A Comparative Study of Recursive Identification Methods

Söderström, Torsten; Ljung, Lennart; Gustavsson, Ivar

1974

Document Version:
Publisher's PDF, also known as Version of record

REPORT 7427
DECEMBER 1974

A Comparative Study Of Recursive Identification Methods

T. Söderström
L. Ljung
I. Gustavsson

Lund Institute of Technology
Department of Automatic Control

A COMPARATIVE STUDY OF RECURSIVE IDENTIFICATION METHODS

T. Söderström, L. Ljung and I. Gustavsson

ABSTRACT.

Recursive identification methods are of great interest in several contexts, e.g. for construction of self-tuning regulators and other adaptive controllers.

The usual way of comparing and analysing different recursive identification methods is to use simulations. Making use of recently developed theory for asymptotic analysis of recursive stochastic algorithms, it is also possible to examine them from a theoretical viewpoint. This is done for five different methods. They are shown to be very similar in structure and need of computer storage and time. Possible convergence points and their stability are examined for the different methods. The theoretical analysis is illustrated and supplemented by simulations.

TABLE OF CONTENTS.

1. INTRODUCTION

In many applications it is highly desirable to obtain the result of the identification of a process recursively as the process develops. For example it might be of interest to proceed with the experiment until a specified parameter accuracy is obtained. Another situation requiring the model to be updated is when the process or noise characteristics are gradually changing. Recursive estimation is also necessary in many adaptive control systems.

An identification scheme which is recursive and does not require all input-output data at each step is called an on-line method. If the parameters of the process are truly time-varying the parameters should be tracked in real-time. This is called real-time identification. In this report algorithms for on-line and real-time identification are studied.

In Åström - Eykhoff (1971) the field of on-line and real-time identification was called "a fiddler's paradize". Many different ways of obtaining recursive algorithms have been proposed. However, the properties of most of the presented methods have only been illustrated by a few simulations. There are a few comparisons available but only few results on unification and convergence have been presented. In general on-line identification algorithms have a smaller need of computer storage and are faster than off-line methods. However, on the other hand they are generally not as accurate as off-line methods.

In this report five commonly used recursive identification techniques are analysed. The methods are outlined in Section 2 and it is shown that they can be described by essentially the same algorithm, only the involved quanti-

ties are defined differently. Also real-time versions of the algorithms are discussed. In Section 3 a new tool for analysis of convergence of recursive algorithms is discussed. The uniqueness of convergence points is treated in Section 4. In Section 5 the convergence of the methods is analysed. Extensive simulations are reviewed in Section 6. It will in particular be shown that the commonly used extended matrix method can fail to converge for certain systems, which is in contrast to what earlier simulations indicate and also to earlier presented convergence proofs for this method.

2. SOME DIFFERENT METHODS

2.1 Introduction.

There are a great number of methods for recursive identification. In this report the analysis will be limited to five different methods. It will be shown that they have great similarities as far as their analysis is concerned. Some of them are very well known and discussed in several papers. They will be applied to linear single-input single-output discrete-time systems.

There is considerable confusion concerning the names of the methods. The names used in the report are given below, along with some names used by others. The five methods will be described in detail in Sections 2.2-2.6. They are:

- RLS - the recursive least squares method
- RGLS - the recursive generalized least squares method
- RIV - the recursive instrumental variables method (also called the bootstrap estimator)
- RML1 - the recursive maximum likelihood method, version 1 (also called the extended least squares method, the extended matrix method, or the approximate maximum likelihood method)
- RML2 - the recursive maximum likelihood method, version 2.

In Table 2.1 some references dealing with one or several of these recursive identification methods are given.

Reference	RLS	RGLS	RIV	RML1	RML2
Åström (1974)				x	
Eykhoff (1974)		x		x	
Finigan-Rowe (1973)			x		
Fuhrt (1973)				x	x*
Gertler-Bányász (1974)		x			
HastingsJames-Sage (1969)	x	x*			
Isermann et al (1974)	x	x	x		
James et al (1974)	x			x	
Johannesson-Wesström (1974)	x		x	x	x
Kashyap (1974)				x	
Koreman (1973)	x		x		
Landau (1974)	x	x	x		
Mayne (1967)			x*		
Pandya (1972), Pandya-Pagurek (1973)			x		
Panuska (1968,1969)				x*	
Rowe (1970)			x		
Saridis (1974)	x				
Sinha-Sen (1972)	x		x		
Smets (1970)			x	x	
Söderström (1973b)				x	x*
Söderström (1974)			x		
Talmon (1971), Talmon-van den Boom (1973)				x	
Tzafestas (1970)			x		
Wong-Polak (1967)			x*		
Young (1968)			x*	x*	
Young (1970a,b,1972)			x	x	
Young-HastingsJames (1970)			x	x	
Young et al (1971)			x	x	

Table 2.1 Some references treating the discussed recursive identification methods. The notation x* means that the method seems to be originally suggested in that reference.

In the references listed in Table 2.1 the method RML1 is applied for different model structures. In several of the papers by Young the methods RIV and RML1 are combined. The deterministic part of the system is estimated with RIV, and RML1 is used to estimate the influence of the noise.

Before going into the details of the different methods, some general conventions and assumptions will be made.

Applying identification techniques means that a model is fitted to measured data u(1),...,u(N) of the input signal and y(1),..., y(N) of the output signal. The model has for all the treated methods the following general structure

$$\hat{A}(q^{-1})\, y(t) = \hat{B}(q^{-1})\, u(t) + \hat{H}(q^{-1})\, \varepsilon(t) \qquad (2.1.1)$$

where $\varepsilon(t)$ denotes the residual. $\hat{A}(q^{-1})$ and $\hat{B}(q^{-1})$ are polynomials in the backward shift operator q^{-1} (i.e. $q^{-1}\, y(t) = y(t-1)$),

$$\hat{A}(q^{-1}) = 1 + \hat{a}_1 q^{-1} + \ldots + \hat{a}_{n_a} q^{-n_a}$$

$$\hat{B}(q^{-1}) = \hat{b}_1 q^{-1} + \ldots + \hat{b}_{n_b} q^{-n_b}$$

The transfer function $\hat{H}(q^{-1})$ has different meaning for the different methods. For RLS and RIV

$$\hat{H}(q^{-1}) = 1 \qquad (2.1.2)$$

for RGLS

$$\hat{H}(q^{-1}) = \frac{1}{\hat{C}(q^{-1})} = \frac{1}{1 + \hat{c}_1 q^{-1} + \ldots + \hat{c}_{n_c} q^{-n_c}} \qquad (2.1.3)$$

and for RML1 and RML2

$$\hat{H}(q^{-1}) = \hat{C}(q^{-1}) = 1 + \hat{c}_1 q^{-1} + \ldots + \hat{c}_{n_c} q^{-n_c} \qquad (2.1.4)$$

It would be straightforward to include several input signals or a larger time delay in the model. No principal difficulties occur but these generalizations are not treated in order to keep the notations simple.

In the forthcoming analysis of uniqueness and consistency it will be assumed that the actual process, also called the system, can be described by an equation of the form

$$A(q^{-1})\, y(t) = B(q^{-1})\, u(t) + H(q^{-1})\, e(t) \qquad (2.1.5)$$

where $e(t)$ is white noise. The polynomials $A(q^{-1})$ and $B(q^{-1})$ are given as

$$A(q^{-1}) = 1 + a_1 q^{-1} + \ldots + a_{n_a} q^{-n_a}$$

$$B(q^{-1}) = b_1 q^{-1} + \ldots + b_{n_b} q^{-n_b}$$

For RLS it is assumed that

$$H(q^{-1}) = 1$$

for RGLS that

$$H(q^{-1}) = 1/C(q^{-1}) = 1/(1+c_1 q^{-1}+\ldots+c_{n_c} q^{-n_c})$$

and for RML1 and RML2 that

$$H(q^{-1}) = C(q^{-1}) = 1 + c_1 q^{-1} + \ldots + c_{n_c} q^{-n_c}$$

Note that $H(q^{-1})$ is not specified for RIV.

It is generally assumed that the description of the system is such that the pair of polynomials A(z), B(z) as well as, if applicable, the pair A(z), C(z) has no common factors. This implies in particular that the process is controllable. It is also assumed that the C(z)-polynomial has all zeros outside the unit circle. This is only a slight restriction, cf the spectral factorization theorem, Åström (1970). This assumption is made, since for the models,the polynomials $\hat{C}(z)$ will be restricted to have all zeros outside the unit circle in order to assure finite variance of the residuals.

The input signal can be determined in several ways. In this report three different cases will be covered,

- open loop operation
- closed loop operation with a time invariant regulator or shifts between some regulators of this type
- closed loop operation where the regulator is time variable and determined by the present model of the process (an adaptive control system).

An example of the third type is the so-called self-tuning regulator, Åström-Wittenmark (1972). In Ljung-Gustavsson-Söderström (1974) it is shown that closed loop operation will not cause extra identifiability problems if e.g. appropriate shifts between feedback laws are made (in the present case it is sufficient to shift between two feedback laws).

It is natural to require that the process is uniquely identifiable (parameter identifiable in the terminology used in Ljung-Gustavsson-Söderström (1974)). For open

loop operation this will be achieved essentially if the input signal is persistently exciting of an appropriate order, see e.g. Åström-Bohlin (1965) for a definition. It will also be generally assumed that the system is asymptotically stable. For open loop operation this means that the A(z) polynomial has all its zeros outside the unit circle.

Now introduce the parameter vectors θ_o and $\hat{\theta}$ for the different methods as follows

- RLS and RIV

$$\theta_o = [a_1,..,a_{n_a},b_1,..,b_{n_b}]^T$$
$$\hat{\theta} = [\hat{a}_1,..,\hat{a}_{n_a},\hat{b}_1,..,\hat{b}_{n_b}]^T \qquad (2.1.6)$$

- RGLS, RML1 and RML2

$$\theta_o = [a_1,..,a_{n_a},b_1,..,b_{n_b},c_1,..,c_{n_c}]^T$$
$$\hat{\theta} = [\hat{a}_1,..,\hat{a}_{n_a},\hat{b}_1,..,\hat{b}_{n_b},\hat{c}_1,..,\hat{c}_{n_c}]^T \qquad (2.1.7)$$

(Note that the c_i-parameters have different meanings for RGLS on one hand and for RML1 and RML2 on the other hand).

The vectors $\hat{\theta}$ are considered as estimates of the true parameter vectors θ_o. In recursive identification methods the vector $\hat{\theta}$ is updated every time new measurements are available. Note however, that for some methods, not treated here, it is not necessary to update the vector $\hat{\theta}$ but only some other quantities, see e.g. Peterka-Šmuk (1969), Peterka-Haluskova (1970). The estimate $\hat{\theta}$ is then calculated only in the sample points where it is desired.

2.2 The RLS method.

The LS method was used already by Gauss (1809). Its recursive version for process identification is well known and it is difficult and of little interest to specify the origin of the RLS method. The RLS method is described e.g. in the survey paper Åström-Eykhoff (1971).

Introduce

$$\varphi(t) = [-y(t-1)\ldots-y(t-n_a),u(t-1)\ldots u(t-n_b)]^T \qquad (2.2.1)$$

$$\Phi = \begin{bmatrix} \varphi(1)^T \\ \vdots \\ \varphi(N)^T \end{bmatrix} \qquad Y = \begin{bmatrix} y(1) \\ \vdots \\ y(N) \end{bmatrix} \qquad (2.2.2)$$

Then the LS estimate of θ_o can be written as

$$\hat{\theta} = (\Phi^T\Phi)^{-1}\Phi^T Y \ ^{\dagger} \qquad (2.2.3)$$

This can be made recursive through

$$\hat{\theta}(t+1) = \hat{\theta}(t) + K(t+1)\,\varepsilon(t+1) \qquad (2.2.4)$$

$$K(t+1) = \frac{P(t)\varphi(t+1)}{1 + \varphi(t+1)^T P(t)\varphi(t+1)} \qquad (2.2.5)$$

$$P(t+1) = P(t) - \frac{P(t)\varphi(t+1)\varphi(t+1)^T P(t)}{1 + \varphi(t+1)^T P(t)\varphi(t+1)} \qquad (2.2.6)$$

$$\varepsilon(t+1) = y(t+1) - \varphi(t+1)^T\,\hat{\theta}(t) \qquad (2.2.7)$$

In (2.2.4) $\hat{\theta}(t)$ denotes the estimate of θ based on t pair of measurements. The residual $\varepsilon(t+1)$ is a prediction error, since it is the difference between the

†Initial value effects are neglected.

measured value $y(t+1)$ and the optimal prediction $\hat{y}(t+1|t) = \varphi(t+1)^T\hat{\theta}(t)$. The vector $K(t+1)$ is to be considered as a weighting factor.

It is well known, see e.g. Åström-Eykhoff (1971) that the RLS method is a simple and easily applicable method. There is only one real disadvantage, namely the consequences of the assumption $H(q^{-1})=1$. If this is not fulfilled, the LS, as well as the equivalent RLS method, will in general not give a consistent estimate. This drawback is in fact the motivation for use of other methods.

The RLS method can be interpreted in several ways. These interpretations are related to the other methods treated here, which can all be considered as extensions of the RLS method. The interpretations are given in Table 2.2.

INTERPRETATION OF RLS	EXTENSION
———	A system of two coupled RLS algorithms → RGLS
Solution of $(\Phi^T\Phi)\hat{\theta} = \Phi^T Y$	Solution of $(Z^T\Phi)\hat{\theta} = Z^T Y$ → RIV
Kalmanfilter for $\theta(t+1) = \theta(t)$ $y(t) = \varphi(t)^T\theta(t) + e(t)$	Kalmanfilter for a larger system structure → RML1
Recursive minimization of $\Sigma\varepsilon^2(t)$	Recursive minimization for another model structure → RML2

Table 2.2 Interpretations and extensions of the RLS method.

2.3 The RGLS method.

The off-line GLS method was suggested in Clarke (1967) and later analysed in Söderström (1972). It can be interpreted as a special way of minimizing the loss function $\sum_1^N \varepsilon^2(t)$, where the residuals $\varepsilon(t)$ are given by (2.1.1).

Inspired by Clarke's algorithm, a recursive method was suggested in HastingsJames-Sage (1969). It consists of two RLS estimators combined via filtering. Introduce the notations

$$\hat{\theta}_1 = [\hat{a}_1 \dots \hat{a}_{n_a}\ \hat{b}_1 \dots \hat{b}_{n_b}]^T \tag{2.3.1}$$

$$\hat{\theta}_2 = [\hat{c}_1 \dots \hat{c}_{n_c}]^T \tag{2.3.2}$$

$$\varphi_1(t) = [-\hat{C}(q^{-1})y(t-1) \dots -\hat{C}(q^{-1})y(t-n_a), \hat{C}(q^{-1})u(t-1) \dots \hat{C}(q^{-1})u(t-n_b)]^T \tag{2.3.3}$$

$$\varphi_2(t) = [-\tilde{\varepsilon}(t-1) \dots -\tilde{\varepsilon}(t-n_c)]^T \tag{2.3.4}$$

$$\tilde{\varepsilon}(t) = \hat{A}(q^{-1})y(t) - \hat{B}(q^{-1})u(t) \tag{2.3.5}$$

The algorithm can be written as

$$\hat{\theta}_1(t+1) = \hat{\theta}_1(t) + K_1(t+1)\ \varepsilon_1(t+1) \tag{2.3.6}$$

$$K_1(t+1) = \frac{P_1(t)\varphi_1(t+1)}{1 + \varphi_1(t+1)^T P_1(t)\varphi_1(t+1)} \tag{2.3.7}$$

$$P_1(t+1) = P_1(t) - \frac{P_1(t)\varphi_1(t+1)\varphi_1(t+1)^T P_1(t)}{1 + \varphi_1(t+1)^T P_1(t)\varphi_1(t+1)} \tag{2.3.8}$$

$$\varepsilon_1(t+1) = \hat{C}(q^{-1})y(t+1) - \varphi_1(t+1)^T\hat{\theta}_1(t) \tag{2.3.9}$$

$$\hat{\theta}_2(t+1) = \hat{\theta}_2(t) + K_2(t+1)\ \varepsilon_2(t+1) \tag{2.3.10}$$

$$K_2(t+1) = \frac{P_2(t)\varphi_2(t+1)}{1 + \varphi_2(t+1)^T P_2(t)\varphi_2(t+1)} \qquad (2.3.11)$$

$$P_2(t+1) = P_2(t) - \frac{P_2(t)\varphi_2(t+1)\varphi_2(t+1)^T P_2(t)}{1 + \varphi_2(t+1)^T P_2(t)\varphi_2(t+1)} \qquad (2.3.12)$$

$$\varepsilon_2(t+1) = \tilde{\varepsilon}(t+1) - \varphi_2(t+1)^T \hat{\theta}_2(t) \qquad (2.3.13)$$

In the computations for $\hat{\theta}_1(t+1)$ the polynomial $\hat{C}(q^{-1})$ is required. It is obtained via the estimate $\hat{\theta}_2(t)$.

Different kinds of approximations are possible. E.g. in computation of $\varphi_1(t)$ one can filter all relevant data with the $\hat{C}(q^{-1})$ polynomial given by $\hat{\theta}_2(t)$. Another possibility is to use the previously computed estimates of $\hat{C}(q^{-1})y(t-2),\ldots,\hat{C}(q^{-1})y(t-n_a),\hat{C}(q^{-1})u(t-2),\ldots,\hat{C}(q^{-1})u(t-n_b)$ and to make a new filtering only to compute $\hat{C}(q^{-1})y(t-1)$ and $\hat{C}(q^{-1})u(t-1)$. Such minor differences will not influence the result of the coming analysis. Nevertheless, it can in some cases have a significant influence on the transient behaviour of the algorithm.

2.4 The RIV method.

The principle of instrumental variables, see e.g. Kendall-Stuart (1961), has been applied to system identification in several ways. The off-line description of such methods is as follows. Let Z be a matrix satisfying

$$\lim \frac{1}{N} Z^T\Phi = R \quad \text{a non-singular (quadratic) matrix} \qquad (2.4.1)$$

$$\lim \frac{1}{N} Z^T(Y-\Phi\theta_o) = 0 \qquad (2.4.2)$$

Normally lim is understood as limes in probability, although the stronger concept limes with probability one can be applied without difficulties. The conditions above imply that the estimate

$$\hat{\theta} = (Z^T\Phi)^{-1}Z^TY \tag{2.4.3}$$

becomes consistent.

There are of course a great number of ways of satisfying the conditions (2.4.1), (2.4.2). It is not difficult to show that

$$Z = \begin{bmatrix} z(1)^T \\ \vdots \\ z(N)^T \end{bmatrix} \tag{2.4.4}$$

$$z(t) = \left[-\frac{B(q^{-1})}{A(q^{-1})}u(t-1) \ldots -\frac{B(q^{-1})}{A(q^{-1})}u(t-n_a), u(t-1) \ldots u(t-n_b) \right]^T \tag{2.4.5}$$

will meet the assumptions provided that the system is operating in <u>open loop</u>. Note that the difference between Φ, given in (2.2.1), (2.2.2), and Z is just that in Z the influence of the disturbances on the output is subtracted.

The IV estimate (2.4.3) is easily rewritten into an equivalent recursive algorithm of the form

$$\hat{\theta}(t+1) = \hat{\theta}(t) + K(t+1)\varepsilon(t+1) \tag{2.4.6}$$

$$K(t+1) = \frac{P(t)z(t+1)}{1 + \varphi(t+1)^TP(t)z(t+1)} \tag{2.4.7}$$

$$P(t+1) = P(t) - \frac{P(t)z(t+1)\varphi(t+1)^TP(t)}{1 + \varphi(t+1)^TP(t)z(t+1)} \tag{2.4.8}$$

$$\varepsilon(t+1) = y(t+1) - \varphi(t+1)^T\hat{\theta}(t) \qquad (2.4.9)$$

The vectors $\{z(t)\}$ are defined from (2.4.4)

There is one problem in using the vectors $\{z(t)\}$ given by (2.4.5). Their computation requires knowledge of $A(q^{-1})$ and $B(q^{-1})$. On the other hand the goal of the whole procedure is just to obtain estimates of these parameters. This problem is solved by the use of

$$z(t) = [-x(t-1)..-x(t-n_a),u(t-1)...u(t-n_b)]^T \qquad (2.4.10)$$

where

$$x(t) = z(t)^T\hat{\theta}(t) \qquad (2.4.11)$$

The equation (2.4.11) for computing $x(t)$ is often modified. Usual variants are

$$x(t) = z(t)^T\hat{\theta}(t-\tau) \qquad (2.4.12)$$

where τ is a small positive integer, or

$$x(t) = z(t)^T\bar{\theta}(t) \qquad (2.4.13)$$

$$\bar{\theta}(t) = (1-\gamma)\bar{\theta}(t-1) + \gamma\hat{\theta}(t) \qquad (2.4.14)$$

γ being a number slightly less than one. However, these variants do not change the analysis made in this report.

For this type of algorithm original work seems to have been done by Mayne (1967), Wong-Polak (1967) and Young (1968). It should be mentioned that there are other

recursive identification methods similar to this approach.† One of them is the so-called tally principle treated in Peterka - Smuk (1969), Peterka - Haluskova (1970).

2.5 The RML1 method.

As mentioned earlier the RLS method can be interpreted as a Kalman filter for the system

$$\theta(t+1) = \theta(t) \quad [=\theta] \tag{2.5.1}$$

$$y(t) = \varphi(t)^T\theta(t) + e(t) \tag{2.5.2}$$

This interpretation is described e.g. in Åström-Eykhoff (1971). This idea can be exploited also for the model structure given by (2.1.1), (2.1.4).

Assume for a moment that the noise sequence $\{e(t)\}$ is known (measurable). Then the estimation problem could be treated as an ordinary LS problem but with two inputs u(t) and e(t). This would lead to exactly the algorithm (2.2.4) - (2.2.7) with the exceptions that $\varphi(t)$ is given by

$$\varphi(t) = [-y(t-1)\ldots-y(t-n_a),u(t-1)\ldots u(t-n_b),e(t-1)\ldots e(t-n_c)]^T \tag{2.5.3}$$

and $\hat{\theta}$ as (2.1.7). Since the noise e(t) is in fact not known, some approximation must be made. A natural way to get a useful algorithm is to substitute e(t) with $\varepsilon(t)$ i.e. to take

$$\varphi(t) = [-y(t-1)\ldots-y(t-n_a),u(t-1)\ldots u(t-n_b),\varepsilon(t-1)\ldots\varepsilon(t-n_c)]^T \tag{2.5.4}$$

Thus the algorithm becomes

---

†The analysis made here for the RIV method can easily be generalized to other choices of Z, if the elements of z(t) are formed by filtering the input signal u(t).

$$\hat{\theta}(t+1) = \hat{\theta}(t) + K(t+1)\varepsilon(t+1) \tag{2.5.5}$$

$$K(t+1) = \frac{P(t)\varphi(t+1)}{1 + \varphi(t+1)^T P(t)\varphi(t+1)} \tag{2.5.6}$$

$$P(t+1) = P(t) - \frac{P(t)\varphi(t+1)\varphi(t+1)^T P(t)}{1 + \varphi(t+1)^T P(t)\varphi(t+1)} \tag{2.5.7}$$

$$\varepsilon(t+1) = y(t+1) - \varphi(t+1)^T\hat{\theta}(t) \tag{2.5.8}$$

and $\varphi(t)$ defined by (2.5.4).

This algorithm seems to be originally proposed by Panuska (1968, 1969) and by Young (1968). It is often used to model time series i.e. used without b_i-parameters and sometimes also processes without a_i-parameters. In Talmon (1971) it is used for the model structure given by

$$\hat{A}(q^{-1})y(t) = \hat{B}(q^{-1})u(t) + \frac{\hat{C}(q^{-1})}{\hat{D}(q^{-1})}\varepsilon(t) \tag{2.5.9}$$

2.6 The RML2 method.

The origin for this method is the off-line maximum likelihood method developed by Åström-Bohlin (1965). This off-line method has been shown to have nice properties both in theory and in practice. Thus it is a challenge to try to attain them for a recursive method as well. The off-line ML method consists of minimizing the function $\sum_1^N \varepsilon^2(t)$ where $\varepsilon(t)$ is defined by (2.1.1). A recursive (approximative) version of this minimization is described in Söderström (1973b). This algorithm will be called the RML2 method. A similar algorithm has been proposed by Fuhrt (1973). The RML1 can be interpreted as an approximation of the RML2 method.

The RML2 method is described by the equations

$$\hat{\theta}(t+1) = \hat{\theta}(t) + K(t+1)\varepsilon(t+1) \qquad (2.6.1)$$

$$K(t+1) = \frac{P(t)\varphi(t+1)}{1 + \varphi(t+1)^T P(t)\varphi(t+1)} \qquad (2.6.2)$$

$$P(t+1) = P(t) - \frac{P(t)\varphi(t+1)\varphi(t+1)^T P(t)}{1 + \varphi^T(t+1)P(t)\varphi(t+1)} \qquad (2.6.3)$$

The prediction error $\varepsilon(t)$ is computed from

$$\hat{C}(q^{-1})\varepsilon(t) = \hat{A}(q^{-1})y(t) - \hat{B}(q^{-1})u(t) \qquad (2.6.4)$$

To obtain the exact prediction error, eq. (2.6.4) has to be solved from t = 0 for every new measurement. This will require too much computation, and some suitable approximation has to be made.

One way is to make use of (2.5.4) and (2.5.8), i.e. to take

$$\varepsilon(t) = y(t) - [-y(t-1)\ldots-y(t-n_a),u(t-1)\ldots u(t-n_b)\varepsilon(t-1)\ldots\varepsilon(t-n_c)]\hat{\theta}(t-1) \qquad (2.6.5)$$

This equation is iterated only once for every measurement.

An alternative way, described in Söderström (1973b), is to consider†

$$x(t+1) = \begin{bmatrix} -\hat{c}_1 & 1 & & \\ -\hat{c}_2 & & \ddots & \\ \vdots & & & 0 \\ & \ddots & & \\ -\hat{c}_{n_c} & 0 & & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} y(t+1) + \begin{bmatrix} \hat{a}_1 \\ \vdots \\ \hat{a}_{n_a} \end{bmatrix} y(t) - \begin{bmatrix} \hat{b}_1 \\ \vdots \\ \hat{b}_{n_b} \end{bmatrix} u(t) \qquad (2.6.6)$$

† It is assumed, for notational simplicity only, that $n_a = n_b = n_c$ in (2.6.6) and (2.6.8).

which is quite equivalent to (2.6.4) with $\varepsilon(t)=x_1(t)$. Similarly to (2.6.5), the state equation (2.6.6) is solved only once. In the computation of $x(t+1)$ from $x(t)$ the matrix elements are formed from the estimate $\hat{\theta}(t)$.

In (2.6.2) and (2.6.3) the vector $\varphi(t)$ contains the derivatives of $\varepsilon(t)$ with respect to $\hat{\theta}$ and is defined by

$$\varphi(t) = \left[-\frac{1}{\hat{C}(q^{-1})} y(t-1) \ldots -\frac{1}{\hat{C}(q^{-1})} y(t-n_a), \right.$$

$$\frac{1}{\hat{C}(q^{-1})} u(t-1) \ldots \frac{1}{\hat{C}(q^{-1})} u(t-n_b), \frac{1}{\hat{C}(q^{-1})} \varepsilon(t-1) \ldots$$

$$\left. \frac{1}{\hat{C}(q^{-1})} \varepsilon(t-n_c) \right]^T \qquad (2.6.7)$$

The vector $\varphi(t)$ can be shown to satisfy

$$\varphi(t+1) = \begin{bmatrix} -\hat{c}_1 \ldots -\hat{c}_{n_c} & & & \\ 1 \;\ddots\; 0 & 0 & 0 \\ 0 \;\ddots\; 1 & & \\ \hline & -\hat{c}_1 \ldots -\hat{c}_{n_c} & & \\ 0 & 1 \;\ddots\; 0 & 0 & \\ & 0 \;\ddots\; 1 & & \\ \hline & & -\hat{c}_1 \ldots -\hat{c}_{n_c} & \\ 0 & 0 & 1 \;\ddots\; 0 & \\ & & 0 \;\ddots\; 1 & \end{bmatrix} \varphi(t) + \begin{bmatrix} y(t) \\ 0 \\ \vdots \\ \hline -u(t) \\ 0 \\ \vdots \\ \hline -\varepsilon(t) \\ 0 \\ \vdots \end{bmatrix}$$

(2.6.8)

Analogously to (2.6.6), the equation (2.6.8) is solved only once and with the present values $\hat{\theta}(t)$ in the matrix elements when $\varphi(t+1)$ is computed from $\varphi(t)$.

In Fuhrt (1973) another and somewhat less accurate approximation of the computation of $\varphi(t)$ is given.

2.7 Stochastic approximation versions.

There are several recursive identification methods based on stochastic approximation, see e.g.Sakrison (1967), Saridis-Stein (1968), Lobbia-Saridis (1972), and Saridis (1974).

The methods treated above can be simplified to yield algorithms of this type by essentially substituting the matrix $P(t)$ by a scalar $p(t)$, e.g. c/t or $1/\text{tr } P(t)^{-1}$. This means that the general description of the algorithms is transformed into

$$\hat{\theta}(t+1) = \hat{\theta}(t) + p(t+1)z(t+1)\varepsilon(t+1) \qquad (2.7.1)$$

Such a simplification will of course reduce the computation time considerably. However, in general the convergence will be slower than if the original algorithm is applied.

Algorithms of the type (2.7.1) are not considered explicitly in this report. There are,however, no difficulties in extending the analysis to such methods also. In fact, the theoretical results of such simplified algorithms when uniqueness and stability are concerned will be analogous to the results for the original algorithms.

2.8 Initial conditions.

In order to start up the algorithms it is necessary to have some initial values $\hat{\theta}(0)$ and $P(0)$. With use of the interpretation of the RLS method as a Kalman filter it can be seen that $\hat{\theta}(0)$ can be

interpreted as an a priori estimate of θ_0 with the covariance matrix P(0). If no a priori information is available, then the algorithm can be started with $\hat{\theta}(0) = 0$ and with $P(0) = \alpha \cdot I$ where α is a large number. The choice of a suitable value of α depends on the variances of the signals. It is often sufficient to take it 10 times the variance of y(t).

The choice $\hat{\theta}(0) = 0$, $P(0) = \alpha \cdot I$ is applicable for all the methods. It is not very difficult to show, cf Section 3, that if the methods converge properly, then the initial values will have no influence asymptotically. Note, however, that the choice of initial values can have a large influence on the transient behaviour of the methods.

2.9 Real time versions.

Real time identification means that time variable parameters can be tracked. It is possible to modify the general algorithms to handle such systems. Essentially two kinds of extensions are used in the literature.

