is maintained constant. In table (6) the results show insensitivity to the change in the ratio of the hysteresis damping coefficients. The damping at the intermediate articulation point in all the cases were very weak in comparison to the strong damping at the fixed articulation hinge which might explain the slow change in the results. However Hagedorn, ref /20/, concludes that the critical loading coincides for $b_{hi} = 0$ and $b_{hi} \Rightarrow \infty$ with those of a linearly damped system.

In the event of one or more of μ_2, τ_2 and β_2 is 0, then one must calculate μ_4, τ_4, β_4, (tables 1 and 3). The hand calculations are cumbersome and the method of normal form has been relied upon in those cases.

Table 1. Effect of internal damping on the stability of the system

F_c	Amu2	Tau2	Beta2	b_1/b_2
1.469286	0.443266	−0.1317021	−0.0203905	1.0
1.801567	0.8721479	−0.286852	−0.1337239	2.0
1.945556	1.323841	−0.473116	−0.478738	3.0
2.020	1.811652	−0.688839	−1.207822	4.0
2.062879	2.336991	−0.933049	−2.315393	5.0
2.089524	2.899150	−1.2033668	−3.5607143	6.0
2.107115	3.496927	−1.498744	−4.680442	7.0
2.119365	4.128765	−1.817736	−5.576004	8.0
2.128333	4.793003	−2.158962	−6.274476	9.0
2.135227	5.487918	−2.520903	−6.835324	10
2.190136	18.388513	0.0	0.5269954	25

Table 2. Effect of Coriolis force on the stability of the system $b_1/b_2 = 0$

F_c	Amu2	Tau2	Beta2	τ
2.000	0.7499612	−0.1250157	−0.749964	1.0
2.000	0.312498	0.118055	−0.1736111	2.5
2.000	0.220587	0.169170	−0.064878	5.0
2.000	0.202367	0.179240	−0.0401169	7.5
2.000	0.195894	0.182834	−0.029382	10.0
2.000	0.192882	0.184509	−0.023072	12.5

Table 3. Effect of cubic damping on the stability of the system $b_1/b_2 = 1.0$

F_c	Amu2	Tau2	Beta2	b_{c1}/b_{c2}
1.801667	0.862287	−0.281425	−0.1322120	1
"	0.9141806	−0.307854	−0.140154	10
"	0.971739	−0.337132	−0.148978	20
"	1.432207	−0.57150728	−0.2195733	10^2
"	0.006124	0.020822	0.0137667	10^3
"	0	0	0.955700	10^5

Table 4. Effect of initial deflection on stability of damped systems $b_1/b_2 = 1.0$

F_c	Amu2	Tau2	Beta2	ϕ_{10}	ϕ_{20}
1.469286	0.444326	−0.131762	−0.0203905	0.0	0.0
1.469286	0.4443405	−0.1317062	−0.0203905	0.1	0.1
1.48446	0.49897995	−0.1361530	−0.0228096	0.25	0.1
1.58361	1.0450381	−0.232959	−0.046355	0.5	0.1
1.809897	3.8253977	−0.887019	−0.1560687	0.75	0.1
2.275186	9.086790−2	0.028614	−0.3112677	1.0	0.1

Table 5. Effect of external damping on the stability of the system $b_1/b_2 = 1.0$

F_c	Amu2	Tau2	Beta2	Beta
2.0	1.828909	−0.728811	−0.718040	0.1
2.0	1.624999	−0.0638888	0.44444	0.25
2.0	1.124999	−0.041666	0.499999	0.5
2.0	0.3749108	−0.083350	−0.499912	1.0

Table 6. Effect of hysteresis damping on the stability of the system $b_1/b_2 = 1.0$

F_c	Amu2	Tau2	Beta2	b_{h1}/b_{h2}
1.469286	0.436099	−0.128865	−0.020012	0.1
1.469286	0.436096	−0.128864	−0.020012	1.0
1.469286	0.436099	−0.128865	−0.020012	10
1.469286	0.436099	−0.128865	−0.020012	100
1.469286	0.436099	−0.128865	−0.020012	1000
1.469286	0.436099	−0.128865	−0.020012	10000
1.469286	0.436099	−0.128865	−0.020012	100000

Table 7. Effect of stiffness coefficients on the stability of the system for $b_1/b_2 = 1.0$

F_c	Amu2	Tau2	Beta2	c_1	c_2
1.801668	0.971637	−0.337084	−0.148978	1.0	1.0
3.743095	0.687978	−0.0977537	−0.0280407	2.0	1.0
6.308571	0.8911043	−0.893746	−0.0415708	3.0	1.0

Table 8. Effect of end mass on the stability of the system μ =1.0. b_1/b_2 = 1.0

F_c	Amu2	Tau2	Beta2	M_2
3.204545	0.3671728	−0.043208	−0.046301	0.5
1.79749	0.4028528	−0.116627	−0.059982	1.0
1.797619	0.402852	−0.116627	−0.059982	1.5
1.644231	0.389450	−0.153600	−0.054561	2.0

Table 9. Effect of coefficients α_1 and α_2 on the stability of the damped system b_1/b_2 = 1.0.

F_c	Amu2	Tau2	Beta2	α_1	α_2
1.973590	0.410673	−0.102141	−0.0720136	0.25	0.25
2.002155	0.407260	−0.096965	−0.069244	0.5	0.5
2.125	0.397157	−0.0799058	−0.0598072	1.0	1.0

4. THE METHOD OF NORMAL FORM

The method of normal form and the two theorems related to Brjuno and Poincaré has been discussed in detail in Refs /24, 26, 27, 28/.

