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A Robust Approach to the Synthesis of Dolph-Chebyshev Beampatterns for Uniform Circular Arrays

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ABSTRACT

The Davies transformation is a method to transform the steering vector of a uniform circular array to another vector with Vandermonde form. However, this transformation can be highly sensitive to perturbations of the underlying ideal array model. In this paper, we describe a method for deriving a more robust transformation using optimisation techniques. The effectiveness of the method is illustrated through a numerical example.

1. INTRODUCTION

By virtue of their geometry, uniform circular arrays (UCAs) can provide a natural 360° of coverage in the azimuthal plane (see Fig. 1). Accordingly, they are eminently suitable for applications such as radar, sonar and mobile wireless communications where one desires all-azimuth coverage [1]. This innate advantage of UCAs is offset, however, by the unfriendly mathematical structure of their steering vectors. In particular, many useful techniques that have been developed for uniform linear arrays (ULAs), such as Dolph-Chebyshev beampattern design [2-4], and spatial smoothing for direction-of-arrival estimation in a correlated signal environment [5-6] and adaptive and optimum beamforming in a correlated signal environment [7], cannot be applied directly to UCAs. The reason for this is because the aforesaid techniques exploit the Vandermonde structure of a ULA's steering vector in order to work while the steering vector of a UCA is not Vandermonde [8,9].

In [10], Davies proposes a method to transform the sensor element outputs of a UCA to derive the so-called *virtual array*. The key feature of the virtual array is that its steering vector is Vandermonde, or approximately so. In [8], we use the Davies transformation to design Dolph-Chebyshev beampatterns for UCAs, while in [1] and [9], it is used to enable, respectively, direction-of-arrival estimation and optimum beamforning for UCAs in a correlated signal environment.

Fig. 1. A uniform circular array

The Davies transformation is not, however, without problems. Specifically, Davies [10] tacitly assumes that (i) all antenna elements have the same omnidirectional response, (ii) the electronics associated with each antenna element are identical, (iii) the antenna elements are located at their correct positions, and (iv) there is no mutual coupling between the antenna elements. In a real system, it is certain that none of the above assumptions will hold. Although in [1, 11], it is pointed out that these real-world effects can be ameliorated somewhat by calibration, there remains the problem of residual calibration errors. In [12], it is shown through simulation that when errors are introduced into the model of an ideal UCA, as represented by a perturbation of its steering vector, the performance of the UCA can degrade appreciably. The yardstick used in [12] to assess performance is the Dolph-Chebyshev beampattern obtained through the method of [8].

Our aim is to find an alternative transformation that has the desirable properties of the Davies transformation, i.e., the ability to transform the steering vector of a UCA to Vandermonde form, but
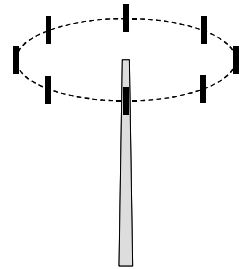
is more robust with respect to perturbations to the steering vector of an ideal UCA. In this paper, we derive such a robust transformation using optimisation techniques.

The paper is organized as follows. In Section 2, we formulate the optimisation problem, and in Section 3, we describe the solution method. In Section 4, we demonstrate the effectiveness of our robust transformation through a simulation example. Finally, Section 5 concludes the paper.

2. PROBLEM FORMULATION

2.1 The Davies Transformation

Consider a UCA with N elements and radius r. The nth component of the N-dimensional array response (or *steering*) vector $\mathbf{a}(\theta)$, $n = 1, \ldots, N$, to a narrowband signal of wavelength λ arriving from angle θ, $\theta \in [-\pi, \pi]$, is given by

$$\left[\mathbf{a}(\theta) \right]_n = G_n(\theta) \exp\left[j \tfrac{2\pi r}{\lambda} \cos\left(\theta - \tfrac{2\pi(n-1)}{N} \right) \right] \tag{1}$$

where $G_n(\theta)$ is the complex gain pattern of the nth array element.
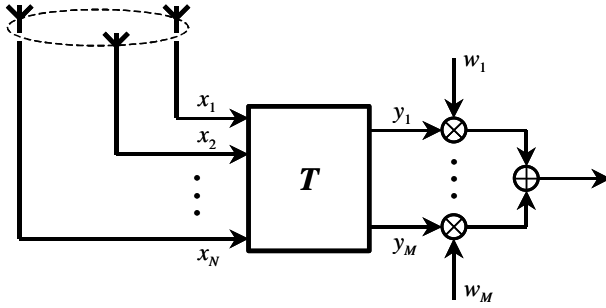

