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IDENTIFICATION OF DYNAMICS OF A ONE DIMENSIONAL HEAT DIFFUSION PROCESS.

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IDENTIFICATION OF DYNAMICS OF A ONE DIMENSIONAL HEAT DIFFUSION PROCESS.

B. Leden

ABSTRACT.

Parametric linear models of a one dimensional heat diffusion process are determined using the maximum likelihood method. A theoretical model of the process is given by the infinite partial fraction expansion of the transfer function

$$G(x,s) = \sum_{k=1}^{\infty} \frac{K_k(x)}{1 + T_k s} \qquad (*)$$

relating the temperature at a point x on the rod to the end point temperature. The models obtained from the identification are compared with this theoretical model. The slowest mode of the model (*) is found in all estimated models. The representation of the other modes of the model (*) is closely related to the sign of the gain factors $K_k(x)$. The study shows that adjacent modes of the model (*) which have gain factors of the same sign are represented by a single mode in the estimated model. The sum of the gain factors of the adjacent modes is the gain factor of the single mode. The modes of the model (*) which are close to the frequency $-\pi/T$ are in some cases represented by a pair of complex modes. The Nyquist frequency is π/T. The estimated models are of 4:th and 5:th order. Models are also estimated from simulated samples of a diffusion process.

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1. INTRODUCTION.

Parametric linear models of a one dimensional heat diffusion process are estimated by the maximum likelihood procedure. The process which consists of a long homogeneous copper rod is described by the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{a^2}\frac{\partial u}{\partial t} = 0 \qquad (1.1)$$

where a^2 is the thermal diffusivity. The system input variables are the end temperatures of the rod. The output variables are the temperatures in 7 equally spaced points on the rod. The partial fraction expansion of the transfer function relating temperature at a point x on the rod to the left end point temperature is given by

$$G(x,s) = \sum_{k=1}^{\infty} \frac{K_k(x)}{1 + T_k s} \qquad (1.2)$$

The right end point of the rod is kept at zero temperature. The models (1.1) or (1.2) have an infinite number of negative real modes. The representation of the infinite number of modes in the estimated finite dimensional discrete models is studied extensively. The slowest mode of the model (1.2) is found in all estimated models. The representation of the other modes of the model (1.2) is closely related to the sign of the gain factors $K_k(x)$. The study shows that adjacent modes of the model (1.2) which have gain factors of the same sign are represented by a single mode in the estimated model. The sum of the gain factors of the adjacent modes is the gain factor of the single mode. The phenomenon explains the relative low order of the estimated models. The modes of the model (1.2) which are close to the frequency $-\pi/T$ are in some cases represented by a pair of complex modes in the estimated models. The Nyquist frequency is π/T, where T is the sampling period. The estimated models do not contain modes $s = a + ib$ outside the closed region $\sqrt{a^2 + b^2} \leq \pi/T$, $a \leq 0$ in the complex s-plane.

Models are estimated from 2 series. The sampling period of the series are 2 and 10 sec. A statistical F-test indicates that the appropriate orders of the models are 4 and 5. The higher order is obtained from the series with 2 sec. sampling period. The small values of the standard deviations of the coefficients of the characteristic equation of the discrete models and the distinct drops in the test quantities $F_{4n,4(n-1)}$ should be noticed. The prediction errors of the one-step ahead predictor for the models are extremely small. The errors are $0.0003^{\circ}C$ - $0.0008^{\circ}C$. The short time drifts of the sensors recording the temperature of the rod are $0.0002^{\circ}C$. The output swings are $0.4^{\circ}C$ - $-1.6^{\circ}C$. The slowest time constant of the process is 177 sec. Models are also estimated from simulated samples of a diffusion process. The models of the simulated and the experimental process agree well.

The diffusion process, the measurements and the analysed series are described in Section 2. In Section 3 the model to be identified and comments on the maximum likelihood method are found. The results of identification on the series are given in Sections 4 and 5. The consistency of the estimated models is discussed in Section 6. The results of identification on the simulated samples are found in Section 7.

2. DIFFUSION PROCESS AND THE MEASUREMENTS.

A one dimensional heat diffusion process is investigated. The process is described in [5], [6]. A simplified block diagram of the process is found in Fig. 2.1.

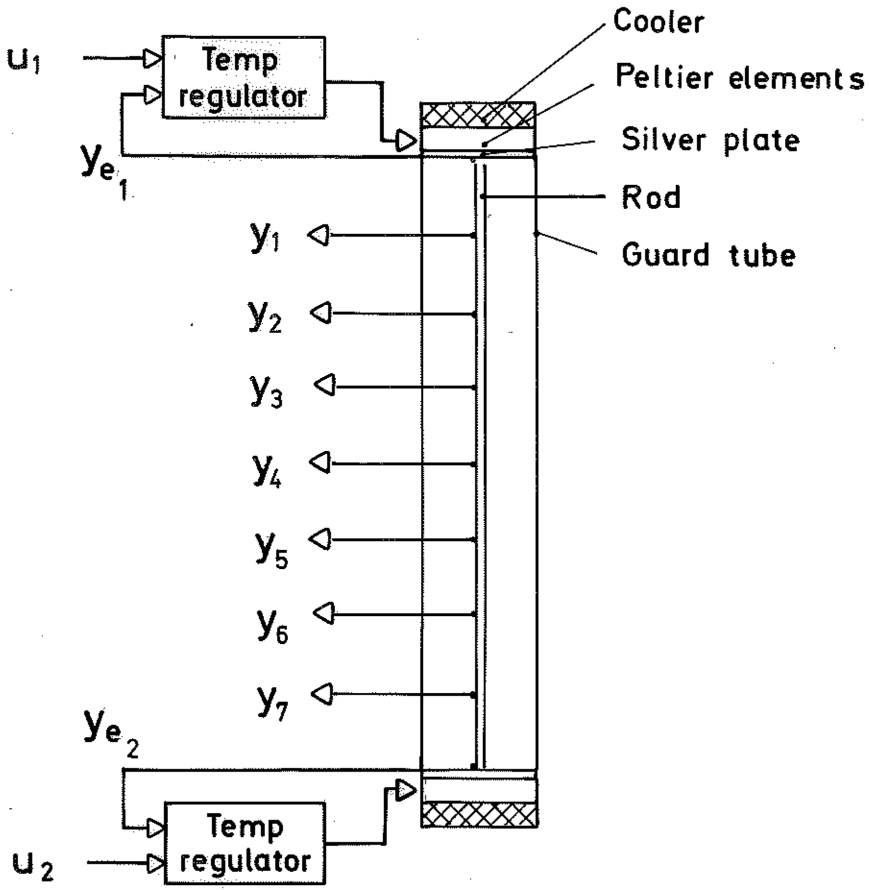

Fig. 2.1 - A simplified block diagram of the investigated process.

The diffusion process consists of a long homogeneous copper rod of constant cross section. The rod is enclosed in a copper tube. The ends of the tube and the rod are thermally connected by a silver plate. The tube yields an almost unidirectional flow of heat in the rod. The Peltier elements supply the power necessary to control the end temperature of the rod.

The cooler serves as a reference temperature to the elements. The tube is surrounded by a conventional heat insulation.

The system input variables are the end surface temperatures controlled by the inputs u_1 and u_2. The output variables are the temperatures in 7 equally spaced points on the rod. The temperatures are converted to outputs $y_1, y_2, \ldots, y_7$. The process operates in the temperature range $20^\circ C$ - $30^\circ C$. The relation between the temperature T [$^\circ$C] and the input or output voltage u[V] is

$$u = 25 + T \tag{2.1}$$

The end surface temperature may be recorded. The signal is denoted u_{servo}.

The measurement of the input and output variables of the system has been performed with a data logger. A start command connects the input signals of the logger through a multiplexer to the voltmeter of the logger at each sampling event. Aitken's scheme for Lagrange interpolation is employed in order to synchronize the readings of the different channels within the same sampling event. The time displacement between the readings of 2 consecutive channels is 0.180 sec. The interpolation is carried out so that data are related to the input u_1. The long term accuracy of the logger is 0.01% of full scale and 0.02% of reading.

We analyse 2 series denoted S1 and S2. The input signals u_1 of both series are a pseudo random binary signal of maximum period, PRBS. The second input u_2 is kept constant at $25^\circ C$. The input temperature swing is approximately $1.8^\circ C$. The temperature profile of the rod is stationary but nonzero when the input is applied to the process. A perfect synchronism is achieved between the start commands of the logger and the clock pulses of the PRBS sequence. The sampling period of series S1 is 10 sec. The series contain 862 data points. The minimum pulse length of the PRBS sequence is 60 sec. and the length of the shift register generating the sequence 12. The signals y_1, y_2, ..., y_7, u_{servo_1}, u_{servo_2}, u_1 are recorded in the order mentioned.

The sampling period of series S2 is 2 sec. The series contain 1828 data points. The minimum pulse length of the PRBS sequence is 20 sec. and the length of register generating the sequence 10. The slowest time constant of process is 177 sec. The number of data points are limited by the length of paper tape. During the 2 experiments a 5°C rise in the room temperature occured. By convection the room temperature influences the temperature profile of the rod. A rise in the profile of approximately 0.01°C should be expected according to [6]. The choice of the sampling rate for the considered process has been discussed briefly in [4].

3. MODEL TO BE IDENTIFIED AND COMMENTS ON THE MAXIMUM LIKELIHOOD METHOD.

We consider the problem of determining an appropriate model of a process from which input-output samples are available. The process is assumed to be linear of n:th order and subjected to disturbances that are stationary normal random processes with rational power spectra. Thus we choose the model

$$A^*(q^{-1})y(t) = B^*(q^{-1})u(t) + \lambda C^*(q^{-1})e(t) \qquad (3.1)$$

where $\{u(t), y(t), t = 1, 2, \ldots, N\}$ is the input-output sequence and $\{e(t), t = 1, 2, \ldots, N\}$ is a sequence of independent normal (0.1) random variables. The shift operator is denoted q and $A^*(q^{-1})$, $B^*(q^{-1})$, $C^*(q^{-1})$ are polynomials

$$A^*(q^{-1}) = 1 + a_1q^{-1}, \ldots, a_nq^{-n}$$

$$B^*(q^{-1}) = 1 + b_1q^{-1}, \ldots, b_nq^{-n}$$

$$C^*(q^{-1}) = 1 + c_1q^{-1}, \ldots, c_nq^{-n} \qquad (3.2)$$

The parameter estimate is chosen so that the likelihood function is maximized according to [1]. The likelihood function is a function of θ and λ where θ is a vector whose components are $a_1, \ldots, a_n, b_1, \ldots, b_n, c_1, \ldots, c_n$ and n initial conditions of eq. (3.4). The parameter vector θ is determined so that the loss function

$$V(\theta) = \frac{1}{2}\sum_{t=1}^{N} \varepsilon(t)^2 \qquad (3.3)$$

is minimized. The sequence $\{\varepsilon(t)\}$ is given by eq. (3.4)

$$C^*(q^{-1})\varepsilon(t) = A^*(q^{-1})y(t) - B^*(q^{-1})u(t) \tag{3.4}$$

and the observed input-output sequence $\{u(t), y(t)\}$. The parameter λ is determined from

$$\lambda^2 = \frac{2}{N} \min_{\theta} V(\theta) \tag{3.5}$$

The maximum likelihood estimate is consistent, efficient and asymptotically normal $(\theta_0, \lambda^2 V^{-1}_{\theta\theta\theta_0})$ under mild conditions given in [1]. The vector θ_0 stands for the correct value of θ.

In order to determine the order of the system we fit models of different orders to the input-output sequence and analyse the reduction of the loss function. To test if the loss function is significantly reduced when the number of parameters is increased from n_1 to n_2 we use the test quantity

$$F_{n_1,n_2} = \frac{V_{n_1} - V_{n_2}}{V_{n_2}} \frac{N - n_2}{n_2 - n_1} \tag{3.6}$$

which has an $F(n_2-n_1, N-n_2)$ distribution under the null hypothesis

$$H_0:\ a_{n_1+1} = a_{n_1+2} \cdots a_{n_2} = b_{n_1+1} = b_{n_1+2} \cdots b_{n_2} =$$

$$= c_{n_1+1} = c_{n_1+2} \cdots c_{n_2} = d_{n_1+1} =$$

$$= d_{n_1+2} \cdots d_{n_2} = 0$$

Most often the test is used with $n_2 - n_1 = 4$, i.e. we test a model of order n+1 against a model of order n. No parameters are then omitted in the models considered. Provided that the test quantity is greater than 2.4 we conclude at a risk level of 5% that the order of the model is at least n+1. Fortran programs for the identification procedure have been available. The programs which can handle the multiple input, single output case are described in [3].

