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To my parents

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Preface

My interest in robustness analysis by use of integral quadratic constraints started in the fall 1993. Docent Anders Rantzer and I had started to look for a new stability result for systems with slowly time-varying parameters. The intention was to obtain tools that could be used for robust adaptive control and gain scheduling. We had a result that could be viewed as a generalization of the upper bound for the complex structured singular value. The next step was to exploit the fact that the parameters were real-valued. To do this we needed something that could be viewed as a generalization of the G-scale for the computation of the real structured singular value. It turned out to be hard. The main reason for this is that it is nontrivial to interpret the G-scale as a multiplier in the feedback loop. In other words, it was hard to derive the desired result in the classical framework for stability theory.

At this point we started to consider Professor Megretskis interesting paper "Power Distribution Approach in Robust Control" that had been published at the IFAC World Congress the same year. It was easy to derive the desired result using Megretski's approach. However, many questions arose at the same time. In particular, how did the power distribution approach compare to the classical stability theory, why was it only formulated for the linear case, and what did the proof look like? Rantzer and Megretski started to develop the approach further, now under the name of integral quadratic constraints (IQC).

The next step in the work on slowly time-varying systems was to find a method to compute the multipliers that appeared in our stability criterion. Professor Pascal Gahinet at INRIA in France provided us with an early version of LMI-Lab, a software package for solution of convex optimization problems in terms of linear matrix inequalities. At this point the ideas for this thesis started to take form. The main effort of my research has since then been focused on the problem of computing the multipliers that appear in the IQC approach for robustness analysis.

Outline of the Thesis

The main contribution of this thesis is contained in Chapters 2 through 5. Every chapter is a self-contained unit and can be read independently of the others. One reason for this is that the thesis has grown out from four different papers. The price paid is that some material appears at several places in the thesis.

Chapter 1 (Introduction): This chapter contains a presentation of the IQC approach for stability and robustness analysis. Some effort is also made to put the methodology in perspective with other approaches for stability and robustness analysis. The chapter ends with a brief description of the main ideas of the thesis. The work on slowly time-varying parameters is summarized in one section of the chapter. That work will be published in:

JÖNSSON, U. and A. RANTZER (1996): "Systems with uncertain parameters– time-variations with bounded derivatives." *International Journal of Robust and Nonlinear Control.* Accepted for publication.

An early version of the paper was presented at the conference:

JÖNSSON, U. and A. RANTZER (1994): "Systems with uncertain parameters– time-variations with bounded derivatives." In *Proceedings of the 33rd IEEE Conference on Decision and Control*, pp. 3074–3079, Lake Buena Vista, Florida.

Chapter 2 (Popov Multipliers): In this chapter it is shown how the Popov criterion can be used in a natural way in the IQC framework. As a result of this we can show that in stability analysis of systems with slope restricted nonlinearities it is possible to combine the Popov criterion and a famous stability criterion by Zames and Falb. An example shows that the combined criterion is superior to any of the two criteria alone. We also obtain a new Popov criterion for systems with slowly time-varying parameters. The chapter has grown out of the report:

JÖNSSON, U. (1996): "Stability analysis with Popov multipliers and integral quadratic constraints." Technical Report TFRT–7546, Department of Automatic Control, Lund Institute of Technology, Lund, Sweden. Submitted to a journal.

The main result of the chapter is applied in:

JÖNSSON, U. and M. LAIOU (1996): "Stability analysis of systems with nonlinearities." In *Proceedings of the 35th IEEE Conference on Decision and Control*, Kobe, Japan. Accepted for publication.

JÖNSSON, U. (1996):. "A popov criterion for systems with slowly time-varying parameters.". Submitted.

Chapter 3 (Computation of Multipliers): This is the central chapter of the thesis. A format for computing the multipliers that appear in IQC based robustness and performance analysis is presented. The idea is to parametrize a finite-dimensional and convex subset of the multipliers in such a way that the corresponding restricted robustness test can be formulated as a convex optimization problem. We give several examples on how multiplier descriptions of nonlinearities, dynamic uncertainties, parametric uncertainties, performance specifications, and signal specifications can be treated in our format. Early work in the direction of this chapter can be found in:

JÖNSSON, U. and A. RANTZER (1995): "A unifying format for multiplier optimization." In *Proceedings of the American Control Conference*, pp. 3859–3860, Seattle, Washington.

JÖNSSON, U. and A. RANTZER (1995): "A format for multiplier optimization." Technical Report TFRT–7530, Department of Automatic Control, Lund Institute of Technology,, Lund, Sweden.

JÖNSSON, U. (1995): "Robustness analysis based on integral quadratic constraints." In *Euraco Workshop: Recent Results in Robust and Adaptive Control*, pp. 171–189, Florence, Italy.

Chapter 4 (Duality Bounds in Multiplier Computation): Analysis problems that appear in IQC based robustness analysis are generally infinite-dimensional but convex. By restricting the analysis to a finite-dimensional subspace, as discussed in Chapter 3, we obtain a problem that is tractable for numerical solutions. However, the quality of the solution is critically dependent on the choice of subspace. We show in this chapter how duality theory can be used to investigate the quality of a particular finite-dimensional restriction. A considerable effort is invested in a discussion on computational issues for the dual robustness problem. Preliminary ideas along the direction of this chapter were presented in:

JÖNSSON, U. and A. RANTZER (1995): "On duality in robustness analysis." In *Proceedings of the 34th IEEE Conference on Decision and Control*, pp. 1443–1448, New Orleans, Louisiana.

An improved and more elegant format for the dual was then derived in the following report, which also is the basis for the chapter:

JÖNSSON, U. (1996): "Duality in analysis via integral quadratic constraints." Technical Report TFRT–7543, Department of Automatic Control, Lund Institute of Technology, Lund, Sweden.. Submitted to a journal.

Chapter 5 (Duality in Analysis with Mixed Multipliers): It is shown in this chapter that the dual problem derived in Chapter 4 is particularly attractive when the set of multipliers consists of a constant part and a frequency varying part. The frequency varying part is defined by a frequency independent constraint. The chapter is based on the report:

JÖNSSON, U. and A. RANTZER (1996): "Duality bounds in robustness analysis." Technical Report TFRT–7544, Department of Automatic Control, Lund Institute of Technology, Lund, Sweden. Accepted to Automatica.

The result has also been presented in:

JÖNSSON, U. and A. RANTZER (1996): "Duality bounds in robustness analysis." In *IFAC World Congress, Preprints*, volume H, pp. 35–40, San Fransisco, California.

Chapter 6 (Concluding Remarks): This sections contains some concluding remarks and suggestions for future research.

Acknowledgements

First, I would like to thank my supervisor Anders Rantzer. His optimism, enthusiasm, and many ideas made my thesis work joyful and exciting. He is also the co-author of several papers behind this thesis. His confidence, friendship and astonishing creativity have been invaluable.

My co-advisor Per Hagander has always been available for interesting discussions and valuable help. His critical thinking has been very valuable and I really appreciate his constructive criticism of the material in Chapter 2, which led to many improvements.

I am indebted to Karl Johan Åström who made it all possible. His dynamic and enthusiastic leadership makes the Department of Automatic Control a very stimulating and exciting place to work at. He has been a constant source of inspiration and his exceptional overview of the field of Automatic Control has been of great benefit to me.

I am delighted to acknowledge the help from Bo Bernhardsson, who has offered many suggestions and corrections to early versions of the manuscript. Bo has been a mentor and a good friend during my time as a graduate student.

Henrik Olsson read early versions of the first three chapters. He contributed with many improvements and corrections. Henrik is a good friend of mine since many years. I would also like to thank Lennart Andersson, who suggested several improvements on the organization of Chapter 3. Many thanks to Jörgen Malmborg, Karl Henrik Johansson, and Johan Nilsson, with whom I have spent a lot of time both at work and in leasure hours.

Most of all I would like to thank all the outstanding people at the department. It has been a privilege to work in the friendly and creative atmosphere they all contribute to. I would in particular wish to express my gratitude to Eva Schildt, for always being very helpful, and to Leif Andersson for providing excellent computer and typesetting facilities.

I would also like to express my gratitude to several persons outside the department. Andy Packard and Ian Rhodes stimulated my interest in control and system theory when I was a student at University of California, Santa Barbara. Special thanks to Alexander Megretski for numerous discussions about my research and to Pascal Gahinet for providing an early version of LMI-Lab. Maria-Christina Laiou is co-author of a paper that contains two of the examples in this thesis. She also wrote some software that was used to produce those examples.

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Finally, I would like to thank my parents and my brother for all their help and support throughout the years.

U.J.

1

Introduction

1.1 Background

Stability and robustness analysis has played an important role in the development of theory for automatic control systems and other feedback systems. There are many different definitions of stability in the literature. However, regardless of if we discuss stability in terms of boundedness and convergence of the systems state vector or boundedness of the input–output map for the closed loop system, stability is always a property that assures that the system does not *explode* in some sense. It is thus the least one must require of a system in practical applications. The next step is to require satisfaction of robustness or performance specifications. Robustness means that we require some form of margin for stability. This could, for example, mean that stability holds not only for the nominal system but also for the class of systems obtained by including an uncertainty model to the nominal system. Performance specifications are typically a measure of the disturbance rejection in the system.

This thesis is devoted to the theory and application of so called integral quadratic constraints (IQCs) in stability and robustness analysis. The focus of the thesis is the particular approach for using IQCs that was outlined by Megretski (1993b) and further developed in Rantzer and Megretski (1994) and Megretski and Rantzer (1995).

Many methods for stability and robustness analysis can be regarded as being IQC-based. A typical feature of these methods is that multipliers are used in the analysis and that the stability criterion can be formulated as a feasibility problem in terms of a set of multipliers. We will shortly review some of the most important multiplier-based methods for stability and robustness analysis in the next section. The review is by no means complete but it treats some of the most used and popular methods and it is sufficiently informative to allow us to put these methods in perspective with the IQC framework. We also introduce various convex problems involving linear matrix inequalities (LMIs). These LMI methods will be important tools in later chapters of the thesis.

The organization of the remaining part of this chapter is as follows. Section 1.3 introduces some of the notation and mathematical preliminaries that will be used in the first three chapters of the thesis. The IQC method, as it was proposed in Megretski and Rantzer (1995), is introduced in Section 1.4. Some of the most important features of this methodology is discussed in some depth. An application of the IQC method to systems with slowly time-varying parameters is given in Section 1.5. A detailed treatment of the classical multiplier method for input–output systems is given in Section 1.6. We show that the IQC framework has several advantages over the classical multiplier technique. The Lyapunov function method for deriving absolute stability results is presented in Section 1.7 and some of its potential and limitations are discussed. Finally, Section 1.8 shortly presents the main ideas of the thesis.

1.2 Related Work

We will here shortly review several well-known methods for stability and robustness analysis.

Absolute Stability Theory

Lure and Postnikov (1944) proposed the absolute stability problem. They considered the system

$$\begin{aligned} \dot{x} &= Ax + bu, \quad x(0) = x_0, \\ y &= cx, \\ u &= \varphi(y), \end{aligned} \tag{1.1}$$

where $A \in \mathbf{R}^{n\times n}$ is Hurwitz, $b, c^T \in \mathbf{R}^n$, and where φ is a memoryless nonlinearity that belongs to the sector $[0, \infty)$, i.e., $\varphi(y)y \geq 0$ for all y. The system can also be represented as in Figure 1.1, where $G(s) = c(sI - A)^{-1}b$. The system is said to be absolutely stable if the state vector is bounded and converges asymptotically to zero for all possible nonlinearities from the stated class and for all initial conditions. Lure and Postnikov obtained such a condition by applying Lyapunov's direct method with

$$V(x) = x^T Px + \lambda \int_0^y \varphi(\sigma)d\sigma, \tag{1.2}$$

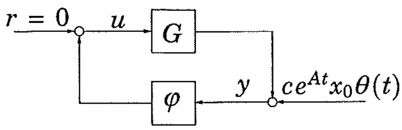

Figure 1.1 A block diagram description of the system in (1.1). The plant has the transfer function $G(s) = c(sI - A)^{-1}b$ and $ce^{At}x_0\theta(t)$ denotes the response due to the initial condition.

where $P = P^T > 0$ and where $\lambda \geq 0$. The integral in the definition of this Lyapunov function is nonnegative due to the assumption on φ.

A breakthrough in absolute stability theory came with Popov (1961). Popov showed that the system in (1.1) is absolutely stable if there exists $\lambda \geq 0$ such that

$$\mathrm{Re}\left[(1 + j\omega\lambda)G(j\omega)\right] < 0, \quad \forall\omega \in [0, \infty].$$

The importance of this result is that the stability condition is formulated in terms of the frequency response of G. The existence of a suitable λ can be determined from a Popov plot of G, see, for example, any of the books Khalil (1992), Slotine and Li (1991) or Vidyasagar (1993).

Shortly after the advent of the Popov criterion, Yakubovich derived the famous Kalman-Yakubovich-Popov lemma (KYP). This lemma gives an equivalence between Lure's and Popov's solutions to the absolute stability problem, see Yakubovich (1962) and Kalman (1963).

The development of the absolute stability theory was rapid after the advent of the KYP lemma. In the Soviet Union, Yakubovich derived results for systems with slope restrictions on the nonlinearities in Yakubovich (1965a), for systems with hysteresis nonlinearities in Yakubovich (1965b), and for systems with several nonlinearities in Yakubovich (1967).

In the west stability results for systems with time-varying gains, slope restricted nonlinearities and multiple nonlinear gains were developed. Many of these results are collected in the monograph Narendra and Taylor (1973).

We will in Section 1.7 compare the approach for absolute stability theory that we have discussed here with the IQC methodology that is the topic of the thesis.

Input–Output Stability

In parallel with the development of absolute stability theory during the 1960s there was also intensive work on the input–output approach for stability analysis. This work is based on operator theory and functional

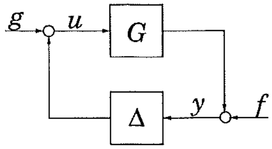

Figure 1.2 The system under study in the input-output approach for stability analysis.

analysis. The most powerful results are obtained when the system is defined on a Hilbert space. We assume that this is the case and that the system under study is on the form

$$\begin{aligned} y &= Gu + f, \\ u &= \Delta y + g, \end{aligned} \tag{1.3}$$

where G and Δ are operators on the Hilbert space. See also Figure 1.2. We use positive feedback interconnections in contrast to most of the reference literature on input–output stability, which consider negative feedback interconnections. We do this in order to allow for easy comparison with the IQC methodology.

The first important step in the input–output theory was taken by Zames and Sandberg. They introduced a framework that allows consideration of unbounded signals in the loop and that ensures that the system equations in (1.3) make sense. Extended spaces and the issue of well-posedness are the corner stones of this framework. We will discuss these notions in some more detail further on. With the initial questions settled it was possible to derive the two most fundamental results in the input-output theory, namely the small gain theorem and the passivity theorem, see, for example, Zames (1966a), Sandberg (1965b) and Sandberg (1965a).

The small gain theorem and the passivity theorem give conditions for stability that are too conservative in many applications. However, the introduction of loop transformations and multipliers can reduce the conservatism. It was shown in Zames (1966b) that several important results such as the circle criterion and the Popov criterion can be obtained in this way. The most powerful technique uses noncausal multipliers for the analysis. This technique was developed in Zames and Falb (1968) and it is also treated in the monographs Willems (1971a), and Desoer and Vidyasagar (1975).

Let us illustrate how the stability conditions look when we use multipliers. We assume that G is a stable linear time invariant operator with rational transfer function. The idea is to find a multiplier M that makes

Δ look positive in the sense that

$$\int_{-\infty}^{\infty} (M(j\omega)\widehat{y}(j\omega))^* \widehat{\Delta(y)}(j\omega) d\omega \geq 0, \quad \forall y \in \mathbf{L}_2[0,\infty). \tag{1.4}$$

Here $\widehat{y}$ and $\widehat{\Delta(y)}$ denote the Fourier transforms of y and $\Delta(y)$. Assuming that M is a rational transfer function and that certain technical conditions hold, the stability condition reduces to the frequency domain condition

$$\mathrm{Re}\,[M(j\omega)G(j\omega)] < 0, \quad \forall \omega \in [0,\infty]. \tag{1.5}$$

The multiplier-based stability theory will be discussed in more detail in Section 1.6, where we investigate its relation to the IQC approach developed in Megretski and Rantzer (1995).

Dissipativity Theory

The theory for dissipative dynamic systems developed in Willems (1972) and Hill and Moyland (1980) gives an abstract framework for stability analysis of a general class of systems. Dissipativity of a system means that it in a general sense absorbs more energy than it supplies. This theory has been used successfully for stability analysis of nonlinear systems in, for example, Hill and Moyland (1976), and Byrnes *et al.* (1991), and for large scale systems in Moyland and Hill (1978). The dissipativity theory can be interpreted in terms of IQCs. However, the class of multipliers that has been considered are memoryless and thus very restricted.

Topological Separation

Safonov introduced an abstract approach for stability analysis in Safonov (1982). The idea is to consider the topological separation of the graphs of the systems G and Δ in Figure 1.2. He could use this method to capture many classical stability results. Safonov also applied his abstract framework to obtain a multiplier-based multivariable circle criterion, see also Safonov and Athans (1981). However, the set of multipliers that was used is somewhat restricted due to the use of what we will call hard IQCs.

Postmodern Robustness Analysis

The issues of stability robustness and performance robustness of feedback systems have gained a lot of attention since the late 1970s. The question is whether a feedback control system will maintain stability and performance in spite of plant variations or plant uncertainties.

The $\mathbf{H}_\infty$ methodology was introduced in Zames (1981). The idea was to use the $\mathbf{H}_\infty$-norm to quantify frequency domain plant uncertainty and

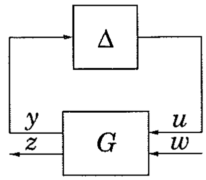

Figure 1.3 System setup for robust performance analysis.

system performance in terms of the signal gain from disturbances to the error output. The system under consideration is on the form in Figure 1.3. Here G denotes a linear time invariant nominal plant and Δ denotes the plant uncertainty. It is assumed that the system is scaled such that $\|\Delta\|_\infty \leq 1$, where $\|\cdot\|_\infty$ denotes the $\mathbf{H}_\infty$-norm. The disturbance signals are denoted by w and the error output is denoted by z.

It is in general conservative to consider full block uncertainties. The idea of taking the structure of the uncertainty into consideration was independently proposed by Safonov (1982) (the multivariable stability margin) and Doyle (1982) (the structured singular value, μ). A lot of attention has since then been directed towards the problem of developing numerical methods for the robustness analysis. As an example, let us consider the computation of the complex structured singular value.

It is assumed that the uncertainty block satisfies $\Delta(j\omega) \in \mathbf{\Delta}$, where the spatial structure, $\mathbf{\Delta}$, is defined as

$$\mathbf{\Delta} = \{\mathrm{diag}(\delta_1 I_{r_1}, \ldots, \delta_S I_{r_S}, \Delta_1, \ldots, \Delta_F) : \delta_i \in \mathbf{C},\ \Delta_j \in \mathbf{C}^{m_j \times m_j}\}, \tag{1.6}$$

Here it is assumed that

$$\sum_{i=1}^{S} r_i + \sum_{j=1}^{F} m_j = m$$

for consistency among the dimensions. The structured singular value for a matrix $M \in \mathbf{C}^{m \times m}$ is defined as

$$\mu_\Delta(M) = \left(\min\{\overline{\sigma}(\Delta) : \det(I - M\Delta) = 0,\ \Delta \in \mathbf{\Delta}\}\right)^{-1}.$$

The optimization problem

$$\gamma_\mu = \sup_{\omega \in [0,\infty]} \mu_\Delta(G(j\omega)).$$

gives a bound for assuring robust stability of a system with structured uncertainty. In fact, the feedback interconnection of G and Δ is stable if and only if $\gamma_\mu \|\Delta\|_\infty < 1$, see, for example, Zhou *et al.* (1996).

Exact computation of μ is in general a hard problem. Indeed, it can be shown to be NP-hard, see Toker and Özbay (1995). Instead of exact μ computation one usually resorts to the computation of upper and lower bounds. An upper bound can be obtained in the following way.

Let $\mathbf{D}$ be a set of matrices that commute with the structure of Δ, i.e., every $D \in \mathbf{D}$ satisfies $D\Delta = \Delta D$. Then an upper bound of γ_μ can be obtained by solving the optimization problem

$$\gamma_{\hat{\mu}} = \sup_{\omega \in [0,\infty]} \inf_{D \in \mathbf{D}} \overline{\sigma}(DG(j\omega)D^{-1}). \tag{1.7}$$

In practice, this problem is solved by choosing a finite frequency grid over which the optimization is performed. This is the way $\gamma_{\hat{\mu}}$ is computed in the software package Balas *et al.* (1993). We will in Example 1.5 see how the computation of $\gamma_{\hat{\mu}}$ can be formulated in the IQC framework.

An interesting relationship for $\gamma_{\hat{\mu}}$ was obtained in Poola and Tikku (1995). There it was shown that $\gamma_{\hat{\mu}}$ gives a nonconservative bound against arbitrarily slowly varying structured linear perturbations.

The μ theory was later extended to include real-valued uncertainty (parametric uncertainty) in Doyle (1985) and Fan *et al.* (1991). We will not discuss the computation of upper bounds for real and mixed complex/real μ here. However, a simple case that is easily extended into full generality is discussed in Example 1.6.

The term *postmodern robustness analysis* in the title of this section is adopted from Doyle *et al.* (1991). Several reasons for using this term were given in that paper. The one that deserves particular attention is the use of linear matrix inequalities (LMIs) in modern robust control.

Robustness Analysis and LMIs

The recent development of efficient algorithms for the solution of convex problems involving LMIs has opened up new perspectives for the analysis and synthesis of control systems. It is in particular various interior point methods that have proved to be successful both in theory and applications. We refer to the monograph Nesterov and Nemirovski (1993) and the survey Vandenberghe and Boyd (1996) for details and further references on interior point methods.

One reason for the popularity of the LMI methods is due to the availability of reliable software packages. Several of these have been developed by control groups around the world, see, for example, Gahinet *et al.* (1995), El Ghaoui (1995), and Wu and Boyd (1996).

We will in Chapter 3 show that a large class of problems in analysis of robust stability and robust performance can be formulated as, or at least be approximated by, either of the following problems.

Feasibility problem for LMIs (LMIP): Find $x \in \mathbf{R}^n$ such that

$$F(x) = F_0 + \sum_{i=1}^{n} x_i F_i < 0,$$

where $F_i = F_i^T \in \mathbf{R}^{n \times n}$ are given matrices. This means that F is an affine function of $x \in \mathbf{R}^n$.

Eigenvalue problem (EVP): Solve the optimization problem

$$\begin{aligned} &\inf \gamma \quad \text{subject to} \\ &A(x,\gamma) < 0, \end{aligned}$$

where A depends affinely on x and γ and takes values in the space of symmetric matrices.

Generalized Eigenvalue Problem (GEVP): Solve

$$\begin{aligned} &\inf \gamma \quad \text{subject to} \\ &\gamma B(x) < A(x), \quad B(x) < 0, \quad C(x) < 0, \end{aligned}$$

where A, B and C depend affinely on x and take values in the space of symmetric matrices.

The LMI problems presented above involve only strict LMIs. A non-strict LMI is an inequality on the form $F(x) \leq 0$. Strict LMIs are easier to treat numerically. Control oriented treatments of LMIPs, EVPs, and GEVPs can be found in the surveys Boyd *et al.* (1993), Boyd *et al.* (1994), and Packard *et al.* (1991).

1.3 Notation and Preliminaries

This section contains the notation and some preliminary results that will be used in the first three chapters of the thesis. First follows a list of notation.

M^*	Hermitian conjugate of a matrix.
$\lvert \cdot \rvert$	The Euclidean norm $\lvert x \rvert = \sqrt{x^T x}$.
$\overline{\sigma}(M)$	The largest singular value of a real or complex matrix M.
$\mathbf{RL}_\infty^{l \times m}$	The space consisting of proper real rational matrix functions with no poles on the imaginary axis. For $H \in \mathbf{RL}_\infty^{l \times m}$ we define $H^*(s) = H(-s)^T$.

$\mathbf{RH}_\infty^{l\times m}$ The subspace of $\mathbf{RL}_\infty^{l\times m}$ consisting of functions with no poles in the closed right half plane. The subspace $\mathbf{RH}_\infty^{l\times m}(\alpha)$ consists of functions that satisfy the property $H(s-\alpha) \in \mathbf{RH}_\infty^{l\times m}$.

$\theta(\cdot)$ The unit step function defined as $\theta(t) = 1$ for $t \geq 0$ and $\theta(t) = 0$ for $t < 0$.

P_T The truncation operator on the vector space of functions mapping $\mathbf{R}$ into $\mathbf{R}^m$. It is defined by $P_T u(t) = u(t)\theta(T-t)$.

$\mathbf{L}_2^m[0,\infty)$ The Lebesgue space of $\mathbf{R}^m$ valued signals with norm defined by

$$\|u\|^2 = \int_0^\infty |u(t)|^2 dt.$$

The space $\mathbf{L}_2^m(-\infty,\infty)$ is defined in a similar fashion.

$\mathbf{L}_{2e}^m[0,\infty)$ The vector space of functions f satisfying the condition $P_T f \in \mathbf{L}_2^m[0,\infty)$ for all $T > 0$. This space contains signals that are unbounded in the $\mathbf{L}_2$ norm. For example, the exponential function $f(t) = e^t\theta(t)$ and the unit step function $\theta(t)$ are included in this space.

$\text{diag}(\cdot,\cdot)$ If $\Delta_i : \mathbf{L}_{2e}^{m_i}[0,\infty) \to \mathbf{L}_{2e}^{m_i}[0,\infty)$, for $i = 1, 2$, then the operator $\text{diag}(\Delta_1, \Delta_2) : \mathbf{L}_{2e}^{m_1+m_2}[0,\infty) \to \mathbf{L}_{2e}^{m_1+m_2}[0,\infty)$ is defined by the input–output relation

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \Delta_1(u_1) \\ \Delta_2(u_2) \end{bmatrix}$$

where $y_i, u_i \in \mathbf{L}_{2e}^{m_i}[0,\infty)$ for $i = 1, 2$.

Preliminaries

We next present some of the mathematical and system theoretic preliminaries that will be used. We refer to Desoer and Vidyasagar (1975), Willems (1971a), Francis (1987), and Luenberger (1969) for more detailed treatments.

Causality of an Operator: An operator $H : \mathbf{L}_{2e}^m[0,\infty) \to \mathbf{L}_{2e}^l[0,\infty)$ is said to be causal if $P_T H P_T = P_T H$ for all $T \geq 0$. This means that the value at a certain time instant does not depend on future values of the argument. The operator H is said to be anticausal if $(I - P_T)H = (I - P_T)H(I - P_T)$, for all $T \geq 0$. This means that the value at a certain time does not depend on past values of the argument.

Boundedness of an Operator: A causal operator $H : \mathbf{L}_{2e}^m[0,\infty) \to \mathbf{L}_{2e}^l[0,\infty)$ is bounded if $H(0) = 0$ and if the gain defined as

$$\|H\| = \sup_{\substack{u \in \mathbf{L}_2^m[0,\infty) \\ u \neq 0}} \frac{\|Hu\|}{\|u\|} \tag{1.8}$$

is finite. Note that the gain is defined in terms of functions in $\mathbf{L}_2^m[0,\infty)$ and not the corresponding extended space. However, the definition in (1.8) implies boundedness on $\mathbf{L}_{2e}^m[0,\infty)$, since $\|P_T Hu\| \leq \|H\| \cdot \|P_T u\|$ for all $u \in \mathbf{L}_{2e}^m[0,\infty)$ and all $T \geq 0$. It can be shown that $\|H\|$ is the smallest such bound.

We will use noncausal operators as multipliers for stability analysis in this thesis. A noncausal operator $H : \mathbf{L}_2^m(-\infty,\infty) \to \mathbf{L}_2^l(-\infty,\infty)$ is bounded if $H(0) = 0$ and if the gain

$$\|H\| = \sup_{\substack{u \in \mathbf{L}_2^m(-\infty,\infty) \\ u \neq 0}} \frac{\|Hu\|}{\|u\|}$$

is finite.

Convolution Operators: A convolution operator with transfer function $H \in \mathbf{RL}_\infty^{l\times m}$ is defined in the following way. Let h be the weighting function defined as the inverse Laplace transform of H, i.e., $h(t) = \mathcal{L}^{-1}\{H(s)\}$. Then for any $u \in \mathbf{L}_2^m(-\infty,\infty)$ we define

$$(Hu)(t) \stackrel{\text{def}}{=} h * u(t) = \int_{-\infty}^{\infty} h(t-\tau)u(\tau)d\tau \tag{1.9}$$

The notation $*$ refers to convolution. In this way H defines a (possibly noncausal) map of $\mathbf{L}_2^m(-\infty,\infty)$ into $\mathbf{L}_2^l(-\infty,\infty)$. This operator is linear time-invariant and bounded. It can be shown that the $\mathbf{L}_2$-gain becomes

$$\|H\| = \sup_\omega \overline{\sigma}(H(j\omega)). \tag{1.10}$$

The restriction of H to $\mathbf{L}_2^m[0,\infty)$ is defined in the same way.

A linear time invariant convolution operator with transfer function $H \in \mathbf{RH}_\infty^{l\times m}$ is causal and bounded. The corresponding weighting function satisfies $h(t) = 0$ for all $t \leq 0$. For any $u \in \mathbf{L}_2^m[0,\infty)$ the convolution in (1.9) reduces to

$$(Hu)(t) = \int_0^t h(t-\tau)u(\tau)d\tau,$$

from which the causality follows. The gain is defined as in (1.10).

Inner Products, Adjoints, and Quadratic Forms: The vector spaces $\mathbf{L}_2^m[0,\infty)$ and $\mathbf{L}_2^m(-\infty,\infty)$ are Hilbert spaces and we can define the following inner product

$$\langle u, v\rangle = \int_{-\infty}^{\infty} u(t)^T v(t) dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{u}(j\omega)^* \widehat{v}(j\omega) d\omega.$$

The last inequality follows from Parseval's theorem and the notation $\widehat{u}$ refers to the Fourier transform of u defined as

$$\widehat{u}(j\omega) = \int_{-\infty}^{\infty} u(t) e^{-j\omega t} dt.$$

Let $H : \mathbf{L}_2^m(-\infty,\infty) \to \mathbf{L}_2^l(-\infty,\infty)$ be a bounded linear operator. The adjoint operator H^* satisfies

$$\langle u, H^* v\rangle = \langle Hu, v\rangle$$

for all $u \in \mathbf{L}_2^m(-\infty,\infty)$ and all $v \in \mathbf{L}_2^l(-\infty,\infty)$. The adjoint of any $H \in \mathbf{RL}_\infty^{l\times m}$ is defined by $H^*(s) = H^T(-s)$. It can be shown that the adjoint of a bounded convolution with transfer function $H \in \mathbf{RH}_\infty^{l\times m}$ is a bounded anticausal operator.

Any $H = H^* \in \mathbf{RL}_\infty^{m\times m}$ defines a quadratic form on $\mathbf{L}_2^m[0,\infty)$ in the following way. Let $u \in \mathbf{L}_2^m[0,\infty)$, then

$$\langle u, Hu\rangle = \int_0^{\infty} u(t)^T (Hu)(t) dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{u}(j\omega)^* H(j\omega)\widehat{u}(j\omega) d\omega,$$

defines a quadratic form. The last inequality follows from Parseval's theorem. We will at some places in the thesis allow quadratic forms to be defined in terms of more general functions. It is not an essential restriction to assume that these functions satisfy the properties presented in this section.

Nullspace, Range, and Orthogonal Complements: Two vectors x and y in $\mathbf{R}^m$ are said to be orthogonal if $x^T y = 0$. This is also denoted $x \perp y$. The orthogonal complement of $S \subset \mathbf{R}^m$ is denoted $S^\perp$. It consists of all vectors $x \in \mathbf{R}^m$ orthogonal to every vector in S.

The nullspace $\mathcal{N}(A)$ of a matrix $A \in \mathbf{R}^{l\times m}$ is defined as $\mathcal{N}(A) = \{x : Ax = 0\}$. The range is defined as $\mathcal{R}(A) = \{Ax : x \in \mathbf{R}^m\}$. We have the important relation $\mathcal{R}(A^T)^\perp = \mathcal{N}(A)$.

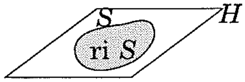

Figure 1.4 Illustration of the relative interior of a set $S \subset \mathbf{R}^3$. The affine hull of S is the hyperlane, H,that is illustrated in the figure. The relative interior of S consists of the grey area in the figure. It does not contain the solid contour.

Convexity and Relative Interior: The relative interior of a set $S \subset \mathbf{R}^m$ is denoted ri S. It is defined in terms of the affine hull of S, which is denoted aff S. The affine hull is defined as the set of all linear combinations on the form $\sum \alpha_i x_i$, where $x_i \in S$, and $\sum \alpha_i = 1$. We can now define ri S as the set of points of S, which are interior relative aff S. This means that for any $x \in$ ri S, there exists $\varepsilon > 0$ such that all $y \in$ aff S with $|x - y| < \varepsilon$ are also members of S. The definition is illustrated in Figure 1.4.

A set C in a linear vector space is *convex* if $\alpha x_1 + (1-\alpha)x_2 \in C$, for all $x_1, x_2 \in C$ and $\alpha \in [0,1]$. Furthermore, C is a *convex cone* if $\alpha x_1 + \beta x_2 \in C$, whenever $x_1, x_2 \in C$ and $\alpha, \beta \geq 0$. The *convex polytope*, C, with vertices at $x_1, \ldots, x_n \in \mathbf{R}^m$ is defined as the convex hull of these points, i.e.,

$$C = \mathrm{co}\{x_1, \ldots, x_n\} = \left\{ \sum_{i=1}^{n} \alpha_i x_i : \alpha_i \geq 0, \sum_{i=1}^{n} \alpha_i = 1 \right\}.$$

Let $F(x) = F_0 + \sum_{i=1}^{n} x_i F_i$, where $F_i = F_i^T \in \mathbf{R}^{n \times n}$. We say that the nonstrict LMI $F(x) \leq 0$ is *strictly feasible* if the strict LMI $F(x) < 0$ is feasible. The set $\{x : F(x) \leq 0\}$ is convex and even a convex cone if $F_0 = 0$.

In terms of LMIs we have the following relation. Assume that $F(x) \leq 0$ is strictly feasible and consider

$$C_1 = \{x \in \mathbf{R}^n : F(x) < 0\}, \quad \text{and} \quad C_2 = \{x \in \mathbf{R}^n : F(x) \leq 0\}.$$

Then $C_1 =$ ri C_2 and $\overline{C_1} = C_2$, i.e., the closure of C_1 is equal to C_2. This fact will be used in Chapter 3.

Useful Facts on Factorization: We will use the following facts several times in the thesis.

LEMMA 1.1
(*i*) Any $X \in \mathbf{RL}_\infty^{m \times m}$ satisfying $X(j\omega) = X^*(j\omega) \geq 0$ can be represented as $X = R^* R$, for some $R \in \mathbf{RH}_\infty^{m \times m}$.

(*ii*) Any $Y \in \mathbf{RL}_\infty^{m \times m}$ satisfying $Y(j\omega) = -Y^*(j\omega)$ can be represented as $Y = S - S^*$ for some $S \in \mathbf{RH}_\infty^{m \times m}$.

Proof: Part (i) is well-known and easy to see. We prove the second part. Let us assume that $Y = Y_c + Y_{ac}$, where $Y_c, Y_{ac}^* \in \mathbf{RH}_\infty^{m\times m}$. Then the condition of the lemma implies that

$$Y_c + Y_{ac} = -Y_c^* - Y_{ac}^*.$$

Since the causal and the anticausal part of both sides must be equal we get $Y_{ac} = -Y_c^*$. This concludes the proof. □

Absolute Continuity: Absolute continuity is a necessary and sufficient condition for x to be the indefinite integral of its derivative, i.e., the relation $x(t) = x_0 + \int_0^t \dot{x}(\tau)d\tau$ holds for all $t \geq 0$, see Royden (1988) or Riesz and Sz.-Nagy (1953). It is thus a reasonable assumption on the solution of a system of differential equations. It can be shown that if $x, \dot{x} \in \mathbf{L}_{2e}^m[0,\infty)$, then x is absolutely continuous on any finite time interval $[0,T]$.

The next lemma will be used frequently in Chapter 2. It is formulated in, for example, Desoer and Vidyasagar (1975).

LEMMA 1.2
If a function satisfies $x, \dot{x} \in \mathbf{L}_2^m[0,\infty)$, then x is bounded and $x(t) \to 0$ as $t \to \infty$. □

1.4 Robust Stability Theory Based on IQCs

The purpose of this section is to give a brief overview of the framework for stability and robustness analysis that was developed in Megretski (1993b) and Megretski and Rantzer (1995). We will pay particular attention to the underlying assumptions and characteristics of this method. This will allow us to compare the IQC method with other approaches for stability theory.

We consider the system shown in Figure 1.5, i.e.,

$$\begin{aligned} y &= Gu + f, \\ u &= \Delta(y) + g, \end{aligned} \tag{1.11}$$

where G and Δ are assumed to be bounded causal operators. It is further assumed that G is linear time-invariant with transfer function in $\mathbf{RH}_\infty^{l\times m}$. The injected signals f and g are assumed to be in $\mathbf{L}_{2e}^l[0,\infty)$ and $\mathbf{L}_{2e}^m[0,\infty)$, respectively. The operator Δ will sometimes be called perturbation. The reason is that our system can be considered as consisting of a nominal linear plant G in feedback interconnection with a perturbation

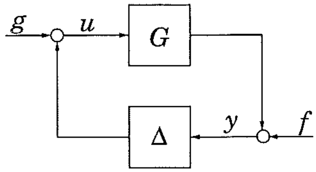

Figure 1.5 Linear time invariant plant G in positive feedback interconnection with a bounded perturbation Δ.

Δ that contains, for example, all nonlinearities, time-varying components, unmodeled dynamics, and parametric uncertainties in the system.

A well-posedness assumption is typically needed in operator-based stability theory. The following definition of well-posedness and stability was suggested in Megretski and Rantzer (1995).

DEFINITION 1.1—WELL-POSEDNESS AND STABILITY
The feedback interconnection of G and Δ in (1.11) is *well-posed* if the map $(u, y) \mapsto (f, g)$ has a causal inverse on $\mathbf{L}_{2e}^{l+m}[0, \infty)$. The system is *stable* if in addition there exists a constant $c > 0$ such that

$$\int_0^T (|y|^2 + |u|^2)dt \leq c \int_0^T (|f|^2 + |g|^2)dt, \quad \forall T \geq 0.$$

□

This well-posedness assumption demands the existence of a unique solution $(u, y) \in \mathbf{L}_{2e}^{l+m}[0, \infty)$ for every pair of injected signals, $(f, g) \in \mathbf{L}_{2e}^{l+m}[0, \infty)$. A well-posed system is thus nice enough to ensure a solution that is defined and reasonably regular on any finite time interval. However, note that this does not mean that the system is stable. The truncated norms $\|P_T u\|$ and $\|P_T y\|$ of the signals in a well-posed system can tend to infinity as $T \to \infty$ and the growth may be arbitrarily fast. It is possible to infer well-posedness from conditions on the incremental gain of the system, see, for example, Desoer and Vidyasagar (1975) or Willems (1971a).

We will next give some examples of systems that do not satisfy the well-posed assumption. We say that these systems are ill-posed.

EXAMPLE 1.1
Consider the feedback interconnection of $G(s) = 1/(s + 1)$ and the nonlinearity $\varphi(x) = x + x^2$. Let the injected signals be $f = 0$ and $g(t) = \theta(t)$. The closed loop system is described by the differential equation

$$\dot{x} = x^2 + 1, \quad t \geq 0$$

The solution $\arctan(x) = t\theta(t)$ or equivalently $x(t) = \tan(t)\theta(t)$ is clearly not in $\mathbf{L}_{2e}[0,\infty)$ since it goes to infinity as $t \to \pi/2$. Note that this system is not on the form studied in the thesis since φ is not bounded on $\mathbf{L}_{2e}[0,\infty)$. □

The next two examples are taken from Willems (1971a).

EXAMPLE 1.2
Let $G(s) = 1$ and let $\Delta = 1 - e^{-sT}$. In this case the closed loop operator becomes $(I - G\Delta)^{-1}G = e^{sT}$. Hence, the system is not causal and thus not well-posed. □

EXAMPLE 1.3
Consider the case when $G(s) = 1$, $\Delta = k$ and $f = 0$. If $k = 1$, then the return ratio $(I - G\Delta)$ is not invertible and the system is clearly not well-posed. For all other cases of k we get $(I - G\Delta)^{-1}G = 1/(1-k)$. However, even here it is questionable if the system is well-posed or not in the case $|k| > 1$. For example, if the system is a model of two interconnected physical systems then there will always be some small delay in the loop. In this case it can be shown that the step response for the physical system is unstable, i.e., $s(t) \to \infty$ as $t \to \infty$. This is in conflict with the expected solution from the model $s(t) = 1/(1-k)\theta(t)$. Hence, for some applications this system should be regarded as ill-posed. □

The examples above illustrate how ill-posedness can be a consequence of algebraic loops.

The idea behind the IQC approach for stability analysis is to find a description of the perturbation Δ in terms of integral quadratic constraints. Every IQC is defined in terms of a matrix function Π, which we call multiplier in this thesis. The following formal definition was given in Megretski and Rantzer (1995).

DEFINITION 1.2—IQC
Let $\Pi : j\mathbf{R} \to \mathbf{C}^{(l+m)\times(l+m)}$ be a bounded measurable function that takes Hermitian values. A bounded operator $\Delta : \mathbf{L}_{2e}^l[0,\infty) \to \mathbf{L}_{2e}^m[0,\infty)$ is said to satisfy the IQC defined by Π if

$$\int_{-\infty}^{\infty} \begin{bmatrix} \widehat{y}(j\omega) \\ \widehat{\Delta(y)}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \widehat{y}(j\omega) \\ \widehat{\Delta(y)}(j\omega) \end{bmatrix} \geq 0 \qquad (1.12)$$

for all $y \in \mathbf{L}_2^l[0,\infty)$. □

As an example we note that the IQCs defined by the multipliers

$$\Pi(j\omega) = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \quad \text{and} \quad \Pi(j\omega) = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

defines Δ to be contractive and positive, respectively. More general multipliers are discussed later.

The main stability criterion in Megretski and Rantzer (1995) is formulated as follows.

THEOREM 1.1
Assume that

(*i*) for all $\tau \in [0,1]$, the interconnection of G and $\tau\Delta$ is well-posed,

(*ii*) for all $\tau \in [0,1]$, $\tau\Delta$ satisfies the IQC defined by Π,

(*iii*) there exists $\varepsilon > 0$ such that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \le -\varepsilon I, \quad \forall \omega \in \mathbf{R}.$$

Then the feedback interconnection of G and Δ in (1.11) is stable. □

REMARK 1.1
It is important to make sure that the system satisfies the well-posedness condition in (*i*) when applying the theorem. As an example, let $G = -1$ and $\Delta = k > 1$. We can then use the positivity multiplier to satisfy condition (*ii*) and (*iii*), still the system is not reasonable in some applications, see Example 1.3. □

REMARK 1.2
The reason that the first two conditions of the theorem need to be satisfied not only at $\tau = 1$ but for the whole interval $[0,1]$ can be found in the proof that uses an homotopy argument. First show that if the feedback interconnection of G and $\tau\Delta$ is stable for some $\tau \in [0,1]$, then it is also stable when τ is perturbed into $\tau + \tau_\Delta$, where $|\tau_\Delta| \le \gamma_\Delta$. Here γ_Δ is a positive constant that is independent of τ. The proof of this property uses conditions (*i*) and (*ii*). The conclusion of the theorem follows by using this argument from $\tau = 0$, where the system is stable, up to $\tau = 1$ in steps of size γ_Δ.

It turns out to be important that the system under consideration is stable in each step of this iteration. It is this that allows us to define the IQCs in terms of signals in $\mathbf{L}_2[0,\infty)$. A similar homotopy argument will be used for the proof of Theorem 2.2 in Chapter 2. □

REMARK 1.3
It should be noted that in many applications the validity of (*i*) and (*ii*) at $\tau = 1$ ensures their validity for all other $\tau \in [0,1]$. This is the case in all examples of this section. □

We will next interpret some examples from Section 1.2 in the IQC framework.

EXAMPLE 1.4
The positivity condition in (1.4) is equivalent to saying that Δ satisfies the IQC defined by

$$\Pi(j\omega) = \begin{bmatrix} 0 & M(j\omega)^* \\ M(j\omega) & 0 \end{bmatrix}.$$

The stability condition in (1.5) is equivalent to the condition

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0, \quad \forall \omega \in [0,\infty].$$

Note that this condition is equivalent to the third condition in Theorem 1.1. This follows since M was assumed to be a rational transfer function. □

EXAMPLE 1.5
Let us consider the computation of the bound $\gamma_{\hat{\mu}}$ in (1.7). We first notice that the constraint $\overline{\sigma}(DMD^{-1}) < \rho$, where $M \in \mathbf{C}^{m\times m}$, is equivalent to the constraint $M^*XM - \rho^2 X < 0$, where $X = D^*D$. It follows that $\gamma_{\hat{\mu}}$ is equal to $\gamma_{\text{opt}}^{1/2}$, where γ_{opt} is the solution to the following optimization problem.

$$\inf \gamma \quad \text{subject to} \tag{1.13}$$
$$\begin{cases} \exists X \in \mathbf{RL}_\infty^{m\times m} \text{ such that} \\ X(j\omega) \in \{D^*D : D \in \mathbf{D}\}, \quad \forall \omega \in [0,\infty], \\ \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \begin{bmatrix} X(j\omega) & 0 \\ 0 & -\gamma X(j\omega) \end{bmatrix} \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0, \quad \forall \omega \in [0,\infty]. \end{cases}$$

This would have been an obvious statement if X was any bounded function on the extended imaginary axis. However, it can be shown that the constraint that X is rational is nonrestrictive due to the continuity of G.

Let us define the following set of matrices, where $\boldsymbol{\Delta}$ is the spatial structure defined in (1.6),

$$\mathbf{B}_\Delta(\gamma) = \{\Delta \in \boldsymbol{\Delta} : \overline{\sigma}(\Delta) \le \gamma^{-1/2}\} \subset \mathbf{C}^{m\times m}.$$

Any dynamic uncertainty with $\Delta(j\omega) \in \mathbf{B}_\Delta(\gamma)$ satisfies the IQCs defined by the multipliers in the set

$$\Pi_\Delta(\gamma) = \left\{ \begin{bmatrix} X(j\omega) & 0 \\ 0 & -\gamma X(j\omega) \end{bmatrix} : X(j\omega) \in \{D^*D : D \in \mathbf{D}\} \right\}.$$

To see this we notice that (we suppress the arguments $j\omega$)

$$\begin{bmatrix} I \\ \Delta \end{bmatrix}^* \Pi \begin{bmatrix} I \\ \Delta \end{bmatrix} = D^*D - \gamma\Delta^*D^*D\Delta = D^*D - \gamma D^*\Delta^*\Delta D \ge 0,$$

where the last equality follows since $D\Delta = \Delta D$ and the inequality follows since $\Delta \in \mathbf{B}_\Delta(\gamma)$, which implies that $\Delta^*\Delta \le \gamma^{-1}I$. Thus, we have shown that the structured uncertainty with $\Delta(j\omega) \in \mathbf{B}_\Delta(\gamma)$ satisfies the IQCs defined by the multipliers in $\Pi_\Delta(\gamma)$. The optimization problem in (1.13) can now be formulated as

$$\begin{aligned} &\inf \gamma \quad \text{subject to} \\ &\begin{cases} \exists \Pi \in \Pi_\Delta(\gamma) \quad \text{such that} \\ \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0, \quad \forall\omega \in [0,\infty]. \end{cases} \end{aligned}$$

The optimal solution γ_{opt} can be used to give a bound on the maximal norm on the uncertainty Δ, i.e., $\|\Delta\|_\infty \le \gamma_{\text{opt}}^{-1/2}$ assures stability. □

The optimization problem in this example is a special case of a much more general class of problems that will be considered from a computational perspective in Chapters 3, 4, and 5. Next example is a simple case of mixed real/complex μ analysis.

EXAMPLE 1.6
Let $G \in \mathbf{RH}_\infty^{m\times m}$ be the nominal system and let $\Delta = \text{diag}(\Delta_1, \delta_2 I_{m_2})$, where $\Delta_1 \in \mathbf{H}_\infty^{m_1\times m_1}$ is a dynamic uncertainty block with $\|\Delta\|_\infty \le 1$, and where δ is an uncertain parameter with $\delta \in [-1, 1]$. We assume that $m_1 + m_2 = m$. The perturbation Δ satisfies the IQCs defined by the multipliers

$$\Pi(j\omega) = \left[\begin{array}{cc|cc} x(j\omega)I_{m_1} & 0 & 0 & 0 \\ 0 & X(j\omega) & 0 & Y(j\omega) \\ \hline 0 & 0 & -x(j\omega)I_{m_1} & 0 \\ 0 & Y(j\omega)^* & 0 & -X(j\omega) \end{array}\right] \tag{1.14}$$

where $x(j\omega) = \overline{x(j\omega)} \geq 0$, and where $X, Y : j\mathbf{R} \to \mathbf{C}^{m_2 \times m_2}$ satisfy $X(j\omega) = X(j\omega)^* \geq 0$ and $Y(j\omega) = -Y(j\omega)^*$, respectively. To see this we note that direct calculation shows that the integrand in (1.12) is positive at every ω. We notice that for every $\tau \in [0,1]$, $\tau\Delta$ is in the same class of operators as Δ. This gives the following two properties.

- For every $\tau \in [0,1]$, $\tau\Delta$ satisfies the IQC defined by Π in (1.14).
- Well-posedness of the feedback interconnection of G and Δ implies well-posedness of the interconnection of G and $\tau\Delta$ for $\tau \in [0,1]$.

□

We next discuss how multipliers can be combined to improve the description of the perturbation Δ. Then follows a short discussion on the difference between hard and soft IQCs.

Combination of Multipliers

One of the most distinguished properties of the IQC methodology is the ease with which multipliers can be combined in order to obtain an as accurate as possible description of a perturbation Δ. Indeed, any conic combination $\sum \alpha_i \Pi_i$, $\alpha_i \geq 0$ of multipliers that define valid IQCs for Δ gives a new multiplier that describes Δ. From this discussion it is clear that the set of all multipliers that describe a perturbation in terms of IQCs is a convex cone.

We will see in Section 1.6 that it is cumbersome to combine the multipliers that appear in the classical input-output theory. The reason is that invertibility conditions and factorization conditions on the multipliers must be taken care of. This gives the IQC framework an advantage over the input–output method.