Fig. 2. Modal transformation for UCAs

Suppose the array elements are all identical and isotropic, i.e., $G_n(\theta) = 1$ for $n = 1, \ldots, N$. Suppose further that the antenna element outputs are processed as shown in Fig. 2 where $x_1, \ldots, x_N$ represent the baseband complex output signals of the "real" array, $y_1, \ldots, y_M$, $M < N$, represent the baseband complex output signals of the virtual array, and $w_1, \ldots, w_M$ are a set of weighting coefficients whose function is to shape the spatial response of the antenna array.

In [10] (see also [1]), it is shown that if the transformation matrix $\mathbf{T}$ is defined as follows

$$\mathbf{T} = \mathbf{JF} \tag{2}$$

where the matrices $\mathbf{J} \in \mathbb{C}^{M \times M}$ and $\mathbf{F} \in \mathbb{C}^{M \times N}$ are given by

$$\mathbf{J} = \mathrm{diag}\left\{ \left[j^{m-1-H} \sqrt{N} J_{m-1-H}\left(\tfrac{2\pi r}{\lambda} \right) \right]^{-1} \right\} \tag{3}$$

and

$$\left[\mathbf{F} \right]_{mn} = \frac{1}{\sqrt{N}} e^{j 2\pi(m-1-H)(n-1)/N}, \tag{4}$$

and where $m = 1, \ldots, M$, $n = 1, \ldots, N$, $J_k(\cdot)$ denotes a kth order Bessel function of the first kind,

and

$$H = (M-1)/2 \in \mathbb{Z}, \tag{5}$$

then the M-dimensional steering vector of the virtual array will take on, approximately, the Vandermonde form

$$\mathbf{b}(\theta) = \mathbf{T}\mathbf{a}(\theta) \cong \left[e^{-jH\theta} \ \cdots \ 1 \ \cdots \ e^{+jH\theta} \right]. \tag{6}$$

Note, in view of Eq. (5), M is odd. Define k such that $m = k + 1 + H$. The mth elements of $\mathbf{b}(\theta)$,

$$b_m(\theta) = e^{-j(m-1-H)\theta} = e^{-jk\theta} \tag{7}$$

is called the kth mode of $\mathbf{b}(\theta)$ [1, 10].

2.2 Robustness

In [12], it was shown that for some array radii r, the Davies transformation is highly susceptible

to perturbations in $\mathbf{a}(\theta)$. Fig. 3 shows the Dolph-Chebyshev beampattern obtained from a UCA with $N = 29$ elements, radius 2.1273λ, and a virtual array of $M = 23$ elements. The look direction was $0°$ and the mainlobe to sidelobe level was set to -30 dB. Fig. 3 also shows the resultant beampattern when the gain and phase responses of the antenna elements, and their locations are perturbed. We also introduced mutual coupling between adjacent antenna elements. All perturbations were drawn from a uniform random number generator. The limits for gain perturbation are ± 0.005 (relative to 1); for phase perturbation, $\pm 1°$; for radial perturbation, $\pm 0.005\lambda$; for angular position perturbation, $\pm 1°$; and for mutual coupling, $\pm 0.01 \pm j0.01$ (relative to 1).
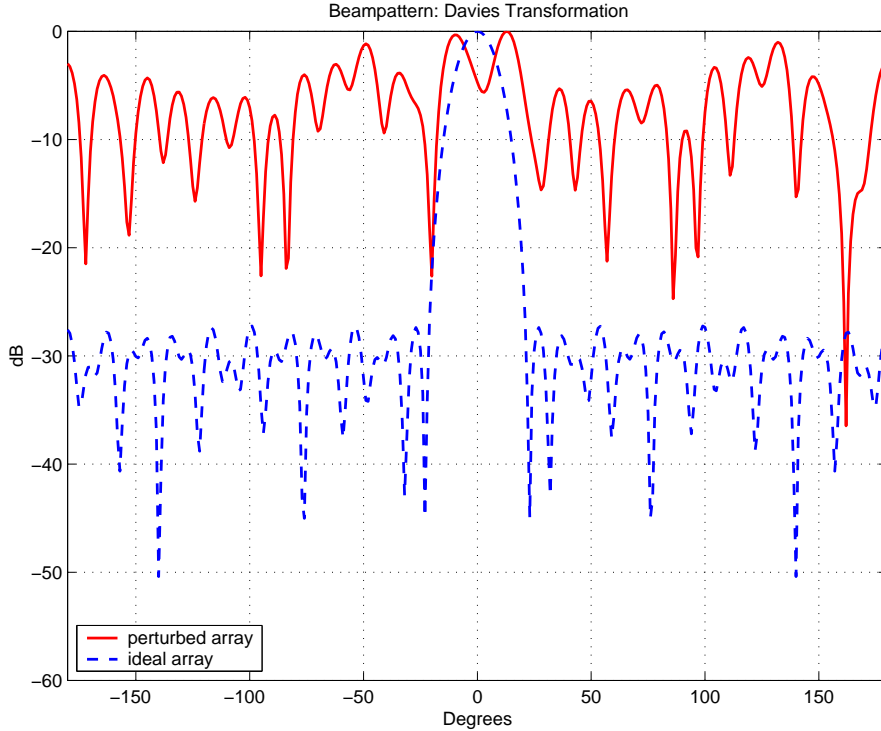

