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Abstract

In this paper we analyse an extremum controller based on a pulse technique. The idea is to superimpose probing pulses to the control signal and use the size of the pulse response in the output for feedback. The probing control strategy has been used with success for the control of the substrate feeding in *E. coli* cultivations. The stability analysis is done here for systems of Hammerstein type with a piecewise affine nonlinearity. Stability regions of the closed-loop system are derived by solving suitable linear matrix inequalities. Some robustness results with respect to uncertainty in the plant are also given.

Key words
extremum control, piecewise affine system, Hammerstein system, fed-batch cultivation, glucose feeding

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1. Introduction

In extremum control, the task is to find and track the best operating point of a nonlinear process. The optimal setpoint is usually given by a nonlinear static input-output map presenting an extremum. There are also applications where it is desirable to drive the process to a saturation instead of an extremum. The classical approach consists in adding a known time-varying signal to the process input and correlating the output with the perturbation signal to get information about the nonlinearity gradient. The controller adjusts continuously the control signal towards the optimum.

In [8], the authors presented the stability analysis of an extremum seeking scheme for a general nonlinear dynamical system. Stability of the seeking scheme was proven under restrictive conditions: small adaptation gain and fast plant dynamic. In [7] they developped a tighter analysis where the process was modelled by a Wiener-Hammerstein system. No stability region was provided.

In [1] and [2] a probing controller based on a pulse technique is described. The main difference with the classical scheme is the separation in time of the correlation/probing phase and the control phase. Pulses are periodically introduced at the process input and a control action is taken at the end of every pulse. The control algorithm has been implemented and tested on real plants where good performance could be achieved, see [1].

The objective of this paper is to provide a rigorous analysis for stability and robustness of the probing controller. The analysis is local but it is possible to characterize the region where stability is guaranteed. The paper is organized as follows. In Section 2, we present the control strategy and formulate the problem in a piecewise affine framework. Stability and performance analysis is carried out in Section 3, using linear matrix inequalities. We finally examine in Section 4 an example that was at the origin of the probing controller.

2. Problem formulation

We assume that the process is a Hammerstein model: a static nonlinearity followed by a dynamical linear process, see Figure 1. The analysis would be similar for an output nonlinearity.

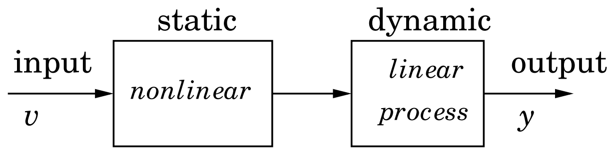

Figure 1 Hammerstein model

The control objective is to drive the system to the extremum of the static input-output map of the process. It may also be desirable to control the process to a saturation instead of an optimum.

The probing controller gets information about the nonlinearity from pulses that are periodically superimposed to the control signal. The size of the pulse response, which depends on the local gain of the nonlinearity, is used to adjust the control signal.

The controller law can be written as:

$$u_{k+1} = u_k + K[y((k+1)T) - y(kT + T_c) - y_r]$$
$$v(t) = u_k + u_p(t) \qquad t \in [kT,\ kT + T]$$

where v is the control signal and $u_p(t)$ is the T-periodic perturbation signal defined by:

$$u_p(t) = 0 \qquad\qquad t \in [kT,\ kT + T_c)$$
$$u_p(t) = u_p^0 \qquad\qquad t \in [kT + T_c, (k+1)T)$$

A reference value y_r is used for the desired pulse response. It is usefull when the nonlinearity does not present an extremum but a saturation. During the control phase with length T_c, the process input is kept constant. In [1], this phase is used to control the output by manipulating a second input variable. The duration of the probing phase is $T_p = T - T_c$.