Hard versus Soft IQCs

The IQC condition that appears in Definition 1.2 is called a *soft* IQC, see Megretski and Rantzer (1995). This refers to the fact that the integral in (1.12) corresponds to a energy constraint over the positive time-axis. The term *hard* IQC refers to integral quadratic constraints that are required to hold over every finite time interval $[0, T]$. Most of the approaches for stability theory reviewed in Section 1.2 are based on hard IQCs that can be obtained in the following way. Suppose we have the factorization $\Pi = \Psi^* M \Psi$, where Ψ is a bounded causal with linear transfer function and where M is a symmetric matrix. Then use

$$\left\langle P_T \Psi \begin{bmatrix} y \\ \Delta(y) \end{bmatrix}, M P_T \Psi \begin{bmatrix} y \\ \Delta(y) \end{bmatrix} \right\rangle \geq 0, \quad \forall T \geq 0,\ \forall y \in \mathbf{L}_{2e}^{l}[0,\infty), \quad (1.15)$$

as a hard IQC. The inner product in (1.15) can equivalently be formulated as

$$\int_0^T w(t)^T M w(t) dt \geq 0, \tag{1.16}$$

where

$$w(t) = \left(\Psi \begin{bmatrix} y \\ \Delta(y) \end{bmatrix} \right)(t). \tag{1.17}$$

We will use this type of hard IQC when we discuss absolute stability theory in some more detail in Section 1.7.

It is clear that every hard IQC implies the validity of the corresponding soft IQC. The other implication does not hold in general. This means that methods that are based on hard IQCs are restrictive in the sense that the number of available multipliers in general is smaller than in the IQC approach. It is in particular the use of so called noncausal multipliers that becomes complicated in a framework based on hard IQCs. An inventive way to partially overcome the difficulty was developed in Zames and Falb (1968). This will be discussed in Section 1.6.

1.5 An Application: Slowly Time-Varying Parameters

We will in this section apply the IQC methodology to derive a stability result for systems with slowly time-varying parameters. The key to this result is two new sets of multipliers to be used in the stability condition. The application is taken from Jönsson and Rantzer (1996).

We consider the system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(t_0) = x_0, \\ y(t) &= Cx(t) + Du(t), \\ u(t) &= \delta(t)y(t), \end{aligned} \tag{1.18}$$

where $A \in \mathbf{R}^{n\times n}, B \in \mathbf{R}^{n\times m}, C \in \mathbf{R}^{m\times n}$, and $D \in \mathbf{R}^{m\times m}$. We assume that A is Hurwitz and that $\delta(t)$ is a real-valued time-varying parameter satisfying $|\delta(t)| \leq 1$ and $|\dot{\delta}(t)| \leq d$. We consider uniform exponential stability defined as follows.

Definition 1.3—Uniform Exponential Stability
The system in (1.18) is uniformly exponentially stable if there exist $m, \alpha > 0$ such that for any initial condition $x_0 \in \mathbf{R}^n$

$$|x(t)| \leq me^{-\alpha(t-t_0)}|x_0|, \quad \forall t \geq t_0,$$

and for all possible parameter variations, i.e., for all possible functions $\delta(t)$ satisfying the conditions above. □

Our first step is to derive two new multipliers Π_R and Π_S that define valid IQCs for $\delta(t)I$. The multipliers are defined in terms of transfer functions $R, S \in \mathbf{RH}_\infty^{m\times m}$. If these have the realizations $R(s) = C_R(sI - A_R)^{-1}B_R + D_R$ and $S(s) = C_S(sI - A_S)^{-1}B_S + D_S$, respectively, then we define

$$\begin{aligned} R_C(s) &= C_R(sI - A_R)^{-1}, \\ R_B(s) &= (sI - A_R)^{-1}B_R, \\ S_C(s) &= C_S(sI - A_S)^{-1}, \\ S_B(s) &= (sI - A_S)^{-1}B_S. \end{aligned}$$

We will use the derivative bound $|\dot{\delta}(t)| \leq d$ to show that the operator defined by multiplication with $\delta(t)I$ in the time domain satisfies the IQCs defined by

$$\Pi_R = \begin{bmatrix} R^*R + dR_B^*R_B & 0 \\ 0 & -R^*R + dR^*R_C R_C^* R \end{bmatrix},$$

and

$$\Pi_S = \begin{bmatrix} d(S_B^*S_B + S_C S_C^*) & S - S^* \\ S^* - S & 0 \end{bmatrix}.$$

To prove this we use the so called *swapping lemma*, which has previously been used in adaptive control theory, see, for example, Morse (1980).

LEMMA 1.3—SWAPPING LEMMA
Suppose that $\delta(t)$ has the derivative $\dot{\delta}(t)$ and let δ and $\dot{\delta}$ denote the corresponding multiplication operators. Let the convolution operator H have the transfer function $H(s) = C(sI - A)^{-1}B + D \in \mathbf{RH}_\infty^{m\times m}$. Then for all $f \in \mathbf{L}_2^m(-\infty, \infty)$

$$H\delta f = \delta H f - H_C \dot{\delta} H_B f,$$

where $H_C = C(sI - A)^{-1}$ and $H_B = (sI - A)^{-1}B$.

Proof: Let $p = d/dt$. We note that

$$(pI - A)(\delta H_B f) = \dot{\delta} H_B f + B\delta f.$$

Let H_C operate from the left on the equation above. After addition of $\delta D = D\delta$, we get

$$\delta H f = H_C \dot{\delta} H_B f + H \delta f,$$

from which the lemma follows. □

The swapping lemma shows that permutation of the convolution operator H and the multiplication operator δ gives an extra term that contains the time derivative of δ.

The following inner product space relations will be used. Let $x, y \in \mathbf{L}_2^m[0,\infty)$ and let H be a bounded operator on $\mathbf{L}_2^m[0,\infty)$, then

1. $\langle x, y \rangle = \langle y, x \rangle$,
2. $\pm 2 \langle x, y \rangle \le \langle x, x \rangle + \langle y, y \rangle$,
3. $\langle Hy, Hy \rangle \le \|H\|^2 \langle y, y \rangle$.

The same relations hold on $\mathbf{L}_2^m(-\infty,\infty)$. We can now prove the following lemmas.

LEMMA 1.4
A multiplication operator δ, with $|\delta(t)| \le 1$ and derivative bound $|\dot{\delta}(t)| \le d$, satisfies the IQC defined by Π_R, for any $R \in \mathbf{RH}_\infty^{m\times m}$.

Proof: Let $v = \delta u$, where $u \in \mathbf{L}_2^m[0,\infty)$, then

$$\begin{aligned} 2\langle Rv, Rv \rangle &= 2\langle Rv, \delta R u - R_C \dot{\delta} R_B u \rangle \\ &= 2\langle Rv, \delta R u \rangle - 2\langle \sqrt{d} R_C^* R v, \frac{1}{\sqrt{d}} \dot{\delta} R_B u \rangle \\ &\le \langle Rv, Rv \rangle + \langle Ru, Ru \rangle + d\langle R_C^* R v, R_C^* R v \rangle + d\langle R_B u, R_B u \rangle. \end{aligned}$$

Hence by using the definition of the adjoint we get

$$\left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \Pi_R \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle \ge 0$$

for all $u, v \in \mathbf{L}_2^m[0,\infty)$, such that $v = \delta u$. □

LEMMA 1.5
A real-valued multiplication operator δ, with $|\delta(t)| \le 1$ and derivative bound $|\dot{\delta}(t)| \le d$, satisfies the IQC defined by Π_S, for any $S \in \mathbf{RH}_\infty^{m\times m}$.

Proof: Let $v = \delta u$, where $u \in \mathbf{L}_2^m[0,\infty)$, then

$$\begin{aligned} 2\langle u, (S - S^*)v\rangle &= 2(\langle u, S\delta u\rangle - \langle u, \delta S u\rangle) \\ &= -2\langle u, S_C\dot{\delta}S_B u\rangle = -2\langle \sqrt{d}S_C^* u, \frac{1}{\sqrt{d}}\dot{\delta}S_B u\rangle \\ &\geq -d(\langle S_C^* u, S_C^* u\rangle + \langle S_B u, S_B u\rangle). \end{aligned}$$

Hence, we have

$$\left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \Pi_S \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle \geq 0$$

for all $u, v \in \mathbf{L}_2^m[0,\infty)$, such that $v = \delta u$. □

We are now in the position to state a stability result for the system in (1.18). Let $G(s) = C(sI - A)^{-1}B + D$, then we have the following result.

THEOREM 1.2
Assume that $\delta(t)$ is real-valued and satisfies $|\delta(t)| \leq 1$, $|\dot{\delta}(t)| \leq d$ and $\det(I - \delta(t)D) \neq 0$, $\forall t$. If there exists $R, S \in \mathbf{RH}_\infty^{m\times m}$ such that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0, \quad \forall\omega \in [0,\infty], \tag{1.19}$$

where $\Pi \in \mathbf{RL}_\infty^{m\times m}$ is defined as

$$\Pi = \begin{bmatrix} R^*R + d\Gamma^*\Gamma & S - S^* \\ S^* - S & -R^*R + d\Upsilon^*\Upsilon \end{bmatrix},$$

with

$$\Gamma = \begin{bmatrix} R_B \\ S_B \\ S_C^* \end{bmatrix}, \quad \text{and} \quad \Upsilon = R_C{}^*R.$$

Then the system in (1.18) is uniformly exponentially stable.

Proof: Since the multiplication operator δI satisfies the IQC defined by Π_R and Π_S it follows that it also satisfies the IQC defined by $\Pi = \Pi_R + \Pi_S$. The condition that $\det(I - \delta(t)D) \neq 0$ implies that the system in (1.18) is well-posed. It is now trivial to verify conditions (i) and (ii) of Theorem 1.1. Hence, an application of Theorem 1.1 proves stability in terms of Definition 1.1. For this system this also implies uniform exponential stability according to a result in Megretski and Rantzer (1995). □

REMARK 1.4
It is easy to extend the result to more general classes of time-varying parameters. For example, block diagonal case $\text{diag}(\delta_1 I_{m_1}, \ldots, \delta_N I_{m_M})$, where $|\delta_k| \leq 1$ and $|\dot{\delta}_k| \leq d_k$, were treated in Jönsson and Rantzer (1996). □

REMARK 1.5
The swapping lemma has been used before to derive stability results for slowly varying parameters. For example, Packard and Teng (1990) derived a result that is related to ours in the case when only Π_R is used. A similar result was later also obtained in Helmersson (1995b). □

REMARK 1.6
For $d = 0$, Π reduces to

$$\Pi(j\omega) = \begin{bmatrix} X(j\omega) & Y(j\omega) \\ Y(j\omega)^* & -X(j\omega) \end{bmatrix},$$

where $X(j\omega) = R(j\omega)^* R(j\omega)$, and $Y(j\omega) = S(j\omega) - S(j\omega)^*$. It follows from Lemma 1.1 that any positive semidefinite X and skew-Hermitian Y can be obtained in this way. This stability condition corresponds to the one obtained from a frequency dependent upper bound of the structured singular value. Note that the multiplier X corresponds to the D scale and the multiplier Y corresponds to the G scale in Fan *et al.* (1991) and Young (1993). □

REMARK 1.7
The terms $d\Gamma^*\Gamma$ and $d\Upsilon^*\Upsilon$ in Theorem 1.2 can be viewed as penalties for taking time-variations into consideration. □

REMARK 1.8
Consider the case with arbitrary rate of variation, i.e.; when $d \to \infty$. Then the penalty terms $d\Gamma^*\Gamma$ and $d\Upsilon^*\Upsilon$ tends to infinity in critical directions. This implies that the condition in Theorem 1.2 can be satisfied only if the multipliers are constant, i.e., $R(s) = D_R$ and $S(s) = D_S$, since then Γ and Υ are both zero. Thus, the stability condition reduces to a search for matrices $X, Y \in \mathbf{R}^{m\times m}$, where $X = X^T \geq 0$ and $Y = -Y^T$, such that (1.19) is satisfied with

$$\Pi = \begin{bmatrix} X & Y \\ Y^T & -X \end{bmatrix}.$$

To see this we just note that any X and Y satisfying the conditions above can be represented as $X = D_R^T D_R$ and $Y = D_S - D_S^T$ for some $D_R, D_S \in \mathbf{R}^{m\times m}$. □

1.6 The Classical Multiplier Theory

The use of multipliers in stability analysis with the small gain theorem or the passivity theorem can generally reduce conservatism of the analysis extensively. We will here discuss the classical multiplier theory and relate it to the IQC approach for stability analysis. We limit our discussion to the methodology that was introduced in Zames and Falb (1968), see also Willems (1971a) and Desoer and Vidyasagar (1975). The theory is restricted to square systems for reasons that will become apparent. The main tool in the derivation of the results is the passivity theorem

THEOREM 1.3—PASSIVITY THEOREM
Assume that the feedback interconnection of G and Δ in (1.11) is well-posed and that the following conditions hold

$$\begin{aligned}\langle u_T, Gu_T\rangle &\le -\varepsilon\|u_T\|^2,\\ \langle u_T, \Delta u_T\rangle &\ge 0,\end{aligned}$$

for all $u \in \mathbf{L}_{2e}^m[0,\infty)$. The system is then stable.

Proof: See, for example, Desoer and Vidyasagar (1975). □

We will next follow the arguments in Zames and Falb (1968) and Desoer and Vidyasagar (1975) that lead to the multiplier theorem. The idea is the following. Assume that we want to study stability of system S_1 in Figure 1.6. We introduce an invertible multiplier M into the system. This results in the system S_2 in Figure 1.6. All multipliers in this section are assumed to be bounded linear operators.

The multiplier M and its inverse are assumed to be bounded but not necessarily causal. The passivity theorem requires causal operators in the feedback interconnection and it can therefore not be applied to system S_2 if M or M^{-1} is noncausal. In this case it is required that there exists a factorization $M = M_-M_+$, where $M_+, M_+^{-1}, M_-^*, (M_-^*)^{-1}$ are bounded and causal. If such a factorization exists we use the following lemma from Zames and Falb (1968).

LEMMA 1.6
The soft IQCs in (i) imply the hard IQCs in (ii), where (i) and (ii) are defined as follows.

(i) For some $\varepsilon > 0$,

$$\begin{aligned}\langle v, MGv\rangle &\le -\varepsilon\|v\|^2,\\ \langle v, M^*\Delta(v)\rangle &\ge 0,\end{aligned} \tag{1.20}$$

for all $v \in \mathbf{L}_2^m[0,\infty)$.

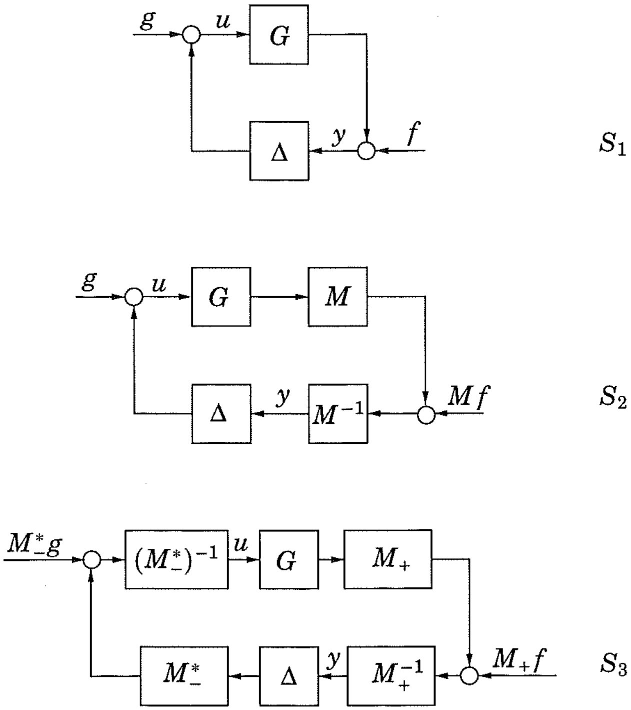

Figure 1.6 In the classical input–output theory a multiplier M is inserted in the loop resulting in system S_2. The passivity theorem cannot be applied if M or M^{-1} is noncausal. In this case it is required that M can be factored into $M = M_-M_+$, where M_-^*, M_+ and their inverses are causal and bounded. If such a parametrization exists, stability of S_1 is equivalent to stability of S_3. The stability conditions can be stated in terms of IQCs involving the multiplier M.

(*ii*) For some $\varepsilon > 0$,

$$\begin{aligned} \langle u_T, M_+G(M_-^*)^{-1}u_T \rangle &\le -\varepsilon \|u_T\|, \\ \langle u_T, M_-^*\Delta(M_+^{-1}u_T) \rangle &\ge 0, \end{aligned} \tag{1.21}$$

for all $u \in \mathbf{L}_{2e}^m[0,\infty)$ and for all $T \ge 0$.

Proof: Let $u \in \mathbf{L}_{2e}^m[0,\infty)$. Then,

$$\begin{aligned} \langle u_T, M_+G(M_-^*)^{-1}u_T \rangle &= \langle M_-^*v, M_+Gv \rangle \\ &= \langle v, MGv \rangle \le -\varepsilon \|(M_-^*)^{-1}\|^2 \|u_T\|^2. \end{aligned}$$

This follows since $v = (M_-^*)^{-1}u_T \in \mathbf{L}_2^m[0,\infty)$ and from the first condition in (1.20). In the same way we get

$$\langle u_T, M_-^*\Delta(M_+^{-1}u_T)\rangle = \langle M_+v, M_-^*\Delta(v)\rangle = \langle v, M^*\Delta(v)\rangle \geq 0,$$

where $v = M_+^{-1}u_T \in \mathbf{L}_2^m[0,\infty)$. □

Consider now system S_3 in Figure 1.6. Stability and well-posedness of system S_1 and S_3 are equivalent conditions. This follows since all the multipliers in S_3 are bounded and causal. We arrive at the multiplier theorem below by applying the passivity theorem to system S_3. The conditions in the passivity theorem follow from the assumptions in the theorem statement and from Lemma 1.6.

THEOREM 1.4—MULTIPLIER THEOREM
Assume that

(*i*) the feedback interconnection of G and Δ is well-posed,

(*ii*) Δ satisfies the IQC defined by

$$\Pi(j\omega) = \begin{bmatrix} 0 & M^* \\ M & 0 \end{bmatrix}, \tag{1.22}$$

(*iii*) M can be factored into $M = M_-M_+$, where M_+, M_-^* and their inverses are causal and bounded,

(*iv*) there exists $\varepsilon > 0$ such that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \leq -\varepsilon I, \quad \forall \omega \in \mathbf{R}.$$

Then the interconnection of G and Δ is stable. □

REMARK 1.9
If we compare this result with the corresponding result obtained with Theorem 1.1 we see that the factorization condition is not needed in the IQC framework. The price paid for this is that well-posedness is required for every feedback interconnection of G and $\tau\Delta$, when $\tau \in [0,1]$. This condition is in most applications weak. Note that $\tau\Delta$ satisfies the IQC defined by (1.22) for every $\tau \in [0,1]$. □

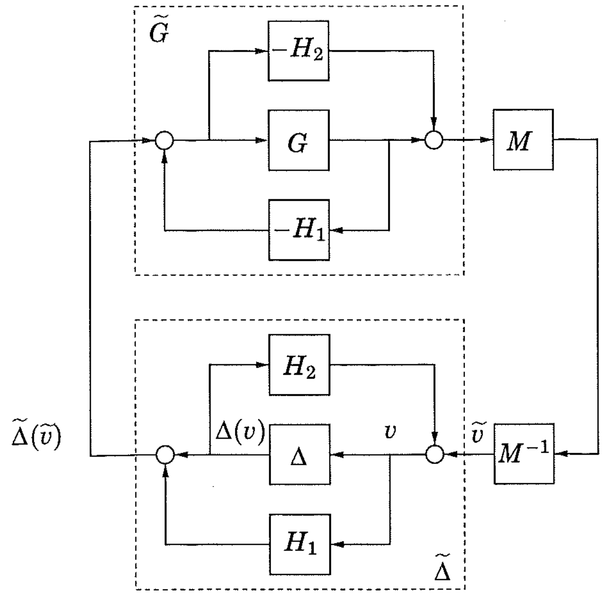

Figure 1.7 Loop transformations can be used to transform Δ into a new perturbation $\widetilde{\Delta}$ that is suitable for application of the multiplier theorem.

REMARK 1.10
The formulation of the multiplier theorem is somewhat different compared to the corresponding formulation in Zames and Falb (1968) and Desoer and Vidyasagar (1975). In particular, they state the stability condition in terms of signals in $\mathbf{L}_2^m(-\infty, \infty)$. This means that stability for any of the systems S_1 through S_3 implies stability of the others. Our definition of stability in Definition 1.1 involves signals in $\mathbf{L}_{2e}^m[0, \infty)$ and we can only have equivalence for stability of S_1 and S_3. Moreover, they define the soft IQCs in terms of signals in $\mathbf{L}_2^m(-\infty, \infty)$. However, this can be shown to be equivalent to the corresponding definitions in terms of signals in $\mathbf{L}_2^m[0, \infty)$, see Theorem 3.1 in Megretski and Treil (1993). □

It is often necessary to transform the feedback loop in order to obtain a system that is suitable for application of the multiplier theorem. Figure 1.7 shows such a loop transformation. Here H_1 and H_2 are bounded causal linear operators. We assume that the loop transformation is well-posed in the sense that the operators

$$\widetilde{G} = (G - H_2)(I + H_1 G)^{-1} \quad \text{and} \quad \widetilde{\Delta} = (\Delta + H_1)(I - H_2 \Delta)^{-1}$$

are well-defined on $\mathbf{L}_{2e}^m[0,\infty)$. We can formulate the following loop transformation result.

PROPOSITION 1.1—LOOP TRANSFORMATION
Assume that

(*i*) the feedback interconnection of G and Δ is well-posed,

(*ii*) Δ satisfies the IQC defined by

$$\Pi = \begin{bmatrix} I & -H_2 \\ H_1 & I \end{bmatrix}^* \begin{bmatrix} 0 & M^* \\ M & 0 \end{bmatrix} \begin{bmatrix} I & -H_2 \\ H_1 & I \end{bmatrix}, \qquad (1.23)$$

where the transformation operator

$$\begin{bmatrix} I & -H_2 \\ H_1 & I \end{bmatrix}$$

is invertible on $\mathbf{L}_2^m[0,\infty)$,

(*iii*) M can be factored into $M = M_- M_+$, where M_+, M_-^* and their inverses are causal and bounded,

(*iv*) there exists $\varepsilon > 0$ such that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \le -\varepsilon I, \quad \forall \omega \in \mathbf{R}.$$

Then the feedback interconnection of G and Δ is stable.

Proof: We need to show that $\widetilde{\Delta}$ and $\widetilde{G}$ satisfy condition (*ii*) and (*iv*) in Theorem 1.4. Let us verify condition (*ii*). We notice that

$$\begin{bmatrix} \widetilde{v} \\ \widetilde{\Delta}(\widetilde{v}) \end{bmatrix} = \begin{bmatrix} I & -H_2 \\ H_1 & I \end{bmatrix} \begin{bmatrix} v \\ \Delta(v) \end{bmatrix},$$

where the notation refers to Figure 1.7. The invertibility of the transformation operator implies that $\widetilde{\Delta}$ is well-defined. It remains to show that assumption (*ii*) in the proposition implies (*ii*) in Theorem 1.4. This follows since

$$2\left\langle \widetilde{v}, M^*\widetilde{\Delta}(\widetilde{v}) \right\rangle = \left\langle \begin{bmatrix} v \\ \Delta(v) \end{bmatrix}, \Pi \begin{bmatrix} v \\ \Delta(v) \end{bmatrix} \right\rangle \ge 0,$$

for all v and hence for all $\widetilde{v}$ in $\mathbf{L}_2^m[0,\infty)$. Condition (*iv*) is verified in a similar way. □

The invertibility condition on the transformation operator and the factorization condition on M is not needed for the corresponding result derived in the IQC framework. The proposition also indicates a very fruitful approach to obtain multipliers for the IQC framework. Loop transformations and multipliers from the classical theory can be used to obtain the IQC multiplier in (1.23). Hence, it is possible to include loop transformations in the IQC multipliers. We illustrate with a simple example.

EXAMPLE 1.7
Consider a nonlinearity satisfying the sector condition $\alpha x^2 \le \varphi(x,t)x \le \beta x^2$ for all $(x,t) \in \mathbf{R} \times \mathbf{R}^+$. We assume that $\beta \ne 0$. In an application of the passivity theorem to a system with such a nonlinearity we use a loop transformation to obtain a resulting nonlinearity $\widetilde{\varphi} \in \text{sector}[0,\infty)$. For this we use $H_1 = -\alpha$ and any $H_2 > 1/\beta$ in Figure 1.7. Note that the choice $H_2 = 1/\beta$ does not guarantee invertibility of $1 - H_2\varphi$. An application of Proposition 1.1 results in the circle criterion.

For the IQC framework we use the multiplier

$$\Pi(j\omega) = \begin{bmatrix} 1 & -\beta^{-1} \\ -\alpha & 1 \end{bmatrix}^T \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -\beta^{-1} \\ -\alpha & 1 \end{bmatrix} = \begin{bmatrix} -2\alpha & 1+\alpha/\beta \\ 1+\alpha/\beta & -2/\beta \end{bmatrix}.$$

Note that we use $H_2 = 1/\beta$. It is straightforward to see that this multiplier defines a valid IQC for φ. □

The next example shows how the system in Example 1.6 can be treated with Proposition 1.1. The example is based on the ideas in Balakrishnan (1995), Balakrishnan *et al.* (1994), Ly *et al.* (1994), Helmersson (1995b), and Goh and Safonov (1995).

EXAMPLE 1.8
We here assume that the dynamic and parametric uncertainty in Example 1.6 satisfies the somewhat stronger conditions $\|\Delta_1\|_\infty \le 1-\rho$ and $\delta \in [-1+\rho, 1-\rho]$, for some small $\rho > 0$. Let $\Delta = \text{diag}(\Delta_1, \delta)$ and let us use Proposition 1.1 with $H_1 = I$, and $H_2 = I$. The resulting transformed multiplier $\widetilde{\Delta} = (I+\Delta)(I-\Delta)^{-1}$ can be shown to be positive semidefinite, i.e., $\widetilde{\Delta}(j\omega) + \widetilde{\Delta}(j\omega)^* \ge 0$. Let us use the multiplier $M = \text{diag}(M_1 I_{m_1}, M_2) \in \mathbf{RL}_\infty^{m\times m}$, where $M_1 \in \mathbf{RL}_\infty^{1\times 1}$ and $M_2 \in \mathbf{RL}_\infty^{m_2\times m_2}$ satisfy the conditions.

$$\begin{aligned} M_1(j\omega) &= \overline{M_1(j\omega)} > 0, \\ M_2(j\omega) + M_2(j\omega)^* &> 0, \end{aligned}$$

for all $\omega \in [0,\infty]$. In this case it is easy to see that

$$M(j\omega)^*\widetilde{\Delta}(j\omega) + \widetilde{\Delta}(j\omega)^*M(j\omega) \ge 0.$$

Furthermore, the conditions on M_1 and M_2 imply that $M(j\omega) + M(j\omega)^* > 0$. This means that there exists a factorization of M according to condition *(iii)* in Proposition 1.1, see, for example, Lemma 2.2 in Helmersson (1995b). Hence, we can use the multiplier

$$\Pi(j\omega) = \begin{bmatrix} M + M^* & M^* - M \\ M - M^* & -(M + M^*) \end{bmatrix}.$$

This is essentially the same multiplier as in Example 1.6. To see this we let $M_1 = x/2$ and $M_2 = (X - Y)/2$, where x, X, and Y are specified as in Example 1.6 except that x and X now should satisfy strict inequalities.

□

We have seen that the IQC formalism and the classical input–output approach to stability analysis are closely related. However the IQC approach has several advantages. In particular, it is easy to combine multipliers and the use of noncausal multipliers has been simplified. The factorization conditions on noncausal multipliers in the input–output framework is in general restrictive. It can also be seen from the discussion above that it generally will be quite cumbersome to obtain multiplier descriptions of complicated Δ structures in the input–output framework. The conditions on the loop transformation and the multipliers can be hard to verify.

To conclude we list the following advantages of the IQC method compared to the input–output method.

- Invertibility and factorizability of the multipliers are not needed.
- It is easy to combine multipliers.
- Non-square systems are treated in a natural way.

1.7 A Lyapunov Function Approach

A large number of results in stability and robustness analysis have been derived by use of a particular Lyapunov function technique. This method has been particularly useful in absolute stability theory during the 1960s and in LMI-based robustness analysis during the 1990s. We will discuss this Lyapunov technique in a notation that allows easy comparison with the IQC framework. The potential and the limitations of the method will be pointed out in the end of the section.

Let us consider the system

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad x(0) = x_0, \\ y &= Cx, \\ u &= \Delta(y), \end{aligned} \tag{1.24}$$

where $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$, and $C \in \mathbf{R}^{l\times n}$. Here A is assumed to be Hurwitz and the operator $\Delta : \mathbf{L}_{2e}^{l}[0,\infty) \to \mathbf{L}_{2e}^{m}[0,\infty)$ is bounded and causal. In most applications of absolute stability theory Δ is memoryless. It could for example be a diagonal structure consisting of nonlinearities and time-varying parameters.

We assume that Δ satisfies the hard IQC defined by the multiplier $\Pi = \Psi^* M \Psi$, where Ψ is a stable rational transfer function and where M is a symmetric matrix. In other words, the condition in (1.16) holds with this multiplier. Let Ψ have the realization $\Psi(s) = C_\Psi(sI - A_\Psi)^{-1}B_\Psi + D_\Psi$, and let us define the matrices

$$\begin{aligned} \widehat{C} &= \begin{bmatrix} C \\ 0 \end{bmatrix}, \qquad \widehat{D} = \begin{bmatrix} 0 \\ I \end{bmatrix}, \qquad A_\Phi = \begin{bmatrix} A_\Psi & B_\Psi \widehat{C} \\ 0 & A \end{bmatrix}, \\ B_\Phi &= \begin{bmatrix} B_\Psi \widehat{D} \\ B \end{bmatrix}, \quad C_\Phi = \begin{bmatrix} C_\Psi & D_\Psi \widehat{C} \end{bmatrix}, \quad D_\Phi = D_\Psi \widehat{D}. \end{aligned} \tag{1.25}$$

It can be verified that this is a realization of the transfer function

$$\Phi(s) = \Psi(s) \begin{bmatrix} G(s) \\ I \end{bmatrix},$$

where $G(s) = C(sI - A)^{-1}B$. The following proposition will be derived by using the Lyapunov technique from absolute stability theory. Well-posedness means in this proposition that there exists an absolutely continuous solution x to (1.24) on any finite time interval.

Proposition 1.2
Assume that

(*i*) the system in (1.24) is well-posed,

(*ii*) Δ satisfies the hard IQC in (1.16),

(*iii*) A_Φ, B_Φ, C_Φ, and D_Φ are defined as in (1.25),

(*iv*) there exists $P = P^T > 0$, with $\dim P = \dim A_\Psi + n$, such that

$$\begin{bmatrix} A_\Phi & B_\Phi \\ I & 0 \\ C_\Phi & D_\Phi \end{bmatrix}^T \begin{bmatrix} 0 & P & 0 \\ P & 0 & 0 \\ 0 & 0 & M \end{bmatrix} \begin{bmatrix} A_\Phi & B_\Phi \\ I & 0 \\ C_\Phi & D_\Phi \end{bmatrix} \le -\varepsilon I \tag{1.26}$$

holds for some $\varepsilon > 0$.

Then the the state vector x in (1.24) is bounded and $x(t) \to 0$ as $t \to \infty$.

Proof: Let us define

$$V_2(t) = \int_0^t w(\tau)^T M w(\tau) d\tau.$$

The convolution in (1.17) can be performed in terms of the following differential equation

$$\begin{aligned} \dot{x}_\psi &= A_\Psi x_\psi + B_\Psi \begin{bmatrix} y \\ \Delta(y) \end{bmatrix}, \quad x_\psi(0) = 0, \\ w &= C_\Psi x_\psi + D_\Psi \begin{bmatrix} y \\ \Delta(y) \end{bmatrix}. \end{aligned}$$

The IQC condition in (1.16) implies that $V_2(t) \geq 0$ for all $t \geq 0$.

Let $x_\phi = [\, x_\psi^T \quad x^T \,]^T$, and $V_1(t) = x_\phi^T P x_\phi$. The function $V(t) = V_1(t) + V_2(t)$, satisfies the following two conditions

(*a*) $V(t) \geq 0$, for all $t \geq 0$,

(*b*) $\dot{V}(t) \leq -\varepsilon(|x_\phi(t)|^2 + |u(t)|^2)$, for all $t \geq 0$,

where the last follows from (*iv*) in the statement of the proposition. If we integrate (*b*) over the interval $[0, T]$ and use (*a*) then we obtain

$$\int_0^T (|x_\phi|^2 + |u|^2) dt \leq \frac{1}{\varepsilon} V(0), \quad \forall T \geq 0$$

This implies that $x_\phi \in \mathbf{L}_2^n[0,\infty)$ and $u \in \mathbf{L}_2^m[0,\infty)$. From the system equation in (1.24) we see that also $\dot{x}_\phi \in \mathbf{L}_2^{n+n_\psi}[0,\infty)$, where $n_\psi = \dim A_\psi$. Hence, by Lemma 1.2 it follows that x_ϕ is bounded and converges to zero as $t \to \infty$. This proves the proposition. □

Here we list some comments on the Lyapunov technique:

- The perturbation Δ has been memoryless and Lipschitz continuous in most applications of this method. Well-posedness is in this case obvious.

- It is not always possible to ensure that the integral for the hard IQC in (1.16) is positive at all time instants $T \geq 0$. However, it is enough to ensure that it is lower bounded. This implies that the function V_2 in the proof of Proposition 1.2 now satisfies $V_2(t) \geq -\rho$ for some $\rho > 0$. The proof is easily adapted to this situation.

- When the Popov criterion is used in connection with this Lyapunov technique then Ψ is nonproper. This does not add any serious complications. Let us assume that we want to describe a memoryless nonlinearity φ in the sector $[0,\infty)$ with a Popov multiplier. The Popov part of the IQC condition in (1.16) will be an integral on the form

$$\lambda \int_0^T \varphi(y)\dot{y}\,dt = \lambda \int_0^{y(T)} \varphi(\sigma)d\sigma - \lambda \int_{y_0}^0 \varphi(\sigma)d\sigma.$$

  If we assume that the Popov parameter λ is nonnegative then the first term on the right hand side is positive for all $T \geq 0$ and the last is lower bounded by a constant that depends on the initial output $y_0 = y(0)$ and the maximal gain of the nonlinearity. This shows that lower boundedness of the hard IQC is ensured only if $\lambda \geq 0$. We extend the IQC framework to include nonproper Popov multipliers in Chapter 2. The use of soft IQCs will allow us to consider also negative Popov parameters.

- The use of hard IQCs and Lyapunov techniques along the lines of Proposition 1.2 have proved to be useful to derive conditions for several interesting robust performance measures. As an example, we mention the generalized $\mathbf{H}_2$-performance that has been treated in, for example, Rotea (1993) and Scherer (1995).

The Lyapunov technique has two major limitations:

- Only hard IQCs are considered which in general limits the number of multipliers that can be used for the analysis.

- It can be conservative to require positive definiteness of the matrix P that appear in the (1.26) in Proposition 1.2. In other words, it is not always the case that this LMI is equivalent to the frequency domain condition

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Psi(j\omega)^* M \Psi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0, \quad \forall \omega \in [0,\infty].$$

  We will in Chapter 3 show how robustness analysis with IQCs leads to LMI conditions on the form in (1.26) but now with no definiteness assumption on P.

1.8 Main Ideas and Summary of the Thesis

The main idea of this thesis is to develop tools for practical application of the IQC approach for robustness analysis. We will in particular focus on the problem of finding the multipliers that appear in the robustness criterion by computational methods. The IQC framework is particularly suitable for this since it allows easy combination of different robustness criteria to obtain the strongest possible result.

We illustrate in Figure 1.8 the steps taken for a typical example. We consider disturbance rejection for the cascade controller in the upper part of the figure, where φ represents a saturation nonlinearity and Δ_1 represents modeling error. It is assumed that the controllers K_1 and K_2 are designed such that the nominal system is stable and that K_1, K_2, G_1, and G_2 are linear and stable. The following four steps are taken in the analysis.

Step 1. Transform the system into the nominal form in second part of the figure. Here $\Delta = \mathrm{diag}(\Delta_1, \varphi)$ and G is the stable transfer function matrix

$$G = \begin{bmatrix} 0 & \dfrac{G_1 G_2 K_2}{I + G_2 K_2} & \dfrac{G_1 G_2}{I + G_2 K_2} & G_1 \\ -K_1 & -\dfrac{K_1 G_2 G_2 K_2}{I + G_2 K_2} & -\dfrac{K_1 G_1 G_2}{I + G_2 K_2} & -K_1 G_1 \\ I & \dfrac{G_1 G_2 K_2}{I + G_2 K_2} & \dfrac{G_1 G_2}{I + G_2 K_2} & G_1 \end{bmatrix}$$

The stability of G can be deduced from the stability of K_1, K_2, G_1, G_2, and the nominal system.

Step 2. Find multiplier descriptions of the perturbation Δ, the spectral properties of the load disturbances, and the performance specification. We do this by choosing appropriate convex cones of multipliers from the *IQC library*. These cones are then combined into a complete multiplier description in terms of the convex cone denoted $\Pi_{\mathrm{Tot}}(\gamma)$. Here the parameter γ denotes the robustness criterion, which in this case is a performance measure or rather a measure of the disturbance rejection.

The IQC library is the collection of all known multipliers used for robustness analysis.

Step 3. Formulate an optimization problem for the computation of the robustness margin. This is in general an infinite-dimensional optimization problem and suboptimal solutions must be considered. This can be done by restricting the original problem to a finite-dimensional subspace.

Step 4. Solve the finite-dimensional optimization problem obtained in step 3 with an LMI solver.

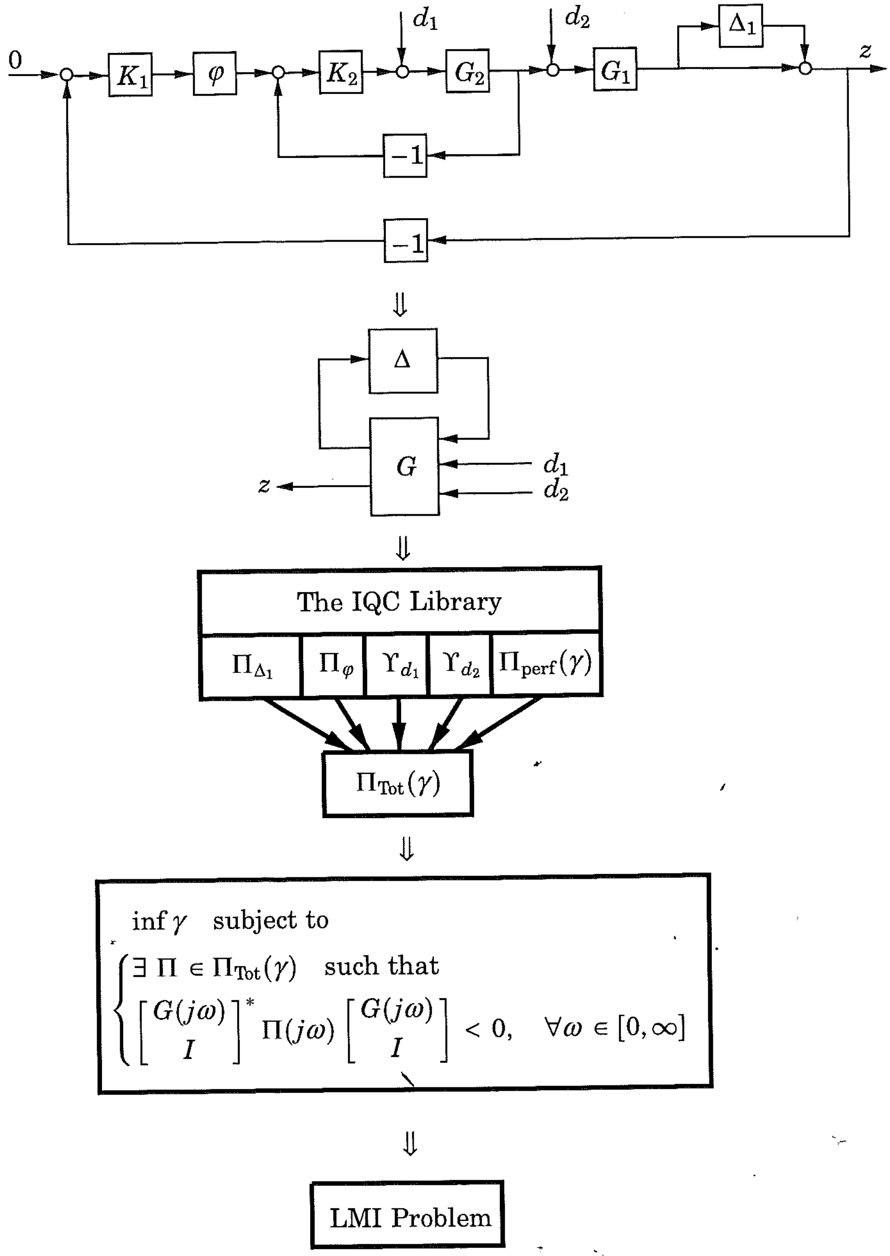

Figure 1.8 The figure shows the steps taken in robustness analysis via IQCs.

The ultimate goal would be to derive a fully automatic procedure for the robustness analysis along the lines of Figure 1.8. This thesis will provide results in this direction together with relevant theoretical justifications. The main contributions of the thesis are

- We show how Popov multipliers can be used in the IQC framework. Popov multipliers are defined in terms of a small number of parameters and are therefore easy to compute. They also add significantly to the accuracy of the stability bounds. This is the topic of Chapter 2.
- We discuss robust performance analysis where multipliers are used to describe signal specifications. This is done in the first part of Chapter 3.
- The cone $\Pi_{\mathrm{Tot}}(\gamma)$ obtained in Figure 1.8 is in most cases infinite-dimensional. We suggest a format for finite-dimensional restrictions of this cone in Chapter 3. An approximate solution to the robustness analysis can then be obtained by use of LMI methods.
- In Chapter 4 and 5 we use duality theory to obtain bounds on the conservatism that is introduced by approximating the robustness problem.

2

Popov Multipliers

Abstract

It is shown how the Popov criterion can be used in stability analysis based on Integral Quadratic Constraints (IQC). A consequence of this is that the popov criterion can be combined with a stability criterion for slope restricted nonlinearities developed by Zames and Falb. An example shows that the combination of these two criteria is useful in applications. Another consequence is that several recent results on stability analysis with Popov multipliers for systems with parametric uncertainty can be extended by the formulation in the IQC framework.

2.1 Introduction

The framework for using integral quadratic constraints (IQCs) in stability analysis was outlined for the linear case by Megretski (1993b) and generalized to treat nonlinear operators in Rantzer and Megretski (1994) and Megretski and Rantzer (1995). In this chapter we show how to use Popov multipliers in stability analysis based on IQCs.

The classical Popov criterion states that a positive feedback interconnection of a stable linear system G and a memoryless nonlinearity in the sector $[0, \infty)$ is stable if there exists $\varepsilon > 0$ such that

$$\operatorname{Re}\left[(1 + j\omega\lambda)G(j\omega)\right] < 0, \quad \forall\omega \in [0, \infty]. \tag{2.1}$$

A result of this chapter is that the stability criterion for odd slope restricted nonlinearities in Zames and Falb (1968) can be generalized by combining it with the Popov criterion. The resulting stability condition becomes

$$\operatorname{Re}\left[(1 + j\omega\lambda + H(j\omega))G(j\omega)\right] < 0, \quad \forall\omega \in [0, \infty],$$

where H is a noncausal multiplier that will be described later. An example will show that there exists systems for which it is useful to use this generalized stability result.

We consider systems consisting of a nominal linear time-invariant operator G in a positive feedback interconnection with a bounded causal perturbation Δ. The idea behind the IQC approach to stability analysis is to find descriptions of Δ in terms of bounded and Hermitian valued matrix functions Π that define valid IQCs in the sense that

$$\int_{-\infty}^{\infty} \begin{bmatrix} \widehat{y}(j\omega) \\ \widehat{\Delta(y)}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \widehat{y}(j\omega) \\ \widehat{\Delta(y)}(j\omega) \end{bmatrix} d\omega \geq 0,$$

for all square integrable y. The matrix function Π is called multiplier.

The IQC methodology gives a unified approach for multiplier-based stability analysis that has several advantages compared to the classical framework described in, for example, Desoer and Vidyasagar (1975), Narendra and Taylor (1973), Willems (1971a), Zames (1966b), and Zames and Falb (1968). It should be noted that the term multiplier is used in a somewhat different meaning in the classical work where it denotes a device that is used to multiply the operators in the feedback loop in order to make them look passive or contractive. The multiplier Π can be used to collect the multipliers from the classical framework in a structure that is suitable for defining integral quadratic constraints.

Some of the most important advantages with the IQC methodology are the following:

- Noncausal multipliers are easy to use since there is no need for factorization conditions.
- It is simple to combine different multipliers that describe a certain operator. This follows because conic combinations of multipliers that define valid IQCs for a specific operator still give valid IQCs for this particular operator.
- There is generally no need to consider the multipliers in terms of loop transformations.

The cost for the increased flexibility is the need for a stronger but still very reasonable assumption on well-posedness of the system.

We will derive a stability result along the lines of Megretski and Rantzer (1995) that allow us to combine the bounded multipliers with Popov multipliers on the form

$$\Pi_P(j\omega) = \begin{bmatrix} 0 & -j\omega\Lambda^T \\ j\omega\Lambda & 0 \end{bmatrix}, \tag{2.2}$$

where Λ is a real matrix. These multipliers are used to define constraints in terms of the integral

$$\int_0^\infty v^T \Lambda \dot{y} dt, \tag{2.3}$$

where $v = \Delta(y)$ and where it is assumed that both $\dot{y}$ and v are square integrable. It is thus necessary that y is differentiable. This is the case if the nominal plant G is strictly proper. However, strict properness is in general only required for the output channels analyzed with Popov multipliers.

The Popov criterion in (2.1) can now be rewritten as

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \begin{bmatrix} 0 & 1 - j\omega\lambda \\ 1 + j\omega\lambda & 0 \end{bmatrix} \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0, \quad \forall \omega \in [0, \infty].$$

Here we added the passivity multiplier to the scalar version of the Popov multiplier in (2.2). The parameter λ is no longer restricted to be positive. This is, as it was noted in Megretski and Rantzer (1995), a consequence of the definition of the IQC as an integral over the positive time axis.

The Popov parameter Λ is in general a full matrix. This is for example the case in Popov multipliers for parametric uncertainty and slowly time-varying parameters. With the results of this chapter it is possible to extend recent stability criteria for systems with parametric uncertainty in Feron *et al.* (1995) and Bernstein *et al.* (1995) into the IQC framework. This is useful in the sense that we get a larger class of multipliers to use for the stability analysis. The Popov parameter is defined by a small number of variables and is thus computationally cheap to determine. This makes Popov multipliers useful in stability analysis of large complex systems.

New stability criteria for systems with slowly time-varying parameters and with parametric uncertainty are given as applications of our main result.

2.2 The Popov IQC

The standard definition of integral quadratic constraints in terms of a bounded matrix function excludes easy use of the Popov criterion. However, it is possible to extend the definition to include Popov multipliers on the form

$$\Pi_P(j\omega) = \begin{bmatrix} 0 & -j\omega\Lambda^T \\ j\omega\Lambda & 0 \end{bmatrix}, \tag{2.4}$$

where $\Lambda \in \mathbf{R}^{m\times l}$. This multiplier will be used to define constraints involving the integral in (2.3). The reason for defining the Popov multiplier in terms of a time domain integral is twofold. Firstly, the Popov descriptions are most naturally derived in the time domain. Secondly, since we consider signals in $\mathbf{L}_2[0,\infty)$ the relation $\mathcal{F}\{\dot{y}\} = j\omega\mathcal{F}\{y\}$ holds only if $y(0) = 0$. The Fourier transform of y is here denoted $\mathcal{F}\{y\}$.

In order to combine Popov multipliers with bounded multipliers we need a definition that includes both. We suggest the following definition.

DEFINITION 2.1—INTEGRAL QUADRATIC CONSTRAINT
Let $\Pi_B : j\mathbf{R} \to \mathbf{C}^{(l+m)\times(l+m)}$ be a bounded and measurable function that takes Hermitian values and let Π_P be the Popov multiplier in (2.4).

We say that Δ satisfies the IQC defined by $\Pi = \Pi_B + \Pi_P$ if there exists a positive constant γ such that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \begin{bmatrix} \widehat{y}(j\omega) \\ \widehat{v}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \widehat{y}(j\omega) \\ \widehat{v}(j\omega) \end{bmatrix} d\omega + \int_0^{\infty} 2v^T\Lambda\dot{y}dt \geq -\gamma|y_0|^2, \quad (2.5)$$

for all y and $v = \Delta(y)$ such that $y, \dot{y} \in \mathbf{L}_2^l[0,\infty)$ and $v \in \mathbf{L}_2^m[0,\infty)$. We use the notation $y_0 = y(0)$. □

The differentiability condition on all components of y is restrictive in many applications. We will therefore extend our definition, in Section 2.4. We also note that the first term in this definition could equivalently be defined in terms of the time domain integral

$$\int_0^{\infty} \begin{bmatrix} y \\ v \end{bmatrix}^T (\pi_B * \begin{bmatrix} y \\ v \end{bmatrix})dt,$$

where π_B is the weighting function corresponding to Π_B (assuming it exists).

As a final remark we note that in most instances Popov multipliers and bounded multipliers define valid IQCs on their own. It could therefore be argued that it would be more natural to give separate definitions. New multipliers could then be obtained from conic combinations $\Pi = \alpha\Pi_B + \beta\Pi_P$, where $\alpha, \beta \geq 0$. However, Example 2.3 shows that it is sometimes necessary to combine bounded multipliers with Popov multipliers in order to obtain valid IQCs. It is then the combination $\Pi = \Pi_B + \Pi_P$ that should be regarded as multiplier in the stability analysis.

Next follows some examples of the use of Popov multipliers for describing nonlinearities, uncertain parameters and slowly time-varying parameters.