One way is to include a weighting factor or forgetting factor λ making the algorithm into (the involved quantities are defined earlier for each of the methods)

$$\hat{\theta}(t+1) = \hat{\theta}(t) + K(t+1)\varepsilon(t+1) \tag{2.9.1}$$

$$K(t+1) = \frac{P(t)z(t+1)}{\lambda + \varphi(t+1)^T P(t) z(t+1)} \tag{2.9.2}$$

$$P(t+1) = \left[P(t) - \frac{P(t)z(t+1)\varphi(t+1)^T P(t)}{\lambda + \varphi(t+1)^T P(t) z(t+1)}\right] / \lambda \tag{2.9.3}$$

For the RLS method this algorithm means that $\sum_{s=1}^{t} \lambda^{t-s}\varepsilon(s)^2$ is minimized. By choosing λ smaller than 1 (e.g. as 0.99) old residuals will have small influence on the estimates. This approach is described e.g. in Wieslander (1969).

Another way, see e.g. Bohlin (1968) is to make use of the interpretation of the RLS method as a Kalman filter. Inclusion of process noise in the model will lead to the algorithm

$$\hat{\theta}(t+1) = \hat{\theta}(t) + K(t+1)\varepsilon(t+1) \qquad (2.9.4)$$

$$K(t+1) = \frac{P(t)z(t+1)}{1 + \varphi(t+1)^T P(t)z(t+1)} \qquad (2.9.5)$$

$$P(t+1) = P(t) - \frac{P(t)z(t+1)\varphi(t+1)^T P(t)}{1 + \varphi(t+1)^T P(t)z(t+1)} + R \qquad (2.9.6)$$

where R is a positive (semi) definite matrix.

Both the extensions have the desired property that the weighting factor K(t) does not tend to zero.

2.10 Improvement of the convergence rate.

In Söderström (1973b) several ways to improve the convergence rate were discussed for the RML2 method. A very common choice, see e.g. Fuhrt (1973) and Talmon-van den Boom (1973), is to consider the algorithm

$$\hat{\theta}(t+1) = \hat{\theta}(t) + K(t+1)\varepsilon(t+1) \qquad (2.10.1)$$

$$K(t+1) = \frac{P(t)z(t+1)}{\lambda(t+1) + \varphi(t+1)^T P(t)z(t+1)} \qquad (2.10.2)$$

$$P(t+1) = [P(t) - \frac{P(t)z(t+1)\varphi(t+1)^T P(t)}{\lambda(t+1)+\varphi(t+1)^T P(t) z(t+1)}] /\lambda(t+1) \qquad (2.10.3)$$

$$\lambda(t+1) = \lambda_o \lambda(t) + (1-\lambda_o) \qquad (2.10.4)$$

The number λ_o is chosen close to 1, e.g. as 0.99, and so is the initial value $\lambda(0)$. Eq. (2.10.4) means that $\lambda(t)$ tends exponentially to 1 as t tends to infinity.

The algorithm can be viewed as a generalization of the real time algorithm (2.9.1) - (2.9.3). The time dependent weighting factor $\lambda(t)$ will cause the influence of the first estimates $\hat{\theta}(1)$, $\hat{\theta}(2)$... on distant estimates of θ_o to decrease. The weighting factor $\lambda(t)$ is constructed to tend to 1. This means that, asymptotically, also the first data points are given positive weights in contrast to the real time algorithms. This will in particular imply that the accuracy of the parameter estimates will be better and that they may really converge.

It is easy to see that (2.10.3) also can be written

$$P(t+1)^{-1} = \lambda(t+1)P(t)^{-1} + z(t+1)\varphi(t+1)^T$$

and that, in the case of the RLS algorithm, this algorithm gives the minimum of the loss function

$$\sum_{k=1}^{t} [\prod_{i=k}^{t} \lambda(i)] \varepsilon(k)^2 \qquad (2.10.5)$$

Here the "forgetting profile" of old data is clearly seen, and the real time algorithm (2.9.11) corresponds to exponential forgetting of previous data. Algorithm (2.10.4) gives the forgetting coefficient

$$\lambda(t) = \lambda_o{}^t \lambda(0) + (1-\lambda_o{}^t) = 1 - \lambda_o{}^t(1-\lambda(0))$$

and asymptotically for t large

$$\prod_k^t \lambda(i) = e^{-\lambda_o{}^k \cdot [\frac{1-\lambda(0)}{1-\lambda_o}(1-\lambda_o{}^{t+1-k})]}$$

For some of the theoretical considerations in the next section, it is convenient to rewrite the algorithm. For a given sequence $\{\lambda(t)\}$, which need not satisfy (2.10.4), define the sequence $\{\gamma(t)\}$ recursively:

$$\gamma(t+1) = \frac{\gamma(t)}{\lambda(t+1) + \gamma(t)} \quad ; \gamma(0) = 1 \qquad (2.10.6)$$

Then

$$\lambda(t+1) = \frac{\gamma(t)}{\gamma(t+1)}(1-\gamma(t+1)) \qquad (2.10.7)$$

Introduce

$$\tilde{P}(t) = \frac{1}{\gamma(t)} P(t)$$

$$\tilde{K}(t) = \frac{1}{\gamma(t)} K(t)$$

Then by straightforward calculations (2.10.1)-(2.10.3) become

$$\hat{\theta}(t+1) = \hat{\theta}(t) + \gamma(t+1)\tilde{K}(t+1)\varepsilon(t+1) \qquad (2.10.8)$$

$$\tilde{K}(t+1) = \frac{\tilde{P}(t)z(t+1)}{1 + \gamma(t+1)[\varphi(t+1)^T\tilde{P}(t)z(t+1)-1]} \qquad (2.10.9)$$

$$\tilde{P}(t+1)^{-1} = \tilde{P}(t)^{-1} + \gamma(t+1)[z(t+1)\varphi(t+1)^T - \tilde{P}(t)^{-1}] \qquad (2.10.10)$$

where (2.10.10) also can be written

$$\hat{P}(t+1) = \frac{1}{1-\gamma(t+1)}\hat{P}(t) - \frac{\gamma(t+1)}{1-\gamma(t+1)}\frac{\hat{P}(t)z(t+1)\varphi(t+1)^T\hat{P}(t)}{1+\gamma(t+1)[\varphi(t+1)^T\hat{P}(t)z(t+1)-1]}$$

This description may be useful, since $\hat{P}$ and $\hat{R}$ do not tend to zero as $t\to\infty$, but to limits that may have physical interpretation. Also, this description gives a more explicit account for the "step length" of the correction at time t. This is given by $\gamma(t)$.

In Ljung (1974a) algorithms of the type (2.10.8) - (2.10.10) are studied, and the choice of $\{\gamma(t)\}$ to improve the convergence rate is discussed. In particular, reasons are given why slowly decreasing sequences $\{\gamma(t)\}$ (corresponding to $\lambda(t) < 1$) are better than $\gamma(t) = 1/t$.

We will also anticipate here some conditions that will be imposed on $\{\gamma(t)\}$ later on, and discuss their interpretation for $\{\lambda(t)\}$.

First, a necessary condition is

$$\sum_{1}^{\infty} \gamma(t) = \infty \qquad (2.10.11)$$

This means that the step sizes must be large enough to allow for any amount of change necessary.

In terms of $\{\lambda(t)\}$ it essentially implies that

$$\lambda(t) \leq c.$$

Secondly, to yield convergence w.p.1 $\{\gamma(t)\}$ must decrease sufficiently fast to reduce the noise influence. The condition will be

$$\sum_{1}^{\infty} \gamma(t)^p < \infty \qquad \text{some large enough } p > 1 \qquad (2.10.12)$$

This is a condition on how fast $\lambda(t)$ approaches 1. Clearly, the real time algorithm (2.9.1) - (2.9.3) does not satisfy (2.10.12).

Finally, two other conditions are imposed for technical reasons:

$$\gamma(t+1) \geq \gamma(t)(1-\gamma(t+1)) \qquad (2.10.13)$$

$$\{\gamma(t)\} \quad \text{decreasing} \qquad (2.10.14)$$

Eq. (2.10.13) is equivalent to $\lambda(t+1) \leq 1$, and it is easy to see that $\{\gamma(t)\}$ corresponding to the λ-sequence generated by (2.10.4) satisfies (2.10.11) - (2.10.14).

2.11 A unified description of the methods.

The description of the discussed methods can now be summarized and unified as follows. The description has also been used in the Sections 2.9 and 2.10. All the methods can be described by

$$\hat{\theta}(t+1) = \hat{\theta}(t) + K(t+1)\varepsilon(t+1) \qquad (2.11.1)$$

$$K(t+1) = \frac{P(t)z(t+1)}{\lambda(t+1) + \varphi(t+1)^T P(t) z(t+1)} \qquad (2.11.2)$$

$$P(t+1) = \left[P(t) - \frac{P(t)z(t+1)\varphi(t+1)^T P(t)}{\lambda(t+1) + \varphi(t+1)^T P(t) z(t+1)}\right] / \lambda(t+1) \qquad (2.11.3)$$

$$\lambda(t+1) = \lambda_0 \lambda(t) + (1-\lambda_0) \qquad (2.11.4)$$

The meaning and the computations of the quantities $\hat{\theta}(t)$, $\varepsilon(t)$, $\varphi(t)$ and $z(t)$ for the different methods are given in Table 2.3.

For part of the analysis in this report the alternative description derived in Section 2.10 will be useful

$$\hat{\theta}(t+1) = \hat{\theta}(t) + \gamma(t+1)\,\tilde{K}(t+1)\varepsilon(t+1) \qquad (2.11.5)$$

$$\tilde{K}(t+1) = \frac{\tilde{P}(t)z(t+1)}{1 + \gamma(t+1)[\varphi(t+1)^T\tilde{P}(t)z(t+1)-1]} \qquad (2.11.6)$$

$$\tilde{P}(t+1) = \frac{1}{1-\gamma(t+1)}\,P(t) - \frac{\gamma(t+1)}{1-\gamma(t+1)}\,\frac{\tilde{P}(t)z(t+1)\varphi(t+1)^T\tilde{P}(t)}{1+\gamma(t+1)[\varphi(t+1)^T\tilde{P}(t)z(t+1)-1]} \qquad (2.11.7)$$

Unified algorithm: $\hat{\theta}(t+1) = \hat{\theta}(t) + K(t+1)\varepsilon(t+1)$; $K(t+1) = \dfrac{P(t)z(t+1)}{\lambda(t+1)+\varphi(t+1)^T P(t)z(t+1)}$;

$$P(t+1) = \left[P(t) - \frac{P(t)z(t+1)\varphi(t+1)^T P(t)}{\lambda(t+1)+\varphi(t+1)^T P(t)z(t+1)}\right]/\lambda(t+1)$$

Method	Model structure	Quantity			
		$\hat{\theta}$	$\varepsilon(t)$	$\varphi(t)$	$z(t)$
RLS	$\hat{A}(q^{-1})y(t) = \hat{B}(q^{-1})u(t)+\varepsilon(t)$	$[\hat{a}_1 \ldots \hat{a}_{n_a}, \hat{b}_1 \ldots \hat{b}_{n_b}]^T$	$\varepsilon(t) = y(t)-\varphi(t)^T\hat{\theta}(t-1)$	$\varphi(t) = [-y(t-1)\ldots-y(t-n_a), u(t-1)\ldots u(t-n_b)]^T$	$z(t) = \varphi(t)$
RGLS	The RGLS method consists of a pair of coupled RLS algorithms, see Section 2.3. $\hat{A}(q^{-1})y(t)=\hat{B}(q^{-1})u(t) + \frac{1}{\hat{C}(q^{-1})}\varepsilon(t)$	$\hat{\theta}_1 = [\hat{a}_1 \ldots \hat{a}_{n_a}, \hat{b}_1 \ldots \hat{b}_{n_b}]^T$ $\hat{\theta}_2 = [\hat{c}_1 \ldots \hat{c}_{n_c}]^T$	$\varepsilon_1(t)=\hat{C}(q^{-1})y(t)-\varphi_1(t)^T\hat{\theta}_1(t-1)$ $\varepsilon_2(t)=\tilde{\varepsilon}(t)-\varphi_2(t)^T\hat{\theta}_2(t-1)$ $\tilde{\varepsilon}(t)=\hat{A}(q^{-1})y(t)-\hat{B}(q^{-1})u(t)$	$\varphi_1(t) = [-\hat{C}(q^{-1})y(t-1)\ldots-\hat{C}(q^{-1})y(t-n_a), \hat{C}(q^{-1})u(t-1)\ldots\hat{C}(q^{-1})u(t-n_b)]^T$ $\varphi_2(t) = [-\tilde{\varepsilon}(t-1)\ldots-\tilde{\varepsilon}(t-n_c)]^T$	$z_1(t) = \varphi_1(t)$ $z_2(t) = \varphi_2(t)$
RIV	$\hat{A}(q^{-1})y(t) = \hat{B}(q^{-1})u(t)+\varepsilon(t)$ ($\varepsilon(t)$ need not be white noise)	$[\hat{a}_1 \ldots \hat{a}_{n_a}, \hat{b}_1 \ldots \hat{b}_{n_b}]^T$	$\varepsilon(t) = y(t)-\varphi(t)^T\hat{\theta}(t-1)$	$\varphi(t) = [-y(t-1)\ldots-y(t-n_a), u(t-1)\ldots u(t-n_b)]^T$	$z(t) = [-z(t-1)\hat{\theta}(t-1)\ldots -z(t-n_a)\hat{\theta}(t-n_a), u(t-1)\ldots u(t-n_b)]^T$
RML1	$\hat{A}(q^{-1})y(t)=\hat{B}(q^{-1})u(t)+\hat{C}(q^{-1})\varepsilon(t)$	$[\hat{a}_1 \ldots \hat{a}_{n_a}, \hat{b}_1 \ldots \hat{b}_{n_b}, \hat{c}_1 \ldots \hat{c}_{n_c}]^T$	$\varepsilon(t) = y(t)-\varphi(t)^T\hat{\theta}(t-1)$	$\varphi(t) = [-y(t-1)\ldots-y(t-n_a), u(t-1)\ldots u(t-n_b), \varepsilon(t-1)\ldots\varepsilon(t-n_c)]^T$	$z(t) = \varphi(t)$
RML2	$\hat{A}(q^{-1})y(t)=\hat{B}(q^{-1})u(t)+\hat{C}(q^{-1})\varepsilon(t)$	$[\hat{a}_1 \ldots \hat{a}_{n_a}, \hat{b}_1 \ldots \hat{b}_{n_b}, \hat{c}_1 \ldots \hat{c}_{n_c}]^T$	$\varepsilon(t)= y(t)-[-y(t-1)\ldots-y(t-n_a), u(t-1)\ldots u(t-n_b), \varepsilon(t-1)\ldots \varepsilon(t-n_c)]\hat{\theta}(t-1)$	$\varphi(t) = [-\frac{1}{\hat{C}(q^{-1})}y(t-1)\ldots-\frac{1}{\hat{C}(q^{-1})}y(t-n_a), \frac{1}{\hat{C}(q^{-1})}u(t-1)\ldots\frac{1}{\hat{C}(q^{-1})}u(t-n_b), \frac{1}{\hat{C}(q^{-1})}\varepsilon(t-1)\ldots\frac{1}{\hat{C}(q^{-1})}\varepsilon(t-n_c)]^T$ For computation, see (2.5.8)	$z(t) = \varphi(t)$

Table 2.3. Interpretation of the unified description of the algorithms.

3. TOOLS FOR THE ANALYSIS OF RECURSIVE ALGORITHMS

3.1 Introduction.

The algorithms (2.11.1) - (2.11.4) are, in fact, non-linear, time varying stochastic difference equations and the convergence properties of them are difficult to analyse. On the other hand, vital questions like consistency of the estimates and uniqueness of solutions rely upon the question of convergence.

A simple and natural way of studying the algorithms is simulation. This often gives valuable information about the behaviour of the algorithms, and in Section 6 the results from numerous simulations are reported. In fact, simulation seems so far to have been the main tool for studying the asymptotic properties and the only tool for comparisons of the algorithms. However, since the algorithms are stochastic, many simulations are required in order to evaluate the behaviour. Also, from simulations of a number of different systems it is dangerous to make conjectures about any general properties. An important illustration of this will be given (see Examples 5.6.3,5.6.4). Therefore,some analytical tools are required as a complement to simulation.

The RLS method has been analysed e.g. in Åström-Eykhoff (1971). The analysis relies upon the corresponding off-line method, since the obtained estimates are identical. The other recursive methods treated in Section 2 differ from their off-line counterparts, and therefore they must be analysed separately.

In Ljung (1974a,b) and in Ljung-Wittenmark (1974) it has been shown how a deterministic, ordinary differential equation (ODE) can be associated with recursive

stochastic algorithms. In this section these ideas will be applied to the algorithms (2.11.1) - (2.11.4). A heuristic introduction is given in Section 3.2. The main results are stated in Section 3.3 and in Section 3.4 it is discussed how the tools can be used for analysis.

3.2 A heuristic approach.

Consider the algorithm (2.11.5)†

$$\hat{\theta}(t) = \hat{\theta}(t-1) + \gamma(t)\tilde{P}(t-1)z(t)\varepsilon(t) \qquad (3.2.1)$$

For simplicity, in this section a stochastic approximation variant with $\tilde{P}(t) = I$ will be considered. The interpretation of $\hat{\theta}(t)$, $z(t)$ and $\varepsilon(t)$ is given in Table 2.3.

Clearly, for all methods $\varepsilon(t)$ is a function of $\hat{\theta}(t-1)$. For RML1, (2.5.4) - (2.5.8), it is seen that $\varepsilon(t)$ is formed from the difference equation

$$\varepsilon(t) + \hat{c}_1(t)\varepsilon(t-1) + \ldots + \hat{c}_{n_c}(t)\varepsilon(t-n_c) =$$

$$= y(t) + \hat{a}_1(t)y(t-1) + \ldots + \hat{a}_{n_a}(t)y(t-n_a) -$$

$$- \hat{b}_1(t)u(t-1) - \ldots - \hat{b}_{n_b}(t)u(t-n_b) \qquad (3.2.2)$$

Therefore, $\varepsilon(t)$ (and hence $\varphi(t) = z(t)$) will in this case depend on <u>all</u> previous $\hat{\theta}(s)$, $s < t$.

The same holds for RML2 and RGLS. If the input sequence is independent of $\hat{\theta}(t)$, then $\varphi(t)$ is independent of $\hat{\theta}$ for RIV and RLS, and $\varepsilon(t)$ depends only on $\hat{\theta}(t-1)$. However, for RIV $z(t)$ is defined recursively using old estimates. This makes $z(t)$ depend on <u>all</u> old $\hat{\theta}(s)$, $s < t$.

† Since $\gamma(t)$ tends to zero, the denumerator in (2.11.6) has been omitted for this asymptotic analysis.

When needed, such dependences will be denoted through

$$\varepsilon(t) = \varepsilon(t;\ \hat{\theta}(\cdot)) = \varepsilon(t;\ \hat{\theta}(t-1),..,\hat{\theta}(0))$$
$$\varphi(t) = \varphi(t;\ \hat{\theta}(\cdot)) = \varphi(t;\ \hat{\theta}(t-1),...,\hat{\theta}(0))$$
$$z(t) = z(t;\ \hat{\theta}(\cdot)) = z(t;\ \hat{\theta}(t-1),...,\hat{\theta}(0))$$

When, for an adaptive controller, the input is (partly) determined by a feedback law that depends on $\hat{\theta}(t)$:

$$u(t) = u_R(t) + F(\hat{\theta}(t),\ q^{-1})y(t)$$

then it is quite clear that $\varphi(t)$ and $\varepsilon(t)$ for all the different algorithms will depend on all previous $\hat{\theta}(s)$.

These complicated relations between $\hat{\theta}(s)$, $\varepsilon(t)$ and $z(t)$, naturally is a major stumblingblock for the analysis of the algorithms. An evident simplification follows if it is assumed a priori that the estimates converge to some value θ^*.

It is then reasonable to assume that, as $t\to\infty$, $\varepsilon(t\ ;\hat{\theta}(\cdot))$ tends to a variable $\bar{\varepsilon}(t;\theta^*)$, which would be obtained from (3.2.2) for $\hat{\theta}(s) = \theta^*$; $s < t$. Introduce also, analogously the stationary processes $\bar{\varphi}(t;\theta^*)$ and $\bar{z}(t;\theta^*)$.

Such analysis is made in Åström-Wittenmark (1972), Söderström (1973b) and Åström (1974).

Then we have from (3.2.1)

$$\hat{\theta}(t+N) = \hat{\theta}(t) + \sum_{k=t+1}^{t+N} \gamma(k)z(k;\hat{\theta}(\cdot))\varepsilon(k;\hat{\theta}(\cdot))$$

and for large T

$$\hat{\theta}(k) \simeq \theta^* \qquad k \geq T$$

$$\sum_{T+1}^{T+N} \gamma(k)\, \bar{z}(k;\theta^*)\, \bar{\varepsilon}(k;\theta^*) = 0 \qquad (3.2.3)$$

Now

$$\bar{z}(k;\theta^*)\, \bar{\varepsilon}(k;\theta^*)$$

is a stationary stochastic process for given θ^*, and since

$$\sum_{T}^{\infty} \gamma(k) = \infty$$

it should follow that

$$E\, \bar{z}(k;\theta^*)\, \bar{\varepsilon}(k;\theta^*) = 0 \qquad (3.2.4)$$

if (3.2.3) holds with non-zero probability.

Consequently (3.2.4) is a necessary condition on θ^* to be a possible convergence point to (3.2.1).

By considering the number of points that satisfies (3.2.4), an analysis of uniqueness[†] of the method can be made. Such analysis is reported in e.g. Söderström (1973b) and in Section 4 of this report. The main idea of the analysis so far has been that if $\hat{\theta}(k)$ converges to a value θ^*, then the complex functions $\varepsilon(t+1\ ;\ \hat{\theta}(\cdot))$ and $\varphi(t+1\ ;\ \hat{\theta}(\ \cdot\ ))$ tend to functions $\bar{\varepsilon}(t+1;\ \theta^*)$ and $\bar{\varphi}(t+1;\ \theta^*)$, which are easier to handle. This idea can be extended as follows. Since $\varepsilon(t+1\ ;\ \hat{\theta}(\cdot))$ and $\varphi(t+1;\ \hat{\theta}(\cdot))$ depend on old $\hat{\theta}(k)$ only to a rapidly decreasing extent, and also since $\hat{\theta}(k)$ in most cases varies more and more slowly as k increases, it is reasonable to assume that $\varepsilon(t+1;\ \hat{\theta}(\cdot))$ in some respect tends to $\bar{\varepsilon}(t+1;\ \hat{\theta}(t+1))$ and analogously for $\varphi(t+1;\ \hat{\theta}(\cdot))$.

[†] i.e. whether the method can give several limits

Then

$$\hat{\theta}(t+N) = \hat{\theta}(t) + \sum_{t+1}^{t+N} \gamma(k) z(k;\hat{\theta}(\cdot))\varepsilon(k;\hat{\theta}(\cdot)) \simeq$$

$$\simeq \hat{\theta}(t) + \sum_{t+1}^{t+N} \gamma(k)\bar{z}(k;\hat{\theta}(k))\bar{\varepsilon}(k;\hat{\theta}(k)) \simeq$$

$$\simeq \hat{\theta}(t) + E\ \bar{z}(t;\hat{\theta}(t))\bar{\varepsilon}(t;\hat{\theta}(t)) \sum_{t+1}^{t+N} \gamma(k) \qquad (3.2.5)$$

where the last step should follow since $\bar{z}(k;\hat{\theta}(k))\bar{\varepsilon}(k;\hat{\theta}(k))$ $k=t+1,\ldots,t+N$ are a number of random variables with mean approximatively $E\ \bar{z}(t;\hat{\theta}(t))\cdot\bar{\varepsilon}(t;\hat{\theta}(t))$. This would imply that the sequence of estimates more or less follows the trajectories of the ODE

$$\dot{\theta} = f(\theta) \qquad (3.2.6)$$

where

$$f(\theta) = E\ \bar{z}(t;\theta)\bar{\varepsilon}(t;\theta)$$

This result can in fact be shown formally, as will be done in the next section, but it requires that several technical problems first are solved.

The conclusion about the possible convergence points made first in this section then implies that the sequence of estimates $\{\hat{\theta}(k)\}$ can converge only to stationary points of (3.2.6). This result can be strengthened by including 'dynamic properties', i.e. the stability character of the stationary point.

3.3 An ordinary differential equation associated with the recursive algorithms.

Consider a linear, stochastic discrete time system

$$y(t) = G(q^{-1})u(t) + H(q^{-1})e(t) \qquad (3.3.1)$$

where G and H are rational functions of the backward shift operator. $\{e(t)\}$ is a stationary sequence of independent, random variables. Let $H(z)$ have all poles strictly outside the unit circle. Assume that a model of one of the structures described in Section 2 will be fitted to the input - output data provided by (3.3.1). For this purpose a recursive algorithm (2.11.5)-(2.11.7) is used:

$$\hat{\theta}(t+1) = \hat{\theta}(t) + \gamma(t+1)\tilde{K}(t+1)\varepsilon(t+1) \qquad (3.3.2)$$

$$\tilde{K}(t) = \tilde{P}(t)z(t+1)/\{1 + \gamma(t+1)[\varphi(t+1)^T\tilde{P}(t)z(t+1)-1]\} \qquad (3.3.3)$$

$$\tilde{P}(t+1)^{-1} = \tilde{P}(t)^{-1} + \gamma(t+1)[z(t+1)\varphi(t+1)^T-\tilde{P}(t)^{-1}] \qquad (3.3.4)$$

Here $\hat{\theta}(t)$, $\varepsilon(t)$ $\varphi(t)$ and $z(t)$ have the meanings given in Table 2.3.

Let the input to the system be given by

$$u(t) = u_R(t) + F_1(q^{-1})y(t) + F_2(\hat{\theta}(t),q^{-1})y(t) \qquad (3.3.5)$$

where $u_R(t)$ is a stationary, external input signal independent of $\{e(t)\}$. The second term is a linear, time invariant feedback term, and the third term is a feedback element that depends on the current estimate.

Introduce

$$f(\theta) = E\ \bar{z}(t;\theta)\ \bar{\varepsilon}(t;\theta) \qquad (3.3.6a)$$

$$G(\theta) = E\ \bar{z}(t;\theta)\ \bar{\varphi}(t;\theta)^T \qquad (3.3.6b)$$

where the expectation shall be taken with respect to $\{e(t)\}$, assuming that $y(t)$ and $u(t)$ are generated with the feedback

$$u(t) = u_R(t) + F_1(q^{-1})y(t) + F_2(\theta;q^{-1})y(t) \qquad (3.3.7)$$

and that these sequences have reached stationarity. It shall also be assumed that $z(t)$, $\varphi(t)$ and $\varepsilon(t)$ are formed from $\{y(t)\}$, $\{u(t)\}$ according to Table 2.3, using a time invariant estimate θ, and so that they have reached stationarity: $\varepsilon(t) = \bar{\varepsilon}(t;\theta)$, etc.

Stationarity can be achieved, and hence the functions f and G defined only if the linear filters involved in the generation of $\{y(t)\}$ and $\{\bar{\varepsilon}(t;\theta)\}$ are exponentially stable.

Therefore, introduce the area

$$D_s = \{\theta \mid I-G(z)F_1(z)-G(z)F_2(\theta,z) = 0 \Rightarrow |z| > 1\}$$

which consists of all θ that makes the system (3.3.1), together with the feedback (3.3.7), exponentially stable. In the common case when the term F_2 is lacking this area does not have to be considered and can be replaced in the sequel by $R^{n_a+n_b}$ for the RLS and RIV methods and by $R^{n_a+n_b+n_c}$ for the RGLS, RML1 and RML2 methods.

Introduce also the area D_p, which consists of all θ such that the generation of ε, φ from y, u is exponentially stable. This area has relevance only for RML1 and RML2 and consists of

$$D_p = \{\theta \mid \hat{C}(z) \text{ has all zeroes strictly outside the unit circle}\}$$

For RLS and RIV let $D_p = R^{n_a+n_b}$ and for RGLS, RML1 and RML2 let $D_p = R^{n_a+n_b+n_c}$.

In order to let the definitions (3.3.4) make sense, it consequently has to be assumed that

$$\theta \in D_s \cap D_p.$$

Finally, introduce

$$D_I = \{\theta \mid G(\theta) \text{ invertible}\}.$$

Remark: The assumption on linearity of the system (3.3.1) and the feedback (3.3.5) is by no means crucial. The reason for it is that the notation and the stability concepts are somewhat easier to handle.

Consider now the ODE

$$\dot{\theta} = R^{-1} f(\theta) \tag{3.3.8a}$$

$$\dot{R} = G(\theta) - R \tag{3.3.8b}$$

The relationships between the algorithm (3.3.2) - (3.3.4) and the ODE (3.3.8) are given by the following theorem.

Theorem 3.1.

Consider the system (3.3.1) with input (3.3.5) and the algorithm (3.3.2) - (3.3.4). Suppose that, if the term $\Gamma_2(\theta;q^{-1})$ is present, it is Lipschitz continuous in θ. Suppose that $\{\gamma(t)\}$ satisfies (2.10.11)-(2.10.14). Suppose also that $E|e(t)|^p < \infty$ all $p > 0$.

a) Then $\hat{\theta}(t) \to \theta^* \in D_s \cap D_p \cap D_I$ with non-zero probability as $t \to \infty$ only if

i) $f(\theta^*) = 0$ (3.3.9)

ii) all eigenvalues to

$$G(\theta^*)^{-1} \frac{d}{d\theta} f(\theta)\Big|_{\theta=\theta^*} \tag{3.3.10}$$

have non-positive real parts

b) Assume that $\hat{\theta}(t)$ belongs to a compact subset of $D_s \cap D_p$ infinitely often w.p.1. Assume that $(\theta,R) = (\theta^*, G(\theta^*))$ $(\theta^* \in D_I)$ is a globally, asymptotically stable stationary point of the ODE (3.3.8). Then

$\hat{\theta}(t) \to \theta^*$ w.p.1 as $t \to \infty$.

c) Denote

$$\tau_t = \sum_1^t \gamma(s)$$

Assume that the right-hand side of (3.3.8) is continuously differentiable. Denote the solution to (3.3.8) with initial condition $\theta = \theta(t_o), R = \tilde{P}^{-1}(t_o)$ at $\tau = \tau_{t_o}$ by $\tilde{\theta}(\tau;\ \tau_{t_o},\ \theta(t_o), \tilde{P}^{-1}(t_o))$. Consider the ODE (3.3.8) linearized around this solution. Suppose that there exists a quadratic Lyapunov function for this linear, time varying ODE. (See e.g. Brockett (1970)). Let I be a set of integers, such that $\inf |\tau_i - \tau_j| = D > 0$, where $i \neq j$ and $i,j \in I$. Then there exist K, δ_o and ε_o, such that for

$\varepsilon < \varepsilon_o$, $|\gamma(t_o)x(t_o)| < \delta_o$

we have

$$P\{\sup_{\substack{t\in I\\ t>t_o}} |\theta(t) - \tilde{\theta}(\tau_t;\ \tau_{t_o}, \theta(t_o), \tilde{P}^{-1}(t_o))| > \varepsilon\} \leq$$

$$\leq \frac{K^r}{\varepsilon^{4r}} \sum_{j=t_o}^{N} \gamma(j)^r \qquad \text{any } r > 1 \qquad (3.3.11)$$

where $N = \sup_{t\in I} t$, which may be ∞.

