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A DOLPH-CHEBYSHEV APPROACH TO THE SYNTHESIS OF ARRAY PATTERNS FOR UNIFORM CIRCULAR ARRAYS

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ABSTRACT

Uniform circular arrays (UCAs) are naturally suited to provide 360 degrees of coverage in the azimuthal plane. In this paper, we describe a new approach for synthesizing array patterns with guaranteed maximum sidelobe levels for any look direction. The advantage of this approach is that it is computationally efficient which makes it eminently suitable for real-time beamforming and beamsteering applications. The approach is based on the Dolph-Chebyshev method for uniform linear arrays except here, the Dolph-Chebyshev method is applied to the transformed array response vector of the UCAs.

1. INTRODUCTION

In many scenarios for antenna array systems, such as radar, sonar and wireless communications, one desires all-azimuth angle, i.e. 360°, coverage [1]. One method to achieve this is to employ UCAs. We consider here the problem of beamforming with UCAs. In particular, the problem of synthesizing an array pattern whose main lobe is as narrow as possible and whose sidelobes have a guaranteed maximum [2].

One of the standard approaches to the above design problem is the Dolph-Chebyshev approach [3-7]. But this approach applies only to uniform linear arrays (ULAs). In the case of UCAs, methods suited for arbitrary arrays are used to produce the minimax response. These include optimization, iterative weighted least squares and adaptive array approaches (see [8] and the references therein).

In this paper, we show how the Dolph-Chebyshev approach can be adapted to synthesize the desired minimax patterns for UCAs. The new approach employs a transformation technique, first proposed by Davies [9]. Davies' work is further extended by more recent research efforts [10-13]. The technique is basically a pre-processing procedure that transforms the array element space to a mode space, sometimes called spatial harmonics. The result is a *virtual array* in which the spatial response is similar in form (Vandermonde structure) to that of a ULA. This allows the use of techniques such as spatial smoothing previously limited to ULAs to perform high-resolution direction-of-arrival estimation in a coherent signal environment [1].

We show here that the virtual array concept can be used to synthesize Dolph-Chebyshev-like array patterns for UCAs. The advantages of this synthesis technique are as follows:

(1) No complex calculations are necessary for different look angles once the design weights are found. This translates to computational savings when compared to other methods such as the iterative adaptive array approach[1] of [8] where the array weights need to be recalculated for different look angles. The new approach also maintains the same array pattern for all look directions of the main lobe.

(2) The approach in [8] breaks down in cases where the constraint matrices and/or the interference signal covariance matrices are ill conditioned. Even though remedial procedures are available, they require human interventions. In contrast, our approach does not suffer from these problems.

(3) The modal-transformed-data can be used in other applications, such as direction finding of coherent signals with UCAs [1]. This suggests the sharing of computational load.

(4) The approach allows non-isotropic element patterns as it is able to remove the effects of the known element patterns from the virtual array [12,13]. It is also simple to incorporate mutual coupling effects into the formulation, as in [1].

Note, however, that the transformation approach is limited by the accuracy of the approximation involved. This depends on the UCA parameters and the signal scenario, and necessitates some compromises in terms of the size of the virtual array. Hence, it may not yield the narrowest possible main lobe for a given maximum sidelobe level.

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[1] We choose the method of [8] for comparison, as it appears to be the most promising of recent methods in performance and computational complexity.

2. PROPOSED METHOD

For a UCA of N elements and radius r, the ith component of the array response (or steering) vector $\mathbf{a}(\theta)$ for a narrowband signal of wavelength λ arriving at angle θ is given by

$$\left[a(\theta)\right]_i = \alpha_i(\theta) e^{jkr\cos\left(\theta - \frac{2\pi(i-1)}{N}\right)}, i \in \{1,2,\ldots,N\} \qquad (1)$$

where $k = 2\pi/\lambda$, $\alpha_i(\theta)$ is the complex gain pattern of the ith element and $\theta \in [-\pi, \pi]$.

