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ROBUSTNESS OF A DESIGN METHOD BASED ON ASSIGNMENT OF POLES AND ZEROS

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1. INTRODUCTION

It is an empirical fact that complex processes can often be controlled well by surprisingly simple regulators. When the regulators are designed using analytic design methods the complexity of the regulator is uniquely given by the complexity of the model. To obtain a simple regulator it is therefore important to base the design on a simplified model. It is then important to investigate the sensitivity of the closed loop system to variations in the model used for the design. Such a problem is formulated and solved in this paper. The notations used are given in Section 2. The design method used, based on pole-zero assigment and polynomial manipulations, is described in Section 3. The main results are given in Section 4.

2. PRELIMINARIES

It is assumed that the systems considered are linear timeinvariant and that they have one input and one output. The input-output characteristics of such systems can be described by analytic transfer functions. Both continuous time and discrete time systems are considered.

The stability region is a subset of the complex plane. For continuous time systems it is the left half plane excluding the imaginary axis. For discrete time systems the stability region is the interior of the unit circle. A system is stable if all the poles of its transfer function are inside the stability region. The instability region is the complement of the stability region. From a practical point of view it is useful to introduce a restricted stability region Z which is strictly inside the stability region. For continuous time system the region Z could e.g. be characterized by

$$3\pi/4 < \arg s < 5\pi/4$$

$$\mathrm{Re}\ s < -s_0 < 0$$

The image of this region under the transformation $z = \exp(sT)$ could be the restricted stability region for discrete time systems.

The critical curve Γ is the imaginary axis for continuous time systems and the unit circle for discrete time systems.

3. POLE AND ZERO PLACEMENT

The problem of designing a servo with a given closed loop transfer function will now be described and solved.

Formulation

Consider a process characterized by the rational transfer function

$$G = \frac{B}{A} \tag{3.1}$$

where A and B are polynomials. It is assumed that A and B are coprime and that

$$\deg B < \deg A.$$

It is desired to find a controller such that the closed loop is stable and that the transfer function from the command input u_c to the output is given by

$$G_M = \frac{Q}{P} \tag{3.2}$$

where P and Q are coprime and

$$\deg P - \deg Q \geqslant \deg A - \deg B. \tag{3.3}$$

Design procedure

A general linear regulator can be described by

$$Ru = Tu_c - Sy. \tag{3.4}$$

The regulator can be thought of as a combination of a feedback having the transfer function

$$G_{FB} = \frac{S}{R} \tag{3.5}$$

and a feedforward with the transfer function

$$G_{FF} = \frac{T}{R} \tag{3.6}$$

The closed loop transfer function relating y to u_c is given by

$$\frac{TB}{AR + BS}$$

Since this transfer function should equal the desired closed loop transfer function G_M given by (3.2) we get

$$\frac{TB}{AR + BS} = \frac{Q}{P} . \tag{3.7}$$

The design problem is thus equivalent to the algebraic problem of finding polynomials R, S and T such that (3.7) holds. It follows from (3.7) that factors of B which are not also factors of Q must be factors of R. Since factors of B correspond to open loop zeros it means that open loop zeros which are not desired closed loop zeros must be canceled. Factor B as

$$B = B^+B^- , \tag{3.8}$$

where all the zeros of B^+ are in the restricted stability region Z and all zeros of B^- outside Z. This means that all zeros of B^+ correspond to well damped modes and all zeros of B^- correspond to unstable or poorly damped modes.

A necessary condition for solvability of the servo problem is thus that the specifications are such that

$$Q = Q_1B^- . \tag{3.9}$$

Since deg P is normally less than deg(AR + BS) it is clear that there are factors in (3.7) which cancel. In state space theory it can be shown that the regulator (3.4) corresponds to a combination of an observer and a state feedback. See [1]. It is natural to assume that the observer is designed in such a way that changes in command signals do not generate errors in the observer. This means that the factor which cancels in the right hand side of (3.7) can be interpreted as the observer polynomial T_1.

The design procedure can be formulated as follows.

Data: Given the desired response specified by the polynomials P and Q, subject to deg P = deg A and the conditions (3.3), (3.9), and and the desired observer polynomial T_1. It is assumed that P and T_1 have all their zeros in Z.

Step 1: Solve the equation

$$AR_1 + B^- S = PT_1 \tag{3.10}$$

with respect to R_1 and S.

