Design of low order approximately linear phase IIR filters

Lau, Buon Kiong; Sreeram, Victor; Leung, Yee Hong; Teo, Kok Lay

Published in:
IEEE Symposium on Advances in Digital Filtering and Signal Processing (DFSP), 1998

DOI:
10.1109/ADFSP.1998.685702

1998

Document Version:
Peer reviewed version (aka post-print)

DESIGN OF LOW ORDER APPROXIMATELY LINEAR PHASE IIR FILTERS

B.K. Lau and V. Sreeram

Department of Electrical
and Electronic Engineering
University of Western Australia
Nedlands, WA 6907, Australia.
e-mail: sreeram@ee.uwa.edu.au

Y.H. Leung and K.L. Teo

Australian Telecommunications
Research Institute
Curtin University of Technology
Bently, WA 6102, Australia.
e-mail: leung@atri.curtin.edu.au

ABSTRACT

In this paper, a new method for the design of approximately linear phase IIR filters is proposed. This method is based on first designing an asymmetrical FIR filter which is approximately linear phase and then deriving a low order approximately linear phase IIR filter by model reduction techniques.

1. INTRODUCTION

Linear phase digital filters are important in the distortion free transmission of signals. This means that all signal components in the frequency domain are delayed by the same amount by the filtering system. This requirement can be easily satisfied by FIR filters whereas IIR filters can at best achieve approximately linear phase response. Although FIR filters are superior to IIR filters in this respect, FIR filters tend to be of much higher order than IIR filters satisfying the same magnitude characteristics, especially for the case of highly frequency selective filters. Highly selective filters are characterised by narrow transition regions or equivalently, sharp rolloffs between passband and stopband regions. In order to take advantage of this characteristic of IIR filters, many approaches have been proposed to design low order IIR filters with approximately linear phase in the passband.

One earlier approach is known as the equalization technique. This involves first designing an IIR filter that meets the magnitude specifications. Optimization techniques are then applied to design allpass equalizer stages connected in cascade with the IIR filter in order to correct the phase response [3, 2]. Another approach is to design IIR filters directly using optimization methods to meet both the magnitude and phase specifications simultaneously [4]. More recent approach makes use of model reduction techniques from control theory to design such filters [9, 6]. In this approach, a linear phase FIR filter which satisfies the magnitude specifications is first designed. This filter is then reduced using model reduction techniques to obtain a low order IIR filter which meets the original magnitude response specifications while maintaining an approximately linear phase characteristic in the passband. This technique gives better results than the equalization approach in terms of the filter order. The order reduction can improved slightly further by using frequency weighted model reduction techniques [8] as shown in [6]. Signficant order reduction is very to difficult to achieve for original FIR filters with symmetrical impulse response characteristics. This is because the energy in the impulse response of such filters is concentrated away from the discrete time instant $k = 0$. Model reduction techniques works well when the energy of the impulse response of the original system is close to $k = 0$. This is the case for systems we come across in control systems. Significant order reduction can be achieved if the original FIR filter has asymmetrical filter coefficients with most of its energy concentrated near the time instant $k = 0$. In this paper we propose a method for designing such filters and we then use model reduction techniques to obtain low order IIR filters which satisfy the magnitude specifications while maintaining approximately linear phase in the passband.

2. PRELIMINARIES

Consider a stable single-input, single-output (SISO) discrete system having the following minimal realization

$$\begin{aligned} x(k+1) &= Ax(k) + bu(k) \\ y(k) &= cx(k) + du(k) \end{aligned}$$

where $x(k) \in R^n$.

This work was supported by ARC Large Grants

An FIR filter with the following transfer function

$$H(z) = \sum_{i=0}^{N} h_i z^i$$

can be represented by the following state-space model (known as controllability canonical form [7]):

$$\hat{A} = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}$$

$$\hat{b} = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}^T$$

$$\hat{c} = \begin{bmatrix} h_1 & h_2 & \dots & h_N \end{bmatrix}$$

$$\hat{d} = h_0$$

Definition 1: The impulse-response Gramian for a stable discrete SISO system is given [9] by

$$P = \sum_{k=0}^{\infty} \begin{bmatrix} h_{k+1}^2 & \dots & h_{k+1}h_{k+N} \\ \vdots & \vdots & \vdots \\ h_{k+1}h_{k+N} & \dots & h_{k+N}^2 \end{bmatrix}$$

where

$$h_k = cA^{k-1}b$$

is the impulse response of the system. This Gramian can be computed easily by solving the following Lyapunov equation:

$$P - \hat{A}^T P \hat{A} = \hat{c}^T \hat{c}$$

where $\{\hat{A}, \hat{b}, \hat{c}\}$ is in controllability canonical form [7] given by

$$\begin{aligned} \hat{A} &= T^{-1}AT \\ \hat{b} &= T^{-1}b \\ \hat{c} &= cT \end{aligned}$$

and T is the controllability matrix given by

$$T = [b \; Ab \; \dots A^{n-1}b]$$

Given the original system $\{A, b, c, d\}$ the reduced order model $\{A_r, b_r, c_r, d_r\}$ can be obtained using the following steps:

Step 1: Compute the impulse-response Gramian, P, for the system $\{A, b, c, d\}$

Step 2: Compute the orthogonal matrix L which diagonalizes P.

$$L^T P L = \Sigma = \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \sigma_n \end{bmatrix}$$

where

$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n$$

are the eigenvalues of P.