Fig. 3. Dolph-Chebyshev beampatterns of an ideal and a non-ideal UCA

Clearly, the beampattern of the perturbed (or non-ideal) array is entirely unacceptable. The beampattern of the unperturbed (or ideal) array is also not quite satisfactory since it violates the design target of -30 dB mainlobe to sidelobe ratio. This is just another indication of the poor behaviour of the transformation matrix.

At this point, it is worthwhile reiterating that, for a given M and λ, the robustness (or lack of robustness) of the $\mathbf{T}$ matrix depends only on the array radius [12]. For some radii, $\mathbf{T}$ can be quite robust. For other radii, it can be totally non-robust.

2.3 Problem Statement

The lack of robustness of the Davies transformation can be traced to the construction of $\mathbf{J}$, Eq. (3). As can be seen, for some choices of m, h, and r/λ, the magnitude of one or more of the diagonal elements of $\mathbf{J}$ can approach infinity as the corresponding value of $J_{m-1-h}(2\pi r/\lambda)$ approaches zero. Accordingly, the norm of $\mathbf{T}$ can become very large. But the square of the norm of $\mathbf{T}$ gives a measure of the noise amplification of the transformation matrix. Therefore, for a $\mathbf{T}$ with large norm, small perturbations in $\mathbf{a}(\theta)$ will translate to large perturbations in $\mathbf{b}(\theta)$.

Based on the above observation, we formulate the following semi-infinite optimisation problem to find a more robust transformation matrix. The basic idea is to trade-off the approximation error in the transformation of $\mathbf{a}(\theta)$ to a vector with Vandermonde form, for robustness.

Denote the robust transformation matrix by $\mathbf{U} \in \mathbb{C}^{M \times N}$. We find $\mathbf{U}$ as follows:

$$\min_{\mathbf{U}} \|\mathbf{U}\|_F^2 \tag{$\mathcal{P}$1}$$

subject to $\quad |\mathbf{U}\mathbf{a}(\theta) - \mathbf{b}(\theta)| \leq \mathbf{e}, \quad \forall \theta \in [-\pi, \pi]$

where $\|\cdot\|_F$ denotes Frobenius norm, $|\cdot|$ is the absolute value norm

$$|\mathbf{U}\mathbf{a}(\theta) - \mathbf{b}(\theta)| = \begin{bmatrix} \max\left\{\left|\text{Re}\left\{\mathbf{u}_1^T\mathbf{a}(\theta) - b_1(\theta)\right\}\right|, \left|\text{Im}\left\{\mathbf{u}_1^T\mathbf{a}(\theta) - b_1(\theta)\right\}\right|\right\} \\ \vdots \\ \max\left\{\left|\text{Re}\left\{\mathbf{u}_M^T\mathbf{a}(\theta) - b_M(\theta)\right\}\right|, \left|\text{Im}\left\{\mathbf{u}_M^T\mathbf{a}(\theta) - b_M(\theta)\right\}\right|\right\} \end{bmatrix}, \tag{8}$$

$$\mathbf{e} = \begin{bmatrix} \varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_M \end{bmatrix}^T, \quad \varepsilon_m \in \mathbb{R}_+, \quad m = 1, \ldots, M. \tag{9}$$

and $\mathbf{u}_m^T \in \mathbb{C}^{1 \times N}$ is the mth row of $\mathbf{U}$.