EXAMPLE 2.1—MEMORYLESS NONLINEARITY
Let $\varphi : \mathbf{R} \to \mathbf{R}$ be a measurable function satisfying $\varphi(0) = 0$ and a sector condition $\alpha x^2 \leq \varphi(x)x \leq \beta x^2$, where $-\infty < \alpha < \beta < \infty$. Then φ satisfies the IQC defined by the Popov multiplier

$$\Pi_P(j\omega) = \begin{bmatrix} 0 & -j\omega\lambda \\ j\omega\lambda & 0 \end{bmatrix},$$

where $\lambda \in \mathbf{R}$. This follows since

$$2\lambda \lim_{\tau\to\infty} \int_0^\tau \varphi(y)\dot{y}dt = 2\lambda \lim_{\tau\to\infty} \int_{y_0}^{y(\tau)} \varphi(\sigma)d\sigma = -2\lambda \int_0^{y_0} \varphi(\sigma)d\sigma \geq -\gamma|y_0|^2,$$

for all $y, \dot{y} \in \mathbf{L}_2^m[0,\infty)$. We can use $\gamma = |\lambda|\max(|\alpha|,|\beta|)$. The second equality follows since Lemma 1.2 in Chapter 1 implies that $y(\tau) \to 0$ as $\tau \to \infty$.
□

The next example shows that the Λ parameter of the Popov multiplier can be a full matrix.

EXAMPLE 2.2—UNCERTAIN PARAMETER
Let $\Delta = \delta I$, where δ is a constant but uncertain real-valued parameter with $\delta \in [-1, 1]$. Then Δ satisfies the IQC defined by

$$\Pi_P(j\omega) = \begin{bmatrix} 0 & -j\omega\Lambda \\ j\omega\Lambda & 0 \end{bmatrix},$$

where $\Lambda = \Lambda^T \in \mathbf{R}^{m\times m}$. This follows since

$$\int_0^\infty 2\delta y^T \Lambda \dot{y} dt = \int_0^\infty \delta (y^T\Lambda y)' dt = \lim_{\tau\to\infty} \delta \left[y^T \Lambda y\right]_0^\tau = -\delta y_0^T \Lambda y_0 \geq -\gamma|y_0|^2,$$

for any $y, \dot{y} \in \mathbf{L}_2^m[0,\infty)$ if we let $\gamma = \overline{\sigma}(\Lambda)^2$. This Popov multiplier can be combined with the bounded multipliers that can be derived from Fan *et al.* (1991). This results in multipliers on the form

$$\Pi(j\omega) = \begin{bmatrix} X(j\omega) & Y(j\omega)^* - j\omega\Lambda \\ Y(j\omega) + j\omega\Lambda & -X(j\omega) \end{bmatrix},$$

where $X(j\omega) = X(j\omega)^* \geq 0$, $Y(j\omega) = -Y(j\omega)^*$, and $\Lambda = \Lambda^T$.

The example can easily be generalized to diagonal structures of uncertain parameters as in Feron *et al.* (1995) or to more general parametric uncertainty as in Bernstein *et al.* (1995). See also Theorem 2.4 for a generalization.
□

The Popov multiplier must sometimes be combined with a bounded multiplier in order to satisfy condition (2.5) in Definition 2.1. This is the case in the next example.

EXAMPLE 2.3—SLOWLY TIME-VARYING PARAMETER
Let $\Delta(t) = \delta(t)I$ be a slowly time-varying parameter with $\delta(t) \in [-1, 1]$ and $\dot{\delta}(t) \in [-b, b]$. Then Δ satisfies the IQC defined by the multiplier $\Pi = \Pi_B + \Pi_P$, where

$$\Pi_B(j\omega) = \begin{bmatrix} b\Lambda & 0 \\ 0 & 0 \end{bmatrix}, \qquad \Pi_P(j\omega) = \begin{bmatrix} 0 & -j\omega\Lambda \\ j\omega\Lambda & 0 \end{bmatrix},$$

and where $\Lambda = \Lambda^T \geq 0$. This follows since integration by parts gives

$$\begin{aligned} \int_0^\infty b y^T \Lambda y dt + \int_0^\infty 2\delta y^T \Lambda \dot{y} dt &= \lim_{\tau\to\infty} \left[\delta y^T \Lambda y\right]_0^\tau + \int_0^\infty (b - \dot{\delta}) y^T \Lambda y dt \\ &\geq -\delta(0) y_0^T \Lambda y_0 \geq -\overline{\sigma}(\Lambda)^2 |y_0|^2, \end{aligned}$$

for any $y, \dot{y} \in \mathbf{L}_2^m[0,\infty)$. Note that $(b - \dot{\delta}) y^T \Lambda y$ is positive for all $t \geq 0$. This multiplier can be used in combination with other multipliers for slowly time-varying parameters. We generalize this multiplier in Theorem 2.3.

□

2.3 A Stability Result

We will here present a stability theorem that uses the multiplier descriptions obtained from Definition 2.1. We consider the system

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad x(0) = x_0, \\ y &= Cx, \\ u &= \Delta(y) + g, \end{aligned} \tag{2.6}$$

where $A \in \mathbf{R}^{n\times n}$ is Hurwitz, $B \in \mathbf{R}^{n\times m}$ and $C \in \mathbf{R}^{l\times n}$. It is further assumed that $g \in \mathbf{L}_{2e}^m[0,\infty)$ and that $\Delta : \mathbf{L}_2^l[0,\infty) \to \mathbf{L}_2^m[0,\infty)$ is a causal and bounded operator. The system in (2.6) can equivalently be represented as the feedback system in Figure 2.1, where $G(s) = C(sI - A)^{-1}B \in \mathbf{RH}_\infty^{l\times m}$, and where $f(t) = Ce^{At}x_0\theta(t)$ represents the response corresponding to the initial condition.

Before we formulate the main stability result, we define well-posedness and stability of the system in (2.6). These definitions are similar to the ones in Megretski and Rantzer (1995).

DEFINITION 2.2—WELL-POSEDNESS
The system in (2.6) is *well-posed* if for any initial condition x_0 and for any input $g \in \mathbf{L}_{2e}^m[0,\infty)$ there exists a solution (x, u) such that $(x, \dot{x}, u) \in \mathbf{L}_{2e}^n[0,\infty) \times \mathbf{L}_{2e}^n[0,\infty) \times \mathbf{L}_{2e}^m[0,\infty)$. Furthermore, the map from g to (x, u) should be causal. □

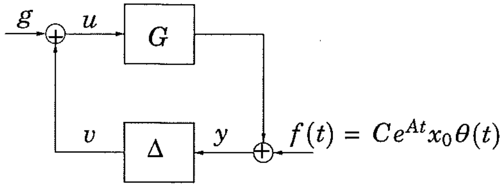

Figure 2.1 Block diagram representation of the system in (2.6). Here f corresponds to the response of the initial condition and $G(s) = C(sI - A)^{-1}B$.

DEFINITION 2.3—STABILITY
The system in (2.6) is *stable* if it is well-posed and if there exist constants $c > 0$ and $\rho > 0$ such that

$$\int_0^T (|y|^2 + |u|^2)dt \le \rho|x_0|^2 + c\int_0^T |g|^2 dt,$$

for all $T > 0$ and for arbitrary $x_0 \in \mathbf{R}^n$ and $g \in \mathbf{L}_{2e}^m[0,\infty)$. □

The next theorem corresponds to the main stability result in Megretski and Rantzer (1995) with the exception that it allows the use of Popov multipliers in the stability criterion. The price paid for this increased flexibility is the additional assumption on strict properness of the nominal system. We will give a generalization of this theorem in the next section where strict properness is necessary only in the channels that correspond to the part of Δ described by Popov multipliers.

THEOREM 2.1
Assume that

(*i*) for all $\tau \in [0,1]$, the system (2.6) with Δ replaced by $\tau\Delta$ is well-posed,

(*ii*) for all $\tau \in [0,1]$, $\tau\Delta$ satisfies the IQC defined by $\Pi = \Pi_B + \Pi_P$,

(*iii*) there exists $\varepsilon > 0$ such that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* (\Pi_B(j\omega) + \Pi_P(j\omega)) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \le -\varepsilon I, \quad \forall\omega \in \mathbf{R}. \quad (2.7)$$

Then the system in (2.6) is stable.

Proof: It follows from the proof of Theorem 2.2 in the next section. □

COROLLARY 2.1
If $g \in \mathbf{L}_2^m[0,\infty)$ and if the assumptions of Theorem 2.1 hold, then x is bounded and $x(t) \to 0$ as $t \to \infty$.

Proof: The conditions of the corollary imply that the system is stable. Since $g \in \mathbf{L}_2^m[0,\infty)$ and A is Hurwitz we conclude that $x, \dot{x} \in \mathbf{L}_2^n[0,\infty)$. Lemma 1.2 in Chapter 1 gives the desired result. □

REMARK 2.1
Corollary 2.1 only assures asymptotic convergence of the state x to zero with no guaranteed rate of decay. It would be more useful to verify exponential convergence. If for example $g = 0$, then the conditions in Theorem 2.1 imply exponential convergence if the operator Δ is memoryless and bounded. This follows from a result in Megretski and Rantzer (1995). □

2.4 Extension of the Stability Result

In order to use Popov multipliers we have required differentiability of y, see Definition 2.1. This enforces a condition on strict properness of the nominal transfer function. This may be overly restrictive in applications, particularly in the case when Λ is a sparse matrix. However, the problem can be overcome. If we introduce $P_\Lambda : \mathbf{R}^l \to \mathbf{R}^l$ as the orthogonal projection onto the range of Λ^T, then we can use the derivative $(P_\Lambda y)' = \frac{d}{dt}(P_\Lambda y)$ in (2.3). This works since $\mathcal{R}(\Lambda^T)^\perp = \mathcal{N}(\Lambda)$. Hence, $P_\Lambda y \perp \mathcal{N}(\Lambda)$ and only the part of y that contributes to the integral in (2.3) is differentiated. We can use the following refined IQC definition.

DEFINITION 2.4—REFINED IQC DEFINITION
Let $\Pi_B : j\mathbf{R} \to \mathbf{C}^{(l+m)\times(l+m)}$ be a bounded and measurable function that takes Hermitian values, and let Π_P be the Popov multiplier in (2.4).

We say that Δ satisfies the IQC defined by $\Pi = \Pi_B + \Pi_P$ if there exists a positive constant γ such that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\begin{bmatrix}\widehat{y}(j\omega)\\ \widehat{v}(j\omega)\end{bmatrix}^*\Pi_B(j\omega)\begin{bmatrix}\widehat{y}(j\omega)\\ \widehat{v}(j\omega)\end{bmatrix}d\omega + \int_0^\infty 2v^T\Lambda(P_\Lambda y)'dt \geq -\gamma|P_\Lambda y_0|^2,$$

for all y and $v = \Delta(y)$ for which $y, (P_\Lambda y)' \in \mathbf{L}_2^l[0,\infty)$ and $v \in \mathbf{L}_2^m[0,\infty)$. □

We illustrate this definition with an example.

EXAMPLE 2.4
Consider the diagonal operator $\Delta = \text{diag}(\varphi, \Delta_1)$. Let φ be a sector bounded nonlinearity satisfying $0 \leq \varphi(x)x \leq x^2$, for all $x \in \mathbf{R}$, and let Δ_1 be a single-input single-output dynamic uncertainty satisfying $\|\Delta_1\|_\infty \leq 1$. Then Δ satisfies the IQC defined by $\Pi = \Pi_B + \Pi_P$, where

$$\Pi_B(j\omega) = \left[\begin{array}{cc|cc} 0 & & 1 & \\ & x(j\omega) & & 0 \\ \hline 1 & & -2 & \\ & 0 & & -x(j\omega) \end{array}\right],$$

with $x(j\omega) = \overline{x(j\omega)} \geq 0$, and where

$$\Pi_P(j\omega) = \left[\begin{array}{cc|cc} 0 & & -\lambda j\omega & \\ & 0 & & 0 \\ \hline \lambda j\omega & & 0 & \\ & 0 & & 0 \end{array}\right],$$

with $\lambda \in \mathbf{R}$. In this case we have

$$P_\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

The nominal system $G \in \mathbf{RH}_\infty^{2\times 2}$ must satisfy the condition $G_{11}(\infty) = G_{12}(\infty) = 0$ in order to ensure differentiability of $P_\Lambda y$. □

Next follows the main result of this chapter. We use a more general parametrization of Δ than in Theorem 2.1. A remark after the theorem gives motivation for this parametrization.

THEOREM 2.2
Assume that there is a parametrization Δ_τ, for $\tau \in [0,1]$, of the operator Δ and a corresponding parametrized system:

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad x(0) = x_0, \\ y &= Cx + Du, \\ u &= \Delta_\tau(y) + g. \end{aligned} \tag{2.8}$$

Let $G(s) = C(sI - A)^{-1}B + D$ and assume that

(*i*) Δ_τ is causal and bounded for all $\tau \in [0,1]$, (*ii*) $\Delta = \Delta_1$,

(*iii*) for some $\kappa > 0$ we have

$$\|\Delta_{\tau_2}(y) - \Delta_{\tau_1}(y)\| \le \kappa|\tau_2 - \tau_1| \cdot \|y\|,$$

for all $y \in \mathbf{L}_2^l[0,\infty)$, and all $\tau_1, \tau_2 \in [0,1]$,

(*iv*) the system in (2.8) is stable when $\tau = 0$,

(*v*) $\Lambda G(\infty) = 0$,

(*vi*) for all $\tau \in [0,1]$, Δ_τ satisfies the IQC defined by $\Pi = \Pi_B + \Pi_P$,

(*vii*) for all $\tau \in [0,1]$, the system in (2.8) is well-posed,

(*viii*) there exists $\varepsilon > 0$ such that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* (\Pi_B(j\omega) + \Pi_P(j\omega)) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \le -\varepsilon I, \quad \forall \omega \in \mathbf{R}. \quad (2.9)$$

Then the system in (2.8) is stable for all $\tau \in [0,1]$.

Proof: The proof is given in Appendix 2.8. □

REMARK 2.2
It often possible to use the parametrization $\Delta_\tau = \tau\Delta$. Condition (*i*) – (*iv*) of the theorem statement can then be omitted. □

REMARK 2.3
An example of the more general parametrization is $\varphi_\tau(x,t) = (1-\tau)\alpha x + \tau\varphi(x,t)$. It is assumed that φ is a sector bounded nonlinearity with $\alpha x^2 \le \varphi(x,t)x \le \beta x^2$, where $0 < \alpha < \beta < \infty$. It follows from the discussion in Example 1.7 in Chapter 1 that φ satisfies the IQC defined by the multiplier

$$\Pi(j\omega) = \begin{bmatrix} -2\alpha & 1+\alpha/\beta \\ 1+\alpha/\beta & -2/\beta \end{bmatrix}.$$

It is easy to see that $\tau\varphi$ does not satisfy the IQC defined by Π when τ is small. However, $\varphi_\tau(x,t) = (1-\tau)\alpha x + \tau\varphi(x,t)$ satisfies this IQC for all $\tau \in [0,1]$. In fact, $\varphi_\tau \in \text{sector}[\alpha,\beta]$ for every $\tau \in [0,1]$. □

2.5 Applications to Systems with Parametric Uncertainty

We will in this section use Popov multipliers to obtain new stability results for systems with slowly time-varying parameters and for systems with parametric uncertainty. The first theorem extends the multiplier in Example 2.3 to give an attractive result on stability for a particular class of slowly time-varying systems.

THEOREM 2.3
Consider the system

$$\dot{x} = (A + B\Delta(t)C)x, \quad x(0) = x_0,$$

where $A \in \mathbf{R}^{n\times n}$ is Hurwitz, $B \in \mathbf{R}^{n\times m}$, and $C \in \mathbf{R}^{l\times n}$. Suppose that Δ is a differentiable matrix function with $\Delta(t) \in C_1$ and $\dot{\Delta}(t) \in C_2$ for all $t \geq 0$, where

$$\begin{aligned} C_1 &= \mathrm{co}\{\Delta_1, \dots, \Delta_N\} \subset \mathbf{R}^{m\times l} \\ C_2 &= \mathrm{co}\{\Omega_1, \dots, \Omega_N\} \subset \mathbf{R}^{m\times l} \end{aligned}$$

In other words, C_1 and C_2 are the polytopes defined as the convex hulls of the vertices $\Delta_1, \dots, \Delta_N$ and $\Omega_1, \dots, \Omega_N$, respectively. We assume that $0 \in C_1$ and $0 \in C_2$.

Let there be $X = X^T \in \mathbf{R}^{m\times m}$, $Z = Z^T \in \mathbf{R}^{l\times l}$, and $Y, \Lambda \in \mathbf{R}^{m\times l}$, satisfying the conditions

(*i*) $X \geq 0$,

(*ii*) $\Delta_i^T \Lambda = \Lambda^T \Delta_i$, for $i = 1, \dots, N$,

(*iii*) for $i, j \in \{1, \dots, N\}$

$$\begin{bmatrix} I \\ \Delta_i \end{bmatrix}^T \begin{bmatrix} Z & Y^T \\ Y & -X \end{bmatrix} \begin{bmatrix} I \\ \Delta_i \end{bmatrix} - \frac{1}{2}(\Lambda^T \Omega_j + \Omega_j^T \Lambda) \geq 0.$$

Under these conditions, if

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \begin{bmatrix} Z & Y^T - j\omega\Lambda^T \\ Y + j\omega\Lambda & -X \end{bmatrix} \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0, \quad \forall \omega \in [0, \infty],$$

holds with $G(s) = C(sI - A)^{-1}B$, then $x(t) \to 0$ exponentially as $t \to \infty$.

Proof: The operator defined by multiplication with $\Delta(t)$ satisfies the IQC defined by

$$\Pi(j\omega) = \begin{bmatrix} Z & Y^T - j\omega\Lambda^T \\ Y + j\omega\Lambda & -X \end{bmatrix}. \tag{2.10}$$

To see this let us define the functions

$$F_1(\Delta) = \begin{bmatrix} I \\ \Delta \end{bmatrix}^T \begin{bmatrix} Z & Y^T \\ Y & -X \end{bmatrix} \begin{bmatrix} I \\ \Delta \end{bmatrix},$$

and

$$F_2(\Delta, \dot{\Delta}) = F_1(\Delta) - \frac{1}{2}(\Lambda^T \dot{\Delta} + \dot{\Delta}^T \Lambda).$$

We note that $X \geq 0$ implies that F_2 is a concave function. Hence, by condition (*iii*) and the assumptions on $\Delta(t)$ we have $F_2(\Delta(t), \dot{\Delta}(t)) \geq 0$.

Evaluation the integrals according to Definition 2.1 gives

$$\int_0^\infty y^T F_1(\Delta(t)) y \, dt + \lim_{\tau\to\infty} \int_0^\tau 2y^T \Delta^T \Lambda \dot{y} dt =$$
$$\lim_{\tau\to\infty} \left[y^T \Delta^T \Lambda y\right]_0^\tau + \int_0^\infty y^T F_2(\Delta(t), \dot{\Delta}(t)) y \, dt \geq -\gamma |y_0|^2$$

for all $y, \dot{y} \in \mathbf{L}_2^l[0, \infty)$, where we can use $\gamma = \max_i \overline{\sigma}(\Delta_i^T \Lambda)^2$. The equality follows from integration by parts and condition (*ii*). The inequality follows since $y(\tau) \to 0$ as $\tau \to 0$ and since the second integral is positive.

To finish the proof we first notice that $\tau\Delta(t) \in C_1$ and $\tau\Delta(t) \in C_2$ for all $\tau \in [0, 1]$. This follows from the convexity of C_1 and C_2 and since 0 is contained in these sets. By use of this property we can prove the following statements.

- The system $\dot{x} = (A + B\tau\Delta(t)C)x$ is well-posed for any $\tau \in [0, 1]$. This follows since the right hand side is Lipschitz continuous for all $t \geq 0$.
- For any $\tau \in [0, 1]$, the operator $\tau\Delta$ satisfies the IQC defined by the multiplier in (2.10).

Hence, the conditions of Corollary 2.1 are satisfied. The exponential stability conclusion follows from the remark following that corollary. □

REMARK 2.4
In the special when $\Delta(t) = \delta(t)I$, where $\delta(t) \in [-1,1]$ and $\dot{\delta}(t) \in [-b,b]$, then we can use $Z = X + b\Lambda$, $X = X^T \geq 0$, $Y = -Y^T$, and $\Lambda = \Lambda^T \geq 0$.

If $\Delta(t) = \text{diag}(\delta_1(t)I_1, \ldots, \delta_N(t)I_N)$, where each δ_i satisfies $\delta_i(t) \in [-1,1]$ and $\dot{\delta}_i(t) \in [-b_i, b_i]$, then we use diagonal matrices X, Y, Z and Λ, where each diagonal element satisfies the conditions above. □

The next theorem gives a stability result for systems with a general form of parametric uncertainty.

THEOREM 2.4
Consider the system

$$\dot{x} = (A + B\Delta C)x, \quad x(0) = x_0,$$

where $A \in \mathbf{R}^{n\times n}$ is Hurwitz, $B \in \mathbf{R}^{n\times m}$, and $C \in \mathbf{R}^{l\times n}$. Suppose that $\Delta \in C = \text{co}\{\Delta_1, \ldots, \Delta_N\} \subset \mathbf{R}^{m\times l}$ is a parametric uncertainty. We assume that $0 \in C$.

Let Π_P be the Popov multiplier

$$\Pi_P(j\omega) = \begin{bmatrix} 0 & -j\omega\Lambda^T \\ j\omega\Lambda & 0 \end{bmatrix},$$

where $\Delta_i^T\Lambda = \Lambda^T\Delta_i$ for $i = 1, \ldots, N$, and let $\Pi_B \in \mathbf{RL}_\infty^{(l+m)\times(l+m)}$ satisfy

$$\begin{bmatrix} I \\ \Delta_i \end{bmatrix}^T \Pi_B(j\omega) \begin{bmatrix} I \\ \Delta_i \end{bmatrix} \geq 0, \text{ and } \begin{bmatrix} 0 \\ I_m \end{bmatrix}^T \Pi_B(j\omega) \begin{bmatrix} 0 \\ I_m \end{bmatrix} \geq 0,$$

for all $\omega \in \mathbf{R}$ and $i = 1, \ldots, N$.

Under these conditions, if

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* (\Pi_B(j\omega) + \Pi_P(j\omega)) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0, \quad \forall\omega \in [0,\infty],$$

holds with $G(s) = C(sI - A)^{-1}B$, then $x(t) \to 0$ exponentially as $t \to \infty$.

Proof: It is straightforward to show that Δ satisfies the IQCs defined by Π_B and Π_P. Hence, it also satisfies the IQC defined by $\Pi = \Pi_B + \Pi_P$. The same arguments that were used to conclude the proof of Theorem 2.3 are also valid here. This proves the Theorem. □

2.6 Application to Systems with Nonlinearities

In this section we give a generalization of the stability criterion in Zames and Falb (1968). We consider the system

$$\begin{aligned} \dot{x} &= Ax + bu, \quad x(0) = x_0, \\ y &= cx, \\ u &= \varphi(y) + g, \end{aligned} \tag{2.11}$$

where $A \in \mathbf{R}^{n\times n}$ is Hurwitz, $b, c^T \in \mathbf{R}^n$, and $g \in \mathbf{L}_{2e}[0,\infty)$. Here φ is an odd and slope restricted nonlinearity satisfying the conditions:

(*i*) φ is odd,

(*ii*) $(y_1 - y_2)(\varphi(y_1) - \varphi(y_2)) \geq 0$, for all $y_1, y_2 \in \mathbf{R}$,

(*iii*) there exists $k > 0$ such that $|\varphi(y)| \leq k|y|$, for all $y \in \mathbf{R}$.

The following corollary extends Zames and Falb's stability criterion for systems with slope restricted nonlinearities. Our contribution is that we add a Popov multiplier to the criterion.

COROLLARY 2.2
Assume that the system in (2.11) is well-posed. Let $Z(j\omega) = 1 + \lambda j\omega + H(j\omega)$, where $\lambda \in \mathbf{R}$, and where

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt,$$

for some real-valued function with

$$\int_{-\infty}^{\infty} |h(t)|dt \leq 1.$$

Under these conditions, if there exists $\varepsilon > 0$ such that

$$\mathrm{Re}\,[Z(j\omega)G(j\omega)] \leq -\varepsilon, \quad \forall \omega \in \mathbf{R}, \tag{2.12}$$

holds with $G(s) = c(sI - A)^{-1}b$, then the system in (2.11) is stable.

Proof: We can use the multiplier

$$\Pi(j\omega) = \begin{bmatrix} 0 & 1 - j\omega\lambda + H(j\omega)^* \\ 1 + j\omega\lambda + H(j\omega) & 0 \end{bmatrix}.$$

This multiplier is the sum of the Popov multiplier in Example 2.1 and the multiplier derived in Zames and Falb (1968). It is clear that $\tau\varphi$ satisfies the IQC defined by this multiplier for all $\tau \in [0,1]$. Hence, the conditions of Theorem 2.1 are satisfied. It is easy to see that the stability condition in (2.7) reduces to (2.12). □

Remark 2.5
Zames and Falb considered a more general class of convolution operators for the nominal system G. They also allowed H to be of a somewhat more general form than we used here. It easy to extend the stability results in this chapter to this class of convolution operators. This is discussed in Appendix 2.9. □

The next example shows that the Popov term $j\omega\lambda$ can be important in applications of Corollary 2.2.

Example 2.5
Let the system in (2.11) have the transfer function

$$G(s) = \frac{(2s^2 + s + 2)(s + 100)}{(s + 10)^2(s^2 + 5s + 20)}.$$

We can apply Corollary 2.2. With $\lambda = 0.04$ and $H(s) = 0.92/(s - 1)$, we get

$$Z(s) = 0.04\frac{s^2 + 24s - 2}{s - 1}.$$

Figure 2.2 shows the Nyquist curves for G and ZG. The stability criterion in (2.12) is satisfied with this Z. This is more easily seen in Figure 2.3. We note that it would not be possible to prove stability for this system with any of the usual stability criteria such as the circle criterion, the Popov criterion or the off axis circle criterion of Cho and Narendra (1968). We also note that it would not be possible to use only the multiplier $1 + H(j\omega)$ in (2.12) since the Popov part $j\omega\lambda$ is needed at $\omega = \infty$. □

2.7 Conclusions

We showed how Popov multipliers can be used in the IQC methodology for stability analysis. Examples show that this allow us to generalize several existing stability criteria into the IQC framework. As a result of our main theorem we also obtained new stability criteria for systems with slowly time-varying parameters and for systems with a general class of parametric uncertainty. The example in Section 2.6 shows that the nonproperness of the Popov multiplier can be very important in applications.

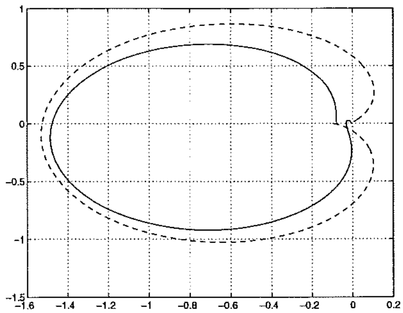

Figure 2.2 The diagram shows the Nyquist curve for ZG in solid line and the Nyquist curve for G in dashed line. It follows that the stability condition in 2.12 is satisfied.

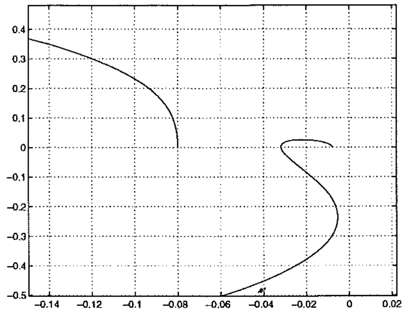

Figure 2.3 Here we have zoomed in the details around the origin of the Nyquist curve for ZG. It is clear that the stability condition in 2.12 is satisfied.

2.8 Appendix: Proof of Theorem 2.2

The following inequalities will be used in the proof

$$\langle x, y \rangle \leq \|y\| \cdot \|x\|, \tag{2.13}$$

$$\|x + y\|^2 \leq 2(\|x\|^2 + \|y\|^2), \tag{2.14}$$

for arbitrary $x, y \in \mathbf{L}_2^m[0, \infty)$. We will also use that

$$\sqrt{x + y} \leq \sqrt{x} + \sqrt{y}, \tag{2.15}$$

for arbitrary $x, y \geq 0$.

The idea for the proof is to use a similar homotopy argument as that

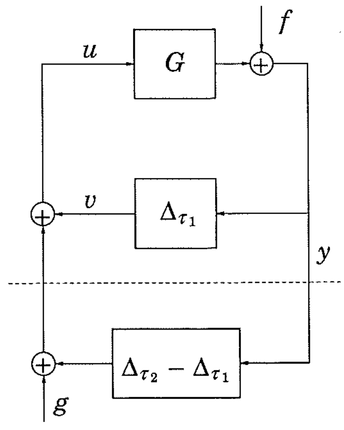

Figure 2.4 Block diagram for illustration of the proof of Theorem 2.2. If the upper loop is stable when it is disconnected from the lower branch, then the total system is also stable if $|\tau_2 - \tau_1|$ is small enough.

in Megretski and Rantzer (1995). Consider the system in Figure 2.4. We will show that if the upper loop is stable when it is disconnected from the lower branch, then the whole system is also stable if $|\tau_2 - \tau_1|$ is small enough. Iterative use of this from $\tau = 0$ up to $\tau = 1$ in small steps of equal size will finish the proof.

Part 1: Assume that the system

$$\begin{aligned} y &= Gu + f, \quad & f(t) &= Ce^{At}x_0\theta(t), \\ u &= v + g, \quad & v &= \Delta_\tau(y) \end{aligned} \tag{2.16}$$

is stable for some $\tau \in [0,1]$. Then for any $x_0 \in \mathbf{R}^n$ and $g \in \mathbf{L}_2^m[0,\infty)$ we have $(y,v) \in \mathbf{L}_2^l[0,\infty) \times \mathbf{L}_2^m[0,\infty)$ and we will show that there exist constants $c_0 > 0$ and $\rho_0 > 0$, which both are independent of τ, such that y is bounded as

$$\|y\| \leq \rho_0|x_0| + c_0\|g\|. \tag{2.17}$$

To prove that (2.17) holds, we notice that $g, v \in \mathbf{L}_2^m[0,\infty)$ implies that also $(P_\Lambda y)' \in \mathbf{L}_2^l[0,\infty)$. We use that the Fourier transform of $\Lambda(P_\Lambda Gv)'$ is $j\omega\Lambda G(j\omega)\widehat{v}(j\omega)$. This follows since $\Lambda G(\infty) = 0$. Use of this together with the relation $y = Gv + Gg + f$, gives

$$2\int_0^\infty v^T\Lambda(P_\Lambda y)'dt = \left\langle v, \begin{bmatrix} G \\ I \end{bmatrix}^* \Pi_P \begin{bmatrix} G \\ I \end{bmatrix} v \right\rangle + 2\int_0^\infty v^T\Lambda(P_\Lambda f)'dt$$
$$+ \frac{1}{\pi}\mathrm{Re}\int_{-\infty}^\infty \widehat{v}(j\omega)^*\Lambda j\omega G(j\omega)\widehat{g}(j\omega)d\omega.$$

Hence, the IQC constraint in Definition 2.4 with the multiplier $\Pi = \Pi_B + \Pi_P$ gives

$$\begin{aligned}
-\gamma|P_\Lambda y_0|^2 &\le \left\langle \begin{bmatrix} y \\ v \end{bmatrix}, \Pi_B \begin{bmatrix} y \\ v \end{bmatrix} \right\rangle + 2\int_0^\infty v^T\Lambda(P_\Lambda y)'dt \\
&= \left\langle v, \begin{bmatrix} G \\ I \end{bmatrix}^* (\Pi_B + \Pi_P) \begin{bmatrix} G \\ I \end{bmatrix} v \right\rangle + 2\mathrm{Re}\left\langle \begin{bmatrix} Gv \\ v \end{bmatrix}, \Pi_B \begin{bmatrix} Gg + f \\ 0 \end{bmatrix} \right\rangle \\
&\quad + \left\langle \begin{bmatrix} Gg + f \\ 0 \end{bmatrix}, \Pi_B \begin{bmatrix} Gg + f \\ 0 \end{bmatrix} \right\rangle + 2\int_0^\infty v^T\Lambda(P_\Lambda f)'dt \\
&\quad + \frac{1}{\pi}\mathrm{Re}\int_{-\infty}^\infty \widehat{v}(j\omega)^*\Lambda j\omega G(j\omega)\widehat{g}(j\omega)d\omega \\
&\le -\varepsilon\|v\|^2 + 2\left(c_1|x_0| + c_2\|g\|\right)\|v\| + c_3|x_0|^2 + c_4\|g\|^2.
\end{aligned}$$

The first term in the last inequality follows from the frequency inequality (2.9) in the theorem statement. The other terms follows by use of (2.13) and (2.14). Let $\|\Pi_{1i}\| = \sup_\omega \overline{\sigma}(\Pi_{1i}(j\omega))$ for $i = 1, 2$ denote the norms of the corresponding blocks of the matrix operator

$$\Pi_B = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^* & \Pi_{22} \end{bmatrix}.$$

Then the constants can be taken as

$$\begin{aligned}
c_1 &= (\|\Pi_{11}\| \cdot \|G\| + \|\Pi_{12}\|)\,\gamma_1 + \gamma_2, \\
c_2 &= \|\Pi_{11}\| \cdot \|G\|^2 + \|\Pi_{12}\| \cdot \|G\| + \|j\omega\Lambda G\|, \\
c_3 &= 2\|\Pi_{11}\|\gamma_1^2, \\
c_4 &= 2\|G\|^2 \cdot \|\Pi_{11}\|.
\end{aligned}$$

The constants γ_1 and γ_2 defined as

$$\begin{aligned}
\gamma_1^2 &= \int_0^\infty \overline{\sigma}(Ce^{At})^2 dt, \\
\gamma_2^2 &= \int_0^\infty \overline{\sigma}(\Lambda CAe^{At})^2 dt
\end{aligned}$$

give the bounds on $\|f\|$ and $\|\Lambda\dot{f}\|$, respectively. After some work and with the use of (2.15) we obtain the bound

$$\|v\| \leq \alpha_1|x_0| + \alpha_2\|g\|, \tag{2.18}$$

where e.g.,

$$\begin{aligned}
\alpha_1 &= \frac{1}{\varepsilon}\left(c_1 + \sqrt{2c_1^2 + \varepsilon(c_3 + \gamma\overline{\sigma}(C)^2)}\right), \\
\alpha_2 &= \frac{1}{\varepsilon}\left(c_2 + \sqrt{2c_2^2 + \varepsilon c_4}\right).
\end{aligned}$$

Hence, using (2.18) we get

$$\|y\| = \|G(v+g) + f\| \leq \rho_0|x_0| + c_0\|g\|,$$

where $\rho_0 = \|G\|\alpha_1 + \gamma_1$ and $c_0 = \|G\|(\alpha_2 + 1)$. This is the bound in (2.17).

Part 2: We now take the key step in the proof. The idea is illustrated in Figure 2.5. Assume that the system in (2.16) is stable when $\tau = \tau_1$, and consider the case when $\tau = \tau_2$. The system equations can be written as

$$\begin{aligned}
y &= Gu + f, \\
u &= \underbrace{\Delta_{\tau_1}(y) + (\Delta_{\tau_2} - \Delta_{\tau_1})(y)}_{v} + g.
\end{aligned}$$

The well-posedness assumption implies that for arbitrary $x_0 \in \mathbf{R}^n$ and $g \in \mathbf{L}_{2e}^m[0,\infty)$ there exists a solution $(y, v) \in \mathbf{L}_{2e}^l[0,\infty) \times \mathbf{L}_{2e}^m[0,\infty)$. Let $T \geq 0$ and let $y_T = P_T y$, and $g_T = P_T g$, and define $g_1 = (\Delta_{\tau_2} - \Delta_{\tau_1})(y_T) + g_T$. It follows from assumption (i) on the parametrization of Δ that $g_1 \in \mathbf{L}_2^m[0,\infty)$. If we let f and g_1 be input signals to the system in (2.16) when $\tau = \tau_1$, then we get the system equations (where the loop signals are denoted y_1, u_1 and v_1)

$$\begin{aligned}
y_1 &= Gu_1 + f, \\
u_1 &= \underbrace{\Delta_{\tau_1}(y_1)}_{v_1} + \underbrace{(\Delta_{\tau_2} - \Delta_{\tau_1})(y_T) + g_T}_{g_1}.
\end{aligned}$$

On the time interval $[0, T]$ it follows by the well-posedness assumption that this equation is satisfied by $P_T y_1 = y_T$ and $P_T u_1 = u_T$. Furthermore, the assumed stability of the system in (2.16) when $\tau = \tau_1$ implies that

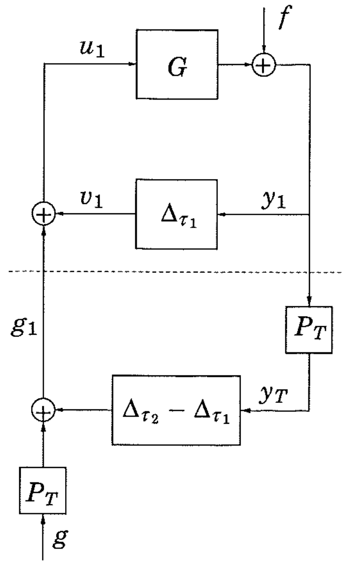

Figure 2.5 Illustration of the key step in the proof. By well-posedness there exists a solution (u, y) to the system equations on any finite time interval $[0, T]$, see Figure 2.4 for notation. Truncate the signals in the lower branch and denote the signals in the upper branch u_1 and y_1. It follows by the well-posedness that $y_T = P_T y_1$ and $u_T = P_T u_1$, i.e., the loop signals in the truncated system and the system in Figure 2.4 can be assumed to be equivalent on the interval $[0, T]$. If the upper loop is stable, then u_1 and y_1 will have finite energy. This is used to prove that the total system is stable when $|\tau_2 - \tau_1|$ is small enough.

$y_1 \in \mathbf{L}_2^l[0,\infty)$, and $v_1 = \Delta_{\tau_1}(y_1) \in \mathbf{L}_2^m[0,\infty)$. From (2.17) and from the assumption on the parametrization of Δ we obtain

$$\begin{aligned}\|y_T\| = \|P_T y_1\| \leq \|y_1\| &\leq \rho_0|x_0| + c_0\|(\Delta_{\tau_2} - \Delta_{\tau_1})(y_T) + g_T\| \\ &\leq \rho_0|x_0| + c_0\kappa|\tau_2 - \tau_1| \cdot \|y_T\| + c_0\|g_T\|.\end{aligned}$$

Let $|\tau_2 - \tau_1| < 1/c_0\kappa$, then we get

$$\|y_T\| \leq \tilde{\rho}_0|x_0| + \tilde{c}_0\|g_T\|, \tag{2.19}$$

where $\tilde{\rho}_0 = \rho_0/(1-c_0\kappa|\tau_2-\tau_1|)$ and $\tilde{c}_0 = c_0/(1-c_0\kappa|\tau_2-\tau_1|)$ are positive.

A similar bound on $u_T = P_T u$ can be obtained as follows. We have

$u_T = P_T u_1 = P_T(v_1 + (\Delta_{\tau_1} - \Delta_{\tau_2})(y_T)) + g_T$. Hence

$$\begin{aligned}\|u_T\| &\le \|v_1\| + \kappa|\tau_2 - \tau_1| \cdot \|y_T\| + \|g_T\| \\ &\le \alpha_1|x_0| + \alpha_2\|g_1\| + \kappa|\tau_2 - \tau_1| \cdot \|y_T\| + \|g_T\| \\ &\le (\alpha_1 + (1+\alpha_2)\kappa|\tau_2 - \tau_1|\widetilde{\rho}_0) \cdot |x_0| + \\ &\quad (1+\alpha_2)(1+\kappa|\tau_2 - \tau_1|\widetilde{c}_0) \cdot \|g_T\|, \end{aligned} \tag{2.20}$$

where the second inequality follows from (2.18) with g replaced by g_1 and the last inequality follows by use of (2.19).

The bounds in (2.19) and (2.20) hold for any $T \ge 0$, which implies that the system in (2.16) is stable when $\tau = \tau_1 + \Delta\tau$, if $\Delta\tau < 1/(c_0\kappa)$. Iterative application of this conclusion from $\tau = 0$, where the system is assumed to be stable, up to $\tau = 1$ in steps $\Delta\tau < 1/(c_0\kappa)$ shows that the system in (2.16) is stable for all $\tau \in [0,1]$.

2.9 Appendix: Extension to General Convolution Operators

It is possible to generalize Theorem 2.2 to systems where the nominal part is from the following class of convolution operators, see Desoer and Vidyasagar (1975).

DEFINITION 2.5

Let $\mathcal{A}^{l\times m}$ be the transfer functions with weighting functions on the form

$$g(t) = \begin{cases} g_a(t) + \sum_{i=1}^{N} g_i\delta(t - t_i), & t \ge 0, \\ 0, & t < 0, \end{cases}$$

where $g_a \in \mathbf{L}_1^{l\times m}[0,\infty)$, $g_i \in \mathbf{R}^{l\times m}$, and $t_i \ge 0$. □

This will allow us to consider the generalization of Zames and Falb's criterion in the same framework as they used. The following properties holds, see Desoer and Vidyasagar (1975).

1. The transfer functions in $\mathcal{A}^{l\times m}$ correspond to bounded and casual convolution operators.
2. If $G \in \mathcal{A}^{l\times m}$, and $f \in \mathbf{L}_{2e}^m[0,\infty)$ then $Gf = g * f \in \mathbf{L}_{2e}^m[0,\infty)$. Furthermore, if $f \in \mathbf{L}_2^m[0,\infty)$, then $\|Gf\| \le \|G\|_\infty \cdot \|f\|$, where $\|G\|_\infty = \sup_\omega \overline{\sigma}(G(j\omega))$.
3. We see that $sP_\Lambda G(s) \in \mathcal{A}^{l\times m}$ if $(P_\Lambda g_a)' \in \mathbf{L}_1^{l\times m}[0,\infty)$ and $P_\Lambda g_i = 0$ for all i. Here P_Λ operates column by column.

4. If $sP_\Lambda G(s) \in \mathcal{A}^{l\times m}$, then it follows that the Fourier transform of $(P_\Lambda Gf)'$ becomes $j\omega P_\Lambda G(j\omega)\widehat{f}(j\omega)$.

As an example the system

$$G(s) = \begin{bmatrix} \frac{1}{s+1}e^{-sT} & \frac{1}{s+2} \\ 0 & \frac{1}{s+2} + e^{-sT} \end{bmatrix}$$

satisfies $sP_\Lambda G(s) \in \mathcal{A}^{2\times 2}$, when

$$P_\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

In fact, we have

$$sP_\Lambda G(s) = \begin{bmatrix} -\frac{1}{s+1}e^{-sT} + e^{-sT} & -\frac{2}{s+2} + 1 \\ 0 & 0 \end{bmatrix}.$$

It is now possible to obtain stability results of the same type as Theorem 2.1 and Theorem 2.2 that includes systems from the class $\mathcal{A}^{l\times m}$. In all definitions below we assume that Λ is the Popov parameter that will be used for stability analysis. We will consider stability of the system

$$\begin{aligned} y &= Gu + f, \\ u &= \Delta(y) + g, \end{aligned} \tag{2.21}$$

where $G \in \mathcal{A}^{l\times m}$, and where $\Delta : \mathbf{L}_{2e}^l[0,\infty) \to \mathbf{L}_{2e}^m[0,\infty)$ is bounded and causal. Furthermore, $g \in \mathbf{L}_{2e}^m[0,\infty)$ and $f, (P_\Lambda f)' \in \mathbf{L}_2^l[0,\infty)$. We can think of f as the initial conditions response of G. The differentiability assumption $P_\Lambda f$ is reasonable in practical applications.

Stability and well-posedness for this system is defined in a similar way was as before.

DEFINITION 2.6
The system in (2.21) is *well-posed* if for every pair (f,g) with $f,(P_\Lambda f)' \in \mathbf{L}_2^l[0,\infty)$ and $g \in \mathbf{L}_{2e}^m[0,\infty)$, there exists a solution $(y,(P_\Lambda y)',u)$ with $y,(P_\Lambda y)' \in \mathbf{L}_{2e}^l[0,\infty)$ and $u \in \mathbf{L}_{2e}^m[0,\infty)$. Furthermore, the map $(f,g) \to (u,y)$ should be causal. The system is *stable* if in addition there exist positive constants ρ_1, ρ_2, and c such that

$$\int_0^T (|y|^2 + |u|^2)dt \le \rho_1\|f\|^2 + \rho_2\|(P_\Lambda f)'\|^2 + c\int_0^T |g|^2 dt,$$

for all $T > 0$. □

We note that for the systems discussed in Section 2.3 and Section 2.4 this definition is implied by the ones in Definition 2.2 and Definition 2.3. This is easy to see since $x, \dot{x} \in \mathbf{L}_{2e}^{n}[0,\infty)$ and $u \in \mathbf{L}_{2e}^{m}[0,\infty)$ implies that $y = Cx + Du$ satisfies the conditions for well-posedness in the new definition. Similarly, with $f = Ce^{At}x_0\theta(t)$, the term $\rho_1\|f\|_2^2 + \rho_2\|(P_\Lambda f)'\|_2^2$ can be bounded by $\rho|x_0|^2$ for some positive constant ρ.

The following proposition corresponds to Theorem 2.2.

PROPOSITION 2.1
Assume that there is a parametrization Δ_τ, for $\tau \in [0,1]$, of the operator Δ and consider the parametrized version of the system in (2.21):

$$\begin{aligned} y &= Gu + f, \\ u &= \Delta_\tau(y) + g. \end{aligned}$$

Assume that $sP_\Lambda G(s) \in \mathcal{A}^{l\times m}$ and condition $(i)-(iv)$, and $(vi)-(viii)$ of Theorem 2.2 holds. Then the system in (2.21) is stable.

Proof: The proof is essentially that same as the proof of Theorem 2.2. The only difference is that we need to replace the terms $const|x_0|$ with terms on the from $const_1\|f\| + const_2\|(P_\Lambda f)'\|$ for suitable constants at suitable places in that proof. □

3

Computation of Multipliers

Abstract

Robustness problems are considered, where structural information about uncertainty is given in terms of integral quadratic constraints. A flexible format for representation of such information is introduced. The format supports the use of numerical software for solution of linear matrix inequalities. Examples are given involving bounded time-variations and nonlinearities.

3.1 Introduction

The computation of multipliers for robustness and stability analysis is a problem with a long history. In fact, it was discussed by Zames in his classical paper Zames (1966b). There he showed that a stable linear system G in a feedback interconnection with a slope restricted nonlinearity is stable if there exists a multiplier $M(j\omega)$ such that

$$\mathrm{Re}[M(j\omega)(G(j\omega)+1)] < 0 \quad \forall \omega \in [0,\infty]$$

The multiplier can be selected from the class of so called RL multipliers, see Narendra and Taylor (1973). Zames stated that the main difficulty in applying this stability condition was to find a suitable multiplier from this class.

Several more sophisticated multiplier-based stability results were developed during the period 1965–1975. We mention in particular the criterion for slope restricted nonlinearities in Zames and Falb (1968) and the criterion for slowly time-varying parameters in Sundareshan and Thathachar (1972). See also the books Willems (1971a) and Desoer and Vidyasagar (1975). These results had their main limitation in the computability of the multipliers. This was mainly due to the lack of powerful computers and suitable numerical software.

Since the early 80s a lot of work focused on robustness analysis of uncertain systems. In particular, multiplier-based methods for diagonally perturbed linear time invariant systems gained a lot of interest through the work by Safonov (1982) and Doyle (1982). The theory was extended to treat also parametric uncertainty in Fan *et al.* (1991) and Young (1993). Since only time-invariant uncertainties are considered it is possible to compute the optimal value of the corresponding multiplier at each frequency of a preselected frequency grid, see Balas *et al.* (1993), Young *et al.* (1992) and Packard and Doyle (1993). This method for multiplier computation has been successful in applications. However, there are two problems involved in the approach. First, there is the possibility that the worst frequency is missed in the frequency grid. This is particularly serious since the optimal value can be a discontinuous function of frequency if robustness to real uncertain parameters is studied, see Barmish *et al.* (1990) and Packard and Pandey (1993). Methods to overcome this problem was presented in, for example, Packard and Pandey (1993) and Helmersson (1995a). The second problem is that this approach is not easily extended to analysis of systems that contain time-varying or nonlinear components. In this case there exists couplings between the frequencies. It is then no longer possible to perform a frequency-by-frequency optimization of the multipliers.

The development of efficient algorithms for the solution of convex optimization problems involving linear matrix inequalities (LMIs) and the appearance of software implementations of these algorithms have greatly increased the possibilities for efficient computations in system analysis. For references, see Nesterov and Nemirovski (1993), Gahinet *et al.* (1995), El Ghaoui (1995), and Boyd *et al.* (1994). The use of LMI optimization for computation of multipliers corresponding to different forms of dynamic and parametric uncertainty was treated in Ly *et al.* (1994), Balakrishnan *et al.* (1994), Balakrishnan (1995), and Helmersson (1995b). The idea behind these papers is to find a finite-dimensional affine parametrization of a subset of the multipliers. The stability criterion restricted to this subset of multipliers can then by the Kalman-Yakubovich-Popov Lemma (KYP) be formulated as an equivalent LMI condition. LMI methods for the computation of Popov multipliers have also gained a lot of recent interest, see, for example, Feron *et al.* (1995), Bernstein *et al.* (1995).

Characteristic for all the LMI-based methods for multiplier optimization mentioned so far is that each approach only considers a restricted class of problems. One reason for this is that they are based on traditional frameworks for stability analysis. Stability analysis based on passivity theory, dissipativity theory and parametrized Lyapunov functions have restrictions that limit or at least complicates their application. For example, in multiplier-based passivity analysis there are invertibility and factorizability conditions on the multipliers that need to be considered.

A unified approach for multiplier-based robustness analysis was suggested by Megretski (1993b). The idea is to use integral quadratic constraints (IQCs) to describe uncertainties, time-varying parameters, performance criteria, and frequency characteristics for signals. The methodology was later extended in Megretski and Rantzer (1995) to give a theory that allow us to combine multipliers according to various robustness criteria without any need for invertibility or factorizability conditions.

In this chapter we suggest a unifying format for multiplier computation by LMI optimization. Our format is based on the IQC framework. The main idea is to find finite-dimensional parametrizations of the multipliers in terms of a basis. The stability condition is then transformed to an equivalent finite-dimensional convex feasibility test that can be solved by LMI methods. All the examples mentioned above can be treated in our format.

The first three sections of the chapter discuss robustness analysis with IQCs and in particular robust performance analysis with IQCs.

3.2 IQC-Based Robustness Analysis

We consider systems consisting of a nominal transfer function G in a positive feedback interconnection with a bounded causal operator Δ. We will for simplicity of notation only consider square systems in this and the remaining chapters of the thesis.

The basic idea behind the IQC approach for robustness analysis is to find a description of Δ in terms of bounded and Hermitian valued multipliers, Π, that satisfy

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\begin{bmatrix}\widehat{y}(j\omega)\\ \widehat{v}(j\omega)\end{bmatrix}^*\Pi(j\omega)\begin{bmatrix}\widehat{y}(j\omega)\\ \widehat{v}(j\omega)\end{bmatrix}d\omega, \tag{3.1}$$

for all square integrable y and $v = \Delta(y)$. Here $\widehat{y}$ and $\widehat{v}$ denotes the Fourier transforms of y and v, respectively.