The method can be regarded as a generalization of Jordan's canonical form applied to nonlinear systems.

Consider the following system

$$\dot{z} = F(z,\eta) = f(z,\eta) + Q(\eta)z \qquad (12)$$

where F is a vector function analytic in its arguments, $z \in \mathbb{R}^n$, $\eta \in \mathbb{R}$

and Q is strictly nonlinear in z; $f(o,\eta)$. For flutter analysis, we assume that η is in a small vicinity of the critical controlling parameter η_c, the matrix Q posses a pair of non–zero imaginary eigenvalues. Without loss of generality, it is assumed that the eigenvalues are complex with negative real parts for $\eta < \eta_c$ (damped system). As η is increased, the real part of at least one pair of complex conjugate eigenvalues vanishes and then becomes positive.

The equation system (12) has been reduced by Hsu by means of a regular analytic transformation to the following normal form, see Ref /24/

$$\dot{y}_1 = y_1[\lambda_1(\epsilon) + \sum_{k=1}^{\infty} g_{1k}(\epsilon)(y_1 y_2)^k]$$
$$y_2 = y_1^* \tag{13}$$

where $\lambda_1(\epsilon) = j\omega + o(\epsilon^0)$ and $g_{ik}(\epsilon)$ are complex power series in ϵ.

The flutter bifurcation of the system (12) can be classified into benign and explosive flutter pending on the sign of the real part of the resonant term $g_{11}(0)$:

a) if $\mathrm{Re}\left[g_{11}(0)\right] < 0$, $z=0$, then the system experiences subcritical (benign flutter) bifurcations

b) if $\mathrm{Re}\left[g_{11}(0)\right] > 0$ $z=0$, then explosive flutter occurs (supercritical bifur–cations).

The amplitude of vibration $\bar{a}$ and the periodic solutions z(t) are given by the following expressions from Ref /27/

$$\bar{a}^2 = -\left[\mathrm{Re}\left[\lambda_1'(0)\right]/\mathrm{Re}\left[g_{11}(0)\right]\right]\epsilon$$

$$z(t) = 2\mathrm{Re}\left[u_1 \bar{a} e^{j\omega t}\right] \tag{14}$$

$$\text{where } \lambda_1'(o) = v_1^T\left[\frac{\partial A}{\partial \epsilon}\right]_{\epsilon=0}\Bigg| u_1 \tag{15}$$

and u_1, v_1 are the left and right eigenvalues of Q(0) respectively found with respect to the critical eigenvalue $\lambda_1 = i\omega$.

The equations of motion (1) have been recasted in the system form (12) (local coordinates) and expanded into power series up to third order.

The program NORFOR2 has been utilized in this section which calculates the normal transformation and the reduced form. The input data for the program is given in Appendix 2 (reproduced from professor Hsu's notes without change) and a typical example of output is given in Appendix 3.

The results are given in tables 10–14 and the coefficient of the resonent term are given as FI, 2, 122 $\rightarrow$ $\varphi_{2(1,2)}$ = $g_2(1,2)$, see Appendix 2.

Table 10. Effect of initial deflection on the stability of the system $b_1/b_2 = 1.0$

F_c	Re $g_2(1,2)$	Im $g_2(1,2)$	ϕ_{10}	ϕ_{20}
1.469286	1.4950	7.4120	0.0	0.0
1.475996	1.4915	7.3919	0.0	0.1
1.484460	1.4871	7.3670	0.25	0.1
1.58361	1.4395	7.1106	0.5	0.1
1.809897	1.3528	6.68	0.75	0.1
2.275186	1.2340	6.0978	1.0	0.1
7.869113	0.93399	4.1761	1.50	0.1

Table 11. Effect of cubic damping on the stability of the system $b_1/b_2 = 1.0$

F_c	Re $g_2(1,2)$	Im $g_2(1,2)$	b_{c1}/b_{c2}
1.469286	2.2267	7.3928	1.0
1.469286	2.0727	7.3931	10
1.469286	5.3276	7.3961	10^2
1.469286	–14.867	7.4263	10^3
1.469286	–16.886	7.7281	10^4
1.469286	–17.088	107.46	10^5

Table 12. Effect of internal damping on the stability of the system

F_c	Re $g_2(1,2)$	Im $g_2(1,2)$	b_1/b_2
1.469286	1.4950	7.4120	1.0
1.801667	1.8699	6.6151	2.0
1.945556	2.2124	5.6989	3.0
2.02	2.5067	4.8727	4.0
2.062879	2.7531	4.1533	5.0
2.089524	2.9548	3.5221	6.0
2.107115	3.1138	2.9596	7.0
2.119365	3.2313	2.4513	8.0
2.128333	3.3075	1.9877	9.0
2.135227	3.3427	1.5636	10.0
2.190136	–0.09279	1.0370	25.0

Table 13. Effect of end mass on the stability of the system $b_1/b_2=1.0$

F_c	Re $g_2(1,2)$	Im $g_2(1,2)$	M_2
2.654615	0.359	1.2016	0.25
1.815556	0.70115	3.1335	0.5
1.575507	1.0809	5.271	0.75
1.469286	1.4950	7.4120	1.0
1.37614	2.4061	11.403	1.5
1.335833	3.4002	14.836	2.0

Table 14. Effect of stiffness coefficients on the stability of the system $b_1/b_2=1.0$

F_c	Re $g_2(1,2)$	Im $g_2(1,2)$	c_1	c_2
0.376428	6.1233	1.4468	1.0	0.25
0.624076	2.4212	7.9251	1.0	0.5
1.005	1.7401	7.2518	1.0	0.75
1.469286	1.4950	7.4120	1.0	1.0
2.547857	1.3276	8.3334	1.0	1.5
3.743095	1.2856	9.4469	1.0	2.0

4.1 DISCUSSION OF THE RESULTS

In table (10) the effect of initial equilibrium angles on the stability of Herrmann's model is shown with (μ = 1.0, M_1 = 2.0, M_2 = 1.0, and b_1/b_2 = 1.0), see Appendix 3. The coefficient $g_2(1,2)$ obtained from NORFOR2 program for $\phi_{10} = \phi_{20} = 0$ has a positive real part and one concludes that supercritical bifurcation occurs. Other combinations of initial deflections give similar results.

