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DIRECTION OF ARRIVAL ESTIMATION IN THE PRESENCE OF CORRELATED SIGNALS AND ARRAY IMPERFECTIONS WITH UNIFORM CIRCULAR ARRAYS

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ABSTRACT

The Davies transformation is a method to transform the steering vector of a uniform circular array (UCA) to a vector with Vandermonde form. As this form is similar to that of the steering vector of a uniform linear array (ULA), we can apply to UCAs the many tools that have been developed for ULAs. However, the Davies transformation can be highly sensitive to perturbations of the underlying ideal array model. In this paper, we present a method for deriving a more robust transformation using novel optimization techniques. In particular, we consider its application to direction of arrival estimation in the presence of correlated signals. The effectiveness of the method is illustrated through a numerical example.

1. INTRODUCTION

By virtue of their geometry, uniform circular arrays (UCAs) are eminently suitable for applications such as radar, sonar and mobile wireless communications where one desires 360° of coverage in the azimuthal plane [1]. This innate advantage of UCAs is counterbalanced, however, by the uncooperative mathematical structure of their steering vectors. In particular, many important techniques that have been developed for uniform linear arrays (ULAs), such as Dolph-Chebyshev beampattern design [2], and spatial smoothing for direction of arrival estimation [3] and adaptive and optimum beamforming [4] in a correlated signal environment, cannot be applied directly to UCAs. In [5, 6], it is observed that the reason for this is because the aforesaid techniques exploit the Vandermonde structure of a ULA's steering vector while the steering vector of a UCA is not Vandermonde.

In [7], Davies proposes a method to transform the sensor element outputs of a UCA to derive the so-called *virtual array*. The key feature of the virtual array is that its steering vector is Vandermonde, or approximately so. In [5], we use the Davies transformation to design Dolph-Chebyshev beampatterns for UCAs, while in [1] and [6], it is used to enable, respectively, direction of arrival estimation and optimum beamforming for UCAs in a correlated signal environment.

Though attractive, the Davies transformation is not without

problems. Specifically, Davies [7] tacitly assumes that (i) all antenna elements have the same omnidirectional response, (ii) the electronics associated with each antenna element are identical, (iii) the antenna elements are located at their correct positions, and (iv) there is no mutual coupling between the antenna elements. Clearly, in a real system none of the above assumptions will hold. Although in [1], it is pointed out that these real-world effects can be ameliorated somewhat by calibration, there still remains the issue of residual calibration errors. In [8], it is shown through simulation that when errors are introduced into the model of an ideal UCA, as represented by a perturbation of its steering vector, the performance of the UCA can degrade appreciably. The criterion used in [8] to assess performance is the Dolph-Chebyshev beampattern obtained through the method of [5]. The aim of this paper is to find, through global optimization techniques [9, 10], an alternative transformation that has the desirable property of the Davies transformation, i.e., transforming the steering vector of a UCA to Vandermonde form, but is more robust with respect to perturbations to the steering vector of an ideal UCA.

2. PROBLEM FORMULATION

2.1. The Davies Transformation
Consider a UCA with N elements and radius r. The nth component of the N-dimensional array response (or *steering*) vector $\mathbf{a}(\theta)$, $n = 1, \ldots, N$, to a narrowband signal of wavelength λ arriving from angle θ, $\theta \in [-\pi, \pi]$, is given by

$$a_n(\theta) = G_n(\theta) \exp\left[j \frac{2\pi r}{\lambda} \cos\left(\theta - \frac{2\pi(n-1)}{N} \right) \right] \quad (1)$$

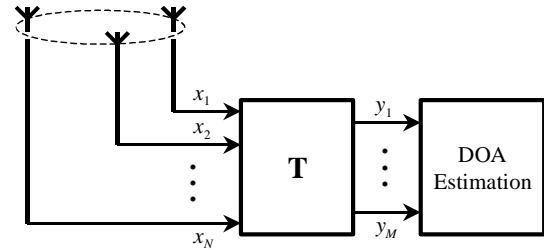where $G_n(\theta)$ is the complex gain pattern of the nth element.

