Extension of the Combined Multiplier-Penalty Function Method to Optimal Control Problems

Glad, Torkel

1975

Document Version:
Publisher's PDF, also known as Version of record

Link to publication

Citation for published version (APA):
Glad, T. (1975). *Extension of the Combined Multiplier-Penalty Function Method to Optimal Control Problems*. (Research Reports TFRT-3094). Department of Automatic Control, Lund Institute of Technology (LTH).

Total number of authors:
1

EXTENSION OF THE COMBINED MULTIPLIER - PENALITY FUNCTION METHOD TO OPTIMAL CONTROL PROBLEMS

TORKEL GLAD

Report 7511 June 1975
Department of Automatic Control
Lund Institute of Technology

EXTENSION OF THE COMBINED MULTIPLIER-PENALTY FUNCTION METHOD TO OPTIMAL CONTROL PROBLEMS. †)

T. Glad

ABSTRACT.

The multiplier method of Hestenes and Powell is extended to control problems. It is shown that the value of the parameter c which is needed to obtain a minimum is determined by a Riccati equation.

†) This work has been supported by the Swedish Institute of Applied Mathematics.

TABLE OF CONTENTS

		Page
1.	Introduction	1
2.	Formulation of the Optimal Control Problem	3
3.	The Optimal Control Problem as a Non-dynamic Problem	5
4.	Extension to Fixed End Point Problems	15
5.	Examples	18
6.	References	24

APPENDIX - Properties of the Riccati Equation

1. INTRODUCTION.

An interesting method for minimizing a function of finitely many variables under equality constraints

$$\begin{cases} \min f(x) \\ \text{subject to } g(x) = 0 \end{cases}$$

has been given by Hestenes (1969) and Powell (1969). The idea is to form an augmented function

$$F(x,p,c) = f(x) + p^T g(x) + c g(x)^T g(x)$$

It can be shown, Hestenes (1969), that under fairly general conditions there exists a p such that F(x,p,c) has a minimum at the solution to the original problem, provided c is large enough. Since the method appears to work quite well when applied to numerical problems, Glad (1973), Glad (1975), it seems natural to extend it to optimal control problems. One of the simplest versions of this problem is

$$\text{Minimize } J = \int_0^T L(x,u,t)\,dt$$

$$\text{subject to } \begin{cases} \dot{x} = f(x,u,t) \\ x(0) = a \end{cases}$$

It has been suggested by Hestenes (1969) that the augmented criterion $\bar{J}$ should be formed

$$\bar{J}(x,u) = \int_0^T \Big\{ L(x,u,t) + p^T\big(f(x,u,t) - \dot{x}\big) +$$
$$+ c\big(f(x,u,t) - \dot{x}\big)^T \big(f(x,u,t) - \dot{x}\big) \Big\} dt$$

It is proved already in Hestenes (1947) that $\bar{J}$ has a minimum with respect to arbitrary (x,u) (not necessarily satisfying the differential equation) if the parameter c is large enough. Di Pillo et al (1974) have proved further results. In this report it will be shown that $\bar{J}$ has a minimum, by using an approach which is different from the one used in the above references. This has the advantage that one can see more explicitly what value of c is the smallest possible. It turns out that this is determined by a certain Riccati equation.

2. FORMULATION OF THE OPTIMAL CONTROL PROBLEM.

To begin with the optimal control problem without constraints will be studied. This is defined as follows.

Minimize

$$J = \int_0^T L(x(t),\ u(t),\ t)dt + F(x(T))$$

where x satisfies

$$\begin{cases} \dot{x}(t) = f(x(t),\ u(t),\ t) \\ x(0) = a \end{cases}$$

It is assumed that x is continuously differentiable and u continuous. The functions L, F and f are assumed to be three times continuously differentiable. Furthermore it will be assumed that the above problem has a so called normal solution $(\bar{x},\bar{u})$.

Then there exists a function p(t) such that the following conditions hold at the optimal solutions $\bar{x}$ and $\bar{u}$.

First order necessary conditions:

$$\begin{cases} \dot{\bar{x}} = f(\bar{x},\bar{u},t) \\ \dot{p} = -L_x^T(\bar{x},\bar{u},t) - f_x^T(\bar{x},\bar{u},t)p \\ \bar{x}(0) = a \\ p(T) = F_x^T(\bar{x}(T)) \\ L_u(\bar{x},\bar{u},t) + p^T f_u(\bar{x},\bar{u},t) = 0 \end{cases}$$

Introducing $H(x,u,p,t) = L(x,u,t) + p^T f(x,u,t)$ this can be written as

$$\begin{cases} \dot{\bar{x}} = \dfrac{\partial H}{\partial p} & \bar{x}(0) = a \\[2ex] \dot{p} = -\dfrac{\partial H}{\partial x} & p(T) = F_x^T\big(x(T)\big) \\[2ex] \dfrac{\partial H}{\partial u} = 0 & \end{cases}$$

A sufficient condition for $(\bar{x},\bar{u})$ to be a local minimum is that the conditions above hold and that furthermore

(1) $H_{uu}(\bar{x},\bar{u},p,t) > 0 \qquad 0 \le t \le T$

(2) The Riccati equation

$$(**)\begin{cases} -\dot{P} = P(f_x - f_u H_{uu}^{-1} H_{ux}) + (f_x - f_u H_{uu}^{-1} H_{ux})^T P + \\[1ex] \qquad + H_{xx} - H_{xu} H_{uu}^{-1} H_{ux} - P f_u H_{uu}^{-1} f_u^T P \\[2ex] P(T) = F_{xx}\big(\bar{x}(T)\big) \end{cases}$$

where f_x, f_u, H_{ux} etc. are evaluated along $(\bar{x},\bar{u})$, has a solution that exists over the whole interval $0 \le t \le T$. A discussion of these conditions can be found in Bryson and Ho (1969).