4. RESULTS FROM IDENTIFICATION BY THE MAXIMUM LIKELIHOOD METHOD ON THE SERIES WITH 10 SECONDS SAMPLING PERIOD.

In this section typical results from the identification on the series S1 are presented. The ideal input is considered as input variable. A study shows that the interesting features of the dynamics of the process may be exposed by showing the results for $x = \ell/4, \ell/2, 3\ell/4$. The results for the points $x = \ell/8, 3\ell/8, 5\ell/8, 7\ell/8$ are therefore omitted.

First we give an example showing the results of identification for increasing order of the model. The results are found in Tables A.1, A.2, A.3. The small values of the estimated standard deviations of the a-parameters should be observed. The relative errors of the b-parameters, i.e. the ratios between the standard deviations and the absolute value of the b-parameters, decrease with x. The standard deviations of the c-parameters are almost equal for model of identical order. The large values of the d-parameters are due to the nonzero steady-state temperature profile of the rod. The standard deviations λ of the residuals are extremely small. For the 4:th order models the standard deviations are $0.0003^\circ C$ - $0.0008^\circ C$. The residuals are the errors of the one-step ahead predictor. The static gain K given by

$$K = \frac{\sum_{i=1}^{n} b_i}{1 + \sum_{i=1}^{n} a_i} \qquad (4.1)$$

is estimated accurately in the 3 points. The correct values are 0.2500, 0.5000, 0.7500.

The models in Tables A.1, A.2, A.3 contain redundant parameters at a risk level of 5%†. The roots of the characteristic equations and the poles of the continuous processes are shown in Table A.4. Notice the complex roots and the roots of the 5:th order models on the negative real axis. A sampled system with roots on the negative real axis has no corresponding continuous system.

Below we show the values of the F-test quantities $F_{4n,4(n-1)}$ when testing the reduction of the loss function for a model of order n compared to a model of order n-1.

$x = \ell/4$	$x = \ell/2$	$x = 3\ell/4$	
$F_{12,8} = 875$	$F_{12,8} = 2398$	$F_{12,8} = 476$	
$F_{16,12} = 758$	$F_{16,12} = 430$	$F_{16,12} = 266$	
$F_{20,16} = 4.7$	$F_{20,16} = 6.4$	$F_{20,16} = 1.4$	(4.2)

At a risk level of 5% we conclude that the models for $x = \ell/4$, $\ell/2$ are at least of 5:th order. Since the test quantities $F_{4n,4(n-1)}$, for n = 5 drop drastically we, however, choose the 4:th order models. Notice that the 5:th order models have no continuous correspondence. Thus the orders of the estimated models series S1 are chosen 4.

In order to account for constant levels present in the outputs the parameters of the extended model (4.3) are estimated for the 4:th order models. The extended model is

† This is approximately the 2σ limit provided that the residuals are independent and normally distributed. The standard deviation of the coefficients is denoted σ.

$$A^{*}(q^{-1})y(t) = B_1^{*}(q^{-1})u_1(t) + B_2^{*}(q^{-1})u_2(t) + \\ + \lambda C^{*}(q^{-1})e(t) \,^{\dagger} \tag{4.3}$$

where the parameters $b_{22}, b_{23}, \ldots, b_{2n}$ are zero. The second input is a step of magnitude, $1^{o}C$. The calculation of the residuals $\varepsilon(t)$ is modified. The residuals are limited to 3λ, i.e.

$$\varepsilon(t) = 3\lambda \cdot \text{sign}(\varepsilon(t)), \quad |\varepsilon(t)| \geq 3\lambda \tag{4.4}$$

The function sign denotes the signum function. The residuals are calculated from the observed input-output sequence and eq. (3.4). The modification (4.4) affects the loss function V, the gradient with respect to the parameters V_{θ} and the matrix of second order partial derivatives $V_{\theta\theta}$. Further the redundant parameters b_{11} are put equal to zero in the models for $x = \ell/2$, $3\ell/4$.

The results of identification are shown in Table 4.1. A significant reduction of the loss functions occur when we use the extended model (4.3). The modified calculation of the residuals (4.4) reduces the loss function for the point $x = 3\ell/4$ significantly. The constant levels in the outputs partly originate from the zero adjustment errors of the transducers partly from the fact that data are transformed to have zero mean values. The zero adjustment errors k_{index} defined by

† The model $A^{*}(q^{-1})y(t) + k = B^{*}(q^{-1})u(t) + \lambda C^{*}(q^{-1})e(t)$ should be preferred as the second input also involves a transient part.

	$x = \ell/4$	$x = \ell/2$	$x = 3\ell/4$
a_1	-1.9634 ± 0.0010	-2.0018 ± 0.0007	-2.9563 ± 0.0017
a_2	1.2222 ± 0.0020	1.3338 ± 0.0015	3.2694 ± 0.0049
a_3	-0.2829 ± 0.0015	-0.3464 ± 0.0014	-1.6134 ± 0.0047
a_4	0.0359 ± 0.0006	0.0284 ± 0.0006	0.3025 ± 0.0015
b_{11}	0.0107 ± 0.0001	0.0000	0.0000
b_{12}	0.0527 ± 0.0001	$0.5152\cdot10^{-3}\pm0.0221\cdot10^{-3}$	$0.1297\cdot10^{-3}\pm0.0124\cdot10^{-3}$
b_{13}	-0.0462 ± 0.0001	$3.8709\cdot10^{-3}\pm0.0411\cdot10^{-3}$	$-0.4166\cdot10^{-3}\pm0.0252\cdot10^{-3}$
b_{14}	-0.0080 ± 0.0001	$2.4768\cdot10^{-3}\pm0.0266\cdot10^{-3}$	$0.7942\cdot10^{-3}\pm0.0138\cdot10^{-3}$
b_{21}	$0.0673\cdot10^{-3}\pm0.0043\cdot10^{-3}$	$0.1117\cdot10^{-3}\pm0.0047\cdot10^{-3}$	$0.0115\cdot10^{-3}\pm0.0004\cdot10^{-3}$
b_{22}	0.0000	0.0000	0.0000
b_{23}	0.0000	0.0000	0.0000
b_{24}	0.0000	0.0000	0.0000
c_1	-0.5145 ± 0.0347	-1.0345 ± 0.0345	-2.4919 ± 0.0365
c_2	-0.6301 ± 0.0418	0.5208 ± 0.0517	2.3331 ± 0.0940
c_3	0.0921 ± 0.0331	-0.1885 ± 0.0509	-0.9918 ± 0.0917
c_4	0.2053 ± 0.0352	0.1410 ± 0.0317	0.1804 ± 0.0344

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	$x = \ell/4$	$x = \ell/2$	$x = 3\ell/4$
d_1	-0.6919	-0.4603	-0.2266
d_2	0.6768	0.4611	0.4428
d_3	-0.1217	-0.1523	-0.2960
d_4	0.0304	0.0105	0.0669
λ	$0.8286 \cdot 10^{-3}$	$0.3170 \cdot 10^{-3}$	$0.4279 \cdot 10^{-3}$
V	$0.29594 \cdot 10^{-3}$	$0.43313 \cdot 10^{-4}$	$0.78926 \cdot 10^{-4}$
K	0.7408	0.4920	0.2440

Table 4.1 - The final results from maximum likelihood identification on series S1, $x = \ell/4,\ \ell/2,\ 3\ell/4$. The extended model $A^*(q^{-1})y(t) = B_1^*(q^{-1})u_1(t) + B_2^*(q^{-1})u_2(t) + \lambda C^*(q^{-1})e(t)$ is employed where the second input is a step of magnitude, $1^\circ C$. The model is used to account for a constant level in the output. The residuals are limited to 3λ, i.e. $\varepsilon(t) = 3\lambda \text{ sign} \cdot \{\varepsilon(t)\},\ |\varepsilon(t)| \geq 3\lambda$. Further the redundant parameters b_{11} are put equal to zero in the models for $x = \ell/2,\ 3\ell/4$.

$$y(t) + k_{index} = \frac{B_1^*(q^{-1})}{A^*(q^{-1})} u_1(t) \qquad (4.5)$$

may be estimated from the mean values of the input-output sequences and the estimated models. A short calculation shows that

$$k_{index} = \frac{B_1^*(1)}{A^*(1)} \bar{u}_1 - \frac{B_2^*(1)}{A^*(1)} u_2 - \bar{y} \qquad (4.6)$$

where $\bar{u}_1$ and $\bar{y}$ denote the mean values of the input and output sequence respectively. The definition (4.5) yields that the transducer indicates too low temperature provided that k_{index} is positive. The zero adjustment errors of the transducers are

$$k_{\ell/4} = 0.0132^{\circ}C$$

$$k_{\ell/2} = 0.0060^{\circ}C$$

$$k_{3\ell/4} = 0.0055^{\circ}C \qquad (4.7)$$

The models in Table 4.1 are the final results from maximum likelihood identification on series S1. The models are denoted S1/ℓ/4, S1/ℓ/2, S1/3ℓ/4. The roots of the A, B and C polynomials and the poles of the continuous processes are shown in Table 4.2 for the models.

The assumption of independence and normality made on the sequence $\{\varepsilon(t)\}$ should be tested for the models S1/ℓ/4, S1/ℓ/2, S1/3ℓ/4. The sample covariance functions $r(\tau)$ given by

$$r(\tau) = \frac{1}{N - \tau} \sum_{t=1}^{N-\tau} \varepsilon(t)\varepsilon(t+\tau), \quad \tau \geq 0 \qquad (4.8)$$

	$x = \ell/4$	$x = \ell/2$	$x = 3\ell/4$
A	$0.1408 \pm i0.1778$	0.1523	$0.6017 \pm i0.1843$
	0.7411	0.3642	0.8097
	0.9406	0.5417	0.9433
		0.9435	
A'	$-0.1484 \pm i0.0901$	-0.1882	$-0.4631 \cdot 10^{-1} \pm i0.2973 \cdot 10^{-1}$
	$-0.2996 \cdot 10^{-1}$	-0.1010	$-0.2111 \cdot 10^{-1}$
	$-0.6119 \cdot 10^{-2}$	$-0.6130 \cdot 10^{-1}$	$-0.5842 \cdot 10^{-2}$
		$-0.5814 \cdot 10^{-2}$	
B	-5.6697	-6.8075	$-1.6060 \pm i1.8827$
	-0.1308	-0.7062	
	0.8912		
C	$-0.5116 \pm i0.2814$	$-0.1153 \pm i0.4878$	$0.3720 \pm i0.2904$
	$0.7688 \pm i0.1057$	$0.6325 \pm i0.4014$	$0.8740 \pm i0.2149$

Table 4.2 - Roots of the A, B and C polynomials and the poles of the continuous processes for the models $S1/\ell/4$, $S1/\ell/2$, $S1/3\ell/4$. The characteristic polynomial of the continuous process is A'.

are plotted in Fig. A.1. The residuals of the models S1/ℓ/2, S1/3ℓ/4 are almost uncorrelated but the residuals of the model S1/ℓ/4 are strongly correlated. The sample covariance function of the model S1/3ℓ/4 contains spikes appearing for τ = 6, 12, 18. The integer 6 is the ratio between the minimum pulse length if the PRBS sequence and the sampling period. The model S1/ℓ/4 does not describe the physical process perfectly during the first sampling events after that a shift has occured in the input signal.

The residuals are tested for normality using a chi-square goodness-of-fit test. The test quantities χ^2_{index} are given below:

$$\chi^2_{\ell/4} = 239$$

$$\chi^2_{\ell/2} = 203$$

$$\chi^2_{3\ell/4} = 56 \tag{4.9}$$

The number of degrees of freedom is 33. Provided that χ^2_{index} is less than 48 the hypothesis that the residuals are normally distributed is accepted at a risk level of 5%. In Fig. A.2 the cummulative frequencies of the residuals are plotted. The verticale scale of the diagram is chosen so that a perfectly normally distributed variable yields a straight line. The assumptions made on the residuals are fulfilled by the model S1/3ℓ/4 at a risk level of 0.5%.

In Fig. A.3, A.4, A.5 we show

1. the input signal $u_1(t)$,
2. the output signals $y_i(t)$,
3. the model outputs $y_{m_i}(t)$ defined by

$$y_{m_i}(t) = \frac{B_1^*(q^{-1})}{A^*(q^{-1})} u_1(t) \tag{4.10}$$

The deterministic model is $B_1^*(q^{-1})/A^*(q^{-1})$.