□

Remark. The matrix in (3.3.10) is the system matrix of the linear ODE obtained from (3.3.8) by linearization around $(\theta^*, G(\theta^*))$, cf Section 5.2.

Proof. The proof is heuristically outlined in the previous section. The formal treatment is given in Appendix 1.

□

Theorem 3.1 can be applied in several ways to the analysis of the algorithm (3.3.2) -(3.3.4), and the analysis of the rest of this report will be based upon this theorem. The next section contains a discussion of the applicability of the theorem.

3.4 How to use Theorem 3.1.

It is naturally desirable that the recursive identification algorithm has the property that it converges w.p.1 to the "true" parameter value. According to Theorem 3.1b such a result can be shown if, essentially, a certain ODE is globally asymptotically stable. In many instances this is not the case, and even if it is, it may be difficult to prove rigorously.

However, in Section 5 some examples of convergence results are given.

A simpler, but still very important, task is to determine the set of possible convergence points.

According to Theorem 3.1a this can be made in two steps. First the set

$$D_s = \{\theta | f(\theta) = 0\}$$

is determined. This is done in Section 4 for the various methods. Then these candidates for limit values can be tested further, and those values that yield unstable linearized equations can be eliminated. Such analysis is made in Section 5. It may happen that then the resulting set of possible convergence points is empty, and that thus non-convergence of the method has been proved.

The third result of Theorem 3.1c is more technical. Let us first describe the result intuitively. Suppose that the estimates $\{\hat{\theta}(t)\}$ are plotted against the fictitious time τ_t, Figure 3.1.

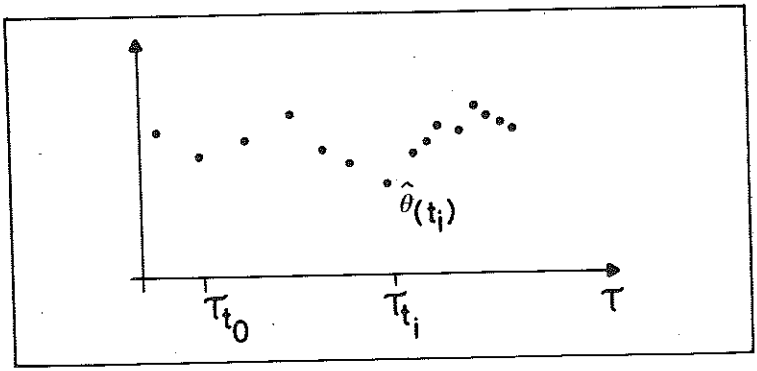

Figure 3.1 The estimates $\{\hat{\theta}(t)\}$ plotted against the fictitious time τ_t.

Plot the solution $\tilde{\theta}(\tau, \tau_{t_o}, \theta(t_o), \tilde{P}^{-1}(t_o))$ in the same diagram, Figure 3.2.

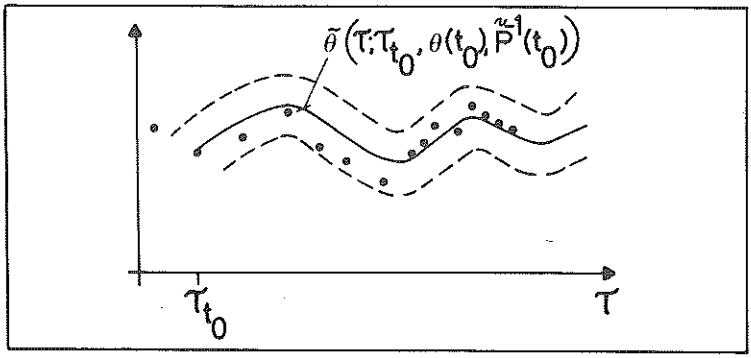

Figure 3.2 The estimates $\{\hat{\theta}(t)\}$ and the solution $\tilde{\theta}(\tau,\tau_{t_o},\theta(t_o),\tilde{P}^{-1}(t_o))$ plotted against the fictitious time τ_t.

Let I be a set of integers. The probability that <u>all</u> points $\theta(t)$; $t\in I$ simultaneously are within a certain distance ε from the trajectory is then estimated in (3.3.11).

Notice that since the sum

$$\sum_{1}^{\infty} \gamma(j)^p$$

is convergent, the RHS of (3.3.11) can, for fixed ε, be chosen arbitrarily small by taking t_o sufficiently large. Thus, the theorem states that the trajectories of the ODE (3.3.8) describe the behaviour of the algorithm (3.3.2) - (3.3.2) arbitrarily well for sufficiently large time points.

It should be remarked that, although the proof of Theorem 3.1c provides an estimate of K, it is not prac-

tically feasible to use the theorem to obtain numerical bounds for the probability. The estimates are too crude. The main value of the theorem is that a basic relationship between the trajectories and the algorithm is established.

In Ljung (1974a) it is discussed how the shape of the trajectories of (3.3.8) influences the behaviour of (3.3.2) - (3.3.4), rate of convergence, etc. Numerical solution of (3.3.8) therefore is a very valuable complement to simulation of the algorithm. Examples of this are given in Section 6.

4. UNIQUENESS

4.1 Introduction.

It was shown in Section 3 that the possible convergence points for the recursive algorithms are the solutions of the equation

$$E\ \bar{z}(t;\theta^*)\bar{\varepsilon}(t;\theta^*) = 0 \qquad (4.1.1)$$

In this section the purpose will be to examine the number of solutions of (4.1.1) for the different methods. It is generally assumed that the process is operating in open loop. To analyse systems operating in closed loop it might be necessary to know the feedback laws.

The question whether there is a unique solution for the recursive identification methods is very closely related to the same question for the corresponding off-line methods. The question is treated e.g. in Söderström (1972, 1973a) and Åström - Söderström (1974) for different off-line methods and the results in these references will be strongly utilized.

For all the methods (i.e. RLS, RGLS, RIV, RML1, RML2) it is seen that the true parameter vector θ_o is a solution of (4.1.1), i.e.

$$E\ \bar{z}(t;\theta_o)\bar{\varepsilon}(t;\theta_o) = 0 \qquad (4.1.2)$$

The interesting question is thus if eq. (4.1.1) has more solutions than θ_o.

4.2 The RLS method.

Since the RLS method is equivalent to the off-line LS method, it follows from a well-known analysis that the result of this method is unique. A straightforward analysis of (4.1.1) would be to rewrite it as

$$0 = E\,\varphi(t)\bar{\varepsilon}(t;\theta^*) = E\,\varphi(t)[y(t)-\varphi(t)^T\theta^*]$$

$$= E\,\varphi(t)[\varphi(t)^T\theta_o + e(t)-\varphi(t)^T\theta^*]$$

$$= [E\,\varphi(t)\varphi(t)^T][\theta_o-\theta^*] \qquad (4.2.1)$$

This is a system of linear equations, and due to assumptions made in Section 3, (here essentially open loop operation + stationary processes) it implies that $E\,\varphi(t)\varphi(t)^T$ is positive definite and thus non-singular. This shows that $\theta^* = \theta_o$ is the only solution.

4.3 The RGLS method.

The analysis of the RGLS method given in the following lemma shows that in some cases (4.1.1) can have several solutions.

For the RGLS method eq. (4.1.1) becomes two systems of equations, namely (cf (2.3.1) - (2.3.13)).

$$\begin{cases} E\,\bar{\varepsilon}_1(t;\theta^*)\bar{\varphi}_1(t;\theta^*) = 0 \\ E\,\bar{\varepsilon}_2(t;\theta^*)\bar{\varphi}_2(t;\theta^*) = 0 \end{cases} \qquad (4.3.1)$$

Lemma 4.1 The equations (4.3.1) can equivalently be written as

$$V'_{\theta^*}(\theta^*) = 0^{\dagger} \qquad (4.3.2)$$

†In the following $\frac{d}{d\theta}V(\theta)$ and $V'_\theta(\theta)$ will both be used to denote the derivative of V with respect to θ (analogously for other functions).

where the function $V(\theta^*)$ is given by[†]

$$V(\theta^*) = \frac{1}{2} E\, \varepsilon(t)^2;\ \varepsilon(t) = A^*(q^{-1})C^*(q^{-1})y(t) - B^*(q^{-1})C^*(q^{-1})u(t) \quad (4.3.3)$$

Proof. It is easy to see that

$$\bar{\varepsilon}_1(t;\theta^*) = A^*(q^{-1})[C^*(q^{-1})y(t)] - B^*(q^{-1})[C^*(q^{-1})u(t)] = \varepsilon(t)$$

$$\bar{\varepsilon}_2(t;\theta^*) = C^*(q^{-1})[A^*(q^{-1})y(t) - B^*(q^{-1})u(t)] = \varepsilon(t)$$

Moreover

$$V'_{\theta^*}(\theta^*) = E\, \varepsilon(t)\, \varepsilon'_{\theta^*}(t)$$

Now

$$\varepsilon'_{\theta^*}(t) = [C^*(q^{-1})y(t-1)\ldots C^*(q^{-1})y(t-n_a) - C^*(q^{-1})u(t-1)\ldots - C^*(q^{-1})u(t-n_b)\{A^*(q^{-1})y(t-1) - B^*(q^{-1})u(t-1)\}\ldots \{A^*(q^{-1})y(t-n_c) - B^*(q^{-1})u(t-n_c)\}]$$

$$= [-\varphi_1(t;\theta^*)^T \mid -\varphi_2(t;\theta^*)^T]$$

and the assertion of the lemma easily follows. □

Remark. The function $V(\theta^*)$ given by (4.3.3) has the following interpretation. The off-line GLS method can be interpreted as a special way of minimizing the function $\Sigma\varepsilon^2(t)$, see Söderström (1972). $V(\theta^*)$ is an asymptotic normalized version of this loss function.

[†]The polynomials associated with the parameter vector θ^* are denoted by A^*, B^* and C^*. This convention is also used in the forthcoming proof of Lemma 4.2.

Corr Suppose that the system is controllable and that the input signal is persistently exciting of order (n_a+n_b). Then there are two constants S_o and S_1, $0 < S_1 < S_o < \infty$ such that

i) if the signal-to-noise ratio S satisfies $S_o < S$, then $\theta^* = \theta_o$ is the only solution of (4.3.1).

ii) if S satisfies $0 < S < S_1$, then (4.3.2) has multiple solutions.

Proof: The proof follows immediately from the analysis of equation (4.3.2) made in Söderström (1972). □

4.4 The RIV method.

The analysis of the RIV method is almost a repetition of Section 4.2. The equation (4.1.1) can be written as

$$0 = E\ \bar{z}(t;\theta^*)\bar{\varepsilon}(t;\theta^*) = E\ \bar{z}(t;\theta^*)[y(t)-\varphi(t)^T\theta^*] =$$

$$= E\ \bar{z}(t;\theta^*)[\varphi(t)^T\theta_o+e(t)-\varphi(t)^T\theta^*] =$$

$$= [E\ \bar{z}(t;\theta^*)\varphi(t)^T][\theta_o-\theta^*] = G(\theta^*)(\theta_o-\theta^*) \qquad (4.4.1)$$

The procedure has degenerated in some way if $G(\theta^*)$ becomes singular. If only solutions of (4.1.1) such that $G(\theta^*)$ is non-singular is considered, it is clear that $\theta^*=\theta_o$ is the only solution. This does not prevent (4.1.1) from also having other solutions in some cases. However, since it is possible to test on singularity of G during the identification, an examination of such solutions is of minor interest.

Remark. Notice that it is important that the system operates in open loop. For closed loop systems $E(\bar{z}(t;\theta^*)e(t)) \neq 0$ in general, unless e(t) is white noise. This means that in such a case θ_o is not a possible convergence point.

4.5. The RML1 method.

It is not easy to give a complete analysis of equation (4.1.1) for the RML1 method. An independent analysis of this method is given in Åström (1974). Some partial results are given in the following lemma.

Lemma 4.2 In each of the following cases sufficient conditions for θ to be the unique solution of (4.1.1) are given.

i) Consider ARMA processes, i.e. $A(q^{-1})y(t)=C(q^{-1})e(t)$. Then $\theta^* = \theta_o$ follows.

ii) Assume that the input signal is persistently exciting of order (n_a+n_b). Then there is a constant S_1 such that if the signal-to-noise ratio $S < S_1$, then $\theta^* = \theta_o$ follows.

Proof

Part i) is proved as follows. The equation (4.1.1) can be written as

$$\begin{cases} E\, y(t-i)\, \dfrac{A^*(1^{-1})C(q^{-1})}{A(q^{-1})C^*(q^{-1})}\, e(t) = 0 \quad 1 \leq i \leq n_a \\ E\, \dfrac{A^*(q^{-1})}{C^*(q^{-1})}\, y(t-i)\, \dfrac{A^*(q^{-1})C(q^{-1})}{A(q^{-1})C^*(q^{-1})}\, e(t) = 0 \quad 1 \leq i \leq n_c \end{cases} \tag{4.5.1}$$

It may be possible that A* and C* have common factors. For this reason introduce new polynomials $\bar{A}$, $\bar{C}$ and $\bar{D}$ by

$$A^* = \bar{A}\bar{D}$$

$$C^* = \bar{C}\bar{D}$$

where $\bar{A}$ and $\bar{C}$ are relatively prime and $\bar{D}$ is a unitary polynomial $\bar{D}(z) = 1 + d_1 z + \dots$.

Then the equations (4.5.1) can be rewritten as

$$\begin{bmatrix} 1 & \bar{c}_1 \dots \bar{c}_{\bar{n}_c} & & \\ & \ddots & \ddots & \\ & & \ddots & \bar{c}_{\bar{n}_c} \\ \hline 1 & \bar{a}_1 \dots & \bar{a}_{\bar{n}_a} & \\ & \ddots & \ddots & \\ & & 1 \dots & \bar{a}_{\bar{n}_a} \end{bmatrix} E \begin{bmatrix} \frac{C(q^{-1})}{A(q^{-1})\bar{C}(q^{-1})} e(t-1) \\ \vdots \\ \frac{C(q^{-1})}{A(q^{-1})\bar{C}(q^{-1})} e(t-n_a-\bar{n}_c) \end{bmatrix} \frac{\bar{A}(q^{-1})C(q^{-1})}{A(q^{-1})\bar{C}(q^{-1})} e(t)=0$$

It follows from the theory of resultants, see e.g. van der Waerden (1937) that the matrix with dimensions $(n_a+n_c)(n_a+\bar{n}_c)$ has rank $n_a+\bar{n}_c$. Thus

$$E \frac{C(q^{-1})}{A(q^{-1})\bar{C}(q^{-1})} e(t-j) \frac{\bar{A}(q^{-1})C(q^{-1})}{A(q^{-1})\bar{C}(q^{-1})} e(t) = 0 \quad 1 \leq j \leq n_a+\bar{n}_c$$

which can be written as integrals around the unit circle as

$$\frac{1}{2\pi i} \oint \frac{C(z)}{A(z)\bar{C}(z)} \frac{\bar{A}(z^{-1})C(z^{-1})}{A(z^{-1})\bar{C}(z^{-1})} z^j \frac{dz}{z} = 0 \qquad 1 \leq j \leq n_a+\bar{n}_c$$

Note that the function $\bar{A}(z^{-1})C(z^{-1})/A(z^{-1})\bar{C}(z^{-1})$ also can be written as a fraction between two polynomials in z and of degree $(n_a+\bar{n}_c)$. Then it follows from Åström-Söderström (1974) first that

$$\bar{A}(z^{-1})C(z^{-1})/A(z^{-1})\bar{C}(z^{-1}) \equiv 1$$

and then that this implies

$$A^*(z) \equiv A(z),\ C^*(z) \equiv C(z)$$

i.e. $\theta^* = \theta_o$

Part ii) of the lemma follows from theorems proved in Söderström (1972).

□

4.6 The RML2 method.

The analysis of equation (4.1.1) for the RML2 method consists of two steps. The first step is to realise that this equation can be rewritten as $V'_{\theta^*}(\theta^*)=0$ where $V(\theta^*)$ is a loss function. This loss function has in fact been studied in Söderström (1973a) and Åström-Söderström (1974) and some results on the uniqueness of the RML2 method are then easily obtained.

Lemma 4.3 The equation (4.1.1) can be equivalently expressed as

$$V'_{\theta^*}(\theta^*) = 0 \qquad (4.6.1)$$

where

$$V(\theta^*) = \frac{1}{2} E\, \varepsilon(t)^2;\ \varepsilon(t) = \frac{A^*(q^{-1})}{C^*(q^{-1})} y(t) - \frac{B^*(q^{-1})}{C^*(q^{-1})} u(t) \qquad (4.6.2)$$

Proof Since $V'_{\theta^*}(\theta^*) = E\, \varepsilon(t)\varepsilon'_{\theta^*}(t)$ it is sufficient to show that $\bar{\varphi}(t;\theta^*) = -\varepsilon'_{\theta^*}(t)$. However, this can be shown by straightforward calculations, cf Åström-Bohlin (1965).

□

Corr 1 Consider the case of ARMA processes, i.e. $n_b=0$. Then $\theta^* = \theta_o$ is the unique solution of (4.6.1).

Proof See Åström-Söderström (1974). □

Corr 2 Assume that the input signal is persistently exciting of order (n_a+n_b). Then there exist two constants S_o and S_1, $0 < S_1 < S_o < \infty$ such that

i) if the signal-to-noise ratio S is larger than S_o then $\theta^* = \theta_o$ is the only solution of (4.6.1)

ii) if S is smaller than S_1 then $\theta^* = \theta_o$ is the only solution of (4.6.2).

Proof See Söderström (1973a). □

4.7 Summary of the section.

The obtained results are now summarized in Table 4.1. It is generally assumed that the process is operating in open loop and that the input signal is persistently exciting of a proper order and that only points giving a non-singular $G(\theta)$ matrix are of interest. Let S denote the signal-to-noise ratio.

Method	Sufficient conditions for a unique solution	Sufficient conditions for multiple solutions
RLS	No additional conditions are needed	There are never multiple solutions
RGLS	$S \gg 1$	$S \ll 1$
RIV	No additional conditions are needed	There are never multiple solutions
RML1	i) ARMA processes or ii) $S \ll 1$ or iii) $n_a=n_b=n_c=1$ and $u(t)$ white noise, see Åström (1974)	(Not examined)
RML2	i) ARMA processes or ii) $S \ll 1$ or iii) $S \gg 1$	(Not examined)

Table 4.1 Summary of the results of Section 4.

5. CONVERGENCE

5.1 Introduction.

The purpose of this section is to further analyse the ordinary differential equations introduced in Section 3. The stationary points were examined in Section 4. Stability of these points will now be considered. The differential equations are written as

$$\dot{\theta} = R^{-1} f(\theta)$$
$$\dot{R} = G(\theta) - R \qquad (5.1.1)$$

The stationary points are determined by the solutions of $f(\theta^*) = 0$ and then also characterized by $R = G(\theta^*)$.

In Section 5.2 the equations (5.1.1) are linearized around stationary points. In the following Sections 5.3 - 5.7 linearization is applied to the different recursive identification methods. Thus the possible convergence points according to the analysis of Section 4 can be tested further. In Section 5.8 global stability is considered and finally in Section 5.9 a summary of the results in this section is given.

5.2 Linearization.

Introduce $r = \mathrm{col}(R)$, i.e. let r be a column vector containing the elements of R. If R is symmetric then it is sufficient to include equal elements once.

When the system (5.1.1) is linearized around a stationary point $(\theta^{*T}, r^{*T})^T$ it becomes

$$\frac{d}{dt}\begin{bmatrix} \theta - \theta^* \\ r - r^* \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \theta - \theta^* \\ r - r^* \end{bmatrix} \qquad (5.2.1)$$

It is easy to see that

$$A_{11} = G(\theta^*)^{-1} f'_\theta(\theta^*)$$

$$A_{12} = 0$$

$$A_{21} = \frac{d}{d\theta} \operatorname{col}[G(\theta)]\Big|_{\theta=\theta^*}$$

$$A_{22} = -I$$

To derive the expression for A_{12}, utilize $f(\theta^*) = 0$. Thus the matrix becomes triangular and the stability is determined by the eigenvalues of the matrix

$$A = G(\theta^*)^{-1} f'_\theta(\theta^*) \qquad (5.2.2)$$

For a general stationary point

$$f'_\theta(\theta^*) = E\, \bar{z}'_\theta(t;\theta^*)\bar{\varepsilon}(t;\theta^*) + E\, \bar{z}(t;\theta^*)\bar{\varepsilon}'_\theta(t;\theta^*) \qquad (5.2.3)$$

Consider now in particular the true parameter vector. It will be allowed that the system is operating in closed loop. Adaptive systems where the regulator depends on the estimates will also be included in the analysis.

For linearization around θ_o the first term will vanish. For the methods RLS, RGLS, RML1 and RML2 this follows since then $\bar{\varepsilon}(t;\theta_o) = e(t)$ and the matrix $\bar{z}'_\theta(t;\theta_o)$ depends only on older values of the noise and eventually on an independent input signal. For the RIV method the residual $\bar{\varepsilon}(t;\theta_o)$ becomes equal to the noise $H(q^{-1})e(t)$, while the matrix $\bar{z}'_\theta(t;\theta_o)$ contains only elements depending on the input signal. If for this method the additional assumption of open loop operation is made then the first term of (5.2.3) vanishes.

Some caution has to be taken in the interpretation of $\bar{\varepsilon}'_\theta(t;\theta^*)$ for adaptive systems. Let the regulator be written as

$$u(t) = -F(\hat{\theta}(t);q^{-1})y(t) \qquad (5.2.4)$$

The cases when the regulator depends explicitly on also older estimates is straightforwardly obtained by generalization. With use of (5.2.4) the stationary residuals can be written symbolically as

$$\bar{\varepsilon}(t;\theta,F(\theta))$$

and they obviously depend on θ both explicitly, and implicitly via the regulator. Thus the derivative written in a symbolic way becomes

$$\bar{\varepsilon}'_\theta(t;\theta) = \frac{\partial}{\partial\theta}\bar{\varepsilon}(t;\theta,F(\theta)) + \frac{\partial}{\partial F}\bar{\varepsilon}(t;\theta,F(\theta))\frac{\partial F}{\partial\theta} \qquad (5.2.5)$$

However, since $\bar{\varepsilon}(t;\theta_o,F(\theta_o))$ is pure noise, (in fact the white noise $e(t)$ for all methods but RIV and equal to the noise $H(q^{-1})e(t)$ for RIV) and thus independent of F, it follows that the second term in (5.2.5) can be dropped when linearization around θ_o is made.

To summarize, for the true parameter vector θ_o the matrix A becomes (with $\theta^*=\theta_o$)

$$A = [E\,\bar{z}(t;\theta^*)\bar{\varphi}(t;\theta^*)]^{-1}\,E\,\bar{z}(t;\theta^*)\frac{\partial}{\partial\theta}\bar{\varepsilon}(t;\theta^*) \qquad (5.2.6)$$

The only restriction has to be made for the RIV method, for which it is assumed that the process is operating in open loop or that $H(q^{-1})=1$.

In the following sections the matrix A will be examined for the different methods.

5.3 The RLS method.

It is known from Section 4.2 that $\theta^* = \theta_o$ is the only stationary point. Thus the matrix A is always given by (5.2.6). Moreover for the RLS method

$$\bar{z}(t;\theta^*) = \bar{\varphi}(t;\theta^*) = \varphi(t) \tag{5.3.1}$$

$$\frac{\partial \bar{\varepsilon}}{\partial \theta}(t;\theta^*) = -\varphi(t)$$

$$A = -I \tag{5.3.2}$$

Thus θ_o is a stable solution of the ODE. Moreover, all the eigenvalues of A are in -1 and they are not coupled.

5.4 The RGLS method.

For the RGLS method the analysis becomes more difficult. First there will be two systems of the form (5.1.1) to consider, and these two systems are coupled. The analog of (5.2.1) will be

$$\frac{d}{dt}\begin{bmatrix} \theta_1-\theta_1^* \\ \theta_2-\theta_2^* \\ r_1-r_1^* \\ r_2-r_2^* \end{bmatrix} = \left[\begin{array}{cc|cc} & & 0 & 0 \\ & A & 0 & 0 \\ \hline x & x & -I & 0 \\ x & x & 0 & -I \end{array}\right] \begin{bmatrix} \theta_1-\theta_1^* \\ \theta_2-\theta_2^* \\ r_1-r_1^* \\ r_2-r_2^* \end{bmatrix} \tag{5.4.1}$$

The blocks marked x are without interest when the stability is examined. The stability is entirely determined by the properties of the matrix A.

The matrix A can be written as

$$A = \begin{bmatrix} [E\ \bar{\varphi}_1(t;\theta^*)\bar{\varphi}_1(t;\theta^*)^T]^{-1}\{E\ \bar{\varphi}_1(t;\theta^*)\bar{\varepsilon}_{1\theta}'(t;\theta^*) + \\ \qquad + E\ \bar{\varphi}_1'_{\theta}(t;\theta^*)\bar{\varepsilon}_1(t;\theta^*)\} \\ [E\ \bar{\varphi}_2(t;\theta^*)\bar{\varphi}_2(t;\theta^*)^T]^{-1}\{E\ \bar{\varphi}_2(t;\theta^*)\bar{\varepsilon}_2'_{\theta}(t;\theta^*) + \\ \qquad + E\ \bar{\varphi}_2'_{\theta}(t;\theta^*)\bar{\varepsilon}_2(t;\theta^*)\} \end{bmatrix} \qquad (5.4.2)$$

But, see Section 4.3,

$$\bar{\varepsilon}_1(t;\theta^*) = \bar{\varepsilon}_2(t;\theta^*)$$

$$\bar{\varepsilon}_1'_{\theta}(t;\theta^*) = [-\bar{\varphi}_1(t;\theta^*)^T - \bar{\varphi}_2(t;\theta^*)^T]$$

Utilizing these relations the matrix A can be written as

$$A = \begin{bmatrix} -I & -[E\ \bar{\varphi}_1(t;\theta^*)\bar{\varphi}_1(t;\theta^*)^T]^{-1}E\ \bar{\varphi}_1(t;\theta^*)\bar{\varphi}_2(t;\theta^*)^T \\ -[E\ \bar{\varphi}_2(t;\theta^*)\bar{\varphi}_2(t;\theta^*)^T]^{-1}E\ \bar{\varphi}_2(t;\theta^*)\bar{\varphi}_1(t;\theta^*)^T & -I \end{bmatrix}$$

$$+ \begin{bmatrix} [E\ \bar{\varphi}_1(t;\theta^*)\bar{\varphi}_1(t;\theta^*)^T]^{-1}E\ \bar{\varphi}_1'_{\theta}(t;\theta^*)\bar{\varepsilon}_1(t;\theta^*) \\ [E\ \bar{\varphi}_2(t;\theta^*)\bar{\varphi}_2(t;\theta^*)^T]^{-1}E\ \bar{\varphi}_2'_{\theta}(t;\theta^*)\bar{\varepsilon}_2(t;\theta^*) \end{bmatrix} \qquad (5.4.3)$$

Consider first the true parameter vector θ_o. Then the second matrix in (5.4.3) will vanish.

Lemma 5.1 The matrix A, given by (5.4.3) and evaluated at the true parameter vector θ_o, has all eigenvalues strictly inside the left half plane.

Proof Introduce the square matrix Q by

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} = E \begin{bmatrix} \bar{\varphi}_1(t;\theta_o) \\ \bar{\varphi}_2(t;\theta_o) \end{bmatrix} [\bar{\varphi}_1(t;\theta_o)^T \ \bar{\varphi}_2(t;\theta_o)^T]$$

This matrix is positive definite. Introduce the matrix $\bar{Q}_{12}$ by

$$Q_{12} = Q_{11}^{1/2} \bar{Q}_{12} Q_{22}^{1/2}$$

Thus

$$Q = \begin{bmatrix} Q_{11}^{1/2} & 0 \\ 0 & Q_{22}^{1/2} \end{bmatrix} \begin{bmatrix} I & \bar{Q}_{12} \\ \bar{Q}_{12}^T & I \end{bmatrix} \begin{bmatrix} Q_{11}^{1/2} & 0 \\ 0 & Q_{22}^{1/2} \end{bmatrix}$$

Thus the matrix in the middle must be positive definite and in particular the matrix

$$I - \bar{Q}_{12}^T \bar{Q}_{12}$$

as well.

Now, the matrix A can be written as

$$A = \begin{bmatrix} -I & -Q_{11}^{-1} Q_{12} \\ -Q_{22}^{-1} Q_{12}^T & -I \end{bmatrix}$$

The characteristic equation of A is then examined as follows. Assume for simplicity that $n_a+n_b \geq n_c$ (the opposite case can be treated analogously)

$$0 = \det(sI-A) = \det \begin{bmatrix} sI + I & Q_{11}^{-1}Q_{12} \\ Q_{22}^{-1}Q_{12}^{T} & sI + I \end{bmatrix} =$$

$$= \det[(s+1)I_{n_a+n_b}]\det[(s+1)I_{n_c} - Q_{22}^{-1}Q_{12}^{T}\frac{1}{s+1}I_{n_a+n_b}^{-1}Q_{11}^{-1}Q_{12}]$$

$$=(s+1)^{n_a+n_b-n_c}\det[(s+1)^2 I - Q_{22}^{-1/2}\bar{Q}_{12}^{T}\bar{Q}_{12}Q_{22}^{1/2}]$$

$$=(s+1)^{n_a+n_b-n_c}\det[(s+1)^2 I - \bar{Q}_{12}^{T}\bar{Q}_{12}]$$

Let now the matrix $I - \bar{Q}_{12}^{T}\bar{Q}_{12}$ have eigenvalues $0 < \lambda_1 \leq \lambda_2 \ldots \leq \lambda_{n_c}$, which are positive due to previous conclusions. The matrix A has obviously $n_a+n_b-n_c$ eigenvalues in -1 and the remaining $2n_c$ are given by

$$(s+1)^2 - 1 + \lambda_i = 0 \qquad 1 \leq i \leq n_c$$

or

$$s = -1 \pm \sqrt{1-\lambda_i} \qquad 1 \leq i \leq n_c$$

which all have strictly negative real parts. This proves the lemma.

□

Assume now in the rest of this section that the process is operating in open loop. It was found that there are other stationary points when the signal-to-noise ratio S is very small. The proof, given in Söderström (1972), shows the existence of stationary points satisfying

$$A^*(z)C^*(z) = A(z)C(z) + O(S)^{\dagger} \qquad (5.4.4)$$

which e.g. is obtained as

$$A^*(z) = C(z) + O(S)$$

$$C^*(z) = A(z) + O(S)$$

It will now be analysed if points of this type will correspond to stable stationary solutions of the ODE.