For cubic damping (table 11) the results obtained from NORFOR2 program ($b_{c1}/b_{c2} = 10^3$) differs from those obtained by BIFOR2 program (Table 5). In the first, the bifurcation is supercritical and in the later subcritical. Further increase in the value of the coefficient (b_{c1}/b_{c2}) requires the calculation of μ_4, τ_4 and β_4 for Hopf bifurcation as Amu2 = Tau2 = 0 however the calculation tends to be a tedious procedure. For details see ref /25/ pp. 97.

5. CONCLUDING REMARKS

The investigation carried out in this paper is divided essentially into two parts. The first part deals with the stability analysis of equations (1) by means of Hopfs bifurcation theorem. This was achieved by using program BIFOR2. The second was to give an analysis of the same system by means of normal form method; employing the program NORFOR2.

The paper illustrates the applicability of BIFOR2 and NORFOR2 to flutter instability in imperfect structural systems. The program NORFOR2 is cheaper to run, but the system of equation (1) must be expressed in power series, which tends to be tedious and error prone for systems with three or more degrees of freedom. Some discrepancies have been found between the results obtained by NORFOR2 and BIFOR2. These differences have been noticed by Hsu as well, when applying the method of normal form to similar problems.

For example, in Lorenz equations analysis, Hsu found unstable regions where the Hopf bifurcation occurs which were contrary to the stable regions findings of Marsden and McCracken /25/, using an analytical method.

The effect of eccentricity on the Hopf bifurcations is similar to that of initial misalignments. The results are displayed in table 15 with small vanishing damping ($b_1 = b_2 = 0.001$).

Table 15 Effect of eccentricity on the stability of the damped system $b_1/b_2 = 1.0$

Eccentricity	Amu2	Tau2	Beta2
0.1	0.52737	−0.142722	−0.00024
0.2	0.94915	−0.213674	−0.000398

These results show that the bifurcation is supercritical.

6. ACKNOWLEDGEMENT

The author is indebted to Professors L Hsu and B D Hassard for sending him a copy of their programs and for consultation. Assistance of Stefan Burg in implementing the programs at Lund University Computing Centre is gratefully acknowledged. This research have been financed by a grant from Lund University.

7. NOTATIONS

c_1, c_2 stiffness of the springs at lower and upper hinges, respectively

b_1, b_2 linear damping at the lower and upper hinges, respectively

b_{c1}, b_{c2} cubic damping at the lower and upper hinges, respectively

b_{h1}, b_{h2} hysteresis damping at the lower and upper hinges, respectively

b external damping coefficient

β dimensionless external damping coefficient

τ coriolis coefficient

ϕ_1, ϕ_2 configuration angles from the vertical

$\phi = \phi_1 - \phi_2$

e eccentricity of the applied load

ϕ_{10}, ϕ_{20} initial rotations of the lower and upper link, respectively, in the unstrained configuration

$\bar{\Phi}_1 = \phi_1 - \phi_{10}$

$\bar{\Phi}_2 = \phi_2 - \phi_{20}$

m_1, m_2	masses of the links
M_1,M_2	concentrated masses
L_1, L_2	length of the rigid links
I_1, I_2	moment of inertia of the lower and upper link, respectively
I_3	moment of inertia of the concentrated mass at the end of the upper link
P	follower force acting at the end of the upper bar
μ	ratio of distributed masses
a_1	distance of centre of gravity of the lower link from lower hinge
a_2	distance of the centre of gravity from the upper hinge for the 2nd link
$\bar{a}$	distance of the centre of gravity of the end mass from the end of the 2nd link
Ω, ω	frequency
α	length ratio
γ	length ratio
$\lambda_{1,2}$	eigenvalues
D	external damping force at the free end
q	coriolis force

8. REFERÈNCES

/1/ Caesari L, Asymptotic Behaviour and Stability Problems in Ordinary Differential Equations, 3rd edn, Springer Verlag, Berlin, 1970

/2/ Bogolinbov N N and Mitropolski P A, Asymptotic Method in the Theory of Non–linear Oscillations, Gordon and Breach, New York 1961

/3/ Poincaré H, Les Methodes Nouvelles de la Mechanique Celeste, Gauthiers–Villars, Paris 1891

/4/ Hsu L, Sur l'Obtention d'une Transformation Ponctuelle Associée à un Système d'Equations Diffèrentielles à Coefficients Périodiques, Comptes Rendus Acad. Sc. Paris

/5/ Brjuno A D, Normal Form for Nonlinear Oscillations. Analytical Methods in the Theory of Nonlinear Oscillations (Proc. Fifth Internat. Conf. Nonlin. Osc.), Kiev, Inst. Matem. Academ. Nank Ukrain SSR, 1970

/6/ Brjuno A D, Transactions of the Moscow Mathematical Society 25, 132–198. Analytical form of differential equations (English translation edited by the American Mathematical Society 1971)

/7/ Brjuno A D, Transactions of the Moscow Mathematical Society 26, 199–299. Analytical form of differential equations (English translation edited by the American Mathematical Society 1972)