Now, since the rows of $\mathbf{U}$ are not related in the above formulation, ($\mathcal{P}$1) can be solved, row-by-row, as follows:

For $m = 1, \ldots, M$, $\qquad\qquad\qquad \min_{\mathbf{u}_m} \|\mathbf{u}_m\|_F^2 \tag{$\mathcal{P}$2}$

subject to $\quad \left|\text{Re}\left\{\mathbf{u}_m^T\mathbf{a}(\theta) - b_m(\theta)\right\}\right| \leq \varepsilon_m \quad$ and $\quad \left|\text{Im}\left\{\mathbf{u}_m^T\mathbf{a}(\theta) - b_m(\theta)\right\}\right| \leq \varepsilon_m, \quad \forall \theta \in [-\pi, \pi]$.

The advantage of ($\mathcal{P}$2) is that it allows the original problem ($\mathcal{P}$1) to be solved efficiently.

2.4 Remarks

1. The robustness of $\mathbf{U}$ depends on the choice of ε_m, $m = 1, \ldots, M$. One method is to set ε_m to be some multiple of the corresponding value in $\mathbf{T}$ where the multiple is greater than 1.

2. If, for a given m, $\varepsilon_m > 1$, then for that m, ($\mathcal{P}$2) has the trivial solution $\mathbf{u}_m = \mathbf{0}$. This follows since $|b_m(\theta)| < 1$. See Eq. (7).

3. As a guide to robustness, the square of the norm of each row of $\mathbf{U}$ should not greatly exceed N/M. The reasoning is as follows. Suppose the output signal from each antenna element contains a complex noise term whose real and imaginary parts are independent with identical variance σ_x^2. Also, suppose the noise terms of all the antenna elements are mutually independent. The total noise from the array of N elements is then given by $2N\sigma_x^2$. Suppose the transformation matrix has Frobenius norm $\|\mathbf{U}\|_F$. The total noise at the output of the virtual array is then given by $2\|\mathbf{U}\|_F^2 \sigma_x^2$. If the transformation is required to not increase noise, then we require $2N\sigma_x^2 \geq 2\|\mathbf{U}\|_F^2 \sigma_x^2$, or $\|\mathbf{U}\|_F^2 \leq N$. Finally, suppose the noise gain is distributed uniformly over the elements of the virtual array. We then get $\|\mathbf{u}_m\|_F^2 = \|\mathbf{U}\|_F^2 / M \leq N/M$.

3. QUADRATIC SEMI-INFINITE PROGRAMMING

3.1 The Dual Parameterisation Method

Consider the mth sub-problem of ($\mathcal{P}$2). Denote this sub-problem by ($\mathcal{P}_m$). Define the vector of decision variables

$$\mathbf{x} = \begin{bmatrix} \text{Re}\left\{\mathbf{u}_m^T\right\} & \text{Im}\left\{\mathbf{u}_m^T\right\} \end{bmatrix}^T \in \mathbb{R}^{2N}. \tag{10}$$

($\mathcal{P}_m$) can be written as a standard quadratic semi-infinite programming problem as follows.

$$\min_{\mathbf{x}} \tfrac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} \qquad\qquad (\mathcal{P}'_m)$$

$$\text{subject to}\qquad \mathbf{A}(\theta)\mathbf{x} - \mathbf{c}(\theta) \le \mathbf{0}, \quad \forall\theta\in[-\pi,\pi]$$