Quite generally

$$\bar{\varphi}_1'_{\theta}(t;\theta) = \begin{bmatrix} & & -y(t-2).....-y(t-n_c-1) \\ 0 & 0 & \vdots \\ & & -y(t-n_a-1)...-y(t-n_a-n_c) \\ & & u(t-2)..... u(t-n_c-1) \\ & & \vdots \\ & & u(t-n_b-1)....u(t-n_b-n_c) \end{bmatrix}$$

$$\bar{\varphi}_2'_{\theta}(t;\theta) = \begin{bmatrix} -y(t-2).....-y(t-n_a-1),u(t-2).....u(t-n_b-1) & \\ \vdots & 0 \\ -y(t-n_c-1)...-y(t-n_a-n_c),u(t-n_c-1)...u(t-n_b-n_c) & \end{bmatrix}$$

For points satisfying (5.4.4)

$$\bar{\varepsilon}_1(t;\theta^*) = \bar{\varepsilon}_2(t;\theta^*) = e(t) + O(S)$$

---

†The terms $O(S)$ tend to zero as S, i.e. $S^{-1}O(S)$ is bounded when $S \to 0$.

Clearly

$$E\,\bar{\varphi}_1'{}_\theta(t;\theta)e(t) = 0\ ;\ E\,\bar{\varphi}_{2\theta}'(t;\theta)e(t) = 0$$

After some inspection it can also be seen that the relative change from the case $\theta^* = \theta_o$ of the matrix elements in A is of order S. This means that if S is sufficiently small, then the eigenvalues of A will remain in the left hand plane and the solution of the ODE will be stable.

5.5 The RIV method.

The analysis of the RIV is almost a repetition of Section 5.3. Since

$$\frac{\partial \bar{\varepsilon}}{\partial \theta}\ (t;\theta^*) = -\bar{\varphi}(t;\theta^*)$$

it is easily concluded that

$$A = -I \qquad (5.5.1)$$

and stability is proved.

5.6 The RML1 method.

For this method it is difficult to analyse A in the general case. This difficulty is not only a technical problem because it will be shown that sometimes the true parameter vector is an unstable solution to the ODE. The analysis will be limited to simple processes and specific numerical examples.

Example 5.6.1 Consider the case of a pure moving average, i.e. when only C-parameters are to be estimated. Then the matrix A becomes, evaluated for the true

parameter vector (the only stationary point, see Section 4.5)

$$A = -\{E\begin{bmatrix} e(t-1) \\ \vdots \\ e(t-n_c) \end{bmatrix}[e(t-1)\ldots e(t-n_c)]\}^{-1} E \begin{bmatrix} e(t-1) \\ \vdots \\ e(t-n_c) \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} \frac{1}{C(q^{-1})} e(t-1) & \ldots & \frac{1}{C(q^{-1})} e(t-n_c) \end{bmatrix}$$

$$= \begin{bmatrix} -1 & & & \\ x & -1 & & 0 \\ x & & \ddots & \\ \vdots & & & \ddots \\ x & \cdots & x & -1 \end{bmatrix} \qquad (5.6.1)$$

The elements below the diagonal are without interest in determining the stability properties. It is seen that the matrix A has all eigenvalues in -1. Note, however, that in contrast to (5.2.1) and (5.5.1) the eigenvalues are coupled. This may mean that the expected convergence rate is slow for the RML1 method.

□

Example 5.6.2 Consider a first-order ARMA process, i.e.

$$y(t) + a\, y(t-1) = e(t) + c\, e(t-1)$$

Then the matrix A becomes (evaluated at the true parameter vector, which is the only stationary point due to Lemma 4.2)

$$A = -\{E\begin{bmatrix} -y(t-1) \\ e(t-1) \end{bmatrix}[-y(t-1)e(t-1)]\}^{-1} E\begin{bmatrix} -y(t-1) \\ e(t-1) \end{bmatrix}[-\frac{1}{C(q^{-1})}y(t-1),$$

$$\frac{1}{C(q^{-1})} e(t-1)] \qquad (5.6.2)$$

Straightforward, but somewhat tedious calculations, lead to the explicit form

$$A = \begin{bmatrix} \frac{a}{c-a} & -\frac{c(1-a^2)}{(c-a)(1-ac)} \\ \frac{c}{c-a} & \frac{a-2c+ac^2}{(c-a)(1-ac)} \end{bmatrix}$$

which has eigenvalues in -1 and in $-\frac{1}{1-ac}$. Thus θ_o is a stable solution to the ODE in this case.

□

It will now be shown that there exist systems, however, such that the matrix A, evaluated at the true parameter vector, will have eigenvalues with positive real parts. This means that for these systems the true parameter vector is an unstable solution to the ODE, and that the corresponding recursive algorithm cannot converge to the true value. To construct such systems, Ljung-Söderström-Gustavsson (1974), consider

$$A = [E\ \bar{z}(t;\theta^*)\ \bar{\varphi}(t;\theta^*)]^{-1}\ E\ \bar{z}(t;\theta^*)\ \frac{\partial}{\partial\theta}\ \bar{\varepsilon}(t;\theta^*)$$

where

$$\frac{\partial}{\partial\theta}\ \bar{\varepsilon}(t;\theta^*) = \bar{\tilde{\varphi}}(t;\theta^*) = E[\tilde{y}(t-1),\ldots,\tilde{y}(t-n_a),-\tilde{u}(t-1),\ldots,$$

$$-\tilde{u}(t-n_b),-\tilde{e}(t-1),..,-\tilde{e}(t-n_c)]^T$$

with

$$C^*(q^{-1})\tilde{y}(t) = y(t);\ C^*(q^{-1})\tilde{u}(t) = u(t);\ C^*(q^{-1})\tilde{e}(t) = e(t)$$

Furthermore $\bar{z}(t;\theta^*) = \bar{\varphi}(t;\theta^*)$ for RML1

(Here it has been assumed that the generation of u(t) does not depend on $\hat{\theta}(t)$. This assumption does not

hold for adaptive regulators of various types.)

Hence

$$E\ \bar{z}(t;\theta^*)\ \frac{\partial}{\partial\theta}\ \bar{\varepsilon}(t;\theta^*) = E\ \bar{\varphi}(t;\theta^*)\tilde{\bar{\varphi}}(t;\theta^*)$$

It follows that

$$\operatorname{tr} E\ \bar{z}(t;\theta^*)\ \frac{\partial}{\partial\theta}\ \bar{\varepsilon}(t;\theta^*) = n_a[-E\ y(t)\tilde{y}(t)] + n_b[-E\ u(t)\tilde{u}(t)] +$$

$$+ n_c[-E\ e(t)\tilde{e}(t)]$$

Assume that $\{u(t)\}$ is white noise independent of the noise and with variance 1, and that $E\ e^2(t)=1$. Then the covariances are given by, Åström (1970),[+]

$$E\ y(t)\tilde{y}(t) = \frac{1}{2\pi i}\oint \frac{1}{z}\,\frac{1}{C(z)}\,\phi_{yy}(z)dz =$$

$$= \frac{1}{2\pi i}\oint \frac{1}{z}\,\frac{1}{C(z)}\left[\frac{B(z)B(z^{-1})}{A(z)A(z^{-1})} + \frac{C(z)C(z^{-1})}{A(z)A(z^{-1})}\right]dz \qquad (5.6.3)$$

$$E\ u(t)\tilde{u}(t) = E\ e(t)\tilde{e}(t) = 1$$

Hence

$$\operatorname{tr} E\ \bar{z}(t;\theta^*)\ \frac{\partial}{\partial\theta}\ \bar{\varepsilon}(t;\theta^*) = -(n_b+n_c) - n_a\ E\ y(t)\tilde{y}(t)$$

The point is now that $E\ y(t)\tilde{y}(t)$ can be made negative. As seen from (5.6.3), this may be achieved if $C(z)$ is chosen so that

$$\operatorname{Re} C(z_o) < 0 \quad \text{some } z_o;|z_o| = 1 \quad \text{and if } |A(z_o)| \text{ is small.}$$

By taking $|A(z_o)|$ sufficiently small, the trace will be positive and hence

$$E\ \bar{z}(t;\theta^*)\ \frac{\partial}{\partial\theta}\ \bar{\varepsilon}(t;\theta^*)$$

[+] The integration path in (5.6.3) is the unit circle.

has at least one positive eigenvalue. The eigenvalues of A can then be checked by calculation. The same procedure also applies for ARMA-processes with $n_b=0$.

Example 5.6.3. For the system

$$y(t) + 0.9y(t-1) + 0.95y(t-2) = u(t-1) + e(t) + 1.5e(t-1) + \\ + 0.75e(t-2) \qquad (5.6.4)$$

the matrix

$$E\ \bar{z}(t;\theta^*)\ \frac{\partial}{\partial\theta}\ \bar{\varepsilon}(t;\theta^*)$$

has the eigenvalues $(-3.081\pm24.239i,\ 0.864,\ 1.116,\ 1)$, and A has the eigenvalues $(0.162\pm1.383i,\ -1,\ -1,\ -1)$ if $u(t)$ is white noise of unit variance. Simulations of this system are given in Johannesson-Wesström (1974). □

Example 5.6.4. The ARMA-process

$$y(t) + 0.9y(t-1) + 0.95y(t-2) = e(t) + 1.5e(t-1) + \\ + 0.75e(t-2) \qquad (5.6.5)$$

yields the eigenvalues of

$$E\ \bar{z}(t;\theta^*)\ \frac{\partial}{\partial\theta}\ \bar{\varepsilon}(t;\theta^*)$$

$(0.772,\ 1.152,\ -1.177\pm8.285i)$ and the eigenvalues of A $(0.162\pm1.383i,\ -1,\ -1)$. Simulations of this process are given in Section 6.3. □

According to Theorem 3.1 the algorithm (2.5.5)-(2.5.8) will with probability one not converge to the true parameter vector when applied to (5.6.4) or (5.6.5). This

holds also for the stochastic approximation variant described in Section 2.7. In fact, for (5.6.5) it can be shown that $\hat{\theta}(t)$ will not converge to any finite limit at all.

One might ask why this phenomenon has not been revealed previously, in spite of the rather extensive simulations made. One reason evidently is that a randomly picked low-order system has rather small probability of having an unstable ODE. The counterexamples have been constructed with care, and many conditions must be satisfied for a second-order system in order to make

$$\text{tr } E\ \bar{z}(t;\theta^*)\ \frac{\partial}{\partial\theta}\ \bar{\varepsilon}(t;\theta^*) > 0$$

It is also important to notice that Theorem 3.1 in connection with the examples 5.6.3 and 5.6.4 show that the convergence theorems in Panuska (1968) and Kashyap (1974) are not correct.

5.7 The RML2 method.

Consider now the RML2 method. For this method

$$\begin{cases} \bar{z}(t;\theta^*) = \bar{\varphi}(t;\theta^*) \\ \dfrac{\partial \bar{\varepsilon}}{\partial \theta}\ (t;\theta^*) = -\bar{\varphi}(t;\theta^*) \end{cases} \tag{5.7.1}$$

This means that for the true parameter vector θ_o the matrix A becomes

$$A = -I$$

Thus θ_o is a stable solution of the ODE. Also note that in contrast to the RML1 method there is no coupling between the different eigenvalues.

5.8 Global stability.

It is in fact possible to make some analysis concerning the global stability properties. The analysis is carried out for the RML2 method. It is also valid for the RLS and the RGLS methods. Note, however, that it is already well known that θ_o is globally stable for the RLS method.

According to Theorem 3.1 global asymptotic stability for the solution $\theta(\tau) = \theta_o$ of the ODE (5.1.1) implies, together with some additional conditions, convergence with probability one for the algorithm to the true parameter values. It will be shown that there exists a Lyapunov function that assures stability for the ODE (5.1.1). Consider the function (r defined in Section 5.2)

$$V(\theta,r) = \frac{1}{2} E\, \bar{\varepsilon}(t;\theta)^2 \qquad (5.8.1)$$

This function clearly is positive definite. Moreover, its time derivative satisfies

$$\dot{V}(\theta,r) = V'_\theta(\theta,r)\dot{\theta} + V'_r(\theta,r)\dot{r} =$$

$$= [E\, \bar{\varepsilon}(t;\theta)\bar{\varepsilon}'_\theta(t;\theta)]R^{-1}\, E\, \bar{\varepsilon}(t;\theta)\bar{\varphi}(t;\theta)$$

Since

$$\bar{\varepsilon}'_\theta(t;\theta) = -\bar{\varphi}(t;\theta)$$

it follows that

$$\dot{V}(\theta,r) = -f(\theta)R^{-1}f(\theta) \qquad (5.8.2)$$

Since only regions where R is positive definite are of interest, it is clear that $\dot{V}$ is negative definite as long as there is a unique stationary point, i.e. when

$$f(\theta) = 0$$

has a unique solution. Sufficient conditions for this to be true were given in Section 4.6 for the open loop case.

5.9 Summary of the section.

The analysis made in Section 5 is now summarized. In Table 5.1 sufficient conditions for θ_o to be a stable point are given. Note that it is generally assumed that the process may operate in closed loop or even that the regulator may depend on the parameter estimates.

Method	Sufficient conditions for local stability of θ_o	Supplementary characteristics
RLS	always stable	all eigenvalues of A in -1 and decoupled
RGLS	always stable	——
RIV	if the process is operating in open loop θ_o is always stable	all eigenvalues of A in -1 and coupled
RML1	stable for MA process and for first-order ARMA processes unstable for some processes	all eigenvalues of A in -1 and coupled
RML2	always stable	all eigenvalues of A in -1 and decoupled

Table 5.1 Results of local stability for θ_o.

In Section 4 it was shown that the RGLS method has other stationary points when the signal-to-noise ratio is very small. It has been proved that there are such points which also are stable. Other stationary points that are unstable may perhaps exist.

Global stability was treated for the methods RLS, RGLS and RML2. It was shown that θ_o was a globally stable point when the corresponding loss function $\Sigma\ \varepsilon^2(t)$ has a unique local minimum in θ_o. Hence, according to Theorem 3.1 if $\theta(t)$ belongs to a compact subset of $D_s \cap D_p$ infinitely often w.p.1, this implies that $\theta(t)$ converges to θ_o w.p.1 as t tends to infinity. Notice that the requirement that the loss function has a unique local minimum in fact is required also to assure convergence to θ_o for the corresponding off-line methods. An important consequence of Section 5.8 thus is that the convergence (consistency) properties of the ML-method do not deteriorate when it is approximated by the recursive RML2 method. On the other hand, the RML1 method has worse consistency properties than the off-line method.

For processes operating in open loop sufficient conditions for a unique local minimum of the loss function were derived in Section 4. For the RLS method θ_o is always a global, stable solution due to the assumption that the matrix G is nonsingular.

6. NUMERICAL EXAMPLES.

The five recursive identification methods considered in this report have also been subject to extensive simulations. In this section some of the typical results obtained are presented. The methods are compared and the influence of the choice of the time dependence of the weighting factor and of the signal-to-noise ratio is discussed in Section 6.1. In Section 6.2 the ODEs associated with the algorithms are studied by simulation of the ODEs. In Section 6.3 a counterexample to the general convergence of the RML1 method is presented. Several of the results in this section were originally presented in the Master Thesis, Johannesson - Wesström (1974).

6.1 Results of simulations.

For the methods considered in this report different model structures are postulated in order to treat the question of convergence and consistency. It has been generally assumed that the system has a proper (i.e. corresponding) structure. In the simulations, however, one of the main purposes is to compare the behaviour of the algorithms for a given system. Therefore it is in this section in general assumed that the system is described by

$$A(q^{-1})y(t) = B(q^{-1})u(t) + C(q^{-1})e(t) \qquad (6.1.1)$$

In order to get a good fit also models with a higher order than the true system will be tried for the RLS method. Similarly the degree of the $\hat{C}$-polynomial for

the RGLS method was chosen higher than the degree of C.

Results of identification of simulated data from the following systems are presented,

1st order system

$$\begin{aligned} A(q^{-1}) &= 1 - 0.8q^{-1} \\ B(q^{-1}) &= \quad 1.0q^{-1} \\ C(q^{-1}) &= 1 + 0.7q^{-1} \end{aligned} \tag{6.1.2}$$

2nd order system

$$\begin{aligned} A(q^{-1}) &= 1 - 1.5q^{-1} + 0.7q^{-2} \\ B(q^{-1}) &= \quad 1.0q^{-1} + 0.5q^{-2} \\ C(q^{-1}) &= 1 - 1.0q^{-1} + 0.2q^{-2} \end{aligned} \tag{6.1.3}$$

3rd order system (only the RML1 and RML2 methods)

$$\begin{aligned} A(q^{-1}) &= 1 - 1.60q^{-1} + 1.61q^{-2} - 0.776q^{-3} \\ B(q^{-1}) &= \quad 1.20q^{-1} - 0.95q^{-2} + 0.200q^{-3} \\ C(q^{-1}) &= 1 + 0.10q^{-1} + 0.25q^{-2} + 0.873q^{-3} \end{aligned} \tag{6.1.4}$$

In order to test the RLS and RGLS algorithms simulations have also been performed using the same structure for the system and the model. These results are, however, not reviewed here but showed that the algorithms had expected properties.

The input signal has been chosen as a pseudo random binary sequence (PRBS) of period 127 with the bit interval equal the sampling interval or the same signal with the bit interval ten times the sampling interval. In the following only results using the first input signal are presented, since in general this input signal gave the best accuracy. The noise e(t) was generated by a random number

generator, producing a normally distributed sequence with a mean value of zero and a variance of one. The amplitude of the input was chosen so that the signal-to-noise ratio, S, at system output was either 1 or 10 for each simulation. S is here defined by

$$S = E\, y_1(t)^2 / E\, y_2(t)^2$$

where

$$y_1(t) = \frac{B(q^{-1})}{A(q^{-1})}\, u(t)$$

and

$$y_2(t) = \frac{C(q^{-1})}{A(q^{-1})}\, e(t)$$

The weighting factor according to (2.11.4)

$$\lambda(t+1) = \lambda_o \lambda(t) + (1-\lambda_o)$$

has been varied in order to study the influence on the convergence rate. Different choices of λ_o and $\lambda(0)$ have been studied. For the results presented here $\lambda(t)$ was put equal to one for the RLS and RIV methods. The initial value of P has been chosen as $100 \cdot I$ and $\hat{\theta}(0)$ as zero when nothing else is stated.

The results given in the following are the averages of 10 runs of 2000 data points each. In addition an estimate of the standard deviation based on those ten runs is given. All the simulations reviewed in this section were carried out on UNIVAC 1108.

In order to compare the results not only by studying the parameter estimates three different measures of the accuracy are used, cf. Söderström - Ljung - Gustavsson (1974),

(i) $V_1(\hat{\theta}) = E\ \varepsilon(t)^2$

The function $V_1(\hat{\theta})$ expresses the variance of the one step prediction errors that will be obtained using the model. For methods giving efficient estimates the expected value of V_1 is 1+p/N, where p is the number of estimated parameters and N the number of samples.

(ii) $V_2(\hat{\theta}) = E\ [(\frac{\hat{B}(q^{-1})}{\hat{A}(q^{-1})} - \frac{B(q^{-1})}{A(q^{-1})})\ u(t)]^2$

The function $V_2(\hat{\theta})$ expresses how well the model describes the deterministic part of the system. If the input signal u(t) is white noise then

$$V_2(\hat{\theta}) = \sum_{i=1}^{\infty} (\hat{h}_i - h_i)^2$$

where h_i ($\hat{h}_i$) is the discrete-time pulse response of the system (resp. the model).

(iii) Assume that the model is used for construction of a minimum variance regulator, cf Åström (1970). Suppose that the true system is controlled by this regulator and take

$$V_3(\hat{\theta}) = E\ y^2(t)$$

The regulator is given by

$$u(t) = - \frac{\hat{C}(q^{-1}) - \hat{A}(q^{-1})}{\hat{B}(q^{-1})}\ y(t)$$

Thus the closed loop system will be

$$[B(q^{-1})\hat{C}(q^{-1}) + \{A(q^{-1})\hat{B}(q^{-1}) - \hat{A}(q^{-1})B(q^{-1})\}]y(t) =$$

$$= \hat{B}(q^{-1})C(q^{-1})e(t)$$

If the system and the model contain a larger delay the equations will have a more complicated form.

These functions can be used for measuring the accuracy of $\hat{\theta}$ by considering the scalar

$$V_i(\hat{\theta}_N) \qquad i=1,2,3$$

In interpreting the results notice that the RLS and RIV methods only estimate the parameters of the polynomials A and B and that the RLS and RGLS methods give biased estimates for the systems considered.

The results obtained for the five different methods are described in Sections 6.1.1 - 6.1.5. Appendix 2 contains some supplementary results. In Section 6.1.6 a comparison between the methods is made.

In Sections 6.1.1 - 6.1.5 the methods are illustrated with plots of the estimates for one of the realizations.

In the text for the figures and the tables the following notations are used,

- o *S* denotes the system
- o *I* denotes the identification method used
- o *M* denotes the model structure, which will be specified with the integers $\hat{n}_a$, $\hat{n}_b$ and (possibly) $\hat{n}_c$. The cap is used to denote that these integers may not be the same as n_a, n_b and n_c respectively.

In the figures the true values of the parameters are given by dashed lines.

6.1.1. The RLS method.

The RLS algorithm has been used to estimate the parameters of models of different orders for the first and second order systems (6.1.2) and (6.1.3). The numerical results are summarized in Tables A2.2 - A2.8 in Appendix 2. In Figure 6.1.1 the estimates for one of the realizations of the first order system are shown. In Figure 6.1.2 one of the realizations of the second order system is illustrated.

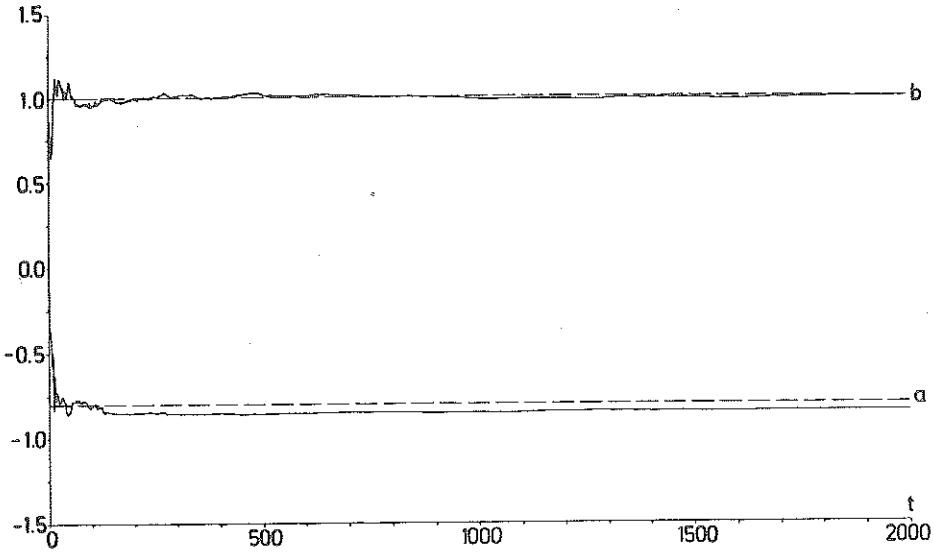

Figure 6.1.1. S: First order (6.1.2), S=1
I: RLS
M: $\hat{n}_a=\hat{n}_b=1$

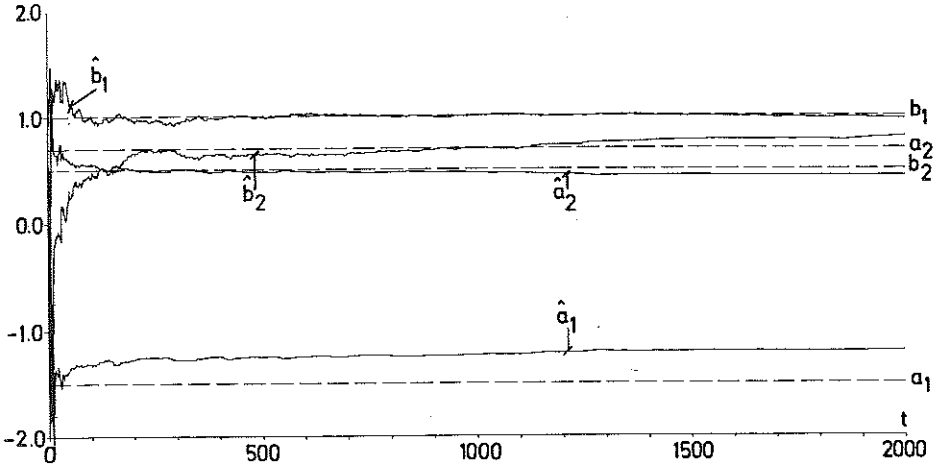

Figure 6.1.2. *S*: Second order (6.1.3), S=10
I: RLS
M: $\hat{n}_a=\hat{n}_b=2$

The main conclusions are:

- For the first order system the estimates are close to the final estimates after less than 100 samples. The convergence is somewhat slower for the second order system. See Figures 6.1.1 and 6.1.2.
- The estimates are biased since the system and the model have different structures. A high signal-to-noise ratio reduces the bias, but may be quite substantial even for a signal-to-noise ratio of 10. See Table A2.8 and Figure 6.1.2.
- V_1, V_2 and V_3 decrease in general with increasing number of samples, see Tables A2.4-A2.6. Notice, however, that since the estimates are biased V_1 and V_3 do not tend to one but to some values greater than one.
- In general V_1 and V_3 decrease with increasing order of the model. This is not true for V_2 which in

general first decreases and then increases again. There seem, however, to be several local minima, which indicates that it is difficult to choose the order of the model using this criterion.

- For the second order system the derived minimum variance strategy often gave an unstable closed-loop system, which made it impossible to use V_3 as a measure of the accuracy of the model.

6.1.2. The RGLS method.

The RGLS method has been used to estimate the parameters of models with different orders of the polynomial $\hat{C}$ for the first and second order systems (6.1.2) and (6.1.3). To improve the transient behaviour of the algorithm it was started up in the following way. For the first 50 samples no filtering was used. More precisely, $\hat{C}(q^{-1})$ was substituted with 1 in (2.3.3) and (2.3.9) during this initial phase of the estimation. The results are summarized in Tables A2.9-A2.13. In Figures 6.1.3 and 6.1.4 the estimates are plotted versus number of samples for one realization of the first and second order systems respectively.

The main conclusions are:

- For the first order system the estimates are close to the final ones after less than 100 samples. The convergence is considerably slower for the second order system, particularly for the case S=1. See Figures 6.1.3 and 6.1.4 and Tables A2.9, A2.10, A2.12 and A2.13.
- By theoretical considerations it can be concluded that the estimates $\hat{a}$ and $\hat{b}$ will be biased. This is

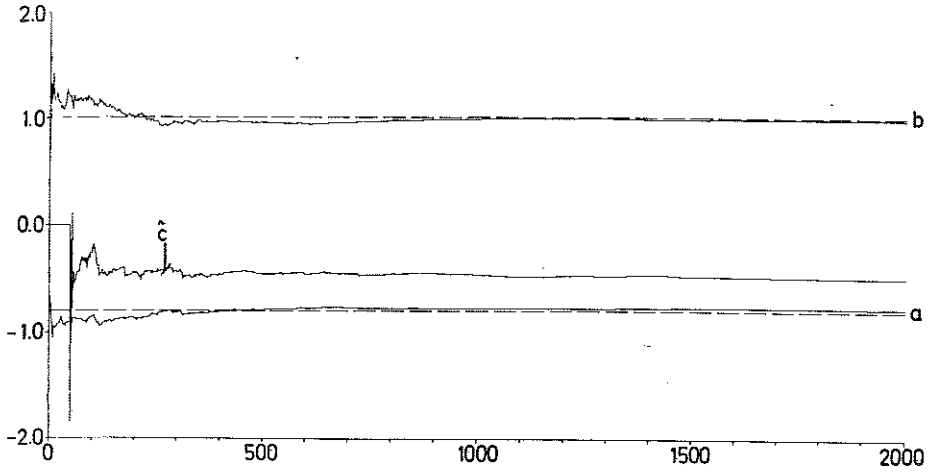

Figure 6.1.3. *S*: First order (6.1.2), S=1
I: RGLS
M: $\hat{n}_a=\hat{n}_b=\hat{n}_c=1$

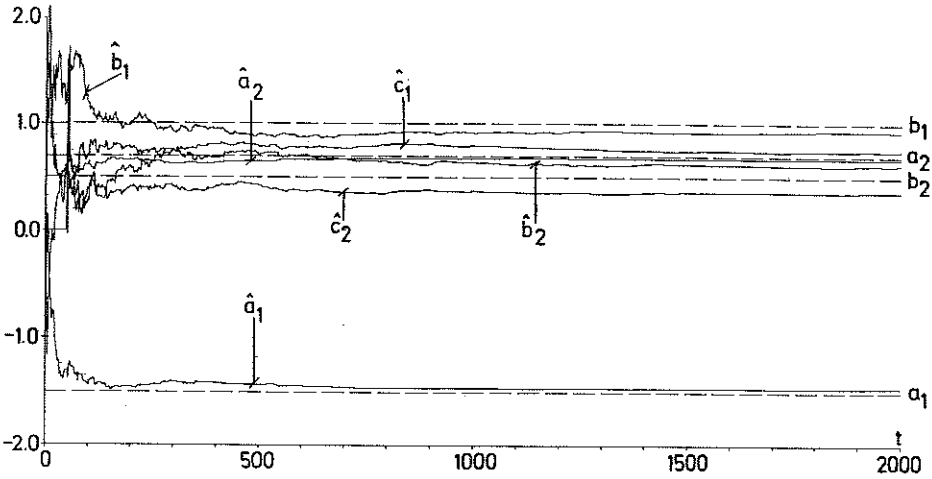

Figure 6.1.4. *S*: Second order (6.1.3), S=10
I: RGLS
M: $\hat{n}_a=\hat{n}_b=\hat{n}_c=2$

difficult to establish from the simulations only. There are, however, indications of bias, see Figures 6.1.13 and 6.1.14 (cf also Section 6.2.2).

- A comparison of V_2 for different orders of the polynomial $\hat{C}$ indicates that V_2 in general increases for model of high orders, indicating that nothing can be gained by increasing the order of $\hat{C}$ too much.

- A comparison of the estimated standard deviations for $\hat{a}$ and $\hat{b}$ for different orders of the polynomial $\hat{C}$ is shown in Figures 6.1.5 and 6.1.6. The standard deviations are generally increasing with the order of $\hat{C}$.

