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L.A.

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Introduction

Mathematical models are frequently used in control engineering for analysis, simulation, and design of control systems. Many of these models are accurate but may for some tasks be too complex. In such situations the model needs to be simplified to a suitable level of accuracy and complexity. There are many simplification methods available for models with known parameters and dynamics. However, for models with uncertainty, which have gained a lot of interest during the last decades, much needs to be done. Such models can be used to capture for example parametric uncertainty and unmodeled components and are important both in theory and applications.

The main contributions of this thesis are, an error bound for comparison and simplification of models with uncertainty, and a general methodology for converting a nonlinear differential-algebraic model with uncertain parameters into a form suitable for robustness analysis. The thesis also contains an overview of model simplification.

The thesis consists of the following five papers.

[1] L. Andersson. "An overview of model simplification." Unpublished. 1998.

[2] L. Andersson, A. Rantzer, and C. Beck. "Model comparison and simplification." *International Journal of Robust and Nonlinear Control* **No 9:3**, pp. 157–181, March 1999.

[3] L. Andersson and A. Rantzer. "Frequency dependent error bounds for uncertain linear models." *IEEE Transactions on Automatic Control*, Accepted for publication. March 2000.

[4] L. Andersson and A. Rantzer. "Robustness of equilibria in nonlinear systems." Submitted to Automatica. 1999.

[5] L. Andersson, A. Rantzer and M. Lantz. "Robustness analysis of large differential-algebraic models with application to power systems." Submitted to Automatica. 1999.

The connection between the different results are illustrated in Figure 1.

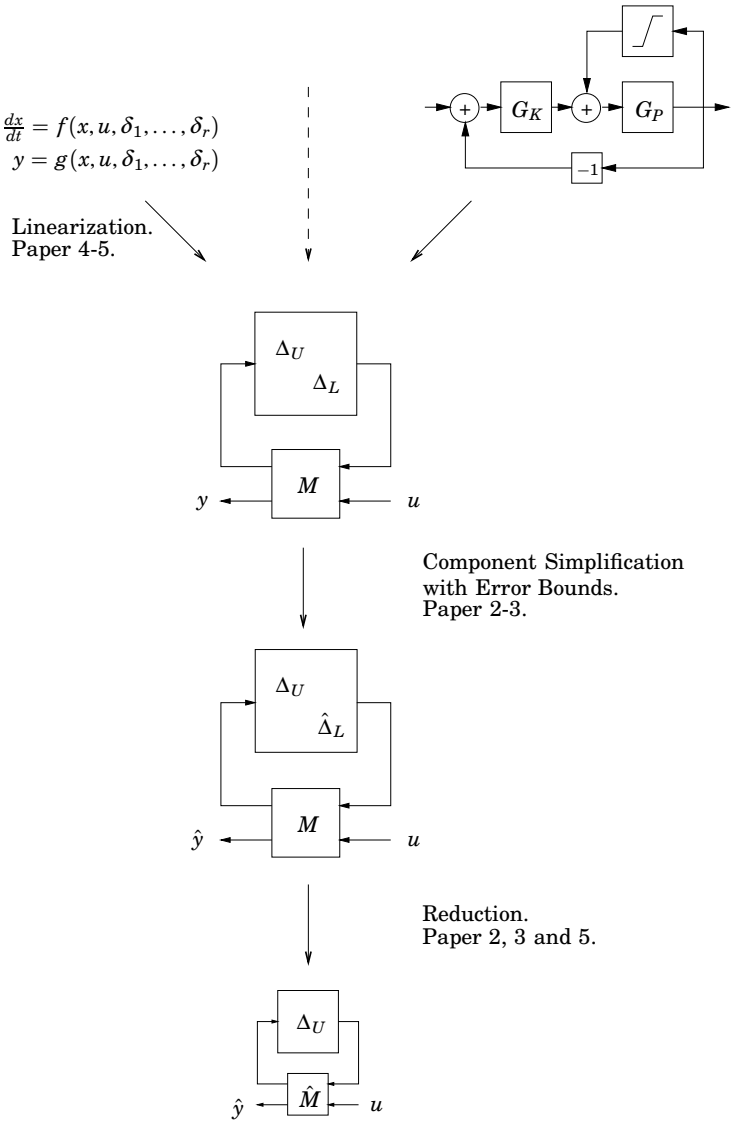

Figure 1. Thesis contribution. Linearization of nonlinear differential equations with uncertain parameters is studied in Paper 4 and Paper 5. Simplification and reduction of models on uncertainty feedback form is considered in Paper 2 and Paper 3, where bounds on the errors also are given. The reduction problem is also considered in Paper 5. The symbols δ and Δ are used to denote uncertainty.

Background

Mathematical models are frequently used to abstract essential properties of a physical process, in order to facilitate reasoning, decision making and problem solving. What properties that are important and how they are modeled depends on the purpose of the model. Different models of the same physical process may be used in different situations. Naturally only the physical process itself represents all aspects.

Typical areas where mathematical models are important are in science, economy and engineering. In control engineering models are widely used for process design, controller design, performance analysis, operator training, diagnosis and prediction. Models can reduce the need for prototype building. They can be used in simulations to reduce the need for real experiments (which may be both expensive and dangerous). They are essential in the design of high performance control systems and for detecting and locating process faults such as sensor failures.

The importance of models in control engineering increases with increasing demands on quality and productivity. The particular choice of model for each task is usually based on a trade off between simplicity and accuracy of the model. Simulation and controller design based on complex models may result in time consuming computations as well as numerical difficulties. The designed controller usually also becomes complex and may therefore be expensive and difficult to implement. Models with low accuracy, on the other hand, may give rise to incorrect simulations, as well as control loops with poor performance or even instability. Therefore, there is a need for a hierarchy of models, each equipped with an accuracy measure. The most appropriate model should then be used for each task. As much effort often is used to obtain an accurate process model, simplification of such a model is a way to obtain a model suitable for a specific task. Model simplification is studied in this thesis.

The choice of modeling framework restricts the type of behaviors that can be described. Mathematical models describing only the linear part of a system are for example more restrictive than models describing also the nonlinear part. Independent of which parts of a system we choose to describe there will always be imperfections in the model. These imperfections may, in more difficult design problems, result in control loops with poor performance or even instability. One way to reduce the effects of imperfect models is to also model the imperfection. Typical sources of imperfection are neglected disturbances, uncertain parameters, unmodeled dynamics and linearization of nonlinear models. Models including bounds on such imperfections are called uncertain. The modeling framework used in this thesis captures both dynamic and nonlinear uncertainty.

Methods for analyzing the importance of uncertain components are

considered in this thesis. Such methods are not only useful in model simplification, but also for detecting what parts of a model that needs to be more accurately described.

Development of modeling tools, see for example [Eborn, 1998; Mattsson and Elmqvist, 1997], are beyond the scope of this thesis. Such tools are important in order to reduce engineering effort and shorten development time. These tools should facilitate the development of accurate models and also be able to automatically simplify these models to a suitable form for problem solving tools. Simplification methods are important in the development of such tools.

Contribution outline

This thesis contributes to the development of methods for simplification of models with uncertainty. The first paper gives an overview of the ideas and methods used to simplify mathematical models in general. Specific problems concerning simplification of models with uncertainty are considered in the other four papers.

One important problem in model simplification is the possibility to identify the less important parts of a model prior to the actual simplification. It is also interesting to identify the more important parts as they may need a more accurate description. In Paper 2 and Paper 3 this problem is solved by the computation of numbers such that the resulting simplification error is bounded by the sum of these numbers for the simplified components. In Paper 2 the uncertainties may be nonlinear and time-varying whereas in Paper 3 they are restricted to be linear and time-invariant. In the latter case the error bounds become frequency dependent, which is useful when the required accuracy is different for different frequencies. A typical example is process models used for controller design where the accuracy close to the cross-over frequency is the most important.

Another important problem is robustness analysis of nonlinear models with uncertainty. Direct analysis of such models is difficult and simplifications are needed to convert the model to a form suitable for analysis. In Paper 4 we consider models with parametric uncertainty and simplify such models using linearization. Attention has to be payed to the fact that the location of the equilibria depends on the uncertain parameters. Ideas are given to cope with this problem. Some of these ideas are only useful for small models. In Paper 5 a general methodology for robustness analysis of large differential-algebraic models is presented. This methodology involves some further steps of simplification in addition to linearization. The methodology is applied to a model of the Nordic power system.

The remaining part of this chapter consists of a more detailed overview

group	properties	
I	Parametric	Non-parametric
II	Continuous time	Discrete time
III	Finite order	Infinite order
IV	Linear	Nonlinear
V	With uncertainty	Without uncertainty
VI	Stable	Unstable
VII	Composite models	Sub-models

Table 1. Some typical properties by which models may be classified.

of the different papers. Possible continuations of the work are presented at the end of this introduction.

Paper 1; An Overview of Model Simplification.

Contribution

This paper gives an overview of model simplification. Both concepts and approaches are given. The paper puts the results of this thesis into a wider perspective, but also serves as an independent overview of model simplification.

Extended abstract

Model simplification is the general process by which a model that is easier to use is obtained. The simplification methods are different for different classes of models and for different purposes. It is therefore natural to classify simplification methods based on the model class and the purpose.

Typical properties by which mathematical models can be classified are shown in Table 1. Models may be simplified within the same class by for example reducing the number of parameters. A typical example is order reduction of linear time-invariant models. Models may also be simplified by obtaining a model in another class. A typical example of this is linearization, where a linear model is obtained based on a nonlinear model.