4. The model errors $e_{m_i}(t)$ defined by

$$e_{m_i}(t) = y_i(t) - y_{m_i}(t) \tag{4.11}$$

5. The residuals $\varepsilon_i(t)$. The model errors e_{m_i} and the residuals ε_i are related according to

$$e_{m_i}(t) = \frac{C^*(q^{-1})}{A^*(q^{-1})} \varepsilon_i(t) \tag{4.12}$$

for the models S1/ℓ/4, S1/ℓ/2, S1/3ℓ/4. The model errors contain small negative parts in the beginning of the experiments. The negative parts may be caused by some nonlinear effect originating from the nonzero steady-state profile of the rod. The 2 spikes appearing in the residuals for the model S1/3ℓ/4 are caused by transients in the AC-mains. The calculation of the residuals is modified according to eq. (4.4) in order to limit the effect of the spikes in the maximum likelihood algorithm. The spikes in the model errors are $0.003^{\circ}C$.

5. RESULTS OF IDENTIFICATION BY THE MAXIMUM LIKELIHOOD METHOD ON THE SERIES WITH 2 SECONDS SAMPLING PERIOD.

In this section results from identification on series S2 are presented. The ideal input is considered as the input variable. The series contain 2 outputs, viz. the outputs for $x = \ell/8$, $\ell/4$. The results are only presented for $x = \ell/4$. Table A.5 shows the results from maximum likelihood identification for increasing order of the model. The estimated standard deviations of the a-parameters are small. The short sampling period of series S2 yields a large decrease in the b-parameters. Compare Tables A.1, A.5. The large values of the d-parameters are due to the nonzero steady-state temperature profile of the rod. The parameters λ are extremely small. The models contain redundant parameters at a risk level of 5%. The roots of the characteristic equations and the poles of the continuous processes are shown in Table A.6 for the models.

Below we show the values of the F-test quantities $F_{4n,4(n-1)}$ when testing the reduction of the loss function for a model of order n compared to a model of order n-1

$$F_{16,12} = 412$$

$$F_{20,16} = 259$$

$$F_{24,20} = 0.5 \qquad (5.1)$$

The appropriate order of the model is 5. This model exhibits several interesting features not found for models of lower order. The model contains the modes $-0.62\cdot10^{-2}$, $-0.31\cdot10^{-1}$, i.e. the slowest modes of the model S1/ℓ/4. Further the static gain of this model is close to the theoretical value, 0.7500. The short minimum pulse length and the short sampling period of series S2 complicate the estimation of the low frequency dynamics of the process.

The parameters of the extended model (4.3) are estimated by the maximum likelihood procedure for the 5:th order model. The second input is a step of magnitude, $1^{o}C$. The calculation of the residuals is modified according to eq. (4.4). Further the redundant parameters b_{11}, b_{12}, c_5 are put equal to zero. The result of identification is shown in Table 5.1.

a_1	-3.5157 ± 0.0002	c_1	-2.4107 ± 0.0229
a_2	4.8648 ± 0.0004	c_2	2.2588 ± 0.0555
a_3	-3.3415 ± 0.0005	c_3	-1.0831 ± 0.0549
a_4	1.1599 ± 0.0005	c_4	0.2362 ± 0.0225
a_5	-0.1673 ± 0.0002	c_5	0.0000
b_{11}	0.0000	d_1	-0.7192
b_{12}	0.0000	d_2	1.8093
b_{13}	$0.5337\cdot10^{-3}\pm0.0144\cdot10^{-3}$	d_3	-1.6896
b_{14}	$1.4695\cdot10^{-3}\pm0.0299\cdot10^{-3}$	d_4	0.7147
b_{15}	$-1.9457\cdot10^{-3}\pm0.0158\cdot10^{-3}$	d_5	-0.1186
b_{21}	$1.9479\cdot10^{-6}\pm0.0142\cdot10^{-6}$	λ	$0.4532\cdot10^{-3}$
b_{22}	0.0000	V	$0.18773\cdot10^{-3}$
b_{23}	0.0000	K	0.7448
b_{24}	0.0000		
b_{25}	0.0000		

Table 5.1 - The final result from maximum likelihood identification on series S2, $x = \ell/4$. The extended model $A^*(q^{-1})y(t) = B_1^*(q^{-1})u_1(t) + B_2^*(q^{-1})u_2(t) + \lambda C^*(q^{-1})e(t)$ is employed where the second input is a step of magnitude, $1^{o}C$. The model is used to account for a constant level in the output. The residu-

als are limited to 3λ, i.e. $\varepsilon(t) = 3\lambda \, \mathrm{sign}\{\varepsilon(t)\}$, $|\varepsilon(t)| \geq 3\lambda$. Further the redundant parameters b_{11}, b_{12}, c_5 are put equal to zero.

A significant reduction of the loss function V occurs when we use the extended model. The modified calculation of the residuals also reduces the loss function significantly. The zero adjustment error of the transducer $k_{\ell/4}$ may be estimated from eq. (4.6). We obtain

$$k_{\ell/4} = 0.0036 \tag{5.2}$$

Eq. (4.7), (5.2) show that the transducer recording the temperature at $x = \ell/4$ indicates 0.0096°C higher temperature during experiment S2 than during experiment S1. The increase in the zero adjustment error is caused by the 5°C rise in the room temperature. The matrix of second order partial derivatives of the model discussed is positive definite but the 5:th order model in Table A.5 has an indefinite matrix of second order partial derivatives. The model presented in Table 5.1 is the final results from the identification on series S2. The model is denoted $S2/\ell/4$. The roots of the A, B and C polynomials and the poles of the continuous process are shown in Table 5.2.

The assumptions of independence and normality made on the sequence $\{\varepsilon(t)\}$ should be tested. Fig. A.6 shows that the residuals are almost independent. The test quantity $\chi^2_{\ell/4}$ of a chi-square goodness-of-fit test is

$$\chi^2_{\ell/4} = 195 \tag{5.3}$$

The number of degrees of freedom is 41. Provided that

A	0.4339±i0.2488	B	-3.7304
	0.7202		0.9772
	0.9397		
	0.9880		
A'	-0.3464±i0.2603	C	0.3686±i0.4624
	-0.1641		0.6796
	$-0.3111\cdot10^{-1}$		0.9939
	$-0.6023\cdot10^{-2}$		

Table 5.2 - Roots of the A, B and C polynomials and the poles of the continuous process for the model S2/ℓ/4. The characteristic polynomial of the continuous process is A'.

the test quantity $\chi^2_{\ell/4}$ is greater than 57 the hypothesis that the residuals are normally distributed is rejected at a risk level of 5%. In Fig. A.7 the cummulative frequencies of the residuals are plotted. The vertical scale of the diagram is chosen so that a perfectly normally distributed variable yields a straight line. In Fig. A.8 the model output, the model error, the residuals and the input-output sequence of the model S2/ℓ/4 are plotted. The 2 spikes appearing in the residuals are caused by transients in the AC-mains. The calculation of the residuals is modified according to eq. (4.4) in order to limit the effect of the spikes in the maximum likelihood procedure. The spikes in the model error are $0.005^{o}C$.

The input-output samples of series S2 for x = ℓ/4 are believed to be a useful test example when comparing the effectiveness of different identification algorithms. The matrix of second order partial derivatives for model S2/ℓ/4 is extremely ill-conditioned, i.e. the ratio between the largest and smallest eigenvalues of the matrix is $0.28\cdot10^{12}$.

6. CONSISTENCY OF THE RESULTS OBTAINED BY THE MAXIMUM LIKELIHOOD METHOD.

In this section the discrete models estimated by the maximum likelihood procedure are compared with the theoretical models of the process. The continuous partial fraction expansion of the models are used for the comparison.

According to [6] a theoretical model of the process is given by the transfer function

$$G(x,s) = \sum_{k=1}^{\infty} \frac{K_k(x)}{1 + T_k s} \tag{6.1}$$

where

$$K_k(x) = \frac{2(-1)^{k+1} \sin \frac{\ell - x}{\ell} \pi k}{\pi k}$$

$$T_k = \tau/(k^2\pi^2), \quad \tau = \ell^2/a^2 \tag{6.2}$$

The length and the thermal diffusivity of the rod are ℓ and a^2 respectively. The gain factors $K_k(x)$ are inversely proportional to k and thus decrease slowly with k. The model (6.1) contains an infinite number of negative real poles

$$s_k = - k^2 s_0 : \quad k \neq k_i, \quad s_0 = 0.56537 \text{ rad/sec}^{\dagger} \tag{6.3}$$

† The employed value of the constant τ is 1746. The values of the length ℓ and the diffusivity a^2 are 45 cm and 1.16 cm^2/sec. respectively. In [6] it is shown that for this value of the thermal diffusivity an optimal fit is obtained between the measured step response and the theoretical step response of the process.

The integers k_i satisfy according to eq. (6.1), (6.2)

$$\sin\left(\frac{\ell-x}{\ell}\pi k_i\right) \neq 0 \tag{6.4}$$

The choice of the sampling period for parametric identification is considered in [2], [4]. The reports show that the variance of the estimation of a mode $s = a + ib$ increases rapidly with increasing $T \geq \pi/\sqrt{a^2 + b^2}$. The effect is analogous to the aliasing effect in spectral analysis. Hence we expect the modes of the estimated parametric models to be located inside the closed region

$$\sqrt{a^2 + b^2} \leq \pi/T, \ a \leq 0 \tag{6.5}$$

in the complex s-plane. The frequency π/T is the well-known Nyquist frequency. In order to obtain a unique relation between the poles of a continuous and a sampled system we require $|b| \leq \pi/T$. The requirement is consistent with eq. (6.5). The step response of a first order system $b/(s+a)$ where $a = -\pi/T$ reaches 95.7% of its final value after one sampling period.

The quantities $K_k(x)$, T_k and s_k of the expansion (6.1) are calculated and shown in Table 6.1 for k such that $|s_k| \leq \pi/T$, $T = 2$. The points $x = \ell/4$, $\ell/2$, $3\ell/4$ are considered. The factors for $x = 3\ell/4$ are obtained from the factors for $x = \ell/4$ by changing sign of the 4:th gain factor from the 2:nd onwards.

The continuous partial fraction expansion of the model $S1/\ell/2$ is

$$GS1(\ell/2,s) = \frac{0.63337}{1 + 172.00s} - \frac{0.37123}{1 + 16.314s} + \frac{0.28546}{1 + 9.9014s} - \frac{0.55608 \cdot 10^{-1}}{1 + 5.3138s} \tag{6.6}$$

k		Time const. T_k	Poles s_k	Gain factors $K_k(x)$ $x = \ell/4$	$x = \ell/2$	$x = 3\ell/4$
1		176.88	$-0.56537 \cdot 10^{-2}$	0.45016	0.63662	0.45016
2		44.214	$-0.22618 \cdot 10^{-1}$	0.31831	xxxx	-0.31831
3		19.653	$-0.50883 \cdot 10^{-1}$	0.15005	-0.21221	0.15005
4	S1	11.055	$-0.90452 \cdot 10^{-1}$	xxxx	xxxx	xxxx
5		7.0750	-0.14134	$-0.90032 \cdot 10^{-1}$	0.12732	$-0.90032 \cdot 10^{-1}$
6		4.9132	-0.20353	-0.10610	xxxx	0.10610
7		3.6097	-0.27703	$-0.64308 \cdot 10^{-1}$	$-0.90946 \cdot 10^{-1}$	$-0.64308 \cdot 10^{-1}$
8		2.7637	-0.36184	xxxx	xxxx	xxxx
9	S2	2.1837	-0.45795	$0.50018 \cdot 10^{-1}$	$0.70736 \cdot 10^{-1}$	$0.50018 \cdot 10^{-1}$
10		1.7688	-0.56537	$0.63662 \cdot 10^{-1}$	xxxx	$-0.63662 \cdot 10^{-1}$
11		1.4618	-0.68410	$0.40923 \cdot 10^{-1}$	$-0.57875 \cdot 10^{-1}$	$0.40923 \cdot 10^{-1}$
12		1.2283	-0.81413	xxxx	xxxx	xxxx
13		1.0466	-0.95547	$-0.34628 \cdot 10^{-1}$	$0.48971 \cdot 10^{-1}$	$-0.34628 \cdot 10^{-1}$
14		0.90243	-1.1081	$-0.45473 \cdot 10^{-1}$	xxxx	$0.45473 \cdot 10^{-1}$
15		0.78611	-1.2721	$-0.30011 \cdot 10^{-1}$	$-0.42441 \cdot 10^{-1}$	$-0.30011 \cdot 10^{-1}$
16		0.69092	-1.4474	xxxx	xxxx	xxxx
17		0.61203	-1.6339	$0.26480 \cdot 10^{-1}$	$0.37448 \cdot 10^{-1}$	$0.26480 \cdot 10^{-1}$

Table 6.1 - Text, see next page.

Table 6.1 (see page 24) - The quantities $K_k(x)$, T_k and s_k of the theoretical partial fraction expansion of the diffusion process for k such that $|s_k| \leq \pi/T$, T = 2. The stars indicate that the gain factor of the corresponding mode is zero. The employed value of the thermal diffusivity of copper is 1.16 cm^2/sec.