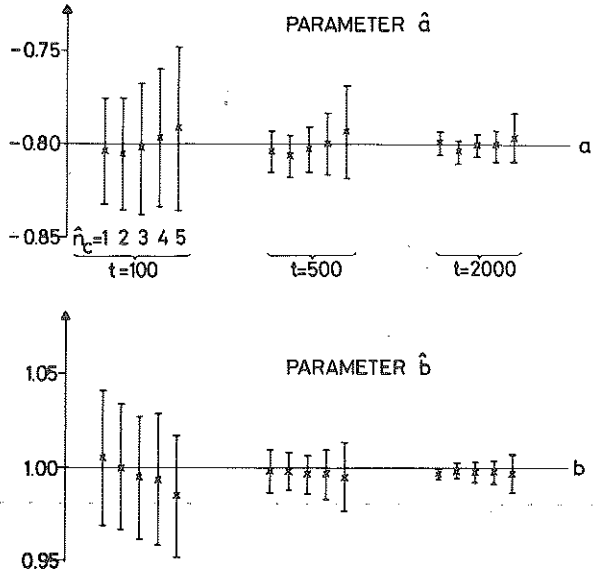

Figure 6.1.5. Influence of the model structure and the number of samples on the accuracy of the estimates. The length of the drawn lines are twice the standard deviation.
S: First order (6.1.2), S=10
I: RGLS
M: $\hat{n}_a=\hat{n}_b=1$, $\hat{n}_c=1,2,3,4,5$

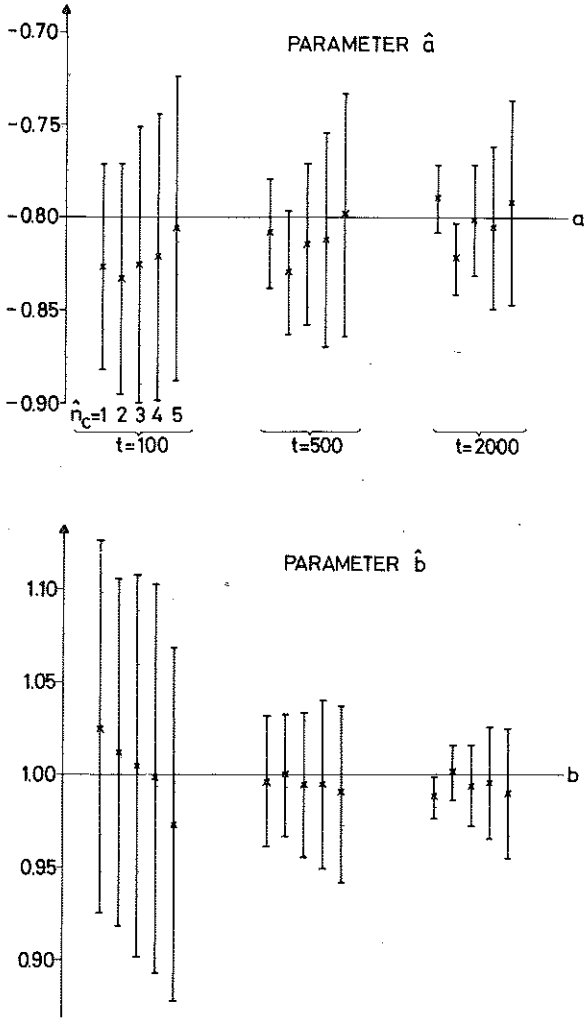

Figure 6.1.6. Influence of the model structure and the number of samples on the accuracy of the estimates. The length of the drawn lines are twice the standard deviation.

S: First order (6.1.2), S=1

I: RGLS

M: $\hat{n}_a=\hat{n}_b=1$, $\hat{n}_c=1,2,3,4,5$

6.1.3. The RIV method.

For the RIV method only simulations with a proper model structure were performed. The effect of varying the delay τ in (2.4.12) has been studied. The results are summarized in Tables A2.14-A2.18. In Figures 6.1.7 and 6.1.8 the estimates are plotted for one realization of the first and second order systems respectively.

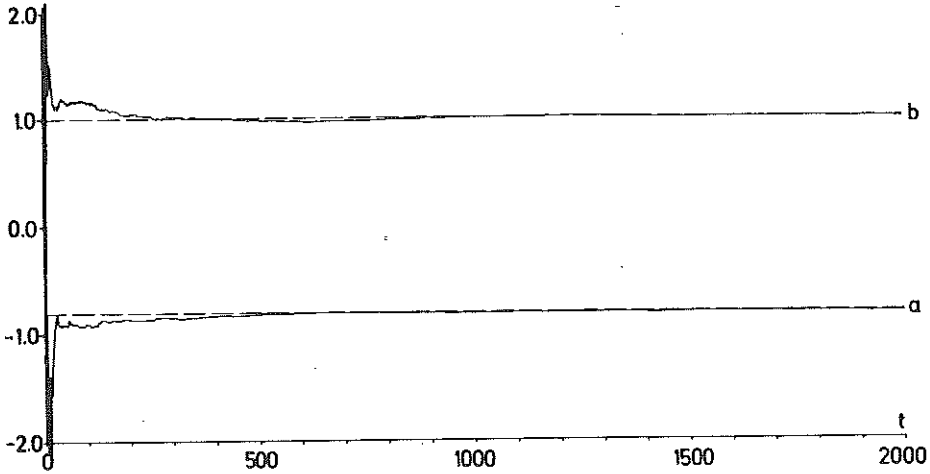

Figure 6.1.7. S: First order (6.1.2), S=1
I: RIV
M: $\hat{n}_a=\hat{n}_b=1$

The main conclusions are:

- The estimates are in general close to the true parameter values after 100 samples, but there are runs for which the estimates took quite unreasonable values, even after 2000 samples, in particular for the second order system, cf Table A2.17.
- The value of τ seems not to be too critical.

o Since the polynomial C is not estimated, V_1 and V_3 tend to constant values greater than one.

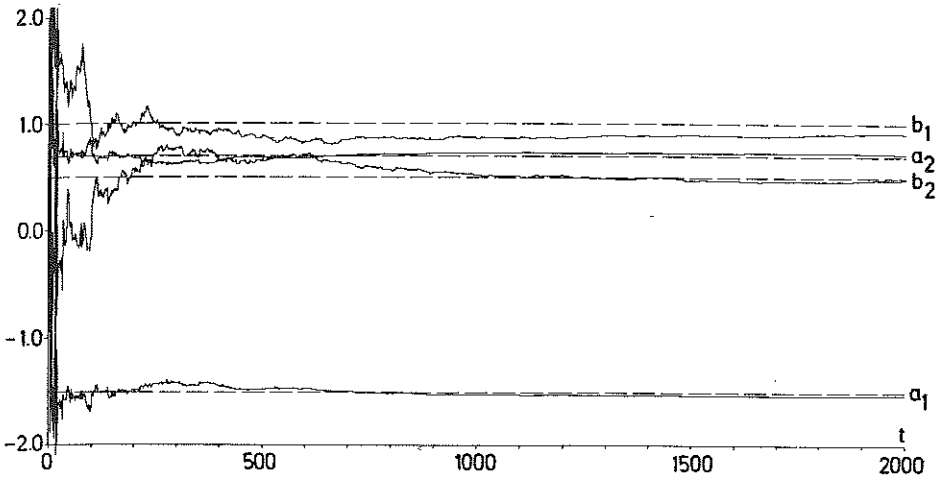

Figure 6.1.8. *S*: Second order (6.1.3), S=10
I: RIV, τ=2
M: $\hat{n}_a=\hat{n}_b=2$

6.1.4. The RML1 method.

The RML1 method has been used for the estimation of models for the first, second and third order systems (6.1.2)-(6.1.4). The results are summarized in Tables A2.18-A2.23. In Figures 6.1.9 and 6.1.10 the estimates are plotted for one realization of the first and second order systems respectively. In Table A2.28 the results for the third order system are given and they are compared to the results obtained by the RML2 method.

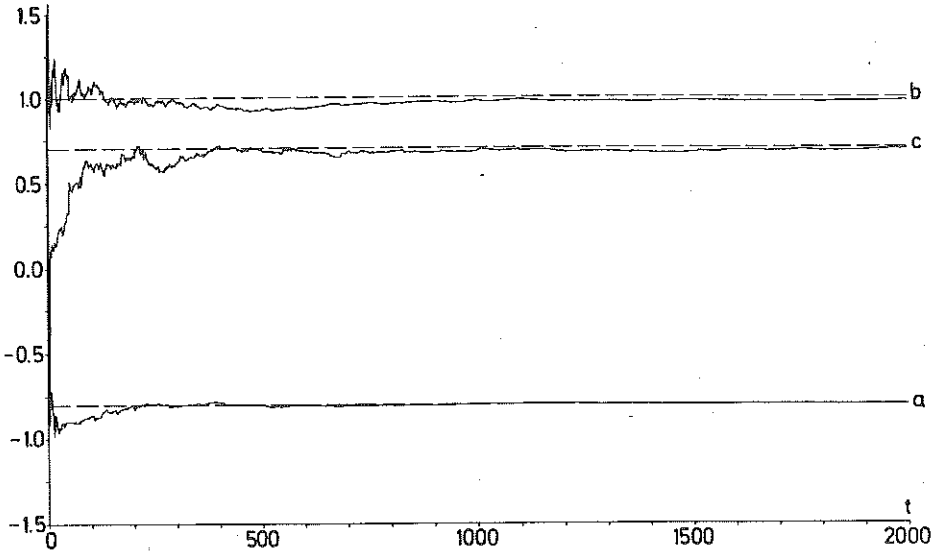

Figure 6.1.9. *S*: First order (6.1.2), S=1
I: RML1, $\lambda(0)=0.95$, $\lambda_o=0.99$
M: $\hat{n}_a=\hat{n}_b=\hat{n}_c=1$

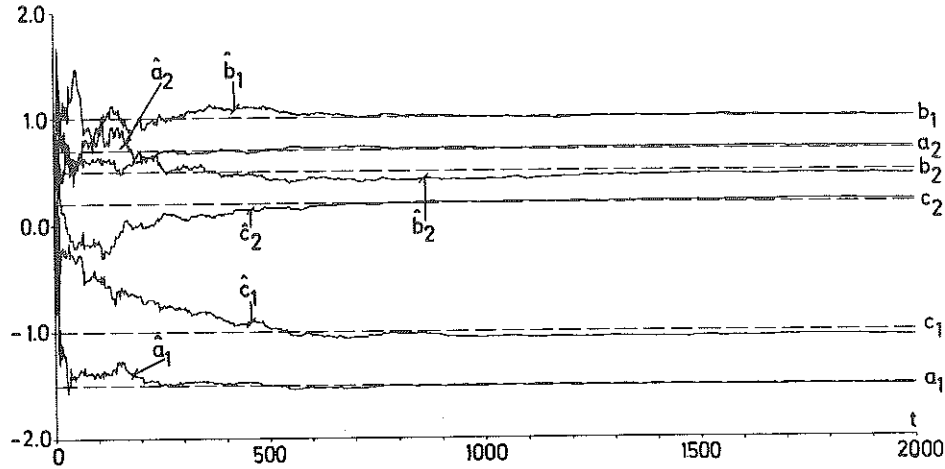

Figure 6.1.10. *S*: Second order (6.1.3), S=10
I: RML1, $\lambda(0)=0.95$, $\lambda_o=0.99$
M: $\hat{n}_a=\hat{n}_b=\hat{n}_c=2$

The main conclusions are:

- o In general the choice $\lambda(0)=0.95$, $\lambda_o=0.99$ gives the best accuracy.
- o A larger $\lambda(0)$ implies a slower convergence of the estimates $\hat{c}_i$.
- o For the first order system the estimates $\hat{a}$ and $\hat{b}$ are close to the true values after 100 samples. The estimate $\hat{c}$ converges much slower. For the second order system the estimates $\hat{a}_i$ and $\hat{b}_i$ also converge slow, but not so slow as the estimates $\hat{c}_i$.

6.1.5. The RML2 method.

The RML2 method has been used for the estimation of models for the same systems as for the RML1 method. The results are summarized in Tables A2.24 - A2.27. In Figures 6.1.11 and 6.1.12 the estimates are plotted for one realization of the first and second order systems respectively. In Table A2.28 the results for the third order system are given. They are compared to the results obtained by the RML1 method.

The main conclusions are:

- o In general the choice $\lambda(0)=0.95$, $\lambda_o=0.99$ is good. Also $\lambda(0)=0.99$, $\lambda_o=0.99$ has given good results.
- o For the first order system the estimates $\hat{a}$ and $\hat{b}$ are close to the true values after 100 samples. The estimate $\hat{c}$ converges slower. For the second order system the convergence of the estimates $\hat{a}_i$ and $\hat{b}_i$ is slower, but still the convergence of the estimates $\hat{c}_i$ is slowest.

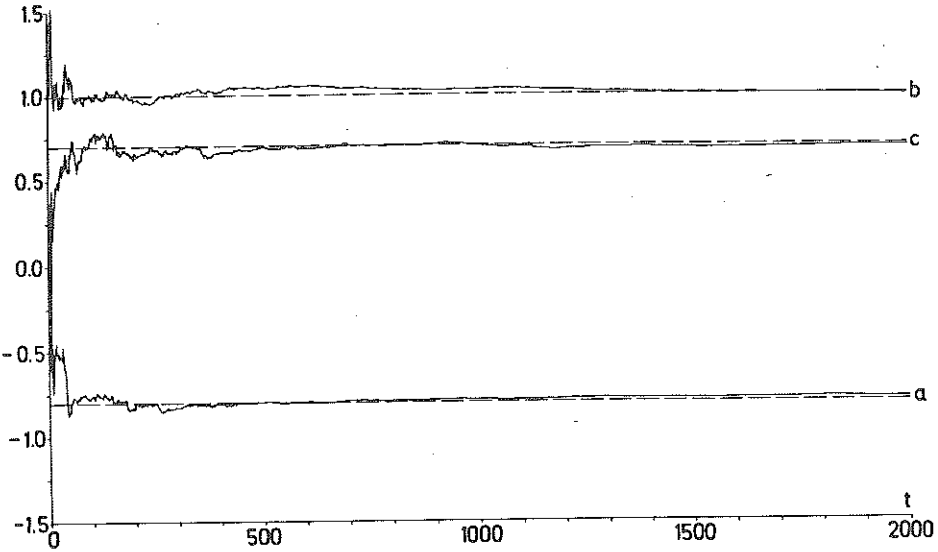

Figure 6.1.11. *S*: First order (6.1.2), S=1
I: RML2, $\lambda(0)=0.95$, $\lambda_o=0.99$
M: $\hat{n}_a=\hat{n}_b=\hat{n}_c=1$

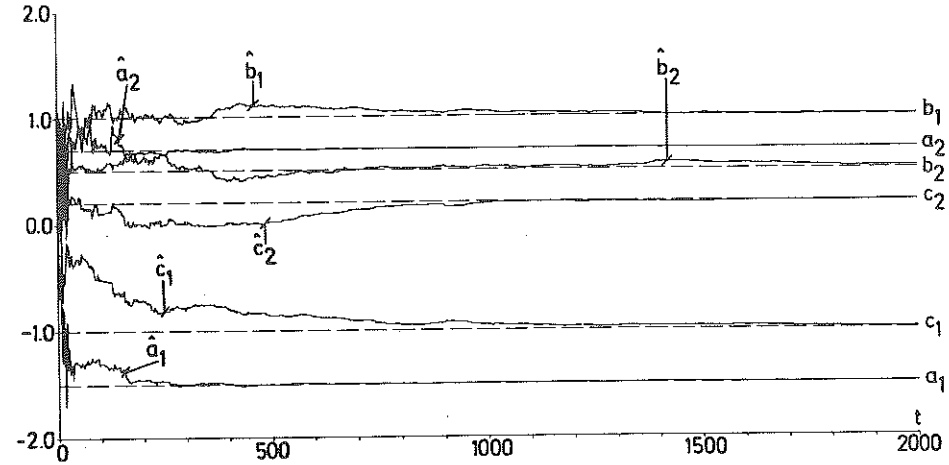

Figure 6.1.12. *S*: Second order (6.1.3), S=10
I: RML2, $\lambda(0)=0.95$, $\lambda_o=0.99$
M: $\hat{n}_a=\hat{n}_b=\hat{n}_c=2$

6.1.6. Comparison of the methods.

In this section the five methods will be compared. In order to compare the results of the recursive methods with the most accurate off-line methods, the asymptotic Cramér-Rao bounds on the standard deviations for different number of samples are given in Table A2.29 for the first and second order systems (6.1.2) and (6.1.3). In Figures 6.1.13 - 6.1.17 the estimates and their accuracy are compared for the five methods used. The Cramér-Rao bounds are also given in these figures.

The main conclusions are:

- o The RLS and RGLS methods give biased estimates.
- o The RIV, RML1 and RML2 methods often give good results. Notice, however, that the RIV method sometimes produces unacceptable estimates.
- o In general the RML2 method is the most accurate method.
- o The RML2 method seems to be superior to the RML1 method, in particular concerning the estimation of the parameters of $\hat{C}$.

For a further comparison between the RML1 and RML2 methods the results of the estimation of the parameters for the third order system (6.1.4) shown in Table A2.28 can be used. The conclusion is that the RML2 method is better for 500 and 2000 samples but that the RML1 method often gives better estimates of the parameters of the polynomials $\hat{A}$ and $\hat{B}$ after 100 samples.

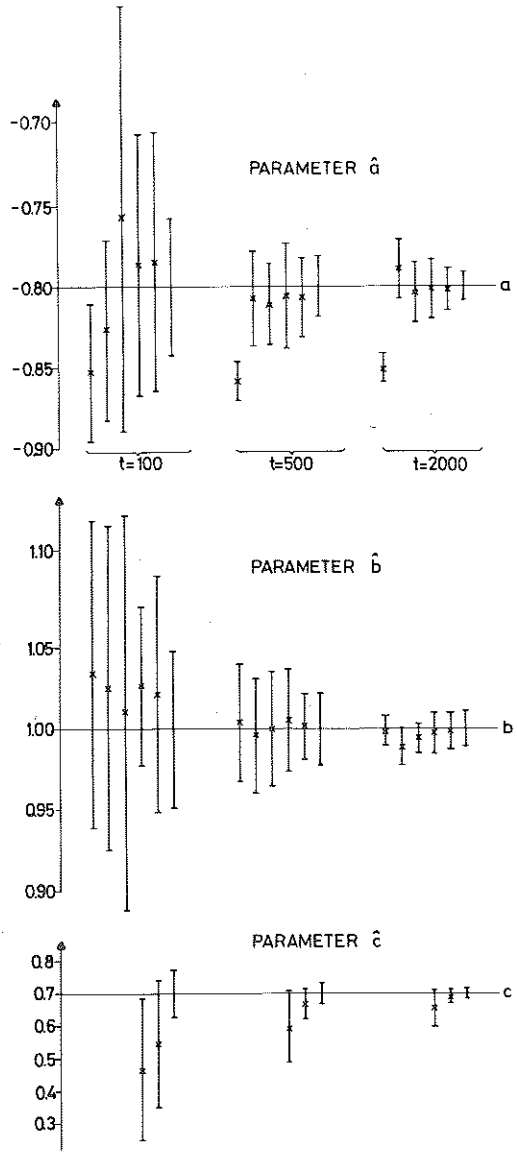

Figure 6.1.13. Comparison between the different methods.
S: First order (6.1.2), S=1
I: RLS, RGLS, RIV, RML1 ($\lambda(0)=0.95$, $\lambda_o=0.99$), RML2 ($\lambda(0)=0.95$, $\lambda_o=0.99$) and the Cramér-Rao lower bound (in order from left to right).
M: $\hat{n}_a=\hat{n}_b=1$. For RGLS, RML1 and RML2 $\hat{n}_c=1$.
The length of the lines is twice the standard deviation.

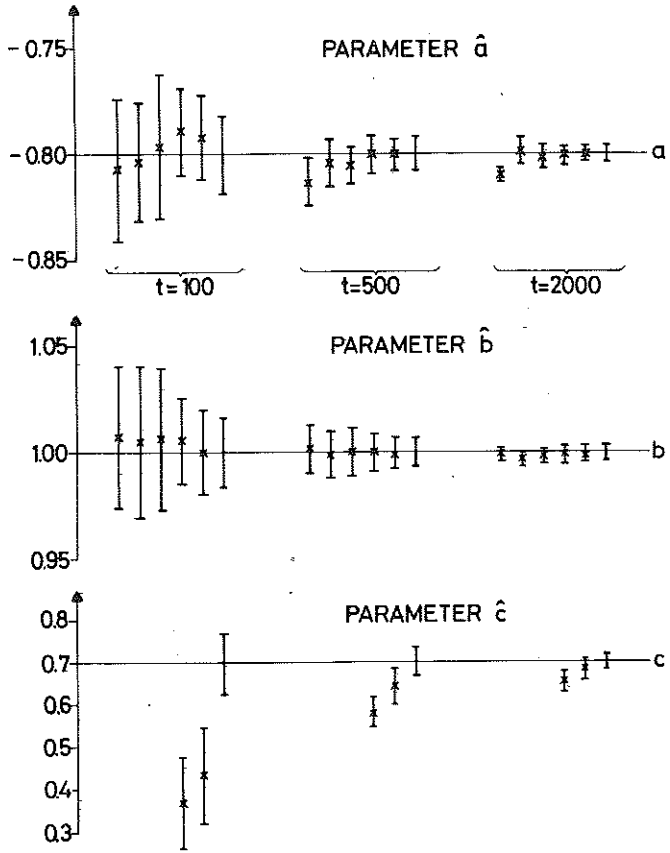

Figure 6.1.14. Comparison between the different methods.
S: First order (6.1.2), S=10
I: RLS, RGLS, RIV, RML1 ($\lambda(0)=0.95$, $\lambda_o=0.99$), RML2 ($\lambda(0)=0.95$, $\lambda_o=0.99$) and the Cramér-Rao lower bound (in order from left to right).
M: $\hat{n}_a=\hat{n}_b=1$. For RGLS, RML1 and RML2 $\hat{n}_c=1$.
The length of the lines is twice the standard deviation.

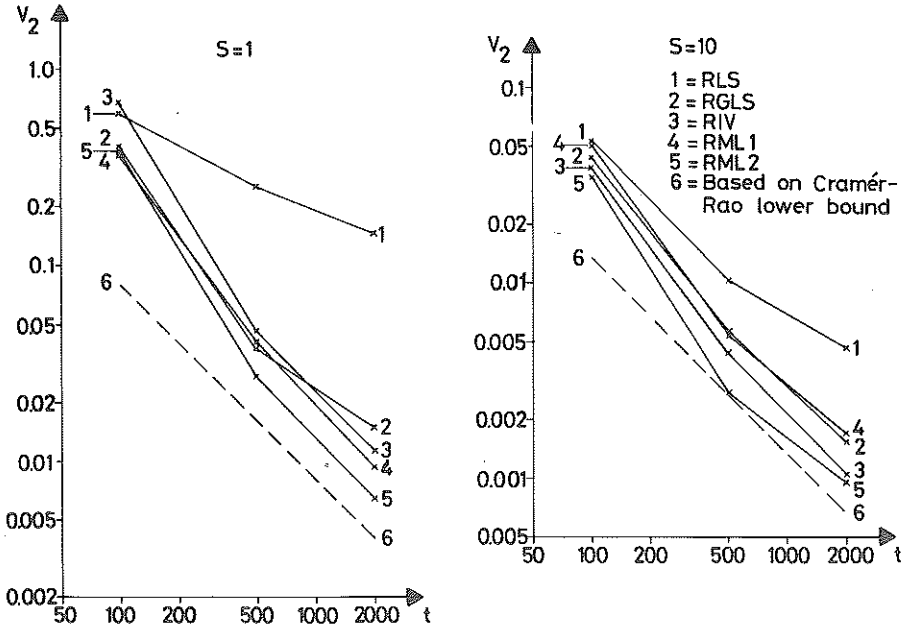

Figure 6.1.15. Comparison between the different methods using the criterion V_2.

S: First order (6.1.2), S=1 and S=10.

I: RLS, RGLS, RIV, RML1, RML2 and the Cramér-Rao lower bound

M: $\hat{n}_a=\hat{n}_b=1$. For RGLS, RML1 and RML2 $\hat{n}_c=1$

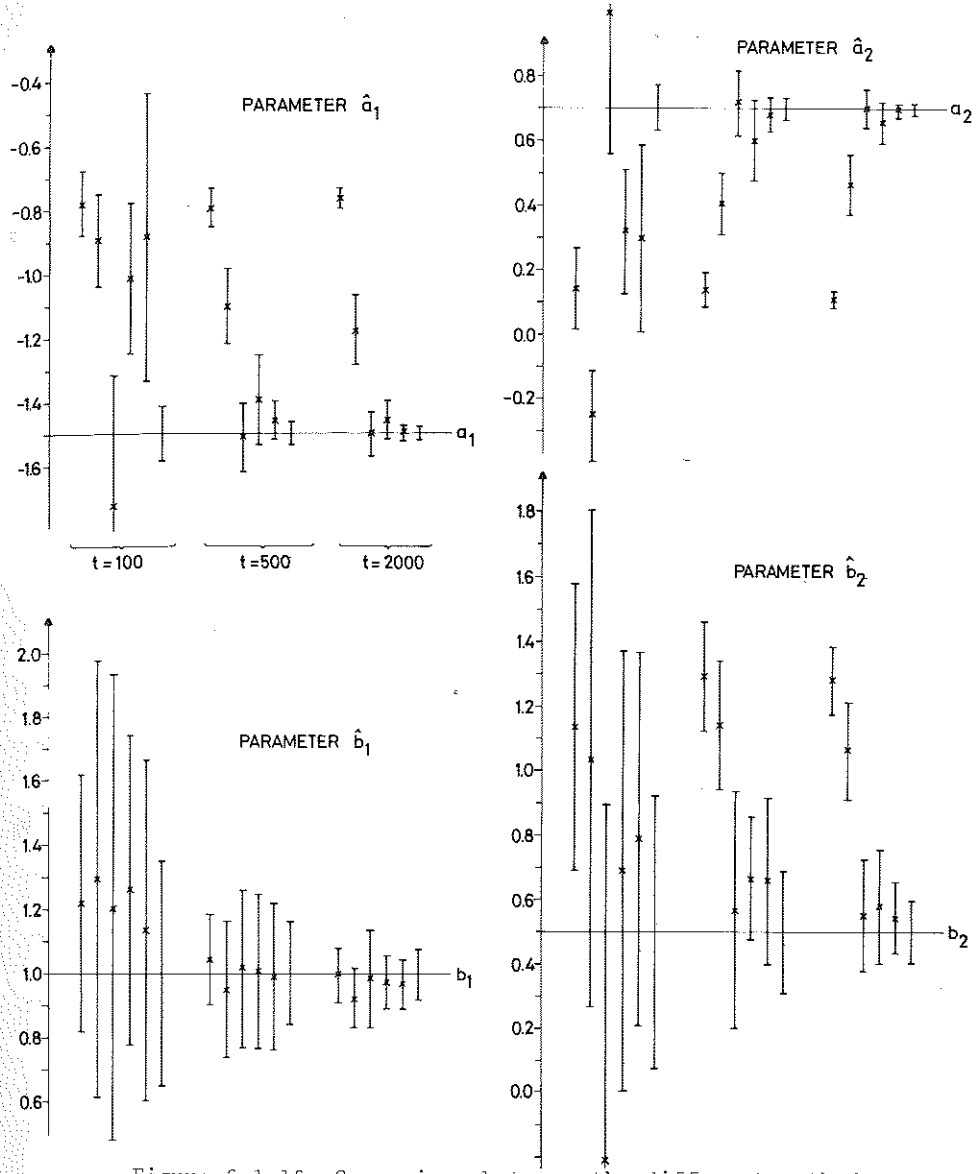

Figure 6.1.16. Comparison between the different methods.
S: Second order (6.1.3), S=1
I: RLS, RGLS, RIV, RML1 ($\lambda(0)=0.95$, $\lambda_o=0.99$), RML2 ($\lambda(0)=0.95$, $\lambda_o=0.99$) and the Cramér-Rao lower bound (in order from left to right).
M: $\hat{n}_a=\hat{n}_b=2$. For RGLS, RML1 and RML2 $\hat{n}_c=2$.
The length of the lines is twice the standard deviation.

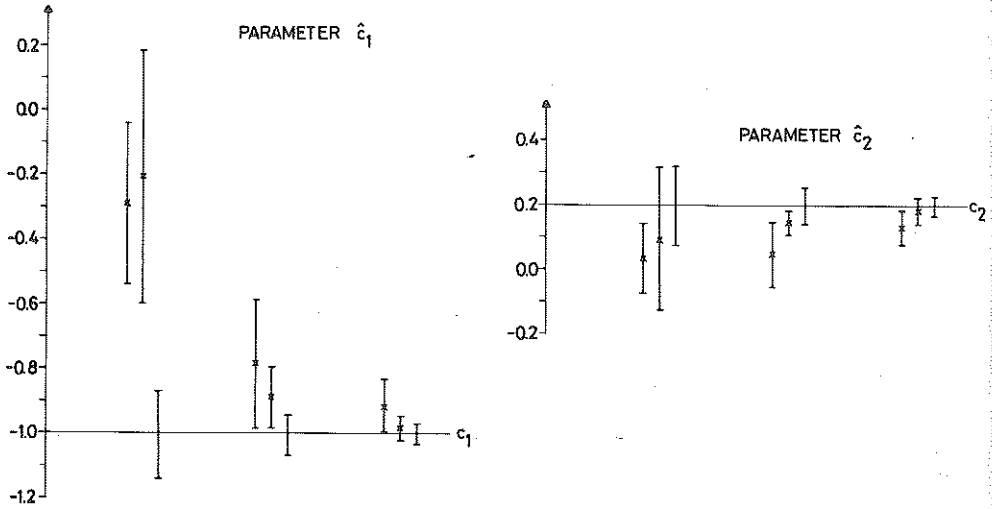

Figure 6.1.16. Continued.

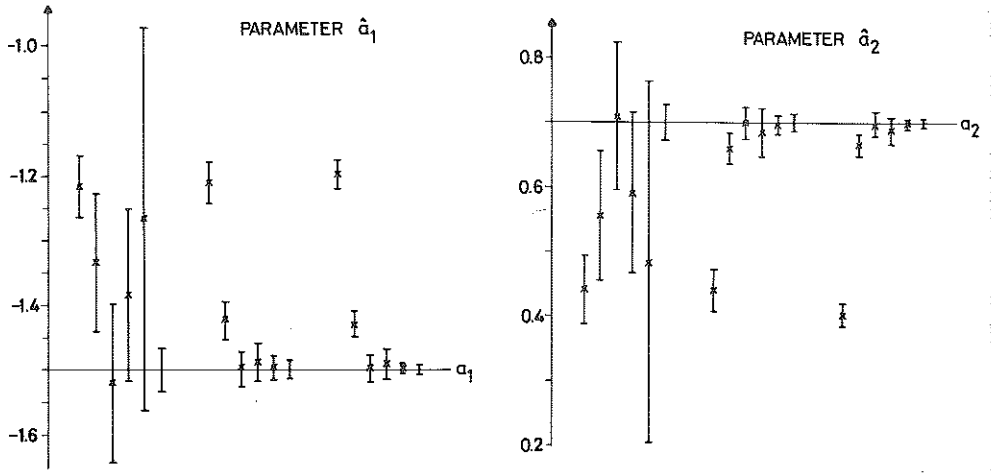

Figure 6.1.17. Comparison between the different methods.
S: Second order (6.1.3), S=10
I: RLS, RGLS, RIV, RML1 ($\lambda(0)=0.95$, $\lambda_o=0.99$), RML2 ($\lambda(0)=0.95$, $\lambda_o=0.99$) and the Cramér-Rao lower bound (in order from left to right).
M: $\hat{n}_a=\hat{n}_b=2$. For RGLS, RML1 and RML2 $\hat{n}_c=2$.
The length of the lines is twice the standard deviation.