Two important purposes of model simplification in control engineering is to reduce simulation times and to simplify controller design. For models used in simulations the required accuracy is often the same at all frequencies, while in controller design this is not at all true. Assume

that the open-loop model $G(s) = G_r(s)G_p(s)$ where $G_r(s)$ is the controller transfer function and $G_p(s)$ the process transfer function. Then the closed-loop model at the frequency ω is given by

$$\frac{G(i\omega)}{1+G(i\omega)} \approx \begin{cases} 1, & |G(i\omega)| >> 1 \\ G(i\omega), & |G(i\omega)| << 1. \end{cases}$$

This shows that the accuracy of the model is less important at frequencies where $|G(i\omega)|$ is far from one. Frequencies where $G(i\omega) \approx -1$ are, however, very important. If the accuracy of the process model is low at such frequencies, controllers designed based on this process model, may result in unstable closed-loop systems. To cope with this problem many standard methods have been extended with frequency weighting, see for example [Zhou *et al.*, 1996]. A further difficulty is that the frequencies at which the process model needs to be accurate will not be known until the controller has been designed. This makes simplification for controller design a difficult problem.

The paper contains references to many different simplification methods. Two standard methods on which the results in this thesis are based are linearization, see [Khalil, 1996] and Balanced truncation, see [Zhou *et al.*, 1996]. Balanced truncation is described below.

Balanced truncation is applicable to linear time-invariant models. The purpose of the method is to reduce the number of states. Consider the discrete time state-space model

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k, \\ y_k &= Cx_k + Du_k. \end{aligned}$$

The first step in Balanced truncation is to transform the given realization into a balanced realization, which is a realization where each of the states is equally observable and controllable. A Balanced realization satisfies

$$\begin{aligned} A\Sigma A^T + \Sigma + BB^T &= 0 \\ A^T\Sigma A + \Sigma + C^TC &= 0 \end{aligned}$$

for some diagonal matrix

$$\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n) \geq 0.$$

The second step is the selection of states to eliminate. The resulting error will be bounded by the sum of the singular values corresponding to the eliminated states

$$\frac{\|y-\hat{y}\|}{\|u\|} \leq \|G(s) - \hat{G}(s)\|_\infty \leq \sum 2\sigma_k.$$

Small elements in Σ thus indicate states with little importance. The Σ-matrix is therefore an important guide when we select which of the states to eliminate.

The last step is to eliminate the chosen states. We truncate the parts of A, B and C corresponding to these states. If we assume that we eliminate the last states and partition the balanced realization correspondingly as

$$\begin{aligned} A &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} & B &= \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \\ C &= \begin{bmatrix} C_1 & C_2 \end{bmatrix} & D, & \end{aligned}$$

then the truncated model is given by

$$\begin{aligned} \hat{x}_{k+1} &= A_{11}\hat{x}_k + B_1 u_k \\ \hat{y}_k &= C_1\hat{x}_k + Du_k. \end{aligned}$$

Paper 2; Model Comparison and Simplification.

Contribution

Simplification of models with uncertainty, together with error bounds computable based on convex optimization, is presented. The results are applicable to models with both dynamic and nonlinear uncertainties. The error bounds hold for a general reduction algorithm including truncation, but may also be used to investigate the importance of model components. Uncertain components that are important may need to be more accurately described in order to obtain a less conservative model. Uncertain components that are less important may be truncated.

Extended abstract

Models with an uncertainty description are important in the design of robust control systems, and for robustness analysis. The efficiency of such methods depends on the complexity of the model. Simplification of models with uncertainty and associated error bounds are presented in this paper.

Modeling framework Consider a model defined by the interconnection of a pair (Δ, M), see for example [Zhou *et al.*, 1996], according to the relations

$$x = \Delta z$$

and

$$\begin{bmatrix} z \\ y \end{bmatrix} = M \begin{bmatrix} x \\ u \end{bmatrix},$$

as illustrated in Figure 2. Here Δ represents nonlinearities, uncertainty

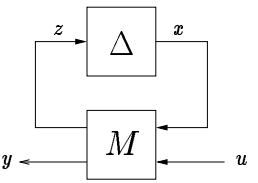

Figure 2. Feedback interconnection representing a model with uncertainty.

and dynamic elements, while M represents linear time-invariant dynamics. The signals in the interconnection are the input u, the output y and the internal signals z and x. The transfer matrix M is partitioned consistently with the signal dimensions as

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

The operator Δ is often assumed to have a block diagonal structure

$$\Delta = \mathrm{diag}(\Delta_1, \ldots, \Delta_r),$$

where each of the blocks satisfies a constraint, for example a norm bound, but otherwise the specific operator Δ need not be known. Thus Δ is particularly useful for describing uncertainty.

There are many examples where Δ described as belonging to a constrained set has proved to be useful. In stability analysis, using for example the Circle criterion and Popov criterion, see for example [Khalil, 1996], it has been useful to describe static nonlinearities using a sector constraint. Such a constraint has the form

$$[\beta z - \Delta(z)] \cdot [\alpha z - \Delta(z)] \leq 0,$$

where z is the input to Δ. The sector is defined by α and β as seen in Figure 3. Another useful type of constraint, in for example μ-analysis, see for example [Zhou *et al.*, 1996], is bounds on a parameter, such as for example

$$\Delta \in [-1, 1].$$

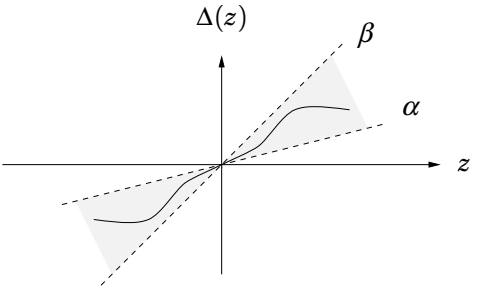

Figure 3. A static nonlinearity in a sector.

The two examples above are examples of constraints that the signals must satisfy at each time. If the operator Δ is dynamic it is often more useful with constraints that in some sense should be satisfied on average. A typical example is bounds on the energy amplification, or equivalently the system norm,

$$\int_0^\infty |\Delta z(t)|^2 dt \leq \int_0^\infty |z(t)|^2 dt \quad \Leftrightarrow \quad \|\Delta\| \leq 1. \tag{1}$$

This type of constraint is used in the small gain theorem. A similar example is passivity constraints used in the passivity theorem. These two theorems can be found in [Khalil, 1996].

All the above commonly used constraints are special cases of integral quadratic constraints. We say that an operator Δ satisfies the integral quadratic constraint defined by Π if

$$\int_{-\infty}^{\infty} \begin{bmatrix} z(i\omega) \\ x(i\omega) \end{bmatrix}^* \Pi(i\omega) \begin{bmatrix} z(i\omega) \\ x(i\omega) \end{bmatrix} d\omega \quad \geq \quad 0,$$

for all vectors $z, x = \Delta z \in L_2$. The definition can be rewritten in the time-domain using Parseval's formula. The norm bound (1) is obtained using

$$\Pi = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}.$$

Stability analysis based on integral quadratic constraints is considered in [Megretski and Rantzer, 1997].

Model simplification A method similar to Balanced truncation can be used to truncate parts of an uncertain model. The first step is to compute σ-values for each of the blocks in Δ satisfying

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \Pi \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -I \end{bmatrix} < 0$$
$$\begin{bmatrix} \Sigma A \\ \Sigma \end{bmatrix}^* \Pi \begin{bmatrix} \Sigma A \\ \Sigma \end{bmatrix} + C^* C < 0, \tag{2}$$

where

$$\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_r).$$

In the special case where

$$\Pi = \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & \Sigma^{-1} \end{bmatrix},$$

the two inequalities reduce to the Lyapunov inequalities

$$A\Sigma A^T + \Sigma + BB^T < 0$$
$$A^T \Sigma A + \Sigma + C^T C < 0.$$

This indicates the relationship with Balanced truncation. A class of operators that satisfies the integral quadratic constraints defined by the above given Π are time varying operators where

$$\Delta = \mathrm{diag}(\delta_1 I_{n_1}, \ldots, \delta_r I_{n_r}), \quad \|\delta_k\| \leq 1, \quad k = 1, \ldots, r.$$

The error bound in this case has been obtained earlier, and is further described in for example [Beck *et al.*, 1996].

The second step is the selection of what blocks to truncate. The resulting error will be bounded by the sum of the σ-values corresponding to the truncated blocks

$$\frac{\|y - \hat{y}\|}{\|u\|} \leq \sum 2\sigma_k.$$

Similar to Balanced truncation, small σ values indicate blocks with little importance. The σ-values is therefore an important guide in the selection of blocks to truncate.

The last step is to truncate the chosen blocks. Assume that the model is partitioned in such a way that the lower blocks of Δ and the corresponding part of M should be truncated. We partition the model consistently as

$$M = \begin{bmatrix} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ C_1 & C_2 & D \end{bmatrix}, \qquad \Delta = \begin{bmatrix} \Delta_U & 0 \\ 0 & \Delta_L \end{bmatrix},$$

where

$$\begin{aligned} \Delta_U &= \operatorname{diag}(\Delta_1, \ldots, \Delta_{\hat{r}}), \\ \Delta_L &= \operatorname{diag}(\Delta_{\hat{r}+1}, \ldots, \Delta_r). \end{aligned}$$

The truncated model is then given by

$$\hat{M} = \begin{bmatrix} A_{11} & B_1 \\ C_1 & D \end{bmatrix}, \qquad \hat{\Delta} = \Delta_U,$$

as illustrated in in Figure 4.

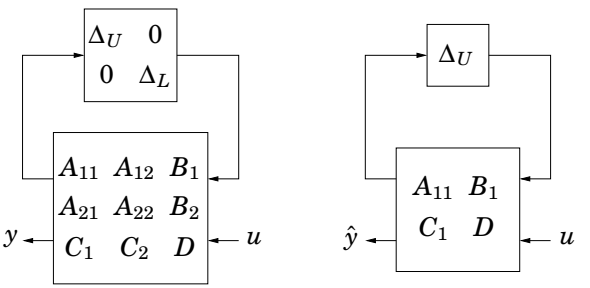

Figure 4. Model truncation.

The error bounds in this paper can also be used for other simplification methods than truncation. One of these methods is singular perturbation approximation.

The error bound above is useful for analyzing what parts of a closed-loop model that are important. The less important parts may be truncated. The simplified model is, however, not always suitable for controller design, since also less important components may influence the closed-loop significantly.

Publication history

This paper is based on the following publications.