Suppose the array elements are isotropic[2], i.e. $\alpha_i(\theta) = 1$, $\forall i \in \{1,2,\ldots,N\}$. In [1], Wax showed that if the sensor outputs are transformed by the matrix $\mathbf{JF}$ as illustrated in Fig. 1 where the matrices $\mathbf{J}$ and $\mathbf{F}$ are defined as follows:

$$\mathbf{F} = \frac{1}{\sqrt{N}}\begin{bmatrix} 1 & \omega^{-h} & \omega^{-2h} & \cdots & \omega^{-(N-1)h} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \omega^{-1} & \omega^{-2} & \cdots & \omega^{-(N-1)} \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{1} & \omega^{2} & \cdots & \omega^{(N-1)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \omega^{h} & \omega^{2h} & \cdots & \omega^{(N-1)h} \end{bmatrix} \qquad (2)$$

$$\mathbf{J} = \mathrm{diag}\left\{\left(j^m\sqrt{N}J_m(kr)\right)^{-1}\right\}, m = -h,\ldots,0,\ldots,h. \qquad (3)$$

where $\omega = e^{j2\pi/N}$, the size of the virtual array is $N_v = 2h + 1$, and $J_m(\cdot)$ is the Bessel function of the first kind of order m, then the array response vector of the resultant virtual array will take approximately the Vandermonde form

$$\mathbf{a}_v(\theta) = \mathbf{JFa}(\theta) \approx \left[e^{-jh\theta},\ldots,1,\ldots,e^{jh\theta}\right]^T. \qquad (4)$$
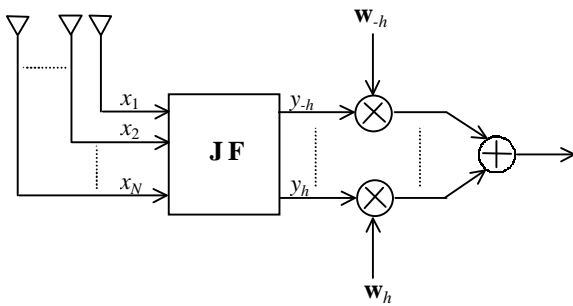
Figure 1. Modal transformation for uniform circular arrays

Appropriate choice of h has been discussed extensively in [1,11] and so is not repeated here.

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[2] We can easily extend the following derivations to the general case as in [12,13].

Consider now the output signals of the virtual array, $y_{-h}, \ldots, y_h$. To steer the look direction of the array to θ_l, one method is to set the weight vector of the virtual array to the form

$$\mathbf{w} = \mathbf{Da}_v(\theta_l), \qquad (5)$$

where $\mathbf{D}$ is a diagonal matrix that specifies the shading that is applied to the array pattern. Thus, the key design task is to find the matrix $\mathbf{D}$.

Suppose $\mathbf{D}$ has the form

$$\mathbf{D} = \mathrm{diag}(I_{-h},\ldots,I_{-1},2I_0,I_1,\ldots,I_h) \qquad (6)$$

The array response is given by

$$\mathbf{w}^H\mathbf{a}_v(\theta) = 2I_0 + \sum_{p=1}^{h} I_{-p}e^{-jp(\theta_l - \theta)} + \sum_{p=1}^{h} I_p e^{jp(\theta_l - \theta)}. \qquad (7)$$

Hence, if the elements of $\mathbf{D}$ have mirror symmetry, i.e. $I_{-p} = I_p$, then

$$\mathbf{w}^H\mathbf{a}_v(\theta) = 2\sum_{p=0}^{h} I_p \cos\left[2pu(\theta)\right] \qquad (8)$$

where

$$u(\theta) = (\theta_l - \theta)/2. \qquad (9)$$

However, we can write $\cos\left[2pu(\theta)\right]$ as a polynomial in $x(\theta)$, as follows [3]

$$\cos[2pu(\theta)] = \sum_{q=0}^{p} b_{2q}^{2p} x^{2q}(\theta) \qquad (10)$$

where

$$x(\theta) = \cos\left[u(\theta)\right], \qquad (11)$$

and

$$b_{2q}^{2p} = (-1)^{p-q}\sum_{s=p-q}^{p}\binom{s}{s-p+q}\binom{2p}{2s}. \qquad (12)$$