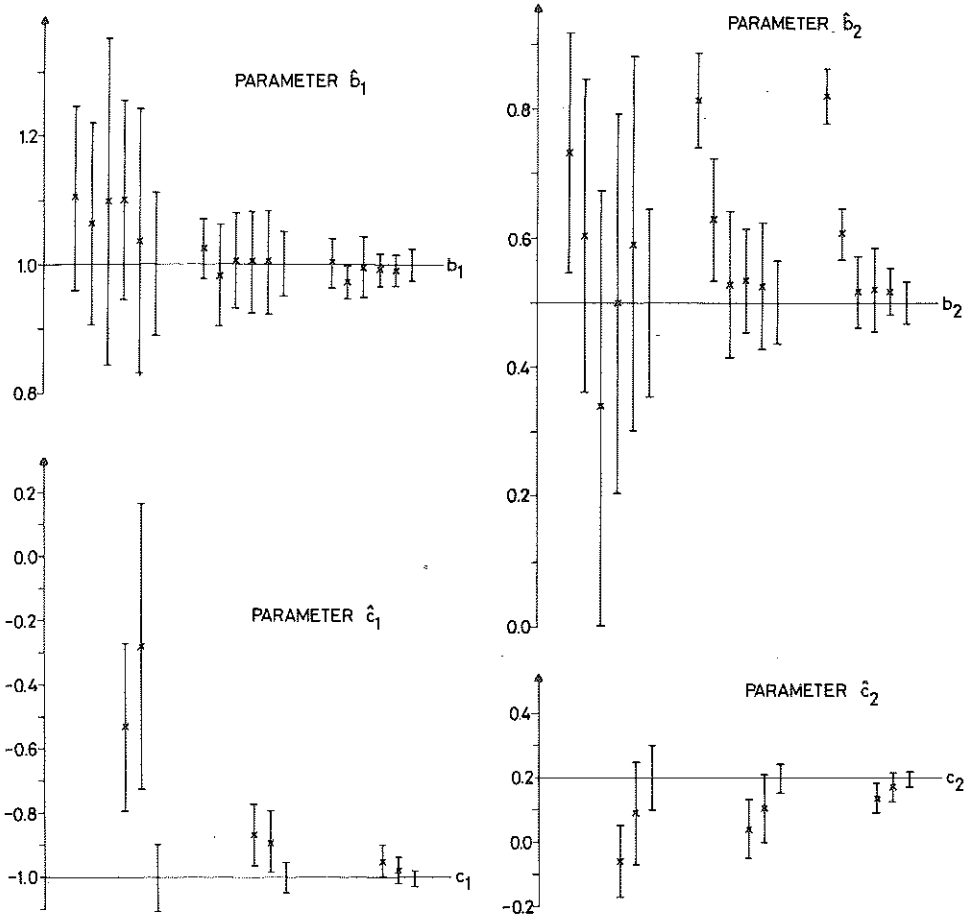

Figure 6.1.17. Continued.

6.2 Simulations of the ODEs for the different algorithms.

In order to study the asymptotic properties of the different recursive identification algorithms treated in this report the ODEs (3.3.8)

$$\dot{\theta} = R^{-1}f(\theta)$$

$$\dot{R} = G(\theta) - R$$

associated with the algorithms were simulated for the first order system

$$(1+aq^{-1})y(t) = bq^{-1}u(t) + (1+cq^{-1})e(t) \qquad (6.2.1)$$

In all simulations in this section it is assumed that $u(t)$ is white noise with zero mean and variance σ_u^2. It is also assumed that the input signal is independent of $e(t)$, which is white noise with zero mean and variance σ_e^2.

The simulations were made possible through the use of an efficient interactive simulation program package, SIMNON, see Elmquist (1975). They were all run on a PDP/15.

The simulations of the ODEs for the different methods were carried out for a number of different conditions,

- o different signal-to-noise ratios S
- o different systems, in particular different values of c
- o different initial values $\hat{a}(0)$, $\hat{b}(0)$ and $\hat{c}(0)$.

In the simulations the initial value of the R-matrix was generally chosen as the unit matrix. Also notice that the time scale in the diagrams shown is a ficticious time, cf Ljung (1974a). Let τ denote the time variable in the ODEs. Then roughly $\tau = \ln(t) + \text{constant}$, if $\lambda=1$.

6.2.1. The RLS method.

The complexity of the right hand side of the differential equations in (3.3.8) for this type of problem will be illustrated by the simplest case, the RLS method. In this case the model is

$$(1+\hat{a}q^{-1})y(t) = \hat{b}q^{-1}u(t) + \varepsilon(t) \qquad (6.2.2)$$

For the RLS method

$$z(t) = \varphi(t) = [-y(t-1),\ u(t-1)]$$

and if y(t) in (6.2.1) is substituted into (6.2.2)

$$\varepsilon(t) = \frac{(b-\hat{b})q^{-1}+(\hat{a}b-a\hat{b})q^{-2}}{1+aq^{-1}}\,u(t) + \frac{(1+\hat{a}q^{-1})(1+cq^{-1})}{1+aq^{-1}}\,e(t)$$

Then

$$f(\theta) = [f_1(\theta),\ f_2(\theta)]^T$$

$$G(\theta) = \begin{bmatrix} G_{11}(\theta) & G_{12}(\theta) \\ G_{21}(\theta) & G_{22}(\theta) \end{bmatrix}$$

and (the integration paths are the unit circle)

$$f_1(\theta) = E[-y(t-1)\varepsilon(t)] = -\frac{\sigma_u^2}{2\pi i}\int \frac{bz^{-2}}{1+az^{-1}}\,\frac{(b-\hat{b})z+(\hat{a}b-a\hat{b})z^2}{1+az}\,\frac{dz}{z} -$$

$$-\frac{\sigma_e^2}{2\pi i}\int \frac{z^{-1}(1+cz^{-1})}{1+az^{-1}}\,\frac{(1+\hat{a}z)(1+cz)}{1+az}\,\frac{dz}{z} =$$

$$= \frac{b^2\sigma_u^2}{1-a^2}(a-\hat{a}) + \frac{(a-c)(1-ac)-\hat{a}(1+c^2-2ac)}{1-a^2}\,\sigma_e^2$$

$$f_2(\theta) = E[u(t-1)\varepsilon(t)] = \frac{\sigma_u^2}{2\pi i}\int z^{-1}\,\frac{(b-\hat{b})z+(\hat{a}b-a\hat{b})z^2}{1+az}\,\frac{dz}{z} = (b-\hat{b})\sigma_u^2$$

$$G_{11}(\theta) = E[y(t-1)^2] = E[y(t)^2] = \frac{\sigma_u^2}{2\pi i}\int \frac{bz^{-1}}{1+az^{-1}}\frac{bz}{1+az}\frac{dz}{z} +$$

$$+ \frac{\sigma_e^2}{2\pi i}\int \frac{1+cz^{-1}}{1+az^{-1}}\frac{1+cz}{1+az}\frac{dz}{z} = \frac{b^2\sigma_u^2}{1-a^2} + \frac{1+c^2-2ac}{1-a^2}\sigma_e^2$$

$$G_{12}(\theta) = G_{21}(\theta) = 0$$

$$G_{22}(\theta) = E[u(t-1)^2] = E[u(t)^2] = \sigma_u^2$$

Since for this method $z(t)=\varphi(t)$ the P-matrix will be symmetric and the system of differential equations will be of fifth order. This can further be reduced to a fourth order system observing $G_{12}(\theta) = G_{21}(\theta) = 0$ and when $R(0)$ is chosen equal to a diagonal matrix, which corresponds to the practical situation where $P(0)$ is commonly chosen to be a diagonal matrix. Then

$$\begin{cases} \dot{\theta}_1 = \dot{\hat{a}} = f_1(\theta)/R_{11} \\ \dot{\theta}_2 = \dot{\hat{b}} = f_2(\theta)/R_{22} \\ \dot{R}_{11} = G_{11}(\theta) - R_{11} \\ \dot{R}_{22} = G_{22}(\theta) - R_{22} \end{cases} \qquad (6.2.3)$$

with $f_1(\theta)$, $f_2(\theta)$, $G_{11}(\theta)$ and $G_{22}(\theta)$ given above.

The results of the simulations of the system (6.2.3) with

$a=-0.8$, $b=1$, $c=0$, $S=1$, $\hat{a}(0)=\hat{b}(0)=0$

are presented in Figure 6.2.1. The convergence to the true values is illustrated and it can be seen that the estimate $\hat{a}$ converges faster than the estimate $\hat{b}$. In

Figure 6.2.2 the trajectories for the case

$a=-0.8$, $b=1$, $c=0.7$, $S=1$, $\hat{a}(0)=\hat{b}(0)=0$

are shown. It is immèdiately observed that the estimate $\hat{a}$ will be biased.

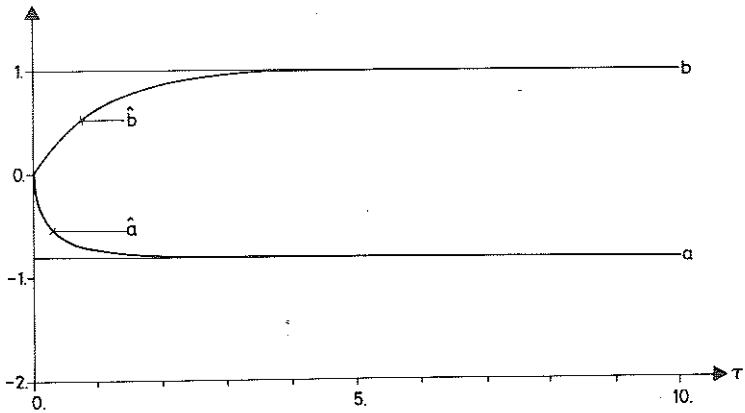

Figure 6.2.1. Trajectories for the RLS case.
$a=-0.8$, $b=1$, $c=0$, $S=1$, $\hat{a}(0)=\hat{b}(0)=0$

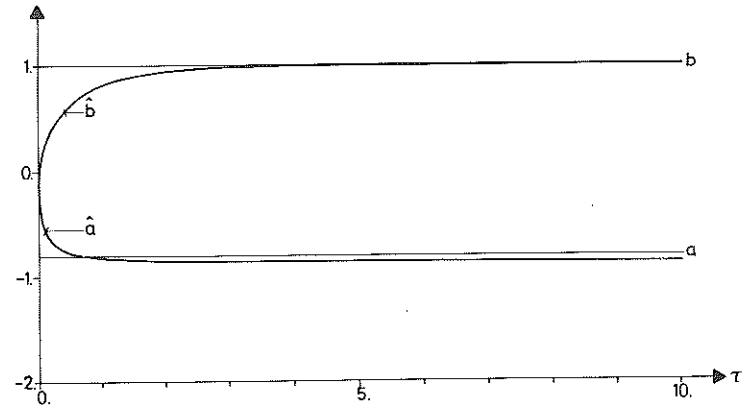

Figure 6.2.2. Trajectories for the RLS case.
$a=-0.8$, $b=1$, $c=0.7$, $S=1$, $\hat{a}(0)=\hat{b}(0)=0$

Of course the behaviour of the ODE can be illustrated by phase planes instead. However, when interpreting the results one must bear in mind that a phase plane only relate two variables but that all systems of differential equations simulated in this section have a greater order than two. In Figure 6.2.3 the phase plane of $\hat{a}$ and $\hat{b}$ for the case

$a=-0.8$, $b=1$, $c=0$, $S=1$

is given as an example. This illustrates the fact that for systems with $C(q^{-1})=1$ the estimates obtained by the RLS method always converge to the true values independent of the initial values.

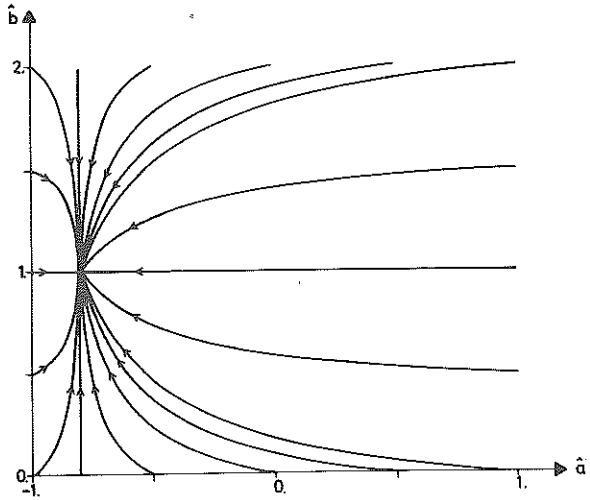

Figure 6.2.3. Phase plane of $\hat{a}$ and $\hat{b}$ for the RLS method. $a=-0.8$, $b=1$, $c=0$, $S=1$

6.2.2. The RGLS method.

For the RGLS method two cases are studied:

i) the system is assumed to be

$$(1+aq^{-1})y(t) = bq^{-1}u(t) + \frac{1}{1+cq^{-1}}\,e(t) \qquad (6.2.4)$$

ii) the system is assumed to be (6.2.1)

Case i) corresponds to a "proper" form of the system for this method. The model is assumed to be

$$(1+\hat{a}q^{-1})y(t) = \hat{b}q^{-1}u(t) + \frac{1}{1+\hat{c}q^{-1}}\,\varepsilon(t) \qquad (6.2.5)$$

for both cases.

Let us first study case i). According to Section 4, Table 4.1, a unique solution can be proved to exist for $S >> 1$, but multiple solutions exist if $S << 1$. The ODE was first simulated for the case S=1 to see what happens. It turned out that there were two stable solutions in this case and that it depended on the initial values which solution was obtained, cf Söderström (1972). The results are illustrated by Figures 6.2.4 and 6.2.5, which show the trajectories for the cases

$$a=-0.8,\ b=1,\ c=0.7,\ S=1,\ \hat{b}(0)=0 \quad \begin{cases} \hat{a}(0)=\hat{c}(0)=0 \\ \hat{a}(0)=-0.5,\ \hat{c}(0)=-0.99 \end{cases}$$

For the first case the estimates converge to the true values, for the second case, however, to another point, $(\hat{a},\hat{b},\hat{c}) = (0.492, 0.366, -0.821)$.

The behaviour of the ODE in this case can also be illustrated by phase planes. As an example the phase plane

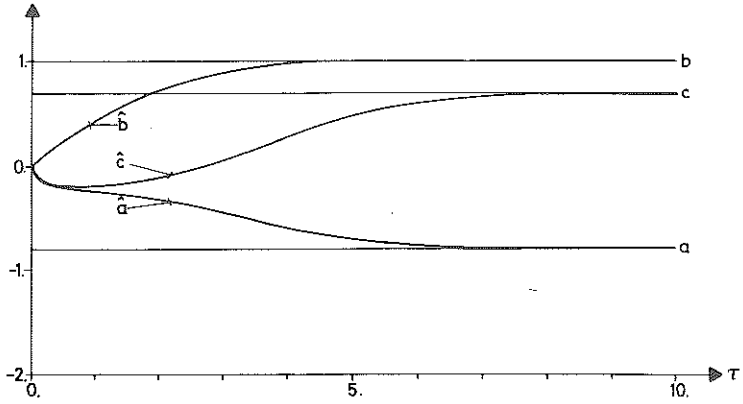

Figure 6.2.4. Trajectories for the RGLS method, case i). a=-0.8, b=1, c=0.7, S=1, $\hat{a}(0)=\hat{b}(0)=\hat{c}(0)=0$

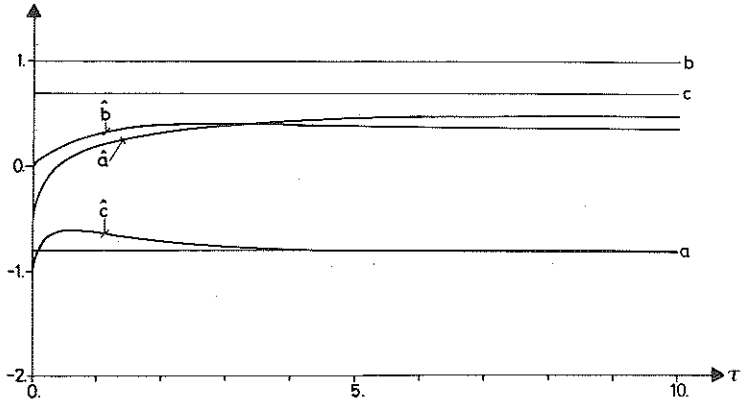

Figure 6.2.5. Trajectories for the RGLS method, case i). a=-0.8, b=1, c=0.7, S=1, $\hat{a}(0)=-0.5$, $\hat{b}(0)=0$, $\hat{c}(0)=-0.99$

of $\hat{a}$ and $\hat{c}$ is given in Figure 6.2.6 which clearly shows the two convergence points found. In Figure 6.2.7 the corresponding phase plane is given for the signal-to-noise ratio S=10, in which case only one convergence point, the true one, exists.

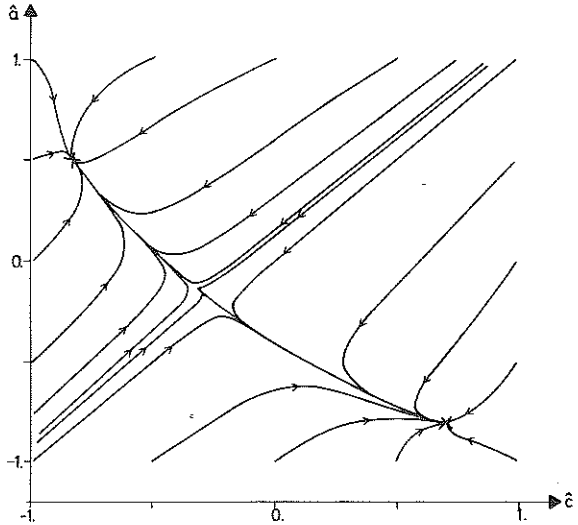

Figure 6.2.6. Phase plane of $\hat{a}$ and $\hat{c}$ for the RGLS method, case i). a=-0.8, b=1, c=0.7, S=1, $\hat{b}(0)$=0.

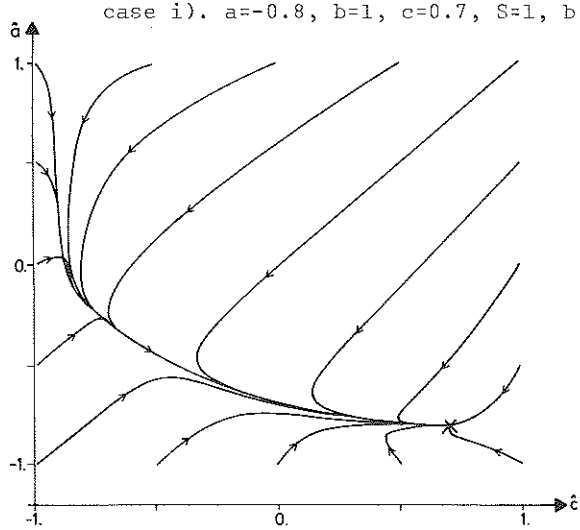

Figure 6.2.7. Phase plane of $\hat{a}$ and $\hat{c}$ for the RGLS method, case i). a=-0.8, b=1, c=0.7, S=10, $\hat{b}(0)$=0.

Then consider case ii), which corresponds to the simulations presented in Section 6.1. Only the case

$a=-0.8$, $b=1$, $c=0.7$, $S=1$, $\hat{a}(0)=\hat{b}(0)=\hat{c}(0)=0$

is illustrated, Figure 6.2.8. The asymptotic estimate is $(\hat{a},\hat{b},\hat{c}) = (-0.772,0.989,-0.498)$ which means that the estimates $\hat{a}$ and $\hat{b}$ will be biased, cf the simulations in Section 6.2.1. For the case $S=10$ the convergence point is $(\hat{a},\hat{b},\hat{c}) = (-0.796,0.999,-0.474)$.

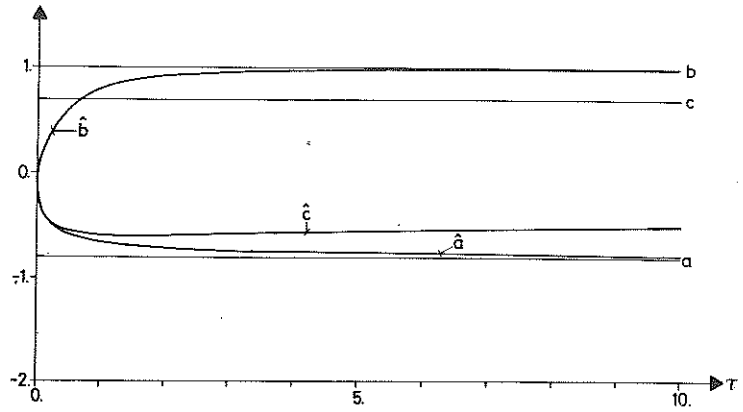

Figure 6.2.8. Trajectories for the RGLS method, case ii). $a=-0.8$, $b=1$, $c=0.7$, $S=1$, $\hat{a}(0)=\hat{b}(0)=\hat{c}(0)=0$

6.2.3. The RIV method.

The different equations corresponding to the RIV algorithm given in Section 2.4 were simulated, giving the results illustrated by Figure 6.2.9. Convergence to the true values occurs as expected. As a comparison the case

where the instrumental variables all are chosen as delayed input signal values, i.e.

$$z(t) = [u(t-1),\ldots,u(t-n_a-n_b)]^T$$

is studied in Figure 6.2.10. The convergence rates seem to be approximately the same for this example.

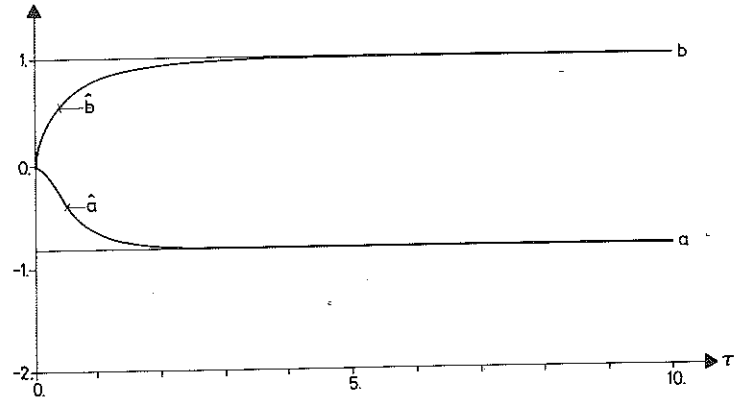

Figure 6.2.9. Trajectories for the RIV method.
a=-0.8, b=1, c=0.7, S=1, $\hat{a}(0)=\hat{b}(0)=0$

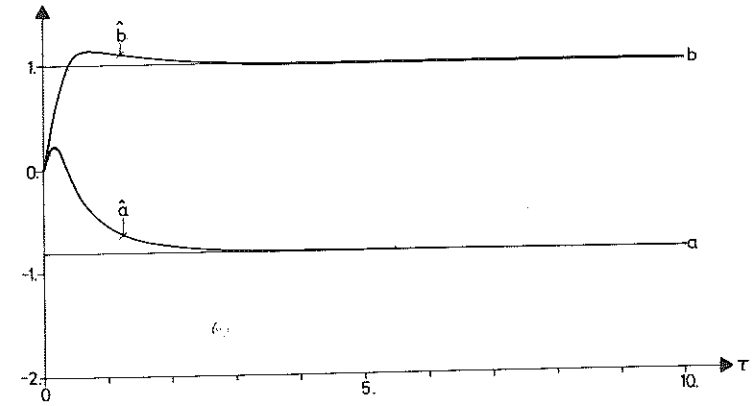

Figure 6.2.10. Trajectories for the RIV method using input signals only as instrumental variables.
a=-0.8, b=1, c=0.7, S=1, $\hat{a}(0)=\hat{b}(0)=0$

6.2.4. The RML1 and RML2 methods.

The simulation of the differential equations corresponding to the RML1 and RML2 algorithms are exemplified in Figures 6.2.11 and 6.2.12. It is readily observed that the estimate $\hat{b}$ follows approximately the same trajectory for the two algorithms. The convergence of the estimate $\hat{c}$ is much faster for the RML2 method, possibly in exchange to a slightly slower convergence of the estimate $\hat{a}$. For this example the convergence of the estimate $\hat{c}$ is slower than the convergence of the other estimates. This fact is highly pronounced if a higher signal-to-noise ratio S is used.

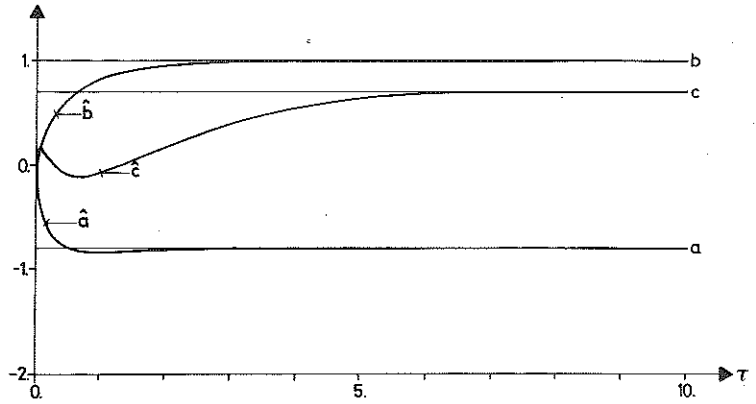

Figure 6.2.11. Trajectories for the RML1 method.
a=-0.8, b=1, c=0.7, S=1, $\hat{a}(0)=\hat{b}(0)=\hat{c}(0)=0$

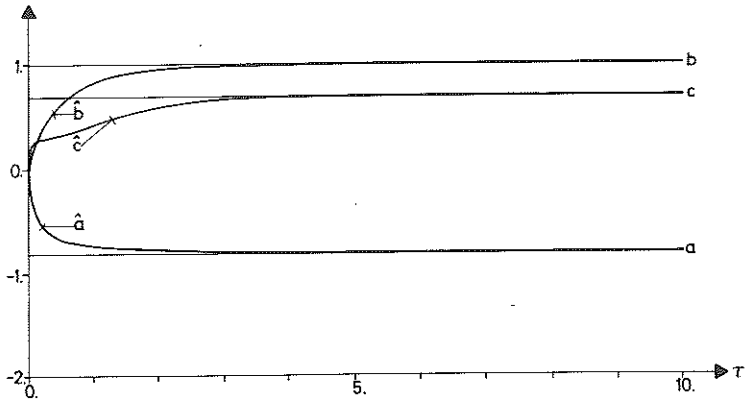

Figure 6.2.12. Trajectories for the RML2 method.
$a=-0.8$, $b=1$, $c=0.7$, $S=1$, $\hat{a}(0)=\hat{b}(0)=\hat{c}(0)=0$

6.2.5. Summary.

A relevant question is how well the trajectories for the estimates as shown in this section describe the real behaviour of the algorithms. The trajectories illustrate the asymptotic behaviour but in practice it turns out that the simulated trajectories often describe the behaviour well also for the first estimates. This is, however, dependent also on factors which are not taken into account in the differential equations, such as the weighting factor $\lambda(t)$.

6.3. Simulations of a counterexample to general convergence of the RML1 method.

In this section simulations for the autoregressive-moving average system (5.6.5)

$$y(t) + 0.9y(t-1) + 0.95y(t-2) = e(t) + 1.5e(t-1) +$$

$$+ 0.75e(t-2)$$

are presented. These simulations were exemplified in Ljung -Söderström - Gustavsson (1974). The results using the RML1 method will be compared to what is obtained using the RML2 algorithm instead. Also the ODE associated with the RML1 estimation problem will be simulated to give a deeper insight into the asymptotic properties of the estimates.

In Table 6.3.1 the averaged estimates together with an estimated standard deviation, based on ten runs with 2000 samples each, are given. Both the RML1 method and the RML2 method have been used. The initial values were chosen in two ways, $\theta(0)=0$, $P(0)=100\cdot I$ and $\theta(0)=\theta_o$, $P(0)=0.0005\cdot I$ respectively. The latter case was simulated to test if the true parameter vector θ_o is a possible limit of the estimates $\hat{\theta}(t)$. In all simulations the weighting factor was chosen as $\lambda(0)=0.95$, $\lambda_o=0.99$ in order to allow the estimates to vary sufficiently. If no weighting factor is used there is a danger that the estimates will remain practically unchanged for any reasonable number of samples, since in such a case $K(t)$ in the algorithm (2.11.1) will reach small values.

The results are also illustated by Figure 6.3.1, showing one of the ten runs. In Figure 6.3.2 the same run using the RML2 method is illustrated. The RML2 algorithm clear-

Parameter	True value	RML1		RML2	
		$\theta(0)=0$ $P(0)=100\cdot I$	$\theta(0)=\theta_o$ $P(0)=0.0005\cdot I$	$\theta(0)=0$ $P(0)=100\cdot I$	$\theta(0)=\theta_o$ $P(0)=0.0005\cdot I$
a_1	0.90	0.898±0.037	0.912±0.049	0.901±0.007	0.901±0.007
a_2	0.95	0.944±0.041	0.941±0.060	0.953±0.006	0.953±0.006
c_1	1.50	1.260±0.129	1.338±0.120	1.490±0.021	1.491±0.021
c_2	0.75	0.407±0.246	0.559±0.153	0.741±0.017	0.742±0.017
V_1	1.002	1.207	1.206	1.002	1.002

Table 6.3.1. Estimates for the system (5.6.5) using the RML1 and RML2 methods respectively.

ly behaves superiorly to the RML1 algorithm for this example.