Andersson, L. and C. Beck (1996): "Simplification Methods for Uncertain Models." Reglermöte 96, pp. 49–50, Luleå, Sweden,

Andersson, L. and C. Beck (1996): "Model Comparison and Simplification." 35th IEEE CDC Proceeding, pp. 3958–3963, Kobe, Japan.

Andersson, L. (1997): "Comparison and Simplification of Uncertain Models." Licentiate Thesis, TFRT-3216, Department of Automatic Control, Lund Institute of Technology, Lund, Sweden.

Paper 3; Frequency Dependent Error Bounds for Uncertain Linear Models.

Contribution

This paper presents a frequency dependent error bound for comparison and simplification of models with uncertainty. This error bound can be used for models with uncertainty that are linear and time-invariant. The proof is based on the result in the previous paper.

Extended abstract

The error bounds obtained using the results in the previous paper can be improved for linear time-invariant models. The idea is to compute frequency dependent error bounds. The maximum over frequency of this error bound will coincide with the error bound obtained in the previous paper, see Figure 5.

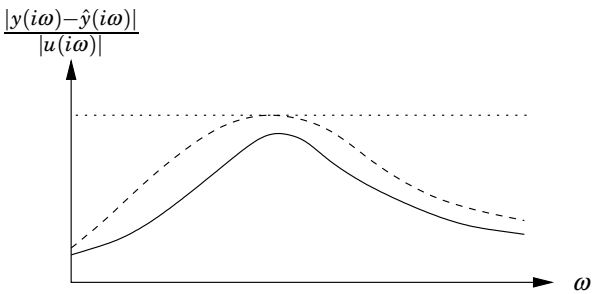

Figure 5. Truncation error. The solid line illustrates a worst case error, the dashed line a frequency dependent bound obtained using the results in Paper 3 and the dotted line a bound obtained using the results in Paper 2.

The idea is that no energy is transfered between different frequencies when a system is linear and time-invariant. This implies that the integral quadratic constraint can be reduced to the quadratic constraint

$$\begin{bmatrix} I \\ \Delta(i\omega) \end{bmatrix}^* \Pi(i\omega) \begin{bmatrix} I \\ \Delta(i\omega) \end{bmatrix} \geq 0, \quad \forall \omega \in [0, \infty), \tag{3}$$

when Δ is linear time-invariant. It also implies that frequency dependent error bounds can be obtained, by solving the inequalities (2) frequency by frequency. A graphical description of quadratic constraints for a scalar Δ is given in Figure 6.

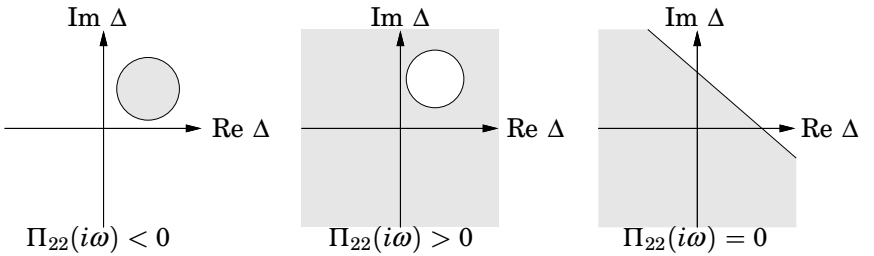

Figure 6. Quadratic constraints for a scalar transfer function Δ at a given frequency ω. The transfer function should be within the shaded region.

The obtained frequency dependent error bounds can be used to analyze the importance of different components for different frequencies. Less important components may be truncated.

Publication history

Andersson, L. and A. Rantzer (1997): "Frequency dependent error bounds for uncertain linear models." American Control Conference, pp. 181–185, Albuquerque, New Mexico.

Andersson, L. and A. Rantzer (1998): "Frequency Dependent Error Bounds for Uncertain Linear Models." Technical Report TFRT-7575, Department of Automatic Control, Lund Institute of Technology, Lund, Sweden.

Andersson, L. (1997): "Comparison and Simplification of Uncertain Models." Licentiate Thesis, TFRT-3216, Department of Automatic Control, Lund Institute of Technology, Lund, Sweden.

Paper 4; Robustness of Equilibria in Nonlinear Systems.

Contribution

This paper investigates robust stability of equilibria in nonlinear systems having uncertain parameters. The main difficulty is that the location of the equilibria depends on the uncertain parameters. A problem formulation is given to cope with this problem. Examples and ideas of how the resulting problem can be solved are given.

Extended abstract

Nonlinear models with uncertain parameters can be described in different ways. One way, as discussed previously, is by the interconnection of a linear time-invariant part and an uncertain nonlinear part. Another well known class of nonlinear models with uncertainty is nonlinear differential equations with uncertain parameters,

$$\frac{dx}{dt} = f(x, \delta).$$

Local stability of such models can be analyzed based on the linearized model. Linearization removes all nonlinear terms while keeping the main stability properties. It also involves the selection of which equilibrium to analyze. The location of this equilibrium depends on the uncertain parameters as an equilibrium x^0 must satisfy the equation

$$f(x^0, \delta) = 0.$$

The linearized model therefore becomes

$$\frac{d\tilde{x}}{dt} = \frac{\partial f}{\partial x}\left(x^0(\delta), \delta\right)\tilde{x}, \quad \text{where} \quad \tilde{x} = x - x^0(\delta).$$

The robust stability problem can therefore be formulated as determining if the eigenvalues of the Jacobian

$$J(\delta) = \frac{\partial f}{\partial x}\left(x^0(\delta), \delta\right)$$

stay in the left half plane for all values of δ. This problem can be solved using μ-analysis if the Jacobian is rational in the elements of δ. If it is not rational it may be possible to change the variables in order to make it rational. Another idea is to approximate the Jacobian by a rational function.

Publication history

Andersson, L. and A. Rantzer (1999): "Robustness of Equilibria in Nonlinear Systems." 14th IFAC World Congress, vol E, pp. 129–134, Beijing, China.

Paper 5; Robustness Analysis of Large Differential-Algebraic Models with Application to Power Systems.

Contribution

This paper presents a general methodology for robustness analysis of large differential-algebraic models. Different steps of simplification are used to convert a nonlinear model into a model on uncertainty feedback form. The resulting model can be used for μ-analysis. The approach is applied to the Nordel power system.

Extended abstract

Stability analysis of large differential-algebraic models are in general difficult. We therefore suggest that simplifications are used to convert the model into uncertainty feedback form with matrices of reasonable dimensions. Robust stability of such models can be investigated using μ-analysis.

We describe the ideas based on a model without algebraic variables. Therefore consider a model of the form

$$\frac{dx}{dt} = f(x, \delta).$$

The first step of simplification is linearization. In this step it is important to include the effects of perturbed equilibria, as pointed out in Paper 4. For large models it is then likely that the resulting Jacobian will depend on the uncertain parameters in a complex way. To avoid this problem we suggest that the Jacobian be approximated. Neglecting that the equilibrium location depends on the uncertain parameters is often a crude approximation. A better way is to approximate the functions $x^0(\delta)$ or $A(\delta)$ using series expansions. Using a first order approximation we obtain a model of the form

$$\frac{d\tilde{x}}{dt} = (A_0 + \sum A_k \delta_k)\tilde{x}.$$

Note, however, that this model, in general, does not preserve the stability properties of the original model. The next step is to rewrite the model

on the form described in the second and third paper, that is as a feedback interconnection describing a model with uncertainty. This is done by extracting the uncertain parameters.

In order to do so we use rank factorizations

$$A_k = S_k \delta_k I_{n_k} T_k$$

where n_k is the rank of A_k. Introducing

$$S = [\, A_1 \quad \dots \quad A_r \,], \quad T = \begin{bmatrix} I \\ \vdots \\ I \end{bmatrix} \text{ and } \Delta = \begin{bmatrix} \delta_1 I & & \\ & \ddots & \\ & & \delta_r I \end{bmatrix},$$

it holds that

$$\sum A_k \delta_k = S \Delta T.$$

Using this and Laplace transformation we obtain the model

$$(I - \underbrace{T(sI - A_0)^{-1} S}_{M(s)} \Delta) \underbrace{T\tilde{x}}_{z} = 0.$$

This model is on a form suitable for robustness analysis using available methods such as μ-analysis. If the matrix dimensions are high, then such analysis may be time-consuming or impossible. One way to reduce the matrix dimension is by singular value decompositions. Neglecting small singular values makes it possible to do rank factorization with lower dimensions n_k.

The methodology is applied to investigate robustness of the Nordel power system. Robustness analysis of power systems is a difficult problem. First of all, power system models are usually nonlinear with thousands of states and parameters, see for example [Machowski *et al.*, 1997]. Secondly the parameter values and the model structure varies with time. The parameter variations are often caused by variations in power demand and power consumption and a possible cause of model structure changes is failures.

It is of course not possible to investigate all possible situations. Instead some interesting cases are studied. For each of these cases it is interesting to vary some of the more important parameters. Typical examples of interesting parameters to vary are reference values, power demand, power generation, and controller parameters. If we assume that the parameter changes appear abruptly and seldom then it is possible to assume that the parameters are constant but uncertain.

In this paper we apply the general methodology to investigate robustness when some of the aggregated loads of the model are uncertain. We conclude that robust stability of a reasonably large model may be investigated using the proposed methodology.

Future work

Model simplification is an important part of modeling and there is a need to develop good simplification methods in particular for large and more advanced models. The need for further research in the area of model simplification is therefore strong. There are also many interesting continuations of the specific work presented in this thesis.

The error bounds in Paper 2-3 are described by inequalities. The problem of finding the best error bound satisfying these inequalities are in general non-convex. It would therefore be interesting to investigate other sub-optimal ways to compute the error bounds, than the one presented in this thesis. Another interesting continuation would be to compare different methods for selecting which components to simplify. It would in particular be interesting to compare the method presented in Paper 2-3 with a method where a set of candidate models are obtained followed by a computation of the error. These two strategies were discussed in Paper 1.

Properties of the fast and simple reduction method presented in Paper 5 needs to be further investigated. A priori computable error bounds would also be valuable.