The estimated model has 4 real modes. The modes of the theoretical and the estimated models agree well. The gain factors of the model (6.1) form an alternating series in the considered point. The model S1/3ℓ/4 has the continuous expansion

$$GS1(3\ell/4,s) = \frac{0.45268}{1 + 171.18s} - \frac{0.27613}{1 + 47.380s} + \frac{1.03115s + 0.67392 \cdot 10^{-1}}{1 + 30.584s + 330.18s^2} \qquad (6.7)$$

The 2 slowest modes of the theoretical and the estimated models agree well. The estimated model also contains a second order term with complex modes. The damping factor of the modes is 0.84. The gain factors of the model (6.1) form an alternating series in the considered point. Model S1/ℓ/4 may be fractioned as

$$GS1(\ell/4,s) = \frac{0.47022}{1 + 163.40s} + \frac{0.44255}{1 + 33.373s} - \frac{0.57793s + 0.17208}{1 + 9.8491s + 33.195s^2} \qquad (6.8)$$

The 2:nd, 3:rd modes of the model (6.1) are represented by a single mode in the estimated model. The sum of the gain factors of the 2 theoretical modes is the gain factor of the single mode. The estimated model also contains a second order term with complex modes. The damping factor of the modes is 0.87. Notice that the 2:nd and 3:rd modes of the model (6.1) have gain factors of the same sign. The model S2/ℓ/4 has the continuous fraction expansion

$$GS2(\ell/4,s) = \frac{0.44998}{1 + 163.81s} + \frac{0.46523}{1 + 32.251s} - \frac{0.25794}{1 + 6.0921s} + \frac{0.12884s + 0.66419 \cdot 10^{-1}}{1 + 3.6896s + 5.3260s^2} \qquad (6.9)$$

The 2:nd, 3:rd and the 5:th, 6:th, 7:th modes of the model (6.1) are represented by the 2:nd and 3:rd modes respectively in the estimated model. The estimated model also contains a second order term with complex modes. The damping factor of the modes is 0.80. Notice that the 2:nd, 3:rd and the 5:th, 6:th, 7:th modes of the model (6.1) have gain factors of the same sign.

In a case where the gain factors $K_k(x)$ form an alternating series the slowest modes of the model (6.1) are present in the estimated model. The modes s_k close to the frequence $-\pi/T$ are represented by either real modes or a pair of complex modes. The damping factor of the complex modes is large, i.e. the overshoot of the step response of the corresponding second order term is small. The step responses of the second order terms of the models (6.7), (6.8), (6.9) and the corresponding residual terms of the model (6.1) agree well, e.g. Fig. A.9. We conclude that modes of the model (6.1) which have gain factors of the same sign are represented by a single mode in the estimated model.

The models S1/ℓ/4, S1/3ℓ/4, S2/ℓ/4 contain a pair of complex modes. The statistical relevance of the pairs is investigated by studying the root loci of the estimated A polynomials subjected to random perturbations of 2 standard deviations in the coefficients. Provided that the pairs remain complex we conclude at a risk level of 5% that the pairs have statistical relevance.

Consider the sensitivity of the roots of a polynomial

$$f(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2}, \ldots, a_n \tag{6.10}$$

with respect to a change δa_i in the coefficient a_i. Let the zeros of the polynomial be $z_1, z_2, \ldots, z_n$. Then f(z) may also be written

$$f(z) = \prod_{i=1}^{n} (z-z_i) \tag{6.11}$$

Assume that z_r is an isolated zero. The perturbed polynomial $f_p(z)$ is

$$f_p(z) = z^n + a_i z^{n-1} \ldots (a_i+\delta a_i) z^{n-i} \ldots a_n =$$

$$= f(z) + \delta a_i z^{n-i} = f(z) + \delta a_i g(z) \tag{6.12}$$

where

$$g(z) = z^{n-i} \tag{6.13}$$

A Taylor expansion of eq. (6.12) for $z = z_r$ yields

$$f_p(z_r+\delta z_r) = f(z_r) + \delta z_r f'(z_r) + \delta a_i g(z_r) + O(\delta z_r^2) \tag{6.14}$$

Equating eq. (6.14) to zero and remembering $f(z_r) = 0$ we have

$$\left| \delta z_r - \frac{g(z_r)}{f'(z_r)} \delta a_i \right| \leqslant O(\delta a_i^2) \tag{6.15}$$

Derivation of eq. (6.11) yields

$$f'(z_r) = \prod_{\substack{i=1 \\ i \neq r}}^{n} (z_r - z_i) \tag{6.16}$$

By substitution eq. (6.13), (6.16) into (6.15) we finally get

$$\delta z_r \sim - \frac{z_r^{n-i}}{(z_r - z_1) \ldots (z_r - z_{r-1})(z_r - z_{r+1}) \ldots (z_r - z_n)} \delta a_i,$$

$$\delta a_i \to 0 \tag{6.17}$$

For a stable system $| z_r | \leqslant 1$. The sensitivity of a root z_r with respect to a given perturbation δa_i will thus have its maximum for $i = n$ †.

The root loci of the A polynomials subjected to perturbations of 2 standard deviations in the last coefficient appear in Fig. A.10 for the models S1/3ℓ/4, S2/ℓ/4. The model S1/ℓ/4 is less sensitive to variations in the coefficients of the A polynomial. The

† Notice that in making this assessment we consider the effect of the same error in each coefficient. In a stable system we are more likely to have δa_i decreasing for increasing i.

inequalities (6.18)

$$0.1372 \leq \mathrm{Re}\, s_{3,4} \leq 0.1447$$
$$0.1714 \leq \mathrm{Im}\, s_{3,4} \leq 0.1838$$
$$0.7143 \leq s_2 \leq 0.7738$$
$$0.9099 \leq s_1 \leq 0.9636 \qquad (6.18)$$

show the region where the roots are located when the A polynomial is subjected to a perturbation of 2 standard derivations in one arbitrary coefficient. The root distributions of the A polynomials of the models S1/3ℓ/4, S2/ℓ/4 are more clustered than the root distribution of the model S1/ℓ/4. The study shows that the models S1/ℓ/4, S1/3ℓ/4, S2/ℓ/4 contain a pair of complex poles even if we account for the uncertainty of the coefficients of the A polynomials.

7. RESULTS OF IDENTIFICATION BY THE MAXIMUM LIKELIHOOD METHOD ON SIMULATED SAMPLES OF A DIFFUSION PROCESS.

In this section we present a method to generate input-output samples of a one dimensional heat diffusion process. The method is exact for a piecewise constant input signal, i.e. the samples corresponding to series S1 and S2 may be simulated.

Consider a linear system excited by a unit impulse of length T. Let the response of the system be h(t). The unit impulse is decomposed into a positive step applied at time 0 and a negative step applied at time T. The linearity of the system implies that

$$h(t) = v(t) - v(t-T), \qquad t \geq 0 \tag{7.1}$$

where v(t) is the unit step response of the system. Eq. (7.1) gives

$$v(t) = \sum_{n=0}^{t} h(n), \qquad t \geq 0 \tag{7.2}$$

Let y(t) be the response of the system to a piecewise constant input signal. The input signal is decomposed into a series of positive and negative steps applied at the sampling events, i.e. $t = nT$, $n \geq 0$. The linearity of the system yields

$$y(t) = \sum_{n=0}^{t} h(n)u(t-n), \qquad t \geq 0 \tag{7.3}$$

The unit step response of the system is given by

$$v(x,t) = \frac{\ell - x}{\ell} - \frac{2}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin \frac{\ell - x}{\ell} \pi n}{n} e^{-\frac{\pi^2 n^2}{\tau} t} \qquad (7.4)$$

according to [6]. The input signals of series S1 and S2 are given by eq. (7.5)

$$X_1(t) = c_1X_1(t-1) \oplus c_2X_2(t-1) \oplus c_3X_3(t-1),$$

$$\ldots\ c_mX_m(t-1)$$

$$X_{n+1}(t) = X_n(t-1), \qquad n = 1, 2, \ldots, m-1$$

$$u(t) = A(2X_1(t) - 1) \qquad t = 1, 2, 3, \ldots, N \quad (7.5)$$

The fictive shift register is X_n. The length of the register is m and the feedback logic c_n. The amplitude of the input signal is A. The symbol $\oplus$ denotes addition modulo 2.

The input-output samples corresponding to series S1, $x = \ell/4$, $\ell/2$, $3\ell/4$ and series S2, $x = \ell/4$ are simulated using eq. (7.1), (7.3), (7.4), (7.5). The parameters of the model (3.1) are estimated from the samples. The order of the model is 4 and 5 for series S1 and S2 respectively. The estimate of the parameter vectors θ have an indefinite matric of second order partial derivatives for the simulated samples. This indicates that numerical problems appear in the maximum likelihood algorithm. The problems are passed by adding correlated noise $\varepsilon(t)$ to the simulated outputs. The noise is generated by the system

$$A^*(q^{-1})\varepsilon(t) = \mu C^*(q^{-1})e(t) \qquad (7.6)$$

where $\{e(t)\}$ is a sequence of independent normal (0.1) random variables. The employed A polynomials of eq. (7.6) are the A polynomials of the models S1/ℓ/4, S1/ℓ/2, S1/3ℓ/4, S2/ℓ/4.

The C polynomials are chosen to yield correlated noise $\varepsilon(t)$. The standard deviation μ is 0.0001 and 0.0002 for series S1 and S2 respectively.

The results of identification are found in Table A.7. The deterministic models $B^*(q^{-1})/A^*(q^{-1})$ of the simulated and the experimental process agree well for series S1. The B polynomials of the simulated and the experimental process differ considerably for series S2. The C polynomials and the standard deviations μ of eq. (7.6) are well estimated from the simulated samples except for series S1, $x = \ell/4$, where the C polynomial is poorly estimated. The estimates of the deterministic models $B^*(q^{-1})/A^*(q^{-1})$ do not depend critically on the choice of the A and C polynomial of eq. (7.6).

The roots of the A polynomials and the poles of the continuous processes are found in Table A.8 for the simulated process. The pole configurations of the simulated and the experimental process agree well. Thus imperfections in the experimental process, e.g. imperfect thermal insulation and servos of finite solution time do not affect the pole configuration of the rod.

8. CONCLUSIONS.

The study shows that the pole configurations of the estimated models are closely related to the sign of gain factors of the partial fraction expansion of the theoretical model. The relative low orders of the estimated models are explained by 2 facts:

1. the representation of adjacent modes of the theoretical model which have gain factors of the same sign,
2. the small residual term of the theoretical partial fraction expansion obtained when the theroetical gain factors form an alternating series.

The study also shows that the experimental process is a very accurate representation of the theoretical system, i.e. the diffusion process. The obtained models are well suited for prediction and control. The prediction errors are extremely small. The model errors are also small. Thus the deterministic models $B^{*}(q^{-1})/$ $/A^{*}(q^{-1})$ describe the physical process well. The deterministic models are valuable when implementing deterministic control strategies, e.g. dead-beat strategies.