In order to study the convergence properties in more detail the ODE (3.3.8) associated with the estimation of the parameters of the system (5.6.5) using the RML1 method was simulated. The system of nonlinear ordinary differential equations is in this case of fourteenth order. The results of the simulations are presented in Figure 6.3.3, where the states corresponding to the estimates $\hat{a}_1$, $\hat{a}_2$, $\hat{c}_1$ and $\hat{c}_2$ are shown. The simulation is initialized by $\hat{\theta}(0) = (0.91, 0.94, 1.49, 0.74)$ and it is seen that the estimates tend to limit cycles, illustrating the fact that the estimates in this case will never converge. Notice that the oscillations in the estimates can hardly be observed in the simulations, cf Figure 6.3.1, using 2000 samples only. The time scale in Figure 6.3.3 is fictitious and it is not immediate to relate it to the number of samples. However, general tendencies such as too small $\hat{c}_i$-estimates and possibly better $\hat{a}_i$-estimates can be noticed.

a)

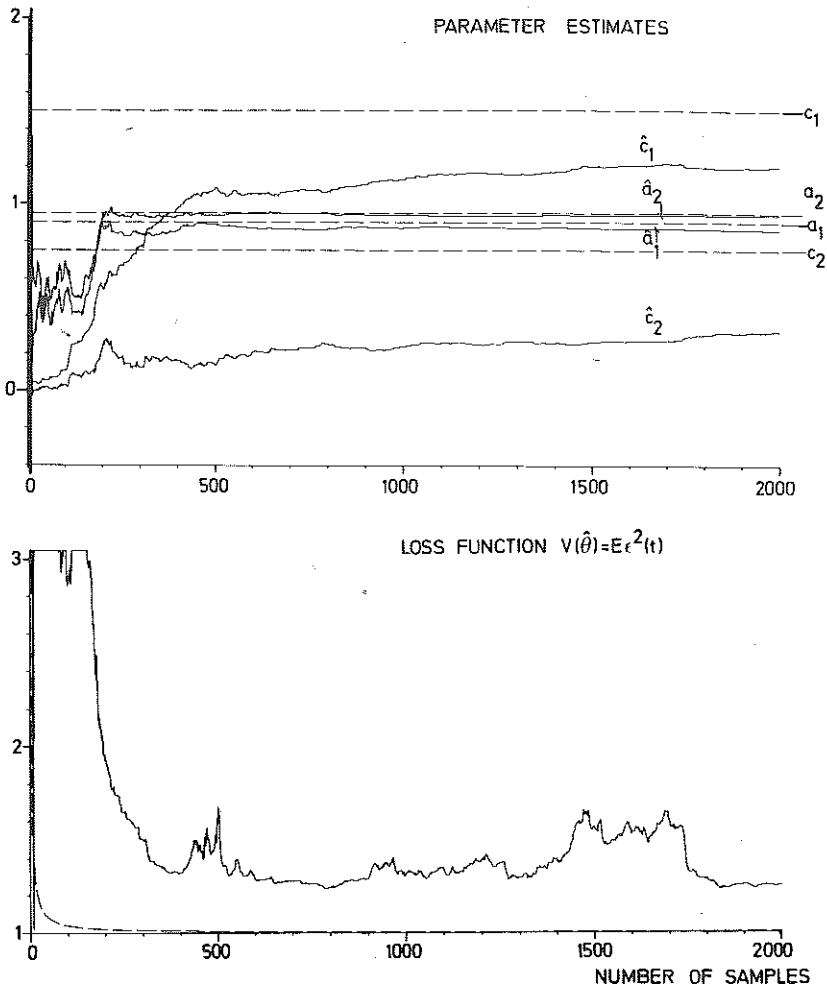

Figure 6.3.1. Results of a recursive identification using the RML1 algorithm, initialized by

a) $P(0)=100\cdot I \qquad \hat{\theta}(0)=0$

b) $P(0)=0.0005\cdot I \quad \hat{\theta}(0)=\theta_o$

The dashed curve indicates the asymptotically expected loss, assuming that $\hat{\theta}$ is asymptotically gaussian distributed with mean value θ_o and variance equal to the Cramér-Rao lower bound.

b)

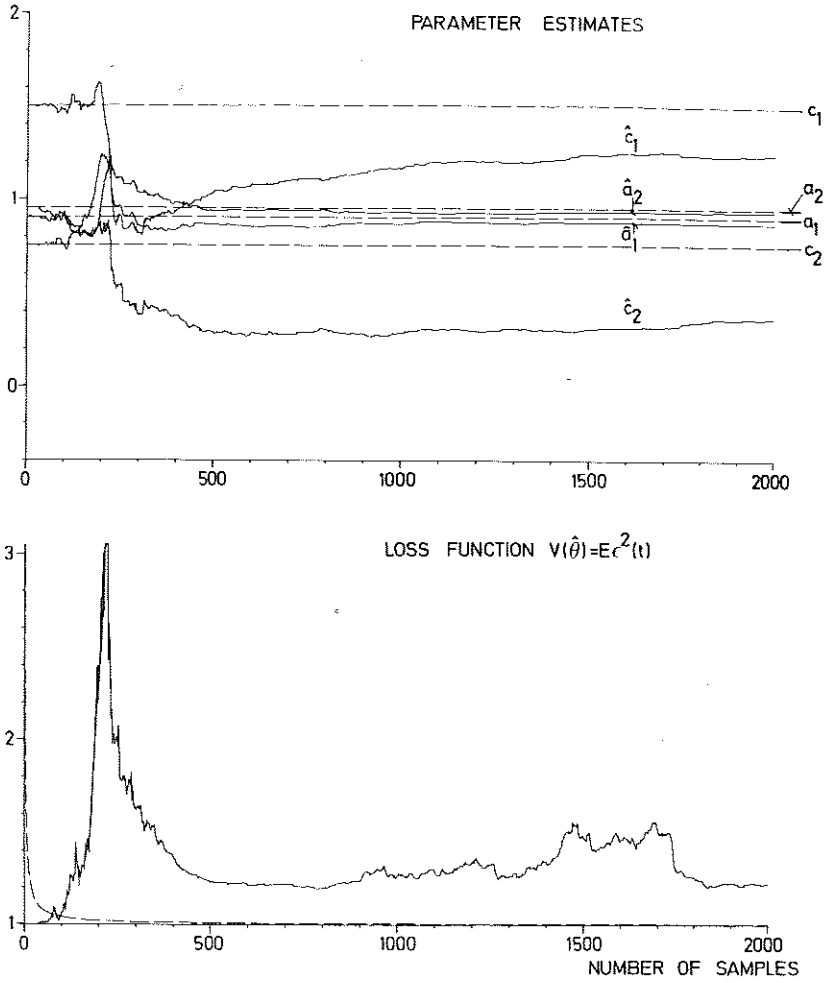

Figure 6.3.1. Continued

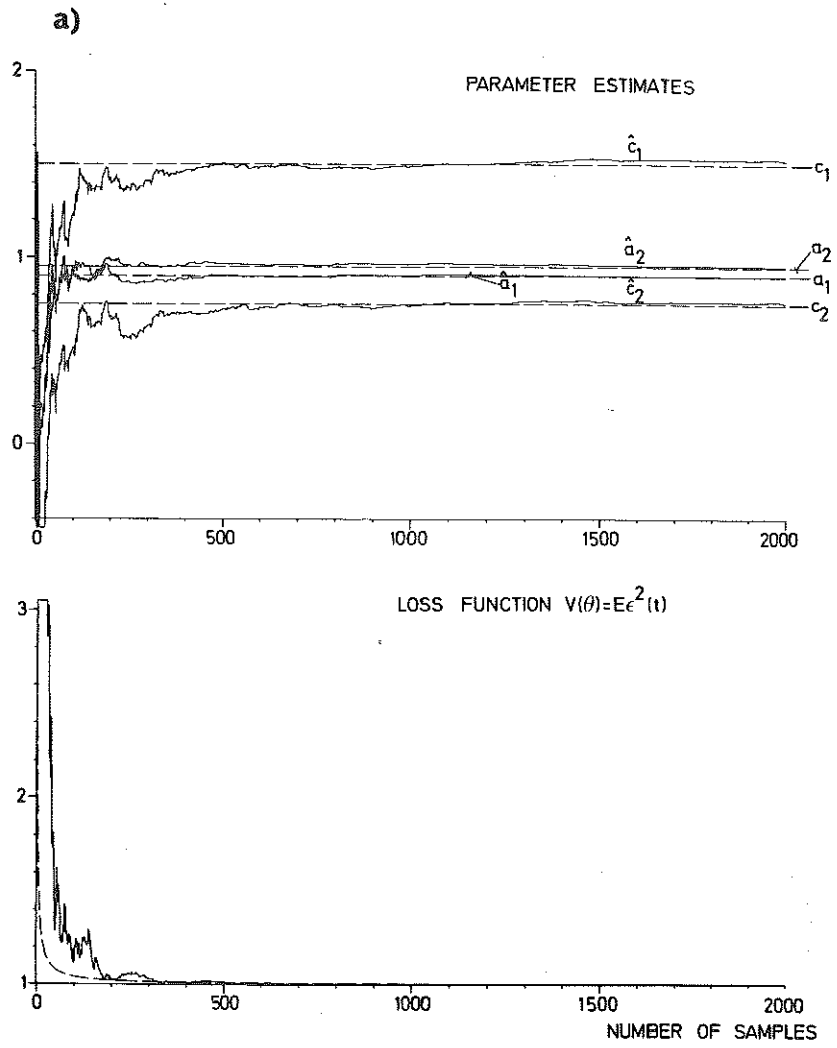

Figure 6.3.2. As in Figure 6.3.1 but using the RML2 algorithm.

b)

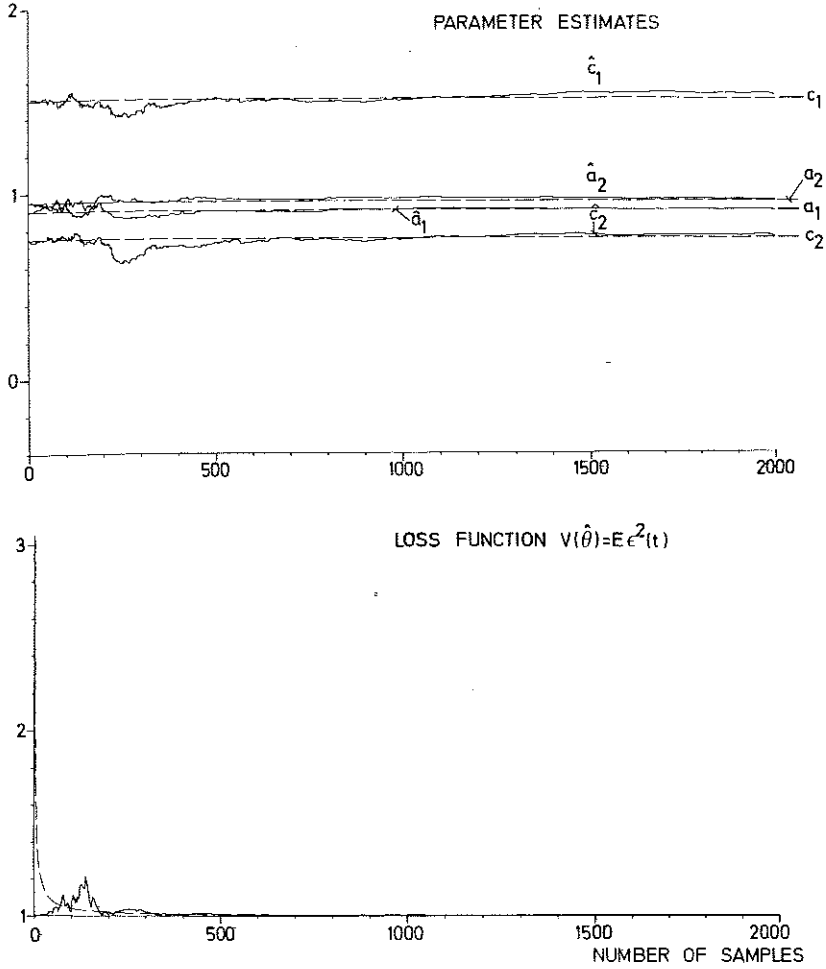

Figure 6.3.2. Continued.

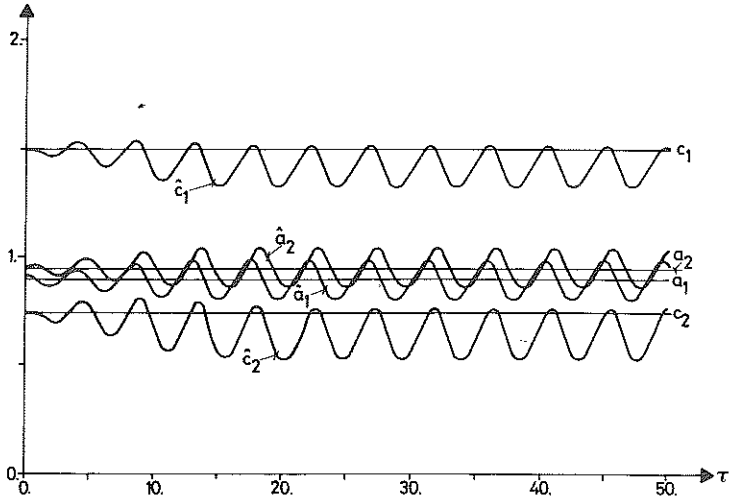

Figure 6.3.3. Trajectories for $\hat{a}_1$, $\hat{a}_2$, $\hat{c}_1$ and $\hat{c}_2$ for the ODE associated with identification of the system (5.6.5) by the RML1 method.

7. CONCLUSIONS

Five different recursive identification methods have been examined, namely the recursive versions of the least squares method (RLS), of the generalized least squares method (RGLS), and of the instrumental variables method (RIV) and two recursive versions of the maximum likelihood method (RML1 and RML2). The methods can be described by very similar algorithms as shown in Section 2.11.

The algorithms are nonlinear and stochastic difference equations which make a direct analysis of convergence extremely difficult. However, it has been shown that a theoretical examination is possible by studying the stationary solutions of a certain system of nonlinear ordinary differential equations. It is then not too difficult to get the possible limit points as well as to examine their stability properties.

The basic results are as follows: It is generally assumed that the system may be operating in closed loop, and even that the regulator may depend on old parameter estimates as is often the case for adaptive controllers. It is also assumed that the model structure is of proper form for the system under consideration. For technical reasons it is assumed that the matrix $G(\theta^*)$, defined in (3.3.6b) is nonsingular. It is in principle not difficult to test if this matrix is singular or not.

For the RLS and the RIV methods the parameter estimates will always converge to the true values. This is a well-known result and follows from the properties of the off-line methods.

For the RGLS method the parameter estimates will converge to the true values if the signal-to-noise ratio is large enough. On the other hand, if this ratio is small, it is

possible that the estimates converge to false values. In such a case the limit depends on the initial values for the estimation algorithms as well as on the realization.

For the RML1 method convergence to the true values is achieved for moving average processes and for first order processes of autoregressive - moving average type. Constructed counterexamples show that there are systems with, as well as without, input signal for which the estimates never converge.

For the RML2 method the parameter estimates will always converge to the true values for autoregressive - moving average processes. For systems with an input signal a very small or a very large signal-to-noise ratio will imply convergence. The convergence under more general conditions is closely related to an examination of the off-line ML method.

The results have been illustrated by several numerical examples.

8. ACKNOWLEDGEMENTS

The authors are grateful to Marianne Moore, who typed the manuscript, and to Britt-Marie Carlsson, who prepared the figures.

The work has been partly supported by the Swedish Board for Technical Development under Contract No. 74-3476.

9. REFERENCES

K. J. Åström (1970)
"Introduction to Stochastic Control Theory". Academic Press, New York.

K. J. Åström (1974)
"A Self-Tuning Estimator". Publication No. 74/55, Dept. of Computing and Control, Imperial College, London.

K. J. Åström and T. Bohlin (1965)
"Numerical Identification of Linear Dynamic Systems from Normal Operating Records". IFAC Symposium on Self-Adaptive Systems, Teddington, England. Also in "Theory of Self-Adaptive Control Systems" (ed. P. H. Hammond), Plenum Press, New York.

K. J. Åström and P. Eykhoff (1971)
"System Identification - A Survey". Automatica, Vol. 7, pp. 123-162.

K. J. Åström and T. Söderström (1974)
"Uniqueness of the Maximum Likelihood Estimates of the Parameters of an ARMA Model". IEEE Trans. AC-19, 769-773.

K. J. Åström and B. Wittenmark (1972)
"On Self-Tuning Regulators". Automatica, Vol. 8, pp. 185-199.

T. Bohlin (1968)
"Real Time Estimation of Time Variable Process Characteristics". Technical paper TP 18.190, IBM Nordic Lab., Lidingö, Sweden.

R. W. Brockett (1970)
"Finite Dimensional Linear Systems". J. Wiley & Sons, New York.

D. W. Clarke (1967)
"Generalized Least Squares Estimation of Parameters of a Dynamic Model." 1st IFAC Symposium on Identification in Automatic Control Systems, Prague.

H. Elmquist (1975)
"SIMNON - An Interactive Simulation Program for Nonlinear Systems. User's guide", Report, Div. of Automatic Control, Lund Inst. of Tech. To appear.

P. Eykhoff (1974)
"System Identification - Parameter and State Estimation." J. Wiley & Sons, London.

B. Finigan and I. H. Rowe (1973)
"On the Identifiability of Linear Discrete Time System Models Using the Instrumental Variable Method". 3rd IFAC Symposium on Identification and System Parameter Estimation, the Hague/Delft.

B. P. Fuhrt (1973)
"New Estimator for the Identification of Dynamic Processes." IBK Report, Institut Boris Kidric Vinca, Belgrade, Yugoslavia.

K. F. Gauss (1809)
"Theoria Motus Corporum Coelestium." A translation into English: Theory of the Motion of the Heavenly Bodies. Dover, New York (1963).

J. Gertler and Cs. Bányász (1974)
"A Recursive (On-line) Maximum Likelihood Identification Method." IEEE Trans. AC-19, 816-820.

R. HastingsJames and M. W. Sage (1969)
"Recursive Generalized Least Squares Procedure for On-line Identification of Process Parameters." IEE Proceedings, Vol. 116, pp.2057-2062.

R. Isermann, U. Baur, W. Bamberger, P. Knepo and H. Siebert (1974)
"Comparison of Six On-line Identification and Parameter Estimation Methods." Automatica, Vol. 10, pp 81-103.

P. N. James, P. Souter and D. C. Dixon (1974)
"A Comparison of Parameter Estimation Algorithms for Discrete Systems". Chemical Engineering Sciene, Vol. 29, pp 539-547.

H. Johannesson and J. O. Wesström (1974)
"Jämförelser mellan några rekursiva identifieringsmetoder." (Comparison of some Recursive Identification Methods). Master Thesis, RE-144, Div. of Automatic Control, Lund Inst. of Tech. (In Swedish).

R. L. Kashyap (1974)
"Estimation of Parameters in a Partially Whitened Representation of a Stochastic Process." IEEE, Trans. AC-19, (1), pp 13-21.

M. G. Kendall and A. Stuart (1961)
"The Advanced Theory of Statistics." Vol. 2, Griffin, London.

R. J. L. Koreman (1973)
"Recursive Parametric Estimation Methods for Multivariable Systems." M.Sc. Dissertation, Control Systems Centre, Univ. of Manchester, Inst. of Science and Tech.

I. D. Landau (1974)
"Algorithme d'identification utilisant le concept de positivite".Presented at le Colloque International IRIA/IFIP sur la Théorie du Contrôle, Rocquencourt, France.

L. Ljung (1974a)
"Convergence of Recursive Stochastic Algorithms." Report 7403, Div. of Automatic Control, Lund Inst. of Tech.

L. Ljung (1974b)
"Convergence of Recursive Stochastic Algorithms." Preprints, IFAC Symposium on Stochastic Control, Budapest.

L. Ljung, I. Gustavsson and T. Söderström (1974)
"Identification of Linear, Multivariable Systems Operating Under Linear Feedback Control." IEEE Trans. AC-19, pp. 836-840.

L. Ljung, T. Söderström and I. Gustavsson (1974)
"Counterexamples to General Convergence of a Commonly Used Recursive Identification Method." Submitted to IEEE Trans. AC.

L. Ljung and B. Wittenmark (1974)
"Asymptotic Properties of Self-Tuning Regulators." Report 7404, Div. of Automatic Control, Lund Inst. of Tech.

R. N. Lobbia and G. N. Saridis (1972)
"On-line Identification of Multivariable Stochastic Feedback Systems." Proc. JACC.

D. Q. Mayne (1967)
"A Method for Estimating Discrete Time Transfer Functions" In 'Advances in Computer Control', Second UKAC Control Convention, The University of Bristol.

R. N. Pandya (1972)
"A Class of Bootstrap Estimators for Identification of Linear Discrete Time Models." Technical report SE 72-3, Div. of Syst. Eng., Carleton University, Ottawa, Canada.

R. N. Pandya and B. Pagurek (1973)
"Two Stage Least Squares Estimators and their Recursive Approximations." 3rd IFAC Symposium on Identification and System Parameter Estimation, the Hague/Delft.

V. Panuska (1968)
"A Stochastic Approximation Method for Identification of Linear Systems Using Adaptive Filtering." Proc. JACC.

V. Panuska (1969)
"An Adaptive Recursive Least Squares Identification Algorithm." IEEE Symp. on Adaptive Processes, Decision and Control.

V. Peterka and K. Šmuk (1969)
"On-line Estimation of Dynamic Model Parameters from Input-Output Data." 4th IFAC Congress, Warsaw.

V. Peterka and A. Halouskova (1970)
"Tally Estimate of Åström Model for Stochastic Systems." 2nd IFAC Symposium on Identification and Process Parameter Estimation, Prague.

I. H. Rowe (1970)
"A Bootstrap Method for the Statistical Estimation of Model Parameters." Int. J. Ctrl.,Vol. 12, No. 5, p 721-738.

D. J. Sakrison (1967)
"The Use of Stochastic Approximation to Solve the System Identification Problem." IEEE Trans. AC-12 , p 563-567.

G. N. Saridis and G. Stein (1968)
"Stochastic Approximation Algorithms for Linear Discrete Time System Identification." IEEE Trans. AC-13, p 515-523.

G. N. Saridis (1974)
"Comparison of Six On-Line Identification Algorithms." Automatica, Vol. 10, p 69-79.

N. K. Sinha and A. Sen (1972)
"On-Line System Identification: A Critical Survey." Internal report SOC-14, Faculty of Eng.,McMaster University, Hamilton, Ontario, Canada.

A. J. Smets (1970)
"The Instrumental Variable Method and Related Identification Schemes." Dept. of El. Eng., Eindhoven Univ. of Technology, Netherlands.

T. Söderström (1972)
"On the Convergence Properties of the Generalized Least Squares Identification Method." Report 7228, Div. of Automatic Control, Lund Inst. of Tech.

T. Söderström (1973a)
"On the Uniqueness of Maximum Likelihood Identification for Different Structures." Report 7307, Div. of Automatic Control, Lund Inst. of Tech.

T. Söderström (1973b)
"An On-line Algorithm for Approximate Maximum Likelihood Identification of Linear Dynamic Systems." Report 7308, Div. of Automatic Control, Lund Inst. of Tech.

T. Söderström (1973c)
"On the Asymptotic Estimates of Least Squares Identification." Report 7327C, Div. of Automatic Control, Lund Inst. of Tech.

T. Söderström (1974)
"Convergence of Identification Methods Based on the Instrumental Variable Approach." Automatica, Vol. 10, 685-688.

T. Söderström, L. Ljung and I. Gustavsson (1974)
"On the Accuracy of Identification and the Design of Identification Experiments." Report 7428, Div. of Automatic Control, Lund Inst. of Tech.

J. L. Talmon (1971)
"Approximated Gauss-Markov Estimators and Related Schemes." TH-Report 71-E-17, Dept. of El. Eng., Eindhoven, Univ. of Technology, Netherlands.

J. L. Talmon and A. J. W. van den Boom (1973)
"On the Estimation of Transfer Function Parameters of Process and Noise Dynamics Using a Single-Stage Estimator." 3rd IFAC Symposium on Identification and System Parameter Estimation, the Hague/Delft.

S. G. Tzafestas (1970)
"Some Computer-Aided Estimators in Stochastic Control Systems Identification." Int. J. Ctrl., Vol. 12, No. 3 p 385-399.

B. L. van der Waerden (1937)
"Moderne Algebra I." Springer, Berlin.

J. Wieslander (1969)
"Real Time Identification - Part I." Report 6908, Div. of Automatic Control, Lund Inst. of Tech.

K. Y. Wong and E. Polak (1967)
"Identification of Linear Discrete Time System Using the Instrumental Variable Method." IEEE Trans. AC-12, p 707-718.

P. C. Young (1968)
"The Use of Linear Regression and Related Procedures for the Identification of Dynamic Processes." Proc. 7th IEEE Symposium on Adaptive Processes, UCLA.

P. C. Young (1970a)
"An Instrumental Variable Method for Real-Time Identification of a Noisy Process." Automatica, Vol. 6, p 271-287.

P. C. Young (1970b)
"An Extension of the Instrumental Variable Method for Identification of a Noisy Dynamic Process." Univ. of Cambridge, Dept. of Eng., Technical Note CN/70/1.

P. C. Young and R. HastingsJames (1970)
"Identification and Control of Discrete Dynamic System Subject to Disturbances with Rational Spectral Density." IEEE Symp. on Adaptive Processes, Decision and Control, Univ. of Texas, Austin.

P. C. Young, S. H. Shellswell and C. G. Neethling (1971)
"A Recursive Approach to Time Series Analysis."
CUED/B - Control/TR 16, Dept. of Eng., Univ. of Cambridge.

P. C. Young (1972)
Comments on "On-Line Identification of Linear Dynamic Systems with Application to Kalman Filtering." IEEE Trans. AC-17, p 269-270.

APPENDIX 1 - Proof of Theorem 3.1.

All three assertions of the theorem follow basically from the following lemma:

Lemma A.1 Suppose that $\theta(n)$ and $\bar{\theta}$ belong to $D_p \cap D_s$. Let $m(n,\Delta\tau)$ satisfy

$$\sum_{n}^{m(n,\Delta\tau)} \gamma(k) \to \Delta\tau \quad \text{as } n \to \infty$$

Suppose that $|\varphi(n)| < C$ and $|z(n)| < C$ (C may depend on the realization). Then for sufficiently small $\Delta\tau$ and $(\theta(n),\hat{P}^{-1}(n))$ sufficiently close to $(\bar{\theta},\bar{R})$

$$\theta(m(n,\Delta\tau)) = \theta(n) + \Delta\tau\, \bar{R}^{-1} f(\bar{\theta}) + q_1(n,\Delta\tau,\bar{\theta},\bar{R}) + q_2(n,\Delta\tau,\bar{\theta},\bar{R}) \qquad (A.1)$$

$$\hat{P}^{-1}(m(n,\Delta\tau)) = \hat{P}^{-1}(n) + \Delta\tau[G(\bar{\theta}) - \bar{R}] + q'_1(n,\Delta\tau,\bar{\theta},\bar{R}) + q'_2(n,\Delta\tau,\bar{\theta},\bar{R}) \qquad (A.2)$$

where

$$|q_1^{(')}(n,\Delta\tau,\bar{\theta},\bar{R})| \leq \Delta\tau \cdot K\{|\theta(n) - \bar{\theta}| + |\hat{P}^{-1}(n) - \bar{R}|\} + A(\Delta\tau)^2$$

and

$$q_2^{(')}(n,\Delta\tau,\bar{\theta},\bar{R}) \to 0 \quad \text{w.p.1} \quad \text{as } n \to \infty.$$

□

Proof of Lemma A.1

This lemma is proved for the RLS algorithm (with $\theta(t)$-dependent feedback) in Ljung-Wittenmark (1974). Since the main difficulty in handling the complicated relationships between $\theta(s)$ and $\varepsilon(t)$, $z(t)$ is exactly the same for the general algorithm, we will use this proof as a basis and discuss the modifications which have to be made.

The counterpart of what is called $x(t)$ in Ljung-Wittenmark (1974) is here $\varphi(t)$, $z(t)$ and $\varepsilon(t)$. These vectors satisfy recursions such as

$$\varphi(t+1) = A(\theta(t))\,\varphi(t) + B\begin{bmatrix} e(t+1) \\ u_R(t) \end{bmatrix} \qquad (A.3)$$

where all feedback effects have been removed. They are included in $A(\theta(t))$. This matrix depends on θ for two reasons. Firstly, because of the removal of $\theta(t)$-dependent feedback terms ($F_2(\theta, q^{-1})$) and secondly, because of the recursive computation of $\varepsilon(t)$ being made through $\hat{C}$-dependent matrices, cf (2.6.8).

This latter effect is not present for the algorithms treated in Ljung-Wittenmark (1974), but the only thing that matters is that

$A(\bar{\theta})$

is a stable matrix. This is assumed since $\bar{\theta} \in D_s \cap D_p$. After this observation the proof of Lemma 4.1 in Ljung-Wittenmark (1974) goes through with only notational changes. □

The conclusions b) and c) follow from Lemma A.1 exactly as Theorems 4.1 and 4.2 in Ljung-Wittenmark (1974).

We will thus turn to assertion a). Let Ω denote the sample

space and ω its points. Suppose that $\theta(t) \to \theta^*$ as $t\to\infty$ for $\omega \in \Omega_1$, where $P(\Omega_1) > 0$. It is a direct consequence of Lemma A.1 that $f(\theta^*) = 0$ then must hold. Consider for notational simplicity the stochastic approximation variant of (3.3.2) with $\hat{P}(t) = I$, (which implies $G(\theta^*) = I$) applied to an ARMA-process (no input). Assume that

$$\frac{d}{d\theta} f(\theta)\Big|_{\theta=\theta^*}$$

has an eigenvalue λ, with positive real part and corresponding left eigenvector L. Denote the unstable mode $\zeta(t) = L[\theta(t) - \theta^*]$. From the proof of Lemma A.1 it follows that

$$|\zeta(t)| \geq C_2/\gamma(t)^{1/2} \quad \text{i.o. for } \omega \in \Omega_2 \text{ where } P(\Omega_2) = 1 \tag{A.4}$$

The assumption that all moments of the noise sequence $\{e_n\}$ are finite, implies via Chebychev's inequality and the Borel-Cantelli lemma that

$$|e(t)| < C\cdot t^{\varepsilon} \quad |\varepsilon(t)| < C\cdot t^{\varepsilon} \quad |\varphi(t)| < C\cdot t^{\varepsilon} \tag{A.5}$$

w.p.1, i.e. for $\omega\in\Omega_3$, where $P(\Omega_3) = 1$, for any $\varepsilon > 0$. The constant C may depend on the realization.