/8/ Moté C D, Nonconservative Stability by Finite Element Method, Procee– dings of ASCE, Journal of Engineering Mechanics Division 97, 645–656, 1971

/9/ Wu J J, A Numerical Method for Missile Stability Analysis, Proceedings US Army Numerical Analysis and Computers Conference, St Louis, 529–556, 1975

/10/ Bolotin V V, Nonconservative Problems of the Theory of Elastic Stabi– lity, Pergamon Press, Oxford, 1963

11/ Hermann G, Stability of Equilibrium of Elastic Systems Subjected to Nonconservative Forces, Appl. Mech. Rev., Vol 20, 103–108, 1967

/12/ Leipholz H, Stability of Elastic Systems, Sijthoff & Noordhoff, the Netherlands 1980

/13/ Plaut R H, Post–Buckling Behaviour of a Double Pendulum With Partial Follower Load, Report No VPI–E–76–2, Virginia Polytech. Institute and State University, Blacksbury, Virginia, February 1976

/14/ Plaut R H, Branching Analysis at Coincident Buckling Loads of Non––Conservative Elastic Systems, Report No VPI–E–76–15, Virginia Polytechnic Institute and State University, Blacksbury, Virginia, August 1976

/15/ Plaut, R H, Post–Buckling Behaviour of Continuous, Nonconservative Elastic Systems, Acta Mechanica 30, 51–64, 1978

/16/ Wiley J C, Non–Linear Behaviour of Elastic Systems Subjected to Follower Forces, Thesis, Purdue Univ., 1968

/17/ Burgess I W and Levinson M, The Post–Flutter Oscillations of Discrete Symmetric Structural Systems With Circulatory Loading, Int. J. Mech. Sci 14, 471–488, 1972

/18/ Burgess P, Non–Linear Analysis of Non–Conservative Systems in the Dynamic Instability Range, M.S. Thesis, Division of Engineering, Brown University, 1976

/19/ Mandadi V, On the General Theory of Stability and Its Applications, PhD, Thesis, Department of Systems Design, University of Waterloo, Waterloo, Dec 1976

/20/ Hagedorn P, On the Destabilizing Effect of Non–Linear Damping in Nonconservative Systems with Follower Forces, Int. J. Non–Linear Mechanics, Vol 15, June 1970

/21/ Sethna P R and Shapiro S M, Nonlinear Behaviour of Flutter Unstable Dynamical Systems With Gyroscopic and Circulatory Forces, J. Appl. Mech. 44, 755–762, Dec 1977

/22/ Mandadi V and Huseyin K, Nonlinear Bifurcation Analysis of Non–Gradient Systems, Int. J. Non–Linear Mech., Vol 15, 159–172 1980

/23/ Scheidl R, Troger H and Zeman K, Coupled Flutter and Divergence Bifurcation of A Double Pendulum, Int. J. Non–Linear Mech., Vol 19, No 2, 163–176, 1983

/24/ Hsu L, Analysis of Critical and Post–Critical Behaviour of Non–Linear Dynamical Systems by the Normal Form Method, Part II: Divergence and Flutter, J. of Sound and Vibrations 89,l No 2, 183–194, 1983

/25/ Hassard B D, Kazarinoff N D and Wan Y A, Theory and Applications of Hopf Bifurcations, Cambridge University Press, 1981

/26/ Hsu L and Favretto, Recursive Bifurcation Formulae for Normal Form and Centre Manifold Theory, J. Math. Anal. and Applic., Vol 101, No 2, pp 562–574, July 1984

/27/ Hsu L, Analysis of Critical and Post–Critical Behaviour of Non–Linear Dynamical Systems by the Normal Form Method, Part I, J. Sound and Vibrations, Vol 82, No 2, pp 169–181, 1983

/28/ Hsu L, On the Analysis of Fourth Order Systems Near a Bifurcation, Mech. Res. Comm., Vol 4, No 4, pp 219–227, 1977