3. THE OPTIMAL CONTROL PROBLEM AS A NON-DYNAMIC PROBLEM.

The following augmented loss function is studied

$$\bar{J}(x,u) = \int_0^T \Big\{ L(x,u,t) + p^T\big(f(x,u,t) - \dot{x}\big) + \frac{c}{2}\big(f(x,u,t) - \dot{x}\big)^T \big(f(x,u,t) - \dot{x}\big)\Big\} dt + F(x(T))$$

Here c is a positive real number and p is the function defined in the previous section. $\bar{J}$ is now regarded as a function of x and u, not necessarily satisfying the differential equation $\dot{x} = f(x,u,t)$. (Note that if the differential equation is satisfied then $J = \bar{J}$.) If $\bar{J}$ has a minimum at $(\bar{x},\bar{u})$, the original problem has been transformed into a non-dynamic one. To see when this occurs, $\bar{J}$ is expanded to second order around $(\bar{x},\bar{u})$.

We write $x = \bar{x} + h$, $u = \bar{u} + k$. Since we only study x which satisfy $x(0) = a$, we have $h(0) = 0$. Perturbations (h,k), where h is continuously differentiable with $h(0) = 0$ and k is continuous, will be called admissible. The norms $\| h \|$ and $\| k \|$ are defined by

$$\| h \| = \sup_{0\le t\le T} |h(t)| + \sup_{0\le t\le T} |\dot{h}(t)|$$

$$\| k \| = \sup_{0\le t\le T} |k(t)|$$

In what follows we write H_{xx} etc for $H_{xx}\big(\bar{x}(t), \bar{u}(t), p(t), t\big)$. Since $\bar{x}$ and $\bar{u}$ are continuous, these matrices are continuous functions of time.

$$J(\bar{x}+h,\ \bar{u}+k) = \int_0^T \Big\{ H(\bar{x}+h,\ \bar{u}+k,\ p,\ t) - p^T(\dot{\bar{x}}+\dot{h}) +$$

$$+ \frac{c}{2}\big(f(\bar{x}+h,\ \bar{u}+k,\ t) - \dot{\bar{x}} - \dot{h}\big)^T \big(f(\bar{x}+h,\ \bar{u}+k,\ t)$$

$$- \dot{\bar{x}} - \dot{h}\big)\Big\} dt + F\big(\bar{x}(T) + h(T)\big)$$

$$\bar{J}(\bar{x}+h,\ \bar{u}+k) = \bar{J}(\bar{x},\bar{u}) + \int_0^T (H_x h + H_u k - p^T \dot{h})\,dt + F_x h(T) +$$

$$+ \frac{1}{2}\int_0^T (h^T H_{xx} h + 2h^T H_{xu} k + k^T H_{uu} k)\,dt +$$

$$+ \frac{c}{2}\int_0^T (h^T f_x^T f_x h + k^T f_u^T f_u k + \dot{h}^T \dot{h} + 2h^T f_x^T f_u k - 2h^T f_x^T \dot{h} -$$

$$-2k^T f_u^T \dot{h})\,dt + \frac{1}{2}\, h^T(T) F_{xx} h(T) + R(h,k)$$

where

$$|R(h,k)| \le \varepsilon(h,k) \int_0^T (h^T h + \dot{h}^T \dot{h} + k^T k)\,dt$$

and $\varepsilon(h,k) \to 0$ as $(h,k) \to 0$.

From the properties of p it follows that the linear term disappears. Then

$$\bar{J}(\bar{x}+h,\ \bar{u}+k) = \bar{J}(\bar{x},\bar{u}) + \frac{1}{2}\int_0^T \Big\{ h^T(H_{xx} + c f_x^T f_x) h +$$

$$+ 2h^T(H_{xu} + c f_x^T f_u) k + k^T(H_{uu} + c f_u^T f_u) k +$$

$$+ c\dot{h}^T \dot{h} - 2ch^T f_x^T \dot{h} - 2ck^T f_u^T \dot{h} \Big\} dt +$$

$$+ \frac{1}{2} h^T(T)F_{xx}h(T) + R(h,k) =$$

$$= \delta^2\bar{J}(h,k) + R(h,k)$$

In order to study $\delta^2\bar{J}$ we transform it into a perfect square. To do this observe that, if $S(t)$ is a continuously differentiable matrix function, then

$$\int_0^T \left\{h^T\dot{S}h + 2h^TS\dot{h}\right\}dt - h^T(T)S(T)h(T) = 0$$

Adding this quantity to the expression for $\delta^2\bar{J}$ gives

$$\delta^2\bar{J} = \frac{1}{2}\int_0^T \left\{h^T(H_{xx}+cf_x^Tf_x+\dot{S})h + 2h^T(H_{xu}+cf_x^Tf_u)k + \right.$$