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APPENDIX

	n=3	n=4	n=5
a_1	-1.9603±0.0024	-1.9421±0.0010	-1.5215±0.0012
a_2	1.2422±0.0037	1.1814±0.0019	0.2957±0.0015
a_3	-0.2620±0.0019	-0.2564±0.0015	0.3784±0.0014
a_4		0.0295±0.0007	-0.1676±0.0014
a_5			0.0315±0.0007
b_1	0.0115±0.0002	0.0107±0.0001	0.0107±0.0001
b_2	0.0507±0.0004	0.0527±0.0001	0.0570±0.0001
b_3	-0.0480±0.0003	-0.0462±0.0001	-0.0247±0.0001
b_4		-0.0797±0.0001	-0.0308±0.0001
b_5			0.0000±0.0001
c_1	-0.5109±0.0467	-0.3913±0.0347	0.0250±0.0337
c_2	-0.3277±0.0638	-0.6079±0.0407	-0.8599±0.0361
c_3	0.4647±0.0666	0.0877±0.0313	-0.1299±0.0378
c_4		0.2461±0.0353	0.3448±0.0337
c_5			0.0627±0.0286
d_1	-0.6918	-0.6918	-0.6920
d_2	0.6767	0.6622	0.3714
d_3	-0.1412	-0.1080	0.2182
d_4		0.0270	-0.0669
d_5			-0.0213
λ	$0.1867 \cdot 10^{-2}$	$0.8720 \cdot 10^{-3}$	$0.8624 \cdot 10^{-3}$
V	$0.15023 \cdot 10^{-2}$	$0.32772 \cdot 10^{-3}$	$0.32054 \cdot 10^{-3}$
K	0.7091	0.7394	0.7407

Table A.1 - Results from maximum likelihood identification on series S1, $x=\ell/4$. The estimated values of the parameters and the estimated values of the standard deviation of the parameters are given. Further the estimated values of the standard deviation of the residuals λ, the minimal loss function V and the static gain K are given.

	n=3	n=4	n=5
a_1	-2.1237 ± 0.0011	-2.0342 ± 0.0067	-1.9329 ± 0.0007
a_2	1.4979 ± 0.0020	1.4034 ± 0.0015	1.1847 ± 0.0015
a_3	-0.3632 ± 0.0010	-0.3963 ± 0.0015	-0.2282 ± 0.0018
a_4		0.0408 ± 0.0006	0.0170 ± 0.0015
a_5			0.0082 ± 0.0006
b_1	$-0.1196\cdot10^{-3}\pm0.0701\cdot10^{-3}$	$0.0220\cdot10^{-3}\pm0.0260\cdot10^{-3}$	$0.0237\cdot10^{-3}\pm0.0256\cdot10^{-3}$
b_2	$-0.2136\cdot10^{-3}\pm0.1351\cdot10^{-3}$	$0.4643\cdot10^{-3}\pm0.0516\cdot10^{-3}$	$0.4595\cdot10^{-3}\pm0.0490\cdot10^{-3}$
b_3	$5.6447\cdot10^{-3}\pm0.0814\cdot10^{-3}$	$3.9034\cdot10^{-3}\pm0.0533\cdot10^{-3}$	$3.9506\cdot10^{-3}\pm0.0561\cdot10^{-3}$
b_4		$2.2986\cdot10^{-3}\pm0.0301\cdot10^{-3}$	$2.7232\cdot10^{-3}\pm0.0499\cdot10^{-3}$
b_5			$0.1290\cdot10^{-3}\pm0.0296\cdot10^{-3}$
c_1	-1.0332 ± 0.0358	-0.8620 ± 0.0359	-0.7653 ± 0.0351
c_2	0.4407 ± 0.0697	0.5382 ± 0.0518	0.4371 ± 0.0412
c_3	0.1057 ± 0.0503	-0.1518 ± 0.0498	-0.0248 ± 0.0403
c_4		0.2445 ± 0.0312	0.0681 ± 0.0413
c_5			0.1846 ± 0.0311
d_1	-0.4605	-0.4604	-0.4603
d_2	0.5172	0.4762	0.4297
d_3	-0.1721	-0.1695	-0.1153
d_4		0.0168	-0.0065
d_5			0.0036
λ	$0.6303\cdot10^{-3}$	$0.3620\cdot10^{-3}$	$0.3565\cdot10^{-3}$
V	$0.17125\cdot10^{-3}$	$0.56465\cdot10^{-4}$	$0.54789\cdot10^{-4}$
K	0.5003	0.4919	0.4917

Table A.2 - Results from maximum likelihood identification on series S1, $x=\ell/2$. The estimated values of the parameters and the estimated values of the standard deviation of the parameters are given. Further the estimated values of the standard deviation of the residuals λ, the minimal loss function V and the static gain K are given.

	n=3	n=4	n=5
a_1	-2.5868 ± 0.0022	-2.9647 ± 0.0018	-2.2107 ± 0.0020
a_2	2.2439 ± 0.0043	3.2909 ± 0.0050	1.0425 ± 0.0053
a_3	-0.6546 ± 0.0021	-1.6323 ± 0.0048	0.8836 ± 0.0065
a_4		0.3081 ± 0.0016	-0.9533 ± 0.0052
a_5			0.2415 ± 0.0018
b_1	$0.5086\cdot10^{-3}\pm0.0511\cdot10^{-3}$	$-0.0154\cdot10^{-3}\pm0.0330\cdot10^{-3}$	$-0.0120\cdot10^{-3}\pm0.0364\cdot10^{-3}$
b_2	$-1.7476\cdot10^{-3}\pm0.1045\cdot10^{-3}$	$0.1666\cdot10^{-3}\pm0.0934\cdot10^{-3}$	$0.1375\cdot10^{-3}\pm0.0953\cdot10^{-3}$
b_3	$1.8677\cdot10^{-3}\pm0.0586\cdot10^{-3}$	$-0.4393\cdot10^{-3}\pm0.0962\cdot10^{-3}$	$-0.2845\cdot10^{-3}\pm0.1163\cdot10^{-3}$
b_4		$0.7897\cdot10^{-3}\pm0.0363\cdot10^{-3}$	$0.4475\cdot10^{-3}\pm0.0985\cdot10^{-3}$
b_5			$0.5856\cdot10^{-3}\pm0.0402\cdot10^{-3}$
c_1	-1.5925 ± 0.0364	-2.3374 ± 0.0372	-1.5773 ± 0.0363
c_2	0.6325 ± 0.0677	2.1375 ± 0.0910	0.3189 ± 0.0621
c_3	0.0688 ± 0.0411	-0.9249 ± 0.0882	0.8192 ± 0.0526
c_4		0.2004 ± 0.0341	-0.6335 ± 0.0653
c_5			0.2056 ± 0.0342
d_1	-0.2267	-0.2267	-0.2266
d_2	0.3602	0.4450	0.2736
d_3	-0.1498	-0.2996	0.0396
d_4		0.0686	-0.1629
d_5			0.0541
λ	$0.7689\cdot10^{-3}$	$0.5119\cdot10^{-3}$	$0.5101\cdot10^{-3}$
V	$0.25483\cdot10^{-3}$	$0.11292\cdot10^{-3}$	$0.11216\cdot10^{-3}$
K	0.2548	0.2437	0.2435

Table A.3 - Results from maximum likelihood identification on series S1, $x=3\ell/4$. The estimated values of the parameters and the estimated values of the standard deviation of the parameters are given. Further the estimated values of the standard deviation of the residuals λ, the minimal loss function V and the static gain K are given.

n		x=ℓ/4		x=ℓ/2		x=3ℓ/4
3	A	0.5213±i0.1172	A	0.5887±i0.1929	A	0.8166±i0.1400
		0.9178		0.9463		0.9537
	A'	$-0.6269\cdot10^{-1}\pm i0.2211\cdot10^{-1}$	A'	$-0.4788\cdot10^{-1}\pm i0.3167\cdot10^{-1}$	A'	$-0.1882\cdot10^{-1}\pm i0.1698\cdot10^{-1}$
		$-0.8580\cdot10^{-2}$		$-0.5523\cdot10^{-2}$		$-0.4739\cdot10^{-2}$
4	A	0.1344±i0.1592	A	0.2656±i0.0797	A	0.6040±i0.1916
		0.7354		0.5594		0.8137
		0.9395		0.9434		0.9429
	A'	−0.1569±i0.0870	A'	−0.1283±i0.0292	A'	$-0.4562\cdot10^{-1}\pm i0.3072\cdot10^{-1}$
		$-0.3073\cdot10^{-1}$		$-0.5809\cdot10^{-1}$		$-0.2062\cdot10^{-1}$
		$-0.0246\cdot10^{-2}$		$-0.5822\cdot10^{-2}$		$-0.5878\cdot10^{-2}$
5	A	−0.5495	A	−0.1534	A	−0.7654
		0.1943±i0.2109		0.2881±i0.1302		0.6083±i0.1921
		0.7422		0.5667		0.8166
		0.9403		0.9434		0.9427
	A'	* * * *	A'	* * * *	A'	* * * *
		−0.1249±i0.0826		−0.1151±i0.0424		$-0.4460\cdot10^{-1}\pm i0.3104\cdot10^{-1}$
		$-0.2982\cdot10^{-1}$		$-0.5679\cdot10^{-1}$		$-0.2026\cdot10^{-1}$
		$-0.6153\cdot10^{-2}$		$-0.5826\cdot10^{-2}$		$-0.5902\cdot10^{-2}$

Table A.4 - Roots of the A polynomials and the poles of the continuous processes for series S1. The characteristic polynomial of the continuous process is A'. The stars indicate a pole which has no continuous correspondance.

	n=4	n=5 †	n=6
a_1	-2.5101±0.0008	-3.5155±0.0023	-2.5177±0.0009
a_2	2.2999±0.0019	4.8639±0.0083	1.3577±0.0023
a_3	-0.9219±0.0018	-3.3402±0.0012	1.5070±0.0016
a_4	0.1362±0.0008	1.1590±0.0078	-2.1651±0.0017
a_5		-0.1671±0.0020	0.9836±0.0022
a_6			-0.1653±0.0008
b_1	$-0.0404\cdot10^{-3}\pm0.0399\cdot10^{-3}$	$0.0420\cdot10^{-3}\pm0.0322\cdot10^{-3}$	$0.0529\cdot10^{-3}\pm0.0322\cdot10^{-3}$
b_2	$0.0132\cdot10^{-3}\pm0.0834\cdot10^{-3}$	$-0.1142\cdot10^{-3}\pm0.1084\cdot10^{-3}$	$-0.1109\cdot10^{-3}\pm0.0803\cdot10^{-3}$
b_3	$0.5438\cdot10^{-3}\pm0.0843\cdot10^{-3}$	$0.6463\cdot10^{-3}\pm0.1543\cdot10^{-3}$	$0.5935\cdot10^{-3}\pm0.0784\cdot10^{-3}$
b_4	$2.0224\cdot10^{-3}\pm0.0416\cdot10^{-3}$	$1.4163\cdot10^{-3}\pm0.1131\cdot10^{-3}$	$2.0017\cdot10^{-3}\pm0.0770\cdot10^{-3}$
b_5		$-1.9316\cdot10^{-3}\pm0.0360\cdot10^{-3}$	$-0.4797\cdot10^{-3}\pm0.0830\cdot10^{-3}$
b_6			$-1.9405\cdot10^{-3}\pm0.0363\cdot10^{-3}$
c_1	-0.9927±0.0245	-2.3828±0.0235	-1.3843±0.0234
c_2	0.7059±0.0378	2.2453±0.0605	-0.1316±0.0400
c_3	-0.1246±0.0369	-1.0767±0.0780	1.1566±0.0372
c_4	0.1976±0.0221	0.2403±0.0645	-0.8210±0.0360
c_5		-0.0027±0.0252	0.2255±0.0433
c_6			0.0016±0.0257
d_1	-0.7192	-0.7192	-0.7192
d_2	1.0861	1.8092	1.0916
d_3	-0.5680	-1.6893	0.1148
d_4	0.0959	0.7142	-0.9681
d_5		-0.1185	0.5911
d_6			-0.1173
λ	$0.6566\cdot10^{-3}$	$0.5236\cdot10^{-3}$	$0.5233\cdot10^{-3}$
V	$0.39410\cdot10^{-3}$	$0.25056\cdot10^{-3}$	$0.25029\cdot10^{-3}$
K	0.6139	0.7429	0.7432

Table A.5 - Results from maximum likelihood identification on series S2, $x=\ell/4$. The estimated values of the parameters and the estimated values of the standard deviation of the parameters are given. Further the estimated values of the standard deviation of the residuals λ, the minimal loss function V and the static gain K are given.

† The estimate of the parameter vector θ has an indefinite matrix of second order partial derivatives $V_{\theta\theta}$.

	n=4	n=5	n=6
A	0.4153	0.4333±i0.2484	-0.9961
	0.5642±i0.1452	0.7225	0.4329±i0.2463
	0.9665	0.9386	0.7214
		0.9879	0.9388
			0.9879
A'	-0.4393	-0.3471±i0.2603	* * * *
	-0.2702±i0.1260	-0.1625	-0.3485±i0.2587
	$-0.1705\cdot 10^{-1}$	$-0.3169\cdot 10^{-1}$	-0.1633
		$-0.6099\cdot 10^{-2}$	$-0.3159\cdot 10^{-1}$
			$-0.6084\cdot 10^{-2}$
B	-2.4083±i0.1259	-2.1408	-1.7279
	0.3757	0.9767	-1.1163
		1.9419±i4.2706	0.9768
			1.9816±i3.9423
C	-0.1279±i0.4908	0.0119	-0.9926
	0.6243±i0.6152	0.3699±i0.4505	-0.0068
		0.8155±i0.0743	0.3746±i0.4532
			0.8172±i0.0756

Table A.6 - Roots of the A, B and C polynomials and the poles of the continuous processes for series S2. The characteristic polynomial of the continuous process is A'. The stars indicate a pole which has no continuous correspondance.