where
$$\mathbf{Q} = 2\mathbf{I}_{2N\times2N}, \qquad\qquad (11)$$

$$\mathbf{A}(\theta) = \begin{bmatrix} \mathrm{Re}\{\mathbf{a}^T(\theta)\} & -\mathrm{Im}\{\mathbf{a}^T(\theta)\} \\ -\mathrm{Re}\{\mathbf{a}^T(\theta)\} & \mathrm{Im}\{\mathbf{a}^T(\theta)\} \\ \mathrm{Im}\{\mathbf{a}^T(\theta)\} & \mathrm{Re}\{\mathbf{a}^T(\theta)\} \\ -\mathrm{Im}\{\mathbf{a}^T(\theta)\} & \mathrm{Re}\{\mathbf{a}^T(\theta)\} \end{bmatrix} \in \mathbb{R}^{4\times2N}, \qquad (12)$$

and $\quad \mathbf{c}(\theta) = \begin{bmatrix} \varepsilon_m + \mathrm{Re}\{b_m(\theta)\} & \varepsilon_m - \mathrm{Re}\{b_m(\theta)\} & \varepsilon_m + \mathrm{Im}\{b_m(\theta)\} & \varepsilon_m - \mathrm{Im}\{b_m(\theta)\} \end{bmatrix}^T \in \mathbb{R}^4.$ (13)

We use the so-called *dual parameterisation method* of [13, 14] to solve ($\mathcal{P}_m$). The parameterised dual problem of ($\mathcal{P}_m$) with k parameters is defined as follows.

$$\min_{\mathbf{x},\,\mathbf{y}_k,\,\mathbf{z}_k} \tfrac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} + \sum_{i=1}^{k}\mathbf{c}^T(\theta_i)\mathbf{y}_i \qquad\qquad (\mathcal{P}'_{m,k})$$

$$\text{subject to}\qquad \mathbf{Q}\mathbf{x} + \sum_{i=1}^{k}\mathbf{A}^T(\theta_i)\tilde{\mathbf{y}}_i = \mathbf{0},$$

$$\tilde{\mathbf{y}}_i \ge \mathbf{0},$$

$$\text{and}\qquad 0 \le \theta_i \le 2\pi, \quad i = 1,\ldots,k,$$

where
$$\tilde{\mathbf{y}}_i = \begin{bmatrix} \tilde{y}_{i,1} & \tilde{y}_{i,2} & \tilde{y}_{i,3} & \tilde{y}_{i,4} \end{bmatrix}^T, \qquad\qquad (14)$$

$$\mathbf{y}_k = \begin{bmatrix} \tilde{\mathbf{y}}_1^T & \tilde{\mathbf{y}}_2^T & \cdots & \tilde{\mathbf{y}}_k^T \end{bmatrix}^T,$$

and
$$\mathbf{z}_k = \begin{bmatrix} \theta_1 & \theta_2 & \cdots & \theta_k \end{bmatrix}^T.$$

The main results relating ($\mathcal{P}_m$) and ($\mathcal{P}'_{m,k}$) are stated in the following theorem.

Theorem 1

(a) There exists a k^* satisfying $0 \le k^* \le 2N$ such that the optimal value sequence $\left\{V\left(\mathcal{P}'_{m,k}\right)\right\}_{k=1}^{k^*}$ is strictly decreasing and for $k > k^*$, $V\left(\mathcal{P}'_{m,k}\right) = V\left(\mathcal{P}'_{m,k^*}\right)$.

(b) The number k^* in **(a)** is the smallest whole number such that for $k \ge k^*$, the global solution of ($\mathcal{P}'_{m,k}$) provides the solution of ($\mathcal{P}_m$) in the sense that, if $(\mathbf{x}^*,\mathbf{y}^*,\mathbf{z}^*)$ is a solution of ($\mathcal{P}'_{m,k}$), then $\mathbf{x}^*$ is the solution of ($\mathcal{P}_m$).

Proof See [13].

3.2 The Algortihm

Based on Theorem 1, the following adaptive algorithm is developed in [14]. Define first the following problem.

$$\min_{\mathbf{x}, \mathbf{y}_k} \tfrac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} + \sum_{i=1}^{k} \mathbf{c}^T(\theta_i)\tilde{\mathbf{y}}_i \qquad\qquad (\mathcal{P}'_{m,\mathbf{z}_k})$$

subject to $\qquad \mathbf{Q}\mathbf{x} + \sum_{i=1}^{k} \mathbf{A}^T(\theta_i)\tilde{\mathbf{y}}_i = \mathbf{0} \quad$ and $\quad \tilde{\mathbf{y}}_i \geq \mathbf{0}, \quad i = 1, \ldots, k$

where $\mathbf{z}_k = \begin{bmatrix} \theta_1 & \theta_2 & \cdots & \theta_k \end{bmatrix}^T$ is a fixed vector.