Thus, substituting (10) into (8), we can rewrite (8) as

$$\mathbf{w}^H\mathbf{a}_v(\theta) = 2\sum_{q=0}^{h}\sum_{p=q}^{h} I_p b_{2q}^{2p} x^{2q}(\theta). \qquad (13)$$

Now, we wish to express (13) as a Chebyshev polynomial with a given β where β is the ratio of the main lobe level to the sidelobe level of the Dolph-Chebyshev pattern. A Chebyshev polynomial of degree $2h$ in z with all its roots in the range $-1 \le z \le 1$ has the following form

$$T_{2h}(z) = \sum_{q=0}^{h} b_{2q}^{2h} z^{2q}. \qquad (14)$$

Accordingly, defining

$$z = z_0 x(\theta), -z_0 \le z \le z_0 \qquad (15)$$

where z_0 is defined by $T_{2h}(z_0) = \beta > 1$ and $z_0 > 0$, we obtain after substituting (15) into (14) and then equating the resulting polynomial in $x(\theta)$ with (13)

$$2\sum_{p=q}^{h} I_p b_{2q}^{2p} = b_{2p}^{2h} z_0^{2q}. \qquad (16)$$

The coefficients I_p (of matrix $\mathbf{D}$) can now be found from (16) using one of the many methods described, for example, in [3-6].

Finally, it follows from (13) to (16) that

$$\mathbf{w}^H \mathbf{a}_v(\theta) = T_{2h}\left[z_0 x(\theta) \right] \qquad (17)$$

and the array pattern is given by

$$\left| \mathbf{w}^H \mathbf{a}_v(\theta) \right|^2 = \left| \mathbf{a}_v^H(\theta_l) \mathbf{D} \mathbf{a}_v(\theta) \right|^2 = \left| T_{2h}[z_0 x(\theta)] \right|^2 . \qquad (18)$$

Note that (18) is in the same form as the Dolph-Chebyshev pattern of a ULA involving Chebyshev polynomials.

We next show that as a result of the linear dependence of $u(\theta)$ on θ_l [see (9)], the half power main lobe width [3,4] will remain the same regardless of θ_l.

The maximum amplitude of the main lobe is given by

$$T_{2h}(z_0) = \beta = \cosh\left(2h \cosh^{-1}(z_0) \right) \qquad (19)$$

or $z_0 = \cosh\left(\dfrac{1}{2h} \cosh^{-1}(\beta) \right).$ $\qquad (20)$

At the half power points,

$$T_{2h}(z_1) = \frac{\beta}{\sqrt{2}} \Rightarrow z_1 = \cosh\left(\frac{1}{2h} \cosh^{-1}(\beta/\sqrt{2}) \right). \qquad (21)$$

Now,

$$z_1 = z_0 \cos u_1 \qquad (22)$$

$$u_1 = u(\theta_1) = (\theta_l - \theta_1)/2 . \qquad (23)$$

This means that

$$\theta_1 = \pm \cos^{-1} \frac{z_1}{z_0} + \theta_l . \qquad (24)$$

Denoting θ_1^a, θ_1^b as the half power points, the half power main lobe width is

$$\left| \theta_1^a - \theta_1^b \right| = 2 \cos^{-1} \frac{z_1}{z_0} \qquad (25)$$

which does not depend on θ_l.

It is easy to extend the above result to show that the array pattern over the entire range of azimuth angles also remains the same (albeit by a shift of θ_l and wrapping by 2π) regardless of θ_l. This is not surprising since the approximation involved in the modal transformation requires $N \gg kr$ [1]. This translates to the need of having "enough" UCA elements so that the behavior between the elements closely approximates that of a continuous aperture circular array where there is no change in array pattern over all look angles.

In contrast, for a ULA, even though the spatial response is also in the Vandermonde form, the main lobe width and the array pattern change with different look angles. This is because in the ULA case,

$$u(\theta) = \frac{\pi d}{\lambda}\left(\sin\theta_l - \sin\theta \right) \qquad (26)$$

which is non-linear in θ_l. [4,7].