All multipliers used in this chapter are assumed to be rational transfer functions. The reason for this becomes apparent when we discuss multiplier computation in Section 3.5. We will generaly consider multipliers consisting of a proper part $\Pi_B = \Pi_B^* \in \mathbf{RL}_\infty^{2m\times 2m}$ and a nonproper Popov multiplier on the form

$$\Pi_P(j\omega) = \begin{bmatrix} 0 & -j\omega\Lambda^T \\ j\omega\Lambda & 0\end{bmatrix}, \tag{3.2}$$

where $\Lambda \in \mathbf{R}^{m\times m}$. The Popov multiplier gives useful descriptions of static nonlinearities, parametric uncertainties and of slowly varying parameters.

We use the Popov multipliers in (3.2) to define constraints involving the integral

$$\int_0^\infty 2(\Delta(y))^T \Lambda \dot{y} dt. \tag{3.3}$$

We note that in order to use the Popov multiplier we must ensure differentiability of y. This enforces a condition on strict properness of the nominal transfer function, which would be overly restrictive in applications when Λ is a sparse matrix. This problem can be overcome. If we introduce $P_\Lambda : \mathbf{R}^m \to \mathbf{R}^m$ as the orthogonal projection onto the range of Λ^T. Then we can use the derivative $(P_\Lambda y)' = \frac{d}{dt}(P_\Lambda y)$ in (3.3) instead of $\dot{y}$. This works since $\mathcal{R}(\Lambda^T)^\perp = \mathcal{N}(\Lambda)$. Hence, $P_\Lambda y \perp \mathcal{N}(\Lambda)$ and only the part of y that contributes to the integral in (3.3) needs to be differentiated.

The following definition of IQC was suggested in Chapter 2.

DEFINITION 3.1—IQC
Let $\Pi_B = \Pi_B^* \in \mathbf{RL}_\infty^{2m\times 2m}$ and let Π_P be the Popov multiplier in (3.2).

We say that Δ satisfies the IQC defined by $\Pi = \Pi_B + \Pi_P$ if there exists a positive constant γ such that

$$\int_0^\infty \begin{bmatrix} y \\ v \end{bmatrix}^T \left(\pi_B * \begin{bmatrix} y \\ v \end{bmatrix} \right) dt + \int_0^\infty 2v^T \Lambda (P_\Lambda y)' dt \ge -\gamma |P_\Lambda y_0|^2,$$

for all y and $v = \Delta(y)$ such that $y, (P_\Lambda y)', v \in \mathbf{L}_2^m[0,\infty)$. Here π_B denotes the impulse response matrix corresponding to Π_B, $*$ denotes convolution, and $y_0 = y(0)$. □

We note that the first integral in the definition above is by Parseval's theorem equivalent to the frequency domain integral in (3.1)

In order to obtain the most accurate robustness condition we need to find as many multipliers for the perturbation Δ as possible. The next two properties can be used to combine multipliers for this purpose.

Property 1 Assume that Δ satisfies the IQCs defined by $\Pi_1, \dots, \Pi_n$. Then Δ also satisfies the IQC defined by the conic combination $\sum_{i=1}^n \alpha_i \Pi_i$, where $\alpha_i \ge 0$, $i = 1, \dots, n$.

Property 2 Assume that $\Delta = \mathrm{diag}(\Delta_1, \dots, \Delta_n)$, and that, for $i = 1, \dots, n$, Δ_i satisfies the IQC defined by

$$\Pi_i = \begin{bmatrix} \Pi_{i(11)} & \Pi_{i(12)} \\ \Pi_{i(12)}^* & \Pi_{i(22)} \end{bmatrix},$$

where the block structures are consistent with the size of the perturbations Δ_i. Then Δ satisfies the IQC defined by

$$\operatorname{daug}(\Pi_1, \ldots, \Pi_n) = \left[\begin{array}{ccc|ccc} \Pi_{1(11)} & & & \Pi_{1(12)} & & \\ & \ddots & & & \ddots & \\ & & \Pi_{n(11)} & & & \Pi_{n(12)} \\ \hline \Pi_{1(12)}^* & & & \Pi_{1(22)} & & \\ & \ddots & & & \ddots & \\ & & \Pi_{n(12)}^* & & & \Pi_{n(22)} \end{array}\right].$$

This is easily seen by writing out the expression for the IQC in Definition 3.1.

It follows from the first property that the set of multipliers that describe Δ is a convex cone. We use the notation Π_Δ for any such convex cone of multipliers that describe Δ. In applications it is important to find an as accurate as possible description of Δ. A description of Δ can be improved by addition and augmentation of convex cones of multipliers.

Addition: If Δ is described by the convex cones $\Pi_{i\Delta}$, $i = 1, \ldots, n$, then Δ is also described by $\sum_{i=1}^n \Pi_{i\Delta} = \{\sum_{i=1}^n \Pi_i : \Pi_i \in \Pi_{i\Delta}\}$.

Diagonal Augmentation: If, for $i = 1, \ldots, n$, Δ_i is described by the convex cone Π_{Δ_i}, then Δ is described by the convex cone

$$\operatorname{daug}(\Pi_{\Delta_1}, \ldots, \Pi_{\Delta_n}) = \{\operatorname{daug}(\Pi_1, \ldots, \Pi_n) : \Pi_i \in \Pi_{\Delta_i}\}.$$

Robust Stability Analysis Based on IQCs

We consider stability of a system consisting of a nominal system G in a positive feedback interconnection with a perturbation Δ as illustrated in the block diagram in Figure 3.1. The system equations are given by

$$\begin{aligned} y &= Gu + f, \\ u &= \Delta(y) + g. \end{aligned} \tag{3.4}$$

We assume that G is a bounded causal operator with transfer function $G \in \mathbf{RH}_\infty^{m\times m}$. The perturbation Δ is a bounded causal operator on $\mathbf{L}_{2e}^m[0,\infty)$. The input g is arbitrary in $\mathbf{L}_{2e}^m[0,\infty)$ and f contains the response due to the initial condition of the nominal system. We assume that $f = Ce^{At}x_0\theta(t)$ for some matrices A and C and initial conditions vector x_0. A is assumed to be Hurwitz.

Stability and well-posedness for the system in (3.4) is defined as follows.

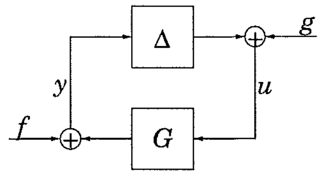

Figure 3.1 System consisting of a nominal plant G and a bounded causal perturbation Δ.

DEFINITION 3.2
The feedback interconnection of G and Δ is *well-posed* if the map $g \mapsto (y,u)$ is causal and if there exists a solution $(y,(P_\Lambda y)',u) \in \mathbf{L}_{2e}^m[0,\infty) \times \mathbf{L}_{2e}^m[0,\infty) \times \mathbf{L}_{2e}^m[0,\infty)$ for any x_0 and for any $g \in \mathbf{L}_{2e}^m[0,\infty)$. The feedback interconnection is *stable* if in addition there are positive constants ρ and c such that

$$\int_0^T (|y|^2 + |u|^2)dt \le \rho|x_0|^2 + c\int_0^T |g|^2 dt, \quad \forall T \ge 0.$$

□

The next theorem gives conditions for stability of the system in (3.4). It is a special case of Theorem 2.2 in Chapter 2.

THEOREM 3.1
Let $G \in \mathbf{RH}_\infty^{m\times m}$ and let Δ be a bounded causal operator on $\mathbf{L}_{2e}^m[0,\infty)$. Assume that

(*i*) for all $\tau \in [0,1]$, the interconnection of G and $\tau\Delta$ is well-posed,

(*ii*) for all $\tau \in [0,1]$, $\tau\Delta$ satisfies the IQC defined by $\Pi = \Pi_B + \Pi_P$,

(*iii*) $\Lambda G(\infty) = 0$, where Λ is the Popov parameter in Π_P,

(*iv*) the inequality

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0$$

holds for all $\omega \in [0,\infty]$.

Then the feedback interconnection of G and Δ is stable. □

REMARK 3.1
It is sometimes useful to consider more general parametrizations than $\tau\Delta$. See Theorem 2.2 in Chapter 2 for more details. □

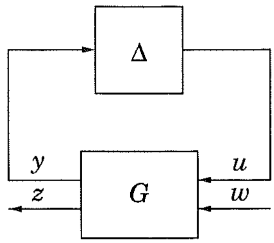

Figure 3.2 System setup for robust performance analysis.

REMARK 3.2
The third condition of the theorem statement ensures that $P_\Lambda y$ is differentiable. It can be omitted when $\Pi_P = 0$. □

Robust Performance Analysis Based on IQCs

For robust performance analysis we consider the system described by the equations

$$\begin{bmatrix} y \\ z \end{bmatrix} = G \begin{bmatrix} u \\ w \end{bmatrix}, \qquad u = \Delta(y), \tag{3.5}$$

where Δ is a bounded causal operator on $\mathbf{L}_{2e}^m[0,\infty)$ and where G is a bounded causal operator with rational transfer function. We assume that G is block partitioned according to the size of the signals, i.e.,

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \in \mathbf{RH}_\infty^{(m+l)\times(m+q)}.$$

The system (3.5) can be illustrated with the block diagram in Figure 3.2.

Performance of the system in (3.5) is generally measured in terms of disturbance attenuation. A measure of this attenuation can, for example, be obtained in terms of the energy ratio of the *error signal* z and the *disturbance* w. It is important to exploit the spectral characteristics of the disturbance w when studying such a performance measure. We will as it has been suggested in, for example, Megretski (1992), Megretski (1993b), and Paganini (1995), study performance criteria in terms of signals from a set $\mathcal{W}_{\text{inp}} \subset \mathbf{L}_2^q[0,\infty)$. This set is assumed to be defined by the convex cone $\Upsilon_{\text{inp}} \subset \mathbf{RL}_\infty^{q\times q}$ in the following way

$$\mathcal{W}_{\text{inp}} = \{w \in \mathbf{L}_2^q[0,\infty) : \int_{-\infty}^{\infty} \widehat{w}(j\omega)^* \Upsilon(j\omega)\widehat{w}(j\omega)d\omega \geq 0,\ \forall \Upsilon \in \Upsilon_{\text{inp}}\}.$$

We note that $\mathcal{W}_{\text{inp}} = \mathbf{L}_2^q[0,\infty)$ if $\Upsilon_{inp} = \{0\}$. If we in a certain application have a set of signals $\mathcal{W}$ then we need to find a convex cone $\Upsilon_{\text{inp}} \subset \mathbf{RL}_\infty^{q\times q}$ such that $\mathcal{W} \subset \mathcal{W}_{\text{inp}}$. We refer to the next section for a discussion and for examples. We define robust performance as follows.

DEFINITION 3.3
Assume that the signal w is in the set $\mathcal{W}_{\text{inp}} \subset \mathbf{L}_2^q[0,\infty)$ defined by the convex cone Υ_{inp}. The system in (3.5) is said to have robust performance according to the performance specification Π_{perf} if

(*i*) the feedback interconnection of G_{11} and Δ is stable.

(*ii*) the inequality

$$\int_{-\infty}^{\infty} \begin{bmatrix} \widehat{z}(j\omega) \\ \widehat{w}(j\omega) \end{bmatrix}^* \Pi_{\text{perf}}(j\omega) \begin{bmatrix} \widehat{z}(j\omega) \\ \widehat{w}(j\omega) \end{bmatrix} \le 0$$

holds for all $z = (G_{22} + G_{21}\Delta(I - G_{11}\Delta)^{-1}G_{12})w$, with $w \in \mathcal{W}_{\text{inp}}$.

□

The next theorem gives conditions that ensure robust performance for the system in (3.5).

THEOREM 3.2
Assume that

(*i*) the interconnection of G_{11} and Δ is stable,

(*ii*) Δ satisfies the IQC defined by $\Pi = \Pi_B + \Pi_P$,

(*iii*) $\Lambda\,[\,G_{11}(\infty) \quad G_{12}(\infty)\,] = 0$, where Λ is the Popov parameter in Π_P,

(*iv*) there exists $\Upsilon \in \Upsilon_{\text{inp}}$ such that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \operatorname{daug}\left(\Pi, \Pi_{\text{perf}} + \begin{bmatrix} 0 & 0 \\ 0 & \Upsilon \end{bmatrix}\right)(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0, \qquad (3.6)$$

for all $\omega \in [0,\infty]$.

Then the system in (3.5) has robust performance.

Proof: The proof is given in Appendix 3.10. □

REMARK 3.3
The third condition of the theorem can be omitted if $\Pi_P = 0$. □

REMARK 3.4
The stability assumption (i) in Theorem 3.2 can be replaced by the conditions:

(ia) for any $\tau \in [0,1]$, the interconnection of G_{11} and $\tau\Delta$ is well-posed,

(ib) $[\, I_q \quad 0_{q\times q}\,]\,\Pi_{\text{perf}}\,[\, I_q \quad 0_{q\times q}\,]^T \geq 0$,

$(ii)'$ for any $\tau \in [0,1]$, $\tau\Delta$ satisfies the IQC defined by $\Pi = \Pi_B + \Pi_P$.

This is also proved in Appendix 3.10. □

3.3 Signal Specifications for Robust Performance Analysis

Spectral information on the disturbance should be used to reduce conservatism in robust performance analysis. The multiplier set Υ_{inp}, which defines the input signal to be in a conic subset $\mathcal{W}_{\text{inp}} \subset \mathbf{L}_2^q[0,\infty)$, can be used to describe various types of spectral information.

In order to get an idea of how Υ_{inp} should be chosen we consider the case when we investigate the $\mathbf{L}_2$-performance and we know that the input signal is constrained to be in the set $\mathcal{W}_{\text{inp}}$ defined by Υ_{inp}. For any $\Upsilon \in \Upsilon_{\text{inp}}$ we obtain the combined multiplier

$$\Pi_{\text{perf}}(j\omega) + \begin{bmatrix} 0 & 0 \\ 0 & \Upsilon(j\omega) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & -I + \Upsilon(j\omega) \end{bmatrix}.$$

We can see from the frequency domain criterion in (3.6) that Υ is only contributing in a useful way at frequencies where it is negative semidefinite. We give two examples that show how Υ_{inp} can be chosen for two important signal classes. The first was suggested in Megretski (1993b).

Dominant Harmonics: Let $w \in \mathbf{L}_2^q[0,\infty)$ be a bandpass signal with $\operatorname{supp}\widehat{w} \in [-b,-a] \cup [a,b]$, where $\operatorname{supp}\widehat{w}$ denotes the support of the Fourier transform of w. Then w is in the set $\mathcal{W}_{\text{inp}}$ defined by

$$\begin{aligned}\Upsilon_{\text{inp}} = \{\Upsilon = \Upsilon^* \in \mathbf{RL}_\infty^{q\times q} : \Upsilon(j\omega) \geq 0,\ |\omega| \in [a,b] \\ \Upsilon(j\omega) \leq 0,\ \text{otherwise}\}. \end{aligned} \tag{3.7}$$

It is clear from the robust performance condition in (3.6) that we should choose a multiplier in Υ_{inp} which is as close as possible to

$$\Upsilon(j\omega) = \begin{cases} 0, & |\omega| \in [a,b], \\ -\infty I, & \text{otherwise.} \end{cases}$$

Signals with a Given Spectral Characteristic: Noise signals in a system are often modeled as stochastic white noise or filtered stochastic white noise. It is more convenient to consider so called deterministic white signals in the IQC framework. A realization of a stochastic white signal has infinite energy and it would for that reason be appropriate to consider the deterministic white signals to be in $\mathbf{L}_{2e}^{q}[0,\infty)$. This was the approach in Megretski (1992). A recent approach for deterministic white signals with finite energy for the discrete time case was developed in Paganini (1996). We will consider a class of signals with given spectral characteristics that gives analysis results that are similar to the ones of Paganini. We consider only scalar signals for simplicity. Let

$$\mathcal{W}_H = \left\{ w \in \mathbf{L}_2^1[0,\infty) : |\widehat{w}(j\omega)|^2 = \frac{\|w\|^2}{\|H\|_2^2}|H(j\omega)|^2 \right\}, \tag{3.8}$$

where $H \in \mathbf{RH}_\infty^{1\times 1}$ is strictly proper and where the $\mathbf{H}_2$-norm is defined by

$$\|H\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(j\omega)|^2 d\omega.$$

We can regard $\mathcal{W}$ as a set of filtered deterministic noise signals. We have the following lemma

LEMMA 3.1
We have $\mathcal{W}_H \subset \mathcal{W}_{\text{inp}}$ when

$$\Upsilon_{\text{inp}} = \left\{ \Upsilon \in \mathbf{RL}_\infty^{1\times 1} : \int_{-\infty}^{\infty} \Upsilon(j\omega)|H(j\omega)|^2\, d\omega \geq 0 \right\}.$$

Proof: Let $w \in \mathcal{W}_H$ and $\Upsilon \in \Upsilon_{\text{inp}}$, then

$$\int_{-\infty}^{\infty} \Upsilon(j\omega)|\widehat{w}(j\omega)|^2 d\omega = \frac{\|w\|^2}{\|H\|_2^2}\int_{-\infty}^{\infty} \Upsilon(j\omega)|H(j\omega)|^2\, d\omega \geq 0,$$

which proves that $\mathcal{W}_H \subset \mathcal{W}_{\text{inp}}$. □

3.4 A Robust Performance Example

We will here illustrate the ideas from the last two sections with a simple example. Consider the system in Figure 3.3. The perturbation Δ is assumed to be a dynamic uncertainty with $\|\Delta\|_\infty \leq 1$, where $\|\cdot\|$ denotes

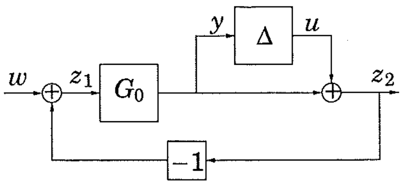

Figure 3.3 System for Example 3.4.

the $\mathbf{H}_\infty$-norm. We assume that w belongs to the set in $\mathcal{W}_H$ in (3.8) for some strictly proper filter H. We want to compute the performance measure

$$\sup_{\substack{w\in\mathcal{W}_H,\ \|w\|\le 1\\ \|\Delta\|_\infty\le 1}} \left\| \begin{bmatrix} W_1 S_\Delta \\ W_2 T_\Delta \end{bmatrix} \right\|. \tag{3.9}$$

The sensitivity function S_Δ and the complementary sensitivity functions T_Δ are defined as

$$\begin{aligned} S_\Delta &= \frac{1}{1+(1+\Delta)G_0}, \\ T_\Delta &= \frac{(1+\Delta)G_0}{1+(1+\Delta)G_0}. \end{aligned}$$

If we assume that the nominal system is stable then the system in Figure 3.3 can be put into the form in Figure 3.2 with

$$G = \frac{1}{1+G_0}\begin{bmatrix} -G_0 & G_0 \\ -1 & 1 \\ 1 & G_0 \end{bmatrix} \in \mathbf{RH}_\infty^{3\times 2}.$$

We will now use Theorem 3.2 to formulate an optimization problem for the computation of an upper bound of (3.9). The spectral characteristic of w can be described by Υ_{inp} in Lemma 3.1 and Δ is described by the following multipliers

$$\Pi_\Delta = \left\{ \begin{bmatrix} x(j\omega) & 0 \\ 0 & -x(j\omega) \end{bmatrix} : x(j\omega) \ge 0,\ \forall\omega \right\}.$$

We can ensure that the optimal value of (3.9) is less that $\gamma^{1/2}$ if the condition (iv) in Theorem 3.2 holds for some $\Pi \in \Pi_\Delta$, and $\Upsilon \in \Upsilon_{\text{inp}}$ and with

$$\Pi_{\text{perf}}(\gamma) = \begin{bmatrix} W_1(j\omega)^* W_1(j\omega) & & \\ & W_2(j\omega)^* W_2(j\omega) & \\ & & -\gamma I \end{bmatrix}.$$

The best upper bound is obtained by solving the optimization problem

$$\inf \gamma \quad \text{subject to} \tag{3.10}$$
$$\begin{cases} \exists \Pi_1 \in \Pi_\Delta,\ \Pi_2 = \Pi_{\text{perf}}(\gamma),\ \Upsilon \in \Upsilon_{\text{inp}}, \quad \text{such that} \\ \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \text{daug}\left(\Pi_1, \Pi_2 + \begin{bmatrix} 0 & 0 \\ 0 & \Upsilon \end{bmatrix}\right)(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0,\ \forall \omega \in [0, \infty]. \end{cases}$$

This is an infinite-dimensional optimization problem. We will in the next two sections formulate a methodology that can be used to obtain suboptimal solutions to this optimization problem.

3.5 LMI Formulations of Robustness Tests

In this section we derive a method for practical application of the robust stability and performance results in Section 3.2. The first step in the analysis is to derive a multiplier description of the perturbation Δ. The multipliers are collected in a set Π_Δ that can be assumed to be a convex cone. In case of performance analysis we also need multipliers for the performance criterion and multipliers that constrain the range of the input to the system. In order to keep the presentation as simple as possible we treat only the case of robust stability analysis. Extension to robust performance analysis is straightforward.

Robust stability analysis based on IQCs can now be formulated as the following convex feasibility test.

ROBUSTNESS TEST 3.1—INFINITE-DIMENSIONAL FEASIBILITY TEST
Find $\Pi \in \Pi_\Delta$ such that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0,$$

for all $\omega \in [0, \infty]$. □

The convex cone Π_Δ is generally infinite-dimensional. It consists of multipliers on the form $\Pi = \Pi_B + \Pi_P$, where $\Pi_B \in \mathbf{RL}_\infty^{2m\times 2m}$ and where Π_P

is a Popov multiplier. The Popov term Π_P involves only a finite number of parameters, namely the parameters in Λ. However, Π_B can generally be chosen from a set of rational transfer functions of arbitrary order. The infinite-dimensionality of the robustness test makes it hard to implement directly as a numerical algorithm.

We will introduce a format for a finite-dimensional parametrization of a convex subcone of Π_Δ. If the robustness test is restricted to this subcone we obtain a finite-dimensional convex feasibility test that generally can be transformed to an equivalent feasibility problem for LMIs. The subcone is defined in terms of a basis multiplier and a convex cone of parameters according to the following definition.

DEFINITION 3.4
Let $\Psi = [\Psi_a \quad \Psi_b]$, where Ψ_a and Ψ_b are $N \times m$ rational transfer functions, and let $M_\Delta \subset \mathbf{R}^{N\times N}$ be a convex cone of symmetric matrices. Then $\mathcal{P}_\Delta(\Psi, M_\Delta) \subset \Pi_\Delta$ denotes the subcone defined as

$$\mathcal{P}_\Delta(\Psi, M_\Delta) = \{\Psi^* M \Psi : M \in M_\Delta\}.$$

We call Ψ a *basis multiplier* and the set M_Δ is called the *parameter set*.

□

We note that Ψ in general is nonproper due to the Popov multipliers. If we for a given $M \in M_\Delta$ write out the expression for the corresponding multiplier we see that it has the structure

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^* & \Pi_{22} \end{bmatrix} = \begin{bmatrix} \Psi_a^* M \Psi_a & \Psi_a^* M \Psi_b \\ \Psi_b^* M \Psi_a & \Psi_b^* M \Psi_b \end{bmatrix}.$$

Before we give an example we notice that addition and diagonal augmentation of subcones on the the form in Definition 3.4 can be done in a simple way.

Addition: Let $\mathcal{P}_{1\Delta} = \mathcal{P}_{1\Delta}(\Psi_1, M_{1\Delta})$ and $\mathcal{P}_{2\Delta} = \mathcal{P}_{2\Delta}(\Psi_2, M_{2\Delta})$, then

$$\mathcal{P}_{1\Delta} + \mathcal{P}_{2\Delta} = \mathcal{P}_\Delta\left(\begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}, \mathrm{diag}(M_{1\Delta}, M_{2\Delta})\right),$$

where

$$\mathrm{diag}(M_{1\Delta}, M_{2\Delta}) = \{\mathrm{diag}(M_1, M_2) : M_1 \in M_{1\Delta}, M_2 \in M_{2\Delta}\}.$$

Augmentation: Let $\mathcal{P}_{\Delta_1} = \mathcal{P}_{\Delta_1}(\Psi_1, M_{\Delta_1})$ and $\mathcal{P}_{\Delta_2} = \mathcal{P}_{\Delta_2}(\Psi_2, M_{\Delta_2})$, then

$$\mathrm{daug}(\mathcal{P}_{\Delta_1}, \mathcal{P}_{\Delta_2}) = \mathcal{P}_\Delta\left(\begin{bmatrix} \Psi_{1a} & 0 & \Psi_{1b} & 0 \\ 0 & \Psi_{2a} & 0 & \Psi_{2b} \end{bmatrix}, \mathrm{diag}(M_{\Delta_1}, M_{\Delta_2})\right),$$

where $\mathrm{diag}(M_{\Delta_1}, M_{\Delta_2})$ is defined analogously with the same operation in the case of addition.

We next give an example that illustrates how parametrizations on the form above can be obtained from well-known multiplier descriptions for a repeated uncertain parameter. More examples are given in Appendix 3.11.

EXAMPLE 3.1
An uncertain real parameter δI, where $\delta \in [-1,1]$ can be described by the following sets of multipliers

$$\begin{aligned}
\Pi_{1\Delta} &= \left\{ \begin{bmatrix} X(j\omega) & 0 \\ 0 & -X(j\omega) \end{bmatrix} : X \in \mathbf{RL}_\infty^{m\times m},\ X(j\omega) = X(j\omega)^* \geq 0 \right\}, \\
\Pi_{2\Delta} &= \left\{ \begin{bmatrix} 0 & Y(j\omega) \\ Y(j\omega)^* & 0 \end{bmatrix} : Y \in \mathbf{RL}_\infty^{m\times m}, Y(j\omega) = -Y(j\omega)^* \right\}, \\
\Pi_{3\Delta} &= \left\{ \begin{bmatrix} 0 & -j\omega\Lambda \\ j\omega\Lambda & 0 \end{bmatrix} : \Lambda = \Lambda^T \right\}.
\end{aligned}$$

We can use the parametrizations

$$X(j\omega) = R(j\omega)^* U R(j\omega), \quad \text{where} \quad U = U^T \geq 0,$$

and

$$Y(j\omega) = S(j\omega) - S(j\omega)^*, \quad \text{where} \quad S = VS_0,$$

where $R \in \mathbf{RH}_\infty^{N_1\times m}, S_0 \in \mathbf{RH}_\infty^{N_2\times m}$, and $U \in \mathbf{R}^{N_1\times N_1}, V \in \mathbf{R}^{m\times N_2}$. It follows from Lemma 1.1 in Chapter 1 that any X and Y satisfying the stated properties can be factorized in this way. We can now use the finite-dimensional convex cones $\mathcal{P}_{1\Delta}(\Psi_1, M_{1\Delta}) \subset \Pi_{1\Delta}$, $\mathcal{P}_{2\Delta}(\Psi_2, M_{2\Delta}) \subset \Pi_{2\Delta}$ and $\mathcal{P}_{3\Delta}(\Psi_3, M_{3\Delta}) = \Pi_{3\Delta}$, where

$$\begin{aligned}
\Psi_1 &= \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}, & M_{1\Delta} &= \left\{ \begin{bmatrix} U & 0 \\ 0 & -U \end{bmatrix} : U = U^T \geq 0 \right\}, \\
\Psi_2 &= \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & S_0 \\ -S_0 & 0 \end{bmatrix}, & M_{2\Delta} &= \left\{ \begin{bmatrix} 0 & 0 & V & 0 \\ 0 & 0 & 0 & V \\ V^T & 0 & 0 & 0 \\ 0 & V^T & 0 & 0 \end{bmatrix} : V \in \mathbf{R}^{m\times N} \right\}, \\
\Psi_3 &= \begin{bmatrix} j\omega I & 0 \\ 0 & I \end{bmatrix}, & M_{3\Delta} &= \left\{ \begin{bmatrix} 0 & \Lambda \\ \Lambda & 0 \end{bmatrix} : \Lambda = \Lambda^T \right\}.
\end{aligned}$$

We note that the constraints that specify the parameter sets are given in terms of nonstrict LMIs. □

Assume that we have obtained a finite-dimensional cone $\mathcal{P}_\Delta(\Psi, M_\Delta) \subset \Pi_\Delta$. If we restrict Robustness Test 3.1 to $\mathcal{P}_\Delta$ we get the following finite-dimensional test.

ROBUSTNESS TEST 3.2—FINITE-DIMENSIONAL FEASIBILITY TEST
Find $M \in M_\Delta$ such that

$$\Phi(j\omega)^* M \Phi(j\omega) < 0, \quad \forall \omega \in [0, \infty], \tag{3.11}$$

where

$$\Phi(s) = \Psi(s) \begin{bmatrix} G(s) \\ I \end{bmatrix} \tag{3.12}$$

is a rational transfer function. □

We note that assumption *(iii)* in Theorem 3.1 implies that Φ must be a proper transfer function. We will next show how the frequency domain inequality in (3.11) can be transformed to an equivalent LMI. The means for doing this is to obtain a state space realization of Φ and then invoke the KYP lemma, which we state next.

LEMMA 3.2—KALMAN-YAKUBOVICH-POPOV
Let M be a symmetric matrix. For the system $\Phi(s) = C(sI - A)^{-1}B + D$, where $\det(j\omega - A) \neq 0$, $\forall \omega$, the following inequalities are equivalent:

a.

$$\Phi^*(j\omega) M \Phi(j\omega) < 0, \quad \forall \omega \in [0, \infty].$$

b. There exists a matrix $P = P^T \in \mathbf{R}^{n\times n}$, where $n = \dim(A)$, such that

$$\left[\begin{array}{cc} A & B \\ I & 0 \\ \hline C & D \end{array}\right]^T \left[\begin{array}{cc|c} 0 & P & 0 \\ P & 0 & 0 \\ \hline 0 & 0 & M \end{array}\right] \left[\begin{array}{cc} A & B \\ I & 0 \\ \hline C & D \end{array}\right] < 0.$$

The equivalence holds for nonstrict inequalities if in addition (A, B) is controllable.

Proof: The lemma follows from Willems (1971b) in the case when (A, B) is controllable. The formulation of the case with strict inequalities does not require controllability. An alternative proof is given in Rantzer (1996). □

Introduce a realization $\Phi(s) = C_\Phi(sI - A_\Phi)^{-1}B_\Phi + D_\Phi$. It is by the assumptions on G and Ψ no restriction to assume that A_Φ has no eigenvalues on the imaginary axis. Hence, it follows by the KYP lemma that Robustness Test 3.2 is equivalent to the following LMI test.

ROBUSTNESS TEST 3.3—FEASIBILITY TEST WITH LMIS
Find $P = P^T$ and $M \in M_\Delta$ such that

$$\left[\begin{array}{cc} A_\Phi & B_\Phi \\ I & 0 \\ \hline C_\Phi & D_\Phi \end{array}\right]^T \left[\begin{array}{cc|c} 0 & P & 0 \\ P & 0 & 0 \\ \hline 0 & 0 & M \end{array}\right] \left[\begin{array}{cc} A_\Phi & B_\Phi \\ I & 0 \\ \hline C_\Phi & D_\Phi \end{array}\right] < 0.$$

The last constraint is an LMI condition. □

The parameter set M_Δ can often can be specified in terms of strict or nonstrict LMIs, see, for example, Example 5.1 and the examples in Appendix 3.11. Strict LMIs are easier to solve numerically. We will here formulate conditions that allow us to solve the nonstrict LMIs as strict LMIs.

Since M_Δ is a convex cone we have $\overline{\text{ri } M_\Delta} = M_\Delta$, where ri M_Δ denotes the relative interior of M_Δ. This proves the equivalence of the following two problems.

1. Find $M \in M_\Delta$ such that $\Phi(j\omega)^* M \Phi(j\omega) < 0, \ \forall\omega \in [0, \infty]$.
2. Find $M \in \text{ri } M_\Delta$ such that $\Phi(j\omega)^* M \Phi(j\omega) < 0, \ \forall\omega \in [0, \infty]$.

This means that every constraint in M_Δ that is formulated in terms of a strictly feasible nonstrict LMI can be solved in terms of its strict version.

3.6 Computation of Robustness Criteria

In applications we often want to compute the stability margin or a performance measure for the system. We consider the computation of such robustness criteria in this section.

We need to parametrize the multipliers in terms of γ, the robustness criterion. This gives us a set valued function, $\Pi_\Delta(\gamma)$, of multipliers. The robustness criterion can then be computed as an optimization problem on the following form.

PRIMAL 3.1—PRIMAL OPTIMIZATION PROBLEM

$$\inf \gamma \qquad \text{subject to}$$

$$P: \begin{cases} \exists \Pi \in \Pi_\Delta(\gamma), \text{ such that} \\ \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0, \quad \forall\omega \in [0, \infty], \end{cases}$$

where $\Pi_\Delta(\gamma)$ is a convex cone of multipliers for every value γ. □

We assume that the following monotonicity condition holds.

DEFINITION 3.5—MONOTONICITY OF $\Pi_\Delta(\gamma)$
If $\gamma_2 \geq \gamma_1$, then $\forall \Pi_1 \in \Pi_\Delta(\gamma_1)$ there exists $\Pi_2 \in \Pi_\Delta(\gamma_2)$ such that $\Pi_1(j\omega) \geq \Pi_2(j\omega)$ for all $\omega \in [0,\infty]$. This ensures that the constraint P is satisfied for all $\gamma > \gamma_{\text{opt}} = \inf_P \gamma$. □

The next example illustrate how such optimization problems can appear.

EXAMPLE 3.2
Consider the system in Figure 3.1 with $G \in \mathbf{RH}_\infty^{3\times 3}$ and a structured perturbation $\Delta = \text{diag}(\Delta_1, \delta_2, \delta_3)$. Here $\Delta_1 \in \mathbf{H}_\infty$ corresponds to a dynamic uncertainty block with $\|\Delta_1\|_\infty \leq 1$ and $\delta_i : \mathbf{L}_{2e}[0,\infty) \to \mathbf{L}_{2e}[0,\infty)$ are nonlinear operators satisfying the norm conditions $\|\delta_2\| \leq 1$ and $\|\delta_3\| \leq \rho$. We want to find an upper bound for ρ such that stability is guaranteed. We can use the multipliers in

$$\Pi_\Delta(\gamma) = \{\text{diag}(x_1(j\omega), x_2, x_3, -x_1(j\omega), -x_2, -\gamma x_3) : \\ x_1(j\omega) \geq 0, \forall\omega, \text{and } x_2, x_3 \geq 0\}.$$

It is easy to see that $\Pi_\Delta(\gamma)$ satisfies the monotonicity condition. An upper bound can be found by solving Primal 3.1 and then use $\rho_{\max} = \gamma_{\text{opt}}^{-1/2}$ as the bound. □

A suboptimal solution to the optimization problem in Primal 3.1 can be found in the following way. Restrict the problem to a finite-dimensional convex cone $\mathcal{P}_\Delta(\Psi, M_\Delta(\gamma)) \subset \Pi_\Delta(\gamma)$, where $M_\Delta(\gamma) \subset \mathbf{R}^{N\times N}$ is a convex cone of symmetric matrices for every $\gamma \in \mathbf{R}$. We assume that $\mathcal{P}_\Delta(\Psi, M_\Delta(\gamma))$ satisfies the monotonicity condition in Definition 3.5

The constraint in Primal 3.1 can then be transformed to an equivalent LMI condition in the same way as before. The corresponding restricted optimization problem can be solved in either of the following ways.

1. By bisection on γ.
2. As an eigenvalue problem for LMIs (EVP).
3. As a generalized eigenvalue problem for LMIs (GEVP).

The last two alternatives are not always possible to use. We restrict our discussion to the case when $M_\Delta(\gamma)$ has the form

$$M_\Delta(\gamma) = \left\{ \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & \gamma M_{22} \end{bmatrix} : \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix} \in M_\Delta \right\}. \tag{3.13}$$

Here $M_\Delta \subset \mathbf{R}^{N\times N}$ is assumed to be a convex cone of symmetric matrices. This means that $M_\Delta(\gamma)$ consists of matrices that depend affinely on γ.

This form for $M_\Delta(\gamma)$ is common in applications. Assume that we have the realization $\Phi(s) = C(sI - A)^{-1}B + D$ of

$$\Phi(s) = \Psi(s)\begin{bmatrix} G(s) \\ I \end{bmatrix}.$$

If we use $\mathcal{P}_\Delta(\Psi, M_\Delta(\gamma))$ in Primal 3.1 with $M_\Delta(\gamma)$ as in (3.13) then by the KYP lemma we have the following equivalent optimization problem.

$$\begin{aligned} &\inf \gamma \quad \text{subject to} \\ &P_{\text{FD}} : \begin{cases} \exists P = P^T,\ M \in M_\Delta, \text{ such that} \\ \mathcal{A}(P, M, \gamma) < 0, \end{cases} \end{aligned} \tag{3.14}$$

where

$$\mathcal{A}(P,M,\gamma) = \left[\begin{array}{cc} A & B \\ I & 0 \\ \hline C_1 & D_1 \\ C_2 & D_2 \end{array}\right]^T \left[\begin{array}{cc|cc} 0 & P & 0 & 0 \\ P & 0 & 0 & 0 \\ \hline 0 & 0 & M_{11} & M_{12} \\ 0 & 0 & M_{12}^T & \gamma M_{22} \end{array}\right] \left[\begin{array}{cc} A & B \\ I & 0 \\ \hline C_1 & D_1 \\ C_2 & D_2 \end{array}\right]. \tag{3.15}$$

The C and D matrices have been structured consistently with the structure of M in (3.13).

Eigenvalue Problems for LMIs

We will here give a condition on M_Δ in (3.13) that allow us to formulate (3.14) as an EVP. To be more precise we will assume that the matrix block M_{22} in (3.13) depends on one parameter only.

PROPOSITION 3.1—EIGENVALUE PROBLEM
Assume that there exist a matrix M_{22}^0 and a parameter α such that

(*i*) $M_{22} = \alpha M_{22}^0$, for all $M \in M_\Delta$, ,

(*ii*) α is restricted to be positive.

Then (3.14) can be formulated as the EVP

$$\begin{aligned} &\inf \gamma \quad \text{subject to} \\ &\begin{cases} \exists P = P^T,\ M \in M_\Delta^0, \text{ such that} \\ \mathcal{A}(P, M, \gamma) < 0, \end{cases} \end{aligned} \tag{3.16}$$

where $M_\Delta^0 = \{M \in M_\Delta : M_{22} = M_{22}^0\}$.

Proof: Consider the optimization problem in (3.14). Assume that $\gamma > \inf_{P_{FD}} \gamma$ and that $P = P^T$ and $M \in M_\Delta$ are feasible for the constraint, i.e., $\mathcal{A}(P, M, \gamma) < 0$. By assumption we have $M_{22} = \alpha M_{22}^0$, for some $\alpha > 0$. It is clear that $\mathcal{A}(P/\alpha, M/\alpha, \gamma) < 0$ and that $M/\alpha \in M_\Delta^0$. This shows that it is nonrestrictive to formulate the optimization problem as in (3.16). This is an EVP since $\mathcal{A}(P, M, \gamma)$ is an affine function of P, $M \in M_\Delta^0$, and γ. Note that M_Δ^0 is a convex set. □

EXAMPLE 3.3—EXAMPLE 3.2 CONTINUED

In Example 3.2 let $x_1 = R^*UR$, where $R \in \mathbf{RH}_\infty^{N\times 1}$ is a basis multiplier and $U = U^T \geq 0$ is the parameter. Hence, we can use the parametrization $\mathcal{P}_\Delta(\Psi, M_\Delta(\gamma))$, where $\Psi = \mathrm{diag}(R, 1, 1, R, 1, 1)$, and where $M_\Delta(\gamma) = \{\mathrm{diag}(U, x_2, x_3, -U, -x_2, -\gamma x_3) : x_2, x_3 \geq 0,\ U = U^T \geq 0\}$. In this example $M_{22} = x_3$ and it is no restriction to fix its value to $x_3^0 = 1$. We can therefore use

$$M_\Delta^0 = \{\mathrm{diag}(U, x_2, 1, -U, -x_2, -1) : x_2 \geq 0, U = U^T \geq 0\}$$

in the EVP in Proposition 3.1. Let G be

$$G = \frac{200}{s+1000}\begin{bmatrix} G_{11} & G_{12} & 0 \\ 0 & G_{22} & G_{23} \\ 0 & G_{32} & G_{33} \end{bmatrix},$$

where

$$G_{11} = 4, \qquad G_{12} = \frac{1}{s+1}, \qquad G_{22} = \frac{1}{s+1},$$
$$G_{23} = \frac{100}{s^2+0.2s+100}, \qquad G_{32} = \frac{1}{s^2+0.2s+1}, \qquad G_{33} = \frac{10}{s+10}.$$

We obtained the results in Table 3.1 using LMI-Lab, Gahinet *et al.* (1995). Here n_{dec} denotes the number of (decision) variables in the optimization problem and $\mathrm{Ritz}(p, n)$ denotes the multiplier

$$\mathrm{Ritz}(p, n) = \begin{bmatrix} 1 & \frac{s-p}{s+p} & \cdots & \frac{(s-p)^n}{(s+p)^n} \end{bmatrix}^T.$$

We did not find a basis R that gave a substantially better result. In Section 3.7 we use duality theory to show that the optimal value of γ in fact is close to 103. □

n_{dec}	Ritz(p,n)	γ_{opt}
58	1	103.44
83	Ritz$(0.1,1)$	103.29

Table 3.1 Numerical results for Example 3.3. Here n_{dec} denotes the number of unknown (decision) variables in the problem. It gives a measure of the complexity of the computations.

Generalized Eigenvalue Problems for LMIs

The next proposition states conditions that allow us to formulate the optimization problem in (3.14) as a GEVP problem on the following form.

$$\inf \gamma \quad \text{subject to} \\ \begin{cases} \exists P = P^T,\ Q = Q^T < 0,\ M \in M_\Delta \text{ such that} \\ \gamma \mathcal{B}(M) < Q, \quad \mathcal{B}(M) < 0, \quad C(P,M,Q) < 0. \end{cases} \tag{3.17}$$

Here $\mathcal{B}$ and C are linear functions of M, P, and Q.

PROPOSITION 3.2—GENERALIZED EIGENVALUE PROBLEM FOR LMIS
Assume that we have matrices N_1 and N_2 satisfying the conditions

(*i*) $[\,C_2 \quad D_2\,]\,N_1 = 0$,

(*ii*) for any $M \in \text{ri}\ M_\Delta$,

$$N_2^T [\,C_2 \quad D_2\,]^T M_{22} [\,C_2 \quad D_2\,] N_2 < 0,$$

(*iii*) the matrix $[\,N_1 \quad N_2\,]$ is square and invertible.

Then the optimization problem in (3.14) can be formulated as the GEVP in (3.17) with

$$\mathcal{B}(M) = N_2^T [\,C_2 \quad D_2\,]^T M_{22} [\,C_2 \quad D_2\,] N_2,$$

and

$$C(P,M,Q) = [\,N_1 \quad N_2\,]^T \mathcal{A}(P,M,0) [\,N_1 \quad N_2\,] + \begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix},$$

where $\mathcal{A}(P,M,\gamma)$ is defined in (3.15).

Proof: To see this we note that the constraint $\mathcal{A}(P, M, \gamma) < 0$ in (3.14) holds if and only if

$$[\,N_1 \quad N_2\,]^T \mathcal{A}(P, M, \gamma)\,[\,N_1 \quad N_2\,] < 0.$$

This constraint can be formulated as

$$C(P, M, 0) + \gamma \begin{bmatrix} 0 & 0 \\ 0 & \mathcal{B}(M) \end{bmatrix} < 0.$$

Consider an arbitrary $P = P^T$ and $M \in M_\Delta$ such that this inequality holds. We first note that it is no restriction to assume that $M \in \text{ri}\, M_\Delta$, which implies that $\mathcal{B}(M) < 0$. It is clear that there exists a $Q = Q^T < 0$ such that the last three constraints in the GEVP hold. For the other direction we note that when the constraint conditions of (3.17) holds then we clearly have $\mathcal{A}(P, M, \gamma) < 0$. This proves the proposition. □

REMARK 3.5
Assume that $M_{22} < 0$ for all $M \in \text{ri}\, M_\Delta$ and that $[\,C_2 \quad D_2\,]$ has full row rank. Then the conditions of Proposition 3.2 are satisfied if we let the columns of N_1 be a basis for $\mathcal{N}([\,C_2 \quad D_2\,])$ and the columns of N_2 be a basis for $\mathcal{N}([\,C_2 \quad D_2\,])^\perp$. These conditions on M_{22} and $[\,C_2 \quad D_2\,]$ are often satisfied in applications and they also ensure that the monotonicity condition on $\mathcal{P}_\Delta(\Psi, M_\Delta(\gamma))$ is satisfied. □

EXAMPLE 3.4
Consider again the system in Figure 3.1 with the same G as in Example 3.3 but with $\Delta = \text{diag}(\delta_1, \delta_2, \Delta_3)$, where δ_1 and δ_2 are nonlinear operators on $\mathbf{L}_{2e}[0, \infty)$ with $\|\delta_i\| \leq 1$, for $i = 1, 2$, and where $\Delta_3 \in \mathbf{H}_\infty$ has the norm bound $\|\Delta_3\|_\infty \leq \rho$. Again we want to find an upper bound for ρ such that stability is guaranteed. We use the multipliers

$$\Pi_\Delta(\gamma) = \{\text{diag}(x_1, x_2, x_3(j\omega), -x_1, -x_2, -\gamma x_3(j\omega)) : \\ x_1, x_2 \geq 0, x_3(j\omega) \geq 0, \ \forall \omega\}.$$

If we use $x_3 = R^* U R$, where R and U are as in Example 3.3 then we get $\mathcal{P}_\Delta(\Psi, M_\Delta(\gamma))$, with $\Psi = \text{diag}(1, 1, R, 1, 1, R)$ and

$$M_\Delta(\gamma) = \left\{\text{diag}(x_1, x_2, U, -x_1, -x_2, -\gamma U) : x_1, x_2 \geq 0, \ U = U^T \geq 0\right\}.$$

In this case $M_{22} = -U$. Hence, the conditions in Remark 3.5 are satisfied for every reasonable choice of R and its state space realization. By using

n_{dec}	Ritz(p, n)	γ_{opt}
58	1	103.3
86	Ritz(5, 1)	5.13
119	Ritz(5, 2)	0.4789
158	Ritz(5, 3)	0.1918
203	Ritz(5, 4)	0.1918

Table 3.2 Numerical results for Example 3.4. Here Ritz(p, n) is defined as in Example 3.3.

LMI-Labs GEVP routine we obtained the numerical results in Table 3.2. The dual to this optimization problem will in the next section be used to show that the last two basis functions Ritz(5, 3) and Ritz(5, 4) are close to optimal. We obtain the upper bound $\rho_{\max} = (0.1918)^{-1/2}$. □

The Robust Performance Example

We will here briefly show the robust performance example in Section 3.4 can be transformed to a suboptimal EVP problem. We first find finite dimensional cones.

Π_Δ: We can use $\mathcal{P}_\Delta(\Psi, M_\Delta)$, where

$$\Psi = \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}, \quad \text{and} \quad M_\Delta = \left\{ \begin{bmatrix} U & 0 \\ 0 & -U \end{bmatrix} : U = U^T \geq 0 \right\},$$

and where $R \in \mathbf{RH}_\infty^{N\times 1}$ and $U \in \mathbf{R}^{N\times N}$.

Π_{perf}: We use $\mathcal{P}_{\text{perf}}(\Psi, M_{\text{perf}}(\gamma))$, where

$$\Psi = \begin{bmatrix} W_1 & 0 & 0 \\ 0 & W_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad M_{\text{perf}}(\gamma) = \left\{ \begin{bmatrix} xI_2 & 0 \\ 0 & -\gamma x \end{bmatrix} : x \geq 0 \right\}.$$

Υ_{inp}: We use $\mathcal{P}_{\text{inp}}([\,0 \quad 0 \quad \Psi\,], M_\Delta)$, where $\Psi \in \mathbf{RH}_\infty^{N\times 1}$, and where M_{inp} is defined as in (3.29) in Appendix 3.11.

We obtain a finite-dimensional parametrization $\mathcal{P}_{\text{Tot}}(\Psi_{\text{Tot}}, M_{\text{Tot}}(\gamma))$ for the total multiplier in (3.10) as

$$\mathcal{P}_{\text{Tot}}(\gamma) = \text{daug}(\mathcal{P}_\Delta, \mathcal{P}_{\text{inp}} + \mathcal{P}_{\text{perf}}(\gamma)) \subset \text{daug}\left(\Pi_\Delta, \Pi_{\text{perf}}(\gamma) + \begin{bmatrix} 0 & 0 \\ 0 & \Upsilon_{\text{inp}} \end{bmatrix}\right).$$

With this order of addition of $\mathcal{P}_{\text{inp}}$ and $\mathcal{P}_{\text{perf}}(\gamma)$ we get $M_{\text{Tot}}(\gamma)$ on the form as in (3.13) with $M_{22} = x$, and where x must be positive. This means that we can use Proposition 3.1 to formulate a suboptimal version of (3.10) as an EVP.

3.7 Duality Bounds

The key problem with the LMI approach for multiplier computation is to find a suitable basis multiplier Ψ for the cone $\mathcal{P}_\Delta(\Psi, M_\Delta(\gamma) \subset \Pi_\Delta(\gamma)$. A large basis affects the speed of the computations severely. It is therefore useful to have tools that help us to choose a good basis multiplier and tools that can help us to evaluate the quality of a particular basis multiplier. We will discuss briefly how the dual to Primal 3.1 can be used to give a lower bound on the objective value γ. This gives the desired quality measure of the basis multipliers.

Figure 3.4 illustrates the use of the dual. The left hand side of the figure illustrates that the primal constraint P is satisfied for all $\gamma > \gamma_{\text{opt}}$. The upper half of the right hand side illustrates the assumption that the constraint of the restricted primal is satisfied for all γ larger than its objective value γ_p. A similar monotonicity condition will hold for the dual optimization problem. The dual is infinite-dimensional and we have to consider suboptimal solutions to it. Any such suboptimal dual objective gives a lower bound on the objective value, γ_{opt}, of the primal. A small duality gap indicates that the basis multiplier used for the primal is close to optimal.

A detailed treatment of the dual to Primal 3.1 will be given in Chapter 4 and 5. We will here state and solve the dual to the optimization problem in Example 3.3. The multipliers in that example are of the form

$$\Pi_\Delta(\gamma) = \{\Phi(j\omega) + \Psi : \Phi(j\omega) \in \Phi_\Delta(\gamma),\ \forall\omega \in [0,\infty], \Psi \in \Psi_\Delta(\gamma)\}, \tag{3.18}$$

where $\Phi_\Delta(\gamma) \subset \mathbf{C}^{2m\times 2m}$ is a closed convex cone of Hermitian matrices for all $\gamma \in \mathbf{R}$, and $\Psi_\Delta(\gamma) \subset \mathbf{R}^{2m\times 2m}$ is a closed convex cone of symmetric matrices for all $\gamma \in \mathbf{R}$. Optimization problems involving such multipliers are treated in Chapter 5.