	Series S1			Series S2
	$x=\ell/4$	$x=\ell/2$	$x=3\ell/4$ †	$x=\ell/4$ †
a_1	-1.9200 ± 0.0093	-1.9599 ± 0.0113	-2.9336 ± 0.0008	-3.4813 ± 0.0022
a_2	1.1332 ± 0.0175	1.2616 ± 0.0236	3.2069 ± 0.0021	4.7927 ± 0.0035
a_3	-0.2228 ± 0.0102	-0.3193 ± 0.0162	-1.5567 ± 0.0017	-3.3084 ± 0.0018
a_4	0.0224 ± 0.0018	0.0316 ± 0.0038	0.2855 ± 0.0005	1.1742 ± 0.0034
a_5				-0.1771 ± 0.0016
b_1	0.0195 ± 0.0000	0.0000	0.0000	0.0000
b_2	0.0417 ± 0.0002	$0.9542\cdot10^{-3}\pm0.0073\cdot10^{-3}$	$0.0604\cdot10^{-3}\pm0.0044\cdot10^{-3}$	0.0000
b_3	-0.0509 ± 0.0004	$4.1729\cdot10^{-3}\pm0.0135\cdot10^{-3}$	$-0.2004\cdot10^{-3}\pm0.0099\cdot10^{-3}$	$2.4156\cdot10^{-3}\pm0.0160\cdot10^{-3}$
b_4	-0.0008 ± 0.0004	$1.8714\cdot10^{-3}\pm0.0593\cdot10^{-3}$	$0.6496\cdot10^{-3}\pm0.0063\cdot10^{-3}$	$-1.9960\cdot10^{-3}\pm0.0456\cdot10^{-3}$
b_5				$-0.3390\cdot10^{-3}\pm0.0298\cdot10^{-3}$
c_1	0.0410 ± 0.0406	-0.7924 ± 0.0358	-2.4988 ± 0.0350	-2.3767 ± 0.0350
c_2	-0.1689 ± 0.0416	0.4876 ± 0.0445	2.2933 ± 0.0919	2.2889 ± 0.0782
c_3	0.3254 ± 0.0600	-0.1515 ± 0.0435	-0.9118 ± 0.0953	-1.1903 ± 0.0781
c_4	0.3437 ± 0.0511	0.2403 ± 0.0358	0.1409 ± 0.0382	0.2937 ± 0.0336
c_5				0.0000
d_1	$0.0539\cdot10^{-3}$	$0.3648\cdot10^{-3}$	$0.0564\cdot10^{-3}$	$0.6454\cdot10^{-3}$
d_2	$-0.0702\cdot10^{-3}$	$-0.4052\cdot10^{-3}$	$-0.1554\cdot10^{-3}$	$-2.1888\cdot10^{-3}$
d_3	$0.1269\cdot10^{-3}$	$0.0363\cdot10^{-3}$	$0.1954\cdot10^{-3}$	$3.0770\cdot10^{-3}$
d_4	$-0.0526\cdot10^{-3}$	$0.0616\cdot10^{-3}$	$0.0947\cdot10^{-3}$	$-2.5661\cdot10^{-3}$
d_5				$1.0195\cdot10^{-3}$

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	Series S1			Series S2
	$x=\ell/4$	$x=\ell/2$	$x=3\ell/4$ †	$x=\ell/4$ †
λ	$0.1605\cdot10^{-3}$	$0.1032\cdot10^{-3}$	$0.1162\cdot10^{-3}$	$0.2490\cdot10^{-3}$
V	$0.10307\cdot10^{-4}$	$0.42624\cdot10^{-5}$	$0.54020\cdot10^{-5}$	$0.30989\cdot10^{-4}$
K	0.7420	0.4996	0.2496	0.7258

Table A.7 - Results from maximum likelihood identification on the simulated samples. The samples corresponding to series S1 and S2 contain 800 and 1000 data points respectively. The employed value of the diffusivity constant of copper is 1.16 cm^2/sec.

† The estimate of the parameter vector θ has an indefinite matrix of second order partial derivatives $V_{\theta\theta}$.

	Series S1			Series S2
	$x=\ell/4$	$x=\ell/2$	$x=3\ell/4$	$x=\ell/4$
A	$0.1217 \pm i0.1322$	$0.2036 \pm i0.1166$	$0.5891 \pm i0.1604$	$0.4236 \pm i0.2998$
	0.7383	0.6079	0.8108	0.7132
	0.9383	0.9449	0.9445	0.9350
				0.9859
A'	$-0.1716 \pm i0.0827$	$-0.1450 \pm i0.0520$	$-0.4934 \cdot 10^{-1} \pm i0.0266 \cdot 10^{-1}$	$-0.3280 \pm i0.3080$
	$-0.3034 \cdot 10^{-1}$	$-0.4977 \cdot 10^{-1}$	$-0.2097 \cdot 10^{-1}$	-0.1690
	$-0.6372 \cdot 10^{-2}$	$-0.5672 \cdot 10^{-2}$	$-0.5706 \cdot 10^{-2}$	$-0.3363 \cdot 10^{-1}$
				$-0.7100 \cdot 10^{-2}$

Table A.8 - Roots of the A polynomials and the poles of the continuous processes for the simulated samples. The characteristic polynomial of the continuous process is A'.

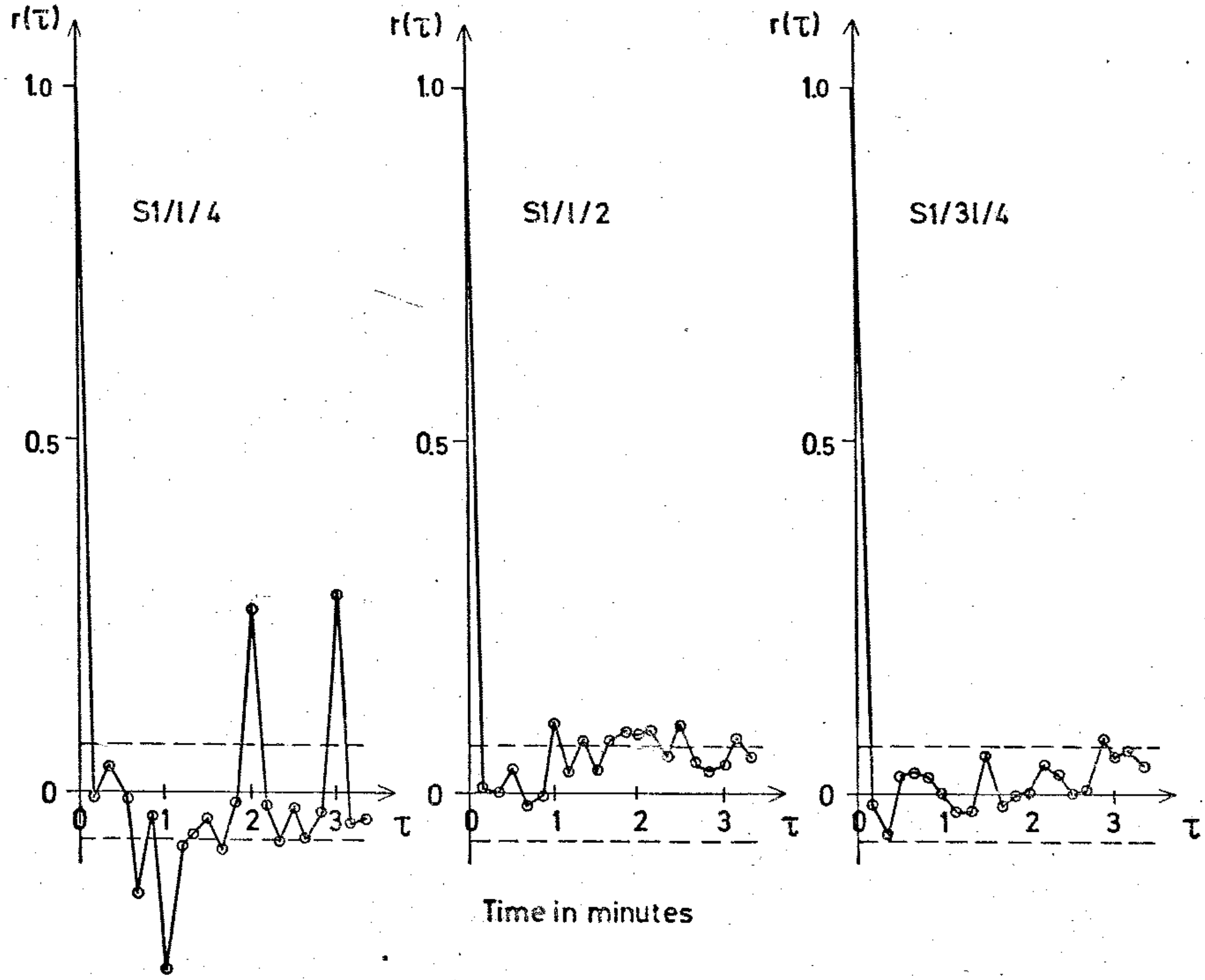
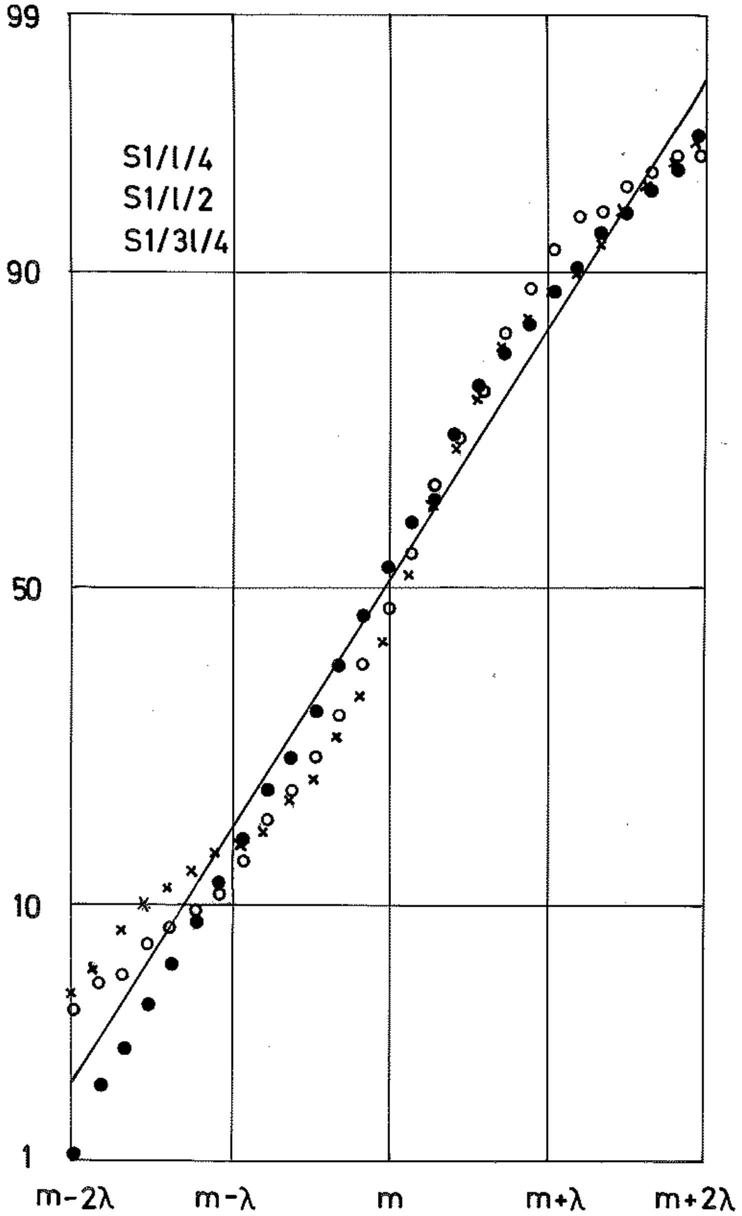


Fig.A.1 - Normalized sample covariance function $r(\tau)$ of the residuals for the models S1/ℓ/4, S1/ℓ/2, S1/3ℓ/4. The dashed lines give the 5% confidence interval for $r(\tau), \tau \neq 0$.



<u>Fig. A.2</u> - The cummulative frequencies of the residuals for the models S1/ℓ/4, S1/ℓ/2, S1/3ℓ/4. The vertical scale of the diagram is chosen so that a perfectly normally distributed variable yields a straight line. The mean value and the standard deviation of the residuals are m and λ respectively.