3. EXAMPLES

In the first example, we considered a circular array of 35 elements with $d/\lambda = 0.3$ and $\beta = 100$ (i.e. 40dB peak sidelobe attenuation). We used the criterion given in [9] to choose the size of virtual array $N_v = 2h + 1$, i.e.

$$\max\left\{ h \;\middle|\; h \le \frac{N-1}{2} \text{ and } \frac{J_{h-N}(kr)}{J_h(kr)} < \varepsilon \right\}$$

and choose $\varepsilon = 0.05$. This gives $N_v = 33$. We also used Stegen's formula [4] to calculate the array weights. Fig. 2 compares the array pattern obtained using our method against that obtained using the method of [8] for $\theta_l = 30°$. We see that the widths of the 2 main lobes are very similar. Here the number of virtual array elements is close to that of the real array.
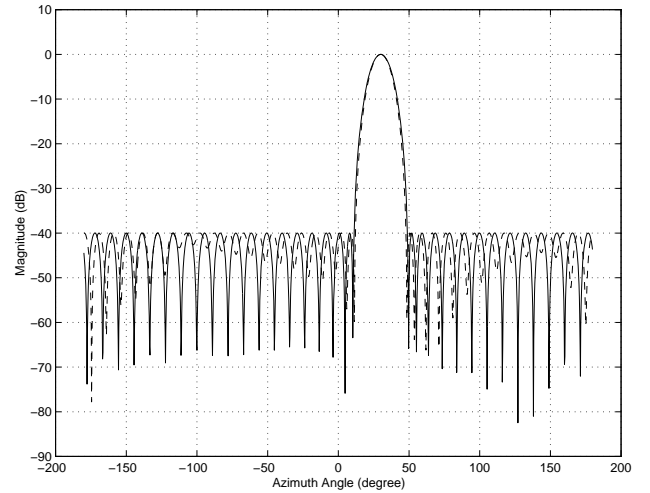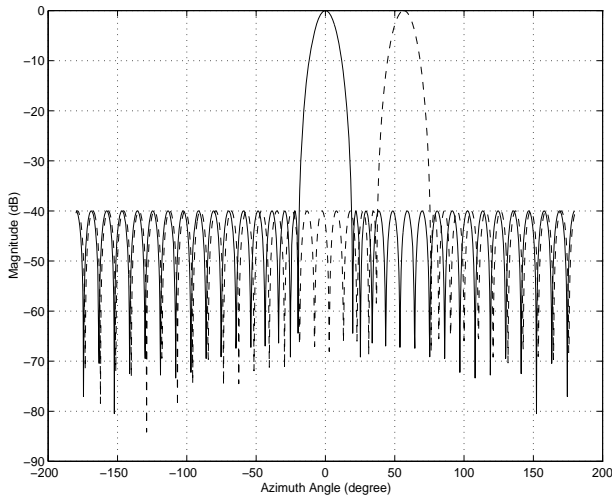

Figure 2. The solid line (——) is for our method, the dashed line (- -) for the method of [8].

In the second example, we compared the array patterns obtained using our method, for the same array as in example 1, for $\theta_l = 0°$ and 56.6°. As can be seen from Fig. 3, the two array patterns are essentially the same, except for a shift.

In the third example, we increased d/λ to 0.6 which gives more peaks than those that can be directly controlled by the method of [8]. As a result, the iterative algorithm suffers from look-angle-dependency and a much longer convergence time. Fig. 4 compares the array patterns obtained with our method ($N_v = 17$) against that using the method of [8] for $\theta_l = 30°$.

4. CONCLUSIONS

The transformation technique as presented in this paper has distinct advantages over the existing method of [8]. Aside from its computational efficiency when different look directions are desired (**D** is fixed, i.e. independent of θ_l), it represents a simple approach to designing minimax array patterns making use of the Vandermonde structure and the Chebyshev polynomials. However, this approach is limited by the proper choice of the virtual array size to ensure a good approximation in the modal transformation.