EXAMPLE 3.5—DUAL OF THE OPTIMIZATION PROBLEM IN EXAMPLE 3.3
The multipliers in Example 3.3 are of the form in (3.18) with

$$\begin{aligned}\Phi_\Delta(\gamma) &= \{\text{diag}(x_1, 0, 0, -x_1, 0, 0) : x_1 \ge 0\},\\ \Psi_\Delta(\gamma) &= \{\text{diag}(0, x_2, x_3, 0, -x_2, -\gamma x_3) : x_2, x_3 \ge 0\}.\end{aligned}$$

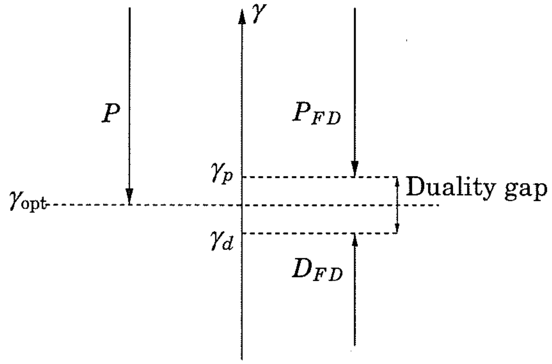

Figure 3.4 The vertical axis of the diagram corresponds to the objective value γ of the optimization problem. The left hand side of the diagram illustrates the monotonicity condition on the infinite-dimensional primal optimization problem. The right hand side illustrates the same monotonicity condition on the restricted primal. It also shows how the dual can be used to obtain a lower bound on the objective value γ_{opt}.

It follows from the results in Chapter 5 that a, possibly suboptimal, solution to the dual can be solved in the following way. Choose a frequency grid $\Omega = \{\omega_1, \omega_2\}$ and solve the optimization problem:

$$\sup \gamma \quad \text{subject to}$$
$$\begin{cases} \exists Z_1, Z_2 \geq 0, \text{ at least one nonzero, such that} \\ [\,G_{11}(j\omega_1) \quad G_{12}(j\omega_1) \quad 0\,]\, Z_1\, [\,G_{11}(j\omega_1) \quad G_{12}(j\omega_1) \quad 0\,]^* - Z_{1_{11}} \geq 0, \\ [\,G_{11}(j\omega_2) \quad G_{12}(j\omega_2) \quad 0\,]\, Z_2\, [\,G_{11}(j\omega_2) \quad G_{12}(j\omega_2) \quad 0\,]^* - Z_{2_{11}} \geq 0, \\ \displaystyle\sum_{k=1}^{2} [\,0 \quad G_{22}(j\omega_k) \quad G_{23}(j\omega_k)\,]\, Z_k\, [\,0 \quad G_{22}(j\omega_k) \quad G_{23}(j\omega_k)\,]^* - Z_{k_{22}} \geq 0, \\ \displaystyle\sum_{k=1}^{2} [\,0 \quad G_{32}(j\omega_k) \quad G_{33}(j\omega_k)\,]\, Z_k\, [\,0 \quad G_{32}(j\omega_k) \quad G_{33}(j\omega_k)\,]^* - \gamma Z_{k_{33}} \geq 0. \end{cases}$$

The constraints in this optimization problem involve complex-valued linear matrix inequalities. We used LMI-Lab for various choices of frequency grid. The results are presented in Table 3.3. We see that the last value in the table is close to the solution of the primal in Example 3.3. □

ω_1	ω_2	γ_d
10	–	0.026
1	–	0.18
1	10	102.1

Table 3.3 Solution to the dual optimization problem in Example 3.5.

ω_1	ω_2	γ_d
10	–	0.026
5	–	0.032
1	10	0.183
1	5	0.183
0.98	–	0.186

Table 3.4 Solution to the dual optimization problem in Example 3.6.

EXAMPLE 3.6—DUAL OF EXAMPLE 3.4
For Example 3.4 we have

$$\Phi_\Delta(\gamma) = \{\text{diag}(0,0,x_3,0,0,-\gamma x_3) : x_3 \geq 0\},$$
$$\Psi_\Delta(\gamma) = \{\text{diag}(x_1,x_2,0,-x_1,-x_2,0) : x_1,x_2 \geq 0\}.$$

The dual will in this case be similar to the dual in Example 3.5. Numerical solution with LMI-Lab for different frequency grids are presented in Table 3.4. The largest value was obtained with only one frequency in the grid. □

3.8 Practical Considerations

This section considers several practical aspects concerning the suggested method for multiplier computation. We discuss issues that arise in a computer implementation of the method and a brief discussion on the problem of choosing suitable basis multipliers is given.

Issues in a Computer Implementation

In a computer implementation all manipulations are performed in terms of state space realizations of the transfer functions involved. In particular, the transfer function $\Psi = [\,\Psi_a \quad \Psi_b\,]$ in $\mathcal{P}_\Delta(\Psi, M_\Delta)$ is represented in terms of a state space realization. It is no restriction to assume that the Popov multipliers contributes only to Ψ_a, see Example 3.1. We use the following notation

$$\Psi(s) = \left[\begin{array}{c|cc|c} A & B_a & B_b & 0 \\ \hline C & D_a & D_b & E_a \end{array}\right] = C(sI - A)^{-1}B + D + s\,[\,E_a \quad 0\,],$$

where $B = [\,B_a \quad B_b\,]$ and $D = [\,D_a \quad D_b\,]$.

Assume that we have the cones $\mathcal{P}_{1\Delta}(\Psi_1, M_{1\Delta})$ and $\mathcal{P}_{2\Delta}(\Psi_2, M_{2\Delta})$, where Ψ_1 and Ψ_2 have the realizations

$$\Psi_1 = \left[\begin{array}{c|cc|c} A_1 & B_{1a} & B_{1b} & 0 \\ \hline C_1 & D_{1a} & D_{1b} & E_{1a} \end{array}\right], \quad \Psi_2 = \left[\begin{array}{c|cc|c} A_2 & B_{2a} & B_{2b} & 0 \\ \hline C_2 & D_{2a} & D_{2b} & E_{2a} \end{array}\right]. \tag{3.19}$$

Then the transfer function Ψ resulting from the addition $\mathcal{P}_{1\Delta} + \mathcal{P}_{2\Delta}$ has the realization

$$\Psi = \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix} = \left[\begin{array}{cc|c|c} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & B_2 & 0 \\ \hline C_1 & 0 & D_1 & E_{1a} \\ 0 & C_2 & D_2 & E_{2a} \end{array}\right]. \tag{3.20}$$

Let us consider diagonal augmentation in the same way. Assume that we have the cones $\mathcal{P}_{\Delta_1}(\Psi_1, M_{\Delta_1})$ and $\mathcal{P}_{\Delta_2}(\Psi_2, M_{\Delta_2})$, where Ψ_1 and Ψ_2 have the realizations in (3.19). Then diagonal augmentation $\operatorname{daug}(\mathcal{P}_{\Delta_1}, \mathcal{P}_{\Delta_2})$ gives

$$\begin{aligned} \Psi &= \begin{bmatrix} \Psi_{1a} & 0 & \Psi_{1b} & 0 \\ 0 & \Psi_{2a} & 0 & \Psi_{2b} \end{bmatrix} \\ &= \left[\begin{array}{cc|cccc|cc} A_1 & 0 & B_{1a} & 0 & B_{1b} & 0 & 0 & 0 \\ 0 & A_2 & 0 & B_{2a} & 0 & B_{2b} & 0 & 0 \\ \hline C_1 & 0 & D_{1a} & 0 & D_{1b} & 0 & E_{1a} & 0 \\ 0 & C_2 & 0 & D_{2a} & 0 & D_{2b} & 0 & E_{2a} \end{array}\right]. \end{aligned} \tag{3.21}$$

We next show how to find a realization of

$$\Phi = \Psi \begin{bmatrix} G \\ I \end{bmatrix}.$$

Assume that we have obtained a realization $\Psi(s) = C_\Psi(sI - A_\Psi)^{-1}B_\Psi + D_\Psi + sE_\Psi$ and let the realization of the nominal system be $G(s) = C(sI -$

$A)^{-1}B + D$. By assumption (*iii*) in Theorem 3.1 and Theorem 3.2 it follows that it is no restriction to assume

$$E_\Psi \begin{bmatrix} D \\ I \end{bmatrix} = 0.$$

Hence, with

$$\widehat{C} = \begin{bmatrix} C \\ 0 \end{bmatrix}, \quad \widehat{D} = \begin{bmatrix} D \\ I \end{bmatrix},$$

we obtain the state space realization $\Phi(s) = C_\Phi(sI - A_\Phi)^{-1}B_\Phi + D_\Phi$, where

$$\begin{aligned} A_\Phi &= \begin{bmatrix} A_\Psi & B_\Psi \widehat{C} \\ 0 & A \end{bmatrix}, & B_\Phi &= \begin{bmatrix} B_\Psi \widehat{D} \\ B \end{bmatrix}, \\ C_\Phi &= \begin{bmatrix} C_\Psi & D_\Psi \widehat{C} + E_\Psi \widehat{C} A \end{bmatrix}, & D_\Phi &= D_\Psi \widehat{D} + E_\Psi \widehat{C} B. \end{aligned} \tag{3.22}$$

These matrices can then be used in Robustness Test 3.3.

The constraints that define M_Δ can in many examples be formulated in terms of LMIs or frequency domain inequalities that have an equivalent LMI formulation. We refer to Appendix 3.11 for examples. All manipulation are also here performed in terms of the state space realizations.

Minimal Realizations: A very important issue regarding the efficiency of the LMI computations is that the realization of the transfer function Ψ is of low order since this affects the order of P in Robustness Test 3.3. The order of this matrix determines the number of variables that the LMI solver has to optimize.

Assuming that Ψ_1 and Ψ_2 above have minimal realizations, we see from (3.21) that diagonal augmentation results in a Φ with minimal realization. However, when we add $\mathcal{P}_{1\Delta}$ and $\mathcal{P}_{2\Delta}$ then, as can be observed from (3.20), we obtain a realization that is observable but not necessarily controllable. By the PBH rank test, see Kailath (1980), we have controllability if and only if

$$\text{rank} \begin{bmatrix} sI - A_1 & 0 & B_1 \\ 0 & sI - A_2 & B_2 \end{bmatrix} = n_1 + n_2, \quad \forall s \in \mathbf{C},$$

where $n_i = \dim(A_i)$, $i = 1, 2$. We can see that the realization is controllable if for example $\text{eig}(A_1) \cap \text{eig}(A_2) = \emptyset$. To conclude we have the following properties.

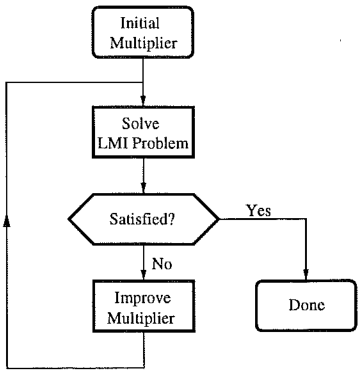

Figure 3.5 Algorithm for multiplier optimization.

1. If the basis multipliers supplied by the user have minimal realizations then the realization of Ψ can lose minimality only when adding multiplier sets.

2. The suggested realization of

$$\Phi = \Psi \begin{bmatrix} G \\ I \end{bmatrix} \tag{3.23}$$

   in (3.22) may lose minimality even if both G and Ψ have minimal realizations.

It would be possible to use a model reduction algorithm to reduce the order of a non-minimal realization. However, there are some drawbacks with this. First it may introduce unnecessary numerical errors. Secondly, most model reduction algorithms require causal systems. This is a problem if noncausal multipliers are collected in Ψ since this gives an unstable Φ in (3.23). This is sometimes convenient to do but never necessary. Thirdly, the sparsity pattern of the matrices appearing in Robustness Test 3.3 may be lost. This might increase computational complexity.

Choice of Basis Multipliers

The success of the proposed method for multiplier computation is critically dependent on our ability to find a good basis multiplier. In applications we use the algorithm in Figure 3.5. The different steps are explained below.

Step 1. Initial Multiplier: Start with simple multipliers. Popov multipliers and constant multipliers are often sufficient. The corresponding LMI computations will be of low complexity.

Step 2. Solve LMI Problem: Depending if we have a feasibility problem or if we compute a robustness criterion we solve the convex feasibility test in Robustness Test 3.3 or any of the optimization problems in Proposition 3.1 or Proposition 3.2.

Step 3. Satisfied?: For a feasibility problem we stop once we have a feasible solution or if we can prove unfeasibility of the problem by using duality theory. For robustness margin problems we stop when the duality gap is small.

Step 4. Improve Multiplier: This is the hard step. We used ad-hoc arguments for the numerical examples in the thesis. We discuss this below.

Ideas for Improving the Basis Multiplier: In each iteration we obtain a multiplier $\Pi_k = \Psi_k^* M_k \Psi_k$, where M_k is the solution to the LMI problem in the second step of the algorithm. The quality of this multiplier can be investigated by studying the function

$$f(\omega) = \overline{\lambda}\left(\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi_k(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}\right),$$

where $\overline{\lambda}$ denotes the largest eigenvalue. A plot of $f(\omega)$ as a function of ω in a diagram gives information about the worst frequencies. A suitable multiplier that is active at these frequencies should be included to improve the basis. However, the choice of this additional multiplier is a difficult problem. To see this we notice that the multiplier Π_k in general is structured since it has been obtained by a diagonal augmentation $\Pi_k = \text{daug}(\Pi_{k(1)}, \dots, \Pi_{k(n)})$. Every multiplier $\Pi_{k(i)}$ in this combination is also subject to various constraints. It is therefore a nontrivial problem to improve the multiplier such that it contributes in useful directions. However, although it is a nonsmooth and infinite-dimensional problem we believe that it is possible to develop useful gradient methods that indicates how the additional multiplier should be chosen.

It may also be useful to exclude those parts of the basis Ψ_k that does not contribute to the robustness problem. This can be done by investigating elements in M_k that are small in magnitude. The corresponding parts of Ψ_k can most probably be removed without any loss.

The removal and inclusion of multipliers is performed by adding or removing suitable subcones $\mathcal{P}_\Delta(\Psi_{k+1}, M_{(k+1)\Delta})$ to the total multiplier description.

3.9 Conclusions

We have introduced a format for multiplier computation in robustness analys. It is applicable to a large class of systems. The resulting computational problem can be formulated as a convex optimization problem in terms of linear matrix inequalities. There are interesting problems that remains to be solved. It would in particular be useful to have tools that give directions for improvement of the basis multiplier that defines the subspace over which the analysis is performed.

3.10 Appendix: Proof of Theorem 3.2 and Remark 3.4

Let $w \in \mathbf{L}_2^q[0,\infty)$ be in $\mathcal{W}_{\text{inp}}$. The stability assumption implies that $\Delta(y) \in \mathbf{L}_2^m[0,\infty)$. Consider the operator

$$\widetilde{\Pi} = \operatorname{daug}\left(\Pi_B + \Pi_P, \Pi_{\text{perf}} + \begin{bmatrix} 0 & 0 \\ 0 & \Upsilon_{\text{inp}} \end{bmatrix}\right).$$

We note that the condition $\Lambda\,[\,G_{11}(\infty) \quad G_{12}(\infty)\,] = 0$ implies that

$$H = \begin{bmatrix} G \\ I \end{bmatrix}^* \widetilde{\Pi} \begin{bmatrix} G \\ I \end{bmatrix} \in \mathbf{RL}_\infty^{(l+m)\times(l+m)}.$$

Hence, H is a bounded and Hermitian valued operator on $\mathbf{L}_2^{(m+l)}(-\infty,\infty)$. Evaluation of the corresponding quadratic functional for $\left[\,\widehat{\Delta(y)}^* \quad \widehat{w}^*\,\right]^*$ gives

$$0 \ge \int_{-\infty}^{\infty} \begin{bmatrix} \widehat{\Delta(y)} \\ \widehat{w} \end{bmatrix}^* \begin{bmatrix} G \\ I \end{bmatrix}^* \widetilde{\Pi} \begin{bmatrix} G \\ I \end{bmatrix} \begin{bmatrix} \widehat{\Delta(y)} \\ \widehat{w} \end{bmatrix} d\omega = \int_{-\infty}^{\infty} \begin{bmatrix} \widehat{z} \\ \widehat{w} \end{bmatrix}^* \Pi_{\text{perf}} \begin{bmatrix} \widehat{z} \\ \widehat{w} \end{bmatrix} d\omega + \underbrace{2\int_0^{\infty} \Delta(y)^T \Lambda (P_\Lambda y)' dt + \int_{-\infty}^{\infty} \begin{bmatrix} \widehat{y} \\ \widehat{\Delta(y)} \end{bmatrix} \Pi_B \begin{bmatrix} \widehat{y} \\ \widehat{\Delta(y)} \end{bmatrix} d\omega}_{\ge -\gamma |P_\Lambda y_0|^2 = 0} + \underbrace{\int_{-\infty}^{\infty} \widehat{w}^* \Upsilon \widehat{w} d\omega}_{\ge 0}.$$

The equality follows from the observations

$$G \begin{bmatrix} \widehat{\Delta(y)} \\ \widehat{w} \end{bmatrix} = \begin{bmatrix} \widehat{y} \\ \widehat{z} \end{bmatrix},$$

and

$$\mathcal{F}^{-1}\left\{ j\omega\Lambda\,[\,G_{11} \quad G_{12}\,] \begin{bmatrix} \widehat{\Delta(y)} \\ \widehat{z} \end{bmatrix} \right\} = \Lambda(P_\Lambda y)',$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. We note that $P_\Lambda y(0) = 0$ since it is assumed that there are no initial conditions in the system.

$$\int_{-\infty}^{\infty} \begin{bmatrix} \widehat{z} \\ \widehat{w} \end{bmatrix}^* \Pi_{\text{perf}} \begin{bmatrix} \widehat{z} \\ \widehat{w} \end{bmatrix} d\omega \leq 0.$$

This proves the theorem.

The statement of Remark 3.4 follows since the upper left block of the inequality (3.6) implies that

$$\begin{bmatrix} G_{11}(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G_{11}(j\omega) \\ I \end{bmatrix} + \begin{bmatrix} G_{21}(j\omega) \\ 0 \end{bmatrix}^* \Pi_{\text{perf}}(j\omega) \begin{bmatrix} G_{21}(j\omega) \\ 0 \end{bmatrix} < 0,$$

for all $\omega \in [0, \infty]$. By assumption (ib) the second term is nonnegative. Hence, the stability follows from Theorem 3.1.

3.11 Appendix: Finite-Dimensional Multiplier Sets

This appendix gives several examples of how different multiplier sets can be put into the finite-dimensional format $\mathcal{P}_\Delta(\Psi, M_\Delta)$ in Definition 3.4. The list of examples is by no means complete but it shows how a wide variety of constraints can be treated.

We start to show how a conic combination of multipliers can be put into the format in Definition 3.4. A conic combination of multipliers $\Pi = \sum_{i=1}^n \alpha_i(\Psi_i + \Psi_i^*)$, where $\alpha_i \geq 0$, can be represented as $\mathcal{P}_\Delta(\Psi, M_\Delta)$, where

$$\Psi = \begin{bmatrix} \Psi_1 \\ \vdots \\ \Psi_n \\ I \end{bmatrix}, \quad M_\Delta = \left\{ \left[\begin{array}{ccc|c} 0 & & & \alpha_1 I \\ & \ddots & & \vdots \\ & & 0 & \alpha_n I \\ \hline \alpha_1 I & \dots & \alpha_n I & 0 \end{array} \right] : \alpha_i \geq 0 \right\}.$$

We will in the next three subsections treat multipliers for the perturbation Δ, multipliers for performance specifications, and multipliers for signal specifications, respectively.

Multipliers for the Perturbation Δ

We will here show how finite-dimensional parametrizations in the format of Definition 3.4 can be obtained for multiplier descriptions of polytopic uncertainty, slowly time-varying parameters, and slope restricted nonlinearities.

Polytopic Uncertainty: Let $\Delta(t)$ be a measurable real-valued matrix function with $\Delta(t) \in \mathrm{co}\{\Delta_1, \dots, \Delta_N\}$ for all $t \geq 0$ (the polytope defined as the convex hull of the vertices $\Delta_i \in \mathbf{R}^{m\times m}$). Then the operator defined by multiplication by Δ in the time domain satisfies the IQCs defined by the multipliers in $\mathcal{P}_\Delta(\Psi, M_\Delta)$, where $\Psi = I$, and where

$$M_\Delta = \left\{ M \in \mathbf{R}^{2m\times 2m} : M_{22} \leq 0, \ \begin{bmatrix} I \\ \Delta_i \end{bmatrix}^T M \begin{bmatrix} I \\ \Delta_i \end{bmatrix} \geq 0, \ \forall i, \dots, N \right\}.$$

Here M_{22} is the lower right $m \times m$ matrix block. Note that the set M_Δ is defined in terms of a finite number of LMI conditions.

Slowly Time-Varying Parameter: Let us consider a slowly varying multiplication operator. Let $\Delta(t) = \delta(t)I$, where $\delta(t)$ is a slowly time-varying real parameter that satisfies the constraints

$$\underline{k} \leq \delta(t) \leq \overline{k}, \quad \forall t \geq 0,$$
$$-2\alpha\delta(t) \leq \frac{d}{dt}\delta(t) \leq 2\alpha\delta(t), \quad \forall t \geq 0.$$

for some $0 < \underline{k} < \overline{k} < \infty$ and $\alpha > 0$. We can assume that $\underline{k}$ is small and that $\overline{k}$ is large. We can use the multipliers that follows from the stability result in Sundareshan and Thathachar (1972), where the case with a single parameter was treated. The extension to repeated parameters is trivial as was shown in Jönsson (1995). Let $M_1, M_2 \in \mathbf{RH}_\infty^{m\times m}(\alpha)$ with $M_i(j\omega - \alpha) + M_i(j\omega - \alpha)^* \geq 0$ for all $\omega \in \mathbf{R}$ and for $i = 1, 2$. We can use

$$\Pi(j\omega) = \begin{bmatrix} 0 & M(j\omega) \\ M(j\omega)^* & 0 \end{bmatrix},$$

where $M = M_1 + M_2^*$.

A finite-dimensional subcone is obtained by using $M_1 = UM_{10}$ and $M_2 = VM_{20}$, where $M_{10}, M_{20} \in \mathbf{RH}_\infty^{N\times m}(\alpha)$ and $U, V \in \mathbf{R}^{m\times N}$. In the format $\mathcal{P}_\Delta(\Psi, M_\Delta)$ we use

$$\Psi = \begin{bmatrix} I_m & 0 \\ 0 & I_m \\ 0 & M_{10} \\ M_{20} & 0 \end{bmatrix},$$

and

$$M_\Delta = \left\{ M = \begin{bmatrix} 0 & 0 & U & 0 \\ 0 & 0 & 0 & V \\ U^T & 0 & 0 & 0 \\ 0 & V^T & 0 & 0 \end{bmatrix} : \Phi_1(j\omega)^* M \Phi_1(j\omega) \geq 0, \quad \forall \omega \in \mathbf{R} \right\},$$

where

$$\Phi_1(s) = \begin{bmatrix} I_m & 0 \\ 0 & I_m \\ M_{10}(s-\alpha) & 0 \\ 0 & M_{20}(s-\alpha) \end{bmatrix}.$$

We can obtain a controllable realization of Φ_1 from controllable realizations of M_{10} and M_{20}. The frequency domain constraint for M_Δ is then equivalent to a nonstrict LMI. This is a consequence of the nonstrict version of the KYP lemma.

Multiplication by a Harmonic Oscillation Let $\Delta(t) = \cos(\omega_0 t) I_m$ be the multiplication operator defined by $(\Delta u)(t) = \cos(\omega_0 t) u(t)$. We can use the multipliers from Megretski and Rantzer (1995), i.e.,

$$\Pi(j\omega) = \begin{bmatrix} \frac{1}{2}(X(j\omega + j\omega_0) + X(j\omega - j\omega_0)) & 0 \\ 0 & -X(j\omega) \end{bmatrix},$$

where $X(j\omega) = X(j\omega)^* \geq 0$. Let us use the finite-dimensional restriction $X = R^* U R$, where $U \in \mathbf{R}^{N\times N}$ satisfies $U = U^T \geq 0$, and where $R \in \mathbf{RH}_\infty^{N\times m}$ has the realization $R(s) = C(sI - A)^{-1}B + D$. In the format $\mathcal{P}_\Delta(\Psi, M_\Delta)$ we use

$$\Psi = \begin{bmatrix} \widehat{R} & 0 \\ 0 & R \end{bmatrix}, \quad M_\Delta = \left\{ \begin{bmatrix} U & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & -U \end{bmatrix} : U = U^T \geq 0 \right\}.$$

The transfer function $\widehat{R} \in \mathbf{RH}_\infty^{2N\times m}$ is obtained by a spectral factorization. It has the realization

$$\widehat{R}(s) = \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix} \left(sI - \begin{bmatrix} A & \omega_0 I \\ -\omega_0 I & A \end{bmatrix} \right)^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix} + \begin{bmatrix} D \\ 0 \end{bmatrix}. \tag{3.24}$$

We will next prove this spectral factorization. With $X = R^*UR$ we can represent $\frac{1}{2}(X(s+j\omega_0) + X(s-j\omega_0))$ as

$$\begin{bmatrix} \frac{1}{\sqrt{2}}R(s+j\omega_0) \\ \frac{1}{\sqrt{2}}R(s-j\omega_0) \end{bmatrix}^* \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}}R(s+j\omega_0) \\ \frac{1}{\sqrt{2}}R(s-j\omega_0) \end{bmatrix}.$$

There is, however, a problem since

$$\widetilde{R}(s) = \begin{bmatrix} R(s+j\omega_0) \\ R(s-j\omega_0) \end{bmatrix}$$

is not in $\mathbf{RL}_\infty^{2n\times m}$. This is taken care of by making a similarity transformation of $\widetilde{R}$ with

$$T = \begin{bmatrix} I_n & I_n \\ jI_n & -jI_n \end{bmatrix}.$$

This gives the realization

$$\widetilde{R}(s) = \left[\begin{array}{c|c} \begin{pmatrix} A & w_0 I \\ -w_0 I & A \end{pmatrix} & \begin{pmatrix} 2B \\ 0 \end{pmatrix} \\ \hline \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix} T^{-1} & \begin{pmatrix} D \\ D \end{pmatrix} \end{array}\right].$$

With this representation it is easy to verify that

$$\widetilde{R}^*(s) \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix} \widetilde{R}(s) = \widehat{R}^*(s) \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix} \widehat{R}(s),$$

where $\widehat{R}$ is defined as in (3.24). This gives the result.

Odd Slope Restricted Nonlinearity Assume that the nonlinearity $\varphi : \mathbf{R} \to \mathbf{R}$ satisfies the conditions

(*i*) φ is odd,

(*ii*) there exists $k > 0$ such that $|\varphi(y)| \leq k|y|$, for all $y \in \mathbf{R}$,

(*iii*) the slope condition

$$\alpha \leq \frac{\varphi(y_1) - \varphi(y_2)}{y_1 - y_2} \leq \beta$$

holds with $-\infty < \alpha < \beta \leq \infty$.

We assume for simplicity that $\beta \neq 0$. Then φ satisfies the IQCs defined by the multipliers

$$\Pi(j\omega) = T^T \begin{bmatrix} 0 & h_0 + H(j\omega)^* \\ h_0 + H(j\omega) & 0 \end{bmatrix} T, \tag{3.25}$$

see Zames and Falb (1968). Here $h_0 \geq 0$, and H is a strictly proper rational transfer function with a corresponding weighting function h that satisfies the the $\mathbf{L}_1$-norm condition

$$\|h\|_1 = \int_{-\infty}^{\infty} |h(t)| dt \leq h_0.$$

The transformation matrix is defined as

$$T = \begin{bmatrix} 1 & -\beta^{-1} \\ -\alpha & 1 \end{bmatrix}.$$

To obtain a finite-dimensional parametrization we use the ideas suggested in Laiou (1995). Let $H_i \in \mathbf{RL}_\infty^{1\times 1}$, $i = 1, \ldots, N$, be strictly proper *basis functions* for a finite-dimensional parametrization of H and express H as

$$H = \sum_{i=1}^{N} (\lambda_i^+ - \lambda_i^-) H_i,$$

where $\lambda_i^+, \lambda_i^- \geq 0$. We can assume that $H_i = H_{ic} + H_{iac}$, where the causal part has the realization $H_{ic} = C_{ic}(sI - A_{ic})^{-1} B_{ic}$, and where the anticausal part has the realization $H_{iac} = C_{iac}(sI - A_{iac})^{-1} B_{iac}$. The $\mathbf{L}_1$-norm of the weighting function corresponding to H_i can be computed as

$$\|h_i\|_1 = \int_0^{\infty} |C_{ic} e^{A_{ic}t} B_{ic}| dt + \int_0^{\infty} |C_{iac} e^{-A_{iac}t} B_{iac}| dt.$$

Then the constraint

$$\sum (\lambda_i^+ + \lambda_i^-) \|h_i\| \leq h_0 \tag{3.26}$$

ensures that $\|h\|_1 \leq h_0$. To obtain a parametrization $\mathcal{P}_\Delta(\Psi, M_\Delta)$ we define

$$\Psi = \left[\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \\ H_1 & 0 \\ -H_1 & 0 \\ \vdots & 0 \\ H_N & 0 \\ -H_N & 0 \end{array}\right] T, \qquad \Phi_1 = \left[\begin{array}{c} 1 \\ \hline -1 \\ \|h_1\| \\ \|h_1\| \\ \vdots \\ \|h_N\| \\ \|h_N\| \end{array}\right], \qquad \lambda = \begin{bmatrix} h_0 \\ \lambda_1^+ \\ \lambda_1^- \\ \vdots \\ \lambda_N^+ \\ \lambda_N^- \end{bmatrix}.$$

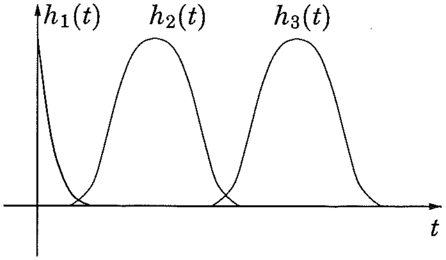

Figure 3.6 Impulse responses with little overlap.

We can now use $\mathcal{P}_\Delta(\Psi, M_\Delta)$, Ψ is as above, and where

$$M_\Delta = \left\{ M = \begin{bmatrix} 0 & \lambda^T \\ \lambda & 0 \end{bmatrix} : \Phi_1^T M \Phi_1 \leq 0,\ h_0 \geq 0,\ \lambda_i^+ \geq 0,\ \lambda_i^- \geq 0 \right\},$$

The $\mathbf{L}_1$-norm constraint in (3.26) is in general conservative. We need to find basis multipliers H_i with impulse responses that overlap as little as possible in order to avoid unnecessary conservativity, see Figure 3.6. This is generally hard since the impulse responses h_i are combinations of exponential functions and it would require multipliers with high order transfer functions to obtain the desired overlap condition. This affects the speed of the LMI conditions. It is for many applications sufficient to use basis functions with impulse responses

$$h_m(t) = \begin{cases} t^m e^{-\alpha t}, & t \geq 0 \\ 0, & t < 0, \end{cases} \quad \text{and} \quad h_n(t) = \begin{cases} 0, & t > 0 \\ t^n e^{\alpha t}, & t \leq 0, \end{cases}$$

where $\alpha \geq 0$. Then use the multiplier

$$H(j\omega) = \sum_{\alpha \in \mathcal{A}} \left(\sum_{m \in \mathcal{M}_\alpha} \frac{\lambda_m^+ - \lambda_m^-}{(j\omega + \alpha)^{m+1}} m! - \sum_{n \in \mathcal{N}_\alpha} \frac{\lambda_n^+ - \lambda_n^-}{(j\omega - \alpha)^{n+1}} n! \right),$$

where $\mathcal{A}$ is a set containing the pole locations, and where $\mathcal{M}_\alpha$ and $\mathcal{N}_\alpha$ contains the degrees of the transfer functions in the basis. These sets are chosen by the user.

We finally note that there is a class of multipliers also for the case when the slope restricted nonlinearity is not odd, see Zames and Falb (1968). They are of exactly the same form as those in (3.25) with the exception for the additional constraint $h(t) \leq 0$. This adds an extra difficulty to the multiplier computation. However, it can still be solved with LMI methods, see Chen and Wen (1995).

Multipliers for Performance Specifications

We here give an example of a multiplier for performance specifications.

Weighted Sensitivity Consider the condition

$$\|WH\| < 1$$

on the weighted induced $\mathbf{L}_2$-norm of the closed loop system. Here $W \in \mathbf{RH}_\infty^{l\times l}$ is a weighting function and $H = G_{22} + G_{21}\Delta(I - G_{11}\Delta)^{-1}G_{12}$ is the closed loop system operator. This performance specification can be represented by the multiplier

$$\Pi_{\text{perf}}(j\omega) = \begin{bmatrix} W(j\omega)^* W(j\omega) & 0 \\ 0 & -I_q \end{bmatrix},$$

which is included in the cone $\mathcal{P}_{\text{perf}}(\Psi, M_{\text{perf}})$, where

$$\Psi = \begin{bmatrix} W & 0 \\ 0 & I_q \end{bmatrix}, \quad M_{\text{perf}} = \left\{ x \begin{bmatrix} I_l & 0 \\ 0 & -I_q \end{bmatrix} : x \geq 0 \right\}.$$

An example that illustrates the use of this performance specification was given in Example 3.4.

Multipliers for Signal Specifications

We recall that a subset $\mathcal{W} \subset \mathbf{L}_2^q[0,\infty)$ can be described in terms of a convex cone Υ_{inp} if $\mathcal{W} \subset \mathcal{W}_{\text{inp}}$, where

$$\mathcal{W}_{\text{inp}} = \left\{ w \in \mathbf{L}_2^q[0,\infty) : \int_{-\infty}^{\infty} \widehat{w}(j\omega)^* \Upsilon(j\omega) \widehat{w}(j\omega)\, d\omega \geq 0, \ \forall \Upsilon \in \Upsilon_{\text{inp}} \right\}.$$

Dominant Harmonics Let $w \in \mathbf{L}_2^q[0,\infty)$ be a bandpass signal with $\text{supp}\, \widehat{w} \in [-b,-a] \cup [a,b]$, where $\text{supp}\, \widehat{w}$ denotes the support of the Fourier transform of w. It was shown in Section 3.3 that it would be optimal to use the multiplier

$$\Upsilon(j\omega) = \begin{cases} 0, & |\omega| \in [a,b], \\ -\infty I, & \text{otherwise.} \end{cases}$$

A finite-dimensional subcone that approximates the optimal multiplier is obtained by using $\Upsilon(j\omega) = (x|H(j\omega)|^2 - d)I$, where H is a given band-pass filter with monotonic slopes around some point in the interval (a,b). The following constraints must be satisfied:

$$x|H(jb)|^2 - d \geq 0, \tag{3.27}$$

$$x|H(ja)|^2 - d \geq 0, \tag{3.28}$$

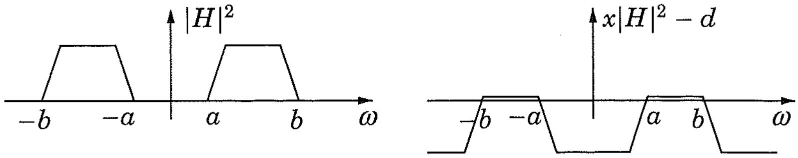

Figure 3.7 Filter specification for band-pass signals.

where $x, d \geq 0$. The idea is illustrated in Figure 3.7. We can obtain multipliers that are close to optimal if H is a sharp filter. In our format we use $\mathcal{P}_{\text{inp}}(\Psi, M_{\text{inp}})$, where

$$\Psi = \begin{bmatrix} HI_q \\ I_q \end{bmatrix}, \quad M_{\text{inp}} = \{\text{diag}(xI_q, -dI_q) : (3.27) \text{ and } (3.28) \text{ holds}\}.$$

Signals with a Given Spectral Characteristic: Consider the following subset of $\mathbf{L}_2^q[0,\infty)$

$$\mathcal{W}_H = \left\{ w \in \mathbf{L}_2^1[0,\infty) : |\widehat{w}(j\omega)|^2 = \frac{\|w\|^2}{\|H\|_2^2} |H(j\omega)|^2 \right\},$$

where $H \in \mathbf{RH}_\infty^{1\times 1}$ is strictly proper and where $\|H\|_2$ denotes the $\mathbf{H}_2$-norm of H. It was shown in Lemma 3.1 that $\mathcal{W}_H \subset \mathcal{W}_{\text{inp}}$ when

$$\Upsilon_{\text{inp}} = \left\{ \Upsilon = \Upsilon^* \in \mathbf{RL}_\infty^{1\times 1} : \int_{-\infty}^{\infty} \Upsilon(j\omega)|H(j\omega)|^2 \, d\omega \geq 0 \right\}.$$

In order to find a finite-dimensional parametrization of Υ_{inp} we let $\Upsilon = \Psi^* U \Psi$, where $\Psi \in \mathbf{RH}_\infty^{N\times 1}$ and where $U = U^T \in \mathbf{R}^{N\times N}$. Assume that we have the realization $\Phi(s) = \Psi(s)H(s) = C_\Phi(sI - A_\Phi)^{-1}B_\Phi$. Then the condition $\Upsilon \in \Upsilon_{\text{inp}}$ becomes

$$\int_{-\infty}^{\infty} \Phi(j\omega)^* U \Phi(j\omega) \, d\omega = B_\Phi^T \underbrace{\int_0^{\infty} e^{A_\Phi^T t} C_\Phi^T U C_\Phi e^{A_\Phi t} dt}_{P_\Phi} B_\Phi.$$

A well-known argument shows that $P_\Phi = P_\Phi^T$ satisfies the Lyapunov equation

$$A_\Phi^T P_\Phi + P_\Phi A_\Phi + C_\Phi^T U C_\Phi = 0.$$

It follows that we can use $\mathcal{P}_{\text{inp}}(\Psi, M_{\text{inp}}) \subset \Upsilon_{\text{inp}}$ with this Ψ and with

$$M_{\text{inp}} = \{U : A_\Phi^T P_\Phi + P_\Phi A_\Phi + C_\Phi^T U C_\Phi = 0, \quad B_\Phi^T P_\Phi B_\Phi \geq 0, \text{ and } P_\Phi = P_\Phi^T\}. \quad (3.29)$$

4

Duality Bounds in Multiplier Computation

Abstract

Frequency domain conditions involving multipliers are useful for robustness analysis. The resulting analysis problem is generally convex, but infinite-dimensional and numerical solutions restricted to finite-dimensional subspaces need to be considered. The resulting finite-dimensional problem can be transformed to a linear matrix inequality, which can be solved with efficient algorithms. This chapter presents a format for the dual of the infinite-dimensional problem. The dual optimization problem can be used to estimate the conservatism of particular finite-dimensional subspaces for the primal.

4.1 Introduction

Many practical systems can be modeled as a feedback interconnection of a linear time-invariant (LTI) plant G and a perturbation Δ. The perturbation consists of everything in the system that cannot be modeled as an LTI plant. For example, it can contain nonlinear elements, time-varying elements, and uncertain elements with various assumptions on the uncertainty.

Several classical results from 1960–1975 give sufficient conditions for stability in terms of the Nyquist curve in the case when G is a single-input single-output (SISO) plant for various nonlinear and/or time-varying perturbations, see for example Desoer and Vidyasagar (1975). Since the early 1980s much progress have been made on computational methods for robustness analysis in the case of multivariable systems with structured uncertainty. For example, Doyle introduced μ-analysis, which can be used for robustness test of a large class of systems with structured LTI perturbations by solving an optimization problem at a preselected grid of frequencies, see Doyle (1982), Packard and Doyle (1993). However, in the case of nonlinear and/or time-varying perturbations there exists coupling between frequencies and the optimization problems at different frequencies cannot be treated separately. One way to overcome this problem is to parametrize the multipliers involved in the optimization problem in terms of a basis of rational transfer functions, Ly *et al.* (1994) and Balakrishnan *et al.* (1994). The corresponding optimization problem can then be transformed into an equivalent linear matrix inequality (LMI) which can be solved by efficient numerical algorithms. The set of multipliers for the original, *primal*, optimization problem is in general infinite-dimensional. Parametrizing this set in terms of a finite-dimensional basis means that we only solve a restricted version of the primal. The effectiveness of this approach is therefore critically dependent on the choice of basis.

The objective of this chapter is to derive a dual optimization problem that can be used to investigate the quality of a particular basis. More precisely, if there is a small duality gap then this shows that we have a good basis for the primal optimization problem.

We will consider robustness analysis in the unified framework based on integral quadratic constraints (IQCs) that was suggested in Megretski (1993b) and Rantzer and Megretski (1994). The computation of a robustness criterion, γ, is an optimization problem of the following type

$$\inf \gamma \quad \text{subject to} \\ \begin{cases} \exists \Pi \in \Pi_\Delta(\gamma), \text{ such that} \\ \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0, \quad \forall \omega \in [0, \infty], \end{cases} \tag{4.1}$$

where $\Pi_\Delta(\gamma)$ is a convex cone of rational transfer functions for every value of the parameter γ. The cone $\Pi_\Delta(\gamma)$ is generally infinite-dimensional and we obtain solutions to (4.1) by introducing a finite-dimensional rational basis for the multipliers in $\Pi_\Delta(\gamma)$. Then we solve an equivalent optimization problem that has LMI conditions in the constraint. The solutions obtained by this method are generally suboptimal.

In order to estimate the conservativeness of a particular basis we derive the dual optimization problem. The dual is an infinite-dimensional optimization problem that can be hard to solve. However, by considering finite-dimensional restrictions, we can solve it for a large number of problems that are of interest in practice. Initial results in the direction of this chapter have been presented in Jönsson and Rantzer (1995).

4.2 Mathematical Preliminaries

This section presents the necessary mathematical preliminaries and notation needed in the chapter. The following standard definitions and results from functional and convex analysis can be found in Luenberger (1969).

- Let X be a normed vector space. The dual of X is the Banach space consisting of all bounded linear functionals on X and it is denoted by X^*. If $x \in X$ and $x^* \in X^*$, then $\langle x, x^* \rangle$ denotes the value of the linear functional x^* at x. We only consider vector spaces defined over the real scalar field and real-valued linear functionals.

- Let $H : X \to Y$ be a bounded linear operator. Then the adjoint operator $H^\times : Y^* \to X^*$ is defined by the relation

$$\langle Hx, y^* \rangle = \langle x, H^\times y^* \rangle,$$

  for all $x \in X$ and $y^* \in Y^*$.

- Let $\Delta_i : \mathbf{L}_{2e}^{m_i}[0,\infty) \to \mathbf{L}_{2e}^{m_i}[0,\infty)$, for $i = 1, 2$. Then the operator $\mathrm{diag}(\Delta_1, \Delta_2) : \mathbf{L}_{2e}^{m_1+m_2}[0,\infty) \to \mathbf{L}_{2e}^{m_1+m_2}[0,\infty)$ is defined by the input-output relation

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \Delta_1(u_1) \\ \Delta_2(u_2) \end{bmatrix},$$

  where $y_i, u_i \in \mathbf{L}_{2e}^{m_i}[0,\infty)$ for $i = 1, 2$.

- A *convex cone* C is a convex subset of a vector space with the property that if $x \in C$, then $\alpha x \in C$ for all $\alpha \geq 0$.

- We will use the following notation for optimization problems with constraints

$$\inf_P \gamma \stackrel{\text{def}}{=} \inf \gamma \ \text{ subject to } P,$$

  where P denotes a constraint definition.

The following separating hyperplane theorem will be a main tool in the chapter.

PROPOSITION 4.1—SEPARATING HYPERPLANE THEOREM
Let C_1 and C_2 be disjoint convex sets in a normed vector space X. Assume further that C_2 is open, then there exists $z \in X^*$ such that $\langle x_1, z \rangle < \langle x_2, z \rangle$ for all $x_1 \in C_1$ and $x_2 \in C_2$.

Proof: This is essentially Theorem 3 on page 133 in Luenberger (1969). In fact: $C = C_1 - C_2$ is an open and convex set such that $0 \notin C$. By the geometric Hahn-Banach theorem there exists an element $z \in X^*$ such that $\langle x, z \rangle < 0$ for all $x \in C$, from which the proposition follows. □

Next is a list of notation and function spaces used in the chapter.

$\overline{M}$ Denotes conjugation of a complex-valued matrix.

M^T Denotes the transpose of a matrix.

M^* $M^* = \overline{M}^T$ denotes Hermitian conjugation of a complex matrix.

I Denotes the identity operator or the identity matrix.

tr The trace of a matrix defined as $\operatorname{tr}(M) = \sum M_{ii}$, i.e., the sum of the diagonal elements.

$|\cdot|_F$ The Frobenius norm of a real or complex matrix M is defined as $|M|_F = \sqrt{\operatorname{tr}(M^*M)}$.

$\mathbf{RL}_\infty^{m\times m}$ The space consisting of proper real rational matrix functions with no poles on the imaginary axis. Note that $x \in \mathbf{RL}_\infty^{m\times m}$ satisfies $x(-j\omega) = \overline{x(j\omega)}$. We also note that x^* generally means the adjoint defined as $x^T(-s)$. The adjoint reduces to the Hermitian conjugate of x when $s = j\omega$. We define the norm on $\mathbf{RL}_\infty^{m\times m}$ as $\|x\| = \max_{\omega\in[0,\infty]} |x(j\omega)|_F$. This is not the usual norm on $\mathbf{RL}_\infty^{m\times m}$.

$\mathbf{RH}_\infty^{m\times m}$ The subspace of $\mathbf{RL}_\infty^{m\times m}$ consisting of functions with no poles in the closed right half plane.

$\mathcal{S}_{\mathrm{R}}^{m\times m}$ The subspace of $\mathbf{R}^{m\times m}$ consisting of symmetric $m\times m$ matrices with the Frobenius norm. The dual space can be identified with $\mathcal{S}_{\mathrm{R}}^{m\times m}$ itself and the linear functionals are defined as $\langle x, z\rangle = \operatorname{tr}(xz)$, where $x, z \in \mathcal{S}_{\mathrm{R}}^{m\times m}$.

$\mathcal{S}_{\mathrm{C}}^{m\times m}$ The subspace of $\mathbf{C}^{m\times m}$ consisting of Hermitian $m\times m$ matrices with the Frobenius norm. The dual space can be identified with $\mathcal{S}_{\mathrm{C}}^{m\times m}$ itself and the linear functionals are defined as $\langle x, z\rangle = \operatorname{tr}(xz)$, where $x, z \in \mathcal{S}_{\mathrm{C}}^{m\times m}$. Note that this linear functional is real-valued.

$\mathcal{S}_\infty^{m\times m}$ The subspace of $\mathbf{RL}_\infty^{m\times m}$ consisting of functions satisfying $x(j\omega) = x(j\omega)^*$ for all ω.

$\geq$ The notation $M \geq 0$ ($M > 0$) means that the matrix $M \in \mathcal{S}_{\mathrm{C}}^{m\times m}$ is positive semidefinite (positive definite). If $x \in \mathcal{S}_\infty^{2m\times 2m}$ then we sometimes use the notation $x \geq 0$ ($x > 0$) to mean that $x(j\omega) \geq 0$ ($x(j\omega) > 0$) for all $\omega \in [0,\infty]$. Similarly, $x_1 \geq x_2$ means that $x_1(j\omega) - x_2(j\omega) \geq 0$ for all $\omega \in [0,\infty]$.

Note that the functions in $\mathcal{S}_{\infty}^{m\times m}$ can be viewed as a subspace of the continuous and Hermitian valued functions defined on the extended imaginary axis. Their limit value $\lim_{\omega\to\infty} x(j\omega)$ is a well-defined matrix in $\mathcal{S}_{\mathrm{R}}^{m\times m}$.

We will need the dual of $\mathcal{S}_{\infty}^{m\times m}$. Proposition 4.2 below shows that we can use the Banach space of functions of bounded variation on the extended real axis. It would be possible to work with functions defined on the unit circle instead of the extended real axis. In fact, we could use a Möbius transform (the so called bilinear transformation) to transform the functions on $\mathcal{S}_{\infty}^{m\times m}$ so that they are defined on the unit circle. The corresponding dual space is also defined on unit circle. We will however transform back to the extended real axis. The main reason for doing this is that it is most natural to work on the extended real axis when analyzing systems that are defined in terms of transfer functions in $\mathbf{RH}_{\infty}$. We use the following definitions.

$\mathcal{S}_{\mathrm{NBV}}^{m\times m}$ The Banach space of functions $\mathbf{R}\cup\{\infty\}\to\mathcal{S}_{\mathrm{C}}^{m\times m}$ of normalized bounded variation. Every $z\in\mathcal{S}_{\mathrm{NBV}}^{m\times m}$ satisfies the properties

(*i*) $z(-\omega) = -\overline{z(\omega)}$ for all $\omega\in[0,\infty]$ and $z(0)=0$,

(*ii*) z is continuous from the left on $(0,\infty)$.

The norm on $\mathcal{S}_{\mathrm{NBV}}^{m\times m}$ is defined as $\|z\| = T.V.(z)$, where

$$T.V.(z) = \int_{-\infty}^{\infty} |dz(\omega)|_F = 2\sup\sum_{k=1}^{N} |z(\omega_k) - z(\omega_{k-1})|_F,$$

where $0=\omega_0\le\omega_1\le\omega_2\le\ldots\le\omega_N=\infty$ is a partition of $[0,\infty]$, and the supremum is taken with respect to all such partitions. Every $z\in\mathcal{S}_{\mathrm{NBV}}^{m\times m}$ has $\|z\|<\infty$.

$P_{\mathrm{NBV}}^{m\times m}$ The positive cone defined as

$$P_{\mathrm{NBV}}^{m\times m} = \{z\in\mathcal{S}_{\mathrm{NBV}}^{m\times m} : z(\omega_1)\ge z(\omega_2),\ \forall\omega_1>\omega_2\ge 0\}.$$

We sometimes use the notation $dZ\ge 0$ to mean that $Z\in P_{\mathrm{NBV}}^{m\times m}$.

