Vertical Flame Spread, Some Notes on Dimensional Analysis and Differential Equations

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Summary

Vertical flame spread is first discussed in terms of a dimensional analysis. This follows Delichatsios´s definition of a characteristic length based on the heat transfer from the flame but another length ℓ_B, characteristic of the burner and the initial condition is also introduced.

A differential equation of second order is deduced for a linear law between flame length and heat release rate. To evaluate the constants in the equations one requires measurements at a minimum of five heights.

1. Introduction

There are essentially two versions of the theory of flame spread up a thick flammable solid. Either the flame and gas phase is discussed in detail beyond its representation by an imposition of a heat source onto the solid or, instead of attempting to calculate the properties of the gas phase, the heat transfer from it is expressed as an independent variable to be measured or assumed. There are two versions of the latter thermal theory based on the two types of correlation between flame length Z_{fl} and heat release rate per unit width of flame front, $Q^{'}$.

$$Z_{fl} = K_o Q^{/\,2/3} \qquad (1)$$

or

$$Z_{fl} = K_1 Q^{/} \qquad (2)$$

where K_o and K_1 are constants, i.e. a power law form or a linear form.

2 Dimensional Analysis

In the linearised theory described by Quintiere [1], Saito, Quintiere and Williams [2], Thomas and Karlsson [3], Thomas [4] and Baraudi and Kokkala [5], equations are given involving various parameters -

1) Z_{po} - the initial length of the pyrolysed zone
2) the burner strength $Q_B^{/}$ - KW per metre width of burner - which is constant ($t>t_{ig}$)
3) K a constant (not a dimensionless one) as in equation (1) or (2)
4) t_{ig} - the ignition time derived from the heat transfer to the fuel
5) t_B - the burn out time. This could be based on another parameter, e.g. $1/\gamma$ here $e^{-\gamma t}$ gives the decrease with time of the pyrolysis rate
6) the heat release rate per unit area from the fuel produced by pyrolysis.

This last quantity is characterised by an initial value $Q_o^{''}$ which is proportional to the mass rate per unit area of pyrolysis $M_o^{''}$ i.e. $Q_o^{''} = qM_o^{''}$ which for a noncharring material may be written as $Q^{''} = q^{''}{}_{net}\ \Delta H_c / \Delta H_v$ where q is the heat release per unit mass (as in references 1-5), i.e. $q = \Delta H_c$.

ΔH_c is an effective calorific value, $M_o^{''}$ is a characteristic pyrolysis rate e.g. $q_{net}^{''}/\Delta H_v$. $q_{net}^{''}$ is an effectively mean net heat transfer rate from the flame and ΔH_v is a characteristic heat of pyrolysis.

Dimensional analysis permits us to write

$$\frac{Z_{fl}}{\ell} = N\, function\left(\frac{qM_o^{''}}{\rho_o C_p T_o \sqrt{g\ell}} \cdot \frac{Z_p}{\ell}\right) \tag{3}$$

where N is a dimensionless constant and ℓ is a dimension characteristic of the fire, ρ_o is density C_p specific heat and T_o is the absolute ambient temperature.

Following Delichatsios and Saito [5], we write

$$\ell = \left(\frac{q''_{net} \dfrac{\Delta H_c}{\Delta H_v}}{\rho_o C_p T_o \sqrt{g}} \right)^2 \tag{4}$$

Hence, assuming a power law formation, we have

$$\frac{Z_{fl}}{\ell} = N \left(\frac{Z_p}{\ell} \right)^{\beta} \tag{5}$$

and $\beta = 1$ or $2/3$ for linearised and natural convection controlled spread respectively.

The development of theory using the 2/3 power instead of the linear law (equation (1) instead of equation (2)) defines a dimensionless parameter (see Appendix).

$$E = Q'_B \frac{(1 + t_{ig} / t_B)^3}{(K_o q M''_o)^3} \tag{6}$$

which with t/t_{ig} , t_{ig}/t_B , and $K_o \left(q M''_o \right)^{2/3}$ defines the behaviour of flame up a thick solid with the 2/3 power flame length law.

If we write for the two notations

$$Z_{fl} = N_1 \left(\frac{Z_p}{\ell} \right)^{2/3} \cdot \ell \tag{7i}$$

$$= K \cdot (q M''_o)^{2/3} Z_p^{2/3} \tag{7ii}$$

where N_1 is a dimensionless constant and $Z_{po} = K_o Q_B'^{2/3}$ at $t = 0$
we have

$$E = \frac{\left(1 + t_{ig}/t_B\right)^3}{N_1^{9/2}} \left(\frac{Z_{po}}{\ell} \right)^{2/3} \tag{8}$$

In summary, whatever power law applies to flame length we have, following Delichatsios and Saito [6]

$$\frac{Z_p}{\ell} = \text{function}\left(\frac{Z_{po}}{\ell}, \frac{t_{ig}}{t_B}, \frac{t}{t_{ig}} \right) \tag{9}$$

If the width of the spreading zone were finite and of width *D* then *D* is a dimension characteristic of the source and ℓ/D is an additional ratio on the right hand side.