Fig. 1. Transformation for UCAs

Suppose the array elements are all identical and isotropic, i.e., $G_n(\theta) = 1$, $n = 1, \ldots, N$. Suppose further the antenna element outputs are processed as shown in Fig. 1 where $x_1, \ldots, x_N$ represent the baseband complex output signals of the "real" array and $y_1, \ldots, y_M$, $M < N$, represent the baseband complex output signals of the virtual array. In [1], it is shown that *if* the transformation matrix $\mathbf{T}$ is defined by

$$\mathbf{T} = \mathbf{JF} \qquad (2)$$

where the matrices $\mathbf{J} \in \mathbb{C}^{M \times M}$ and $\mathbf{F} \in \mathbb{C}^{M \times N}$ are given by

$$\mathbf{J} = \mathrm{diag}\left\{\left[j^{m-1-H} \sqrt{N} J_{m-1-H}\left(\frac{2\pi r}{\lambda}\right)\right]^{-1}\right\} \qquad (3)$$

and

$$[\mathbf{F}]_{mn} = \tfrac{1}{\sqrt{N}} e^{j 2\pi(m-1-H)(n-1)/N}, \qquad (4)$$

and where $m = 1, \ldots, M$, $n = 1, \ldots, N$, $J_k(\cdot)$ denotes a kth order Bessel function of the first kind, and

$$H = (M-1)/2 \in \mathbb{Z}, \qquad (5)$$

then the M-dimensional steering vector of the virtual array will take on, approximately, the Vandermonde form

$$\tilde{\mathbf{b}}(\theta) = \mathbf{T}\mathbf{a}(\theta) \cong \left[e^{-jH\theta} \ \cdots \ 1 \ \cdots \ e^{+jH\theta}\right] \triangleq \mathbf{b}(\theta). \qquad (6)$$

Note, in view of Eq. (5), M is odd.

2.2. Robustness

In [8], it was shown that the Davies transformation can be highly susceptible to perturbations in $\mathbf{a}(\theta)$. Fig. 2 shows the MUSIC spectrum obtained from a UCA with radius 1.118λ, $N = 15$, and $M = 13$. In the signal scenario, there are 5 fully correlated signals, each with SNR of 10 dB, arriving from $-150°$, $-90°$, $0°$, $37°$ and $85°$. Forward-backward spatial smoothing [3] with 5 subarrays, each of 9 elements, is used to restore the rank of the covariance matrix. This is followed by a whitening procedure since the noise in the virtual array is not spatially white [1]. Fig. 2 also shows two realizations of the MUSIC spectrum when the gain and phase responses of the antenna elements and their locations are perturbed, and mutual coupling between adjacent antenna elements are introduced. All perturbations were drawn from a uniform random number generator. The limits for gain perturbation are ± 0.005 (relative to 1); for phase perturbation, $\pm 1°$; for radial perturbation, $\pm 0.005\lambda$; for angular position perturbation, $\pm 1°$; and for mutual coupling, $\pm 0.01 \pm j 0.01$ (relative to 1). Clearly, the MUSIC spectra of the perturbed (or non-ideal) array are unacceptable.

2.3. Problem Statement

The lack of robustness of the Davies transformation can be traced to the construction of $\mathbf{J}$, Eq. (3). As can be seen, for some choices of m, H, and r/λ, the magnitude of one or more of the diagonal elements of $\mathbf{J}$ can approach infinity as the corresponding value of $J_{m-1-H}(2\pi r/\lambda)$ approaches zero. Accordingly, the norm of $\mathbf{T}$ can become very large. But the square of the norm of $\mathbf{T}$ gives a measure of the noise amplification of the transformation matrix. Therefore, for a $\mathbf{T}$ with large norm, small perturbations in $\mathbf{a}(\theta)$ will translate to large perturbations in $\tilde{\mathbf{b}}(\theta)$.