APPENDICES

APPENDIX 1


```
@XQT FHOPD.CPUI

        0:      50:        3010      CPU=      5343

@XQT FHOPD.BIFABS
>>> THE ENTERED INDATA ARE AS FOLLOWS>>>>
 FST=    .2100000+001    DF1=    .0000000        DFM=    .0000000
 FI1=    .1000000+000  FI3=    .1000000+000  C1  =    .1000000+001

 C2  =    .1000000+001  B1   =    .1000000+000    B2 =    .1000000+000
 EKS =    .0000000        BK1 =    .0000000

 BK2 =    .0000000      BH1    =    .0000000        BH2 =    .0000000
 TAU=    .0000000      BETA   =    .0000000        AI1 =    .0000000

 AI2   =    .0000000        AM1   =    .2000000+001    AM2 =    .1000000+001
 GAMA =    .0000000        ALFA =    .1000000+001   AMU=    .0000000

 ALFA1=    .0000000        ALFA2=    .0000000          AI3=    .0000000

 PET=    .0000000     ALAMDA=    .3000000+001

 FFM>>>>    .1469286+001 FFL>>>>    .1469286+001



 CRITICAL DYNAMIC LOADING



 ANU>>>    .1469286+001



 CRITICAL FREQUENCY


 FRQ1>>>>    .0000000        .5345225+000FRQ2>>>>    .0000000      -.5345225+000
```

```
C-   U=     .100000-007
------------------------------------------------------------------------------------------------------------------------
EVALS     N           EPS              NSIG     ITMAX        ANU                      RE EV1                  IM EV1
          4      .1000000000-011         7       20      .1469285714+001      -.1734723476-017      .5345224838+000

 STATIONARY POINT (NIT =  0)    !  F(XS,ANU)                          !  EIGENVALUES
 XS( 1) =      .0000000000      !  F( 1) =      .0000000000           !  EV( 1) =     -.1750000000+000      .1311249404+001
 XS( 2) =      .0000000000      !  F( 2) =      .0000000000           !  EV( 2) =     -.1750000000+000     -.1311249404+001
 XS( 3) =      .0000000000      !  F( 3) =      .0000000000           !  EV( 3) =     -.1734723476-017      .5345224838+000
 XS( 4) =      .0000000000      !  F( 4) =      .0000000000           !  EV( 4) =     -.1734723476-017     -.5345224838+000

************************************************************************************************************************
BIFOR2     N           U                 MTH      JJOB     IPRINT     CRITICAL ANU
           4       .1000000000-007        1        1         2      .1469285714+001

                             AMU2                  TAU2                  BETA2               RE(C1(O))             IM(C1(O))
 VALUES                 .4443266745+000      -.1317021026+000      -.2039056706-001      -.1019528353-001      -.9398509851-002
 ERROR ESTIMATES        .1104722553-002       .4685848581-003       .1284488222-005       .6422441108-006       .1687845139-004

                            DALPHA                DOMEGA               OMEGA(O)              ENORMX                ENORMV
 VALUES                 .2294546809-001       .1795891389+000       .5345224838+000       .0000000000           .2331392371+001
 ERROR ESTIMATES        .5546562758-004       .7901220923-004

 STATIONARY POINT               !  EIGENVECTOR V1
 XS( 1) =      .0000000000      !  V1( 1) =      .1000000000+001     -.5421010862-019
 XS( 2) =      .0000000000      !  V1( 2) =     -.1131636018-015      .5345224838+000
 XS( 3) =      .0000000000      !  V1( 3) =      .1795544948+001     -.5953313265-001
 XS( 4) =      .0000000000      !  V1( 4) =      .3182179793-001      .9597591456+000

C-   U=     .100000-009
------------------------------------------------------------------------------------------------------------------------
EVALS     N           EPS              NSIG     ITMAX        ANU                      RE EV1                  IM EV1
          4      .1000000000-011         7       20      .1469285714+001      -.1734723476-017      .5345224838+000

 STATIONARY POINT (NIT =  0)    !  F(XS,ANU)                          !  EIGENVALUES
 XS( 1) =      .0000000000      !  F( 1) =      .0000000000           !  EV( 1) =     -.1750000000+000      .1311249404+001
 XS( 2) =      .0000000000      !  F( 2) =      .0000000000           !  EV( 2) =     -.1750000000+000     -.1311249404+001
 XS( 3) =      .0000000000      !  F( 3) =      .0000000000           !  EV( 3) =     -.1734723476-017      .5345224838+000
 XS( 4) =      .0000000000      !  F( 4) =      .0000000000           !  EV( 4) =     -.1734723476-017     -.5345224838+000

************************************************************************************************************************
BIFOR2     N           U                 MTH      JJOB     IPRINT     CRITICAL ANU
           4       .1000000000-009        1        1         2      .1469285714+001
```

```
                          AMU2             TAU2             BETA2            RE(C1(0))         IM(C1(0))
 VALUES              .4443368206+000  -.1317051845+000  -.2039055456-001  -.1019527728-001  -.9398344490-002
 ERROR ESTIMATES     .5118122953-004   .2173492845-004   .6015443877-007   .3007721938-007   .7951526187-006

                         DALPHA           DOMEGA           OMEGA(0)          ENORMX            ENORMV
 VALUES              .2294493008-001   .1795883734+000   .5345224838+000   .0000000000       .2331392371+001
 ERROR ESTIMATES     .2574940163-005   .3670444651-005

 STATIONARY POINT               !  EIGENVECTOR V1
 XS( 1) =      .0000000000      !  V1( 1) =     .1000000000+001    -.5421010862-019
 XS( 2) =      .0000000000      !  V1( 2) =    -.1131636018-015     .5345224838+000
 XS( 3) =      .0000000000      !  V1( 3) =     .1795544948+001    -.5953313265-001
 XS( 4) =      .0000000000      !  V1( 4) =     .3182179793-001     .9597591456+000

C-   U=    .100000-011
--------------------------------------------------------------------------------------------------------------
EVALS    N           EPS             NSIG    ITMAX       ANU                 RE EV1               IM EV1
         4      .1000000000-011        7       20    .1469285714+001    -.1734723476-017     .5345224838+000

 STATIONARY POINT (NIT =  0)    !  F(XS,ANU)                     !  EIGENVALUES
 XS( 1) =      .0000000000      !  F( 1) =     .0000000000       !  EV( 1) =    -.1750000000+000     .1311249404+001
 XS( 2) =      .0000000000      !  F( 2) =     .0000000000       !  EV( 2) =    -.1750000000+000    -.1311249404+001
 XS( 3) =      .0000000000      !  F( 3) =     .0000000000       !  EV( 3) =    -.1734723476-017     .5345224838+000
 XS( 4) =      .0000000000      !  F( 4) =     .0000000000       !  EV( 4) =    -.1734723476-017    -.5345224838+000

**************************************************************************************************************
BIFOR2     N           U                 MTH      JJOB     IPRINT     CRITICAL ANU
           4      .1000000000-011         1         1         2      .1469285714+001

                          AMU2             TAU2             BETA2            RE(C1(0))         IM(C1(0))
 VALUES              .4443372006+000  -.1317053011+000  -.2039055398-001  -.1019527699-001  -.9398336803-002
 ERROR ESTIMATES     .2374906511-005   .1008793674-005   .2797464477-008   .1398732238-008   .3702599837-007

                         DALPHA           DOMEGA           OMEGA(0)          ENORMX            ENORMV
 VALUES              .2294490981-001   .1795883428+000   .5345224838+000   .0000000000       .2331392371+001
 ERROR ESTIMATES     .1194880816-006   .1703457926-006

 STATIONARY POINT               !  EIGENVECTOR V1
 XS( 1) =      .0000000000      !  V1( 1) =     .1000000000+001    -.5421010862-019
 XS( 2) =      .0000000000      !  V1( 2) =    -.1131636018-015     .5345224838+000
 XS( 3) =      .0000000000      !  V1( 3) =     .1795544948+001    -.5953313265-001
 XS( 4) =      .0000000000      !  V1( 4) =     .3182179793-001     .9597591456+000
```

```
C-    U=     .100000-013
-----------------------------------------------------------------------------------------------------------
EVALS     N          EPS                NSIG    ITMAX       ANU                RE EV1              IM EV1
          4     .1000000000-011           7      20     .1469285714+001    -.1734723476-017     .5345224838+000

 STATIONARY POINT (NIT =  0)     !  F(XS,ANU)                      !  EIGENVALUES
 XS( 1) =      .0000000000       !  F( 1) =       .0000000000      !  EV( 1) =     -.1750000000+000     .1311249404+001
 XS( 2) =      .0000000000       !  F( 2) =       .0000000000      !  EV( 2) =     -.1750000000+000    -.1311249404+001
 XS( 3) =      .0000000000       !  F( 3) =       .0000000000      !  EV( 3) =     -.1734723476-017     .5345224838+000