$$+ k^T(H_{uu}+cf_u^Tf_u)k + c\dot{h}^T\dot{h} + 2h^T(S-cf_x^T)\dot{h} -$$

$$\left. - 2ck^Tf_u\dot{h}\right\}dt + \frac{1}{2} h^T(T)[F_{xx} - S(T)]h(T) =$$

$$= \frac{1}{2}\int_0^T \begin{pmatrix} k + H_{uu}^{-1}(H_{ux}+f_u^TS)h \\ \dot{h} + [f_uH_{uu}^{-1}(H_{ux}+f_u^TS) + \frac{1}{c}S - f_x]h \end{pmatrix}^T \cdot$$

$$\cdot \begin{pmatrix} H_{uu} + cf_u^Tf_u & -cf_u^T \\ -cf_u & cI \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} k + H_{uu}^{-1}(H_{ux}+f_u^TS)h \\ \dot{h} + [f_uH_{uu}^{-1}(H_{ux}+f_u^TS) + \frac{1}{c}S - f_x]h \end{pmatrix} dt +$$

$$+ \frac{1}{2}\int_0^T h^T[\dot{S} + H_{xx} - H_{xu}H_{uu}^{-1}H_{ux} + (f_x-f_uH_{uu}^{-1}H_{ux})^TS +$$

$$+ S(f_x - f_u H_{uu}^{-1} H_{ux}) - S f_u H_{uu}^{-1} f_u^T S - \frac{1}{c} S^2] h dt +$$

$$+ \frac{1}{2} h^T(T) [F_{xx} - S(T)] h(T)$$

It is now straightforward to prove the following theorem.

Theorem 1: If $c > 0$, if $H_{uu} > 0$, $0 \leq t \leq T$ and if the Riccati equation

$$(*) \qquad - \dot{S} = H_{xx} - H_{xu} H_{uu}^{-1} H_{ux} + (f_x - f_u H_{uu}^{-1} H_{ux})^T S +$$

$$+ S(f_x - f_u H_{uu}^{-1} H_{ux}) - S(f_u H_{uu}^{-1} f_u^T + \frac{1}{c} I) S$$

$$S(T) = F_{xx}$$

has a solution over the whole interval $0 \leq t \leq T$, then $\delta^2 \bar{J} > 0$ for all admissible h and k not both identically zero.

Proof. We show that the matrix

$$\begin{bmatrix} H_{uu} + c f_u^T f_u & -c f_u^T \\ -c f_u & cI \end{bmatrix}$$

is positive definite for all $c > 0$. Form

$$[z^T w^T] \begin{bmatrix} H_{uu} + c f_u^T f_u & -c f_u^T \\ -c f_u & cI \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} =$$

$$= z^T H_{uu} z + c(f_u z - w)^T (f_u z - w) \geq 0$$

Assume that equality holds. Then $z = 0$ and consequently $w = 0$.

Since the second and third terms of $\delta^2\bar{J}$ disappear we then know that $\delta^2\bar{J} \geq 0$. If $\delta^2\bar{J} = 0$ then

$$k + H_{uu}^{-1}(H_{ux}+f_u^T S)h = 0$$

$$\dot{h} + (f_u H_{uu}^{-1}(H_{ux}+f_u^T S)+\frac{1}{c}S-f_x)h = 0$$

all t, $0 \leq t \leq T$. The lower of these equations is a linear differential equation in h with $h(0) = 0$. From the uniqueness theorem for linear differential equations it follows that h is identically zero. The first equation then gives that k is identically zero.

This result can easily be strengthened.

Theorem 2: Suppose the conditions of Theorem 1 hold. Then there exists a constant $\eta > 0$ such that

$$\delta^2\bar{J}(h,k) \geq \eta \int_0^T (h^T h + \dot{h}^T \dot{h}^T + k^T k)\,dt$$

Proof. Study

$$A(h,k,\eta) = \delta^2\bar{J}(h,k) - \eta \int_0^T (h^T h + \dot{h}^T \dot{h} + k^T k)\,dt$$

where $\eta > 0$. The value of $A(h,k,\eta)$ is the same as the value of $\delta^2\bar{J}(h,k)$ with $h^T(H_{xx} + cf_x^T f_x)h$ replaced by $h^T(H_{xx} + cf_x^T f_x - \eta I)h$, $c\dot{h}^T\dot{h}$ replaced by $(c-\eta)\dot{h}^T\dot{h}$ and $k^T(H_{uu} + cf_u^T f_u)k$

replaced by $k^T(H_{uu} + cf_u^T f_u - \eta I)k$. It then follows from Lemma A.5 in the appendix that, if η is chosen sufficiently small, then the Riccati equation corresponding to $A(h, k,\eta)$ exists over the interval $0 \leq t \leq T$. Since

$$\begin{vmatrix} H_{uu} + cf_u^T f_u - \eta I & -cf_u^T \\ -cf_u & (c-\eta)I \end{vmatrix}$$

is still positive definite, it follows that $A(h,k,\eta) \geq 0$ for all admissible h and k.