Based on the above observation, we formulate the following semi-infinite optimization problem to find a more robust transformation matrix. The basic idea is to trade-off the approximation error in the transformation of $\mathbf{a}(\theta)$ to a vector with Vandermonde form, for robustness.
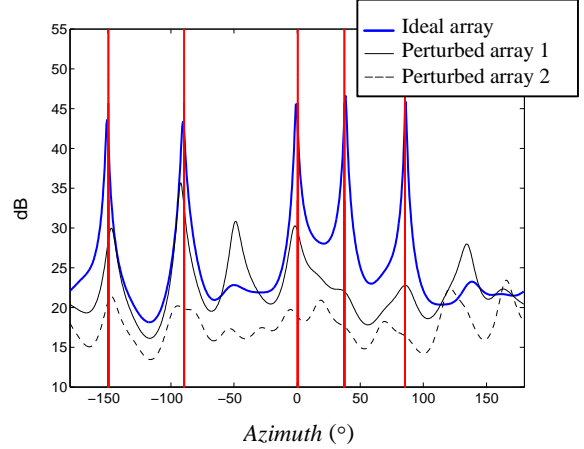

Fig. 2. MUSIC spectra of an ideal vs. non-ideal UCAs

Denote the robust transformation matrix by $\mathbf{U} \in \mathbb{C}^{M \times N}$. We find $\mathbf{U}$ as follows:

$$\min_{\mathbf{U}} \|\mathbf{U}\|_F^2 \qquad (\mathcal{P}1)$$

subject to $\quad |\mathbf{U}\mathbf{a}(\theta) - \mathbf{b}(\theta)| \leq \mathbf{e}$, $\quad \forall \theta \in [-\pi, \pi]$

where $\| \cdot \|_F$ denotes Frobenius norm, $|\cdot|$ is the absolute value norm

$$|\mathbf{U}\mathbf{a}(\theta) - \mathbf{b}(\theta)| =$$

$$\begin{bmatrix} \max\left\{ \left| \mathrm{Re}\left\{ \mathbf{u}_1^T \mathbf{a}(\theta) - b_1(\theta)\right\}\right|, \ \left| \mathrm{Im}\left\{ \mathbf{u}_1^T \mathbf{a}(\theta) - b_1(\theta)\right\}\right|\right\} \\ \vdots \\ \max\left\{ \left| \mathrm{Re}\left\{ \mathbf{u}_M^T \mathbf{a}(\theta) - b_M(\theta)\right\}\right|, \ \left| \mathrm{Im}\left\{ \mathbf{u}_M^T \mathbf{a}(\theta) - b_M(\theta)\right\}\right|\right\} \end{bmatrix}, \quad (7)$$

$$\mathbf{e} = \begin{bmatrix} \varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_M \end{bmatrix}^T, \ \varepsilon_m \in \mathbb{R}_+, \ m = 1, \ldots, M, \quad (8)$$

$\mathbf{u}_m^T \in \mathbb{C}^{1 \times N}$ is the mth row of $\mathbf{U}$ and $b_m(\theta)$ is the mth element of $\mathbf{b}(\theta)$.

Now, since the rows of $\mathbf{U}$ are not related in the above formulation, ($\mathcal{P}1$) can be solved, row-by-row, as follows:

For $m = 1, \ldots, M$, $\qquad \min_{\mathbf{u}_m} \|\mathbf{u}_m\|_F^2 \qquad (\mathcal{P}2)$

subject to $\quad \left| \mathrm{Re}\left\{ \mathbf{u}_m^T \mathbf{a}(\theta) - b_m(\theta)\right\}\right| \leq \varepsilon_m$

and $\quad \left| \mathrm{Im}\left\{ \mathbf{u}_m^T \mathbf{a}(\theta) - b_m(\theta)\right\}\right| \leq \varepsilon_m$, $\quad \forall \theta \in [-\pi, \pi]$.

The advantage of ($\mathcal{P}2$) is that it allows the original problem ($\mathcal{P}1$) to be solved efficiently.

2.4. Remarks

1. The robustness of $\mathbf{U}$ depends on the choice of ε_m, $m = 1, \ldots, M$. One method is to set ε_m to be some multiple of the corresponding value in $\mathbf{T}$ where the multiple is greater than 1.

2. If, for a given m, $\varepsilon_m \geq 1$, then for that m, ($\mathcal{P}2$) has the trivial solution $\mathbf{u}_m = 0$. This follows since $|b_m(\theta)| = 1$.