Simplification of nonlinear models with uncertainty using linearization is important and there remains a lot to be done. This includes the development of methods that capture the approximation errors and methods for non-parametric uncertainty.

There are many possible continuations to robustness analysis of power systems. These include investigation of what the most important parts of a model are when we design a controller, for example a power system stabilizer. The results in Paper 2-3 may be useful in the development of such methods.

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Paper 1

An Overview of Model Simplification

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Abstract

An overview of model simplification in system and control theory is given. Simplification methods for different classes of models and different purposes are reviewed. General concepts in model simplification are also discussed.

1. Introduction

Mathematical models are frequently used in control engineering to abstract essential properties of a physical system. These models are obtained based on physical principles as described in for example [Wellstead, 1979; Nicholson, 1980; Close and Fredrick, 1993] or using experimental data as described in for example [Johansson, 1993; Ljung, 1987]. Combining the two approaches is called semi-physical modeling, see [Lindskog, 1996]. The notations white-box models, black-box models and gray-box models, indicate the amount of physical knowledge used to obtain the model.

The choice of modeling framework, for example linear time-invariant models, restricts the type of behaviors that can be described, and thus the validity of the model. Also within a given modeling framework, the type of behaviors and the accuracy, is restricted by the number of free parameters used. The required accuracy depends on the situation for which the model is intended. Various models of the same physical system may therefore be appropriate for different tasks. Model simplification can be used to simplify a complex and accurate model to a suitable level of accuracy and

complexity.

A control system is often described by a feedback interconnection of a process and a controller as shown in Figure 1. Models are used in both

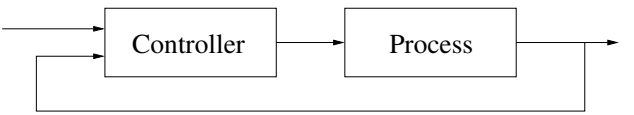

Figure 1. Control system consisting of a process and a controller.

the design and analysis of such control systems.

In process design one of the objectives is to obtain a process which is both easy to control and cheap to build. Time-delays and non-minimum-phase properties should if possible be avoided, as well as superfluous sensors and actuators. Selecting which of the possible sensors and actuators to use can be seen as a model simplification problem, and has been studied in for example [Keller and Bonvin, 1992]. Model based design reduces the need for prototype building. This can reduce development time and costs.

In controller design we assume that the physical process is given, and needs to be controlled. This classical problem was stated in [Åström, 1976] as *"Determine based on the measurements a control signal such that the purpose of the system is fulfilled despite the influence of the environment"*. In the design of control systems with high performance it is usually necessary to use models.

It is not always possible to separate process design, modeling and controller design, when developing high performance control systems. The combination of identification algorithms, modeling methods, model simplification algorithms and controller design algorithms is discussed in [Liu and Skelton, 1993]. In [Balas, 1982] the importance of process design in the development of control systems is considered. This includes for example choice of pre-filters and post-filters as well as location of sensors.

When both the physical process and the controller have been designed, it is important to analyze the behavior of the control system. Models may then be useful for calculation of system properties and for simulations. Model based simulations reduce the need for real experiments. Such experiments are often expensive and may also be dangerous.

For more advanced or dangerous control systems it is important that the operator be well trained. Models can in such cases be used for operator training. Models can also be valuable during operation to guide the operator and to detect and analyze the cause of process faults.

In the problems considered above, there is usually a trade-off between model complexity and model accuracy. Simulation and controller design based on complex models may result in time consuming computations as well as numerical difficulties. The designed controller usually also becomes complex and may therefore be expensive and difficult to implement. Models with low accuracy, on the other hand, may give rise to incorrect simulations, as well as control loops with low performance or even instability. Model simplification is therefore important for extracting the dominant properties of a complex model. Models should as a consequence, preferably, be equipped with a region of validity as well as a quality measure within the specified region.

The remaining part of this paper describes model simplification in the field of control engineering. Important concepts and approaches are given in Section 2, model simplification for analysis is described in Section 3 and for control design in Section 4. Conclusions are given in Section 5.

2. Model simplification concepts

Model simplification has been studied for many years and the problem has been given many different names, where the most commonly used are model reduction, model approximation and model simplification. Based on the meaning of the words simplification, reduction and approximation it seems reasonable to make the following distinctions. Model simplification is the general process in which a model that in some sense is easier to use is obtained. Model reduction is the process of reducing parts of a model, such as the number of states, the dimension of a matrix *etc*. Model approximation means finding a model that approximates the behavior of the original model; this does not necessarily result in a simpler model. Using these definitions, there are many similarities but also differences between reduction, approximation and simplification.

To illustrate the concepts, consider model identification, where for example a linear time-invariant model is obtained based on a sequence of input-output data. The sequence can be seen as a model parameterized by a large number of parameters, accurate for the considered input signal but useless for predicting the behavior for other input signals. The linear model on the other hand often consists of a few parameters, is less accurate and is easily used to predict the behavior for various input-signals. Model identification can therefore be seen as model reduction, model approximation, and model simplification, at the same time.

Model transformation

There are usually many models that describe exactly the same behavior and therefore in some sense are equivalent. It is then desirable to use the simplest of the equivalent models. The procedure of going from one equivalent model to another is called model transformation. A typical example of a model transformation is to go from a state-space model to a transfer-function model. If the obtained model is easier to use, then the model transformation can be seen as a model simplification without loss of accuracy. Note, however, that model simplification in general results in a less accurate model.

Model transformation is used as an intermediate step in some simplification methods. The purpose may be to find a model in which the less important parts can be easily identified. A result of such transformations can be greater simplification at less loss of accuracy.

Realization theory

Realization theory considers the problem of constructing physical systems with a desired behavior. Models with a structure suitable for implementation are used, and can be identified with the realizations. Such models are therefore often called realizations.

An example dating back a few decades, but still important in computer implementations, are realizations based on the interconnection of integrators. In this case state-space models are well suited as they explicitly describe how the integrators should be interconnected. Realization theory for linear time-invariant models therefore traditionally concern the problem of obtaining a state-space model (realization) based on for example a transfer function, see [Brockett, 1970].

Minimality

Within the same modeling framework there may be many models which describe the same behavior. A model is non-minimal if the same behavior can be described using a reduced model. Minimality for state-space models (minimal realizations) usually means as few states as possible. Minimality for a transfer function usually means that the degree of the denominator is as low as possible. For single-input single-output systems a transfer function is minimal if and only if there are no common poles and zeros.

Quality measures

The validity of simplified models needs to be known. Typically a region of validity is needed together with a measure of accuracy within the given region. The region of validity is defined in signal space. Examples of re-

gions are closeness to a given trajectory, frequency intervals and time-scales. A model may for example be valid if the signals are considered at the time-scale of seconds but not if we consider micro seconds.

Typical accuracy measures are trajectory norms and system norms (gains). The trajectory norms measure the distance between a trajectory of the original model and the simplified model. Let $x(t)$ be a trajectory of the original model and $\hat{x}(t)$ a trajectory of the simplified model. The distance $x(t) - \hat{x}(t)$ between these two trajectories can be measured by any trajectory norm, for example the the two-norm or the infinity-norm. If the original and approximating models have an input signal u and output signals y and $\hat{y}$, respectively, then we can form an error model with u as the input and $y - \hat{y}$ as the output and use the system norm (gain) of the error model as a measure of the accuracy. Comparing the original model with the simplified model is called model comparison. A simple introduction to trajectory and system norms is given in [Doyle *et al.*, 1992].

Selecting degree of simplification

When a model needs to be simplified there is usually a number of candidate models to choose among. The particular choice depends on the required accuracy and acceptable complexity. When, after a performed simplification, a candidate model has been obtained it is often easy to calculate the accuracy. By computing all possible candidates and their accuracy, it is easy to select the most suitable candidate. However, this approach would not be possible to use for larger models where the number of combinations is very large. The following example illustrates how this problem has been solved in one of the most successful simplification methods.

EXAMPLE 1
Assume that we have a model with three components Δ_1, Δ_2 and Δ_3 and that we would like to simplify some of these three components. For simplicity we assume that each component only can be simplified in one way.

Assume also that each component is assigned a positive number σ_k such that the error of the simplified model is bounded by the sum of these numbers for the simplified components.

A method of this kind is balanced truncation, see [Green and Limebeer, 1995], where the components are integrators (states). Each integrator is assigned a positive number measuring its importance. An upper bound on the error is obtained by summing these numbers for the truncated states.

We compare the following two approaches:

- Find all possible simplified models. Calculate for each of these models the error between the original model and the simplified model.

Δ_1	Δ_2	Δ_3	error	upper bound
			0	0
x			e_1	σ_1
	x		e_2	σ_2
x	x		e_{12}	$\sigma_1 + \sigma_2$
		x	e_3	σ_3
x		x	e_{13}	$\sigma_1 + \sigma_3$
	x	x	e_{23}	$\sigma_2 + \sigma_3$
x	x	x	e_{123}	$\sigma_1 + \sigma_2 + \sigma_3$

Table 1. The errors and upper bounds when we simplify different components. The components that we simplify are marked with x. The number of σ-values are equal to the number of blocks while the number of errors grow exponentially.

Choose the one with suitable accuracy.

- Select components to simplify so that the sum of their numbers σ_k is less than the acceptable error.

One advantage with direct calculation of the error is that we find a less conservative upper bound. A disadvantage is that the computation time depends exponentially on the number of blocks instead of polynomially, which is the case for the second approach.

Table 1 shows that we have to calculate $2^3 = 8$ errors or 3 σ_k-values. Each of these computations is balanced truncation done in polynomial time with respect to the number of errors and σ_k-values respectively.