Step 1 Choose any $\mathbf{x}_0 \in \mathbb{R}^{2N}$, a small number $\varepsilon > 0$, an integer I, an increasing sequence of integers $\{k_i\}$, and a sequence of parameterisation sets $\Theta_i = \left\{\theta_{i,j} \middle| \theta_{i,j} \in [0, 2\pi], j = 1, \ldots, k_i\right\}$ such that

$$d(\Theta_i, [0, 2\pi]) \triangleq \max_{\theta \in [0, 2\pi]} \min_{1 \leq j \leq k_i} |\theta - \theta_{i,j}| \to 0 \quad \text{as} \quad i \to \infty \qquad (15)$$

Step 2 Let $E_0 = \varnothing$. Set $i = 0$.

Step 3 Set $\qquad\qquad\qquad\qquad\qquad i = i + 1.$

Find $\qquad\qquad\qquad \mathcal{G}_i = \left\{\theta \in \Theta_i \middle| \mathbf{A}(\theta)\mathbf{x}_{i-1} - \mathbf{c}(\theta) \geq \mathbf{0}\right\} \cup E_{i-1}.$

Suppose $\mathcal{G}_i$ has m_i elements. $\qquad \mathcal{G}_i = \left\{\tilde{\theta}_{i,1}, \tilde{\theta}_{i,2}, \ldots, \tilde{\theta}_{i,m_i}\right\}.$

Define $\qquad\qquad\qquad \mathbf{z}_{m_i} = \begin{bmatrix} \tilde{\theta}_{i,1} & \tilde{\theta}_{i,2} & \cdots & \tilde{\theta}_{i,m_i} \end{bmatrix}^T.$

Step 4 Solve problem ($\mathcal{P}'_{m,\mathbf{z}_{m_i}}$) to obtain an optimal solution $(\mathbf{x}_i, \mathbf{y}_i)$.

Step 5 If $\qquad\qquad\qquad i \leq I \quad$ or $\quad \tfrac{1}{2}\mathbf{x}_i^T\mathbf{Q}\mathbf{x}_i - \tfrac{1}{2}\mathbf{x}_{i-1}^T\mathbf{Q}\mathbf{x}_{i-1} \geq \varepsilon,$

find $\qquad\qquad\qquad E_i = \left\{\tilde{\theta} \in \mathcal{G}_i \middle| \mathbf{A}(\tilde{\theta})\mathbf{x}_i - \mathbf{c}(\tilde{\theta}) = \mathbf{0}\right\}.$

Go to **Step 3**.

Step 6 Solve problem ($\mathcal{P}'_{m,m_i}$) starting from $(\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_{m_i})$. Denote the solution by $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{z}^*)$. Take $\mathbf{x}^*$ to be the solution of problem ($\mathcal{P}_m$).

Theorem 2

Let the condition (15) be satisfied. Then, the sequence $\{\mathbf{x}_i\}$ obtained from the Algorithm will converge to the solution of problem ($\mathcal{P}_m$). Therefore, if ε and I are suitably chosen, the $\mathbf{x}^*$ obtained in Step 6 is the optimal solution of ($\mathcal{P}_m$).

Proof See [14].

4. NUMERICAL EXAMPLE

Consider the UCA of Fig. 3. Table 1 summarises the squared-norm and maximum real and imaginary errors of each row of the Davies matrix for this UCA. As can be seen, the squared-norms of rows 3 and 21, 6 and 18, and 11 and 13 greatly exceed the maximum desirable squared-norm of $N/M = 29/23 = 1.2609$. Indeed, it is the very presence of these rows that render the Davies matrix non-robust.

For the robust transformation matrix, our strategy is to retain as many rows of the Davies matrix as possible except for rows with large squared-norms. Accordingly, we replace rows 3 and 21, 6 and 18, and 11 and 13 with rows found by solving ($\mathcal{P}2$) with ε_m set to 1, 0.5 and 0.7 respectively. (Note, rows 3 and 21 can be solved trivially as remarked in Section 2.4.) The Dolph-Chebyshev beampattern obtained from the robust transformation matrix is shown in Fig. 4. The perturbation on the ideal array is the same as that in Fig. 3. The characteristics of the robust transformation matrix are summarised also in Table 1. Note the increase in approximation error in rows 3 and 21, 6 and 18, and 11 and 13 of the robust transformation matrix.