3. As a rule of thumb to robustness, the square of the norm of

each row of $\mathbf{U}$ should not greatly exceed N/M. The reasoning is as follows. Suppose the output signal from each antenna element contains a complex noise term whose real and imaginary parts are independent with identical variance σ_x^2. Also, suppose the noise terms of all the antenna elements are mutually independent. The total noise from the array of N elements is then given by $2N\sigma_x^2$. Suppose the transformation matrix has Frobenius norm $\|\mathbf{U}\|_F$. The total noise at the output of the virtual array is then given by $2\|\mathbf{U}\|_F^2\,\sigma_x^2$. If the transformation is required to not increase noise, then we require $2N\sigma_x^2 \geq 2\|\mathbf{U}\|_F^2\,\sigma_x^2$, or $\|\mathbf{U}\|_F^2 \leq N$. Finally, suppose the noise gain is distributed uniformly over the elements of the virtual array. We then get $\|\mathbf{u}_m\|_F^2 = \|\mathbf{U}\|_F^2\big/M \leq N/M$.

3. QUADRATIC SEMI-INFINITE PROGRAMMING

3.1. The Dual Parameterization Method

Consider the mth sub-problem of ($\mathcal{P}2$). Denote this sub-problem by ($\mathcal{P}_m$). Define the vector of decision variables

$$\mathbf{x} = \left[\mathrm{Re}\left\{\mathbf{u}_m^T\right\} \quad \mathrm{Im}\left\{\mathbf{u}_m^T\right\} \right]^T \in \mathbb{R}^{2N}. \qquad (9)$$

($\mathcal{P}_m$) can be written as a standard quadratic semi-infinite programming problem as follows.

$$\min_{\mathbf{x}} \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} \qquad\qquad (\mathcal{P}'_m)$$

subject to $\quad \mathbf{A}(\theta)\mathbf{x} - \mathbf{c}(\theta) \leq \mathbf{0}, \quad \forall \theta \in [-\pi, \pi]$

where $\qquad\qquad \mathbf{Q} = 2\mathbf{I}_{2N \times 2N}, \qquad\qquad (10)$

$$\mathbf{A}(\theta) = \begin{bmatrix} \mathrm{Re}\left\{\mathbf{a}^T(\theta)\right\} & -\mathrm{Im}\left\{\mathbf{a}^T(\theta)\right\} \\ -\mathrm{Re}\left\{\mathbf{a}^T(\theta)\right\} & \mathrm{Im}\left\{\mathbf{a}^T(\theta)\right\} \\ \mathrm{Im}\left\{\mathbf{a}^T(\theta)\right\} & \mathrm{Re}\left\{\mathbf{a}^T(\theta)\right\} \\ -\mathrm{Im}\left\{\mathbf{a}^T(\theta)\right\} & \mathrm{Re}\left\{\mathbf{a}^T(\theta)\right\} \end{bmatrix} \in \mathbb{R}^{4 \times 2N}, \quad (11)$$

and

$$\mathbf{c}(\theta) = \big[\varepsilon_m + \mathrm{Re}\left\{b_m(\theta)\right\} \quad \varepsilon_m - \mathrm{Re}\left\{b_m(\theta)\right\}$$
$$\varepsilon_m + \mathrm{Im}\left\{b_m(\theta)\right\} \quad \varepsilon_m - \mathrm{Im}\left\{b_m(\theta)\right\} \big]^T \in \mathbb{R}^4. \qquad (12)$$

We use the so-called *dual parameterization method* of [9, 10] to solve ($\mathcal{P}_m$). The parameterized dual problem of ($\mathcal{P}_m$) with k parameters is defined as follows.

$$\min_{\mathbf{x},\,\mathbf{y}_k,\,\mathbf{z}_k} \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} + \sum_{i=1}^{k}\mathbf{c}^T(\theta_i)\mathbf{y}_i \qquad (\mathcal{P}'_{m,k})$$

subject to $\quad \mathbf{Q}\mathbf{x} + \sum_{i=1}^{k}\mathbf{A}^T(\theta_i)\tilde{\mathbf{y}}_i = \mathbf{0}$,

$$\tilde{\mathbf{y}}_i \geq \mathbf{0},$$
$$\text{and} \quad 0 \leq \theta_i \leq 2\pi, \quad i = 1, \ldots, k,$$

where $\qquad \tilde{\mathbf{y}}_i = \left[\tilde{y}_{i,1} \quad \tilde{y}_{i,2} \quad \tilde{y}_{i,3} \quad \tilde{y}_{i,4} \right]^T$,

$$\mathbf{y}_k = \left[\tilde{\mathbf{y}}_1^T \quad \tilde{\mathbf{y}}_2^T \quad \cdots \quad \tilde{\mathbf{y}}_k^T \right]^T,$$

and $\qquad \mathbf{z}_k = \left[\theta_1 \quad \theta_2 \quad \cdots \quad \theta_k \right]^T$.