Figure 3: The solid line (——) is for $\theta_l = 0 \times 360°/N = 0°$ and the dashed line (- -) for $\theta_l = 5.5 \times 360°/N = 56.6°$.
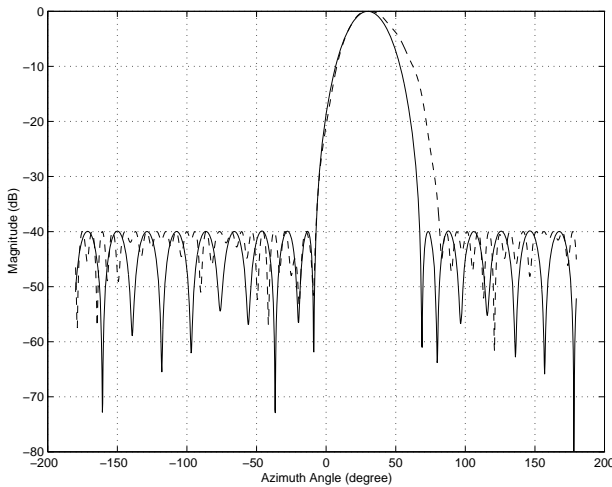
Figure 4. The solid line (——) is for our method, the dashed line (- -) for the method of [8].

5. REFERENCES

[1] M. Wax and J. Sheinvald, "Direction finding of coherent signals via spatial smoothing for uniform circular arrays," *IEEE Trans. Antennas Propagat.*, 42(5), pp. 613-620, 1994.

[2] B. D. van Veen and K. M. Buckley, "Beamforming: a versatile approach to spatial filtering*," IEEE ASSP Magazine*, 5(2), pp. 4-24, 1988.

[3] C. L. Dolph, "A current distribution for broadside arrays which optimizes the relationship between beam width and sidelobe level," *Proc. IRE*, 34(6), pp. 335-348, 1946.

[4] R. J. Stegen, "Excitation coefficients and beam-widths of Tschebyscheff Arrays," *Proc. IRE*, 40(11), pp. 1671-1674, 1953.

[5] T. N. Davidson, "A note on the calculation of the Dolph-Chebyshev shading for a linear array," *ASPL-1991-3*, Dept. Electrical and Electronic Eng, the University of Western Australia, Aug. 1991.

[6] A. Safaai-Jazi, "A new formulation for the design of Chebyshev arrays," *IEEE Trans. Antennas Propagat.*, 42(3), pp. 439-443, 1994.

[7] B. K. Lau, Y. H. Leung, "Analysis of Dolph-Chebyshev patterns for uniform linear arrays," *SPL-TR-013*, ATRI, Curtin University of Technology, Sep. 1999.

[8] C. Y. Tseng and L. J. Griffiths, "A simple algorithm to achieve desired patterns for arbitrary arrays," *IEEE Trans. Signal Processing*, 40(11), pp. 2737-2746, 1992.

[9] D. E. N. Davies, "A transformation between the phasing techniques required for linear and circular aerial arrays," *Proc. IEE*, 112(11), pp. 2041-2045, 1965.

[10] A. H. Tewfik and W. Hong, "On the application of uniform linear array bearing estimation techniques to uniform circular array," *IEEE Trans. Signal Processing*, 40(4), pp. 1008-1011, 1992.

[11] C. P. Mathews, M. D. Zoltowski, "Eigenstructure techniques for 2-D angle estimation with uniform circular arrays," *IEEE Trans. Signal Processing*, 42(9), pp. 2395-2407, 1994.

[12] R. Eiges and H. D. Griffiths, "Mode-space spatial spectral estimation for circular arrays," *IEE Proc. – Radar, Sonar Navig.*, 141(12), pp. 300-306, 1994.

[13] K. M. Reddy and V. U. Reddy, "Analysis of spatial smoothing with uniform circular arrays," *IEEE Trans. Signal Processing*, 47(6), pp. 1726-1730, 1999.