If there is preheating ahead of the flame and it is represented by an additional distance it can be expressed to a first approximation as having two components - one a constant and another proportional to the appropriate scale length. This can be either Z_p or ℓ. Since their ratio is already included in the set of independant ratios in the functional equation we can write the heating as taking place over the distance $\beta + \alpha\left(Z_{fl} - Z_p\right)$ instead of over $Z_{fl} - Z_p$. Thus Z_p/ℓ and Z_{fl}/ℓ are functions of Z_{po}/ℓ, $\mathrm{t_{ig}}/t_B$, $\mathrm{t}/\mathrm{t_{ig}}$, α and β/ℓ.

The form of this functional relation is, in principle, independent of geometry, e.g. a corner, unless this is characterised by a relevant dimension *"D"*. However, in the Appendix, another characteristic length "ℓ_B" is introduced; it is characteristic of the burner and the initial condition.

3 Testing the Theory

Consider

$$Z_{fl}/\ell \text{ as a function of}\left(Z_{po}/\ell,\; t_{ig}/t_B,\; t/t_{ig}\right) \tag{10}$$

Neglecting preheating ahead of the flame, and assuming that we can measure Z_{fl} as a function of *t*, we write, for any given non-charring material,

$$\ell \propto q''^{\,2}_{net} \tag{11}$$

$$t_{ig} \propto \left(T_p - T_s\right)^2 / q''^{\,2}_{net} \tag{12}$$

whilst Z_{po} depends on the burner strength. If we also initially neglect "burn out" for short distances and times of spread the relationship to be tested is therefore

$$Z_{fl}/\ell = function\left(Z_{po}/\ell,\; t/t_{ig}\right) \tag{13}$$

One can experimentally change t_{ig} by preheating the specimen: one cannot easily experimentally vary $q''^{\,2}_{net}$ for a given material over the length of the flame. However, one can measure $q''^{\,2}_{net}$ and calculate ℓ though not precisely in view of the many possible errors in $q''^{\,2}_{net}$, ΔH_v, etc.

Preheating the specimen can, of course, change ΔH_v and ΔH_c! It seems difficult, if not impractical, to test this general result.

4 A Differential Equation for Flame Spread

The two equations we have used for assessing *"V"* the speed of the spread are:

$$V = dZ_p / dt = \left(Z_{fl} - Z_p\right) / t_{ig} \tag{14}$$

and

$$Z_{fl} = KQ_B^{'} + Kq\int_o^t M_o^{''}\left(t - t_p\right)(dZ_p / dt_p)dt_p + KqM_o^{''}Z_{po} \tag{15}$$

These have been solved for $M^{''} = M^{''}e^{-\gamma t}$ and $M^{''}$ = *constant for t < t_B*.

We solve for Z_{fl} instead of Z_p, believing it may be easier to measure Z_{fl}. We consider first $M^{''} = M_o^{''}e^{-\gamma t}$.

For simplicity we consider the most practical case of a constant value for $Q_B^{'}$. Following Thomas & Karlsson [3] we take the Laplace Transforms of each equation and with the notation

$$\bar{y} = \int_o^t y \cdot \bar{e}^{\,pt} \cdot dt$$

we have

$$\left(1 + pt_{ig}\right)\bar{Z}_p = \bar{Z}_{fl} + Z_{po}t_{ig} \tag{16}$$

and

$$\bar{Z}_{fl} = \frac{KqM_o^{''}p\bar{Z}_p}{p + \gamma} + \frac{KQ}{p} \tag{17}$$

where $Q_B^{'}$ is simplified to Q.

Hence

$$\bar{Z}_p = \frac{\left(\frac{KQ}{pt_{ig}} + Z_{po}\right)(p+\gamma)}{p^2 + \left(1/t_{ig} + \gamma - Kq\,M_o^{''}/t_{ig}\right)p + \gamma/t_{ig}} \tag{18}$$

so that

$$\bar{Z}_{fl} = \frac{KQ}{p} + \frac{KqM_o^{''}p\left(\frac{KQ}{pt_{ig}} + Z_{po}\right)}{p^2 + \left(1/t_{ig} + \gamma - KqM_o^{''}/t_{ig}\right)p + \gamma/t_{ig}} \tag{19}$$

Writing the quadratic denominator of equations (19) & (20) as

$$p^2 + Hp + G$$

where

$$H = \frac{1}{t_{ig}} + \gamma - \frac{KqM_o^{''}}{t_{ig}}$$

$$G = \gamma/t_{ig}$$

and $\alpha,\ \beta = \dfrac{-H \pm \sqrt{H^2 - 4G}}{2}$

then $\dfrac{p}{p^2 + Hp + G} = \dfrac{1}{\sqrt{H^2 - 4G}}\left(\dfrac{\alpha}{p-\alpha} - \dfrac{\beta}{p-\beta}\right)$

Hence equation (19) for constant Q leads to

$$Z_{fl} = KQ + \left(ae^{\alpha t} - be^{\beta t}\right)$$

where a and b are constant.