The main results relating ($\mathcal{P}_m$) and ($\mathcal{P}'_{m,k}$) are stated in the following theorem.

Theorem 1

(a) There exists a k^* satisfying $0 \leq k^* \leq 2N$ such that the optimal value sequence $\left\{V\left(\mathcal{P}'_{m,k}\right)\right\}_{k=1}^{k^*}$ is strictly decreasing, and for $k > k^*$, $V\left(\mathcal{P}'_{m,k}\right) = V\left(\mathcal{P}'_{m,k^*}\right)$.

(b) The number k^* in **(a)** is the smallest whole number such that for $k \geq k^*$, the global solution of ($\mathcal{P}'_{m,k}$) provides the solution of ($\mathcal{P}_m$) in the sense that, if $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{z}^*)$ is a solution of ($\mathcal{P}'_{m,k}$), then $\mathbf{x}^*$ is the solution of ($\mathcal{P}_m$).

Proof See [9].

3.2. The Algorithm

Based on Theorem 1, the following adaptive algorithm is developed in [10]. Define first the following problem.

$$\min_{\mathbf{x},\,\mathbf{y}_k} \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} + \sum_{i=1}^{k}\mathbf{c}^T(\theta_i)\tilde{\mathbf{y}}_i \qquad (\mathcal{P}'_{m,\mathbf{z}_k})$$

subject to $\quad \mathbf{Q}\mathbf{x} + \sum_{i=1}^{k}\mathbf{A}^T(\theta_i)\tilde{\mathbf{y}}_i = \mathbf{0}$ and $\tilde{\mathbf{y}}_i \geq \mathbf{0}$, $i = 1, \ldots, k$

where $\mathbf{z}_k = \left[\theta_1 \quad \theta_2 \quad \cdots \quad \theta_k \right]^T$ is a fixed vector.

Step 1 Choose any $\mathbf{x}_0 \in \mathbb{R}^{2N}$, a small number $\varepsilon > 0$, an integer I, an increasing sequence of integers $\{k_i\}$, and a sequence of parameterization sets $\Theta_i = \left\{\theta_{i,j}\,\big|\,\theta_{i,j} \in [0, 2\pi],\ j = 1, \ldots, k_i\right\}$ such that
$$d\left(\Theta_i, [0, 2\pi]\right) \triangleq \max_{\theta \in [0, 2\pi]} \min_{1 \leq j \leq k_i} \left|\theta - \theta_{i,j}\right| \to 0 \text{ as } i \to \infty. \quad (13)$$

Step 2 Let $E_0 = \varnothing$. Set $i = 0$.

Step 3 Set $i = i + 1$.
Find $\mathcal{G}_i = \left\{\theta \in \Theta_i\,\big|\,\mathbf{A}(\theta)\mathbf{x}_{i-1} - \mathbf{c}(\theta) \geq \mathbf{0}\right\} \cup E_{i-1}$.
Suppose $\mathcal{G}_i = \left\{\tilde{\theta}_{i,1}, \tilde{\theta}_{i,2}, \ldots, \tilde{\theta}_{i,m_i}\right\}$.
Define $\mathbf{z}_{m_i} = \left[\tilde{\theta}_{i,1} \quad \tilde{\theta}_{i,2} \quad \cdots \quad \tilde{\theta}_{i,m_i} \right]^T$.

Step 4 Solve problem ($\mathcal{P}'_{m,\mathbf{z}_{m_i}}$) to obtain an optimal solution $(\mathbf{x}_i, \mathbf{y}_i)$.

Step 5 If $i \leq I$ or $\frac{1}{2}\mathbf{x}_i^T\mathbf{Q}\mathbf{x}_i - \frac{1}{2}\mathbf{x}_{i-1}^T\mathbf{Q}\mathbf{x}_{i-1} \geq \varepsilon$,
find $E_i = \left\{\tilde{\theta} \in \mathcal{G}_i\,\big|\,\mathbf{A}(\tilde{\theta})\mathbf{x}_i - \mathbf{c}(\tilde{\theta}) = \mathbf{0}\right\}$.
Go to **Step 3**.