Step 2: The regulator which gives the desired closed loop response is the given by (3.4) with

$$R = R_1 B^+ \tag{3.11}$$

and

$$T = T_1 Q_1. \tag{3.12}$$

The equation (3.10) can always be solved because it was assumed that A and B were coprime. This implies of course that A and B^- are also coprime. Equation (3.10) has infinitely many solutions. The unique solution determined by

$$\deg S < \deg A \tag{3.13}$$

is chosen. If deg P = deg A it then follows that $\deg R_1 = \deg T_1$. If the observer polynomial is chosen in such a way that

$$\deg T_1 = \deg A - \deg B^+ - 1 \tag{3.14}$$

then

$$\deg R = \deg A - 1 \geq \deg S$$

$$\deg T = \deg A - 1 \leq \deg R. \tag{3.15}$$

This means in the continuous time case that the regulator does not include any pure derivatives and in the discrete time case that the regulator is causal. Notice that in special cases the regulator may still have the property (3.15) even if (3.14) does not hold. Also notice that the choice (3.14) corresponds to a Luenberger observer in the state space interpretation. Further discussions including examples are found in [2].

Analysis

A direct calculation gives

$$\frac{TB}{AR + BS} = \frac{T_1 Q_1 B^+ B^-}{B^+(AR_1 + B^- S)} = \frac{T_1 Q_1 B^-}{PT_1} = \frac{Q}{P}$$

which shows that the regulator gives the desired closed loop response. Notice that in this calculation we have divided with the factors B^+ and T_1. This is permitted since it was assumed that all their zeros are well damped.

A direct calculation shows that the closed loop system has the characteristic polynomial B^+T_1P. The polynomial B^+ has all its zeros in the restricted stability region Z by definition. Since the observer polynomial T_1 and the polynomial P were chosen to have all their zeros in Z it follows that the closed loop system has all its poles in Z.

Interpretation as model following

The regulator (3.4) can be interpreted as a model following. It follows from (3.10), (3.11) and (3.12) that

$$\frac{T}{R} = \frac{T_1Q_1}{B^+R_1} = \frac{(AR_1 + B^-S)Q_1}{PB^+R_1} = \frac{AQ_1}{B^+P} + \frac{SB^-Q_1}{B^+R_1P} = \frac{A}{B} \cdot \frac{Q}{P} + \frac{S}{R} \cdot \frac{Q}{P} .$$

The feedback law (3.4) can thus be written as

$$u = \frac{A}{B} y_c + \frac{S}{R} (y_c - y) \tag{3.16}$$

where

$$y_c = \frac{Q}{P} u_c .$$

The signal y_c can be interpreted as the output obtained when the command signal u_c is applied to the model Q/P. When the regulator (3.4) is rewritten as (3.16) it is clear that it can be thougt of as composed of two parts, one feedforward term $(A/B)y_c = (A/B)(Q/P)u_c$ and one feedback term $(S/R)(y_c - y)$. The feedforward is a combination of the ideal model and an inverse of the process model. The feedback term is obtained by feeding the error $y_c - y$ through a dynamical system characterized by the operator S/R. It is thus clear that the regulator can be interpreted as a model following servo. Notice, however, that the system A/B is not realizable although the combination AQ/(BP) is.

4. MAIN RESULTS

It will now be investigated what happens if the design procedure described in Section 2 is applied to a simplified model

$$G = \frac{B}{A}, \tag{4.1}$$

of a process with the transfer function G_0.

The stability of the closed loop system will first be discussed. The sensitivity of the closed loop poles are then considered.

Stability

A sufficient condition for stability is given by

THEOREM 1

Consider the regulator (3.4) obtained by applying the pole-zero assignment design to the stable model $G = B/A$ with the specification that the closed loop transfer function should be $G_M = Q/P$. Let the regulator control a stable system with the transfer function G_0. The closed loop system is then stable if

$$|G - G_0| < \left| \frac{BPT}{AQS} \right| = \left| \frac{G}{G_M} \right| \cdot \left| \frac{G_{FF}}{G_{FB}} \right| \tag{4.2}$$

on the critical curve Γ and at $z = \infty$.