The state space representation of the process can be written as

$$\dot{x} = Ax + Bf(v) \qquad\qquad x \in R^n$$
$$y = Cx$$

Since the controller is time-periodic, one can regard the feedback system as a sampled data system where the sampling time is the period T of the perturbation signal. Denoting by Φ the transition matrix for the linear plant we get

$$x((k+1)T) = \Phi(T_p)x(kT + T_c) + \int_0^{T_p} \Phi(T_p - \sigma)Bf(u_k + u_p^0)d\sigma$$

$$u_{k+1} = u_k + K[C(x((k+1)T) - x(kT + T_c)) - y_r]$$

where

$$x(kT + T_c) = \Phi(T_c)x(kT) + \int_0^{T_c} \Phi(T_c - \sigma)Bf(u_k)d\sigma$$

By augmenting the process state with the controller state, we finally get the closed loop equation

$$\begin{aligned} X_{k+1} &= A_d X_k + B_d \begin{bmatrix} f(u_k) \\ f(u_k + u_p^0) \end{bmatrix} + a_d \\ u_k &= [0\ \ldots 0\quad 1]\, X_k \end{aligned} \tag{1}$$

where

$$X_k = \begin{bmatrix} x(kT) \\ u_k \end{bmatrix} \quad A_d = \begin{bmatrix} \Phi(T) & 0 \\ KC(\Phi(T) - \Phi(T_c)) & 1 \end{bmatrix}$$
$$B_d = \begin{bmatrix} \int_0^{T_c} \Phi(T-\sigma) B d\sigma & \int_0^{T_p} \Phi(T_p - \sigma) B d\sigma \\ KC \int_0^{T_c} g(\sigma) B d\sigma & KC \int_0^{T_p} \Phi(T_p - \sigma) B d\sigma \end{bmatrix}$$
$$a_d = \begin{bmatrix} 0 \\ -Ky_r \end{bmatrix} \quad g(\sigma) = \Phi(T-\sigma) - \Phi(T_c - \sigma)$$

Assumption: the static nonlinearity f is a piecewise affine function defined on $p+1$ intervals.

The functions $f(\cdot)$ and $f(\cdot + u_p^0)$ in (1) induce then a partition of the state space into $2p+1$ polyhedral cells $\{\mathbf{X}_i\}_{i \in I}$. The regions $\mathbf{X}_i$ can be defined by the matrices E_i and e_i such that:

$$\mathbf{X}_i = \{X \in R^{n+1}, \quad E_i X + e_i \succeq 0\}$$

where the inequality $\succeq$ should be taken componentwise.

The closed-loop system has a piecewise affine structure and can be represented by:

$$X_{k+1} = A_i X_k + a_i, \quad \text{for} X_k \in \mathbf{X}_i, \quad i \in I$$

We assume that the closed-loop system has a unique equilibrium, that is shifted such that it coincides with the origin.

3. Stability and performance analysis

Stability analysis of the closed-loop system can be done using piecewise quadratic Lyapunov functions as in [5] or [6] in continuous time and [4] or [3] in discrete time. In the piecewise affine framework, H_∞ performance can also be evaluated. It is then possible to study the effect of uncertainty in the nonlinearity or in the plant dynamics on the probing control strategy.

3.1 Stability

The Lyapunov function candidate is piecewise quadratic:

$$V(X) = \bar{X}^T \bar{P}_i \bar{X} \quad \text{for } X \in \mathbf{X}_i, \ i \in I$$

where $\bar{X}$ denotes the state vector augmented by 1:

$$\bar{X} = \begin{bmatrix} X \\ 1 \end{bmatrix}$$

Stability can be tested by checking that the following linear matrix inequalities are feasible

$$\begin{aligned}
&\bar{P}_i - g_i \bar{G}_i - b_i \bar{B}_r > 0, && i \in I \\
&\bar{A}_i^T \bar{P}_j \bar{A}_i - \bar{P}_i + h_{ij} \bar{G}_i + k_{ij} \bar{G}_{ij} + b_{ij} \bar{B}_r < 0, && i, \ j \in I \\
&g_i > 0, \quad b_i > 0, \quad h_{ij} > 0, \quad k_{ij} > 0, \quad b_{ij} > 0, && i, \ j \in I
\end{aligned}$$

where

$$\bar{A}_i = \begin{bmatrix} A_i & a_i \\ 0 & 1 \end{bmatrix} \qquad \bar{B}_r = \begin{bmatrix} -D & 0 \\ 0 & R^2 \end{bmatrix}$$

and D is a positive diagonal scaling matrix. The so-called S–procedure [9] has been used to reduce the conservativness of the LMIs:

- the domain of analysis is limited to the ball $\bar{X}^T \bar{B}_r \bar{X} > 0$. Global stability cannot be investigated because of the integrators introduced by the control law. Stability can be still proven for a large bounded set of initial conditions.
- the matrix $\bar{G}_i$ is used to restrict the domain of validity for LMIs to the cell $\mathbf{X}_i$. It takes the form

$$\bar{G}_i = \begin{bmatrix} 0 & .5E_i^T \\ .5E_i & e_i \end{bmatrix}$$

- the relaxation term $\bar{G}_{ij}$ describes when a switch from the cell X_i to the cell X_j is possible in one step:

$$\bar{X}_k^T (h_i \bar{G}_i + k_{ij} \bar{G}_{ij}) \bar{X}_k > 0$$
$$\text{when } X_k \in \mathbf{X}_i \text{ and } X_{k+1} \in \mathbf{X}_j$$

We use a matrix $\bar{G}_{ij}$ of the following form:

$$\bar{G}_{ij} = \begin{bmatrix} 0 & .5A_i^T E_j^T \\ .5E_j A_i & e_j + E_j A_i \end{bmatrix}$$

3.2 Robustness

We assume that the model uncertainty can be represented by a multiplicative perturbation as follows:

$$\begin{aligned} X_{k+1} &= A_d X_k + B_d \begin{bmatrix} f(u_k) \\ f(u_k + u_p^0) \end{bmatrix} + B_d w_k + a_d \\ z_k &= \begin{bmatrix} f(u_k) \\ f(u_k + u_p^0) \end{bmatrix} \\ w_k &= (I_2 + \Delta) z_k \end{aligned}$$

where Δ is any stable transfer operator. With this representation, variations in the nonlinearity gain or equivalently in the process gain can be considered. Indeed, using a diagonal matrix for the uncertainty $\Delta = \delta I_2,\ \delta \in R$, we get

$$X_{k+1} = A_d X_k + (1 + \delta) B_d \begin{bmatrix} f(u_k) \\ f(u_k + u_p^0) \end{bmatrix} + a_d$$

Applying Lemma 5 in [3], an upper bound γ of the l_2 gain from w to z can be computed. The small gain theorem can then be used to get a robust stability test:

$$\|z\|_2 < \gamma \|w\|_2 \Rightarrow \text{stable for all } \Delta \text{ with } \|\Delta\|_\infty < \frac{1}{\gamma}$$

4. Illustrative example

In this section, the properties of the probing controller will be illustrated through an example. We will consider the control of glucose feeding in *Escherichia coli* fed-batch cultivations. It should be pointed out that the probing strategy presented here was developped for this specific application. The control problem consists in driving the system to a saturation instead of an optimum. In fed-batch cultivations, substrate is fed into the reactor at a growth limiting rate. High feed rates imply short cultivation times, but overfeeding may cause accumulation of by-products. Above an unknown substrate concentration, the oxygen uptake rate of the cells saturates and the by-product acetate starts to be produced. The manipulated variable is the feeding rate F and the measured variable is the dissolved oxygen concentration O_p. The objective is to keep the substrate concentration G slightly below the critical level G_{crit} by manipulating the feed rate. The closed-loop system can be represented by the block diagram in Figure 2.