Row #	Davies Matrix			Robust Matrix		
	Squared-norm	Max Real Error $(\times 10^{-6})$	Max Imag Error $(\times 10^{-6})$	Squared-norm	Max Real Error $(\times 10^{-6})$	Max Imag Error $(\times 10^{-6})$
1, 23	0.43	31,682	31,682			
2, 22	0.99	19,395	19,395			
3, 21	6,935.23	616,799	61,6799	0.00	1,000,000	1,000,000
4, 20	0.96	2,581	2,581			
5, 19	0.68	732	732			
6, 18	16.38	1,145	1,145	5.86	463,493	464,067
7, 17	1.02	86	86			
8, 16	1.02	25	25			
9, 15	6.28	17	17			
10, 14	0.73	2	2			
11, 13	403.12	10	10	6.96	724,847	728,719
12	0.72	0	0			

Table 1. Characteristics of the Davies and the robust transformation matrices for
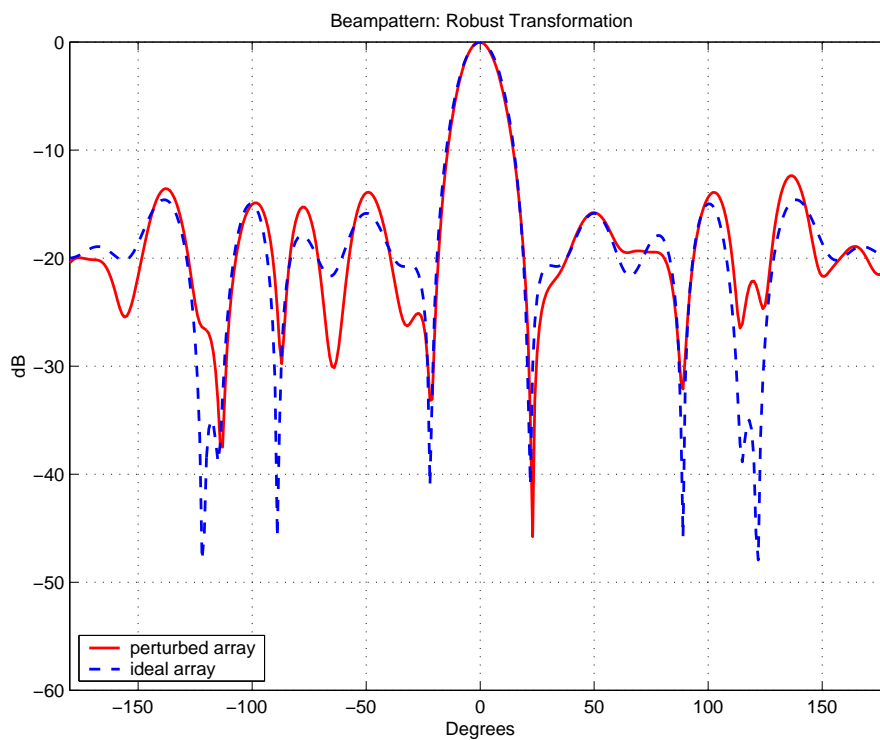$N = 29$, $M = 23$ and $r = 2.1273\lambda$

Fig. 4. Dolph-Chebyshev beampatterns of an ideal and a non-ideal UCA with robust transformation matrix

5. CONCLUSIONS

In this paper, we addressed the important problem of finding a transformation matrix to transform the steering vector of a uniform circular array to one with Vandermonde form, subject to a robustness requirement as demanded by practical considerations. The robust transformation matrix is found by posing and solving a semi-infinite optimisation problem. We showed that, by an appropriate formulation of the optimisation problem, we are able to decompose the problem into a set of much simpler optimisation problems which can then be solved efficiently using the dual parameterization method of [13, 14]. Each sub-problem yields a row of the robust transformation matrix. The robustness of the new transformation matrix is demonstrated by a simulation example. The simulation example also shows that this robustness is gained only through a sacrifice in the accuracy of the resulting steering vector of the virtual array from Vandermonde form.

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