PROPOSITION 4.2
The dual of $\mathcal{S}_{\infty}^{m\times m}$ can be identified with $\mathcal{S}_{\mathrm{NBV}}^{m\times m}$. If $x\in\mathcal{S}_{\infty}^{m\times m}$ and $z\in\mathcal{S}_{\mathrm{NBV}}^{m\times m}$, then the linear functional is defined by the Stieltjes integral

$$\begin{aligned}\langle x,z\rangle &= \int_{-\infty}^{\infty}\mathrm{tr}(x(j\omega)dz(\omega))\\ &= 2\lim_{N\to\infty}\sum_{k=1}^{N}\mathrm{tr}(x(j\omega_{k-1})(z(\omega_k)-z(\omega_{k-1}))),\end{aligned}\tag{4.2}$$

where $0=\omega_0\le\omega_1\le\omega_2\le\ldots\le\omega_N=\infty$ is a partition of $[0,\infty]$, and the limit is considered as $\max_{k\in\{1,\ldots,N-1\}}|\omega_k-\omega_{k-1}|\to 0$ and as $\omega_{N-1}\to\infty$.

Proof: We use the Möbius transform $\psi(z) = (z-1)/(z+1)$ to transform the unit circle, $\partial\mathbf{D}$, to the imaginary axis. For any $x \in \mathcal{S}_\infty^{m\times m}$ let $\tilde{x} = x \circ \psi$. The restriction to the unit circle of the functions $\tilde{x}$ obtained in this way is a subspace of the Banach space of continuous functions $\tilde{x} : \partial\mathbf{D} \to \mathcal{S}_{\mathrm{C}}^{m\times m}$ defined by the following assumptions:

(*i*) $\tilde{x}(e^{-j\omega}) = \overline{\tilde{x}(e^{j\omega})}$, for all $\omega \in [0, \pi]$,

(*ii*) the norm is defined as $\|\tilde{x}\| = \max_{\omega\in[0,\pi]}|\tilde{x}(e^{j\omega})|_F$.

It follows from Theorem 1, on page 113 in Luenberger (1969) that the dual space consists of functions $\tilde{z} : \partial\mathbf{D} \to \mathcal{S}_{\mathrm{C}}^{m\times m}$ of bounded variation. The proposition follows after transformation with the inverse Möbius map ψ^{-1}, i.e., $z(\omega) = \tilde{z}(\psi^{-1}(j\omega))$ for all $\omega \in [-\infty, \infty]$. Note that the symmetry around $\omega = 0$ for the spaces $\mathcal{S}_\infty^{m\times m}$ and $\mathcal{S}_{\mathrm{NBV}}^{m\times m}$ implies that we can compute the Stieltjes integral in (4.2) in terms of positive frequencies. □

The next two propositions will be used in applications of our duality results. The first one is immediate.

PROPOSITION 4.3
Let $z \in \mathcal{S}_{\mathrm{NBV}}^{m\times m}$. The condition

$$\langle x, z\rangle \geq 0, \quad \forall x \in \mathcal{S}_\infty^{m\times m} \text{ with } x(j\omega) \geq 0,$$

implies that $z \in P_{\mathrm{NBV}}^{m\times m}$. □

For the second proposition we consider the space of skew-Hermitian functions in $\mathbf{RL}_\infty^{m\times m}$, i.e., the functions satisfy $y(j\omega)^* = -y(j\omega)$. The dual space consists of functions of bounded variation with the same properties as the functions in $\mathcal{S}_{\mathrm{NBV}}^{m\times m}$ except that they take skew-Hermitian values. The functionals can be defined in terms of the same Stieltjes integral as in (4.2). We have the following proposition.

PROPOSITION 4.4
Let $z \in \mathcal{S}_{\mathrm{NBV}}^{m\times m}$. The condition

$$\langle y, z - z^*\rangle \geq 0, \quad \forall y \in \mathbf{RL}_\infty^{m\times m} \text{ with } y(j\omega)^* = -y(j\omega),$$

implies that $z - z^* \equiv 0$.

Proof: We note that $z - z^*$ takes skew-Hermitian values. The proposition follows from the corresponding property for matrices. Hence, assume that $Z = Z^*$ and that $\mathrm{tr}[Y(Z - Z^*)] \geq 0$ for all $Y = -Y^*$. Let $Y = Z - Z^*$. Then $0 \leq -\mathrm{tr}[(Z - Z^*)^*(Z - Z^*)] \leq 0$. Hence, $Z - Z^* = 0$. □

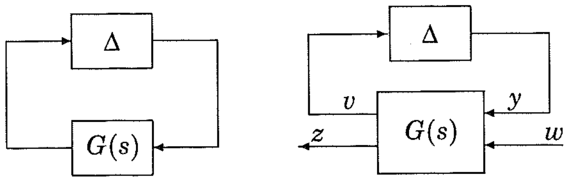

Figure 4.1 System setup for stability analysis and performance analysis respectively.

4.3 The Primal Optimization Problem

A general and unified approach to the use of multipliers was introduced in Megretski and Rantzer (1995). The method is based on the concept integral quadratic constraint (IQC). A bounded operator Δ (possibly nonlinear) on $\mathbf{L}_{2e}^m[0,\infty)$ is said to satisfy the IQC defined by the matrix function $\Pi \in \mathcal{S}_\infty^{2m\times 2m}$, called the *multiplier*, if

$$\int_{-\infty}^{\infty} \begin{bmatrix} \widehat{v}(j\omega) \\ \widehat{\Delta(v)}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \widehat{v}(j\omega) \\ \widehat{\Delta(v)}(j\omega) \end{bmatrix} d\omega \quad \geq \quad 0, \qquad \text{for all } v \in \mathbf{L}_2^m[0,\infty).$$

Here $\widehat{v}$ and $\widehat{\Delta(v)}$ denote the Fourier transforms of v and $\Delta(v)$. Based on this definition, each operator Δ can be described by a set Π_Δ of multipliers that define IQCs satisfied by Δ.

IQCs can be used in robustness analysis of the systems in Figure 4.1. Here G is a bounded causal with transfer function in $\mathbf{RH}_\infty^{m\times m}$ and Δ is a bounded causal operator on $\mathbf{L}_{2e}^m[0,\infty)$. It is possible to analyze the system with respect to robust stability and robust performance. The first step in the analysis consists of finding a description of the perturbation Δ in terms of multipliers. The following properties are convenient when deriving a description of Δ.

Property 1 If Δ is described by the convex cones $\Pi_{1\Delta}$ and $\Pi_{2\Delta}$, then Δ is also described by $\Pi_\Delta = \Pi_{1\Delta} + \Pi_{2\Delta} = \{\Pi_1 + \Pi_2 : \Pi_1 \in \Pi_{1\Delta}, \Pi_2 \in \Pi_{2\Delta}\}$.

Property 2 Assume Δ has the block-diagonal structure $\Delta = \mathrm{diag}(\Delta_1, \Delta_2)$, and that, for $i = 1, 2$, Δ_i satisfies the IQC defined by

$$\Pi_i = \begin{bmatrix} \Pi_{i(11)} & \Pi_{i(12)} \\ \Pi_{i(12)}^* & \Pi_{i(22)} \end{bmatrix},$$

where the block structures are consistent with the size of Δ_1 and Δ_2, respectively. Then Δ satisfies the IQC defined by

$$\mathrm{daug}(\Pi_1, \Pi_2) = \left[\begin{array}{cc|cc} \Pi_{1(11)} & 0 & \Pi_{1(12)} & 0 \\ 0 & \Pi_{2(11)} & 0 & \Pi_{2(12)} \\ \hline \Pi_{1(12)}^* & 0 & \Pi_{1(22)} & 0 \\ 0 & \Pi_{2(12)}^* & 0 & \Pi_{2(22)} \end{array}\right].$$

If, for $i = 1, 2$, Δ_i is described by the cone Π_{Δ_i}, then $\Delta = \mathrm{diag}(\Delta_1, \Delta_2)$ is described by $\mathrm{daug}(\Pi_{\Delta_1}, \Pi_{\Delta_2}) = \{\mathrm{daug}(\Pi_1, \Pi_2) : \Pi_1 \in \Pi_{\Delta_1}, \Pi_2 \in \Pi_{\Delta_2}\}$.

Addition and diagonal augmentation of any finite number of cones can be done in the same way.

In robust performance analysis we also need multiplier descriptions of the performance specification and the characteristics of the input signal. These should be augmented to the multiplier description of Δ. This is discussed in more detail in Chapter 3 and Section 4.8. The robustness analysis can be formulated as an optimization problem on the following form.

PRIMAL 4.1—PRIMAL OPTIMIZATION PROBLEM

$$\inf \gamma \quad \text{subject to}$$

$$P : \begin{cases} \exists \Pi \in \Pi_\Delta(\gamma), \text{ such that} \\ \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0, \quad \forall \omega \in [0, \infty], \end{cases}$$

where $\Pi_\Delta(\gamma) \subset \mathcal{S}_\infty^{2m \times 2m}$ is a convex cone for every $\gamma \in \mathbf{R}$. This implies that $0 \in \Pi_\Delta(\gamma)$. □

The parameter γ corresponds to the robustness criteria which is investigated. It could, for example, correspond to a stability margin. We assume that the following monotonicity condition holds.

DEFINITION 4.1—MONOTONICITY OF $\Pi_\Delta(\gamma)$
If $\gamma_2 \geq \gamma_1$, then $\forall \Pi_1 \in \Pi_\Delta(\gamma_1)$ there exists $\Pi_2 \in \Pi_\Delta(\gamma_2)$ such that $\Pi_1(j\omega) \geq \Pi_2(j\omega)$, for all $\omega \in [0, \infty]$. This ensures that the primal constraint is satisfied for all $\gamma > \inf_P \gamma$. □

The constraint P in the primal optimization problem typically corresponds to an infinite-dimensional convex feasibility test. The following computational algorithm for obtaining a, possibly suboptimal, solution to Primal 4.1 can be used. This is explained in more detail in Chapter 3.

(*i*) Restrict the primal optimization problem to a finite-dimensional subspace by considering the subcone $\mathcal{P}_\Delta(\Psi, M_\Delta(\gamma)) \subset \Pi_\Delta(\gamma)$ defined as

$$\mathcal{P}_\Delta(\Psi, M_\Delta(\gamma)) = \{\Psi^* M \Psi : M \in M_\Delta(\gamma)\}.$$

Here $\Psi \in \mathbf{RL}_\infty^{N\times 2m}$ is a *basis multiplier* and $M_\Delta(\gamma) \subset \mathbf{R}^{N\times N}$ is convex cone of symmetric matrices for every $\gamma \in \mathbf{R}$. It is assumed that $\mathcal{P}_\Delta(\Psi, M_\Delta(\gamma)$ satisfies the same monotonicity assumption as $\Pi_\Delta(\gamma)$.

(*ii*) It follows from the Kalman-Yakubovich-Popov lemma that the constraint of the restricted optimization problem is equivalent with a finite-dimensional LMI test. The restricted optimization problem becomes

$$\begin{aligned} &\inf \gamma \quad \text{subject to} \\ &P_{LMI} : \begin{cases} \exists P_0 = P_0^T,\ M \in M_\Delta(\gamma) \text{ such that} \\ \mathcal{N}^T \mathcal{A}(P_0, M)\mathcal{N} < 0, \end{cases} \end{aligned}$$

Here $\mathcal{N}$ is a constant matrix that depends on the state space realization of

$$\Phi = \Psi \begin{bmatrix} G \\ I \end{bmatrix},$$

and $\mathcal{A}$ depends linearly on P_0 and M.

(*iii*) We can obtain a solution to the restricted primal by either of the following methods.

a. By bisection on γ.

b. As an eigenvalue problem for LMIs.

c. As a generalized eigenvalue problem for LMIs.

LMI, EVP, and GEVP problems can be solved efficiently by numerical software, see for example Boyd *et al.* (1994), Boyd and El Ghaoui (1993), and Nesterov and Nemirovski (1993).

The success of the computational method described above depends on the choice of finite-dimensional subspace for the restricted primal. It is desirable to keep the dimension of this subspace as low as possible since the speed of the LMI computations depends critically on it. In order to evaluate the quality of a particular subspace we would like to have a method to obtain lower bounds on the primal optimization problem. The dual optimization problem derived in the next section can be used for

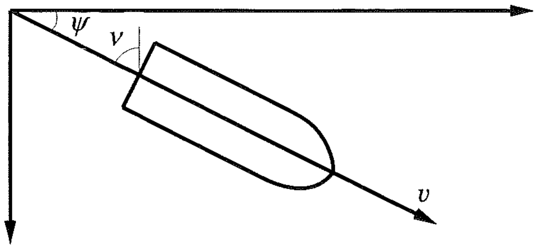

Figure 4.2 Notation used to describe the motion of ships.

exactly this purpose. The dual is infinite-dimensional. However, in applications we may obtain, possibly suboptimal, solutions by restricting attention to finite-dimensional subspaces. The resulting restricted dual is an optimization problem with a finite-dimensional convex constraint. We can then consider the primal optimization problem solved when we have obtained suboptimal solutions of the primal and the dual with a small gap between their corresponding objective values. We illustrate with an example.

EXAMPLE 4.1—SHIP STEERING DYNAMICS
We will consider ship steering dynamics as in Example 9.6 in Åström and Wittenmark (1989). The dynamics for the ship can, with notation as in Figure 4.2, be approximated by the Nomoto model

$$\begin{aligned}\dot{x}(t) &= v(t)(-ax(t) + bv(t)\nu(t)),\\ \dot{\psi}(t) &= x(t),\end{aligned}$$

where ψ denotes the heading of the ship, ν denotes the rudder angle, and v is the speed of the ship. It is assumed that $v(t) \geq 0$. We will as in Åström and Wittenmark (1989) study stability of the ship steering dynamics for an open loop unstable tanker controlled by the PD regulator

$$\begin{aligned}\nu &= -K\psi,\\ K(s) &= k(1 + sT_d),\end{aligned}$$

where $k = 2.5$ and $T_d = 0.86$. It is assumed that $a = -0.3$ and $b = 0.8$. We will investigate the particular case when $v(t) = v_0 + A\cos(\omega_0 t)$, where $v_0 > 0$ and $A > 0$. We want to find a bound $A_{\max}$ such that stability for the ship steering dynamics is guaranteed for all $A < A_{\max}$.

It is possible to represent the system as in Figure 4.1. This can be done with $\Delta = A\cos(\omega_0 t)I_2$ and a transfer function G, which will be in

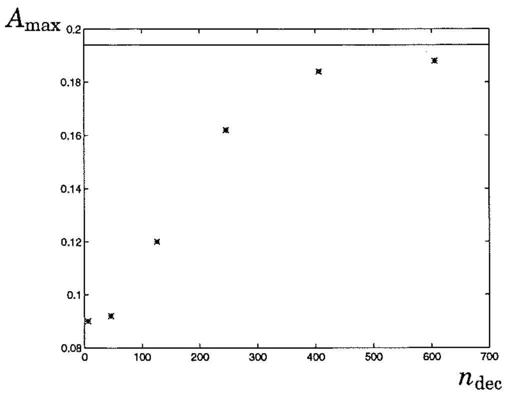

Figure 4.3 The primal optimization problem is solved for six different choices of finite-dimensional subspace (∗). The size of the subspace is given in terms of the number of decision variables in the corresponding LMI constraint. The solid line corresponds to an upper bound obtained from the dual optimization problem described in Section 4.4.

$\mathbf{RH}_\infty^{2\times 2}$ when $v_0 > 0.1744$. Let $\gamma = A^{-2}$. We can then describe Δ with the convex cone

$$\Pi_\Delta(\gamma) = \left\{ \begin{bmatrix} \frac{1}{2}[X(j(\omega+\omega_0)) + X(j(\omega-\omega_0))] & 0 \\ 0 & -\gamma X(j\omega) \end{bmatrix} : X(j\omega) \geq 0 \right\},$$

see Megretski and Rantzer (1995). Let γ_{opt} be the solution to Primal 4.1 with this G and $\Pi_\Delta(\gamma)$. We can then guarantee stability for the ship dynamics when $A < A_{\max} = \gamma_{\text{opt}}^{-1/2}$. We solved the restricted primal for six choices of finite-dimensional subspace for the case when $v_0 = 0.5$ and $\omega_0 = 0.5$. Figure 4.3 shows how the bound for A increases for increasing subspaces. The upper bound given by the dual assures that the two largest finite-dimensional subspaces are close to being optimal. We refer to Section 4.6 for details on the computations and the choice of subspaces. □

4.4 The Dual Optimization Problem

We will in this section discuss the dual to the optimization problem in Primal 4.1. Let $M_G : \mathcal{S}_\infty^{2m\times 2m} \to \mathcal{S}_\infty^{m\times m}$ be the linear operator defined by

$$M_G\Pi = \begin{bmatrix} G \\ I \end{bmatrix}^* \Pi \begin{bmatrix} G \\ I \end{bmatrix},$$

for any $\Pi \in \mathcal{S}_{\infty}^{2m\times 2m}$. Then the primal constraint P can be formulated as: $\exists \Pi \in \Pi_{\Delta}(\gamma)$ such that $M_G\Pi < 0$. The adjoint $M_G^{\times} : \mathcal{S}_{\text{NBV}}^{m\times m} \to \mathcal{S}_{\text{NBV}}^{2m\times 2m}$ is the operator that takes an arbitrary $Z \in \mathcal{S}_{\text{NBV}}^{m\times m}$ into the function $M_G^{\times}Z \in \mathcal{S}_{\text{NBV}}^{2m\times 2m}$ that corresponds to the integral measure

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix} dZ(\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^{*}.$$

The next theorem states the equivalence between the following dual optimization problem and Primal 4.1.

DUAL 4.1—THE DUAL OPTIMIZATION PROBLEM

$$\sup \gamma \quad \text{subject to}$$
$$D : \begin{cases} \exists Z \in P_{\text{NBV}}^{m\times m},\ Z \not\equiv 0, \text{ such that} \\ \begin{bmatrix} G \\ I \end{bmatrix} dZ \begin{bmatrix} G \\ I \end{bmatrix}^{*} \in d\Pi_{\Delta}(\gamma)^{\oplus}, \end{cases}$$

where

$$\Pi_{\Delta}(\gamma)^{\oplus} = \left\{ W \in \mathcal{S}_{\text{NBV}}^{2m\times 2m} : \langle \Pi, W \rangle \geq 0,\ \forall \Pi \in \Pi_{\Delta}(\gamma) \right\}, \tag{4.3}$$

is the positive conjugate cone corresponding to $\Pi_{\Delta}(\gamma)$. Here $d\Pi_{\Delta}(\gamma)^{\oplus}$ is used to denote the set of integral measures corresponding to $\Pi_{\Delta}(\gamma)^{\oplus}$, i.e.,

$$d\Pi_{\Delta}(\gamma)^{\oplus} = \left\{ dW : W \in \Pi_{\Delta}(\gamma)^{\oplus} \right\}.$$

□

The last constraint in the dual optimization problem could equivalently be formulated as $M_G^{\times}Z \in \Pi_{\Delta}(\gamma)^{\oplus}$. The formulation in terms of the integral measure is more suggestive and will be used in most instances in this chapter. The constraint means that the condition

$$\int_{-\infty}^{\infty} \operatorname{tr}\left(\Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} dZ(\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^{*} \right) \geq 0,$$

must hold for all $\Pi \in \Pi_{\Delta}(\gamma)$.

THEOREM 4.1
Primal 4.1 and Dual 4.1 have the same objective value, i.e,

$$\inf_{P} \gamma = \sup_{D} \gamma.$$

Proof: If $\gamma < \inf_P \gamma$ then the convex sets

$$C_1 = \{M_G\Pi : \Pi \in \Pi_\Delta(\gamma)\},$$
$$C_2 = \{X \in \mathcal{S}_\infty^{m\times m} : X(j\omega) < 0, \forall\omega \in [0,\infty]\},$$

are disjoint. By the separating hyperplane theorem and the property that $0 \in \Pi_\Delta(\gamma)$ there exists a nonzero $Z \in \mathcal{S}_{\text{NBV}}^{m\times m}$ such that

$$\langle X, Z\rangle \geq 0, \quad \forall X \in C_1, \tag{4.4}$$
$$\langle X, Z\rangle < 0, \quad \forall X \in C_2. \tag{4.5}$$

For (4.5) to hold we need $Z \in P_{\text{NBV}}^{m\times m}$. Condition (4.4) gives

$$\begin{aligned}
&\langle M_G\Pi, Z\rangle \geq 0, \quad \forall\Pi \in \Pi_\Delta(\gamma) \quad \Longleftrightarrow \\
&\langle \Pi, M_G^\times Z\rangle \geq 0, \quad \forall\Pi \in \Pi_\Delta(\gamma) \quad \Longleftrightarrow \\
&M_G^\times Z \in \Pi_\Delta(\gamma)^\oplus.
\end{aligned}$$

Hence since $\gamma < \inf_P \gamma$ was arbitrary we have $\inf_P \gamma \leq \sup_D \gamma$. For the opposite direction we note that the monotonicity assumption on the γ dependence of $\Pi_\Delta(\gamma)$ implies that for every $\gamma > \inf_P \gamma$ there exists $\Pi \in \Pi_\Delta(\gamma)$ such that $C_1 \cap C_2 \neq \emptyset$. This implies that there is no nonzero $Z \in P_{\text{NBV}}^{m\times m}$ such that $M_G^\times Z \in \Pi_\Delta(\gamma)^\oplus$. Hence, $\sup_D \gamma \leq \inf_P \gamma$ and the theorem is proved. □

Remark 4.1

Furthermore, it follows from the proof of the theorem that the dual constraint D is satisfied for all $\gamma < \gamma_{\text{opt}} = \inf_P \gamma$. This can also be seen from the following argument. If there exits $Z \in P_{\text{NBV}}^{m\times m}$ such that $M_G^\times Z \in \Pi_\Delta(\gamma_2)^\oplus$ or equivalently if $\langle M_G\Pi, Z\rangle \geq 0$, $\forall\Pi \in \Pi_\Delta(\gamma_2)$, then for all $\gamma_1 \leq \gamma_2$ and for all $\Pi_1 \in \Pi_\Delta(\gamma_1)$, there exist $\Pi_2 \in \Pi_\Delta(\gamma_2)$ such that $\langle M_G\Pi_1, Z\rangle \geq \langle M_G\Pi_2, Z\rangle \geq 0$. Hence, $M_G^\times Z \in \Pi_\Delta(\gamma_1)^\oplus$ for all $\gamma_1 \leq \gamma_2$.

The left half of Figure 4.4 illustrates that the primal and dual constraints are satisfied above and below the optimal value γ_{opt}, respectively. In applications we generally find solutions to the primal and dual optimization problem by considering restrictions to finite-dimensional subspaces. The right half of Figure 4.4 illustrates that the resulting primal and dual generally gives suboptimal solutions γ_p and γ_d, respectively. The size of the duality gap $\gamma_p - \gamma_d$ gives an indication on the quality of these solutions. □

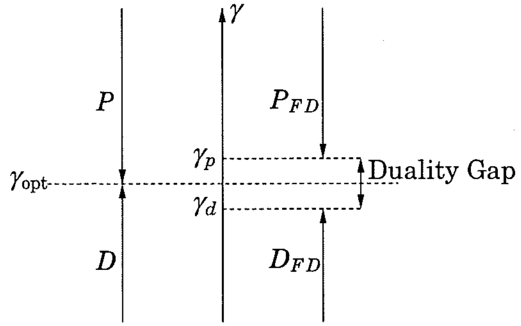

Figure 4.4 The primal constraint P is satisfied for all $\gamma > \gamma_{\text{opt}}$ and the dual constraint D is satisfied for all $\gamma < \gamma_{\text{opt}}$. The size of the duality gap indicates the quality of the suboptimal solutions that are obtained when finite-dimensional restrictions are considered.

THEOREM 4.2—UNFEASIBILITY
Let Π_Δ be a convex set. The following statements are equivalent:

(*i*) There exists no $\Pi \in \Pi_\Delta$ such that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0, \ \forall \omega \in [0, \infty].$$

(*ii*) There exists a nonzero $Z \in P_{\text{NBV}}^{m\times m}$ such that

$$\begin{bmatrix} G \\ I \end{bmatrix} dZ \begin{bmatrix} G \\ I \end{bmatrix}^* \in d\Pi_\Delta^\oplus,$$

where $\Pi_\Delta^\oplus$ is defined as in (4.3).

Proof: (*i*) $\Longrightarrow$ (*ii*): Assume that there is no $\Pi \in \Pi_\Delta$ such that $M_G\Pi < 0$. Then the convex sets C_1 and C_2 defined as in the proof of Theorem 4.1 are disjoint. By the separating hyperplane theorem there exists a nonzero $Z \in \mathcal{S}_{\text{NBV}}^{m\times m}$ and a real constant c such that

$$\begin{aligned} \langle X, Z\rangle &\geq c, \quad \forall X \in C_1, \\ \langle X, Z\rangle &< c, \quad \forall X \in C_2. \end{aligned}$$

Since C_2 is an open cone we see that also in this case we need $Z \in P_{\text{NBV}}^{m\times m}$ and $c = 0$. As in the proof Theorem 4.1 we conclude that $M_G^\times Z \in \Pi_\Delta^\oplus$.

(*ii*) $\Longrightarrow$ (*i*): By assumption there exists a nonzero $Z \in P_{\text{NBV}}^{m\times m}$ such that $\langle M_G\Pi, Z\rangle \geq 0$ for all $\Pi \in \Pi_\Delta$. This implies that there exist frequencies such that $M_G\Pi(j\omega) \not< 0$ since otherwise this linear functional would be negative. □

Combination of Multipliers

We will next derive the dual in the case when the multiplier specification is refined to be $\Pi_\Delta(\gamma) = \text{daug}(\Pi_{\Delta_1}(\gamma), \ldots, \Pi_{\Delta_n}(\gamma))$, where

$$\Pi_{\Delta_i}(\gamma) = \sum_{j=1}^{n_i} \Pi_{j\Delta_i}(\gamma). \tag{4.6}$$

It is assumed that $\Pi_{j\Delta_i}(\gamma) \subset \mathcal{S}_\infty^{m_i \times m_i}$ is a convex cone for all $\gamma \in \mathbf{R}$, which satisfies the monotonicity condition in Definition 4.1. For consistency among the dimensions we require that $\sum_{i=1}^n m_i = m$. Let us define the matrices

$$E_i = \begin{pmatrix} 0_{m_i \times \sum_{k=1}^{i-1} m_k} & I_{m_i} & 0_{m_i \times \sum_{k=i+1}^{n} m_k} \end{pmatrix}, \tag{4.7}$$

and let us partition G consistently with the size of the different Π_{Δ_i}, i.e.,

$$G = \begin{bmatrix} G_1 \\ \vdots \\ G_n \end{bmatrix},$$

where $G_i \in \mathbf{RH}_\infty^{m_i \times m}$. Then we have the following corollary to Theorem 4.1.

COROLLARY 4.1

Let $\Pi_\Delta(\gamma) = \text{daug}(\Pi_{\Delta_1}(\gamma), \ldots, \Pi_{\Delta_n}(\gamma))$, where $\Pi_{\Delta_i}(\gamma)$ is defined as in (4.6). Then

$$\inf_P \gamma = \sup_D \gamma,$$

where the primal and dual constraints are defined as

$$P: \begin{cases} \exists \Pi \in \Pi_\Delta(\gamma) \quad \text{such that} \\ \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0, \quad \forall \omega \in [0, \infty], \end{cases}$$

and

$$D: \begin{cases} \exists Z \in P_{\text{NBV}}^{m \times m},\ Z \not\equiv 0, \quad \text{such that} \\ \begin{bmatrix} G_i \\ E_i \end{bmatrix} dZ \begin{bmatrix} G_i \\ E_i \end{bmatrix}^* \in \bigcap_{j=1}^{n_i} d\Pi_{j\Delta_i}(\gamma)^\oplus, \text{ for } i = 1, \ldots n, \end{cases}$$

respectively. Here the E_i are defined in (4.7).

Proof: By the assumptions on the individual $\Pi_{j\Delta_i}$ it follows that $\Pi_\Delta(\gamma)$ satisfies the monotonicity condition in Definition 4.1. Hence, Theorem 4.1 holds. We need an expression for $\Pi_\Delta(\gamma)^\oplus$. Let the operators $\mathcal{P}_i : \mathcal{S}_{\text{NBV}}^{2m\times 2m} \to \mathcal{S}_{\text{NBV}}^{2m_i\times 2m_i}$ be defined as

$$\mathcal{P}_i W = \begin{bmatrix} E_i & 0 \\ 0 & E_i \end{bmatrix} W \begin{bmatrix} E_i & 0 \\ 0 & E_i \end{bmatrix}^T,$$

for all $W \in \mathcal{S}_{\text{NBV}}^{2m\times 2m}$. Then

$$\begin{aligned}
\Pi_\Delta(\gamma)^\oplus &= \left\{W \in \mathcal{S}_{\text{NBV}}^{2m\times 2m} : \langle \Pi, W\rangle \geq 0,\ \forall \Pi \in \Pi_\Delta(\gamma)\right\} \\
&= \left\{ W : \langle \operatorname{daug}(\Pi_1,\dots,\Pi_n), W\rangle \geq 0,\ \forall \Pi_i = \sum_{j=1}^{n_i} \Pi_{ij},\ \Pi_{ij} \in \Pi_{j\Delta_i}(\gamma) \right\} \\
&= \left\{ W : \sum_{i=1}^{n}\sum_{j=1}^{n_i} \langle \Pi_{ij}, \mathcal{P}_i W\rangle \geq 0,\ \forall \Pi_{ij} \in \Pi_{j\Delta_i}(\gamma) \right\} \\
&= \left\{ W : \mathcal{P}_i W \in \bigcap_{j=1}^{n_i} \Pi_{j\Delta_i}(\gamma)^\oplus,\ i = 1,\dots,n \right\},
\end{aligned}$$

where the last equality follows since each $\Pi_{j\Delta_i}(\gamma)$ is a convex cone. Hence, the condition $M_G^\times Z \in \Pi_\Delta(\gamma)^\oplus$ becomes $\mathcal{P}_i M_G^\times Z \in \bigcap_{j=1}^{n_i} \Pi_{j\Delta_i}(\gamma)^\oplus$, for $i = 1,\dots,n$. The dual constraint, D, in the statement of the corollary follows from the definition of $\mathcal{P}_i$ and the formulation in terms of measures. □

4.5 Computational Issues

Dual 4.1 is defined in terms of functions in $\mathcal{S}_{\text{NBV}}$. This class of functions is very large and the corresponding optimization problem is therefore not tractable for computations. The main purpose of this section is to show how the dual can be restricted to a certain subspace such that the resulting optimization problem in many applications involves a finite number of matrix constraints. This approach for obtaining suboptimal solutions to Dual 4.1 is useful in a large number of practical applications. We introduce a function space $\mathcal{S}_{\text{AM}}^{m\times m}$ that can be identified with the subset of $\mathcal{S}_{\text{NBV}}^{m\times m}$ consisting of functions where the variation corresponds to a finite number of step discontinuities. The following notation will be used

$\mathcal{S}_{\mathrm{AM}}^{m\times m}$ The normed space consisting of step functions of the form

$$z(\omega) = \sum_{k=1}^{N-1}(z_k\theta(\omega-\omega_k) - \overline{z_k}\theta(-\omega-\omega_k)) + z_N\theta_\infty(\omega) - \overline{z}_N\theta_\infty(-\omega),$$

where the unit step functions are defined as

$$\theta(\omega) = \begin{cases} 0, & \omega \le 0 \\ I, & \omega > 0 \end{cases}, \qquad \theta_\infty(\omega) = \begin{cases} 0, & \omega < \infty \\ I, & \omega = \infty \end{cases},$$

and where $z_k \in \mathcal{S}_{\mathrm{C}}^{m\times m}$, $\sum_{k=1}^{N}|z_k|_F < \infty$, N is any finite integer, and $\omega_k \in [0,\infty)$, $k = 1,\dots,N-1$. The norm on $\mathcal{S}_{\mathrm{AM}}$ is defined as $\|z\| = 2\sum_{k=1}^{N}|z_k|_F < \infty$.

$P_{\mathrm{AM}}^{m\times m}$ The positive cone of functions $z \in \mathcal{S}_{\mathrm{AM}}^{m\times m}$ having coefficients satisfying $z_k \ge 0$ for all k.

Every $z \in \mathcal{S}_{\mathrm{AM}}^{m\times m}$ defines an integration measure with atomic support, i.e., it has support only at a finite number of frequencies. This is the reason for the notation $\mathcal{S}_{\mathrm{AM}}$ (Atomic Measure). The restriction of Dual 4.1 to this subspace can be formulated as

$$\sup \gamma \quad \text{subject to}$$
$$D_{AM}: \begin{cases} \exists Z \in P_{\mathrm{AM}}^{m\times m},\ Z \ne 0, \text{ such that} \\ M_G^{\times} Z \in \Pi_\Delta(\gamma)^{\oplus}. \end{cases} \tag{4.8}$$

The restricted adjoint $M_G^{\times} : \mathcal{S}_{\mathrm{AM}}^{m\times m} \to \mathcal{S}_{\mathrm{AM}}^{2m\times 2m}$ is defined as the operator that maps $Z \in \mathcal{S}_{\mathrm{AM}}^{2m\times 2m}$ to a step function with the coefficients

$$\begin{bmatrix} G(j\omega_k) \\ I \end{bmatrix} Z_k \begin{bmatrix} G(j\omega_k) \\ I \end{bmatrix}^*.$$

Using the formulation in terms of integration measures means that the last constraint in (4.8) can be formulated as (we neglect the negative frequencies in order to save space)

$$\sum_{k=1}^{N} \begin{bmatrix} G(j\omega_k) \\ I \end{bmatrix} Z_k \begin{bmatrix} G(j\omega_k) \\ I \end{bmatrix}^* \delta_{\omega_k}(\omega) + \text{neg frequencies} \in d\Pi_\Delta(\gamma)^{\oplus},$$

where δ_{ω_k} denotes the purely atomic measure on $\mathbf{R} \cup \{\infty\}$ with support at ω_k.

Hence, for any given choice of frequency grid $\Omega = \{\omega_1, \dots, \omega_N\}$, which defines the discontinuities of the step function, the constraint definition D_{AM} in (4.8) involves only a finite number of complex valued matrices. In a large number of applications it turns out that the constraint in D_{AM} consists of linear matrix equalities and inequalities. We can solve the restricted dual in (4.8) with a bisection algorithm. The constraints involving complex matrices can be transformed into constraints involving real matrices by use of the following standard procedure.

Let $z = z_r + iz_i \in \mathbf{C}^{m\times m}$ be a complex valued matrix with $z_r, z_i \in \mathbf{R}^{m\times m}$. We can then represent z as the matrix

$$Z = \begin{bmatrix} z_r & z_i \\ -z_i & z_r \end{bmatrix} \in \mathbf{R}^{2m\times 2m}.$$

The following properties hold.

1. The conditions for z to be Hermitian can be stated as $z = z^* \Leftrightarrow Z = Z^T$, which implies that $z_r = z_r^T$ and $z_i = -z_i^T$.
2. If z is Hermitian, then $z \geq 0 \Leftrightarrow Z \geq 0$.
3. Multiplication and addition of complex matrices correspond to multiplication and addition of the corresponding real valued matrices. Hence, we have $z_1 + z_2 \Leftrightarrow Z_1 + Z_2$ and $z_1 z_2 \Leftrightarrow Z_1 Z_2$.

There are cases when the computational approach described above is not successful. For example, there does not always exist a function with a finite number of step discontinuities such that the constraint in the dual is satisfied. In other words, there exists cones Π_Δ such that it is impossible to find $Z \in P_{\text{AM}}^{m\times m}$ satisfying $M_G^\times Z \in \Pi_\Delta^\oplus$. An example will be given in Section 4.8. Note also that for our computational approach to be successful it is necessary that every conjugate cone $\Pi_{j\Delta_i}(\gamma)^\oplus$ in Corollary 4.1 is suited for the approach. If this is not the case then another basis for the restriction of $\mathcal{S}_{\text{NBV}}$ should be considered. In Chapter 5 it is shown that the computational approach discussed above is successful with a small number of frequencies in the grid Ω for a large class of problems of practical interest.

Hard Primal and Soft Dual

It often happens that there are algebraic constraints in the dual that are hard to treat numerically. We illustrate this with a simple example below. We will also show how this problem can be overcome by introducing alternative primal and dual problems.

EXAMPLE 4.2
Consider robust stability for the system in Figure 4.1. Let $\Delta = \delta I$, where δ is an unknown parameter with $\delta \in [-\rho, \rho]$. We want to find an upper bound for the size of ρ. Let $\gamma = \rho^{-2}$, then δI can be described in terms of $\Pi_{1\Delta}(\gamma)$ and $\Pi_{2\Delta}(\gamma)$, where

$$\Pi_{1\Delta}(\gamma) = \left\{ \begin{bmatrix} X & 0 \\ 0 & -\gamma X \end{bmatrix} : X(j\omega) \geq 0, \ \forall \omega \in [0, \infty] \right\},$$

and

$$\Pi_{2\Delta}(\gamma) = \left\{ \begin{bmatrix} 0 & Y \\ Y^* & 0 \end{bmatrix} : Y \in \mathbf{RL}_{\infty}^{m \times m}, \ Y(j\omega)^* = -Y(j\omega), \ \forall \omega \in [0, \infty] \right\}.$$

To find the conjugate cones $\Pi_{1\Delta}(\gamma)^{\oplus}$ and $\Pi_{2\Delta}(\gamma)^{\oplus}$ we consider $W \in \mathcal{S}_{\text{NBV}}^{2m \times 2m}$ with the structure

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{12}^* & W_{22} \end{bmatrix}.$$

By definition, $W \in \Pi_{1\Delta}(\gamma)^{\oplus}$ if for all $\Pi \in \Pi_{1\Delta}(\gamma)$ we have $\langle \Pi, W \rangle \geq 0$. This constraint reduces to $\langle X, W_{11} - \gamma W_{22} \rangle \geq 0$, for all $X(j\omega) \geq 0$. Hence, by Proposition 4.3 we get

$$\Pi_{1\Delta}(\gamma)^{\oplus} = \{ W \in \mathcal{S}_{\text{NBV}}^{2m \times 2m} : W_{11} - \gamma W_{22} \in P_{\text{NBV}}^{m \times m} \}.$$

Similarly, the constraint $\langle \Pi, W \rangle \geq 0$, for all $\Pi \in \Pi_{2\Delta}(\gamma)$ reduces to the condition $\langle Y, W_{12}^* - W_{12} \rangle \geq 0$, for all $Y(j\omega) = -Y(j\omega)^*$. Hence, Proposition 4.4 gives

$$\Pi_{2\Delta}(\gamma)^{\oplus} = \{ W \in \mathcal{S}_{\text{NBV}}^{2m \times 2m} : W_{12} - W_{12}^* \equiv 0 \}.$$

It follows from Corollary 4.1 that the corresponding dual is

$$\sup \gamma \quad \text{subject to}$$

$$D : \begin{cases} \exists Z \in P_{\text{NBV}}^{m \times m}, \ Z \not\equiv 0, \ \text{such that} \\ GdZG^* - \gamma dZ \geq 0 \\ GdZ - dZG^* \equiv 0, \end{cases}$$

where the last constraint is algebraic. For this dual it is no restriction to use the computational method described above with only one frequency in the grid Ω, see Chapter 5. We get

$$\sup \gamma \quad \text{subject to}$$
$$D : \begin{cases} \exists \omega \in [0,\infty],\ Z \in \mathcal{S}_{\mathrm{C}}^{m\times m},\ Z \geq 0,\ Z \neq 0, \text{ such that} \\ G(j\omega)ZG(j\omega)^* - \gamma Z \geq 0 \\ G(j\omega)Z - ZG(j\omega)^* = 0. \end{cases} \tag{4.9}$$

It is shown in Appendix 4.10 that the last constraint in (4.9) corresponds to finding a frequency where $G(j\omega)$ has a real valued eigenvalue. Finding such frequencies is generally a hopeless numerical problem.

In Appendix 4.10 we also prove that the optimization problem in (4.9) is equivalent to finding the frequency where $G(j\omega)$ has a real-valued eigenvalue of maximal modulus. More precisely:

$$\rho_{\max}^{-1} = \gamma_{\text{opt}}^{1/2} = \max\{|\lambda(G(j\omega))| : \lambda(G(j\omega)) \in \mathbf{R},\ \omega \in [0,\infty]\},$$

where $\lambda(G(j\omega))$ denotes an eigenvalue for $G(j\omega)$. The primal optimization problem for this simple example is thus equivalent to the Nyquist criterion. This is of course well-known. What is more interesting is that even for this simple example the dual becomes quite complicated from a computational perspective. □

The example motivates the use of alternative primal and dual optimization problems. Dual problems with algebraic constraints of the type in this example can be avoided by considering a primal with *harder* constraints on the multiplier. This will give a corresponding dual with *softer* constraints. For example, let the primal be

PRIMAL 4.2—HARD PRIMAL OPTIMIZATION PROBLEM

$$\inf \gamma \quad \text{subject to}$$
$$P_H : \begin{cases} \exists \Pi \in \Pi_\Delta(\gamma), \text{ such that} \\ \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < -\varepsilon I, \quad \forall \omega \in [0,\infty], \\ -cI < \Pi(j\omega) < cI,\ \forall \omega \in [0,\infty], \end{cases}$$

where $\varepsilon > 0$ is small and where $c > 0$ is large. □

This restriction of the original primal is reasonable in a computations perspective since only bounded entities can be treated in the computer and the constraints will always be obtained with some marginal ε, if they are obtained at all.

The monotonicity assumption on $\Pi_\Delta(\gamma)$ is no longer enough to ensure that the primal constraint P_H is satisfied for all $\gamma > \inf_{P_H} \gamma$. The following alternative assumption will ensure this. This condition is satisfied in all examples considered in this chapter.

DEFINITION 4.2—MONOTONICITY OF $\Pi_\Delta(\gamma)$ FOR HARD PRIMAL
If $\gamma_2 \geq \gamma_1$, then for all $-cI < \Pi_1 \in \Pi_\Delta(\gamma_1)$ there exists $\Pi_2 \in \Pi_\Delta(\gamma_2)$ such that $-cI < \Pi_2 \leq \Pi_1$. □

Under this assumption on $\Pi_\Delta(\gamma)$ we derive the following dual to Primal 4.2.

DUAL 4.2—SOFT DUAL OPTIMIZATION PROBLEM

$$\sup \gamma \quad \text{subject to}$$
$$D_S : \begin{cases} \exists Z \in P_{\text{NBV}}^{m\times m},\ Z \not\equiv 0, \text{ such that} \\ \begin{bmatrix} G \\ I \end{bmatrix} dZ \begin{bmatrix} G \\ I \end{bmatrix}^* \in d\Pi_\Delta(\gamma)^{\oplus} + d\mathcal{B}(Z,\varepsilon,c), \end{cases}$$

where

$$\mathcal{B}(Z,\varepsilon,c) = \left\{ X \in \mathcal{S}_{\text{NBV}}^{2m\times 2m} : \int_{-\infty}^{\infty} \operatorname{tr}(|dX(j\omega)|) \leq \frac{\varepsilon}{c}\int_{-\infty}^{\infty} \operatorname{tr}(dZ(\omega)) \right\}.$$

Here $|dX|$ denotes the absolute value of the integration measure dX, see Rudin (1987) □

The next proposition states that the hard primal and the soft dual have the same objective value.

PROPOSITION 4.5
Primal 4.2 and Dual 4.2 have the same objective values, i.e.,

$$\inf_{P_H} \gamma = \sup_{D_S} \gamma.$$

Proof: The proof is similar to the proof of Theorem 4.1 and it is given in Appendix 4.11. □

Let us consider Example 4.2 with the soft dual. The constraint in (4.9) need in this case only be satisfied with a precision dependent on c and ε. The situation is illustrated in Figure 4.5.

The suggested soft dual has very much in common with the idea of adding a small complex perturbation to each real parameter in the computations for the mixed real/complex singular value, see Packard and Pandey (1993).

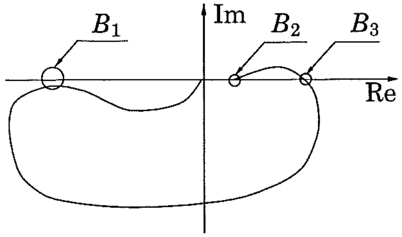

Figure 4.5 If ε/c is sufficiently small then any point of the Nyquist curve within the balls B_2 and B_3 satisfies the constraint of the soft dual. The optimal value will be obtained by a point in B_3. If ε/c becomes to large then also points in B_1 satisfies the soft dual constraint. This will give an optimal value that is far from what should be expected from the original optimization problem.

There are examples when the difference between the objective values for Dual 4.1 and Dual 4.2 is large even when ε is small and c is large, see Figure 4.5. This is often an indication that the system model with G and Δ is ill-conditioned. Problems of this type can be detected in applications. Assume that we find a suboptimal value, γ_p, of Primal 4.1 and a suboptimal solution, γ_{d_S}, of the corresponding soft dual in Dual 4.2. In normal situations we have $\gamma_p \geq \gamma_{d_S}$ if ε and c in the soft dual are chosen suitably small and large, respectively. However, if we obtain $\gamma_{d_S} > \gamma_p$ then this indicates that the system under consideration either lack reasonable robustness or is wrongly modeled. We illustrate this situation in Figure 4.6.

4.6 The Ship Steering Example, cont'd

We will here describe the computations for the ship steering example in Section 4.3. We first discuss the primal optimization problem.

The Primal Optimization Problem: Let $X(j\omega) = R(j\omega)^* U R(j\omega)$, where $R \in \mathbf{RH}_\infty^{N\times 2}$ and $U = U^T \geq 0$. We can then use the finite-dimensional convex cone consisting of the functions

$$\left\{ \left[\begin{array}{c|c} \widehat{R} & 0 \\ \hline 0 & R \end{array}\right]^* \left[\begin{array}{cc|c} U & 0 & 0 \\ 0 & U & 0 \\ \hline 0 & 0 & -\gamma U \end{array}\right] \left[\begin{array}{c|c} \widehat{R} & 0 \\ \hline 0 & R \end{array}\right] : U = U^T \geq 0 \right\} \subset \Pi_\Delta(\gamma).$$

Here $\widehat{R}$ is obtained by spectral factorization of $[R^*(s+j\omega_0)UR(s+j\omega_0) + R^*(s-j\omega_0)UR(s-j\omega_0))]/2$. If R has the realization $R(s) = C(sI-A)^{-1}B +$

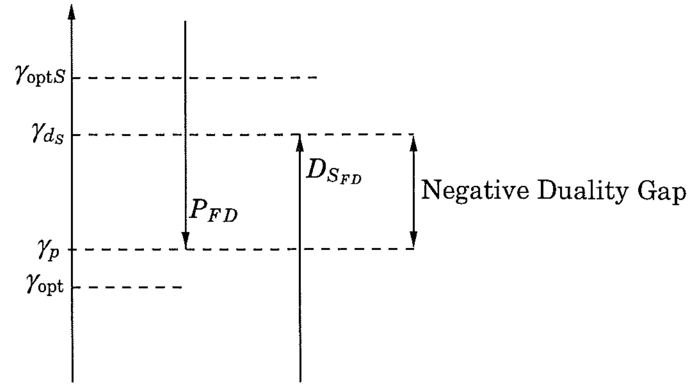

Figure 4.6 System models that lack suitable robustness can be detected in the following way. Find a (suboptimal) solution, γ_p, to Primal 4.1 and a (suboptimal) solution, γ_{d_S}, to the soft dual in 4.2. Then a negative duality gap $\gamma_p - \gamma_{d_S}$ indicates that there is such a lack of robustness. Here γ_{opt} denotes the optimal value of Primal 4.1 and $\gamma_{\text{opt}S}$ denotes the optimal value of the soft dual in Dual 4.2.

$D \in \mathbf{RH}_\infty^{N\times 2}$, then

$$\widehat{R} = \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} sI - A & -\omega_0 I \\ \omega_0 I & sI - A \end{bmatrix}^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix} + \begin{bmatrix} D \\ 0 \end{bmatrix} \in \mathbf{RH}_\infty^{2N\times 2}.$$

Table 4.1 shows numerical results obtained by LMI-lab, see Gahinet *et al.* (1995), when we use R on the form

$$\text{Ritz}(p,n) = \begin{bmatrix} I_2 & \frac{s-p}{s+p} I_2 & \dots & \frac{(s-p)^n}{(s+p)^n} I_2 \end{bmatrix}^T.$$

The results are given in terms of the obtained bound $A_{\max}$.

The Dual Optimization Problem: In order to solve the corresponding dual optimization problem we need the conjugate cone $\Pi_\Delta(\gamma)^\oplus$. In other words we need the set of $W \in \mathcal{S}_{\text{NBV}}^{4\times 4}$ such that for any $\Pi \in \Pi_\Delta(\gamma)$

$$\langle \Pi, W \rangle = \left\langle X(j\omega), \frac{1}{2}(W_{11}(\omega + \omega_0) + W_{11}(\omega - \omega_0)) - \gamma W_{22}(\omega) \right\rangle \geq 0. \tag{4.10}$$

Since (4.10) shall hold for any $X(j\omega) \geq 0$, we get

$$\Pi_\Delta(\gamma)^\oplus = \left\{ W : \frac{1}{2}(W_{11}(\omega + \omega_0) + W_{11}(\omega - \omega_0)) - \gamma W_{22}(\omega) \in P_{\text{NBV}}^{2\times 2} \right\}.$$

If we let S_{ω_0} denote the shift operator defined by $S_{\omega_0}Z(\omega) = Z(\omega + \omega_0)$, then the dual can be formulated as

$$\sup \gamma \quad \text{subject to}$$
$$D : \begin{cases} \exists Z \in P^{2\times 2}_{\text{NBV}},\ Z \not\equiv 0, \text{ such that,} \\ \frac{1}{2}(S_{\omega_0} GdZG^* + S_{-\omega_0} GdZG^*) - \gamma dZ \geq 0 \end{cases} \tag{4.11}$$

The computational ideas in Section 4.5 and the form of the second constraint in (4.11) suggests that we choose a grid $\Omega = \{\omega_1, \ldots, \omega_N\}$ satisfying

$$\omega_{(k-1)L+l} + \omega_0 = \omega_{kL+l} = \omega_{(k+1)L+l} - \omega_0,$$

for $k = 1, \ldots, K-1$ and $l = 1, \ldots, L$, see Figure 4.7. Let

$$Z_k = \begin{bmatrix} Z_{kR} & Z_{kI} \\ -Z_{kI} & Z_{kR} \end{bmatrix},$$

for $k = 1, \ldots, N$, where $N = (K+1)L$. Here $Z_{kR} = Z_{kR}^T \in \mathbf{R}^{2\times 2}$ and $Z_{kI} = -Z_{kI}^T \in \mathbf{R}^{2\times 2}$. Further let

$$G_k = \begin{bmatrix} \text{Re}G(j\omega_k) & \text{Im } G(j\omega_k) \\ -\text{Im } G(j\omega_k) & \text{Re}G(j\omega_k) \end{bmatrix} \in \mathbf{R}^{4\times 4}.$$

We can then formulate the restricted dual optimization problem as

$$\sup_{D_{AM}} \gamma \quad \text{subject to}$$
$$D_{AM} : \begin{cases} \exists Z_{kM+l} \geq 0, \text{ at least one nonzero, such that} \\ \frac{1}{2} G_{L+l} Z_{(L+l)} G_{L+l}^T - \gamma Z_l \geq 0, \\ \frac{1}{2}(G_{(k+1)L+l} Z_{(k+1)L+l} G_{(k+1)L+l}^T \\ + G_{(k-1)M+l} Z_{(k-1)L+l} G_{(k-1)L+l}^T) - \gamma Z_{kL+l} \geq 0, \\ \frac{1}{2} G_{(K-1)L+l} Z_{(K-1)L+l} G_{(K-1)L+l} - \gamma Z_{KL+l} \geq 0, \\ \text{for } k = 1, \ldots, K-1,\ l = 1, \ldots, L. \end{cases}$$

Note that the constraint set is an LMI condition. Numerical results obtained with LMI-Lab are given in Table 4.1.