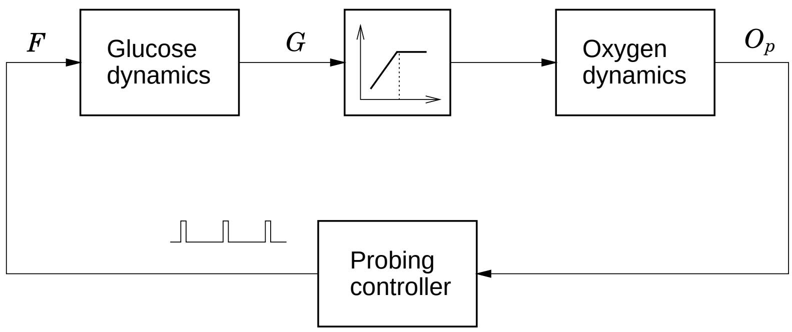

Figure 2 Block diagram of the closed-loop system. By making pulses in the feed rate F, it is possible to determine if the glucose concentration G is below or above the critical value G_{crit} when the oxygen uptake rate q_o start to saturate.

The oxygen dynamics is modelled by a first order transfer function

$$H(s) = \frac{K_o}{1 + T_o s} \qquad \text{with } K_o = 10 \text{ and } T_o = 1$$

Numerical values corresponding to a laboratory scale biorector are used for the simulations. The glucose dynamics is neglected and represented by a static gain $K_g = 5$, that is included in the nonlinearity f

$$\begin{aligned} G &= K_g F \\ f(F) &= \min(\mathrm{G}, G_{crit}) \quad \text{with } G_{crit} = 1 \end{aligned}$$

In [1], tuning rules for the probing controller are given. The pulse duration and the pulse height are chosen such that the pulse response can be clearly seen in the output signal. We choose $T_p = 1$ and $u_p^0 = 0.12$ to get suitable variations in the dissolved oxygen signal. The desired pulse response y_r is taken to be $y_r = 2$. In the original problem, the output O_p is controlled by means of the agitation speed during the control phase of length T_c. Although we do not take this second control loop into account, we follow the rule given in [1], that is $T_c \approx 3T_p = 3$. We choose a rather low controller gain $K = 0.02$, which corresponds to a feed increase of one pulse size when the response in the dissolved oxygen is 10.

The state is composed of the dissolved oxygen concentration O_p and the controller state u:

$$X = \begin{bmatrix} O_p \\ u \end{bmatrix}$$

The functions $f(\cdot)$ and $f(\cdot + u_p^0)$ induce a partition of the state space into 3 regions:

$$\begin{aligned}
\mathbf{X}_1 &= \{X \in R^2, u < G_{crit}/K_g - u_p^0\} \\
&= \{X \in R^2, u < 0.08\} \\
\mathbf{X}_2 &= \{X \in R^2, G_{crit}/K_g - u_p^0 < u < G_{crit}/K_g\} \\
&= \{X \in R^2, 0.08 < u < 0.2\} \\
\mathbf{X}_3 &= \{X \in R^2, u > G_{crit}/K_g\} \\
&= \{X \in R^2, u > 0.2\}
\end{aligned}$$

The dynamics in each region is given by

$$\begin{aligned}
A_1 &= \begin{bmatrix} 0.018 & 49.08 \\ -0.0005 & 1.025 \end{bmatrix} & a_1 &= \begin{bmatrix} 1.726 \\ 0.027 \end{bmatrix} \\
A_2 &= \begin{bmatrix} 0.0183 & 17.47 \\ -0.0005 & 0.519 \end{bmatrix} & a_2 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
A_3 &= \begin{bmatrix} 0.0183 & 0 \\ -0.0005 & 1 \end{bmatrix} & a_3 &= \begin{bmatrix} 1.142 \\ -0.031 \end{bmatrix}
\end{aligned}$$

It can easily be shown that the two dynamics A_1 and A_3 contain an integrator that drives the state to the middle region where the unique equilibrium point is located.

$$\begin{aligned}
O_p &= 8.83 \\
u &= 0.134
\end{aligned}$$

A simulation of the closed-loop system has been carried out and the results are shown in Figure 3. The feed rate starting at 0 is gradually increased by the controller. At time $t \approx 15$, the glucose concentration reaches the critical point and the pulse response becomes smaller. As a consequence, the feed rate is increased carefully until the size of the pulse response equals the setpoint y_r. The stationary feed rate corresponds to a glucose concentration that is just below G_{crit}.