Curves : ⊗ = model S1/ℓ/4

⊙ = model S1/ℓ/2

⦿ = model S1/3ℓ/4

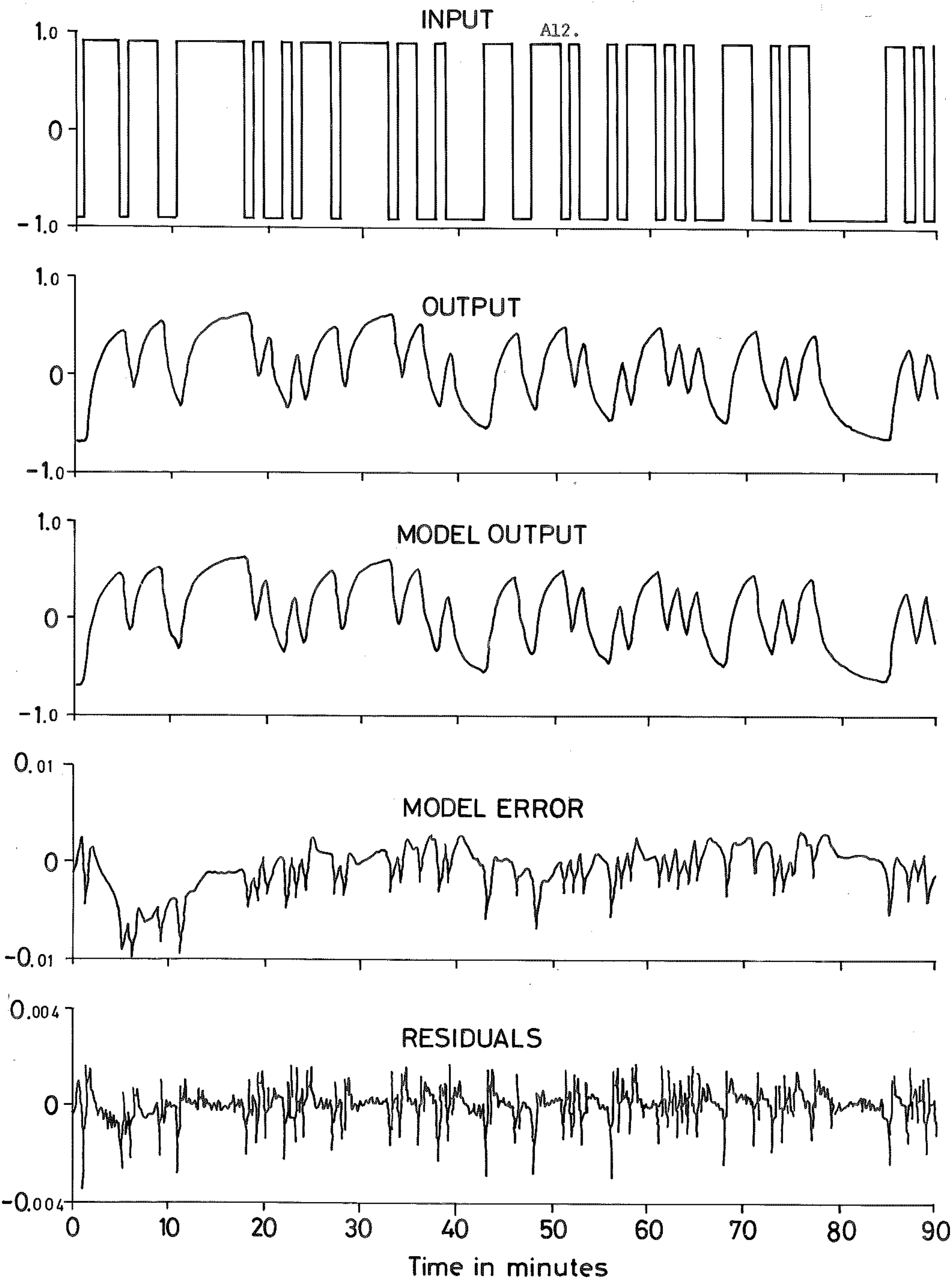


Fig. A.3 - The model output, the model error, the residuals and the input-output sequence of the model S1/ℓ/4.

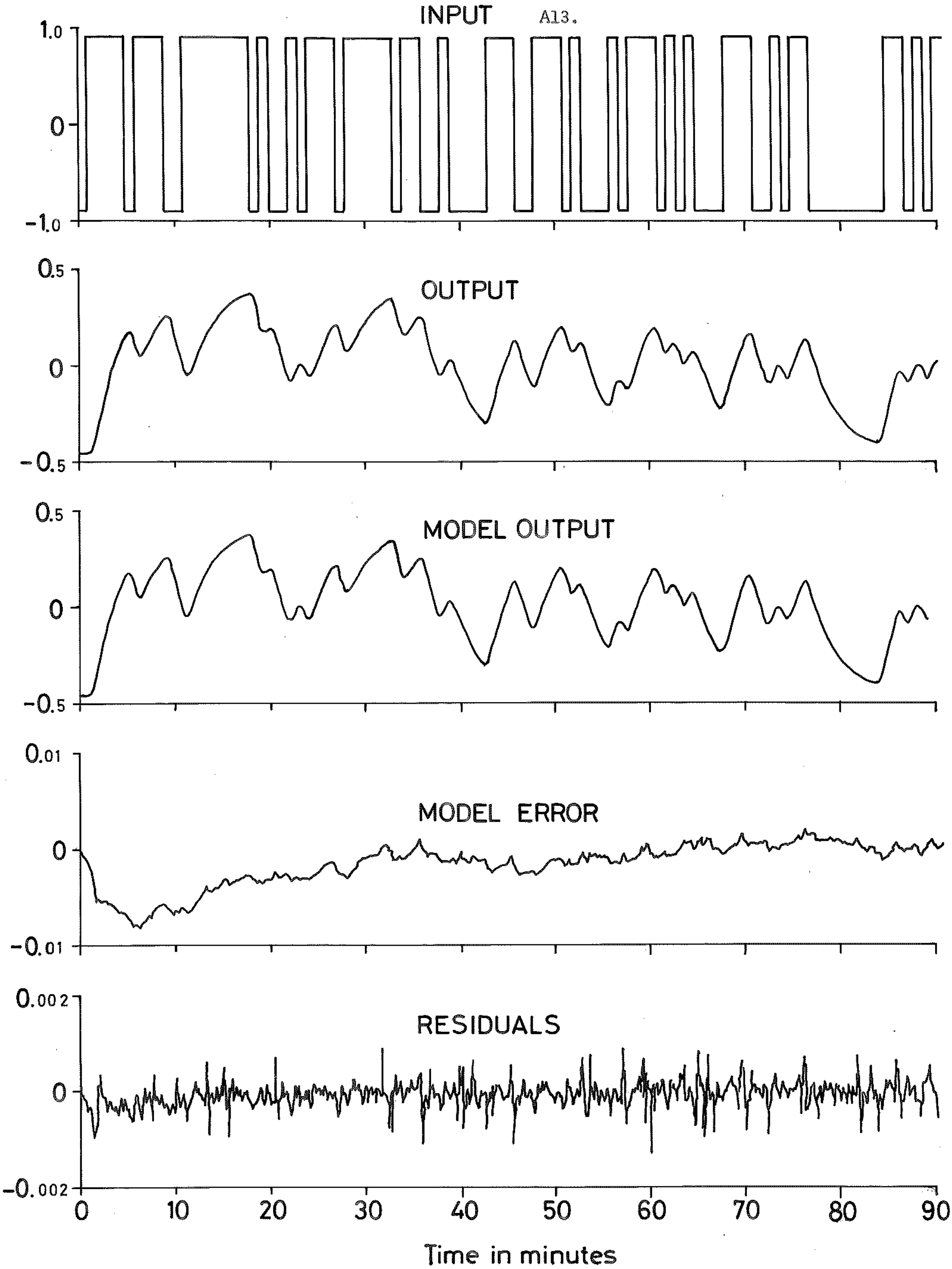


Fig. A.4 - The model output, the model error, the residuals and the input-output sequence of the model S1/ℓ/2.

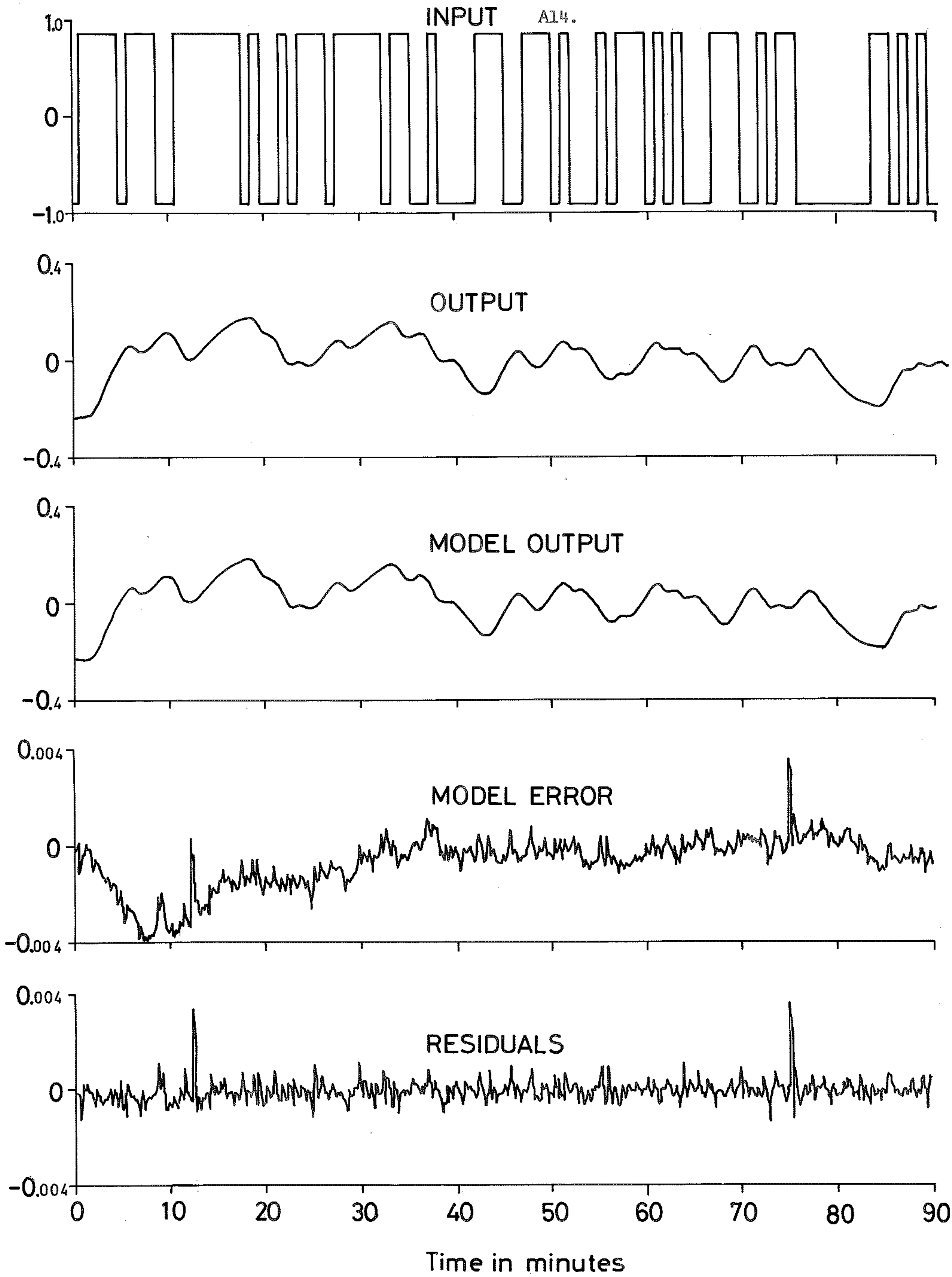


Fig. A.5 - The model output, the model error, the residuals and the input-output sequence of the model S1/3ℓ/4.

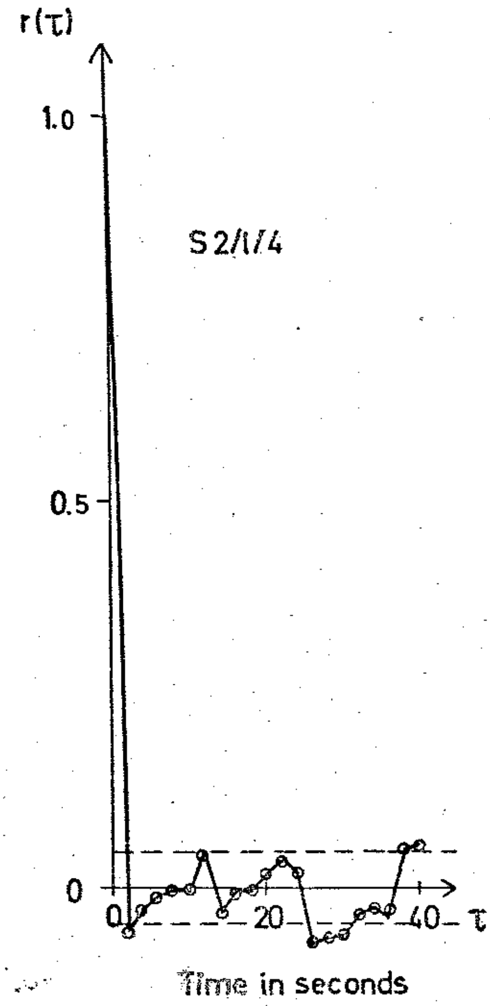


Fig. A.6 - Normalized sample covariance function $r(\tau)$ of the residuals for the model S2/ℓ/4. The dashed line give the 5% confidence interval for $r(\tau),\tau\neq0$.