Step 6 Solve problem ($\mathcal{P}'_{m,m_i}$) starting from $\left(\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_{m_i}\right)$. Denote the solution by $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{z}^*)$. Take $\mathbf{x}^*$ to be the solution of problem ($\mathcal{P}_m$).

Theorem 2

Let the condition (13) be satisfied. Then, the sequence $\{\mathbf{x}_i\}$ obtained from the Algorithm will converge to the solution of problem ($\mathcal{P}_m$). Therefore, if ε and I are suitably chosen, the $\mathbf{x}^*$ obtained in Step 6 is the optimal solution of ($\mathcal{P}_m$).

Proof See [10].

4. NUMERICAL EXAMPLE

Consider the UCA of Fig. 2. Table 1 summarizes the squared-norm and maximum real and imaginary errors of each row of the Davies matrix for this UCA. As can be seen, the squared-norms of rows 6 and 8 greatly exceed $N/M = 15/13 = 1.1538$. Indeed, it is the very presence of these rows that render the Davies matrix non-robust.

For the robust transformation matrix, our strategy is to retain as many rows of the Davies matrix as possible except for rows with large squared-norms. Accordingly, we replace rows 6 and 8 with rows found by solving ($\mathcal{P}2$) with ε_m set to 0.7. The MUSIC spectra obtained from the robust transformation matrix are shown in Fig. 3. The perturbations on the ideal array are the same as those in Fig. 2. The characteristics of the robust transformation matrix are summarized also in Table 1. Note the increase in approximation error in rows 6 and 8 of the robust transformation matrix. Furthermore, we remark that the resulting transformation vectors for rows 6 and 8 are related by a phase rotation. This also holds true when the optimization is carried out for any other pairs of rows in Table 1. Therefore, we only need to compute ($\mathcal{P}2$) once for each such pairs of rows.

Row #	Davies Matrix		Robust Matrix	
	Squared-norm	Max Error	Squared-norm	Max Error
1, 13	0.574	0.1770		
2, 12	0.559	0.0701		
3, 11	2.923	0.0570		
4, 10	2.214	0.0159		
5, 9	0.745	0.0027		
6, 8	6513.343	0.0710	146.15	0.7079
7	0.740	0		

Table 1. Characteristics of the Davies and robust transformation matrices for $N = 15$, $M = 13$ and $r = 1.118\lambda$
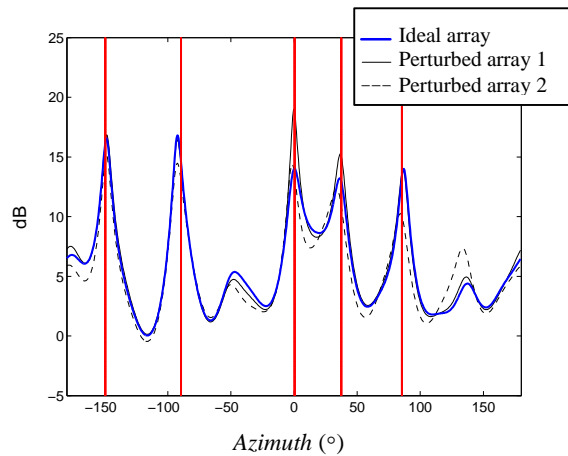

Fig. 3. MUSIC spectra of an ideal and non-ideal UCAs with robust transformation matrix

5. CONCLUSIONS

In this paper, we addressed the important problem of finding a transformation matrix to transform the steering vector of a uniform circular array to one with Vandermonde form, subject to a robustness requirement as demanded by practical considerations. The robust transformation matrix is found by posing and solving a quadratic semi-infinite optimization problem. We showed that, by an appropriate formulation, we can decompose the problem into a set of much simpler optimization problems which can then be solved efficiently using the dual parameterization method of [9, 10]. Each sub-problem yields a row of the robust transformation matrix. The robustness of the new transformation matrix is demonstrated by a simulation example. The simulation example also supports our hypothesis that robustness can be gained by sacrificing the approximation accuracy of the steering vector of the virtual array from its desired Vandermonde form.

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