We can now give conditions for $\bar{J}$ to have a local minimum.

Theorem 3: Suppose the conditions of Theorem 1 are satisfied. Then $\bar{J}$ has a local minimum at $(\bar{x},\bar{u})$ with respect to all (x,u) with x continuously differentiable, $x(0) = a$, and u continuous.

Proof. We have from Theorem 2

$$\bar{J}(\bar{x}+h,\ \bar{u}+k) - J(\bar{x},\bar{u}) = \delta^2\bar{J}(h,k) + R(h,k) \geq$$

$$\geq \left(\eta - |\varepsilon(h,k)|\right)\int_0^T (h^T h + \dot{h}^T\dot{h} + k^T k)\,dt \geq 0$$

if (h,k) is sufficiently small.

It is interesting to note that the magnitude of c that is required only depends on the Riccati equation (*) (provided $c > 0$). The interesting question is, of course: is there any c for which the solution of (*) exists over the whole time interval? We have the following result.

Theorem 4: If (*) has a solution on $0 \leq t \leq T$ for $c = c_1$, then it has a solution for any $c \geq c_1$.

Proof. Let $c_2 > c_1$ and define

$$P_1 = f_u H_{uu}^{-1} f_u^T + \frac{1}{c_1} I$$

$$P_2 = f_u H_{uu}^{-1} f_u^T + \frac{1}{c_2} I$$

Then $P_1 - P_2 \geq 0$. It now follows from Lemma A.2 in the appendix that $S_2(t) \geq S_1(t)$, where S_1 and S_2 are the solutions corresponding to c_1 and c_2 respectively. Since from Lemma A.4 in the appendix, the only way in which the solution S can fail to exist, is by going off to minus infinity, it follows that S_2 exists on any interval where S_1 exists.

Corollary. There exists a number c_0 such that S exists on the whole interval $0 \leq t \leq T$ for $c > c_0$ and S goes off towards minus infinity for some t_1, $0 \leq t_1 \leq T$ when $c < c_0$.

Proof. Take $c_0 = \inf\{\text{all } c > 0 \text{ such that S exists on the whole interval}\}$.

The connection with the sufficiency conditions of Section 2 is given by the following theorem.

Theorem 5: Suppose that the Riccati equation appearing in the sufficiency conditions for a minimum, (**) has a solution over the whole interval. Then there exists a $c_0 > 0$ such that (*) also has a solution over the interval $0 \leq t \leq$ $\leq T$, provided $c > c_0$.

Proof. Since the difference between the matrices $f_u H_{uu}^{-1} f_u^T$ and $(f_u H_{uu}^{-1} f_u^T + \frac{1}{c} I)$ can be made arbitrarily small the result follows from Lemma A.5 in the appendix.

As an immediate consequence we get

Theorem 6: If the sufficiency conditions for a minimum, given in Section 2, hold, then there exists a c_0 such that $\bar{J}$ has a local minimum at $\bar{x}$, $\bar{u}$ for $c > c_0$.

Proof. Follows from Theorems 5 and 1.

If $\bar{J}$ has a minimum at $(\bar{x}, \bar{u})$ with respect to all (x,u), it has obviously a minimum with respect to (x,u) satisfying the differential equation $\dot{x} = f(x,u)$. Since $\bar{J}(x,u) = J(x,u)$ for these (x,u), the theorems of this section form an alternative proof of the sufficiency conditions of section 2.

So far we have shown that the existence of a solution to the the Riccati equation implies that $\bar{J}$ has a local minimum. We will now show that existence of the solution of the Riccati equation is also necessary.

Theorem 7: Suppose that $\bar{J}$ has a local minimum at $(\bar{x},\bar{u})$. Then the Riccati equation (*) has a solution over the interval $\varepsilon \leq t \leq T$ where $\varepsilon > 0$ is arbitrary.

Proof. For $\bar{J}$ to have a local minimum it is necessary that $\delta^2\bar{J}(h,k) \geq 0$ for all admissible h and k. Since the solution of the Riccati equation (*) exists on some interval $t_1 \leq$ $\leq t \leq T$ we have

$$\delta^2\bar{J}(h,k) = \frac{1}{2}\int_0^{t_1} \{h^T(H_{xx} + cf_x^T f_x)h + 2h^T(H_{xu} + cf_x^T f_u)k +$$

$$+ k^T(H_{uu} + cf_u^T f_u)k + c\dot{h}^T\dot{h} - 2ch^T f_x^T\dot{h} - 2ck^T f_u^T\dot{h}\}dt +$$

$$+ \frac{1}{2}\int_{t_1}^{T} \begin{pmatrix} k + H_{uu}^{-1}(H_{ux} + f_u^T S)h \\ \dot{h} + [f_u H_{uu}^{-1}(H_{ux} + f_u^T S) + \frac{1}{c} S - f_x]h \end{pmatrix}^T \cdot$$

$$\cdot \begin{pmatrix} H_{uu} + cf_u^T f_u & -cf_u^T \\ -cf_u & cI \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} k + H_{uu}^{-1}(H_{ux} + f_u^T S)h \\ \dot{h} + [f_u H_{uu}^{-1}(H_{ux} + f_u^T S) + \frac{1}{c} S - f_x]h \end{pmatrix} dt +$$

$$+ \frac{1}{2} h^T(t_1)S(t_1)h(t_1)$$

Now choose

$$\left.\begin{aligned} k(t) &= 0 \\ h(t) &= \frac{t}{t_1} a \end{aligned}\right\} \quad 0 \leq t < t_1$$

where a is an arbitrary vector, and

$$\left.\begin{aligned} k &= -H_{uu}^{-1}(H_{ux} + f_u^T S)h \\ \dot{h} &= -\left[f_u H_{uu}^{-1}(H_{ux} + f_u^T S) + \frac{1}{c} S - f_x\right]h \\ h(t_1) &= a \end{aligned}\right\} t_1 \le t \le T$$