 XS( 4) =      .0000000000       !  F( 4) =       .0000000000      !  EV( 4) =     -.1734723476-017    -.5345224838+000

***********************************************************************************************************
BIFOR2    N           U                  MTH     JJOB     IPRINT    CRITICAL ANU
          4      .1000000000-013          1       1         2      .1469285714+001

                          AMU2              TAU2               BETA2              RE(C1(0))            IM(C1(0))
 VALUES              .4443372185+000   -.1317053066+000   -.2039055396-001   -.1019527698-001   -.9398336446-002
 ERROR ESTIMATES     .9042679742-007    .3929280513-007    .1298983869-009    .6494919344-010    .1719714195-008

                          DALPHA            DOMEGA             OMEGA(0)            ENORMX              ENORMV
 VALUES              .2294490885-001    .1795883413+000    .5345224838+000    .0000000000         .2331392371+001
 ERROR ESTIMATES     .4523332591-008    .6849694992-008

 STATIONARY POINT                !  EIGENVECTOR V1
 XS( 1) =      .0000000000       !  V1( 1) =      .1000000000+001    -.5421010862-019
 XS( 2) =      .0000000000       !  V1( 2) =     -.1131636018-015     .5345224838+000
 XS( 3) =      .0000000000       !  V1( 3) =      .1795544948+001    -.5953313265-001
 XS( 4) =      .0000000000       !  V1( 4) =      .3182179793-001     .9597591456+000

C-    U=     .100000-015
-----------------------------------------------------------------------------------------------------------
EVALS     N          EPS                NSIG    ITMAX       ANU                RE EV1              IM EV1
          4     .1000000000-011           7      20     .1469285714+001    -.1734723476-017     .5345224838+000

 STATIONARY POINT (NIT =  0)     !  F(XS,ANU)                      !  EIGENVALUES
 XS( 1) =      .0000000000       !  F( 1) =       .0000000000      !  EV( 1) =     -.1750000000+000     .1311249404+001
 XS( 2) =      .0000000000       !  F( 2) =       .0000000000      !  EV( 2) =     -.1750000000+000    -.1311249404+001
 XS( 3) =      .0000000000       !  F( 3) =       .0000000000      !  EV( 3) =     -.1734723476-017     .5345224838+000
 XS( 4) =      .0000000000       !  F( 4) =       .0000000000      !  EV( 4) =     -.1734723476-017    -.5345224838+000

***********************************************************************************************************
BIFOR2    N           U                  MTH     JJOB     IPRINT    CRITICAL ANU
          4      .1000000000-015          1       1         2      .1469285714+001
```

```
                          AMU2                TAU2               BETA2              RE(C1(0))           IM(C1(0))
VALUES               .4443372194+000    -.1317053069+000   -.2039055396-001   -.1019527698-001   -.9398336429-002
ERROR ESTIMATES      .4335004019-008     .1877805744-008    .6029503692-011    .3014751846-011    .7964133793-010

                         DALPHA              DOMEGA            OMEGA(0)             ENORMX              ENORMV
VALUES               .2294490880-001     .1795883412+000    .5345224838+000    .0000000000        .2331392371+001
ERROR ESTIMATES      .2170682871-009     .3276157464-009

STATIONARY POINT                   !  EIGENVECTOR V1
XS( 1) =       .0000000000         !  V1( 1) =     .1000000000+001    -.5421010862-019
XS( 2) =       .0000000000         !  V1( 2) =    -.1131636018-015     .5345224838+000
XS( 3) =       .0000000000         !  V1( 3) =     .1795544948+001    -.5953313265-001
XS( 4) =       .0000000000         !  V1( 4) =     .3182179793-001     .9597591456+000

C-   U=     .100000-017
-------------------------------------------------------------------------------------------------------------------
EVALS     N           EPS              NSIG    ITMAX       ANU                  RE EV1               IM EV1
          4     .1000000000-011          7       20     .1469285714+001    -.1734723476-017     .5345224838+000

STATIONARY POINT (NIT =  0)        !  F(XS,ANU)                         !  EIGENVALUES
XS( 1) =       .0000000000         !  F( 1) =      .0000000000          !  EV( 1) =     -.1750000000+000     .1311249404+001
XS( 2) =       .0000000000         !  F( 2) =      .0000000000          !  EV( 2) =     -.1750000000+000    -.1311249404+001
XS( 3) =       .0000000000         !  F( 3) =      .0000000000          !  EV( 3) =     -.1734723476-017     .5345224838+000
XS( 4) =       .0000000000         !  F( 4) =      .0000000000          !  EV( 4) =     -.1734723476-017    -.5345224838+000

*******************************************************************************************************************
BIFOR2    N           U                MTH      JJOB     IPRINT     CRITICAL ANU
          4     .1000000000-017         1        1         2      .1469285714+001

                          AMU2                TAU2               BETA2              RE(C1(0))           IM(C1(0))
VALUES               .4443372195+000    -.1317053069+000   -.2039055395-001   -.1019527698-001   -.9398336429-002
ERROR ESTIMATES      .2292295013-009     .9745788876-010    .3290723271-012    .1645361636-012    .3162256345-011

                         DALPHA              DOMEGA            OMEGA(0)             ENORMX              ENORMV
VALUES               .2294490880-001     .1795883412+000    .5345224838+000    .0000000000        .2331392371+001
ERROR ESTIMATES      .1146677255-010     .1747373405-010

STATIONARY POINT                   !  EIGENVECTOR V1
XS( 1) =       .0000000000         !  V1( 1) =     .1000000000+001    -.5421010862-019
XS( 2) =       .0000000000         !  V1( 2) =    -.1131636018-015     .5345224838+000
XS( 3) =       .0000000000         !  V1( 3) =     .1795544948+001    -.5953313265-001
XS( 4) =       .0000000000         !  V1( 4) =     .3182179793-001     .9597591456+000
```

```
C-   U=     .100000-019
------------------------------------------------------------------------------------------------------------------------
EVALS    N           EPS              NSIG   ITMAX       ANU                RE EV1                IM EV1
         4     .1000000000-011          7      20     .1469285714+001   -.1734723476-017     .5345224838+000

 STATIONARY POINT (NIT =  0)   !  F(XS,ANU)                      !  EIGENVALUES
 XS( 1) =     .0000000000      !  F( 1) =     .0000000000       !  EV( 1) =    -.1750000000+000     .1311249404+001
 XS( 2) =     .0000000000      !  F( 2) =     .0000000000       !  EV( 2) =    -.1750000000+000    -.1311249404+001
 XS( 3) =     .0000000000      !  F( 3) =     .0000000000       !  EV( 3) =    -.1734723476-017     .5345224838+000
 XS( 4) =     .0000000000      !  F( 4) =     .0000000000       !  EV( 4) =    -.1734723476-017    -.5345224838+000

************************************************************************************************************************
BIFOR2    N           U               MTH      JJOB     IPRINT    CRITICAL ANU
          4     .1000000000-019        1        1         2      .1469285714+001

                          AMU2               TAU2               BETA2              RE(C1(0))            IM(C1(0))
 VALUES             .4443405945+000    -.1317062696+000    -.2039055396-001    -.1019527698-001    -.9398336432-002
 ERROR ESTIMATES    .3375084234-005     .1305222924-005     .7666575166-013     .3833287583-013     .3304622879-011

                         DALPHA             DOMEGA              OMEGA(0)            ENORMX               ENORMV
 VALUES             .2294473452-001     .1795881352+000     .5345224838+000     .0000000000         .2331392371+001
 ERROR ESTIMATES    .1742802394-006     .2060174489-006

 STATIONARY POINT              !  EIGENVECTOR V1
 XS( 1) =     .0000000000      !  V1( 1) =     .1000000000+001    -.5421010862-019
 XS( 2) =     .0000000000      !  V1( 2) =    -.1131636018-015     .5345224838+000
 XS( 3) =     .0000000000      !  V1( 3) =     .1795544948+001    -.5953313265-001
 XS( 4) =     .0000000000      !  V1( 4) =     .3182179793-001     .9597591456+000

 PC- EFFECT OF NUMERICAL DIFFERENCING
    U             AMU2            TAU2             BETA2
 .100-007    .444326674521   -.131702102642   -.020390567063
 .100-009    .444336820578   -.131705184516   -.020390554564
 .100-011    .444337200594   -.131705301098   -.020390553983
 .100-013    .444337218531   -.131705306592   -.020390553956
 .100-015    .444337219428   -.131705306864   -.020390553955
 .100-017    .444337219482   -.131705306879   -.020390553955
 .100-019    .444340594508   -.131706269553   -.020390553955
     END

£XQT FHOPD.CPUI
```