α and β may be both positive, both negative or both complex, i.e. Z_{fl} expands asymptotically exponentially or reaches a limit as does Z_p. However there are two terms $e^{\alpha t}$ and $e^{\beta t}$ so the test for compliance with exponentiality is not strictly to check for exponentiality at all Z_{fl}.

One can obtain the differential equation (of which the above is the solution)

viz. $$\frac{d^2 Z_{fl}}{dt^2} + H\frac{dZ_{fl}}{dt} + GZ_{fl} = \frac{\gamma}{t_{ig}} KQ = \frac{\gamma}{t_{ig}} KQ_B^{'} \qquad (20)$$

if Q is constant.

To test this equation one needs to evaluate $\frac{d^2 Z_{fl}}{dt^2}$ which requires at least 3 measurements - one on each side of a particular measuring point for Z_{fl}. These points also define $\frac{dZ_{fl}}{dt}$ and Z_{fl}. The need to define H and G and $KQ_B^{'}$ and their error requires at least 4 points. With one point below and one above one requires <u>at least</u> 6 measurement points.

The three points P_1, P_2 and P_3 in Fig. 1 define Z_{fl} at each point, $\frac{dZ_{fl}}{dt}$ at two points (preferably intermediate) and $\frac{d^2 Z_{fl}}{dt^2}$ at one point. Hence we can get $\frac{d^2 Z_{fl}}{dt^2}$, $\frac{dZ_{fl}}{dt}$ ~ Z_{fl} at one intermediate position, say P_2 . Likewise we get another set of values from P_2, P_3 and P_4 and a third set from P_3, P_4 and P_5. These three sets of values provide 3 equations for H, G and $\frac{\gamma}{t_{ig}} KQ$. To estimate error, departures from constancy of the coefficient etc., at least one more set of values is required, i.e. a minimum of 6 in all.

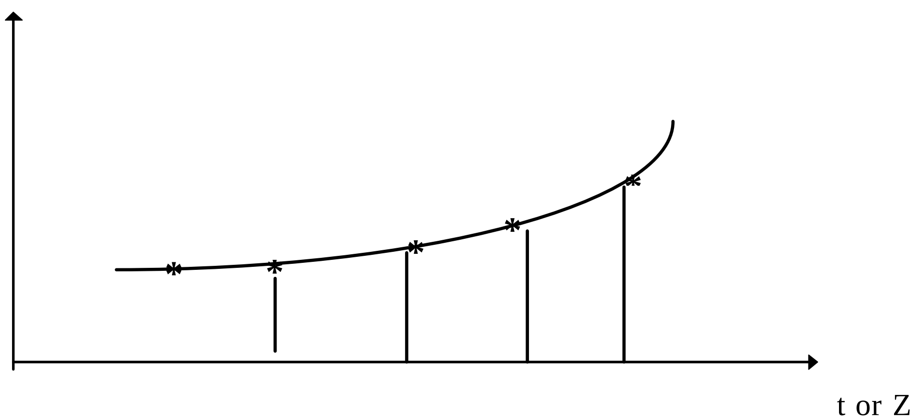

Figure 1

The Quintiere model with

$$
\begin{aligned}
M'' &= M''_o \quad t \le t_B \\
&= 0 \quad\; t \ge t_B
\end{aligned}
$$

requires us to define the locus of the rear Z_R once t exceeds t_B.

Then

$$dZ_R/dt = \left(Z_p - Z_R\right)/t_B \tag{21}$$

Also

$$Z_{fl} - Z_R = KQ + KqM''_o\left(Z_p - Z_R\right) \tag{22}$$

To test the model in full requires testing the three equations (14), (21) and (22).

In principle it should be easier to observe Z_{fl} and Z_R so we eliminate Z_p and from equations (21) and (22) obtain

$$Z_{fl} - Z_R = KQ + KqM''_o t_B\, dZ_R/dt \tag{23}$$

Data on $Z_{fl} - Z_R$ need to be plotted against dZ_R/dt. We need a minimum of two points to evaluate the constants KQ and $KqM'' t_B$ and two are needed to obtain dZ_R/dt. Five points are required to estimate the error.