The result is

$$\delta^2 \bar{J}(h,k) = \frac{1}{2} a^T \int_0^{t_1} \left\{\frac{t^2}{t_1^2} (H_{xx} + c f_x^T f_x) + c \frac{1}{t_1^2} I - \right.$$

$$\left. - c \frac{t}{t_1^2} (f_x + f_x^T)\right\} dt \; a + \frac{1}{2} a^T S(t_1) a \ge 0 \quad \dagger)$$

$$a^T S(t_1) a \ge - a^T \int_0^{t_1} \left\{\frac{t^2}{t_1^2} (H_{xx} + c f_x^T f_x) + \frac{c}{t_1^2} I - \right.$$

$$\left. - \frac{ct}{t_1^2} (f_x + f_x^T)\right\} dt \; a$$

for any vector a. Since the integral exists for any $t_1 > 0$, $S(t)$ is bounded from below on $t_1 \le t \le T$ for any $t_1 > 0$ and the theorem is proved.

---

†) Actually we only know that $\delta^2 J \ge 0$ for h continuously differentiable and k continuous. The (h,k) given above can, however, be approximated arbitrarily well by (h,k) which are continuously differentiable and continuous respectively.

4. EXTENSION TO FIXED END POINT PROBLEMS.

We will suppose that the optimal control problem is formulated as in Section 2 with the additional requirement that certain components of $x(T)$ are fixed. For simplicity the components of x are assumed to be ordered so that the first q values are fixed.

$$x_i(T) = b_i \qquad i = 1, \ldots, q$$

$$x_i(T) \quad \text{free} \qquad i = q+1, \ldots, n$$

It is also natural to assume that F depends only on $x_{q+1}, \ldots, x_n$.

The conditions on the multipliers p in Section 2 are then altered to

$$\dot{p} = -\frac{\partial H}{\partial x} \qquad p_i(T) = \frac{\partial F}{\partial x_i} \qquad i = q+1, \ldots, n$$

We still assume that such a p exists.

The second order expansion of $\tilde{J}$ will be the same. The admissible variations (h,k) will now, in addition to the conditions of Section 3, have to satisfy

$$h_i(T) = 0 \qquad i = 1, \ldots, q$$

From this it follows that the Riccati equation (*) will be the same, except for the end point condition on $S(T)$. This will be

$$S(T) = \left(\begin{array}{c|c} A & B \\ \hline B^T & \tilde{F}_{xx} \end{array}\right)$$

where

$$\tilde{F}_{xx} = \left(\frac{\partial^2 F}{\partial x_i \partial x_j}\right) \qquad \begin{matrix} i = q+1, \ldots, n \\ j = q+1, \ldots, n \end{matrix}$$

and A and B are arbitrary.

The theorems of Section 3 are now modified as follows.

Theorem 1': If $c > 0$, if $H_{uu} > 0$ and if the Riccati equation

$$(***) \quad -\dot{S} = H_{xx} - H_{xu}H_{uu}^{-1}H_{ux} + (f_x - f_u H_{uu}^{-1}H_{ux})^T S +$$

$$+ S(f_x - f_u H_{uu}^{-1} H_{ux}) - S(f_u H_{uu} f_u^T + \frac{1}{c} I)S$$

$$S(T) = \left(\begin{array}{c|c} A & B \\ \hline B^T & \tilde{F}_{xx} \end{array}\right)$$

has a solution over the whole interval $0 \leq t \leq T$ for some value of A and B, then $\delta^2\bar{J} > 0$ for all admissible h and k not both identically zero.

Theorem 2': Suppose the conditions of theorem 1' hold. Then there exists a constant $\eta > 0$ such that

$$\delta^2\bar{J}(h,k) \geq \eta \int_0^T (h^T h + \dot{h}^T \dot{h} + k^T k)\,dt$$

Theorem 3': Suppose the conditions of theorem 1' hold. Then $\bar{J}$ has a local minimum at $(\bar{x},\bar{u})$ with respect to all (x,u) with x continuously differentiable, $x(0) = a$, $x_i(T) = b_i$, $i = 1, \ldots, q$ and u continuous.

Theorem 5': Suppose that the Riccati equation appearing in the sufficiency conditions for a minimum, (**), with boundary conditions

$$S(T) = \left[\begin{array}{c|c} A & B \\ \hline B^T & \tilde{F}_{xx} \end{array}\right]$$

has a solution over the interval $0 \leq t \leq T$ for some A and B. Then there exists a $c_0 > 0$ such that (***) also has a solution over the interval $0 \leq t \leq T$, provided $c > c_0$.