Step 3: Transform and partition the state matrices

$$A_d = L^{-1}\hat{A}L = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$b_d = L^{-1}\hat{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$c_d = \hat{c}L = \begin{bmatrix} c_1 & c_2 \end{bmatrix}$$

to obtain the reduced-order model, $\{A_r, b_r, c_r, d_r\}$ with

$$A_r = A_{11}, \quad b_r = b_1, \quad \text{and} \quad c_r = c_1, \quad d_r = d$$

3. THE MAIN RESULTS

In this section, we propose a technique for designing asymmetrical FIR filters which satisfies the magnitude specification while maintaining approximately linear phase in the passband. As explained in the Introduction, the motivation for designing an asymmetrical FIR filter is to obtain significant order reduction in the model reduction process. Once the asymmetrical filter is designed, a low order IIR filter with approximately linear phase in the passband is obtained using the model reduction technique [9] given in Section 2.

3.1. Asymmetrical FIR filter Design

The design of an asymmetrical FIR filter begins with the design of a symmetrical FIR filter satisfying the given specifications. Many design techniques such as window techniques, optimization techniques are available for the design of symmetrical FIR filter.

Using the symmetrical filter, the asymmetrical filter is obtained with the shift, truncate, and zero pad technique. The shifting process involves shifting the impulse response coefficients of the filter to the left so that the filter becomes anticausal. The truncation process involves truncating the anticausal part of the filter. The zero pad process involves padding the filter to the right by zeros so that the order of the filter is restored to the original order. To explain the above procedure let us suppose the original filter of order $N+1$ is described by

$$H(z) = \begin{bmatrix} h_0 & h_1 & \dots & h_N \end{bmatrix} \begin{bmatrix} 1 \\ z^{-1} \\ \vdots \\ z^{-N} \end{bmatrix}$$

Shifting by k places yields the following anticausal filter

$$H_s(z) = \begin{bmatrix} h_0 & h_1 & \dots & h_N \end{bmatrix} \begin{bmatrix} z^k \\ z^{k-1} \\ \vdots \\ 1 \\ z^{-1} \\ \vdots \\ z^{-N+k} \end{bmatrix}$$

The truncation process yields the following causal filter of order $N-k$:

$$H_{st}(z) = \begin{bmatrix} h_k & h_{k+1} & \dots & h_N \end{bmatrix} \begin{bmatrix} 1 \\ z^{-1} \\ \vdots \\ z^{-N+k} \end{bmatrix}$$

Finally, after the zero pad, the order of the filter is restored to $N+1$ and the filter is now given by

$$H_{stzp}(z) = \begin{bmatrix} h_k & \dots & h_N & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} 1 \\ z^{-1} \\ \vdots \\ z^{-N} \end{bmatrix}$$

It can be easily shown that the asymmetrical filter, $H_{stzp}(z)$ obtained by the above process has approximately linear phase characteristics in the passband if the coefficients corresponding to the first two main lobes of the sinc function are retained. This asymmetrical filter generally does not satisfy the magnitude specifications of the original FIR filter. Using this filter as the initial guess of an optimization procedure, the final asymmetrical filter which satisfies the magnitude constraints and has approximately linear phase characteristics in the passband can be obtained.

The objective of the optimization procedure is to find the asymmetrical filter coefficients h by minimizing the performance criteria given by

$$\begin{aligned} J(h) &= \int_{\Omega_p} \left[g_\epsilon \left([M(h,\omega)]^2 - (1+\delta_1)^2 \right) \right. \\ &\quad \left. + g_\epsilon \left((1-\delta_1)^2 - [M(h,\omega)]^2 \right) \right] d\omega \\ &\quad \int_{\Omega_a} g_\epsilon \left([M(h,\omega)]^2 - \delta_2^2 \right) d\omega \end{aligned}$$

where Ω_p and Ω_a are the passband and stopband regions respectively, $g_\epsilon(.)$ is a smoothing function:

$$g_\epsilon = \begin{cases} 0, & t < \epsilon \\ \frac{t-\epsilon}{4\epsilon}, & -\epsilon \le t \le \epsilon \\ t, & t > \epsilon \end{cases}$$

and $M(h,\omega)$ is the magnitude response of the filter given by:

$$M(h,\omega) = \left[\sum_{l=0}^{N} \sum_{m=0}^{N} h_l h_m cos(l-m)\omega T \right]^{\frac{1}{2}}$$