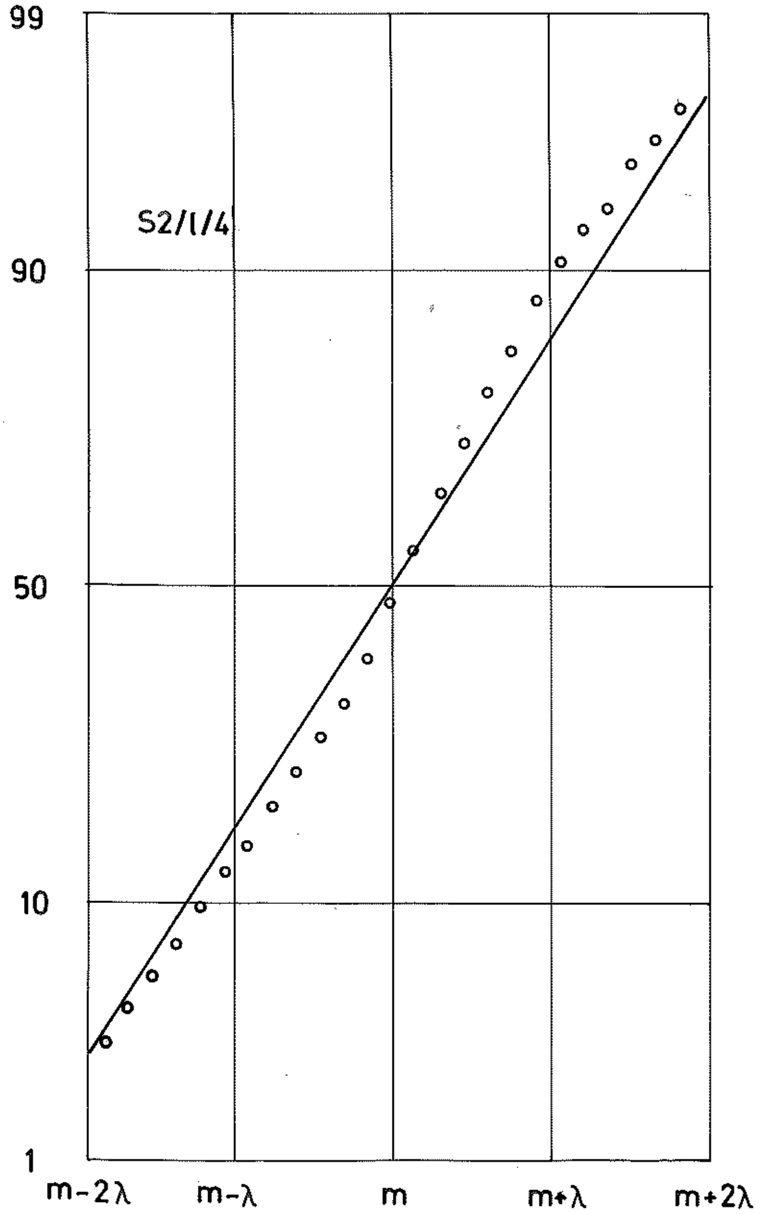


Fig. A.7 - The cummulative frequencies of the residuals for the model S2/ℓ/4. The vertical scale of the diagram is chosen so that a perfectly normally distributed variable yields a straight line. The mean value and the standard deviation of the residuals are m and λ respectively.

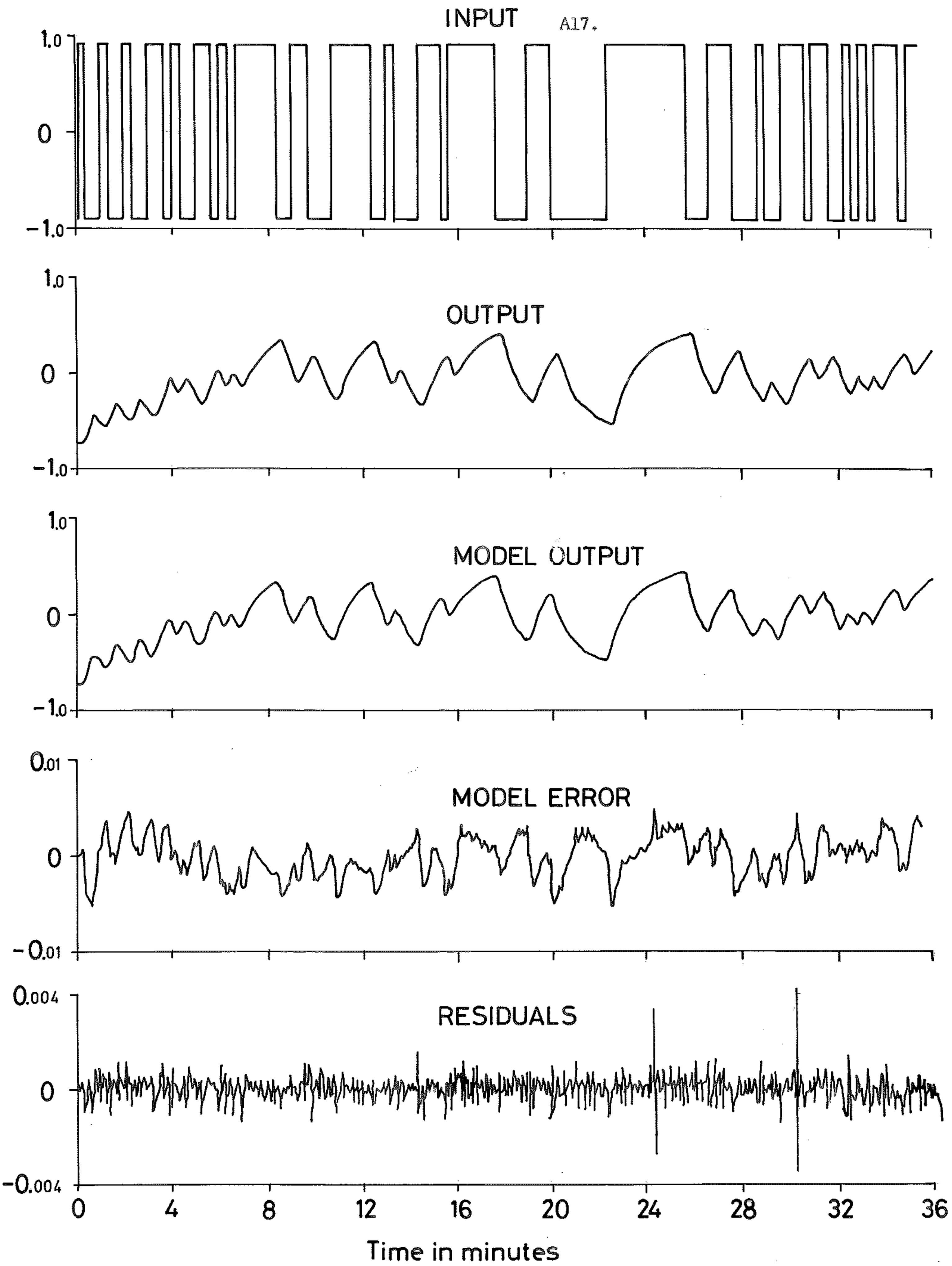


Fig. A.8 - The model output, the model error, the residuals and the input-output sequence of the model S2/ℓ/4.

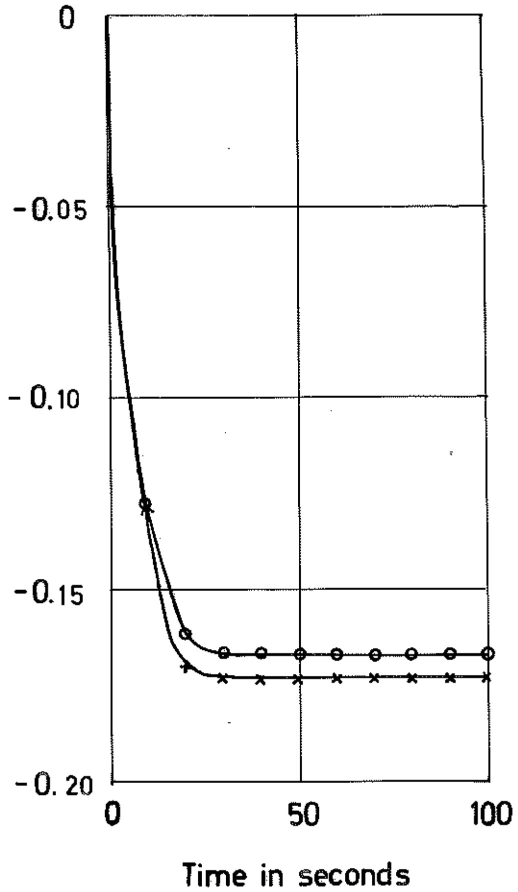
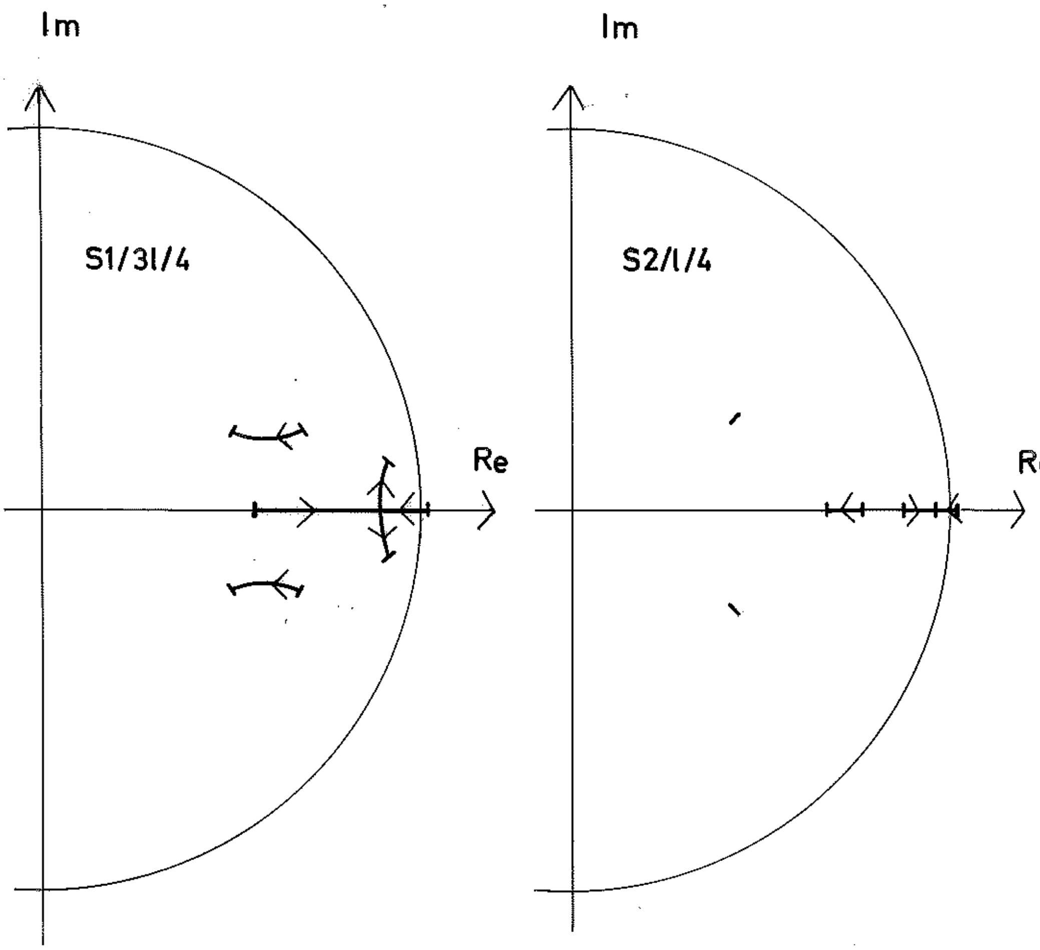


Fig. A.9 - Unit step response of the second-order term of the model GS1(ℓ/4,s) and unit step response of the corresponding residual term of the theoretical model.

Curves :⊗= second-order term $-(0.57793s+0.17208)/(1+9.8491s+33.195s^2)$

◎ = residual term $\sum_{k=5}^{\infty} K_k(\ell/4)/(1+T_k s)$



<u>Fig. A.10</u> - Root loci of the A polynomials for the models S1/3ℓ/4, S2/ℓ/4. The A polynomials are subjected to perturbations of 2 standard deviations in the last coefficient.