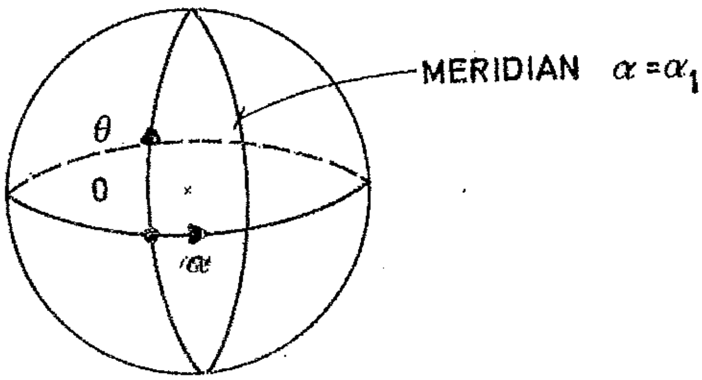
Theorem 6': If $H_{uu} > 0$ and the conditions of Theorem 5' hold, then there exists a c_0 such that $\bar{J}$ has a local minimum at $(\bar{x},\bar{u})$ for $c > c_0$.

The proofs of these theorems are straightforward extensions of Theorems 1-6.

5. EXAMPLES.

To illustrate the results of the previous sections we will give some examples where the conditions can be worked out explicitly. All these examples are from Bryson and Ho (1969).

Example 1. Shortest distance between a point and a great circle on a sphere.



Let the given point be at the origin O of a latitude-longitude coordinate system and let the great circle be the meridian $\alpha = \alpha_1$ (θ is latitude, α is longitude).

Then $ds^2 = (rd\theta)^2 + (r \cos\theta\, d\alpha)^2$ where r is the radius of the sphere. The problem is then to minimize

$$J = \int_0^{\alpha_1} \sqrt{u^2 + \cos^2\theta}\, d\alpha$$

where

$$\frac{d\theta}{d\alpha} = u \qquad \theta(0) = 0$$

The Hamiltonian is given by

$$H = \sqrt{u^2 + \cos^2\theta} + pu$$

The first order necessary conditions are

$$\frac{u}{\sqrt{u^2 + \cos^2\theta}} + p = 0$$

$$\dot{p} = \frac{\cos\theta \sin\theta}{\sqrt{u^2 + \cos^2\theta}} \qquad p(T) = 0$$

These are satisfied by $\bar{u} = 0$, $\bar{\theta} = 0$, $p = 0$. The second order quantities are, evaluated along $\bar{u}$, $\bar{\theta}$.

$$H_{uu} = 1 \qquad H_{u\theta} = 0 \qquad H_{\theta\theta} = -1$$

The Riccati equation (**) is then

$$-\frac{dP}{d\alpha} = -1 - P^2 \qquad P(\alpha_1) = 0$$

with solution

$$P(\alpha) = -\tan(\alpha_1 - \alpha)$$

The second order sufficiency conditions are satisfied if $0 < \alpha_1 < \pi/2$.

The Riccati equation (*) is

$$-\frac{dS}{d\alpha} = -1 - (1 + \frac{1}{c})S^2 \qquad S(\alpha_1) = 0$$

with solution

$$S = -\tan\sqrt{1 + \frac{1}{c}}(\alpha_1 - \alpha) \Big/ \sqrt{1 + \frac{1}{c}}$$

The value of c_0 is then

$$c_0 = \frac{\alpha_1^2}{\frac{\pi^2}{4} - \alpha_1^2}$$

for $0 < \alpha_1 < \frac{\pi}{2}$.

Example 2. Shortest path between two points on a sphere.

The only difference compared with Example 1 is the boundary condition $\theta(\alpha_1) = 0$. The Riccati equation is

$$-\frac{dP}{d\alpha} = -1 - P^2 \qquad P(\alpha_1) \text{ arbitrary}$$

with solution $P = -\tan(\alpha_0 - \alpha)$ where α_0 can be chosen arbitrarily. To prolong the existence of P as far as possible α_0 should be taken close to $\pi/2$ (which corresponds to taking $P(\alpha_1)$ large, as expected). The sufficiency conditions are then satisfied on the interval $0 \le \alpha \le \pi - \varepsilon$ for any $\varepsilon > 0$.