The constants δ_1 and δ_2 are given by

$$\begin{aligned} \delta_1 &= \frac{10^{0.05A_p} - 1}{10^{0.05Ap} + 1} \\ \delta_2 &= 10^{-0.05A_a} \end{aligned}$$

where A_p and A_a are the maximum passband ripple and minimum stopband attenuation respectively. The algorithm proposed in [5] is used here for the minimization. At each iteration $J(h)$ is checked against the satisfaction of the following magnitude constraints (over a fine grid of the regions of interest):

$$\begin{aligned} [M(h,\omega)] - (1+\delta_1) \le 0, &\qquad \omega \in \Omega_p \\ (1-\delta_1) - [M(h,\omega)] \le 0, &\qquad \omega \in \Omega_p \\ [M(h,\omega)] - \delta_2 \le 0, &\qquad \omega \in \Omega_a \end{aligned}$$

There are cases where this optimization procedure finds local solutions which do not satisfy the magnitude constraints. In all such cases, the violations are near or at the stopband edges. In such situations, different local solutions which satisfy the constraints must be sought. Techniques that are used to to do this are: (i) include extra terms together with appropriate weightings at the stopband edges or peak of side lobes to force the satisfaction of constraints, (ii) include extra integration terms together with appropriate weightings where violations occur, (iii) random perturbation of the solution by adding on to the existing solution a vector of generated random numbers (scaled appropriately) and optimize again with this as the new starting guess in order to find another local solution which hopefully would satisfy the constraints, and (iv) additional zeros may be added to the right of the existing filter coefficients and the optimization procedure is repeated. The addition of zeros will allow the magnitude constraints to be satisfied more easily since there are now more coefficients to achieve the same purpose.

4. EXAMPLE

Specifications for a lowpass filter are given in the following Table:

A_p (dB)	0.1
A_a (dB)	40
ω_p (rad/s)	0.3
ω_a (rad/s)	0.4

In the table the frequencies have been normalized with respect to sampling frequency, i.e., $0 \text{ rad/s} \leq \omega \leq 1 \text{ rad/s}$ and sampling frequency is taken to be 2.0 rad/s. In the first step a 54th order linear-phase FIR filter was designed to meet these specifications using the Kaiser window technique [1]. This symmetrical filter was reduced directly using the model reduction technique of Section 2. The lowest order IIR filter to satisfy the filter magnitude specifications, we could obtain was 16. However, using a 54th order asymmetrical filter designed using the proposed technique, we were able to obtain a 10th order IIR filter which satisfied the filter magnitude specifications. Therefore the order reduction obtained using asymmetrical FIR filters is considerably higher than the order reduction obtained using symmetrical FIR filters.

5. CONCLUSION

In this paper, a technique for designing low order approximately linear phase IIR filters is proposed. This technique is based on first designing an approximately linear phase asymmetrical FIR filter which satisfies the magnitude specifications and then obtaining the low order IIR filter using model reduction techniques. Numerical studies indicate that the order of the IIR filters obtained using asymmetrical FIR filters (as proposed in the paper) is considerably lower than those obtained using symmetrical FIR filters.

6. REFERENCES

[1] A. Antoniou, " Digital Filters - Analysis, Design and Applications", Second Edition, McGraw-Hill, U.S.A., 1993.

[2] A. Antoniou, and R. Howell, "Design of phase equalizers for recursive filters", Proceedings of 1993 IEEE Pacific Rim Conference, Victoria, Canada, pp. 104-107, 1993.

[3] C. Charalambous, and A. Antoniou, "Equalization of Recursive Digital Filters", Proceedings of IEE, Part-G, Vol. 127, pp. 219-225, 1980.

[4] G. Cortelazzo, and M. R. Lightner, "Simultaneous Design in Both Magnitude and Group Delay of IIR and FIR filters Based on Multiple Criterion Optimization", IEEE Transactions on Signal Processing, pp. 61-64, 1993.

[5] C.J. Goh, and K.L. Teo, " Alternative Algorithms for Solving Nonlinear Function and Functional inequalities", Applied mathematics and Computation, Vol. 41, pp. 159-177, 1991.

[6] S. Holford, and P. Agathoklis, "The Use of Model Reduction Techniques for Designing IIR Filters with Linear Phase in the Passband", IEEE Transactions on Signal Processing, Vol. 44, pp. 2396-2404, 1996.

[7] T. Kailath. "Linear Systems", Prentice-Hall Inc. Englewood Cliffs., N.J., 1980.

[8] C.-A. Lin, and T.-Y. Chiu, " Model Reduction via Frequency Weighted Balanced Realization", Control-Theory and Advanced Technology, Vol.8, 1992, pp. 341-451.

[9] V. Sreeram, and P. Agathoklis, "Design of Linear-Phase IIR Filters via Impulse-Response Gramians", IEEE Transactions of Signal Processing, Vol. 40, pp. 389-394, 1992.