Proof

Consider the function

$$F = R + G_0 S \tag{4.3}$$

This function is regular outside the stability region because the system G_0 was assumed stable. The zeros of the function F are equal to the closed loop poles. Solving (3.10) for R and insertion into (4.3) gives

$$F = PB^{+}T_1/A - BS/A + G_0S = PB^{+}T_1/A + S(G_0 - G)$$

When $G = G_0$ the zeros of F are thus equal to the zeros of the polynomials B_1, P, and T_1. Since both the system and the model were assumed to be stable the functions $PB^{+}T_1/A$ and $S(G_0 - G)$ are both regular outside the stability region. The functions are thus regular on a contour which encloses the instability region. Notice that

$$\frac{BPT}{AQS} = \frac{B^{+}B^{-}PT_1Q_1}{AQ_1B^{-}S} = \frac{B^{+}PT_1}{AS}.$$

Condition (4.2) thus implies that

$$|S(G_0 - G)| < |PB^{+}T_1/A|$$

on the critical curve and at infinity.

It now follows from Rouchē's theorem [3, p. 119] that the functions F and $PB^{+}T_1/A$ have the same zeros in the instability region. It follows from the design procedure that the polynomial $PB^{+}T_1$ has all its zeros in the restricted stability region. The closed loop system is thus stable. The equality in (4.2) follows from

$$\frac{BPT}{AQS} = \frac{BPTR}{AQRS} = \frac{G\,G_{FF}}{G_M G_{FB}}$$

where (3.5) and (3.6) have been used. □

Theorem 1 gives good insight into the sensitivity of the design to modeling errors. When a model has been obtained and a regulator has been designed, the right hand side of (4.2) can be determined.

It is then easy to establish bounds on the transfer function G which will result in a stable closed loop system. Notice that the bound is proportional to $|G_{FF}/F_{FB}|$. From the point of view of stability the requirements on model precision will thus decrease if the ratio of feedforward to feedback is increased. For single-degree-of-freedom systems $G_{FF} = G_{FB}$ and the bounds are simplified further. Also notice that the bound is proportional to $|G/G_M|$. For a normal servo the desired transfer function G_M is unity for low frequencies. It then remains constant up to frequencies corresponding to the specified bandwidth where it starts to decrease. From the point of view of stability it is thus advantageous to have a high process gain. The ratio $|G/G_M|$ is normally large for low frequencies because the low frequency gain of the process is typically larger than the desired low frequency gain. Reasonable specifications are also often such that $|G/G_M|$ is constant for high frequencies. Since G_0 and G normally are small for high frequencies, this means that the inequality (4.2) can be satisfied even if G and G_0 deviates substantially at high frequencies. Normally it is only in a fairly narrow frequency range around the bandwidth where (4.2) gives critical requirements on the model accuracy.

This agrees well with empirical facts and explains qualitatively why simple models can be useful for pole-placement design. Notice also that it follows from (4.2) that the requirements on model precision will be reduced by reducing the bandwidth of desired closed loop system.

Sensitivity of Closed Loop Poles to Model Errors

So far the discussion has been focussed on the stability problem. Having established that a model is sufficiently accurate to guarantee stability it is of course of interest to analyse the problem further and to investigate the requirements on model precision which are necessary to have the dominating poles close to their specified values.

In the proof of Theorem 1 it was shown that the closed loop poles are the zeros of the function F defined by (4.3), i.e.

$$F = PB_1T_1/A + S(G_0 - G) = H + S(G_0 - G).$$

When $G = G_0$ the system has thus poles at the zeros of P, B_1, and T_1. Consider F as a function of z and G. A Taylor series expansion at $z = p_i$ and $G = G_0$ gives

$$F(z) \approx H(p_i) + H'(p_i)(z - p_i) - S(p_i)[G(p_i) - G_0(p_i)].$$

An approximative formula for the change of the pole p_i due to a modeling error is thus

$$z_i = p_i - [H'(p_i)]^{-1}S(p_i)[G_0(p_i) - G(p_i)].$$

If it is required that a pole p_i change by at most $\alpha|p_i|$ due to a modeling error the following inequality is obtained:

$$|G_0(p_i) - G(p_i)| \leq \alpha|H'(p_i)| \; |S^{-1}(p_i)| \cdot |p_i|. \qquad (4.4)$$

A requirement that certain dominant poles do not change too much will thus lead to a requirement that the values of the model pulse transfer function is close to the process pulse transfer function at the poles of interest. Such a requirement can of course also be satisfied by a fairly simple model provided that the number of dominant poles is not too large.

5. ACKNOWLEDGEMENT

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