For Riccati equation corresponding to $\tilde{J}$ is

$$-\frac{dS}{d\alpha} = -1 - (1 + \frac{1}{c})S^2$$

with solution

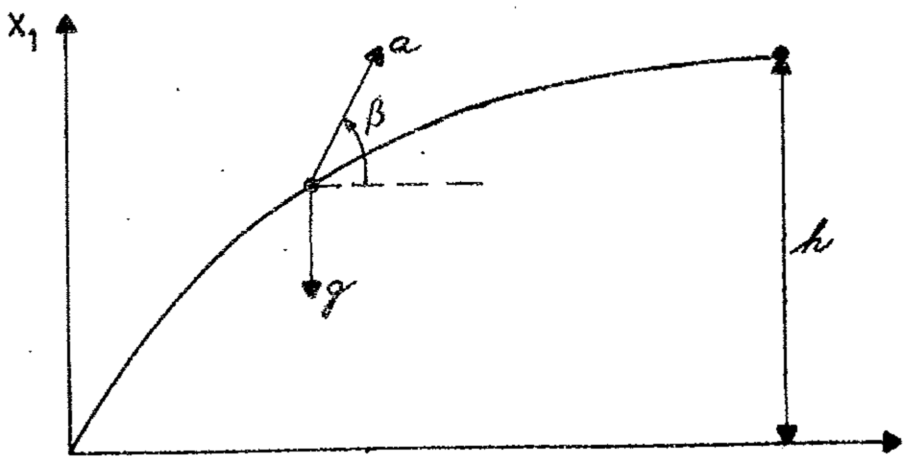
$$S = -\tan\sqrt{1 + \frac{1}{c}}\,(\alpha_0 - \alpha) \Big/ \sqrt{1 + \frac{1}{c}}$$

The value of c_0 is then

$$c_0 = \frac{\alpha_1^2}{\pi^2 - \alpha_1^2}$$

for $0 < \alpha_1 < \pi$.

Example 3.



Study the motion of a rocket in a constant gravitational field. The thrust has a constant magnitude a, but the thrust angle β is a control variable. If x_2 denotes the vertical component of the velocity, the equations of motion are

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = a \sin \beta - g$$

We assume the following boundary conditions

$$x_1(0) = x_2(0) = 0$$

$$x_1(T) = h \qquad x_2(T) = 0$$

The objective is to maximize the horisontal velocity component at the final time T. This gives the loss

$$J = - a \int_0^T \cos \beta \, dt$$

The Hamiltonian is

$$H = - a \cos \beta + p_1 x_2 + p_2 (a \sin \beta - g)$$

The first order necessary condition are

$$\dot{p}_1 = 0$$

$$\dot{p}_2 = - p_1$$

$$\sin \beta + p_2 \cos \beta = 0$$

This gives a control strategy of the form

$$\tan \beta = At + B$$

where the constants A and B are determined from the boundary conditions. The second order terms are

$$H_{xx} = 0 \qquad H_{x\beta} = 0$$

$$H_{\beta\beta} = a \cos \beta - a p_2 \sin \beta = \frac{a}{\cos \beta} > 0$$

Along the trajectory given above we have

$$f_x = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \qquad f_\beta = \begin{vmatrix} 0 \\ a \cos \beta \end{vmatrix}$$

The Riccati equation is

$$-\dot{S} = f_x^T S + S f_x - S(f_\beta H_{\beta\beta}^{-1} f_\beta^T + \frac{1}{c} I)S$$

which is satisfied by $S = 0$.

This means that any $c > 0$ can be used and $c_0 = 0$.

6. REFERENCES.

Anderson, B.D.O., Moore, J.B. (1971):
Linear Optimal Control,
Prentice Hall, New Jersey.

Brockett, R.W. (1970):
Finite Dimensional Linear Systems,
John Wiley and Sons, New York.

Bryson, A.E., Ho, Y.C. (1969):
Applied Optimal Control,
Blaisdell Publishing Company.

Coddington, E.A., Levinson, N. (1955):
Theory of Ordinary Differential Equations,
Mc Graw-Hill, New York.

Di Pillo, G., Grippo, L., Lampariello, F. (1974):
The Multiplier Method for Optimal Control Problems;
Conference on Optimization Problems in Engineering
and Economics,
Naples, Italy, Dec. 16-20, 1974.

Glad, T. (1973):
Lagrange Multiplier Methods for Minimization under
Equality Constraints; Report 7323,
Lund Institute of Technology, Division of Automatic
Control.

Glad, T. (1975):
Algorithms for Nonlinear Minimization with Equality
and Inequality Constraints based on Lagrange Multipliers;
Report 7503,
Lund Institute of Technology, Division of Automatic
Control.

Hestenes, M.R. (1947):
An Indirect Sufficiency Proof for the Problem of Bolza in Nonparametric Form,
Transactions of the American Mathematical Society, Vol. 62, No. 3, pp. 509-535.

Hestenes, M.R. (1969):
Multiplier and Gradient Methods,
Journal of Optimization Theory and Applications, Vol. 4, pp. 303-320.

Mårtensson, K. (1972):
New Approaches to the Numerical Solution of Optimal Control Problems; Report 7206,
Lund Institute of Technology, Division of Automatic Control.

Powell,M.J.D. (1969)
A Method for Nonlinear Constraints in Minimization Problems, in Fletcher, R., ed. : Optimization, Academic Press, London

APPENDIX - Properties of the Riccati Equation.

Here some interesting results about the Riccati equation are collected. Most of them can be found in books about linear quadratic control theory e.g. Brockett (1970) or Anderson and Moore (1971). Another useful reference is Mårtensson (1972).

We will write the Riccati equation in the form

$$-\dot{S}(t) = A^T(t)S(t) + S(t)A(t) + Q\ (t) - S(t)P(t)S(t)$$

$$S(T) = Q_0$$

where A, Q, P and Q_0 are matrices whose elements are continuous functions of t_0 and Q_0, Q and P are symmetric. It follows from standard theorems for differential equations that S(t) exists at least on a sufficiently small interval $t_0 \leq t \leq T$. Moreover, the only way in which S can fail to exist is by having some element which becomes unbounded. In what follows, $M \geq N$, where M and N are symmetric matrices, means that M - N is nonnegative definite and M > N means that M - N is positive definite.