It is worth mentioning that another important feature of the balanced truncation method is that it suggests a realization which is suitable for simplification. □

Computational speed

The aim of model simplification is to obtain a model which is easier to use. There is thus no point in applying model simplification if the cost for obtaining the simplified model together with an error estimate exceeds the benefits of the simplified model. The speed of simplification algorithms is therefore important, and can often be traded off against accuracy of the simplified model and accuracy of the error estimate. In some iterative methods, where the accuracy is improved in each iteration, this trade-off is easy to make, see for example [Grimme, 1997].

group	properties	
I	Parametric	Non-parametric
II	Continuous time	Discrete time
III	Finite order	Infinite order
IV	Linear	Nonlinear
V	With uncertainty	Without uncertainty
VI	Stable	Unstable
VII	Composite models	Sub-models

Table 2. Some typical properties by which models may be classified.

The purpose of model simplification

Depending on the purpose of the model simplification different methods should be applied and different regions of validity are desired. If for example the simplified model will be used for controller design it is important that the simplification preserves as much as possible the accuracy close to the bandwidth of the system to be designed.

This paper will mainly consider two purposes; model simplification for analysis and model simplification for controller design. In simplification for analysis both the process model and controller model are available, whereas in controller design the controller to be designed is unknown.

3. Model simplification in simulations and analysis

Models are often used for simulation and analysis of control systems, as described above. It is often desirable to simplify complex models to a suitable level of accuracy in order to obtain faster simulations and simpler analysis. Different methods for model simplification will be briefly summarized in this section.

The number of available methods is huge. This is a consequence not only of the importance of model simplification but also of the large amount of different model classes, where each class of models often needs its own simplification methods. In this section we consider simplification methods for some important classes of models in control engineering. The classification of models can be done in many ways. We choose to classify the models according to the properties in Table 2. It should directly be pointed out that the chosen way to classify models is not always relevant.

Simplification of non-parametric models

Systems are often described by bode-plots and other input-output data such as a step response. These models are often referred to as non-parametric in the system identification literature, and we will keep this notation, even though it is an abuse of language since the non-parametric models usually consist of hundreds or thousands of parameters (data-points).

The parametric models are often easier to use than non-parametric ones. Simplifying a non-parametric model to a parametric model is the main problem in system identification, see [Bosley and Lees, 1972; Johansson, 1993; Ljung, 1987]. Usually some optimization method is used to determine the parameters of the low order model. One of the most common methods used is least-squares estimation. If the model is assumed to be static, the simplification problem is called curve fitting.

Assume that we have collected the process input signal u and output signal y at each second during N seconds. Also assume that we would like to obtain a model of the form

$$y(t) = ay(t-1) + bu(t-1) = [\, y(t-1) \quad u(t-1) \,] \begin{bmatrix} a \\ b \end{bmatrix} = \varphi^T(t)\theta.$$

By introducing

$$Y = [\, y(2) \quad y(3) \quad \dots \quad y(N) \,]^T, \qquad \Phi = [\, \varphi(2) \quad \varphi(3) \quad \dots \quad \varphi(N) \,]^T$$

the parameters that minimizes $\|Y - \Phi\theta\|_2$ are given by

$$\theta = (\Phi^T\Phi)^{-1}\Phi^T Y.$$

Identification methods are also important for simplification of parametric models. By simulating the complex parametric model we could obtain input-output data which subsequently could be used to identify a less complex parametric model.

Input-output data models capture the system behavior for the given input-signal. It is therefore, when the input-signal is chosen in simulations or experiments, important to use an input signal with properties similar to the situation which the model is intended for. The purpose of any model is always important to have in mind.

Discretization

Continuous time models are often used to describe the behavior of physical processes. However, most controllers today are implemented in computers with algorithms based on discrete time process models. A discrete time

model can be obtained by sampling the continuous time model. Sampling the continuous time model

$$\begin{aligned}\frac{dx(t)}{dt} &= Ax(t) + Bu(t)\\ y(t) &= Cx(t)\end{aligned}$$

at equidistant time instants, with sampling interval h, results in the discrete time model

$$\begin{aligned}x(kh+h) &= \Phi x(kh) + \Gamma u(kh)\\ y(kh) &= Cx(kh)\end{aligned}$$

where $\Phi = e^{Ah}$ and $\Gamma = \int_0^h e^{As}B\,ds$. There are also approximative discretization methods, including Euler's method and Tustin approximation. The main idea in these approximation methods is that

$$\frac{dy(t)}{dt} \approx \frac{y(t+h) - y(t)}{h},$$

if h is small. Further details on discretization of continuous time models can be found in for example [Åström and Wittenmark, 1990].

Simplification of partial differential equations

Models described by partial differential equations are often simplified to ordinary differential equations. Typical methods for doing this are

- Difference approximation
- Finite element methods
- Transfer function approximation

Approximation of transfer functions are treated in Section 3.

Simplification of nonlinear models

Nonlinear models are widely used in the field of control engineering. Theory of nonlinear systems can for example be found in the standard books [Khalil, 1996; Vidyasagar, 1993; Nijmeijer and van der Schaft, 1990; Isidori, 1995]. Simplification of nonlinear models are often necessary for analysis and design of control systems.

Series expansion Approximating nonlinear models using series expansion is the most common approach. Linearization is one such method where the nonlinear model is approximated by a linear model close to a trajectory. Consider the nonlinear system

$$\frac{dx}{dt} = f(x,u)$$

where $x(t) = x_0(t)$ is a solution for $u(t) = u_0(t)$. Typical solutions used are stationary solutions. Such solutions satisfy $f(x_0,u_0) = 0$ for constant x_0 and u_0.

The behavior of the model close to a solution $x_0(t)$ and $u_0(t)$ is obtained by linearization. Introduce the deviations from the given solution as $\tilde{x}(t) = x(t) - x_0(t)$ and $\tilde{u}(t) = u(t) - u_0(t)$, then the linearized model is

$$\frac{d\tilde{x}}{dt} = \frac{\partial f}{\partial x}(x_0(t),u_0(t))\tilde{x}(t) + \frac{\partial f}{\partial u}(x_0(t),u_0(t))\tilde{u}(t)$$

which is a linear time-varying model. If we consider linearization around a stationary solution the expression is

$$\frac{d\tilde{x}}{dt} = \frac{\partial f}{\partial x}(x_0,u_0)\tilde{x}(t) + \frac{\partial f}{\partial u}(x_0,u_0)\tilde{u}(t),$$

which is a linear time-invariant model.

Linearization techniques play a fundamental role in the development of many new simplification methods, see for example [Öhman, 1998].

Another method using series expansion is the describing function method, see [Khalil, 1996] and references therein. In this method the nonlinear model is approximated by a linear model at each frequency. The method is based on Fourier series and is applicable when the signals are almost sinusoidals.

A generalization of this method with application to power networks is presented in the thesis [Möllerstedt, 1998]. The main idea is to describe nonlinear power loads by a linear matrix relation between some of the Fourier coefficients of the voltage and some of the Fourier coefficients of the current.

Perturbation theory Another approach to approximate nonlinear models is based on perturbation theory, see [Khalil, 1996] and references therein. In these methods the model is assumed to depend on a small parameter. Neglecting this parameter may give a nonlinear model which is easier to analyze. Singular perturbation is one such approach, which is applicable to models where some of the states are slowly varying and

others vary rapidly. Details can be found in [Kokotović *et al.*, 1986]. The idea of dividing the system into different time-scales and neglecting parts not corresponding to the time-scale of interest, is commonly used in many ad hoc methods.

To understand the ideas of singular perturbation consider the model

$$\frac{dx}{dt} = f(t, x, z, \varepsilon)$$
$$\varepsilon \frac{dz}{dt} = g(t, x, z, \varepsilon).$$

Here x are the slow states and z the fast states. To eliminate the fast states we let $\varepsilon = 0$ which implies that $g(t, \hat{x}, \hat{z}, \varepsilon) = 0$. Solving this equation for $\hat{z}(t, \hat{x}, \varepsilon)$ and replacing z in the first equation with this expression results in the simplified model

$$\frac{d\hat{x}}{dt} = f(t, \hat{x}, \hat{z}, \varepsilon) = \hat{f}(t, \hat{x}, \varepsilon).$$

Averaging is another perturbation method. This method is applicable when the state behavior consists of high frequency oscillations added to a slowly changing mean value. Adaptive systems excited by a high frequency sinusoidal signal often have parameters with this type of behavior. To simplify the analysis only the average of the solution is considered. The method of averaging to investigate nonlinear oscillations was developed in [Krylov and Bogoliubov, 1937]. Details on averaging can be found in [Åström and Wittenmark, 1995; Khalil, 1996]. The idea is the following. Consider the model

$$\frac{dx}{dt} = \varepsilon f(t, x, \varepsilon)$$

where ε is a small parameter and f is T-periodic. Then a simplified model is given by

$$\frac{d\bar{x}}{dt} = \varepsilon \bar{f}(\bar{x}), \quad \text{where} \quad \bar{f}(\bar{x}) = \frac{1}{T} \int_0^T f(\tau, \bar{x}, 0) d\tau$$

Simplification of linear time-invariant models

In this section we try to give an overview of existing methods for simplification of linear time-invariant models as well as some references to where these methods are described. Further references can be found in

[Bosley and Lees, 1972; Genesio and Milanese, 1976; Hickin and Sinha, 1980; Wortelboer, 1994; Fortuna *et al.*, 1992; Lilja, 1989].

Simplification of time-invariant models in general means reducing the order of the model. For a state-space model this means reducing the dimension of matrices and for a transfer function it means reducing the degree of the denominator polynomial. The amount of available methods for model order reduction is large and different ways to categorize them has been proposed. In [Skelton, 1980] the three categories parameter-optimization methods, polynomial approximation and component truncation are used and in [Wortelboer, 1994] the three categories norm-minimizing methods, parameter-matching methods and projection methods are used. In this paper we use the following two categories:

- Heuristic methods
- Optimization methods

Optimization methods try to minimize an error criteria, for example a norm, while the heuristic methods are based on other ideas such as for example preserving system theoretical properties. Note that a priori computable error bounds may or may not exist for methods in both these categories.

Heuristic methods These methods typically try to preserve some important parameters of the original model. Typical parameters could be poles, zeros, static gain, Markov parameters or the first parameters of a series expansion.