APPENDIX 2

NORFOR 2: INPUT DATA

INTEGERS

ITER: only odd powers ITER=3; even and odd powers ITER=2

LZ: number of critical equations (include parameter equation, $\dot{\epsilon} = 0$)

NZ: total number of equations (NZ≥LZ)

LP: order number of the auxiliary parameter equation. If there is not a parameter equation, set LP>NZ

LMT: total number of equations which have nonlinear terms (at the input)

LEVIT: = 2; no quadratic terms at input
= 3; no cubic terms at input
= 5; no 5th order terms at input
= 1; other cases

Remark: 3 and 5 applies to "odd" system (only odd powers)

NJI: number of equations to be printed at the output

NNLM: upper limit (estimated) of the number of nonlinear terms at input

INTEGRAL VECTORS

ITM (LMT): order number of equations which have nonlinear terms (at input)

IPRT (NJI): order number of equations to be printed at the output

COMPLEX VECTORS

LBD (LZ): vector of critical eigenvalues

COMPLEX MATRICES

U (LZ,NZ): critical right–eigenvectors

V (LZ, NZ): critical left–eigenvectors

A (NZ,NZ): linear part matrix

COEFFICIENTS OF NONLINEAR TERMS

N, JI, Z (VI(J), J=1,5)

N: módulus ($|\nu|$) of the exponent vector VI

JI: order number of corresponding equation

Z: (complex) coefficient of the nonlinear term

VI: exponent vector in δ–notation

A blank line (or card) should follow last nonlinear term in order to stop the reading.