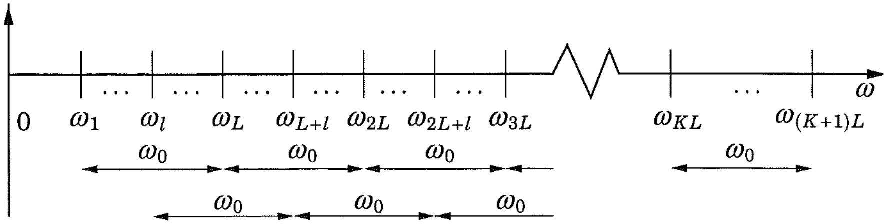

Figure 4.7 Frequency grid for the dual to the ship example.

$R(s)$	$A_{\max}$
I_2	0.090
Ritz(1, 1)	0.092
Ritz(1, 2)	0.120
Ritz(1, 3)	0.162
Ritz(1, 4)	0.184
Ritz(1, 5)	0.188

K	L	ω_1	$A_{\min}$
2	1	0.185	0.195
4	1	0.185	0.194
8	1	0.185	0.194

Table 4.1 Table with results for the primal and dual optimization problem respectively. Here $A_{\max}$ denotes the bound on A obtained from the primal. The bound $A_{\min} = 1/\gamma_d^{-1/2}$ is obtained from the dual. This bound has no interpretation in terms of stability for the ship steering dynamics. It gives an indication of the quality of the solution to the primal optimization problem.

4.7 Slope Restricted Nonlinearities

We will in this section consider a class of multipliers that gives a particularly complicated dual. Let $\varphi : \mathbf{R} \to \mathbf{R}$ be a slope restricted nonlinearity. We assume that φ satisfies the properties

(i) φ is odd,

(ii) there exists $k > 0$ such that $\varphi(x) \leq k|x|$, $\forall x \in \mathbf{R}$,

(iii) $0 \leq \frac{\varphi(x_1)-\varphi(x_2)}{x_1-x_2} \leq c, \quad x_1 \neq x_2$, where $0 < c \leq \infty$.

Let H be a strictly proper transfer function. The corresponding weighting function h and its $\mathbf{L}_1$ norm are defined as

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega)e^{j\omega t}d\omega,$$
$$\|h\|_1 = \int_{-\infty}^{\infty} |h(t)|dt.$$

It follows from Zames and Falb (1968) that the slope restricted nonlinearities satisfies the IQCs from the following convex cone

$$\Pi_\Delta = \left\{ \begin{bmatrix} 0 & h_0 + H(j\omega)^* \\ h_0 + H(j\omega) & -\frac{2}{c}\mathrm{Re}(h_0 + H(j\omega)) \end{bmatrix} : \|h\|_1 \le h_0,\ h_0 \ge 0 \right\}. \tag{4.12}$$

In the next lemma we derive the restriction of the conjugate cone $\Pi_\Delta^\oplus$ to the step functions in $\mathcal{S}_{\mathrm{AM}}^{2\times 2}$.

LEMMA 4.1
Let $W \in \mathcal{S}_{\mathrm{AM}}^{2\times 2}$ be a step function with step discontinuities at $\omega_1, \dots, \omega_{N-1}$, and $\omega_N = \infty$. Each coefficient is structured as

$$W_k = \begin{bmatrix} W_{11,k} & W_{12,k} \\ W_{12,k}^* & W_{22,k} \end{bmatrix}.$$

If $c = \infty$, then the dual cone $\Pi_\Delta^\oplus \cap \mathcal{S}_{\mathrm{AM}}^{2\times 2}$ is defined as

$$\Pi_\Delta^\oplus \cap \mathcal{S}_{\mathrm{AM}}^{2\times 2} = \left\{ W \in \mathcal{S}_{\mathrm{AM}}^{2\times 2} : \mathrm{Re}\sum_{k=1}^{N} W_{12,k} \ge \sup_{t\in\mathbf{R}} \left| \mathrm{Re}\sum_{k=1}^{N-1} W_{12,k} e^{-j\omega_k t} \right| \right\}, \tag{4.13}$$

Proof: By definition $W \in \Pi_\Delta^\oplus \cap \mathcal{S}_{\mathrm{AM}}^{2\times 2}$ if for any $\Pi \in \Pi_\Delta$,

$$\langle \Pi, W \rangle = 4\sum_{k=1}^{N} \mathrm{Re}([h_0 + H(j\omega_k)]W_{12,k}) \ge 0. \tag{4.14}$$

We need to show that the constraint in (4.13) is a necessary and sufficient condition for (4.14) to hold. For the proof of sufficiency we note that $H(j\omega_N) = 0$ and

$$H(j\omega_k) = \int_{-\infty}^{\infty} h(t) e^{-j\omega_k t} dt.$$

A simple argument gives the inequality

$$\begin{aligned} \mathrm{Re}\sum_{k=1}^{N-1} H(j\omega_k) W_{12,k} &= \int_{-\infty}^{\infty} h(t) \cdot \mathrm{Re}\left(\sum_{k=1}^{N-1} W_{12,k} e^{-j\omega_k t} \right) dt \\ &\ge -\|h\|_1 \cdot \sup_{t\in\mathbf{R}} \left| \mathrm{Re}\sum_{k=1}^{N-1} W_{12,k} e^{-j\omega_k t} \right|. \end{aligned} \tag{4.15}$$

It follows from this inequality and the norm condition $\|h\|_1 \leq h_0$ that the constraint in (4.13) is a sufficient condition for (4.14).

In order to prove the necessity we show that the inequality in (4.15) is nonrestrictive. For this we notice that $f(t) = \text{Re}\sum_{k=1}^{N-1} W_{12,k}e^{-j\omega_k t} \in \mathbf{L}_\infty(-\infty,\infty)$. The dual space of $\mathbf{L}_1(-\infty,\infty)$ is $\mathbf{L}_\infty(-\infty,\infty)$ and the linear functionals are defined by the integral

$$\langle h, f\rangle = \int_{-\infty}^{\infty} h(t)f(t)dt.$$

Hence, by the definition of norm for these linear functionals, we get

$$\inf_{\|h\|_1\leq h_0} \langle h, f\rangle = -h_0\|f\|_\infty, \qquad (4.16)$$

where $\|f\|_\infty = \sup_{t\in\mathbf{R}}|f(t)|$. In our application we only consider strictly proper $H \in \mathbf{RL}_\infty^{1\times 1}$. The exponentials $\{t^k e^{-t}\theta(t)\}_{k=0}^\infty$ and $\{t^k e^{t}\theta(-t)\}_{k=0}^\infty$ are dense in $\mathbf{L}_1[0,\infty)$ and $\mathbf{L}_1(-\infty,0]$ respectively, see Szegö (1975). This means that we can approximate an arbitrary $h \in \mathbf{L}_1(-\infty,\infty)$ with any accuracy with a suitable finite linear combination of such exponential functions. The corresponding transfer function will be in $\mathbf{RL}_\infty^{1\times 1}$ and it follows that (4.16) also holds when we consider the optimization over the weighting functions corresponding to this class of rational functions. This proves the necessity. □

It is clear from (4.13) that the frequencies for the step discontinuities must be chosen with care. Actually, for choices most choices of frequency grid the right hand side of (4.13) will be $\sum_{k=1}^{N}|W_{12,k}|$. However, if we let the frequencies be rational numbers then the right hánd side of (4.13) will be periodic and there is a possibility that the condition is satisfied.

We will next consider unfeasibility of a stability test for systems with a linear time invariant plant G in the forward loop and a slope restricted nonlinearity in the feedback loop.

PROPOSITION 4.6—UNFEASIBILITY WITH ZAMES AND FALB'S MULTIPLIER
Assume that there exists $\omega_1,\ldots,\omega_{N-1} \in [0,\infty)$, and $z_1,\ldots,z_N \geq 0$, where at least one z_k is nonzero, such that

$$\sum_{k=1}^{N}\text{Re}\left(z_k(G(j\omega_k) - \frac{1}{c})\right) \geq \sup_{t\in\mathbf{R}}\left|\text{Re}\sum_{k=1}^{N-1} z_k(G(j\omega_k) - \frac{1}{c})e^{-j\omega_k t}\right|, \qquad (4.17)$$

where $\omega_N = \infty$. Then there is no solution to the the stability test: Find a strictly proper $H \in \mathbf{RL}_\infty^{1\times 1}$ with weighting function h such that

a. $\|h\|_1 \leq 1$.

b. $\text{Re}\left((1 + H(j\omega)(G(j\omega) - \frac{1}{c})\right) < 0, \ \forall\omega \in [0,\infty]$.

Proof: The stability test can be formulated as: Find $\Pi \in \Pi_\Delta$ such that $M_G\Pi < 0$, where $M_G : \mathcal{S}_\infty^{2m\times 2m} \to \mathcal{S}_\infty^{m\times m}$ is defined as

$$M_G\Pi = \begin{bmatrix} G - \dfrac{1}{c} \\ I \end{bmatrix}^* \Pi \begin{bmatrix} G - \dfrac{1}{c} \\ I \end{bmatrix},$$

and where Π_Δ corresponds to the convex set in (4.12) when $c = \infty$ and $h_0 = 1$. Hence, the proof follows from Theorem 4.2 and Lemma 4.1. □

It is in most applications hard to find a suitable frequency grid for application of Proposition 4.6. However, the next example shows that it can be done. A different way of treating unfeasibility of the stability test in Proposition 4.6 has been reported in Megretski (1995).

EXAMPLE 4.3
We consider the system in Figure 4.1 when Δ is a slope restricted nonlinearity with slope in $[0, c]$ and when G has transfer function

$$G(s) = \frac{s^2}{(s^2+\alpha)(s^2+\beta) + 10^{-4}(14s^3+21s)},$$

where $\alpha = 0.9997$ and $\beta = 9.0039$. This is a system with two very distinct resonances at $\omega \approx 1$ and $\omega \approx 3$. A similar system was used in Willems (1971a) to give a counterexample to Aizerman's conjecture.

The purpose of the example is to find a bound on c such that stability of the system is guaranteed. The simple multiplier $H(s) = -\frac{6.25}{(s+2.5)^2}$ can be used to prove stability for $c = 0.0048$. If we use Proposition 4.6 with $\omega_1 = 1$ and $\omega_2 = 3$, then the condition in (4.17) is satisfied if $c = 0.0061$. Hence, the duality gap is reasonable small despite the low order of the multiplier H. Figure 4.8 shows the Nyquist curves for $G(j\omega) - \frac{1}{c}$ for $c = 0.0061$ and $c = 0.0048$, respectively. □

4.8 Robust Performance Analysis

We will here investigate the dual that appears in robust performance analysis of the second system in Figure 4.1. We assume that the transfer function has structure

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \in \mathbf{RH}_\infty^{(m+q)\times(m+q)}$$

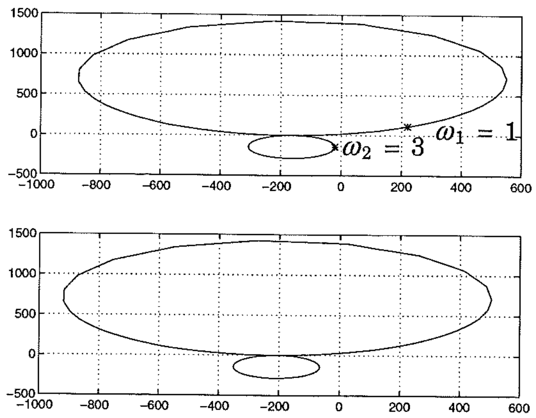

Figure 4.8 The upper plot shows the Nyquist diagram of $G(j\omega) - 1/c$ when $c = 0.0061$. There is no solution to the feasibility test in Theorem 4.6 for this value of c. The lower plot shows the Nyquist diagram of $G(j\omega) - 1/c$ when $c = 0.0048$. The multiplier $H = 6.25/(s + 2.5)^2$ can be used to prove stability for this case.

and that the input signal is in the class of $w \in \mathbf{L}_2^q[0, \infty)$ satisfying the IQC defined by the convex cone $\Upsilon_{\text{inp}} \subset \mathcal{S}_\infty^{q\times q}$, i.e.,

$$\int_{-\infty}^{\infty} \widehat{w}(j\omega)^* \Upsilon(j\omega) \widehat{w}(j\omega) d\omega \geq 0, \ \forall \Upsilon \in \Upsilon_{\text{inp}}. \tag{4.18}$$

It follows from Chapter 3 that robust performance analysis gives optimization problems on the following form.

PRIMAL 4.3—ROBUST PERFORMANCE ANALYSIS

$$\inf \gamma \quad \text{subject to}$$

$$P_{\text{perf}} : \begin{cases} \exists \Pi_1 \in \Pi_\Delta, \ \Pi_2 \in \Pi_{\text{perf}}(\gamma), \ \Upsilon \in \Upsilon_{\text{inp}}, \ \text{such that} \\ \begin{bmatrix} G \\ I \end{bmatrix}^* \text{daug}\left(\Pi_1, \Pi_2 + \begin{bmatrix} 0 & 0 \\ 0 & \Upsilon \end{bmatrix}\right) \begin{bmatrix} G \\ I \end{bmatrix} (j\omega) < 0, \ \forall \omega \in [0, \infty], \end{cases}$$

where Π_Δ is a convex cone of multipliers for Δ, and where $\Pi_{\text{perf}}(\gamma)$ defines the performance criterion. □

We assume that $\Pi_{\text{perf}}(\gamma) \subset \mathcal{S}_\infty^{2q\times 2q}$ is a convex cone for every $\gamma \in \mathbf{R}$, which satisfies the following monotonicity condition.

DEFINITION 4.3—MONOTONICITY FOR $\Pi_{\text{perf}}(\gamma)$
If $\gamma_2 \geq \gamma_1$, then $\forall \Pi_1 \in \Pi_{\text{perf}}(\gamma_1)$ there exists $\Pi_2 \in \Pi_{\text{perf}}(\gamma_2)$ such that $\Pi_1 \geq \Pi_2$. This ensures that the constraint P_{perf} in Primal 4.3 is satisfied for all $\gamma > \inf_{P_{\text{perf}}} \gamma$. □

The dual optimization problem corresponding to Primal 4.3 becomes.

DUAL 4.3—ROBUST PERFORMANCE ANALYSIS

$$\sup \gamma \quad \text{subject to}$$

$$D_{\text{perf}} : \begin{cases} \exists Z \in P_{\text{NBV}}^{(m+q)\times(m+q)}, \; Z \ngeq 0, \text{ such that} \\ \begin{bmatrix} G_{11} & G_{12} \\ I & 0 \end{bmatrix} dZ \begin{bmatrix} G_{11} & G_{12} \\ I & 0 \end{bmatrix}^* \in d\Pi_{\Delta}^{\oplus}, \\ \begin{bmatrix} G_{21} & G_{22} \\ 0 & I \end{bmatrix} dZ \begin{bmatrix} G_{21} & G_{22} \\ 0 & I \end{bmatrix}^* \in d\Pi_{\text{perf}}(\gamma)^{\oplus}, \\ Z_{22} \in \Upsilon_{\text{inp}}^{\oplus}, \end{cases}$$

where

$$\Upsilon_{\text{inp}}^{\oplus} = \{ W \in S_{\text{NBV}}^{q\times q} : \langle \Upsilon, W \rangle \geq 0, \; \forall \Upsilon \in \Upsilon_{\text{inp}} \}.$$

□

We have the following duality result.

PROPOSITION 4.7
Primal 4.3 and Dual 4.3 have the same objective values, i.e.,

$$\inf_{P_{\text{perf}}} \gamma = \sup_{D_{\text{perf}}} \gamma.$$

Proof: Let

$$\Psi = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & \Upsilon \end{bmatrix} : \Upsilon \in \Upsilon_{\text{inp}} \right\}.$$

We notice that the convex cone $\text{daug}(\Pi_{\Delta}, \Pi_{\text{perf}}(\gamma) + \Psi)$ obviously satisfies the required condition on monotonic γ dependence. We can then apply Corollary 4.1. We have

$$\Psi^{\oplus} = \left\{ \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \in S_{\text{NBV}}^{2q\times 2q} : W_{22}^* \in \Upsilon_{\text{inp}}^{\oplus} \right\}.$$

Hence, we see that the constraint

$$\begin{bmatrix} G_{21} & G_{22} \\ 0 & I \end{bmatrix} dZ \begin{bmatrix} G_{21} & G_{22} \\ 0 & I \end{bmatrix}^* \in d\Psi^{\oplus}$$

reduces to $Z_{22} \in \Upsilon^{\oplus}_{\text{inp}}$ and the proposition follows. □

REMARK 4.2
Assume that the condition $[I \quad 0]\Pi_2[I \quad 0]^T \geq 0$ holds for all $\Pi_2 \in \Pi_{\text{perf}}(\gamma)$ and for all $\gamma \in \mathbf{R}$. Then the solution to Dual 4.3 is $\gamma_{\text{opt}} = \infty$ if the following stability test is unfeasible: Find $\Pi_1 \in \Pi_\Delta$ such that

$$\begin{bmatrix} G_{11}(j\omega) \\ I \end{bmatrix}^* \Pi_1(j\omega) \begin{bmatrix} G_{11}(j\omega) \\ I \end{bmatrix} < 0, \quad \forall \omega \in [0, \infty].$$

This follows since for every $\gamma \in \mathbf{R}$ there exists a nonzero $Z \in P_{\text{NBV}}^{(m+q)\times(m+q)}$ with the structure

$$Z = \begin{bmatrix} Z_{11} & 0_{m\times q} \\ 0_{q\times m} & 0_{q\times q} \end{bmatrix}$$

such that the constraints in the dual are satisfied. □

We will illustrate this proposition with two simple examples. The first example shows that restricting the dual to the subspace of step functions can be useless. Not even suboptimal solutions can be obtained in this way for this example.

EXAMPLE 4.4
Let $G \in \mathbf{RH}_\infty^{1\times 1}$. We want to compute

$$\sup_{w \in \mathcal{W}} \frac{\|Gw\|}{\|w\|}, \tag{4.19}$$

where $\|\cdot\|$ denotes the $\mathbf{L}_2$-norm, and where

$$\mathcal{W} = \left\{ w \in \mathbf{L}_2[0,\infty) : |\widehat{w}(j\omega)|^2 = \frac{\|w\|^2}{\|H\|_2^2} |H(j\omega)|^2 \right\}.$$

Here H is a strictly proper transfer function in $\mathbf{RH}_\infty^{1\times 1}$, and the $\mathbf{H}_2$-norm is defined by

$$\|H\|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(j\omega)|^2 \, d\omega.$$

It is shown in Chapter 3 that every $w \in \mathcal{W}$ satisfies the condition in (4.18) when

$$\Upsilon_{\text{inp}} = \left\{ \Upsilon \in \mathcal{S}_{\infty}^{1\times 1} : \int_{-\infty}^{\infty} \Upsilon(j\omega)|H(j\omega)|^2\, d\omega \geq 0 \right\}.$$

Let us try to compute (4.19) by solving Primal 4.3 with this Υ_{inp}, and with $\Pi_{\Delta} = \emptyset$, and

$$\Pi_{\text{perf}}(\gamma) = \left\{ \begin{bmatrix} x & 0 \\ 0 & -\gamma x \end{bmatrix} : x \geq 0 \right\}.$$

We will see that this gives the correct solution, i.e., $\gamma_{\text{opt}}^{1/2} = \|GH\|_2/\|H\|_2$.

It is no restriction to use $x = 1$. We first assume that $\gamma > \inf_{P_{\text{perf}}} \gamma$. Evaluation of the quadratic form corresponding to the operator on the left hand side of the last constraint in P_{perf} for $w \in \mathcal{W}$ gives

$$\begin{aligned} &\int_{-\infty}^{\infty} \widehat{w}^* \begin{bmatrix} G \\ I \end{bmatrix}^* \begin{bmatrix} 1 & 0 \\ 0 & -\gamma + \Upsilon \end{bmatrix} \begin{bmatrix} G \\ I \end{bmatrix} \widehat{w}\, d\omega \\ &\qquad = \frac{\|w\|^2}{\|H\|_2^2} \int_{-\infty}^{\infty} |GH|^2\, d\omega - \gamma 2\pi \|w\|^2 + \int_{-\infty}^{\infty} \Upsilon |\widehat{w}|^2\, d\omega \leq 0. \end{aligned}$$

The last term in this expression is nonnegative by definition of Υ_{inp}. Hence, we can conclude that $\gamma_{\text{opt}} \geq \|GH\|_2^2/\|H\|_2^2$. The question is if equality is obtained at the infimum? We will show that this is the case. Our strategy for proving this is to first show that Dual 4.3 gives the correct value. Then Proposition 4.7 shows that the primal also have this objective value. The conclusion of this is that (4.19) can be computed with arbitrary precision by use of Primal 4.3.

The positive conjugate cones becomes

$$\begin{aligned} \Upsilon_{\text{inp}}^{\oplus} &= \left\{ c W_0 : W_0(\omega) = \int_0^{\omega} |H(j\nu)|^2\, d\nu,\ c \geq 0 \right\} \subset \mathcal{S}_{\text{NBV}}^{1\times 1}, \\ \Pi_{\text{perf}}(\gamma)^{\oplus} &= \left\{ W \in \mathcal{S}_{\text{NBV}}^{2\times 2} : \int_{-\infty}^{\infty} (dW_{11}(\omega) - \gamma dW_{22}(\omega)) \geq 0 \right\}. \end{aligned}$$

The last one is easy to derive. The first follows since Υ_{inp} is the positive half-space determined by W_0, i.e. $\Upsilon_{\text{inp}} = \{\Upsilon : \langle \Upsilon, W_0 \rangle \geq 0\}$. The dual

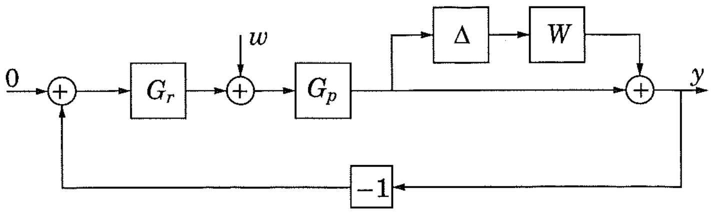

Figure 4.9 Control system for Example 4.5.

becomes (note that $\Pi_\Delta = \emptyset$ and thus $m = 0$)

$$\sup \gamma \quad \text{subject to}$$
$$\begin{cases} \exists Z \in P_{\text{NBV}}^{1\times 1},\ Z \not\equiv 0, \text{ such that} \\ \begin{bmatrix} G \\ I \end{bmatrix} dZ \begin{bmatrix} G \\ I \end{bmatrix}^* \in d\Pi_{\text{perf}}(\gamma)^{\oplus}, \\ Z \in \Upsilon_{\text{inp}}^{\oplus}. \end{cases}$$

This optimization problem reduces to

$$\sup \gamma \quad \text{subject to} \int_{-\infty}^{\infty} (|GH|^2 - \gamma |H|^2)\, d\omega \geq 0,$$

which gives the correct solution $\gamma_{\text{opt}}^{1/2} = \|GH\|_2/\|H\|_2$.

We cannot use step functions for this example. This follows since no $W \in \mathcal{S}_{\text{AM}}^{1\times 1}$ belongs to $\Upsilon_{\text{inp}}^{\oplus}$. For more general examples where, for example, dynamic and parametric uncertainty blocks are included in the model we need to have functions on the form W_0 in the basis. □

EXAMPLE 4.5

Figure 4.5 shows a feedback system consisting of a plant G_p, a controller G_r, and a multiplicative uncertainty represented as $W(s)\Delta(s)$. Here W denotes a weighting filter. We assume that Δ is a linear time invariant with $\|\Delta\|_\infty \leq 1$, where $\|\cdot\|_\infty$ is the usual H_∞-norm. The load disturbance w is assumed to be a low-frequency signal satisfying supp $\widehat{w}(j\omega) \subset [-0.1, 0.1]$, where supp $\widehat{w}$ denotes the support of the Fourier transform of w. The purpose of the example is to compute the worst case induced $\mathbf{L}_2$-norm of the system from w to y. The system can be transformed into the normal form for robust performance problems in Figure 4.1 with

$$G(s) = \frac{1}{1 + G_p G_r} \begin{bmatrix} -G_p G_r W & G_p \\ W & G_p \end{bmatrix}.$$

Assume that the plant has transfer function

$$G_p(s) = \frac{10}{(s+1)^2(s+10)},$$

and that we use a P controller designed with the Ziegler-Nichols frequency method, which gives $G_r(s) = 12.2$. Further assume that the weighting filter is the constant $W(s) = 0.1$. We use the multiplier descriptions

$$\Pi_\Delta = \left\{ \begin{bmatrix} X & 0 \\ 0 & -X \end{bmatrix} : \; X(j\omega) \geq 0 \right\},$$
$$\Pi_{\text{perf}}(\gamma) = \left\{ \begin{bmatrix} x & 0 \\ 0 & -\gamma x \end{bmatrix} : x \geq 0 \right\},$$

and

$$\Upsilon_{\text{inp}} = \left\{ \Upsilon \in \mathcal{S}_\infty^{1\times 1} : \Upsilon(j\omega) \geq 0, \; \omega \in [-a,a], \; \Upsilon(j\omega) \leq 0, \; \omega \notin [-a,a] \right\}.$$

Suboptimal solutions to Primal 4.3 can be obtained after introduction of finite-dimensional restrictions of Π_Δ and Υ_{inp}. We use

$$\left\{ \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}^* \begin{bmatrix} U & 0 \\ 0 & -U \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix} : U = U^T \geq 0 \right\} \subset \Pi_\Delta,$$

for some basis multiplier $R \in \mathbf{RH}_\infty^{N\times 1}$. For the cone Υ_{inp} it is optimal to choose multipliers that are zero in the interval $[-a,a]$ and large negative outside this interval. We use the multipliers

$$\left\{ \lambda_1 |H(j\omega)|^2 - \lambda_2 : \lambda_1 |H(ja)|^2 - \lambda_2 \geq 0, \lambda_1, \lambda_2 \geq 0 \right\} \subset \Upsilon_{\text{inp}},$$

where H is rational low pass filter with monotonically decreasing amplitude function. This is illustrated in Figure 4.10.

With $a = 0.1$ we obtained the results in Table 4.2. Here γ_{opt} denotes the optimal value of the primal optimization problem and the corresponding $\mathbf{L}_2$-performance is $\gamma_{\text{opt}}^{1/2}$. The multiplier Ritz$(p,n)$ is defined as

$$\text{Ritz}(p,n) = \begin{bmatrix} 1 & \frac{s-p}{s+p} & \cdots & \frac{(s-p)^n}{(s+p)^n} \end{bmatrix}.$$

We will next consider the corresponding dual optimization problem. It is easy to see that the positive conjugate cones becomes

$$\Pi_\Delta^\oplus = \left\{ W \in \mathcal{S}_{\text{NBV}}^{2\times 2} : W_{11} - W_{22} \in P_{\text{NBV}}^{1\times 1} \right\},$$
$$\Pi_{\text{perf}}(\gamma)^\oplus = \left\{ W \in \mathcal{S}_{\text{NBV}}^{2\times 2} : \int_{-\infty}^{\infty} (dW_{11}(\omega) - \gamma dW_{22}(\omega)) \geq 0 \right\},$$

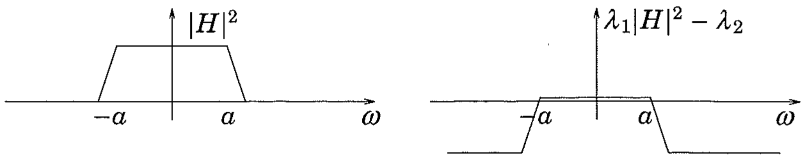

Figure 4.10 The idea for obtaining multipliers for the cone Υ_{inp}. The magnitude of the low pass hilter H should be approximately flat in the interval $(-a, a)$ and strongly decreasing outside this interval.

H	R	γ_{opt}
$\frac{1}{s+1}$	1	0.105
$\frac{1}{s+1}$	Ritz(3, 1)	0.0919
$\frac{1}{s+1}$	Ritz(3, 2)	0.0919
$\frac{10}{s+10}$	1	0.0879
$\frac{10}{s+10}$	Ritz(3, 1)	0.0778
$\frac{10}{s+10}$	Ritz(3, 2)	0.0778

Table 4.2 Numerical results for the primal optimization problem in Example 4.5.

and

$$\Upsilon_{\text{inp}}^{\oplus} = \{W \in \mathcal{S}_{\text{NBV}}^{1\times1} : W(\omega_1) - W(\omega_2) \geq 0, \text{ when } a \geq \omega_1 > \omega_2 \geq 0$$
$$W(\omega_1) - W(\omega_2) \leq 0, \text{ when } \infty \geq \omega_1 > \omega_2 \geq a\}.$$

The primal in this example is of a special type. It could have been solved by frequency-by-frequency optimization over the interval $[0, \infty]$. The dual can for this reason be solved by use of a function in $\mathcal{S}_{\text{AM}}^{2\times2}$ with only one step discontinuity. More general formulations of this type of optimization problems will be treated in Chapter 5. The dual becomes

$$\sup \gamma \quad \text{subject to}$$
$$\begin{cases} \exists \omega \in [0, \infty],\ Z \in \mathbf{C}^{2\times2},\ Z = Z^* \geq 0, Z \neq 0, \text{ s.t.} \\ [\,G_{11}(j\omega) \quad G_{12}(j\omega)\,]\,Z\,[\,G_{11}(j\omega) \quad G_{12}(j\omega)\,]^* - Z_{11} \geq 0, \\ [\,G_{21}(j\omega) \quad G_{22}(j\omega)\,]\,Z\,[\,G_{21}(j\omega) \quad G_{22}(j\omega)\,]^* - \gamma Z_{22} \geq 0, \\ Z_{22} = 0 \text{ if } \omega \notin [0, a]. \end{cases}$$

We obtain a solution to the dual by considering the following two steps.

(*i*) If there exists $\omega \in [0, \infty]$ such that $|G_{11}(j\omega)| \geq 1$, then $\gamma_{\text{opt}} = \infty$. In this case the closed loop system is unstable, see also Remark 4.2.

(*ii*) Otherwise consider the dual over the frequency interval $[0, a]$.

Using LMI-Lab we obtained the solution $\gamma_{\text{opt}} = 0.0764$. The duality gap 0.0014 is mainly due to the low order of the filter H. However, it seems that a filter of very high order is needed in order to obtain a smaller duality gap.

□

4.9 Conclusions

We have derived the format for the dual to optimization problems that appear in robustness analysis based on IQCs. The purpose of the dual is to give lower bounds to optimization problems that corresponds to finding the optimal multiplier in an infinite-dimensional convex set. We have shown how approximate solutions to the dual in many cases can be obtained by solving a finite-dimensional optimization problem at a preselected frequency grid. It is shown in Chapter 5 that this approach is successful with a small number of frequencies in the grid when constant multipliers are combined with frequency dependent multipliers that take independent values at different frequencies.

4.10 Appendix: Proof of the Statement in Example 4.2

Assume that there exists a nonzero $Z \in \mathcal{S}_{\mathbf{C}}^{m\times m}$ with $Z \geq 0$ such that

$$\begin{aligned} GZG^* - \gamma Z &\geq 0, \\ GZ - ZG^* &= 0, \end{aligned} \tag{4.20}$$

where $G \in \mathbf{C}^{m\times m}$. We will first show that the second constraint in (4.20) implies that G has at least one real-valued eigenvalue.

From Lemma 5 of Rantzer (1996) it follows that for (4.20) to hold Z must be on the form $Z = \sum_{k=1}^{m} z_k z_k^*$, where $Gz_k z_k^* = z_k z_k^* G$ for $k = 1, \dots, m$. Assume that $z_k \neq 0$. Then multiplication with z_k from the right in the last identity gives

$$Gz_k = \lambda_k z_k, \quad \lambda_k = z_k^* G z_k / (z_k^* z_k).$$

This implies that

$$0 = Gz_k z_k^* - z_k z_k^* G^* = z_k z_k^* (\lambda_k - \overline{\lambda_k}),$$

from which we conclude that λ_k is real for each k with nonzero z_k.

Next we prove that at least one of these eigenvalues have magnitude greater than or equal to γ. With $Z = \sum_{k=1}^{m} z_k z_k^*$, the first condition in (4.20) becomes

$$\sum_{k=1}^{m} G z_k z_k^* G^* - \gamma z_k z_k^* = \sum_{k=1}^{m} (\lambda_k^2 - \gamma) z_k z_k^* \geq 0.$$

This implies that $|\lambda_k| \geq \gamma$ for at least one k. The largest possible value of γ is obtained when $Z = z_k z_k^*$, where z_k is the eigenvector corresponding to the real-valued eigenvalue of G with largest magnitude. The value becomes λ_k^2. This proves the statements in Example 4.2.

4.11 Appendix: Proof of Proposition 4.5

If $\gamma < \inf_{P_H} \gamma$ then the convex sets

$$\begin{aligned} C_1 &= \{(M_G \Pi + \varepsilon I, \Pi - cI, -\Pi - cI) : \Pi \in \Pi_\Delta(\gamma)\}, \\ C_2 &= \left\{(X_0, X_1, X_2) \in \mathcal{S}_\infty^{m\times m} \times \mathcal{S}_\infty^{2m\times 2m} \times \mathcal{S}_\infty^{2m\times 2m} : X_i(j\omega) < 0, \forall \omega\right\}, \end{aligned}$$

are disjoint. Since the second set is open it follows from the separating hyperplane theorem that there exists a nonzero triple $Z = (Z_0, Z_1, Z_2) \in \mathcal{S}_{\text{NBV}}^{m\times m} \times (\mathcal{S}_{\text{NBV}}^{2m\times 2m})^2$ such that

$$\langle X, Z\rangle \geq 0, \quad \forall X \in C_1, \tag{4.21}$$

$$\langle X, Z\rangle < 0, \quad \forall X \in C_2. \tag{4.22}$$

For (4.22) to hold we need $Z \in P_{\text{NBV}}^{m\times m} \times (P_{\text{NBV}}^{2m\times 2m})^2$. Condition (4.21) can be reformulated as

$$\langle \Pi, M_G^\times Z_0 + Z_1 - Z_2\rangle + c\left\langle I, \frac{\varepsilon}{c} Z_0 - Z_1 - Z_2\right\rangle \geq 0, \ \forall \Pi \in \Pi_\Delta(\gamma). \tag{4.23}$$

Since $\Pi_\Delta(\gamma)$ is a cone containing 0 it is required that both terms need to be positive. Hence, the following constraints need to be satisfied

$$M_G^\times Z_0 + Z_1 - Z_2 \in \Pi_\Delta(\gamma)^\oplus, \tag{4.24}$$

$$\langle I, Z_1 + Z_2\rangle \leq \frac{\varepsilon}{c} \langle I, Z_0\rangle, \tag{4.25}$$

from which it also follows that $Z_0 \neq 0$, since otherwise we get $Z = (Z_0, Z_1, Z_2) = 0$.

We will next show that the constraints in (4.24) and (4.25) can be replaced by the constraint $M_G^{\times} Z_0 \in \Pi_\Delta(\gamma)^{\oplus} + \mathcal{B}(Z_0, \varepsilon, c)$. We notice that if we let $X = Z_1 - Z_2$ for some $Z_1, Z_2 \in P_{\text{NBV}}^{2m\times 2m}$, then we can instead use the minimal decomposition (Jordan decomposition) $X = Z^+ - Z^-$, where $Z^+, Z^- \in P_{\text{NBV}}^{2m\times 2m}$ corresponds to mutually singular positive measures dZ^+ and dZ^-. In this case we have the relation

$$\left\langle I, Z^+ + Z^+ \right\rangle = \int_{-\infty}^{\infty} \operatorname{tr}(|dX(\omega)|) \leq \langle I, Z_1 + Z_2 \rangle .$$

where $|dX|$ denotes the absolute value (total variation) of dX, see Rudin (1987). Hence $\inf_{P_H} \gamma \leq \sup_{D_S} \gamma$. For the opposite direction we note that the monotonicity assumption on $\Pi_\Delta(\gamma)$ implies that for every $\gamma > \inf_{P_H} \gamma$ there exists $\Pi \in \Pi_\Delta(\gamma)$ such that $C_1 \cap C_2 \neq \emptyset$. This implies that there is no triple $(Z_0, Z_1, Z_2) \in P_{\text{NBV}}^{m\times m} \times (P_{\text{NBV}}^{2m\times 2m})^2$ such that the constraints in (4.24) and (4.25) holds. Hence, $\sup_{D_S} \gamma \leq \inf_{P_H} \gamma$.

5

Duality in Analysis with Mixed Multipliers

Abstract

Multipliers are used in stability theory to reduce conservatism and exploit structural information about system components. For an important class of stability problems, which gives a resulting infinite-dimensional multiplier optimization, a corresponding dual problem is stated. The dual gives valuable information about the original problem, in particular error bounds for its finite dimensional approximations.

5.1 Introduction

Absolute stability theory, including passivity and small gain theorems, is an important tool for analysis of systems with nonlinearities, time-variations, and uncertainty. So called multipliers are used to reduce conservatism and exploit structural information about the system components. However, systematic methods for computation and optimization of such multipliers have not been available until recently.

The development of numerical methods for multiplier optimization started with the structured singular value in the early eighties, Safonov and Athans (1981), Doyle (1982). The real breakthrough came with the polynomial time algorithms for convex optimization with constraints defined by linear matrix inequalities, Nesterov and Nemirovski (1993), Boyd *et al.* (1994). This was used in connection with multiplier optimization in Balakrishnan *et al.* (1994) and Ly *et al.* (1994) and in full generality in Rantzer and Megretski (1994), Megretski and Rantzer (1995).

In general, the computation of multipliers becomes a convex optimization problem over an infinite-dimensional space. Such problems can be solved by considering a sequence of finite-dimensional approximations. More specifically, in the search for a rational function that satisfies certain constraints, the finite-dimensional approximation could mean that the denominator is fixed, while the search is restricted to the numerator coefficients. For example, this approach was used in controller design by Boyd and Barratt (1991).

Duality plays an important role in optimization theory, particularly in convex optimization. This chapter aims to demonstrate that multiplier optimization is no exception. For an important class of stability problems, which results in infinite-dimensional convex optimization, we will state a corresponding dual problem. The dual gives valuable information about the original problem, in particular error bounds for the finite dimensional approximations.

A general and unified approach to the use of multipliers was introduced in Rantzer and Megretski (1994), Megretski and Rantzer (1995) based on the concept integral quadratic constraint (IQC). An operator Δ (possibly nonlinear) on $\mathbf{L}_2^m[0,\infty)$ is said to satisfy the IQC defined by the matrix function Π, called *multiplier*, if

$$\int_{-\infty}^{\infty} \begin{bmatrix} \widehat{v}(j\omega) \\ \widehat{(\Delta v)}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \widehat{v}(j\omega) \\ \widehat{(\Delta v)}(j\omega) \end{bmatrix} d\omega \quad \geq \quad 0, \qquad \text{for all } v \in \mathbf{L}_2^m[0,\infty).$$

Here $\widehat{v}$ denotes the Fourier transform of v. Based on this definition, each operator Δ can be described by a set Π_Δ of multipliers Π, that define IQCs satisfied by Δ. For example, a passive operator satisfies the IQC defined by

$$\Pi(j\omega) = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix},$$

while a linear time-invariant operator with H_∞-norm less than one, satisfies any IQC defined by a matrix of the form

$$\Pi(j\omega) = \begin{bmatrix} x(j\omega)I & 0 \\ 0 & -x(j\omega)I \end{bmatrix},$$

where $x(j\omega) \geq 0$ for $\omega \in \mathbf{R}$. Basically, all properties of an operator, that can be expressed by IQCs, can be exploited in stability analysis.

Consider the feedback system in Figure 5.1, where G is a linear time invariant operator with stable transfer function, and where Δ is a bounded causal operator. Robust stability analysis for this system can be formu-

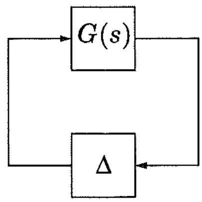

Figure 5.1 Feedback system with perturbation.

lated as optimization problems on the form

$$\inf \gamma \quad \text{subject to}$$
$$P : \begin{cases} \exists \Pi \in \Pi_\Delta(\gamma) \quad \text{such that} \\ \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0, \ \forall \omega \in [0, \infty], \end{cases} \tag{5.1}$$

where γ corresponds to the robustness criterion under consideration. The set $\Pi_\Delta(\gamma)$ is generally infinite-dimensional and approximate solutions to (5.1) can be obtained by considering optimization over a finite-dimensional subset of $\Pi_\Delta(\gamma)$. Conservativeness of this approach can be investigated by means of the dual optimization problem corresponding to (5.1). The dual was considered for general assumptions on the multipliers in $\Pi_\Delta(\gamma)$ in Chapter 4. This is in general a complicated optimization problem. In this chapter we show that the dual is particularly attractive in the case of constant multipliers, frequency dependent multipliers defined by a frequency independent constraint, and for multipliers that are a combination of these two classes of multipliers. We use the term *mixed multipliers* for the last class of multipliers.

5.2 Mathematical Preliminaries

This section presents the necessary mathematical preliminaries and notation needed in the chapter. The following standard definitions and results from functional analysis are available in for example Luenberger (1969).

- Let X be a normed vector space. The dual of X is the normed space consisting of all bounded linear functionals on X and it is denoted by X^*. If $x \in X$ and $x^* \in X^*$, then $\langle x, x^* \rangle$ denotes the value of the linear functional x^* at x. The vector spaces considered in this chapter are defined over the real scalar field and the linear functionals defined by functions from the dual space are real valued.

- The (Cartesian) product of two vector spaces X_1 and X_2, which are defined over the same field of scalars, is denoted $X_1 \times X_2$ and it consists of all ordered pairs $x = (x_1, x_2)$, with $x_1 \in X_1$ and $x_2 \in X_2$. Here x_1 and x_2 are said to be the coordinates of $X_1 \times X_2$. Addition and scalar multiplication is defined as $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$ and $\alpha(x_1, x_2) = (\alpha x_1, \alpha x_2)$.

- The dual of $X_1 \times X_2$ is given as $X_1^* \times X_2^*$, where X_1^* and X_2^* are the duals of X_1 and X_2 respectively. Given $x = (x_1, x_2) \in X_1 \times X_2$ and $x^* = (x_1^*, x_2^*) \in X_1^* \times X_2^*$, we define $\langle x, x^* \rangle = \langle x_1, x_1^* \rangle + \langle x_2, x_2^* \rangle$.

- X^N denotes the Cartesian product of N copies of X.

- Let $H : X \to Y$ be a bounded linear operator. Then the adjoint operator $H^\times : Y^* \to X^*$ is defined by the equation

$$\left\langle Hx, y^* \right\rangle = \left\langle x, H^\times y^* \right\rangle,$$

for all $x \in X$ and $y^* \in Y^*$.

Next is a list of notation and function spaces used in the chapter.

$\overline{M}$ Conjugation of a complex valued matrix.

M^* Hermitian conjugation of a matrix.

$|\cdot|_F$ The Frobenius norm of a real or complex matrix M is defined as $|M|_F = \sqrt{\mathrm{tr}(M^*M)}$.

$\mathbf{RL}_\infty^{m\times m}$ The vector space consisting of proper real rational matrix functions with no poles on the imaginary axis. Note that $H \in \mathbf{RL}_\infty^{m\times m}$ satisfies $H(-j\omega) = \overline{H(j\omega)}$.

$\mathbf{RH}_\infty^{m\times m}$ The subspace of $\mathbf{RL}_\infty^{m\times m}$ consisting functions with no poles in the closed right half plane.

$\mathcal{S}_{\mathrm{R}}^{m\times m}$ The subspace of $\mathbf{R}^{m\times m}$ consisting of symmetric matrices with the topology determined by the Frobenius norm. The dual space can be identified with $\mathcal{S}_{\mathrm{R}}^{m\times m}$ itself. The linear functionals are defined as $\langle X, Z \rangle = \mathrm{tr}(XZ)$, where $X, Z \in \mathcal{S}_{\mathrm{R}}^{m\times m}$.

$\mathcal{S}_{\mathrm{C}}^{m\times m}$ The subspace of $\mathbf{C}^{m\times m}$ consisting of Hermitian matrices with the topology determined by the Frobenius norm. The dual space can be identified with $\mathcal{S}_{\mathrm{C}}^{m\times m}$ itself. The linear functionals are defined as $\langle X, Z \rangle = \mathrm{tr}(XZ)$, where $X, Z \in \mathcal{S}_{\mathrm{C}}^{m\times m}$.

Some Results from Convex Analysis

We will next state some results and definitions from convex analysis. References for this material are Luenberger (1969) and Rockafellar (1970)

- A translated subspace is called an *affine set* (linear variety). The dimension of an affine set is defined as the dimension of this subspace.

- The *affine hull* of a nonempty set S, denoted aff S, is the unique smallest affine set containing S

- The *relative interior* of a nonempty set S, denoted ri S, is the collection of points in S, which are interior points of S relative to aff S. This means that for every x_0 in the relative interior of S, there exists $\varepsilon > 0$ such that all $x \in \text{aff } S$ satisfying $\|x - x_0\| < \varepsilon$ are also members of S. Hence, the relative interior of S is an open subset of aff S.

- The dimension of a convex set C, denoted dim C, is defined as the dimension of the affine hull of C.

- A *convex cone* C is a convex set with the property that if $x \in C$, then $\alpha x \in C$ for all $\alpha \geq 0$.

The following two theorems will be the main tools in this chapter.

THEOREM 5.1—SEPARATING HYPERPLANE THEOREM
Let C_1 and C_2 be disjoint convex sets in a normed vector space X. Assume further that C_2 is open, then there exists $x^* \in X^*$ such that $\langle x_1, x^* \rangle < \langle x_2, x^* \rangle$ for all $x_1 \in C_1$ and $x_2 \in C_2$.

Proof: This is a minor reformulation of Theorem 3 on page 133 in Luenberger (1969). □

THEOREM 5.2—HELLY
Let $\{C_i | i \in I\}$ be a collection of closed bounded convex sets in $\mathbf{R}^n$. I is a set of indices. If $\cap_{i \in I} C_i = \emptyset$, then there exists a subcollection consisting of $n+1$ or fewer sets $\{C_{\alpha_i} | \alpha_i \in I,\ i = 1, \dots, n+1\}$ such that $\cap_{i=1}^{n+1} C_{\alpha_i} = \emptyset$. If the index set I is finite then the result also holds when the C_i are not necessarily closed or bounded.

Proof: See Rockafellar (1970) or Hörmander (1994). Note that this is not the most general formulation of Helly's theorem. □

5.3 Frequency Dependent Multipliers

We will in this section study the case when the multipliers are defined by a frequency independent constraint. More precisely we consider the convex cone

$$\Pi_\Delta(\gamma) = \left\{\Phi \in \mathbf{RL}_\infty^{2m\times 2m} : \Phi(j\omega) \in \Phi_\Delta(\gamma),\ \forall\omega \in [0,\infty]\right\},$$

where $\Phi_\Delta(\gamma) \subset \mathcal{S}_{\mathrm{C}}^{2m\times 2m}$ is a closed convex cone for all $\gamma \in \mathbf{R}$. Note that the values between different frequencies are independent except for the requirement that Φ should be a rational function. We assume that $\Phi_\Delta(\gamma)$ satisfies the following monotonicity condition.

Monotonicity Assumption on $\Phi_\Delta(\gamma)$: If $\gamma_2 \geq \gamma_1$, then $\forall\Phi_1 \in \Phi_\Delta(\gamma_1)$, there exists $\Phi_2 \in \Phi_\Delta(\gamma_2)$ such that $\Phi_1 \geq \Phi_2$.

THEOREM 5.3

$$\inf_P \gamma = \sup_{\omega\in[0,\infty]} \sup_{D(\omega)} \gamma,$$

where primal and dual constraints are defined as

$$P: \begin{cases} \exists\Phi \in \mathbf{RL}_\infty^{2m\times 2m}, \text{ such that} \\ \Phi(j\omega) \in \Phi_\Delta(\gamma),\ \forall\omega \in [0,\infty], \\ \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Phi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0, \quad \forall\omega \in [0,\infty], \end{cases}$$

and

$$D(\omega): \begin{cases} \exists Z \in \mathcal{S}_{\mathrm{C}}^{m\times m},\ Z \geq 0,\ Z \neq 0, \text{ such that} \\ \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} Z \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \in \Phi_\Delta(\gamma)^\oplus, \end{cases}$$

respectively, and where the positive conjugate cone is defined as

$$\Phi_\Delta(\gamma)^\oplus = \left\{W \in \mathcal{S}_{\mathrm{C}}^{2m\times 2m} : \langle\Phi, W\rangle \geq 0, \quad \forall\Phi \in \Phi_\Delta(\gamma)\right\}$$

Proof: It follows from Lemma 5.1 in the appendix that if $\gamma < \inf_P \gamma$ then there exists $\omega \in [0,\infty]$ such that the convex sets

$$C_1 = \left\{ \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Phi \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} : \Phi \in \Phi_\Delta(\gamma) \right\},$$

$$C_2 = \{X \in \mathcal{S}_C^{m\times m} : X < 0\}$$

are disjoint. By the separating hyperplane theorem there exists a nonzero $Z \in \mathcal{S}_C^{m\times m}$ such that

$$\langle X, Z\rangle \geq 0, \quad \forall X \in C_1, \tag{5.2}$$

$$\langle X, Z\rangle < 0, \quad \forall X \in C_2. \tag{5.3}$$

For (5.3) to hold it is required that $Z \geq 0$. The condition in (5.2) can be reformulated in the following way

$$\left\langle \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Phi \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}, Z \right\rangle \geq 0, \qquad \forall \Phi \in \Phi_\Delta(\gamma)$$

$$\Longleftrightarrow \left\langle \Phi, \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} Z \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \right\rangle \geq 0, \qquad \forall \Phi \in \Phi_\Delta(\gamma)$$

$$\Longleftrightarrow \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} Z \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \in \Phi_\Delta(\gamma)^\oplus$$

This proves that $\inf_P \gamma \leq \sup_{D(\omega)} \gamma$. For the opposite direction we note that the assumption on $\Phi_\Delta(\gamma)$ implies that for every $\gamma > \inf_P \gamma$ there exists $\Phi \in \mathbf{RL}_\infty^{2m\times 2m}$ satisfying the conditions of the primal constraint. This implies that for every $\omega \in [0,\infty]$ we have $C_1 \cap C_2 \neq \emptyset$. Hence, there is no separating hyperplane, and therefore no nonzero $Z \geq 0$ such that the second condition in the dual constraint holds. Thus we have $\sup_{\omega\in[0,\infty]} \sup_{D(\omega)} \gamma \leq \inf_P \gamma$. This proves the Theorem. □

REMARK 5.1
In applications we may fail to find the optimal frequency for the dual optimization problem. Howerer, any $\omega \in [0,\infty]$ such that the dual constraint is satisfied can give a useful lower bound for the primal. □

We will next give a simple example that illustrates the theorem. In particular, the example shows that the frequency $\omega = \infty$ needs to be included in the dual.