Reduction based on preserving some poles or eigenvalues is called modal reduction. The connection between different modal reduction methods are described in [Bonvin and Mellichamp, 1982]. Some of the advantages of these methods are that stability properties are preserved and that the reduced-order models are obtained by simple calculations. One of the first methods was the dominant pole method. In this method very fast poles are neglected. Another method is the pole-zero cancellation method. This method suggests that poles and zeros close to each other may be cancelled. Consider the transfer function

$$G(s) = \hat{G}(s)\left(\frac{s+a}{s+a+\varepsilon}\right).$$

The relative error is given by

$$\sup_{\omega}\left|\frac{G(i\omega)-\hat{G}(i\omega)}{G(i\omega)}\right| = \left|\frac{\varepsilon}{\operatorname{Re} a}\right|.$$

Thus, poles and zeros close to the stability boundary should, in general, not be cancelled.

Cancellation problems occur in various situations. Many of these could be avoided. The following example illustrates how the problem occurs in the Control System Toolbox 4.1 of Matlab.

EXAMPLE 2—CANCELLATION PROBLEM IN MATLAB
Assume that the controller and process together form the transfer function

$$G(s) = \frac{(s+1)}{(s+2)(s+3)}.$$

Computing the closed-loop model in Matlab in two different ways gives

$$\frac{G(s)}{1+G(s)} = \frac{s+1}{s^2+5s+6} \cdot \frac{s^2+5s+6}{s^2+6s+7} = \frac{s^3+6s^2+11s+6}{s^4+11s^3+43s^2+71s+42}$$
$$\frac{1}{1+1/G(s)} = \frac{s+1}{s^2+6s+7}.$$

As a consequence we either need to think carefully when we write the expressions or use some reduction method to remove common poles and zeros. □

Eigenvalue preserving methods for state-space models can be found in [Davison, 1966; Marshall, 1966]. These methods are based on diagonalizing the system matrix so that each state corresponds to an eigenvalue. The least important states/modes are then neglected. In the method by Davison the least important states are eliminated by assuming that they are zero, while in the method by Marshall the derivatives of the less important states are assumed to be zero. Selection of eigenvalues to retain are discussed in [Gopal and Mehta, 1982; Skelton, 1980] and references therein. In the early 80's a method called selective modal analysis was introduced. This method, described in [Pérez-Arriaga *et al.*, 1990], uses what is called participation factors to select the least important states of a model. The method has shown to be particularly useful for simplification of large models of electrical power systems.

Some heuristic methods are based on expanding the original model as well as the parameterized approximation into a series of terms and then matching some of the terms. Note that the eigenvalue preserving method [Davison, 1966], discussed above, can be interpreted as truncation of the partial fraction expansion of the transfer function. The Padé-approximation developed in [Padé, 1892] can also be used to approximate a transfer function. The advantage of the Padé approximation is that it can be applied not only to finite order models.

EXAMPLE 3—PADÉ APPROXIMATION
Consider the time-delay

$$G(s) = e^{-sT}$$

and assume that we would like to obtain a first order approximation

$$\hat{G}(s) = \frac{b}{s+a}.$$

The parameters are found from

$$(s+a)e^{-sT} = (s+a)(1 - sT + \frac{(sT)^2}{2!} + \cdots) \approx b$$

using the coefficients of degree zero and one. As a result we find that $a = b = 1/T$ giving

$$\hat{G}(s) = \frac{1}{sT+1}.$$

□

The idea that one could reduce a model by matching lower time-moments (the first Markov parameters) of the impulse response was suggested in the paper [Paynter, 1956]. Truncation of the continued fraction expansion of the transfer function was suggested in [Chen and Shieh, 1968]. These methods do in their original forms not guarantee that stability is preserved. Further developments have been done to cope with this problem. One example for the continued fraction truncation method is the Routh approximation in [Hutton and Friedland, 1975] generalized in [Shamash, 1975]. A huge amount of papers were written on model order reduction using polynomial approximation in the late seventies and early eighties. Many nonsense papers were published including [Gutman *et al.*, 1982], where a transfer function was simplified by differentiating both the numerator and denominator separately.

Model matching methods are important and subject to continued research. The main reason for this is that they often are the only option for very large models. A recent thesis on model matching is [Grimme, 1997].

Another heuristic idea is based on observability, controllability and the closely related minimality property. Moore wrote in his famous paper [Moore, 1981]: *There is a gap between minimal realization theory and the problem of finding a lower order approximation, which we shall refer to as the "model reduction problem".* One of the problems pointed out was that

the controllable and observable subspaces are sensitive to changes in the state-space model parameters. This was called structural instability. To cope with this problem he introduced the "internally balanced realization". This realization was obtained using principal component analysis and an algorithm for computation of singular value decompositions. A similar realization called the "principal axis realization" had been presented earlier in [Mullis and Roberts, 1976].

Any continuous time model

$$\begin{aligned} \frac{dx(t)}{dt} &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$

that is stable can be transformed, by a change of coordinates, into a balanced realization. A realization satisfying the Lyapunov equalities

$$\begin{aligned} A\Sigma + \Sigma A^T + BB^T &= 0 \\ \Sigma A + A^T\Sigma + C^TC &= 0 \end{aligned}$$

where

$$\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_r) > 0$$

is called balanced. For a discrete time model

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned}$$

the Lyapunov equalities are

$$\begin{aligned} A\Sigma A^T + \Sigma + BB^T &= 0 \\ A^T\Sigma A + \Sigma + C^TC &= 0. \end{aligned}$$

The most noteworthy property of balanced realizations is that the states are equally controllable and observable from an energy perspective. This was used by Moore to suggest that the least observable/controllable states of the balanced realization could be truncated. When doing so he stated that asymptotic stability "generically" was preserved as well as the balancing property. However, no formal assumptions that guarantee asymptotic stability and minimality of the truncated realization were given. This

was instead done in the paper [Pernebo and Silverman, 1982]. This paper shows that the truncated realization is asymptotically stable and minimal if Σ_1, the part of $\Sigma = \text{diag}(\Sigma_1, \Sigma_2)$ corresponding to the truncated states, and Σ_2 does not have any entries in common. An example shows that the truncated model otherwise may be non-asymptotically stable and non-minimal. The paper also suggests that parts corresponding to the same Hankel singular value should be treated together. The discrete time case is also treated in the paper.

Truncation of the least significant states of a balanced realization does not in general preserve the static gain. To overcome this problem singular perturbation approximation, [Kokotović *et al.*, 1976], was applied to the continuous time balanced realization in [Fernando and Nicholson, 1982c]. It was shown that the singular perturbation approximation is balanced and that it preserves the static gain. Generalized singular perturbation approximation were used in [Fernando and Nicholson, 1982b]. It was shown that it is possible to choose the frequency where the error between the original model and the simplified model should be zero. The previously used singular perturbation approximation corresponds to choosing this frequency to zero and truncation to letting the frequency go to infinity. In the paper [Fernando and Nicholson, 1982a] it was shown that truncation in the discrete time case, in contrast to the continuous time case, does not result in a balanced realization. However, the singular perturbation approximation was shown to be balanced, minimal and asymptotically stable.

A good introduction to balanced truncation and singular perturbation approximation can be found in [Green and Limebeer, 1995].

Optimization methods These methods try to optimize the quality of the simplified model. The quality can, as discussed in Section 2, be measured in different ways.

Some methods try to minimize the difference between a time or frequency response of the original and simplified model. The difference is measured by some signal norm (or loss function) for a chosen input signal. Typical input signals are an impulses, a step, a sinusoidal or white noise. A typical signal norm is the integrated squared error, that is, the two-norm. Matching parameters for a given input-signal when the signals are given as a data-sequence is a problem considered in model identification. Such methods can therefore be applied if we first simulate the high-order model and then identify the parameters of a low order model based on the obtained data. This was discussed in Section 3. Another approach is to minimize the difference between the responses of the original and simplified models without an intermediate simulation step. Numerical methods are then needed for solving the nonlinear equations defining

the necessary conditions for the optimal solution. These conditions when the input signal is white noise or an impulse are given in [Meier and Luenberger, 1967; Wilson, 1974]. It was shown in [Hyland and Bernstein, 1985] that the optimal solution could be characterized in a simpler way using a projection method. It was also shown that balanced truncation is close to one of the local extrema, but unfortunately the local extremum may be a maximum, which was shown by an example. Projection methods for model reduction in general has been studied in [Wortelboer, 1994]. A characterization of the optimal solution in discrete time can be found in [Aplevich, 1973]. Most optimization methods suffer from numerical problems such as choice of initial starting point, convergence and local minima.

Minimizing the integrated squared output error for a white noise input signal, as in for example [Hyland and Bernstein, 1985], is in fact the same as minimizing the H_2-norm. This naturally leads us to methods that try to minimize a system norm such as the H_∞-norm and the previously mentioned H_2-norm. Unfortunately straight forward solutions to these problems have not been found.

A characterization of the optimal solution in the H_∞-norm case as well as a suboptimal computational algorithm is given in [Kavranoglu and Bettayeb, 1994]. An ad hoc approach to the optimization problem, based on identification methods, was given in [Kavranoglu and Al-Amer, 1996]. The method uses the following iterative scheme

- Simulation: $y^k = Gu^k$
- LS-identification: $u^k, y^k \rightarrow G^k$
- Simulation: $u^{k+1} = (G - G^k)u^k$

where G is the original model and G^k the k-th low order approximation. The simulated signals u^k and y^k are vectors containing the values of the input and output at the considered time-instants.

Another system norm is the Hankel norm. It measures the gain from past input energy to future output energy. The value of this norm lies between the H_2-norm and the H_∞-norm. The Hankel norm is considered less attractive for comparison of models but has the advantage that a closed-form solution to the optimal approximation problem exists. The optimal Hankel-norm approximation for SISO-systems was developed in [Adamjan *et al.*, 1971], based on the results in [Nehari, 1957]. Work on deriving a closed-form optimal solution in the MIMO case was done in the papers [Kung and Lin, 1981; Silverman and Bettayeb, 1980]. The characterization of all optimal Hankel-norm approximations where then given in the paper [Glover, 1984]. There it was also shown that the optimal Hankel-norm approximation had an error that was bounded in the H_∞-norm. The error bound is $2\sigma_{\hat{r}+1} + \cdots + 2\sigma_r$, where the Hankel singular

values σ_k corresponding to the truncated states are added. In addition it was shown that for any optimal Hankel-norm approximation there exists a direct-term D that will make the H_∞-norm of the error bounded by $\sigma_{\hat{r}+1} + \cdots + \sigma_r$.