The stability test from Section 2 consists of 33 LMIs with 22 decision variables. Feasibility of the LMI system, and as a consequence stability, could be established using the *LMI Control Toolbox* for *Matlab*. Convergence to the equilibrium is guaranteed for all initial states in a ball of radius $R = 50$. This allows an error for the initial feed rate of at most $\Delta F = 50u_p$, which is very large.

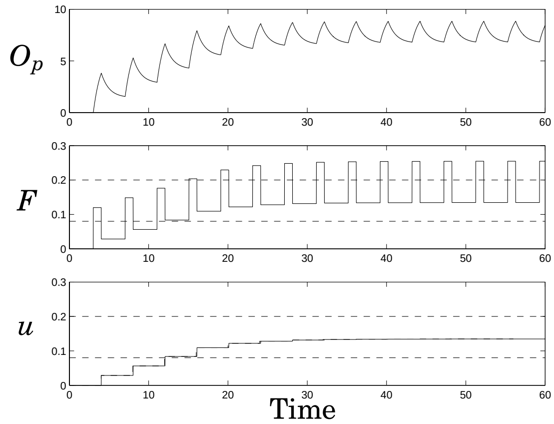

Figure 3 Simulation of the closed-loop system using the probing controller. The feed rate is gradually increased until the response in dissolved oxygen equals the setpoint $y_r = 2$. The dashed lines represent the cell borders.

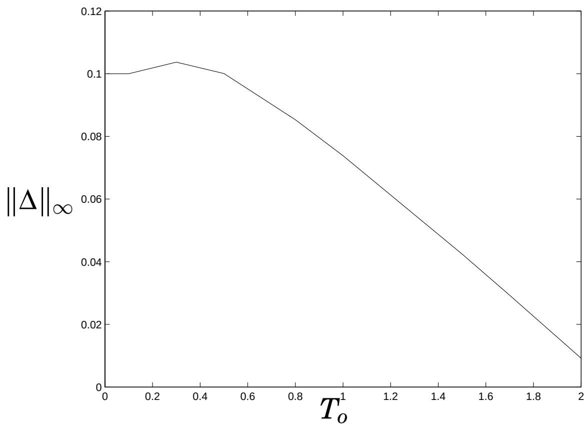

Figure 4 Graph giving the maximal relative variation in the gain K_o as a function of the time constant T_o. For $T_o = 1$, stability of the closed-loop system cannot be guaranteed for variations in K_o larger than 8%.

Since the oxygen dynamics may vary significantly during a cultivation, robust stability tests are valuable for the design evaluation. The norm $\|\Delta\|_\infty$ of the largest allowed uncertainty has been computed and plotted in Figure 4 as a function of the process time constant T_o. The design done for the nominal system allows around 8% variations in the process gain K_o without affecting the closed-loop stability. This robustness result is however rather conservative. Unstructured uncertainty was considered in the analysis although a constant diagonal matrix Δ suffices to model a uncertain gain K_o. Stability was not established for time constants T_o greater than 2.2. The rule of thumb $T_p \approx T_o$ seems to give good stability margin, comparable to the case of a static process $T_o = 0$. The choice of the relative pulse duration is however a compromise between performance and robustness: long pulses result in good stability margins but slow convergence speeds.

5. Conclusion

A probing control strategy has been analysed for linear systems with an output nonlinearity that is piecewise affine. Techniques for piecewise affine systems have been used to derive stability tests of the closed-loop system. The analysis is local but large bounded regions of attraction can be guaranteed. Robustness with respect to uncertainty in the plant dynamics has also been investigated.

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