δ–notation of vector $\nu = (\nu_1, \nu_2 \ldots, \nu_n)$: $(\nu)_\delta$

Let $|\nu| = M$. Then

$\nu = \delta_{s_1} + \delta_{s_2} \ldots + \delta_{s_M}$

where: $\delta_1 = (1,0,0,.,0)$
$\delta_2 = (0,1,0,.,0)$
$\delta_n = (0,\ldots,0,1)$

for some $s_1, s_2, \ldots, s_M \in \{1,2,\ldots,n\}$ and $s_1 \leq s_2 \ldots \leq s_M$.

$(\nu)_\delta = (s_1, s_2, \ldots s_M)$

Ex.: $\nu = (1,2,1) \longleftrightarrow (\nu)_\delta = (1,2,2,3)$ $(|\nu|=M=4)$
$\nu = (1,0,2,1,0)$ $(\nu)_\delta = (1,3,3,4)$ $(|\nu|=M=4)$

The input–data are entered according to the "READ" instructions in the MAIN PROGRAM.

DIMENSION PARAMETERS

(These data are specified in the MAIN PORGRAM by 3 cards after "REAL R5 ( )")

.LIMA: any integer $\geq$ NZ

.LIMF: upper bound for number of nonlinear terms (ϕ_i^{ν}) in each equa– tions. Estimate:

$$\text{LIMF} \geq \sum_{I=2}^{2(\text{ITER})-1} \frac{(NZ+I-1)!}{(NZ-1)!I!}$$

.LIMB: upper bound for number of coefficients B_i^{μ} in each equation of Normal Transformation. Estimate:

$$\text{LIMB} \geq \sum_{I=2}^{\text{ITER}} \frac{(NZ+I-1)!}{(NZ-1)!I!}$$

Other Dimension data to be introduced in MAIN PROGRAM

OR(5,NZ), MM(LIMB); VI(5), ITM(LMT), IPRT(NJI), C1(NZ,NZ), C2(NZ,NZ), C3(NZ,NZ), C4(NZ,NZ), C5(NZ), A(NZ,NZ), LBD(NZ),FV1(LIMF,NZ), FI1(LIMF,NZ), B1(LIMB, NZ), A22(NZ,NZ), Z, V(NZ,NZ), U(NZ, NZ)

NOTE: Actual listed Program is good for Systems of dimension 13. Insert new numerical values in the "Variable type" cards of MAIN PROGRAM if other dimension is desired.

EXTERNAL SUBROUTINES

Form IMSL (International Mathematical and Statistical Library): "SUBROUTINE LEQT 1C".

```
@XQT GAZI1.PCRIT
*** THE ENTERED INDATA ARE AS FOLLOWS***
 FST=    .1000000+001  DF1=    .0000000       DFM=    .0000000
 FI1=    .0000000      FI3=    .0000000       C1  =   .1000000+001

 C2   =    .1000000+001  B1   =    .1000000+000   B2 =    .1000000+000
 EKS  =    .0000000      BK1  =    .0000000

 BK2 =    .0000000     BH1    =    .0000000       BH2 =    .0000000
 TAU=    .0000000      BETA   =    .0000000       AI1  =   .0000000

 AI2   =    .0000000       AM1   =    .2000000+001   AM2 =    .1000000+001
 GAMA  =    .0000000      ALFA  =   .1000000+001   AMU=    .0000000

 ALFA1=    .0000000       ALFA2=    .0000000        AI3   =    .0000000

 FFM>>>>    .1469286+001 FFL>>>>     .1469286+001



 CRITICAL DYNAMIC LOADING



 ANU>>>     .1469286+001



 CRITICAL FREQUENCY


 FRQ1>>>>    .0000000       .5345225+000FRQ2>>>>    .0000000       -.5345225+000

                                    1                         2
      1  .10000+001+I   .00000       .00000    +I   .53452+000
      2  .10000+001+I   .00000       .00000    +I  -.53452+000
      3  .00000      +I   .00000       .00000    +I   .00000
      4  .00000      +I   .00000       .00000    +I   .00000
                                    1                         2
      1  .10000+001+I   .00000       .10000+000+I  -.18708+001
      2  .10000+001+I   .00000       .10000+000+I   .18708+001
```

```
3  .00000    +I  .00000      .00000    +I  .00000
4  .00000    +I  .00000      .00000    +I  .00000
                          1                          2                          3                          4
1  .00000    +I  .00000      .10000+001+I  .00000      .00000    +I  .00000      .00000    +I  .00000
2 -.15000+001+I  .00000     -.15000+000+I  .00000      .10000+001+I  .00000      .10000+000+I  .00000
3  .00000    +I  .00000      .00000    +I  .00000      .00000    +I  .00000      .10000+001+I  .00000
4  .25000+001+I  .00000      .25000+000+I  .00000     -.20000+001+I  .00000     -.20000+000+I  .00000


 .00000      +I   .00000     ====FI,  4  ,   1 2


 .14950+001  +I   .74120+001 ====FI,  2  ,   1 2 2
```