These results were applied to truncation of the balanced realization, [Glover, 1984]. It was shown, which independently also was shown in [Enns, 1984] but using other methods, that the H_∞-norm of the error is bounded by $2\sigma_{\hat{r}+1} + \cdots + 2\sigma_r$ for Balanced truncation. Glover also showed that equal singular values need only be added once. A similar error bound exists for both truncation and singular perturbation approximation of discrete time models, see [Al-Saggaf and Franklin, 1987]. The remaining error bound, that for singular perturbation approximation of continuous time models, were then finally presented in [Liu and Anderson, 1989] where a summary of the four cases also is given. The property that equal singular values need only be counted once also in the discrete time case was shown in [Hinrichsen and Pritchard, 1990].

A good introduction to balanced model reduction and Hankel norm approximation can be found in [Green and Limebeer, 1995].

Simplification of models with uncertainty

The choice of modeling framework restricts the type of behaviors that can be described. Mathematical models describing only the linear part of a system may, in more difficult design problems, result in control loops with low performance or even instability. Independent of which parts of a system we choose to describe there will still always be imperfections in the model. This has lead to the development of modeling frameworks where also the uncertainty is modeled, see for example [Zhou *et al.*, 1996]. Simplification of such models has only recently been considered.

The available methods reduces the number of uncertainties and the number of repeated entries of each uncertainty. One of these methods, [Wang *et al.*, 1991; Beck *et al.*, 1996; Beck, 1996], is closely related to balanced truncation and is useful for simplification of discrete time models where the uncertainties are norm bounded. This method has been generalized in [Andersson *et al.*, 1999b], where the uncertainties are described using integral quadratic constraints, which includes for example norm bounds. The paper considers both discrete time and continuous time models. These methods have associated error bounds similar to the one for standard Balanced truncation. Another method with guaranteed error bounds is presented in [Helmersson, 1995] and is based on μ-analysis. The mentioned methods so far are based on solutions to linear matrix inequalities (LMIs). A method based on singular value decomposition is presented in [Andersson *et al.*, 1999a]. This method reduces the number of repeated entries in an efficient manner. However, there is no guaranteed

bound on the resulting error.

Results on minimal descriptions of models with uncertainty can be found in in [Beck and Doyle, 1996; Cockburn and Morton, 1997; Beck and D'Andrea, 1997].

Simplification of unstable models

Simplification of unstable models can be difficult and occurs when we simplify models without closed-loop considerations.

One of the problems is that special attention has to be payed to initial conditions.

EXAMPLE 4—COMPARING UNSTABLE MODELS
Assume that

$$\frac{dx}{dt} = -x + u, \quad \text{and} \quad \frac{d\hat{x}}{dt} = -\hat{x} + u$$

with $x(0) = x_0$ and $\hat{x}(0) = \hat{x}_0$ then the error for $u = 0$ is given by

$$x - \hat{x} = (x_0 - \hat{x}_0)e^t.$$

Thus $|x - \hat{x}| \to \infty$ as $t \to \infty$ if $x_0 \neq \hat{x}_0$. □

There are some different approaches to deal with unstable process models. One approach is to extract the unstable part and then only simplify the stable part, as suggested in [Enns, 1984]. Two disadvantages with this approach are that dividing the model into a stable and unstable part may be numerically ill-conditioned and that no information about the unstable part is used in the reduction.

Another approach for unstable transfer functions is to use coprime factorizations $G(s) = N(s)D^{-1}(s)$ where N and D are stable transfer matrices and then simplify each of them separately, see [Liu and Anderson, 1986]. In [Meyer, 1988] it was suggested that a normalized coprime factorization should be used.

Simplification of interconnected models

Most technical systems consist of interconnected sub-systems. Control systems usually consist of a (multi-variable) controller and a process. The process itself consists of sub-systems. It is for such systems natural to model each sub-system as well as the interconnections separately.

The sub-models of the process are often put together to form a composite process model. In analysis and simulations it is natural to include both the controller and the process in a composite model for the whole control system.

The resulting composite model is often complex and needs to be simplified. There are some different approaches to this problem. One approach is to simplify groups of sub-models into larger but fewer sub-models. This is called aggregation. Other approaches are to simplify each sub-model on its own, called modular simplification or to consider the composite model and simplify it as a whole, called global simplification.

The approaches have their advantages and disadvantages. If the composite model is very large and complex, as for power systems and chemical plants, it may be difficult or even impossible to do a global simplification while it may be easy to simplify the sub-models. Another advantage of simplifying sub-models is that the structure of the model is left unchanged. This can be important since the sub-models often model a physical component or a sub-system of the process. If such a component is replaced or changed then it will be easy to replace the old sub-model by a new one to obtain a new composite model.

Simplifying a sub-model and at the same time taking all the other sub-models as well as the interconnections into account is a difficult problem. Therefore each sub-model is usually simplified without considering the whole model. This results in more conservative models than necessary. The reason is that the signals for which a considered sub-model should be accurate are determined by all the interconnected sub-models; for the feedback interconnection of a controller and a process it will be seen in a later section that the process model should be accurate close to the cross-over frequency.

Simplification of large scale models have been considered in many publications, see [Vaz and Davison, 1990], and references therein. The cost of different components of linear systems are discussed in [Skelton, 1980]. Weakly coupled sub-systems are considered using singular perturbations in [Kokotović, 1981].

4. Model simplification in controller design

It is often desirable to design low-order controllers. One reason for this is that the computer code should be easy to implement and maintain. Another reason may be limited computational speed. This reason is less significant today when computers are both fast and cheap. The difficulty in obtaining low-order controllers occurs when the design is based on high-order process models since controller design based on high-order models often results in high-order controllers. For example LQG-design results in controllers of the same order as the process model and H^∞-design usually results in controllers of higher order than the process model, see for example [Green and Limebeer, 1995; Anderson and Moore, 1989].

In this section we assume that both the controller and process are linear and time-invariant and give a brief summary of available results for obtaining low-order controllers. Further details can be found in [Anderson and Liu, 1989; Zhou *et al.*, 1996; Wortelboer, 1994] and references therein. Also the theses [Enns, 1984; Wahlberg, 1987] are valuable to read.

The approaches to obtain low-order controllers are illustrated in Figure 2. The indirect approaches divides the problem into two separate steps; design and reduction. Either a high-order controller is designed followed by a controller reduction or a low-order model is obtained followed by a low-order design. The direct approaches combine design and reduction in one step, to obtain an optimal low-order controller based directly on the high-order process model. The three approaches will be discussed in separate sections below.

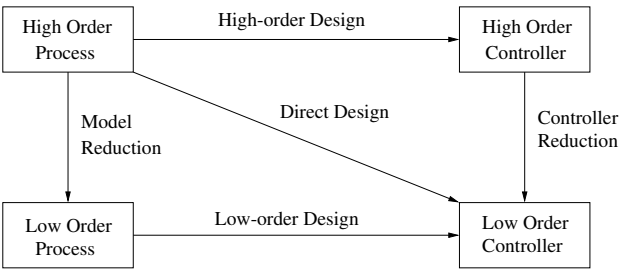

Figure 2. Approaches to obtain low-order controllers based on high-order process models.

Indirect design; model reduction followed by low-order design

The traditional approach is to first reduce the order of the process model and then do the controller design based on the obtained low-order process model. This is a simple method, where the reduction methods in Section 3, such as balanced truncation, can be used to obtain the low-order model. One disadvantage is that the process model may be unstable; order reduction of unstable models is less attractive. The major disadvantage with this approach, however, is that the effects on the closed-loop are neglected. As a result the obtained low-order controller may result in a closed-loop system with low performance or even instability. To avoid such problems one has to check closed-loop stability and performance a posteriori.

The main problem is that the closed-loop approximation error depends not only on the open-loop error but also on the controller to be designed. This makes model simplification for control a difficult problem.

Let $G(s) = G_r(s)G_p(s)$ where $G_r(s)$ is the controller model and $G_p(s)$ the process model, then the closed loop model at the frequency ω is given

by

$$\frac{G(i\omega)}{1+G(i\omega)} \approx \begin{cases} 1, & |G(i\omega)| >> 1 \\ G(i\omega), & |G(i\omega)| << 1. \end{cases}$$

This shows that high accuracy of the process model is less important at frequencies where $|G(i\omega)|$ is far from one. Frequencies where $G(i\omega) \approx -1$ are, however, very important. If the accuracy of the process model is low at such frequencies, then the designed controller, based on this process model, may result in an unstable closed-loop system. To cope with low model accuracy it may be possible to design controllers resulting in control systems with large gain and phase margins at the expense of lower bandwidth.

For a typical control design $|G(i\omega)|$ should be high at low frequencies, to attenuate disturbances and fulfill command response specifications, and low at high frequencies to attenuate sensor noise and satisfy robustness specifications. The frequencies where $|G(i\omega)| \approx 1$ are frequencies close to the cross-over frequency and the bandwidth of the closed-loop system. Note, however, that the cross-over frequency and bandwidth not are known until the controller has been designed.

The relation between the open-loop model and the closed-loop model, means that a small open-loop error at some frequencies may result in a large closed-loop error, while the same open-loop error at another frequency results in a small closed-loop error. We illustrate this with two examples taken from [Skelton, 1989]. First we show that arbitrarily small modeling errors may result in arbitrarily bad closed-loop performance.