EXAMPLE 5.1
Consider the system in Figure 5.1 with

$$G(s) = \frac{2s+1}{s^2+2s+1} - 2$$

and $\Delta = \delta$, where δ is an uncertain real valued parameter, which takes values in $[-\alpha, \alpha]$. We want to find a bound $\alpha_{\max}$ such that the system is stable when $\alpha < \alpha_{\max}$. We can obtain one such bound by considering the primal in Theorem 5.3 with

$$\Phi_\Delta(\gamma) = \left\{ \begin{bmatrix} x & jy \\ -jy & -\gamma x \end{bmatrix} : x \geq 0, y \in \mathbf{R} \right\},$$

and then use $\alpha_{\max} = 1/\sqrt{\gamma_{\text{opt}}}$, where γ_{opt} is the primal objective. The primal optimization problem can be formulated as

$$\inf \gamma \quad \text{subject to}$$
$$\begin{cases} \exists x, y \in \mathbf{RL}_\infty^{1\times 1}, \text{ such that} \\ \sup_{\omega \in [0,\infty]} x(j\omega)[|G(j\omega)|^2 - \gamma] + 2y(j\omega)\text{Im } G(j\omega) < 0, \\ x(j\omega) \geq 0, \quad \forall \omega \in [0,\infty], \\ y(j\omega) \in \mathbf{R}, \quad \forall \omega \in [0,\infty]. \end{cases}$$

It is easy to see that the optimal solution is the maximal value of $|G(j\omega)|^2$, subject to the constraint that Im $G(j\omega) = 0$. From Figure 5.2 we see that $\gamma_{\text{opt}} = \max(|G(0)|^2, |G(j\infty)|^2) = |G(j\infty)|^2 = 4$. We will next see that the dual of Theorem 5.3 gives exactly this solution. It is easy to verify that

$$\Phi_\Delta(\gamma)^\oplus = \{W \in \mathcal{S}_{\mathbf{C}}^{2\times 2} : W_{11} - \gamma W_{22} \geq 0, \quad \text{Im } W_{12} = 0\}.$$

Hence, the dual can be simplified to

$$\sup \gamma \quad \text{subject to}$$
$$\begin{cases} \exists \omega \in [0,\infty], \text{ such that} \\ |G(j\omega)|^2 - \gamma \geq 0, \\ \text{Im } G(j\omega) = 0, \end{cases}$$

which has the solution $\gamma_{\text{opt}} = |G(j\infty)|^2 = 4$. The primal and the dual gives the same bound, $\alpha_{\max} = 2$. This bound is nonconservative and it follows that the primal and the dual optimization problems both corresponds to the Nyquist criterion. □

The next example is also simple but illustrative. It will will be continued in the next section.

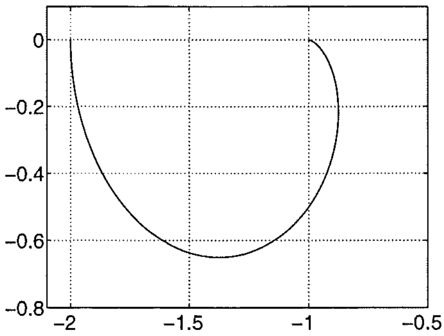

Figure 5.2 Nyquist curve for the system in Example 5.1.

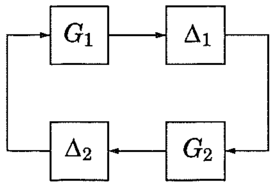

Figure 5.3 System in Example 5.2.

EXAMPLE 5.2
Consider the system in Figure 5.3, where $G_1, G_2 \in \mathbf{RH}_\infty$ and where Δ_1 and Δ_2 are linear time-invariant uncertainties satisfying $\|\Delta_1\|_\infty \leq 1$ and $\|\Delta_2\|_\infty \leq \alpha$. The system can equivalently be described as in Figure 5.1 with $\Delta = \text{diag}(\Delta_1, \Delta_2)$ and

$$G = \begin{bmatrix} 0 & G_1 \\ G_2 & 0 \end{bmatrix} \in \mathbf{RH}_\infty^{2\times 2}.$$

We want to find a bound $\alpha_{\max}$ such that the system is stable if $\alpha < \alpha_{\max}$. Such a bound can be obtained by solving the primal in Theorem 5.3 with

$$\Phi_\Delta(\gamma) = \{\text{diag}(x_1, x_2, -x_1, -\gamma x_2) : x_k \geq 0, k = 1, 2\},$$

and then use $\alpha_{\max} = 1/\sqrt{\gamma_{\text{opt}}}$. It is easy to verify that

$$\Phi_\Delta(\gamma)^\oplus = \left\{W \in \mathcal{S}_{\mathbb{C}}^{4\times 4} : W_{11} - W_{33} \geq 0, W_{22} - \gamma W_{44} \geq 0\right\}.$$

The dual in Theorem 5.3 can be written

$$\sup \gamma \quad \text{subject to}$$
$$\begin{cases} \exists \omega \in [0,\infty],\ z_{11}, z_{22} \geq 0, \quad z_{11} \neq 0 \text{ or } z_{22} \neq 0, \text{ such that} \\ z_{22}|G_1(j\omega)|^2 - z_{11} \geq 0, \\ z_{11}|G_2(j\omega)|^2 - \gamma z_{22} \geq 0, \end{cases}$$

which can be further simplified into

$$\sup_{\omega \in [0,\infty]} |G_1(j\omega)G_2(j\omega)|^2.$$

Hence the dual objective is $\|G_1G_2\|_\infty^2$. This result is of course expected. In the next section we consider the same problem with time-varying parameters δ_1 and δ_2. Then two frequencies will be involved in the dual. □

REMARK 5.2
If we allow Δ_1, Δ_2 to be linear time varying with arbitrarily slow rate of of variation then the primal in Example 5.2 is also a necessary condition for stability, see Poola and Tikku (1995). □

5.4 Constant Multipliers

We will in this section give a similar result as Theorem 5.3, with the multiplier assumed to be a constant matrix from $\Psi_\Delta(\gamma) \subset \mathcal{S}_{\mathbf{R}}^{2m\times 2m}$. In other words, we consider the primal in (5.1) with

$$\Pi_\Delta(\gamma) = \Psi_\Delta(\gamma).$$

We assume that $\Psi_\Delta(\gamma)$ is a closed convex cone for all $\gamma \in \mathbf{R}$, which satisfies the following monotonicity condition.

Monotonicity Assumption on $\Psi_\Delta(\gamma)$: If $\gamma_2 \geq \gamma_1$, then $\forall \Psi_1 \in \Psi_\Delta(\gamma_1)$, there exists $\Psi_2 \in \Psi_\Delta(\gamma_2)$ such that $\Psi_1 \geq \Psi_2$.

THEOREM 5.4
Let $\Omega_N = \{\omega_1, \ldots, \omega_N\}$ denote a grid with $N = \dim\,(\Psi_\Delta(\gamma)) + 1$ frequencies. Then

$$\inf_P \gamma = \sup_{\Omega_N} \sup_{D(\Omega_N)} \gamma,$$

where the primal and dual constraints are defined as

$$P : \begin{cases} \exists \Psi \in \Psi_\Delta(\gamma), \text{ such that} \\ \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Psi \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0, \quad \forall \omega \in [0,\infty], \end{cases}$$

and

$$D(\Omega_N) : \begin{cases} \exists Z_k \in \mathcal{S}_{\mathrm{C}}^{m\times m},\ Z_k \geq 0, \text{ at least one } Z_k \neq 0, \text{ such that} \\ \sum_{\omega_k \in \Omega_N} \mathrm{Re} \begin{bmatrix} G(j\omega_k) \\ I \end{bmatrix} Z_k \begin{bmatrix} G(j\omega_k) \\ I \end{bmatrix}^* \in \Psi_\Delta(\gamma)^\oplus, \end{cases}$$

respectively, and where the positive conjugate cone is defined as

$$\Psi_\Delta(\gamma)^\oplus = \left\{ W \in \mathcal{S}_{\mathrm{R}}^{2m\times 2m} : \langle \Psi, W \rangle \geq 0, \quad \forall \Psi \in \Psi_\Delta(\gamma) \right\}.$$

Furthermore, if $0 \notin \mathrm{ri}\ \Psi_\Delta(\gamma)$, then $N = \dim\ (\Psi_\Delta(\gamma))$ is sufficient.

Proof: This theorem is a special case of Theorem 5.5 in the next section. □

EXAMPLE 5.3

Consider again the system in Figure 5.3 but now with time-varying parameters $\Delta_1 = \delta_1(t)$ and $\Delta_2 = \delta_2(t)$ satisfying $|\delta_1(t)| \leq 1$, $\forall t \geq 0$ and $|\delta_2(t)| \leq \alpha$, $\forall t \geq 0$. Again, we search for the largest $\alpha_{\max}$ such that the system is stable if $\alpha < \alpha_{\max}$. By solving the primal in Theorem 5.4 with

$$\Psi_\Delta(\gamma) = \{\mathrm{diag}(x_1, x_2, -x_1, -\gamma x_2) : x_k \geq 0, k = 1,2\},$$

we get the bound $\alpha_{\max} = 1/\sqrt{\gamma_{\mathrm{opt}}}$. It is easy to verify that

$$\Psi_\Delta(\gamma)^\oplus = \left\{ W \in \mathcal{S}_{\mathrm{R}}^{4\times 4} : W_{11} - W_{33} \geq 0, W_{22} - \gamma W_{44} \geq 0 \right\}.$$

Since $N = \dim\ (\Psi_\Delta(\gamma)) = 2$ and $0 \notin \mathrm{ri}\ \Psi_\Delta(\gamma)$ it is sufficient with two frequencies in the dual. The dual can be formulated as

$$\sup \gamma \quad \text{subject to}$$
$$\begin{cases} \exists \omega_1, \omega_2 \in [0,\infty],\ z_{11_1}, z_{22_1}, z_{11_2}, z_{22_2} \geq 0, \text{ not all zero, s.t.} \\ z_{22_1}|G_1(j\omega_1)|^2 - z_{11_1} + z_{22_2}|G_1(j\omega_2)|^2 - z_{11_2} \geq 0, \\ z_{11_1}|G_2(j\omega_1)|^2 - \gamma z_{22_1} + z_{11_2}|G_2(j\omega_2)|^2 - \gamma z_{22_2} \geq 0. \end{cases} \tag{5.4}$$

It is no restriction to assume that $|G_2(j\omega_2)| \geq |G_2(j\omega_1)$ and $|G_2(j\omega_2)| > 0$. Then the constraint in (5.4) can formulated as

$$\gamma \leq \frac{|G_2(j\omega_2)|^2}{z_{22_1} + z_{22_2}} \left[z_{22_1}|G_1(j\omega_1)|^2 + z_{22_2}|G_1(j\omega_2)|^2 - z_{11_1}\left(1 - \frac{|G_2(j\omega_1)|^2}{|G_2(j\omega_2)|^2}\right) \right].$$

This implies that $\gamma \leq \|G_1\|_\infty^2 \|G_2\|_\infty^2$. Furthermore, we obtain equality by choosing $z_{11_1} = z_{22_2} = 0$. Hence we have shown that the dual objective is $\gamma_{\text{opt}} = \|G_1\|_\infty^2 \|G_2\|_\infty^2$, and furthermore that the dual optimization problem involves two frequencies unless the norms $\|G_1\|_\infty$ and $\|G_2\|_\infty$ are obtained at the same frequency. □

REMARK 5.3
If we in this example consider time-varying operators δ_1 and δ_2 with arbitrary time-variation, which satisfy $\|\delta_1\| \leq 1$ and $\|\delta_2\| \leq \alpha$, then the primal in Example 5.3 is also a necessary condition for stability, see Shamma (1994), Megretski (1993a) and Megretski and Treil (1993). □

5.5 Mixed Multipliers

We will in this section derive the dual of robustness problems involving both frequency dependent multipliers and constant multipliers. More precisely, the multipliers involved are from the set

$$\Pi_\Delta(\gamma) = \left\{ \Phi + \Psi \in \mathbf{RL}_\infty^{2m\times 2m} : \Phi(j\omega) \in \Phi_\Delta(\gamma), \forall\omega \in [0,\infty],\ \Psi \in \Psi_\Delta(\gamma) \right\},$$

where $\Phi_\Delta(\gamma) \subset \mathcal{S}_{\mathbf{C}}^{2m\times 2m}$ and $\Psi_\Delta(\gamma) \in \mathcal{S}_{\mathbf{R}}^{2m\times 2m}$ are closed convex cones for all $\gamma \in \mathbf{R}$, which satisfy the following assumptions.

Monotonicity Assumptions on $\Phi_\Delta(\gamma)$ and $\Psi_\Delta(\gamma)$:

1. If $\gamma_2 \geq \gamma_1$, then $\forall\Phi_1 \in \Phi_\Delta(\gamma_1)$, there exists $\Phi_2 \in \Phi_\Delta(\gamma_2)$ such that $\Phi_1 \geq \Phi_2$.

2. If $\gamma_2 \geq \gamma_1$, then $\forall\Psi_1 \in \Psi_\Delta(\gamma_1)$, there exists $\Psi_2 \in \Psi_\Delta(\gamma_2)$ such that $\Psi_1 \geq \Psi_2$.

Next follows the main result in this chapter.

THEOREM 5.5
Let $\Omega_N = \{\omega_1, \dots, \omega_N\}$ denote a grid with $N = \dim(\Psi_\Delta(\gamma)) + 1$ frequencies. Then

$$\inf_P \gamma = \sup_{\Omega_N} \sup_{D(\Omega_N)} \gamma,$$

where the primal and dual constraints are defined as

$$P: \begin{cases} \exists \Phi \in \mathbf{RL}_\infty^{2m\times 2m},\ \Psi \in \Psi_\Delta(\gamma), \text{ such that} \\ \Phi(j\omega) \in \Phi_\Delta(\gamma),\ \forall \omega \in [0,\infty], \\ \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* (\Phi(j\omega) + \Psi) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0, \quad \forall \omega \in [0,\infty], \end{cases}$$

and

$$D(\Omega_N): \begin{cases} \exists Z_k \in \mathcal{S}_{\mathrm{C}}^{m\times m},\ Z_k \geq 0, \text{ at least one } Z_k \neq 0, \text{ such that} \\ \begin{bmatrix} G(j\omega_k) \\ I \end{bmatrix} Z_k \begin{bmatrix} G(j\omega_k) \\ I \end{bmatrix}^* \in \Phi_\Delta(\gamma)^\oplus, \quad \forall \omega_k \in \Omega_N, \\ \sum_{\omega_k \in \Omega_N} \mathrm{Re} \begin{bmatrix} G(j\omega_k) \\ I \end{bmatrix} Z_k \begin{bmatrix} G(j\omega_k) \\ I \end{bmatrix}^* \in \Psi_\Delta(\gamma)^\oplus, \end{cases}$$

respectively, and where the positive conjugate cones are defined as

$$\Phi_\Delta(\gamma)^\oplus = \left\{ W \in \mathcal{S}_{\mathrm{C}}^{2m\times 2m} : \langle \Phi, W \rangle \geq 0, \quad \forall \Phi \in \Phi_\Delta(\gamma) \right\},$$
$$\Psi_\Delta(\gamma)^\oplus = \left\{ W \in \mathcal{S}_{\mathrm{R}}^{2m\times 2m} : \langle \Psi, W \rangle \geq 0, \quad \forall \Psi \in \Psi_\Delta(\gamma) \right\}.$$

Furthermore, if $0 \notin \mathrm{ri}\, \Psi_\Delta(\gamma)$, then $N = \dim(\Psi_\Delta(\gamma))$ is sufficient.

Proof: The case when $\dim(\Psi_\Delta(\gamma)) = 0$ is treated in Theorem 5.3 so we may assume that $\dim(\Psi_\Delta(\gamma)) > 0$. The proof is based on the idea in Poola and Tikku (1995) and Paganini (1995) to use Helly's Theorem to reduce an infinite number of convex constraints involving the constant multiplier Ψ to a finite number of such constraints. For given $\omega \in [0,\infty]$ we define

$$C_\omega(\gamma) = \{\Psi \in \Psi_\Delta(\gamma) : \exists \Phi \in \Phi_\Delta(\gamma), \text{ s.t. } M_G(\omega)(\Phi + \Psi) < 0\},$$

where $M_G(\omega) : \mathcal{S}_{\mathrm{C}}^{2m\times 2m} \to \mathcal{S}_{\mathrm{C}}^{m\times m}$ denotes the linear operators defined by

$$M_G(\omega)\Phi = \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Phi \begin{bmatrix} G(j\omega) \\ I \end{bmatrix},$$

for any $\Phi \in \mathcal{S}_{\mathrm{C}}^{2m\times 2m}$. We note that $C_\omega(\gamma)$ is a convex set. The proof is based on the following two statements, which are proved in Appendix 5.8.

Statement (*i*): If $\gamma < \inf_P \gamma$, then $\bigcap_{\omega\in[0,\infty]} C_\omega(\gamma) = \varnothing$. Furthermore

$$\inf_P \gamma = \sup_{\bigcap_{\omega\in[0,\infty]} C_\omega(\gamma)=\varnothing} \gamma.$$

Statement (*ii*) The condition $\bigcap_{\omega\in[0,\infty]} C_\omega(\gamma) = \varnothing$ holds if and only if there exists at most $N = \dim(\Psi_\Delta(\gamma)) + 1$ frequencies $\omega_1, \dots, \omega_N \in [0,\infty]$ such that $\cap_{k=1}^N C_{\omega_k}(\gamma) = \varnothing$. Furthermore, if $0 \notin \operatorname{ri} \Psi_\Delta(\gamma)$, then $N \le \dim(\Psi_\Delta(\gamma))$.

From statements (*i*) and (*ii*) above, it follows that $\gamma < \inf_P \gamma$ if and only if there exist at most N frequencies $\omega_1, \dots, \omega_N$ such that the convex sets (here we let $M_G^{\omega_k} = M_G(\omega_k)$)

$$\begin{aligned} C_1 &= \{(M_G^{\omega_1}(\Phi_1 + \Psi), \dots, M_G^{\omega_N}(\Phi_N + \Psi)) : \Phi_k \in \Phi_\Delta(\gamma),\ \Psi \in \Psi_\Delta(\gamma)\}, \\ C_2 &= \{X \in \mathcal{S}_{\mathbb{C}}^{m\times m} : X < 0\}^N \end{aligned}$$

are disjoint. By the separating hyperplane theorem this is equivalent to the existence of a nonzero N-tuple $(Z_1, \dots, Z_N) \in (\mathcal{S}_{\mathbb{C}}^{m\times m})^N$, such that

$$\sum_{k=1}^N \langle X_k, Z_k\rangle \ge 0, \quad \forall (X_1, \dots, X_N) \in C_1, \tag{5.5}$$

$$\sum_{k=1}^N \langle X_k, Z_k\rangle < 0, \quad \forall (X_1, \dots, X_N) \in C_2. \tag{5.6}$$

It is clear that for (5.6) to hold we need $Z_k \ge 0$, for $k = 1, \dots, N$.

The condition in (5.5) gives

$$\begin{aligned} &\sum_{k=1}^N \langle M_G(\omega_k)\Phi_k, Z_k\rangle + \sum_{k=1}^N \langle M_G(\omega_k)\Psi, Z_k\rangle = \\ &\sum_{k=1}^N \langle \Phi_k, M_G(\omega_k)^\times Z_k\rangle + \left\langle \Psi, \sum_{k=1}^N \operatorname{Re} M_G(\omega_k)^\times Z_k \right\rangle \ge 0 \end{aligned}$$

for all $\Phi_k \in \Phi_\Delta(\gamma)$ and for all $\Psi \in \Psi_\Delta(\gamma)$. Hence,

$$\begin{cases} \begin{bmatrix} G(j\omega_k) \\ I \end{bmatrix} Z_k \begin{bmatrix} G(j\omega_k) \\ I \end{bmatrix}^* \in \Phi_\Delta(\gamma)^\oplus, \quad k = 1, \dots, N, \\ \displaystyle\sum_{k=1}^N \operatorname{Re} \begin{bmatrix} G(j\omega_k) \\ I \end{bmatrix} Z_k \begin{bmatrix} G(j\omega_k) \\ I \end{bmatrix}^* \in \Psi_\Delta(\gamma)^\oplus \end{cases}$$

and the theorem follows. □

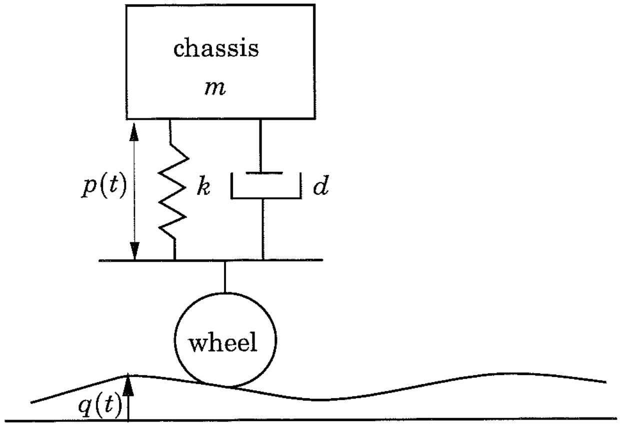

Figure 5.4 Process model of car suspension.

REMARK 5.4
It is in general impossible to find the optimal frequency grid Ω_N for the dual. However, it follows from the proof that any grid Ω_N that allows the dual constraint to be feasible gives a useful lower bound. In applications we use Theorem 5.5 in the following way. Restrict the primal constraint to a finite dimensional subspace and choose a frequency grid Ω_N for the dual. We denote the corresponding primal constraint D_{FD}. If the the duality gap $\inf_{P_{FD}} \gamma - \sup_{D(\Omega_N)} \gamma$ is small then we know that we are close to the optimal solution. Otherwise improve the subspace for the primal and the frequency grid for the dual.

The dual constraint, $D(\Omega_N)$, may contain algebraic conditions that are hard to treat numerically. In this case we have to use approximate solutions of the feasibility problem $D(\Omega_N)$. We refer to Section 4.5 for a discussion. □

5.6 A Numerical Example

In this example we investigate the performance of the suspension of a simple car model. We will follow the approach in Hansson (1995) to obtain a simple model of the system. Figure 5.4 shows one fourth of a car with one wheel and the car suspension equipment consisting of a nonlinear spring with nonlinear spring constant $k(\cdot)$ and damping ratio d. Below follows a list of notation for this example

Notation	Explanation
$p(t)$	spring length
$q(t)$	road profile
p_0	unsprung length of spring
m	mass of car body
g	9.81m/s^2
k	nonlinear spring constant
d	damping ratio of the spring

The differential equation describing the length of the spring $p(t)$ due to the road profile $q(t)$ is

$$m\left(\ddot{p}(t)+\ddot{q}(t)\right) = -k(p(t)-p_0) - d\dot{p}(t) - mg.$$

It is valid as long as the car has contact with the road. In order to obtain a state space equation we use the states $x_1(t) = p(t) - p^0$, and $x_2(t) = \dot{x}_1(t)$, where p^0 is the stationary value of $p(t)$ when $q(t) \equiv 0$. We assume that $k(p(t)-p_0) + mg = k(x_1(t) + p^0 - p_0) + mg = k_l(x_1(t) + \varphi(x_1(t)))$, where φ is a nonlinear function satisfying $k_{\min}x^2 \leq \varphi(x)x \leq k_{\max}x^2$, $\forall x \in \mathbf{R}$.

The mass of the car is uncertain due to varying load. It is assumed that $m \in [\underline{m}, \overline{m}]$. We let the nominal mass be m_0 and we define m by the relation

$$\frac{1}{m} = \frac{1}{m_0} + a\delta,$$

where $m_0 = 2\underline{m}\overline{m}/(\underline{m}+\overline{m})$, $\delta \in [-1, 1]$, and $a = (\underline{m}-\overline{m})/(2\underline{m}\overline{m})$.

With the state $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ and output z as the normal force acting on the compartment, we get the following model for the system

$$\begin{aligned}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} x_2 \\ \frac{1}{m}(-k_l(x_1 + \varphi(x_1)) - dx_2) \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} w, \\ z(t) &= k_l(x_1 + \varphi(x_1)) + dx_2,\end{aligned} \tag{5.7}$$

where $w = \ddot{q}$ is regarded as a disturbance to the system.

The system in (5.7) is equivalent to the system in Figure 5.5 where the nonlinearity φ and the uncertainty δ are collected in a perturbation matrix. This system is equivalent to the system in Figure 5.1 with

$$\begin{aligned} G &= \text{starp}(Q, H), \\ \Delta &= \begin{bmatrix} \varphi & 0 \\ 0 & \delta \end{bmatrix}, \end{aligned}$$

where starp$(\cdot,\cdot)$, denotes the Redheffer star product defined as in Redheffer (1959) and Packard and Doyle (1993)

$$\text{starp}(Q,H) = \begin{bmatrix} F_l(Q,H_{11}) & Q_{12}(I-H_{11}Q_{22})^{-1}H_{12} \\ H_{21}(I-Q_{22}H_{11})^{-1}Q_{21} & F_u(H,Q_{22}) \end{bmatrix},$$

where

$$F_l(Q,H_{11}) = Q_{11} + Q_{12}H_{11}(I-Q_{22}H_{11})^{-1}Q_{21},$$

and

$$F_u(H,Q_{22}) = H_{22} + H_{21}Q_{22}(I-H_{11}Q_{22})^{-1}H_{12}.$$

The matrices H and Q are defined as

$$H = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ -k_l & 0 & -k_l & -d & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ k_l & 0 & k_l & d & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 0_{2\times 2} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix},$$

$$Q_{12} = [\, I_2 \quad 0_{2\times 2} \,], \quad Q_{21} = \begin{bmatrix} 1 & 0 \\ 0 & a \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad Q_{22} = \text{diag}(0, 1/m_0, s^{-1}I_2).$$

We are interested in studying worst case $\mathbf{L}_2$-performance of the system above subject to the nonlinearity φ and the mass uncertainty δ. We assume the following normalized parameter values, $k_l = 1$, $d = 0.2$, $k_{\min} = -0.1$, $k_{\max} = 0.1$ and $m \in [0.8, 1.2]$. In this case $G \in \mathbf{RH}_\infty^{3\times 3}$. An upper bound for the induced $\mathbf{L}_2$-norm of the system is given as $\gamma_{\text{opt}}^{1/2}$, where γ_{opt} is the solution to the following optimization problem

$$\inf \gamma \quad \text{subject to}$$

$$P: \begin{cases} \exists \Psi \in \Psi_\Delta(\gamma), \Phi \in \mathbf{RL}_\infty^{2m\times 2m}, \text{ such that} \\ \Phi(j\omega) \in \Phi_\Delta(\gamma), \quad \forall \omega \in [0,\infty], \\ \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* (\Phi(j\omega) + \Psi) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0, \quad \forall \omega \in [0,\infty], \end{cases}$$

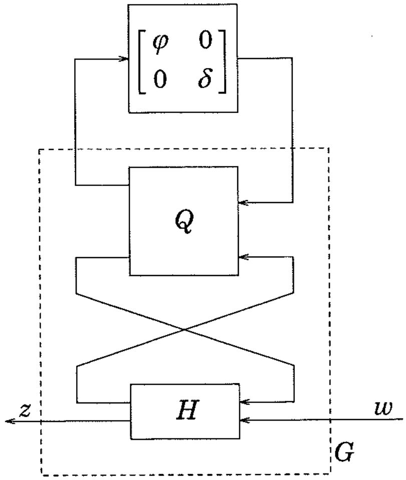

Figure 5.5 Transformed system.

where

$$\Phi_\Delta(\gamma) = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x & 0 & 0 & jy & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -jy & 0 & 0 & -x & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} : \quad x, y \in \mathbf{R}, x \geq 0 \right\},$$

and

$$\Psi_\Delta(\gamma) = \left\{ \begin{bmatrix} 0.1^2 x_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\gamma x_2 \end{bmatrix} : \quad x_1, x_2 \geq 0 \right\}.$$

The performance multiplier has been included in $\Psi_\Delta(\gamma)$.

An approximate solution to the primal can be obtained as suggested in Chapter 3. The idea is to restrict the search of the frequency dependent multiplier to a finite-dimensional subspace. Let $x(j\omega) = R^*(j\omega)UR(j\omega)$

$R_0(s)$	$S_0(s)$	γ_{opt}
1	1	UNFEASIBLE
1	Ritz(0.2, 1)	14.64
1	Ritz(5, 1)	14.64

Table 5.1 Numerical results for the primal optimization problem in the car suspension example.

and $y(j\omega) = VS(j\omega) - S^*(j\omega)V^T$, where $R \in \mathbf{RH}_\infty^{N\times1}$, $S \in \mathbf{H}_\infty^{M\times1}$ are *basis multipliers* and where $U \in \mathbf{R}^{N\times N}$, satisfying $U = U^T \geq 0$ and $V \in \mathbf{R}^{1\times M}$ are the corresponding *coordinates*. The resulting optimization problem can be transformed into an LMI optimization problem in U and V. We obtained the solution in Table 5.1 using LMI-lab, Gahinet *et al.* (1995), where

$$\text{Ritz}(p,n) = \begin{bmatrix} 1 & \frac{s-p}{s+p} & \cdots & \frac{(s-p)^n}{(s+p)^n} \end{bmatrix}^T.$$

Higher order basis functions did not give smaller primal objective value. Does this mean that the last two basis functions in Table 5.1 are close to optimal? We will use the dual in Theorem 5.5 to investigate this question.

In order to solve the dual for the car suspension example we need to determine the cones $\Phi_\Delta(\gamma)^\oplus$ and $\Psi_\Delta(\gamma)^\oplus$. It is simple to verify that

$$\begin{aligned} \Phi_\Delta(\gamma)^\oplus &= \{W \in \mathcal{S}_{\mathrm{C}}^{6\times6} : \quad W_{22} - W_{55} \geq 0, \quad \operatorname{Im} W_{25} = 0\}, \\ \Psi_\Delta(\gamma)^\oplus &= \{W \in \mathcal{S}_{\mathrm{R}}^{6\times6} : \quad 0.1^2 W_{11} - W_{44} \geq 0, \quad W_{33} - \gamma W_{66} \geq 0\}. \end{aligned}$$

We have dim $(\Psi_\Delta(\gamma)) = 2$ and $0 \neq\in \text{ri } \Psi_\Delta(\gamma)$. This means that we need a frequency grid with two frequencies, i.e., $\Omega = \{\omega_1, \omega_2\}$. In order to state the dual we define some transfer functions and matrices

$$\begin{aligned} H_1 &= [\,G_{11} \quad G_{12} \quad G_{13}\,], \\ H_2 &= [\,G_{21} \quad G_{22} \quad G_{23}\,], \\ H_3 &= [\,G_{31} \quad G_{32} \quad G_{33}\,], \\ H_4 &= [1 \quad 0 \quad 0], \\ H_5 &= [0 \quad 1 \quad 0], \\ H_6 &= [0 \quad 0 \quad 1]. \end{aligned}$$

For any choice of grid with two frequencies we get the following, possible suboptimal, dual optimization problem.

$$\sup \gamma \quad \text{subject to}$$

$$D(\Omega): \begin{cases} \exists Z_1, Z_2 \in \mathcal{S}_{\mathrm{C}}^{3\times 3}, \text{ such that} \\ Z_1, Z_2 \geq 0, \text{ at least one nonzero}, \\ H_2(j\omega_k)Z_k H_2^*(j\omega_k) - H_5 Z_k H_5^T \geq 0, \quad k = 1,2, \\ \mathrm{Im}\ (H_2(j\omega_k)Z_k H_5^T) = 0, \quad k = 1,2, \\ \sum_{k=1}^{2} \mathrm{Re}(0.1^2 H_1(j\omega_k)Z_k H_1^*(j\omega_k) - H_4 Z_k H_4^T) \geq 0, \\ \sum_{k=1}^{2} \mathrm{Re}(H_3(j\omega_k)Z_k H_3^*(j\omega_k) - \gamma H_6 Z_k H_6^T) \geq 0, \end{cases}.$$

This optimization problem can be transformed into an optimization problem involving real matrices and 2 algebraic conditions, which corresponds to the constraints $\mathrm{Im}\ (H_2(j\omega_k)Z_k H_5^T) = 0$, $k = 1,2$.

With only the frequency $\omega_1 = 0.91$ in the grid, we get $\gamma_{\mathrm{opt}} = 14.60$ and

$$Z_1 \approx \begin{bmatrix} 0.1491 & -0.1230 & 0.1020 \\ -0.1230 & 14.9706 & -0.1650 \\ 0.1020 & -0.1650 & 0.0704 \end{bmatrix} + i \begin{bmatrix} 0 & -1.4879 & 0.0081 \\ 1.4879 & 0 & 1.0122 \\ -0.0081 & -1.0122 & 0 \end{bmatrix}.$$

It is easy to verify that Z_1 satisfies the conditions for the dual and we see that the dual objective $\gamma_{\mathrm{opt}} = 14.60$ is very close to the value of the primal objective $\gamma_{\mathrm{opt}} = 14.64$. This shows that the chosen basis are indeed close to optimal.

5.7 Concluding Remarks

We have derived duality results for obtaining bounds in robustness analysis with mixed multipliers. Several examples have shown its applicability for simple cases.

It is general impossible to find the the optimal solution to the dual. However, suboptimal solutions can be obtained by chosing a frequency grid $\Omega = \{\omega_1, \ldots, \omega_N\}$. Then solve the optimization problem $\sup_{D(\Omega_N)} \gamma$ in Theorem 5.5. This problem can be solved by bisection on γ. The constraint in the optimization problem corresponds to a finite dimensional convex feasibility problem. For suitable choices of frequency grid this gives a useful lower bound on primal optimization problem. This bound can then be used to investigate the quality of the suboptimal primal.

The suboptimal dual is in general less expensive in terms of computations than the corresponding suboptimal primal. Hence, in many cases little effort is required to obtain useful duality bounds.

5.8 Appendix: Proof of Statement (*i*) and (*ii*).

This appendix contains proofs for statement (*i*) and (*ii*) in the proof of Theorem 5.5.

Proof of statement (i) The next lemma is the main tool for the proof of statement (*i*). The lemma says that it is nonrestrictive to constrain Φ in the primal constraint to be rational transfer function. In other words, we would not gain anything from using an arbitrary function on the extended imaginary axis with values in $\Phi_\Delta(\gamma)$.

LEMMA 5.1
Let $G \in \mathbf{RH}_\infty^{m\times m}$, $\Psi \in \mathcal{S}_{\mathbb{R}}^{2m\times 2m}$, and $\gamma \in \mathbf{R}$. Then the following two statements are equivalent

(*i*) For every $\omega \in [0,\infty]$, there exists $\Phi \in \Phi_\Delta(\gamma)$ such that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* (\Phi + \Psi) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0.$$

(*ii*) There exists $\Phi \in \mathbf{RL}_\infty^{2m\times 2m}$ such that $\Phi(j\omega) \in \Phi_\Delta(\gamma)$ for all $\omega \in [0,\infty]$ and

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* (\Phi(j\omega) + \Psi) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0, \quad \forall \omega \in [0,\infty]. \tag{5.8}$$

Proof: The implication $(ii) \to (i)$ is obvious. For the other direction we use the assumption in (*i*) and the continuity of G to construct a $\Phi \in \mathbf{RL}_\infty^{2m\times 2m}$ that satisfies the conditions in (*ii*). To do this we first transform condition (5.8) to the unit circle. For this we use the Möbius transform $\psi(z) = (z-1)/(z+1)$ and the notation $\widehat{G}(z) = G(\psi(z))$ and $\widehat{\Phi}(z) = \Phi(\psi(z))$. The condition in (5.8) becomes

$$\begin{bmatrix} \widehat{G}(e^{j\omega}) \\ I \end{bmatrix}^* (\widehat{\Phi}(e^{j\omega}) + \Psi) \begin{bmatrix} \widehat{G}(e^{j\omega}) \\ I \end{bmatrix} < 0, \quad \forall \omega \in [0,\pi]. \tag{5.9}$$

By (i) and the continuity of $\widehat{G}$ on the unit circle it follows that every $\nu \in [0,\pi]$ is contained in an open (as a subset of $[0,\pi]$) interval I_ν such that

$$\begin{bmatrix} \widehat{G}(e^{j\omega}) \\ I \end{bmatrix}^* (\Phi_\nu + \Psi) \begin{bmatrix} \widehat{G}(e^{j\omega}) \\ I \end{bmatrix} < 0, \quad \forall \omega \in I_\nu,$$

for some $\Phi_\nu \in \Phi_\Delta(\gamma)$. Compactness of $[0,\pi]$ implies that there exists a finite number of such intervals $I_{\nu_1},\dots,I_{\nu_N}$ that cover $[0,\pi]$. It is no restriction to assume the following

1. Φ_{ν_k} is in the relative interior of $\Phi_\Delta(\gamma)$, for $k = 1,\dots,N$.
2. $\nu_1 = 0$ and $\nu_N = \pi$. Since, $\widehat{G}(e^{j\nu_1}), \widehat{G}(e^{j\nu_N}) \in \mathbf{R}^{m\times m}$ it then follows that it is no restriction to use $\Phi_{\nu_1}, \Phi_{\nu_N} \in \mathcal{S}_{\mathrm{R}}^{2m\times 2m}$.
3. $I_{\nu_k} \cap I_{\nu_{k+2}} = \emptyset$, for $k = 1,\dots,N-2$.

Let us now define the intervals I_k and J_k as

$$\begin{aligned} I_k &= I_{\nu_k} \backslash (I_{\nu_{k+1}} \cup I_{\nu_{k-1}}), \\ J_k &= (\alpha_k, \beta_k) = I_{\nu_k} \cap I_{\nu_{k+1}}. \end{aligned}$$

It is then clear that the continuous function $\widehat{\Phi}_c$ defined as

$$\widehat{\Phi}_c(e^{j\omega}) = \begin{cases} \Phi_{\nu_k}, & \omega \in I_k, \\ \alpha\Phi_{\nu_k} + (1-\alpha)\Phi_{\nu_{k+1}} & \omega = \alpha\alpha_k + (1-\alpha)\beta_k \in J_k \\ \overline{\widehat{\Phi}_c(e^{-j\omega})}, & \omega \in [-\pi, 0) \end{cases}$$

is in the relative interior of $\Phi_\Delta(\gamma)$ for all $\omega \in [0,\pi]$ and it satisfies (5.9).

Next, let us define

$$\widehat{\Phi}_m(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \widehat{\Phi}_c(e^{j\zeta}) K_m(e^{j(\omega-\zeta)}) d\zeta, \tag{5.10}$$

where the Fejér kernel K_m is defined as

$$K_m(z) = \sum_{n=-m}^{m} \frac{m+1-|n|}{m+1} z^n.$$

It is easy to verify that $\widehat{\Phi}_m(e^{-j\omega}) = \overline{\widehat{\Phi}_m(e^{j\omega})}$ and $\widehat{\Phi}_m(e^{j\omega}) = \widehat{\Phi}_m(e^{j\omega})^*$. We note that $\widehat{\Phi}_m$ corresponds to the rational function

$$\widehat{\Phi}_m(z) = \sum_{n=-m}^{m} \widetilde{\Phi}_n z^n,$$

where

$$\widetilde{\Phi}_n = \frac{m+1-|n|}{2\pi(m+1)} \int_{-\pi}^{\pi} \widehat{\Phi}_c(e^{j\zeta})e^{-jn\zeta}d\zeta \in \mathbf{R}^{2m\times 2m}.$$

From Fejèrs Theorem it follows that $\widehat{\Phi}_m \to \widehat{\Phi}_c$ uniformly on the unit circle as $m \to \infty$, see, for example, Young (1988). Hence, the function $\Phi(s) = \widehat{\Phi}_m(\psi^{-1}(s)) \in \mathbf{RL}_\infty^{2m\times 2m}$ satisfies the conditions in (ii) if m is taken sufficiently large. □

Let $\gamma_{\text{opt}} = \inf_P \gamma$. It is clear from Lemma 5.1 that $\gamma < \gamma_{\text{opt}}$ implies that $\cap_{\omega\in[0,\infty]} C_\omega(\gamma) = \emptyset$. Furthermore, it follows from the assumptions on $\Phi_\Delta(\gamma)$ and $\Psi_\Delta(\gamma)$ that if $\gamma > \gamma_{\text{opt}}$ then $\cap C_\omega \neq \emptyset$. This means that the condition $\cap C_\omega = \emptyset$ implies that $\gamma \leq \gamma_{\text{opt}}$, and statement (i) is proved.

Proof of statement (ii) We first reduce the number of frequencies to $N = \dim(\Psi_\Delta(\gamma)) + 1$. This part is proved in the same way as a similar result in Poola and Tikku (1995). For any $\varepsilon > 0$ and $\rho > 0$, define the compact convex set $\widetilde{C}_\omega(\gamma, \varepsilon, \rho)$ as

$$\widetilde{C}_\omega(\gamma,\varepsilon,\rho) = \{\Psi \in \Psi_\Delta(\gamma) : M_G(\omega)(\Phi + \Psi) \leq -\varepsilon I, \\ \Phi \in \Phi_\Delta(\gamma),\ |\Phi| \leq \rho,\ |\Psi| \leq \rho\}.$$

Next assume that $\cap C_\omega(\gamma) = \emptyset$. Since $\widetilde{C}_\omega(\gamma,\varepsilon,\rho) \subset C_\omega(\gamma)$ it follows that $\cap\widetilde{C}_\omega(\gamma,\varepsilon,\rho) = \emptyset$. By Helly's theorem there exists N or fewer distinct frequencies $\omega_1, \ldots, \omega_N \in [0,\infty]$ such that

$$\cap_{k=1}^N \widetilde{C}_{\omega_k}(\gamma,\varepsilon,\rho) = \emptyset \qquad (5.11)$$

Take sequences $\{\varepsilon_i\}$, $\{\rho_i\}$ with $\varepsilon_i \to 0$, and $\rho_i \to \infty$ and let, for each i, ω_k^i be the corresponding frequency such that (5.11) holds. By the sequential compactness of $[0,\infty]$ we can after going to subsequences assume that $\varepsilon_i \to 0$, $\rho_i \to \infty$ and $\omega_k^i \to \omega_k^0$ for some $\omega_k^0 \in [0,\infty]$. This follows since there is either an unbounded subsequence $\omega_k^i \to \infty$ or it stays in a compact interval and convergence follows.

Assume that $\cap_{k=1}^N C_{\omega_k^0}(\gamma)$ is nonempty, i.e., assume that there exists $\Psi_0 \in \cap_{k=1}^N C_{\omega_k^0}(\gamma)$. By continuity of $G \in \mathbf{RH}_\infty^{m\times m}$, there exists $\varepsilon_0 > 0$, $\rho_0 < \infty$ such that for $k = 1, \ldots, N$, $\Psi_0 \in \widetilde{C}_\omega(\gamma, \varepsilon_0, \rho_0)$ for all ω in an interval I_k around ω_k^0. We can assume that the intervals are of either of the forms $[\omega_k^0 - \delta_0, \omega_k^0 + \delta_0]$, $[0, \delta_0]$, or $[1/\delta_0, \infty]$, for some $\delta_0 > 0$.

Next choose index i such that $\varepsilon_i \leq \varepsilon_0$, $\rho_i \geq \rho_0$, and $\omega_k^i \in I_k$, then $\Psi_0 \in \cap_{k=1}^N \widetilde{C}_{\omega_k^i}(\gamma, \varepsilon_i, \rho_i) = \emptyset$, which is a contradiction. Hence, it follows that $\cap_{k=1}^{N} C_{\omega_k^0}(\gamma) = \emptyset$. The reverse implication is trivial.

Next assume that $0 \notin \text{ri } \Psi_\Delta(\gamma)$. We will show that there exists $N+1 = \dim(\Psi_\Delta(\gamma)) + 1$ frequencies ω_k such that $\cap_{k=1}^{N+1} C_{\omega_k}(\gamma) = \emptyset$, if and only if there exists a subcollection consisting of N frequencies ω_{k_j} such that $\cap_{j=1}^{N} C_{\omega_{k_j}}(\gamma) = \emptyset$. We can assume that $C_{\omega_k}(\gamma) \neq \emptyset$, for $k = 1, \ldots, N+1$, since otherwise the result is obvious. The idea is to show that we can fix one element of the matrices $C_{\omega_k}(\gamma)$ to either 1 or -1. This reduces the dimension by one and the result follows from Helly's Theorem.

We will use the following property:

(a) if $\Psi \in C_{\omega_k}(\gamma)$, then $\alpha\Psi \in C_{\omega_k}(\gamma)$, for all $\alpha > 0$.

It follows from the convexity of $C_{\omega_k}(\gamma)$ that if $\Psi \in C_{\omega_k}(\gamma)$, for all k, then there is also $\Psi_0 \in \text{ri } \Psi_\Delta(\gamma)$ such that $\Psi_0 \in C_{\omega_k}(\gamma)$, for all k. Hence, since $0 \notin \text{ri } \Psi_\Delta(\gamma)$ and by property (a) it follows that we can fix one element in all $C_{\omega_k}(\gamma)$ to either 1 or -1. Denote the corresponding sets $C_{\omega_k}^0(\gamma)$. We have $\cap_{k=1}^{N+1} C_{\omega_k}^0(\gamma) = \emptyset$ and it follows from Helly's theorem that there exists a subcollection consisting of N frequencies ω_{k_j} such that $\cap_{j=1}^{N} C_{\omega_{k_j}}^0(\gamma) = \emptyset$. It is clear that also $\cap_{j=1}^{N} C_{\omega_{k_j}}(\gamma) = \emptyset$.

6

Concluding Remarks

Integral quadratic constraints (IQCs) can be used in robustness analysis of feedback systems. This thesis has discussed how. Structural information about uncertainties, nonlinearities, signals, and performance specifications can be characterized and utilized. Four main topics are treated in the thesis.

Stability Theory: The thesis has provided results for robust stability and performance analysis with combinations of bounded multipliers and nonproper Popov multipliers in an IQC framework. Popov multipliers are inexpensive to compute, and they can contribute significantly to the accuracy of the stability conditions. An example in Chapter 2 shows that the nonproperness of such a multiplier can be important in applications.

New Multipliers: The strength of the IQC framework is that multipliers from many different robustness criteria can be combined in a simple and non-restrictive way. It is of interest to extend the set of multipliers available for system analysis. The thesis has provided several classes of multipliers:

- Two new sets of multipliers for slowly time-varying parameters.
- Popov multipliers for slowly time-varying parameters that are defined in terms of a convex polytope, and a similar class of multipliers for the corresponding class of parametric uncertainties.
- Multipliers for signals with a given spectral characterestic.

Computation of Multipliers: Robustness analysis in the IQC framework leads initially to infinite-dimensional optimization problems. Finite-dimensional approximations need to be considered in applications of the method. A flexible format for finite-dimensional restrictions of the original robustness problem has been introduced in Chapter 3. The subspace for the restriction is defined in terms of a basis of rational transfer functions.

Duality Bounds: It is of interest to obtain bounds for the conservatism introduced when restricting the robustness problem to a finite-dimensional subspace. Duality theory has been used to obtain such bounds. The possibilities and limitations of the duality results are discussed in some depth. A large class of problems of practical interest that gives a particularly attractive dual are identified in Chapter 5.

A discussion in Chapter 4 has shown that approximate solutions to the dual also have the potential to identify ill-conditioned problem formulations and system models.

Extensions and Future Research

There are several directions in which the IQC framework can be extended. We will here outline some of them.

Multiplier Computation: More effort needs to be invested in the approach for multiplier computation. The following issues are important.

- Guidelines for the choice of suitable basis multipliers.
- Development of user friendly software interfaces.
- Applications to reasonable sized physical systems.

All numerical examples in the thesis have been solved using a preliminary version of a software package that interfaces LMI-Lab. It consists of commands for combining multipliers, commands that define multiplier sets for various nonlinearities, uncertainties, and signal specifications, and commands for the robustness analysis and for evaluation of the results.

New Theory: Hysteresis phenomena are common in control applications. It appears, for example, in mechanical control components such as valves and transmission mechanisms. Several frequently appearing hysteresis phenomena can be modeled as multivalued functions. This adds an extra difficulty to the stability analysis since, such hysteresis functions often define unbounded operators. Based on ideas from Yakubovich (1963) and Yakubovich (1965b) we recently derived stability results for a class of uncertain systems with hysteresis nonlinearities, see Jönsson (1996). A new result on exponential stability in an IQC framework in Ranzer and Megretski (1996) seems to be useful for extending this work further.

Controller Synthesis: The information provided by multipliers should to the largest possible extent be exploited in controller synthesis. This is the idea used in $D-K$ iteration for controller synthesis in the framework of structured singular values. The optimization problem in a $D-K$ iteration is non-convex and NP-hard. It is thus computationally expensive and convergence to the global optimum cannot be guaranteed. However, in the hands of a good control engineer it can still be a very useful tool.

Synthesis results for robust controllers subject to unstructured uncertainty and various performance objectives have been developed by many authors. The underlying framework for much of this work can be interpreted in terms of IQCs and the design equations can be formulated as an optimization problem over linear matrix inequalities. For references, see for example, Iwasaki (1993), Boyd *et al.* (1994), Gahinet *et al.* (1995), Beran (1995), Scherer (1995), Chilaly *et al.* (1996), and EL Ghaoui and Folcher (1996).

An extension to the case of structured perturbations gives synthesis results in terms of bilinear matrix inequalities (BMI), see, for example, Megretski (1994) and Safonov *et al.* (1994). This means that we again have a nonconvex and potentially NP-hard problem. It has been debated whether it is a good idea to pursue these problems or not. However, although it is unlikely that we will find algorithms that ensure convergence to the global optimal with reasonable computational effort, we believe it is useful to develop suboptimal techniques that take some of the information in the IQCs into consideration in the synthesis of controllers.

An interesting idea for using structural information about uncertainty in the design of gain-scheduled controllers was presented in Packard (1994) and Apkarian and Gahinet (1995). The scheduling parameters are used in the controller and this results in convex synthesis problems. Complex-valued parameters were considered in the original papers. An extension to real valued parameters was obtained in Helmersson (1995b). A further generalization that has connections with the IQC framework was obtained in Scherer (1996).

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