Now consider the algorithm (3.3.2):

$$\hat{\theta}(t+1) = \hat{\theta}(t) + 1/t\ \varphi(t)\ \varepsilon(t+1)$$

From (A.3)

$$\varphi(t+1) = A(\hat{\theta}(t))\ \varphi(t) + B\ y(t+1)$$

$$\varepsilon(t) = C\ \varphi(t) \tag{A.6}$$

Introduce for short $\bar{\varphi}(t) = \bar{\varphi}(t;\hat{\theta}(t))$ and $\bar{\varepsilon}(t) = \bar{\varepsilon}(t;\hat{\theta}(t))$. Then

$$\varphi(t)\varepsilon(t+1) = \bar{\varphi}(t)\bar{\varepsilon}(t+1) + \tilde{\varphi}(t)\bar{\varepsilon}(t+1) + \bar{\varphi}(t)\tilde{\varepsilon}(t+1)$$

where

$$\bar{\varphi}(t) = \sum_{s=0}^{t} [A(\hat{\theta}(t))]^{t-s} B\, y(s) \ ;$$

$$\varphi(t) = \sum_{s=0}^{t} \prod_{k=s}^{t} A(\hat{\theta}(k))\, B\, y(s)$$

$$\tilde{\varphi}(t) = \varphi(t) - \bar{\varphi}(t) \quad ; \quad \tilde{\varepsilon}(t) = \varepsilon(t) - \bar{\varepsilon}(t)$$

The filter $1/C^*(q^{-1})$, corresponding to the estimate θ^*, is stable since $\theta^* \in D_p$. Therefore, the matrix $A(\theta^*)$ has all eigenvalues (= the zeroes of $z^{n_c} C^*(z^{-1})$) strictly inside the unit circle. Hence, if $\theta(t)$ is sufficiently close to θ^*

$$|A(\hat{\theta}(t))|^k \leq C_4 \cdot \mu^k, \ \mu < 1 \ ; \ |A(\hat{\theta}(t)) - A(\hat{\theta}(k))| \leq$$

$$\leq C_5 |\hat{\theta}(t) - \hat{\theta}(k)| \leq C_6^{t-k} \gamma(t)^{1-\varepsilon}$$

where the last inequality follows since

$$|\hat{\theta}(t+1) - \hat{\theta}(t)| = \frac{1}{t}|\varphi(t)\varepsilon(t+1)| \leq C_7 \gamma(t)^{1-\varepsilon}$$

due to (A.5)

This gives, after some calculations,

$$|\tilde{\varphi}(t)| \leq C_8 \gamma(t)^{1-\varepsilon} \qquad \text{(Another } \varepsilon \text{ than above)} \qquad (A.7)$$

Now

$$E\ \bar{\varphi}(t)\bar{\varepsilon}(t+1) = f(\theta(t)) = \left.\frac{df(\theta)}{d\theta}\right|_{\theta=\theta^*}(\theta(t) - \theta^*) +$$

$$+ g(\theta(t) - \theta^*) \qquad (A.8)$$

where $g(x) = o(x)$ as $x \to 0$. Hence

$$\zeta(t+1) = \zeta(t) + \gamma(t)[\lambda z(t) + \xi(t) + \eta(t) + \tilde{g}(t)] \qquad (A.9)$$

where

$$\xi(t) = L[\bar{\varphi}(t)\bar{\varepsilon}(t+1) - E\ \bar{\varphi}(t)\bar{\varepsilon}(t+1)]$$

$$\eta(t) = L\ \tilde{\varphi}(t)\bar{\varepsilon}(t+1) + L\ \bar{\varphi}(t)\tilde{\varepsilon}(t+1)$$

$$\tilde{g}(t) = L\ g(\theta(t) - \theta^*)$$

Introduce

$$\Gamma_{N,M} = \prod_{N}^{M} (1+\lambda\gamma(t)) \ ; \quad \beta_{t,M} = \lambda\gamma(t) \sum_{t+1}^{M} (1+\lambda\gamma(s));$$

$$\tilde{\beta}_{t,N} = \Gamma_{N,M}^{-1}\ \beta_{t,M}$$

Then

$$|\Gamma_{N,M}| \to \infty \quad \text{as } M \to \infty \ ; \quad \sum_{N}^{\infty} \tilde{\beta}_{t,N} = 1$$

Eq. (A.9) implies

$$\zeta(M) = \Gamma_{N,M}\ \zeta(N) + \sum_{t=N}^{M} \beta_{t,M}(\xi(t) + \eta(t) + \tilde{g}(t)) \qquad (A.10)$$

Consider first

$$\alpha_{(N,M)} = \gamma(N)^{1/2} \sum_{N}^{M} \tilde{\beta}_{t,N}\ \xi(t)$$

It is easy to see that $\alpha(N,M) \to \alpha^*(N)$ as $M \to \infty$ for $\omega \in \Omega_3$, and that $\alpha^*(N)$ is a random variable with zero mean value and bounded (in N) variance. It also follows that the correlation between $\zeta(N)$ and $\alpha^*(N)$ tends to zero as N increases. This together with (A.4) implies that

$$|\gamma(N)^{1/2}\,\zeta(N) + \alpha^*(N)| \geq \tfrac{1}{2}\,C_2 \quad \text{i.o. for } \omega \in \Omega_2 \cap \Omega_3 \tag{A.11}$$

Then consider

$$|\psi(N,M)| = |\gamma(N)^{1/2} \sum_{N}^{M} \beta_{t,N}\,\eta(t)| \leq \gamma(N)^{1/2} \max_{t \geq N} |\eta(t)| \leq$$

$$\leq C_9\,\gamma(N)^{1/2-\varepsilon} \tag{A.12}$$

Finally,

$$|\delta(N,M)| = |\gamma(N)^{1/2} \sum_{N}^{M} \beta_{t,N}\tilde{g}(t)| \leq \gamma(N)^{1/2} \max_{t \geq N} |\tilde{g}(t)| \leq$$

$$\leq \gamma(N)^{1/2}\,\tfrac{1}{8}\,C_2|\zeta(N)| \quad \text{i.o. for } \omega \in \Omega_1 \tag{A.13}$$

Eqs. (A.11), (A.12) and (A.13) imply that

$$\chi(N,M) = |\gamma(N)^{1/2}\zeta(N) + \alpha(N,M) + \psi(N,M) +$$

$$+ \delta(N,M)| \geq \tfrac{1}{4}\,C_2 \quad \text{i.o. for } M=M(N) \text{ sufficiently large and for } \omega \in \Omega_1 \cap \Omega_2 \cap \Omega_3.$$

But from (A.10)

$$|\zeta(M)| = \left|\frac{\Gamma_{N,M}}{\gamma(N)^{1/2}}\right| \cdot \chi(N,M)$$

which, with M=M(N) sufficiently large, contradicts the assumed convergence since

$$\Gamma_{N,M} \cdot \gamma(N)^{-1/2} \to \infty \quad \text{as } M \to \infty$$

Hence $\theta(t)$ cannot tend to θ^* if $H(\theta^*)$ has an unstable mode.

APPENDIX 2 - Results of simulations.

In this appendix the results of the simulations are summarized. In all tables the estimates are given together with the estimated standard deviation from ten runs. The estimates are the averaged values from the ten runs. When nothing else is indicated no weighting has been used, i.e. $\lambda(0)=1$ and $\lambda_o=1$.

In some cases theoretically expected values for the estimates and the criteria are given. If such a value includes t it describes how the expected value varies with t. It is then valid only for large t. If no t occurs in the value it is the expected limit when t tends to infinity. In the calculations made for the RLS method the results of Söderström (1973c) have been used.

Table	I(method)	S(system)	S
A2.2	RLS	(6.1.2)	1
A2.3	RLS	(6.1.2)	10
A2.7	RLS	(6.1.3)	1
A2.8	RLS	(6.1.3)	10
A2.9	RGLS	(6.1.2)	1
A2.10	RGLS	(6.1.2)	10
A2.12	RGLS	(6.1.3)	1
A2.13	RGLS	(6.1.3)	10
A2.14	RIV	(6.1.2)	1
A2.15	RIV	(6.1.2)	10
A2.17	RIV	(6.1.3)	1
A2.18	RIV	(6.1.3)	10
A2.19	RML1	(6.1.2)	1
A2.20	RML1	(6.1.2)	10
A2.22	RML1, $\lambda(0)=0.95$, $\lambda_o=0.99$	(6.1.3)	1
A2.23	RML1, $\lambda(0)=0.95$, $\lambda_o=0.99$	(6.1.3)	10
A2.24	RML2	(6.1.2)	1
A2.25	RML2	(6.1.2)	10
A2.26	RML2, $\lambda(0)=0.95$, $\lambda_o=0.99$	(6.1.3)	1
A2.27	RML2, $\lambda(0)=0.95$, $\lambda_o=0.99$	(6.1.3)	10
A2.28	RML1 / RML2, $\lambda(0)=0.95$, $\lambda_o=0.99$	(6.1.4)	1

Table A2.1. Organization of the simulation results. Only tables showing parameter estimates are referred to in this table.

Parameter	True value	Expected value	Number of samples		
			100	500	2000
$\hat{a}$	-0.8	-0.848	-0.853±0.042	-0.859±0.012	-0.851±0.009
$\hat{b}$	1.0	1.0	1.034±0.095	1.004±0.036	0.998±0.009
V_1	-	1.446	1.469	1.448	1.445
V_2	-	0.121	0.594	0.255	0.145
V_3	-	1.426	1.454	1.417	1.420

Table A2.2. *S*: First order (6.1.2), S=1
I: RLS
M: $\hat{n}_a=\hat{n}_b=1$

Parameter	True value	Expected value	Number of samples		
			100	500	2000
$\hat{a}$	-0.8	-0.809	-0.807±0.031	-0.814±0.007	-0.810±0.004
$\hat{b}$	1.0	1.0	1.008±0.033	1.001±0.011	0.999±0.003
V_1	-	1.478	1.490	1.473	1.477
V_2	-	0.0029	0.0538	0.0103	0.0047
V_3	-	1.478	1.490	1.472	1.475

Table A2.3. *S*: First order (6.1.2), S=10
I: RLS
M: $\hat{n}_a=\hat{n}_b=1$

Order of the model	S=1				S=10			
	Expected value	Number of samples			Expected value	Number of samples		
		100	500	2000		100	500	2000
1	1.446	1.469	1.448	1.445	1.478	1.490	1.473	1.477
2	1.153	1.190	1.156	1.153	1.160	1.166	1.160	1.153
3	1.065	1.126	1.072	1.067	1.068	1.100	1.071	1.068
4	1.031	1.105	1.040	1.033	1.032	1.080	1.036	1.033
5	1.014	1.106	1.028	1.018	1.015	1.071	1.024	1.017

Table A2.4. The criterion V_1. *S*: First order (6.1.2)
I: RLS
M: $\hat{n}_a=\hat{n}_b=1,2,3,4,5$

Order of the model	S=1			S=10		
	Number of samples			Number of samples		
	100	500	2000	100	500	2000
1	0.594	0.255	0.145	0.0538	0.0103	0.0047
2	0.442	0.061	0.065	0.0560	0.0045	0.0035
3	0.721	0.070	0.032	0.0685	0.0055	0.0024
4	0.572	0.044	0.019	0.0811	0.0044	0.0021
5	0.872	0.063	0.021	0.0940	0.0059	0.0025

Table A2.5. The criterion V_2. S: First order (6.1.2)
I: RLS
M: $\hat{n}_a=\hat{n}_b=1,2,3,4,5$

Order of the model	S=1				S=10			
	Expected value	Number of samples			Expected value	Number of samples		
		100	500	2000		100	500	2000
1	1.426	1.454	1.417	1.420	1.478	1.490	1.472	1.475
2	1.146	1.168	1.136	1.143	1.158	1.165	1.155	1.157
3	1.063	1.114	1.068	1.065	1.067	1.096	1.070	1.068
4	1.029	1.107	1.039	1.032	1.031	1.075	1.036	1.032
5	1.014	1.126	1.028	1.018	1.015	1.068	1.024	1.017

Table A2.6. The criterion V_3. S: First order (6.1.2)
I: RLS
M: $\hat{n}_a=\hat{n}_b=1,2,3,4,5$

Para-meter	True value	Expected value	Number of samples		
			100	500	2000
$\hat{a}_1$	-1.5	-0.756	-0.785±0.101	-0.798±0.059	-0.771±0.033
$\hat{a}_2$	0.7	0.101	0.143±0.124	0.140±0.052	0.112±0.025
$\hat{b}_1$	1.0	1.000	1.221±0.396	1.045±0.141	0.999±0.086
$\hat{b}_2$	0.5	1.244	1.136±0.445	1.292±0.168	1.276±0.107
V_1	-	2.688	3.150	2.610	2.647
V_2	-	5.219	6.034	4.821	5.017

Table A2.7. S: Second order (6.1.3), S=1
I: RLS
M: $\hat{n}_a=\hat{n}_b=2$

Parameter	True value	Expected value	Number of samples		
			100	500	2000
$\hat{a}_1$	-1.5	-1.185	-1.218±0.047	-1.209±0.034	-1.197±0.022
$\hat{a}_2$	0.7	0.416	0.443±0.052	0.439±0.033	0.426±0.019
$\hat{b}_1$	1.0	1.000	1.103±0.143	1.024±0.047	1.003±0.037
$\hat{b}_2$	0.5	0.815	0.731±0.186	0.816±0.073	0.820±0.042
V_1	-	1.751	1.819	1.757	1.752
V_2	-	2.193	2.031	1.915	2.061

Table A2.8. *S*: Second order (6.1.3), S=10 -
I: RLS
M: $\hat{n}_a=\hat{n}_b=2$

Parameter	True value	Expected value	Number of samples		
			100	500	2000
$\hat{a}$	-0.8	-0.772	-0.827±0.055	-0.808±0.029	-0.789±0.018
$\hat{b}$	1.0	0.989	1.025±0.100	0.996±0.035	0.989±0.011
$\hat{c}$	0.7	-0.498	-0.449±0.092	-0.464±0.039	-0.474±0.026
V_2	-	0.026	0.402	0.038	0.015

Table A2.9. *S*: First order (6.1.2), S=1
I: RGLS
M: $\hat{n}_a=\hat{n}_b=\hat{n}_c=1$

Parameter	True value	Expected value	Number of samples		
			100	500	2000
$\hat{a}$	-0.8	-0.796	-0.804±0.028	-0.804±0.011	-0.799±0.006
$\hat{b}$	1.0	0.999	1.005±0.036	0.999±0.011	0.997±0.003
$\hat{c}$	0.7	-0.474	-0.458±0.080	-0.478±0.032	-0.473±0.013
V_2	-	0.0005	0.0443	0.0056	0.0016

Table A2.10. *S*: First order (6.1.2), S=10
I: RGLS
M: $\hat{n}_a=\hat{n}_b=\hat{n}_c=1$

Order of the C-polynomial	S=1			S=10		
	Number of samples			Number of samples		
	100	500	2000	100	500	2000
1	0.402	0.038	0.015	0.0443	0.0056	0.0016
2	0.494	0.108	0.039	0.0474	0.0064	0.0018
3	0.457	0.087	0.029	0.0521	0.0065	0.0015
4	0.376	0.108	0.055	0.0510	0.0096	0.0023
5	0.249	0.102	0.062	0.0663	0.0200	0.0064

Table A2.11. The criterion V_2. *S*: First order (6.1.2)
I: RGLS
M: $\hat{n}_a=\hat{n}_b=1$, $\hat{n}_c=1,2,3,4,5$

Parameter	True value	Expected value	Number of samples		
			100	500	2000
$\hat{a}_1$	-1.5	-1.332	-0.845±0.134	-0.949±0.153	-1.012±0.161
$\hat{a}_2$	0.7	0.597	0.189±0.185	0.266±0.151	0.319±0.140
$\hat{b}_1$	1.0	0.913	1.300±0.649	1.051±0.274	0.972±0.104
$\hat{b}_2$	0.5	0.891	1.217±0.841	1.254±0.299	1.172±0.156
$\hat{c}_1$	-1.0	0.737	0.180±0.232	0.235±0.198	0.331±0.185
$\hat{c}_2$	0.2	0.307	0.041±0.171	0.037±0.096	0.050±0.077
V_2	-	1.283	8.392	4.148	3.378

Table A2.12. *S*: Second order (6.1.3), S=1
I: RGLS
M: $\hat{n}_a=\hat{n}_b=\hat{n}_c=2$

Parameter	True value	Expected value	Number of samples		
			100	500	2000
$\hat{a}_1$	-1.5	-1.458	-1.310±0.040	-1.416±0.017	-1.443±0.016
$\hat{a}_2$	0.7	0.670	0.535±0.045	0.634±0.013	0.656±0.013
$\hat{b}_1$	1.0	0.978	1.101±0.172	1.010±0.056	0.980±0.022
$\hat{b}_2$	0.5	0.597	0.684±0.277	0.651±0.080	0.623±0.038
$\hat{c}_1$	-1.0	0.793	0.522±0.205	0.661±0.090	0.750±0.035
$\hat{c}_2$	0.2	0.362	0.189±0.198	0.277±0.092	0.324±0.028
V_2	-	0.087	1.712	0.329	0.143

Table A2.13. *S*: Second order (6.1.3), S=10
I: RGLS
M: $\hat{n}_a=\hat{n}_b=\hat{n}_c=2$

Parameter	True value	Expected value	Number of samples		
			100	500	2000
$\hat{a}$	-0.8	-0.8	-0.755±0.137	-0.813±0.027	-0.805±0.018
$\hat{b}$	1.0	1.0	1.009±0.124	1.000±0.035	0.995±0.009
V_1	-	1.490	1.757	1.482	1.487
V_2	-	15.4/t	0.681	0.046	0.012
V_3	-	1.490	1.581	1.473	1.478

Table A2.14. *S*: First order (6.1.2), S=1
I: RIV, τ=2
M: $\hat{n}_a=\hat{n}_b=1$

Parameter	True value	Expected value	Number of samples		
			100	500	2000
$\hat{a}$	-0.8	-0.8	-0.798±0.037	-0.805±0.009	-0.802±0.006
$\hat{b}$	1.0	1.0	1.007±0.033	1.000±0.011	0.998±0.003
V_1	-	1.490	1.506	1.484	1.488
V_2	-	1.54/t	0.0596	0.0044	0.0011
V_3	-	1.490	1.501	1.483	1.486

Table A2.15. *S*: First order (6.1.2), S=10
I: RIV, τ=2
M: $\hat{n}_a=\hat{n}_b=1$

τ	S=1	S=10
2	0.0115	0.00106
4	0.0108	0.00100
6	0.0103	0.00097

Table A2.16. The criterion V_2. *S*: First order (6.1.2)
I: RIV, τ=2,4,6
M: $\hat{n}_a=\hat{n}_b=1$
The theoretical value is 0.0077 for S=1 and 0.00077 for S=10. 2000 data points were used to get the values in the table.

Parameter	True value	Expected value	Number of samples		
			100	500	2000
$\hat{a}_1$	-1.5	-1.5	-1.726±0.409	-1.435±0.314	-1.480±0.100
$\hat{a}_2$	0.7	0.7	1.006±0.438	0.644±0.299	0.681±0.094
$\hat{b}_1$	1.0	1.0	1.203±0.725	1.075±0.183	1.002±0.140
$\hat{b}_2$	0.5	0.5	-0.219±1.103	0.615±0.456	0.568±0.188
V_2	-	0	17.2	3.865	0.420

Table A2.17. *S*: Second order (6.1.3), S=1
I: RIV, τ=2
M: $\hat{n}_a=\hat{n}_b=2$
The estimates after 100 samples are in this case based on seven runs only. The remaining three runs gave quite unacceptable estimates for 100 data points.

Parameter	True value	Expected value	Number of samples		
			100	500	2000
$\hat{a}_1$	-1.5	-1.5	-1.509±0.114	-1.490±0.023	-1.496±0.018
$\hat{a}_2$	0.7	0.7	0.722±0.182	0.692±0.022	0.697±0.016
$\hat{b}_1$	1.0	1.0	1.090±0.159	1.025±0.055	1.001±0.044
$\hat{b}_2$	0.5	0.5	0.372±0.317	0.525±0.083	0.518±0.047
V_2	-	0	2.352	2.029	2.036

Table A2.18. *S*: Second order (6.1.3), S=10
I: RIV, τ=2
M: $\hat{n}_a=\hat{n}_b=2$

Parameter	True value	Expected value	Number of samples		
			100	500	2000
$\hat{a}$	-0.8	-0.8	-0.787±0.081	-0.806±0.032	-0.801±0.018
$\hat{b}$	1.0	1.0	1.027±0.049	1.005±0.031	0.997±0.012
$\hat{c}$	0.7	0.7	0.461±0.215	0.597±0.108	0.656±0.059
V_1	-	1	1.133	1.030	1.008
V_2	-	0	0.3650	0.0404	0.0093
V_3	-	1	1.124	1.025	1.006

Table A2.19. *S*: First order (6.1.2), S=1
I: RML1
M: $\hat{n}_a=\hat{n}_b=\hat{n}_c=1$

Parameter	True value	Expected value	Number of samples		
			100	500	2000
$\hat{a}$	-0.8	-0.8	-0.789±0.037	-0.800±0.013	-0.800±0.008
$\hat{b}$	1.0	1.0	1.006±0.020	1.000±0.009	0.999±0.004
$\hat{c}$	0.7	0.7	0.369±0.109	0.579±0.035	0.653±0.022
V_1	-	1	1.145	1.024	1.005
V_2	-	0	0.0502	0.0055	0.0017
V_3	-	1	1.146	1.023	1.004

Table A2.20. *S*: First order (6.1.2), S=10
I: RML1
M: $\hat{n}_a=\hat{n}_b=\hat{n}_c=1$

S	$\lambda(0)$	λ_o	RML1			RML2		
			V_1	V_2	V_3	V_1	V_2	V_3
1	0.95	0.01	1.0026	0.0054	1.0026	1.0017	0.0053	1.0016
1	0.99	0.01	1.0042	0.0062	1.0032	1.0015	0.0053	1.0010
1	0.999	0.01	1.0078	0.0089	1.0058	1.0016	0.0062	1.0009
1	0.95	0.001	1.0175	0.0172	1.0201	1.0082	0.0276	1.0186
1	0.99	0.001	1.0047	0.0054	1.0059	1.0017	0.0052	1.0024
1	0.999	0.001	1.0051	0.0067	1.0038	1.0016	0.0055	1.0010
1	1.0	0.0	1.0084	0.0093	1.0064	1.0017	0.0064	1.0009
10	0.95	0.01	1.0018	0.0011	1.0017	1.0011	0.0009	1.0010
10	0.99	0.01	1.0020	0.0013	1.0017	1.0011	0.0009	1.0009
10	0.999	0.01	1.0042	0.0017	1.0037	1.0015	0.0009	1.0013
10	0.95	0.001	1.0114	0.0025	1.0113	1.0033	0.0043	1.0043
10	0.99	0.001	1.0032	0.0007	1.0034	1.0008	0.0007	1.0009
10	0.999	0.001	1.0025	0.0013	1.0022	1.0012	0.0009	1.0010
10	1.0	0.0	1.0047	0.0017	1.0042	1.0016	0.0010	1.0013

Table A2.21. Comparison of the criteria V_1, V_2 and V_3 for different choices of $\lambda(0)$ and λ_o. The values are obtained after 2000 samples.
S: First order (6.1.2)
I: RML1 and RML2
M: $\hat{n}_a=\hat{n}_b=\hat{n}_c=1$
The expected values are:
S=1 V_1 1.0015 V_2 0.0040 V_3 1.0010
S=10 V_1 1.0015 V_2 0.0007 V_3 1.0005

Parameter	True value	Expected value	Number of samples		
			100	500	2000
$\hat{a}_1$	-1.5	-1.5	-1.015±0.233	-1.396±0.140	-1.460±0.068
$\hat{a}_2$	0.7	0.7	0.325±0.193	0.600±0.124	0.658±0.065
$\hat{b}_1$	1.0	1.0	1.258±0.478	1.008±0.238	0.975±0.082
$\hat{b}_2$	0.5	0.5	0.692±0.681	0.667±0.188	0.581±0.179
$\hat{c}_1$	-1.0	-1.0	-0.288±0.248	-0.784±0.198	-0.917±0.085
$\hat{c}_2$	0.2	0.2	0.033±0.103	0.049±0.098	0.136±0.053
V_1	-	1	2.995	1.619	1.099
V_2	-	0	5.764	1.832	0.312

Table A2.22. *S*: Second order (6.1.3), S=1
I: RML1, $\lambda(0)=0.95$, $\lambda_o=0.99$
M: $\hat{n}_a=\hat{n}_b=\hat{n}_c=2$

Parameter	True value	Expected value	Number of samples		
			100	500	2000
$\hat{a}_1$	-1.5	-1.5	-1.384±0.132	-1.488±0.031	-1.490±0.024
$\hat{a}_2$	0.7	0.7	0.589±0.126	0.686±0.038	0.689±0.021
$\hat{b}_1$	1.0	1.0	1.099±0.156	1.006±0.078	0.992±0.026
$\hat{b}_2$	0.5	0.5	0.499±0.294	0.535±0.081	0.521±0.066
$\hat{c}_1$	-1.0	-1.0	-0.530±0.261	-0.865±0.096	-0.951±0.049
$\hat{c}_2$	0.2	0.2	-0.060±0.112	0.043±0.092	0.137±0.043
V_1	-	1	1.860	1.089	1.018
V_2	-	0	1.225	0.199	0.037
V_3	-	1	-	1.060	1.009

Table A2.23. *S*: Second order (6.1.3), S=10
I: RML1, $\lambda(0)=0.95$, $\lambda_o=0.99$
M: $\hat{n}_a=\hat{n}_b=\hat{n}_c=2$

Parameter	True value	Expected value	Number of samples		
			100	500	2000
$\hat{a}$	-0.8	-0.8	-0.785±0.080	-0.807±0.024	-0.802±0.013
$\hat{b}$	1.0	1.0	1.021±0.073	1.001±0.020	0.998±0.012
$\hat{c}$	0.7	0.7	0.548±0.195	0.670±0.041	0.689±0.021
V_1	-	1+3/t	1.106	1.006	1.002
V_2	-	7.9/t	0.3820	0.0268	0.0064
V_3	-	1+1.86/t	1.107	1.003	1.001

Table A2.24. *S*: First order (6.1.2), S=1
I: RML2
M: $\hat{n}_a=\hat{n}_b=\hat{n}_c=1$

Parameter	True value	Expected value	Number of samples		
			100	500	2000
$\hat{a}$	-0.8	-0.8	-0.792±0.031	-0.800±0.009	-0.800±0.005
$\hat{b}$	1.0	1.0	1.000±0.020	0.999±0.007	0.999±0.003
$\hat{c}$	0.7	0.7	0.434±0.111	0.641±0.040	0.681±0.024
V_1	-	1+3/t	1.103	1.008	1.002
V_2	-	1.33/t	0.0343	0.0028	0.0010
V_3	-	1+1.09/t	1.100	1.008	1.001

Table A2.25. *S*: First order (6.1.2), S=10
I: RML2
M: $\hat{n}_a=\hat{n}_b=\hat{n}_c=1$

Parameter	True value	Expected value	Number of samples		
			100	500	2000
$\hat{a}_1$	-1.5	-1.5	-0.888±0.445	-1.460±0.060	-1.497±0.023
$\hat{a}_2$	0.7	0.7	0.299±0.287	0.678±0.054	0.698±0.021
$\hat{b}_1$	1.0	1.0	1.135±0.532	0.992±0.231	0.970±0.077
$\hat{b}_2$	0.5	0.5	0.790±0.576	0.661±0.259	0.549±0.110
$\hat{c}_1$	-1.0	-1.0	-0.204±0.389	-0.888±0.093	-0.980±0.038
$\hat{c}_2$	0.2	0.2	0.092±0.219	0.149±0.035	0.182±0.039
V_1	-	1+6/t	3.701	1.155	1.033
V_2	-	0	6.952	0.502	0.087

Table A2.26. *S*: Second order (6.1.3), S=1
I: RML2, $\lambda(0)=0.95$, $\lambda_o=0.99$
M: $\hat{n}_a=\hat{n}_b=\hat{n}_c=2$

Parameter	True value	Expected value	Number of samples		
			100	500	2000
$\hat{a}_1$	-1.5	-1.5	-1.265±0.294	-1.496±0.020	-1.498±0.008
$\hat{a}_2$	0.7	0.7	0.482±0.280	0.699±0.015	0.698±0.007
$\hat{b}_1$	1.0	1.0	1.037±0.205	1.006±0.079	0.991±0.025
$\hat{b}_2$	0.5	0.5	0.588±0.289	0.525±0.098	0.515±0.036
$\hat{c}_1$	-1.0	-1.0	-0.281±0.443	-0.891±0.095	-0.973±0.043
$\hat{c}_2$	0.2	0.2	0.089±0.159	0.101±0.105	0.172±0.044
V_1	-	1+6/t	1.163	1.041	1.006
V_2	-	0	2.240	0.054	0.010
V_3	-	1	-	1.059	1.004

Table A2.27. *S*: Second order (6.1.3), S=10
I: RML2, $\lambda(0)=0.95$, $\lambda_o=0.99$
M: $\hat{n}_a=\hat{n}_b=\hat{n}_c=2$

Parameter	True value	RML1			RML2		
		Number of samples			Number of samples		
		100	500	2000	100	500	2000
$\hat{a}_1$	-1.600	-1.655	-1.594	-1.588	-1.565	-1.603	-1.594
$\hat{a}_2$	1.610	1.607	1.599	1.600	1.464	1.606	1.606
$\hat{a}_3$	-0.776	-0.800	-0.770	-0.764	-0.695	-0.775	-0.770
$\hat{b}_1$	1.200	1.262	1.199	1.188	1.320	1.181	1.193
$\hat{b}_2$	-0.950	-1.057	-0.903	-0.929	-0.955	-0.960	-0.946
$\hat{b}_3$	0.200	0.137	0.235	0.204	0.095	0.204	0.202
$\hat{c}_1$	0.100	0.088	0.145	0.133	0.133	0.110	0.104
$\hat{c}_2$	0.250	0.017	0.162	0.226	-0.019	0.172	0.239
$\hat{c}_3$	0.873	0.232	0.703	0.798	0.176	0.713	0.840
V_1	-	1.692	1.171	1.060	2.140	1.117	1.022

Table A2.28. *S*: Third order (6.1.4), S=1
I: RML1 and RML2, $\lambda(0)=0.95$, $\lambda_o=0.99$
M: $\hat{n}_a=\hat{n}_b=\hat{n}_c=3$

Parameter	S=1			S=10		
	Number of samples			Number of samples		
	100	500	2000	100	500	2000
a	0.042	0.019	0.009	0.018	0.008	0.004
b	0.048	0.022	0.011	0.016	0.007	0.004
c	0.073	0.033	0.016	0.072	0.032	0.016
a_1	0.083	0.037	0.019	0.034	0.015	0.008
a_2	0.071	0.032	0.016	0.028	0.013	0.006
b_1	0.348	0.156	0.078	0.110	0.049	0.025
b_2	0.425	0.190	0.095	0.145	0.065	0.032
c_1	0.134	0.060	0.030	0.105	0.047	0.023
c_2	0.122	0.054	0.027	0.102	0.046	0.023

Table A2.29. The Cramér-Rao lower bounds on the standard deviation of the estimates of the parameters of the first and second order systems (6.1.2) and (6.1.3) respectively.