Alternatively one can obtain from equations (14), (21) and (22)

$$\frac{d^2 Z_{fl}}{dt^2} + \left(\frac{1}{t_{ig}} + \frac{1}{t_B} - \frac{KqM''_o}{t_{ig}}\right)\frac{dZ_{fl}}{dt} = \frac{KQ}{t_B t_{ig}} \tag{24}$$

i.e.

$$\frac{dZ_{fl}}{dt} + \left(\frac{1}{t_{ig}} + \frac{1}{t_B} - \frac{KqM''_o}{t_{ig}}\right)Z_{fl} = \frac{KQt}{t_B t_{ig}} + A \tag{25i}$$

where A is a constant determined by the initial values of $\frac{dZ_{fl}}{dt}$ and Z_{fl} which can be shown to be

$$A = KQ\left(\frac{1}{t_{ig}} + \frac{1}{t_B}\right) + \frac{KqM''_o}{t_B} Z_{po} \tag{25ii}$$

These two equations (24) and (25) also need five points.

Equation (24) is like equation (20) except for the term GZ_{fl} and with t_B instead of $1/\gamma$. The use of the minimum set of data measurement points - 5 points for equation (20), 4 for equation (24) - will define values of H, G and KQ irrespective of whether the models are valid. At least one - preferably several more data points are required for adequate tests to be made of the departures from constancy in H, G and KQ`. We can here suppose these tests have been made of the results acceptable so that from equation (20) we have three equations for $H, G, \frac{\gamma}{t_{ig}} KQ$.

Even so to obtain K, $q\dot{m}''$ and γ we must make use of the known value of Q'_B and an independent measure of t_{ig}. The Quintiere model summarised by equations (14), (21) and (22) is an approximation of what would occur in practice. Equations (14) and (25) are quasi-steady and equation (21) implies that at time zero the rear of the flame moves upward with a speed Z_{po}/t_B. But this is not so for $t < t_B$. For this initial stage the behaviour is obtained from equations (14) and (22) with $Z_R=0$, i.e.

$$\frac{dZ_{fl}}{dt} + \frac{Z_{fl}}{t_{ig}}\left(1 - Kq\dot{m}''\right) = \frac{KQ}{t_{ig}} \tag{26}$$

which agrees with equation (25) for $t_B \to \infty$. This and equations (24) and (25i) show the well known role of $Kq\dot{m}''$ in relation to t_{ig} and t_B in determining indefinite accelerating spread.

5 Conclusion

The flame spread models introduced into the discussion of vertical flame spread require measurements at 5 heights or times. 7 are barely more than the minimum to test for non-constant coefficients in equation (21) or (24).

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Appendix

The equations that are needed to describe upward flame spread are, for ignition

$$\frac{dZ_p}{dt} = \frac{Z_{fl} - Z_p}{\tau_{ig}} \tag{1A}$$

and for burn out

$$\frac{dZ_p}{dt} = \frac{Z_p - Z_R}{\tau_B} \tag{2A}$$

We use these with equation (1) (with Z_{fl} replaced by Z_{fl} - Z_R) and

$$Q^{'}_{.} = Q^{'}_{B} + qM^{''}_{o}\left(Z_p - Z_R\right) \tag{3A}$$

Subtracting equation (2A) from equation (1A) gives

$$\tau_{ig}\frac{d}{dt}\left(Z_p - Z_R\right) = Z_{fl} - Z_p - \frac{\tau_{ig}}{\tau_\beta}\left(Z_p - Z_R\right) \tag{4A}$$

which from equations (1) and (3A) gives

$$\tau_{ig}\frac{d}{dt}Y_p + \left(1 + \frac{\tau_{ig}}{\tau_B}\right)Y_p = K\left(Q^{'}_{B} + qM^{''}_{o}Y_p\right)^{2/3} \tag{5A}$$

where

$$Y_p = Z_p - Z_R$$

i.e.

$$\frac{dy}{d\tau} = \left(E + y\right)^{2/3} - y \tag{6A}$$

where

$$\tau = t\left(\frac{1}{\tau_{ig}} + \frac{1}{\tau_B}\right)$$

$$y = Y_p\left(\frac{1+\tau_{ig}/\tau_B}{KqM_o''}\right)^3 qM_o''$$

and

$$E = Q_B'\left(\frac{1+\tau_{ig}/\tau_B}{KqM_o''}\right)^3$$

$$= \left(1+\tau_{ig}/\tau_B\right)^3 \left(\ell_B/\ell\right)^2$$

where $$\ell_B = \left(\frac{Q_B'}{Z_{po}\rho_o C_o T_o \sqrt{g}}\right)^2$$

ℓ_B is a constant (and as such does not enter into a dimensional analysis) so long as $Z_{po} \propto Q_B'$ which does obtain for a linear relationship between flame length and heat release but does not for the 2/3 power law, in which case ℓ_B is an additional characteristic length.