It is useful to rewrite the Riccati equation as an integral equation. Introduce the fundamental matrix $\phi(t,T)$ satisfying

$$\frac{d}{dt}\phi(t,T) = \left(A(t) - \frac{1}{2}P(t)S(t)\right)\phi(t,T)$$

$$\phi(T,T) = I$$

Then we have

$$S(t) = \int_t^T \phi^T(s,t)Q_1(s)\phi(s,t)dS + \phi^T(T,t)Q_0\phi(T,t)$$

Lemma A.1. For the Riccati equation

$$-\dot{S} = A^T S + SA + Q - SPS$$

let S_1 and S_2 be the solutions corresponding to $S(T) = Q_0^1$ and $S(T) = Q_0^2$ respectively. Then if $Q_0^2 \geq Q_0^1$ it follows that $S_2(t) \geq S_1(t)$ for all $t \in [t_0, T]$ where $[t_0, T]$ is an interval on which both solutions exist.

Proof. We have

$$-\frac{d}{dt}(S_2-S_1) = (A-PS_1)^T(S_2-S_1) + (S_2-S_1)(A-PS_1) - $$
$$- (S_2-S_1)P(S_2-S_1)$$

$$S_2(T) - S_1(T) = Q_0^2 - Q_0^1$$

Regarding this as a Riccati equation in $S_2 - S_1$ we get, using the integral equation representation above

$$S_2 - S_1 = \phi^T(T,t)(Q_0^2-Q_0^1)\phi(T,t)$$

where $\phi(t,T)$ now is the fundamental matrix corresponding to

$$A - PS_1 - \frac{1}{2} P(S_2-S_1)$$

Lemma A.2. Let S_1 and S_2 be the solutions of the Riccati equations

$$-\dot{S} = A^T S + SA + Q - SP_1 S \qquad S(T) = Q_0$$

$$-\dot{S} = A^T S + SA + Q - SP_2 S \qquad S(T) = Q_0$$

respectively. If $P_1 \geq P_2$ then $S_2(t) \geq S_1(t)$ for all $t \in [t_0,T]$, where $[t_0,T]$ is any interval on which both solutions exist.

Proof. We have

$$-\frac{d}{dt}(S_2-S_1) = (A-P_2S_1)^T(S_2-S_1) + (S_2-S_1)(A-P_2S_1) -$$
$$- (S_2-S_1)P_2(S_2-S_1) + S_1(P_1-P_2)S_1$$

$$S_2(T) - S_1(T) = 0$$

Using the integral equation form this can be written

$$S_2(t) - S_1(t) = \int_t^T \phi^T(s,t)S_1(P_1-P_2)S_1\phi(s,t)dS$$

Lemma A.3. Let S_1 and S_2 be the solutions of the Riccati equations

$$-\dot{S} = A^TS + SA + Q_1 - SPS \qquad S(T) = Q_0$$

and

$$-\dot{S} = A^TS + SA + Q_2 - SPS \qquad S(T) = Q_0$$

respectively. Then if $Q_2 \geq Q_1$ it follows that $S_2(t) \geq S_1(t)$ $t \in [t_0,T]$, where $[t_0,T]$ is any interval on which both solutions exist.

<u>Proof</u>. We have

$$S_2(t) - S_1(t) = \int_t^T \phi^T(s,t)(Q_2-Q_1)\phi(s,t)ds$$

where ϕ is the fundamental matrix corresponding to

$$A - PS_1 - \frac{1}{2}P(S_2-S_1)$$

We can now deduce the following result.

<u>Lemma A.4</u>. If $P > 0$ then there exists a continuous matrix $R(t)$ such that $S(t) \le R(t)$ on any interval $[t_0,T]$ where S exists.

<u>Proof</u>. From Lemma A.2 it follows that $S(t) \le R(t)$ where R is the solution to the linear differential equation.

$$-\dot{R} = A^TR + RA + Q \qquad R(T) = Q_0$$

From this lemma it follows that, to prove existence of $S(t)$ on some interval, all that is needed is a lower bound on S on that interval.

<u>Lemma A.5</u>. Let S be the solution of the Riccati equation

$$-\dot{S} = A^TS + SA + Q - SPS \qquad S(T) = Q_0$$

and assume that S exists on the interval $[t_0,T]$. Let $\tilde{S}$ be the solution to the Riccati equation where $\tilde{A}$, $\tilde{Q}$ and $\tilde{P}$ have replaced A, Q and P. Then there exists an $\varepsilon > 0$ such that $\tilde{S}$ also exists on $[t_0,T]$ if $\|\tilde{A} - A\| \leq \varepsilon$, $\|\tilde{Q} - Q\| \leq \varepsilon$ and $\|\tilde{P} - P\| \leq \varepsilon$.

Proof. Since the right hand side of the Riccati equation is a continuous function of S, A, Q and P the result follows from general results for nonlinear differential equations, see Coddington and Levinson (1955).