EXAMPLE 5—SENSITIVE MODELING PROBLEM
Consider the system

$$G(s) = \frac{1}{(s+1)(s\varepsilon+1)}$$

controlled using a P-controller. The closed loop system will then be given by

$$G_{cl}(s) = \frac{K/\varepsilon}{s^2 + (1+1/\varepsilon)s + (K+1)/\varepsilon} = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2},$$

where $\varepsilon > 0$ is small. The parameter $\zeta \geq 0$ is the damping of the closed loop system. When ζ is close to zero then the system is oscillative and when $\zeta \geq 1$ then the system is well damped. We, thus, see that using a large value on K the closed loop system gets arbitrarily oscillative.

Now assume that the design is based on the model

$$\hat{G}(s) = \frac{1}{s+1},$$

which has a small error at all frequencies,

$$\|G - \hat{G}\|_\infty < \varepsilon.$$

The closed loop model using a P-controller is

$$\hat{G}_{cl}(s) = \frac{K}{s+K+1}.$$

The closed loop model will therefore be well damped for all values on K. □

We now show that large open-loop modeling errors do not necessarily lead to large closed-loop errors.

EXAMPLE 6—ROBUST MODELING PROBLEM
Consider the system

$$G(s) = \frac{1}{s+1},$$

controlled using a P-controller. The closed loop system will then be given by

$$G_{cl}(s) = \frac{K}{s+K+1}.$$

Now assume that the design is based on the model

$$\hat{G}(s) = \frac{1}{s},$$

which has an infinite approximation error at low frequencies. The closed loop model using a P-controller is

$$G_{cl}(s) = \frac{K}{s+K}.$$

The closed loop model will therefore be close to the closed loop system when K is large. □

One reduction method where closed-loop effects are considered, generalizes the dominant pole and pole-zero cancellation ideas. It considers the amplitude-diagram of the Bode-plot to determine which break-points to neglect. Poles and zeros close to each other are then ignored as well as break-points where the open-loop gain is very high or very low.

EXAMPLE 7—NEGLECTING BREAK-POINTS
The process transfer function

$$G(s) = \frac{s+1}{(s+0.5)(s+30)(s+900)} \approx \frac{1}{900} \cdot \frac{1}{s+30}$$

if the closed loop bandwidth will be much higher than 1 rad/s and much lower than 900 rad/s. □

The main concern in control design is robustness and if possible good performance. These problems leads to weighted sensitivity problems, see [Doyle *et al.*, 1992] and references therein. These ideas were used in [Enns, 1984] as a motivation for frequency weighted model reduction.

Assume that $G(s)$ is the open-loop transfer function of the controller and process, and that $\hat{G}(s)$ is a simplified model with the same number of unstable poles. Note that this includes both process model reduction and controller reduction. The closed-loop transfer functions are

$$T = G(I+G)^{-1} \quad \text{and} \quad \hat{T} = \hat{G}(I+\hat{G})^{-1}$$

respectively, and the sensitivity functions are $S = I - T$ and $\hat{S} = I - \hat{T}$.

Assuming that $G(s)$ results in a stable closed-loop model, then $\hat{G}(s)$ results in a stable closed-loop model if either

$$\|(G-\hat{G})(I+G)^{-1}\|_\infty < 1, \quad \text{or} \quad \|(I+G)^{-1}(G-\hat{G})\|_\infty < 1.$$

This means that if stability is the only concern then we should minimize a frequency weighted error $\|(G-\hat{G})W_i\|_\infty$ or $\|W_o(G-\hat{G})\|_\infty$ and not for example $\|G-\hat{G}\|_\infty$.

If $G(s)$ not only results in a stable closed-loop model but also achieves the desired performance, that is, has a desirable closed-loop transfer function T, then good performance using $\hat{G}$ is obtained if $\|T-\hat{T}\|_\infty$ is small. Neglecting second order terms in $G-\hat{G}$ we get

$$\|T-\hat{T}\|_\infty = \|S-\hat{S}\|_\infty \approx \|(I+G)^{-1}(G-\hat{G})(I+G)^{-1}\|_\infty,$$

which means that we should minimize a frequency weighted error

$$\|W_o(G-\hat{G})W_i\|_\infty.$$

The above results show that a suitable reduced-order process model is obtained if we minimize

$$\|W_o(G_p-\hat{G}_p)W_i\|_\infty,$$

where

$$W_o = (I + G_p G_r)^{-1}, \quad \text{and} \quad W_i = G_r (I + G_p G_r)^{-1}.$$

Unfortunately these weights are not known since the high-order controller G_r is not known in the model simplification step. Choosing suitable weights a priori is still an open problem in model reduction.

Frequency weighted model reduction was introduced in [Enns, 1984], where balanced truncation was applied. Unfortunately no a priori error bounds on the weighted error are available for this method. An alternative balancing approach with a priori bounds on the weighted error was given in [Al-Saggaf and Franklin, 1988]. A frequency weighted method using Hankel norm approximation was considered in [Latham and Anderson, 1985]. A bound, for this method, on the unweighted error was given in [Anderson, 1986]. However, both the approximation error and the error bound may be conservative. Generalizations of both the balancing and Hankel norm methods were given in [Zhou, 1993]. An alternative Hankel approximation for frequency weighted model reduction was given in [Glover *et al.*, 1992] and is based on rewriting the problem as a so called four-block problem. Further details on this method can be found in [Green and Limebeer, 1995].

A method called balanced stochastic truncation, which is based on balancing, was proposed in [Desai and Pal, 1984]. The reduced-order model obtained using this method satisfies upper bounds on two frequency weighted errors. An upper bound for the case with $W_i = I$ and $W_o = G_p^{-1}$, that is, the relative error, was found in [Green, 1988]. The same upper bound for the case with $W_i = I$ and $W_o = \hat{G}_p^{-1}$, that is, the multiplicative error, was presented in [Wang and Safonov, 1992]. Another relative error reduction method was proposed in [Zhou, 1995]. Explicit error bounds for both the relative and multiplicative error was given. Further references and details can be found in [Zhou *et al.*, 1996].

There has also been an interest in defining measures for the distance between models, in such a way that it reflects the closeness of the corresponding closed-loop models. Examples of this is the graph-metric, gap-metric and the ν-gap metric, see [Zames and El-Sakkary, 1980; El-Sakkary, 1981; Vidyasagar, 1984; Vinnicombe, 1998]. Reduction methods are available see for example [Vinnicombe, 1998].

Indirect design; high-order design followed by controller reduction

Another approach is to design a high-order controller based on the high-order process model and then reduce the controller. A major disadvantage with this approach is that we have to do a high-order controller design, which may be difficult when the order of the process model is very high.

If, however, we can design a high-order controller, then more information about the closed-loop will be available which may be used to improve the quality of the reduced-order controller.

The most straight forward idea used is to design a high-order controller using some design method and then reduce the order of the controller using any of the reduction methods in Section 3. The advantage with this approach, compared to first reducing the model and then do the controller design, is that the reduction is done at a later step in the design procedure. In this step some of the closed-loop properties have been captured by the designed high-order controller to be reduced. However, the main problem still remains that we can not assure that the closed-loop system will be stable using the reduced-order controller.

The frequency weighting methods previously discussed can be used to further take advantage of the available information about the closed-loop. When the high-order controller is available, better approximations of the weights used for stability preservation and performance preservation are available.

The selection of optimal weights is an open problem. In [Lenz *et al.*, 1988; Goddard and Glover, 1998] weights as well as a bound on the weighted error are obtained based on the allowed loss of H_∞-performance. Any reduced-order controller satisfying the weighted error bound can then be used.

Another problem is reduction of unstable controllers. To cope with this problem only the stable part of the controller was suggested to be reduced in [Enns, 1984], while the paper [Liu and Anderson, 1986] suggests that the controller should be rewritten using stable coprime factors which subsequently were reduced. The paper [Liu and Anderson, 1986] considered LQG-designed controllers where the coprime factors were reduced using balancing techniques. The use of frequency weighting methods for this approach was presented in [Liu *et al.*, 1990].

Instead of only basing the controller reduction on the high-order controller and the process model it is also possible to use information from the design method. The LQG-design method was considered in [Jonckheere and Silverman, 1983]. The solution to the Riccati equations was balanced and numbers associated with each state were computed. These numbers, denoted LQG-characteristic values, indicate the importance of each state for filtering and control. States of the controller (or process) corresponding to small numbers are discarded. A similar method for H_∞-design was presented in [Mustafa and Glover, 1991]. In this case a bound on the performance degradation was presented based on the H_∞-characteristic values. A similar result, also based on coprime factorizations and providing a priori bounds on the performance degradation, can be found in [McFarlane *et al.*, 1990].

A disadvantage with these methods is that there may be a low-order controller of the same dimension which results in better closed-loop performance than a low-order controller approximating the high-order controller.

Direct design

In the direct design a low-order controller is designed based directly on the high-order process model.

When the controller structure is fixed then optimization methods have been used to find controller parameters that minimizes some closed-loop performance measure. Optimization based on Lagrange multipliers have been used in the following two papers. In [Hyland and Bernstein, 1984] where the optimal fixed-order LQG-controller is described using coupled matrix equations; two Riccati equations and two Lyapunov equations and in [Bernstein and Haddad, 1989] where similar equations for H_∞-controllers, using four coupled Riccati equations, were given.

5. Conclusion

This paper has given an overview of model simplification in the area of control engineering. Important concepts including quality measures and purpose formulations have been presented. Similarities and differences between model simplification, model reduction, and model approximation have been discussed.

The paper considered two different purposes of model simplification, namely simplification for analysis and simplification for design. In the latter case it was pointed out that the accuracy of the process model is important at frequencies close to the bandwidth of the closed-loop system.

References to simplification methods for different classes of models have been given. Special attention was given to linear time-invariant models.

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MODEL COMPARISON AND SIMPLIFICATION

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SUMMARY

In this paper we consider comparison and simplification of dynamical models. These models may contain non-linearities as well as uncertainty, where both are described using Integral Quadratic Constraints (IQCs). The proposed method includes simplification by truncation and singular perturbation approximation as special cases. The simplification error is defined in terms of the $\mathbf{L}_2$-induced gain. It is shown that each non-linear or uncertain system component can be assigned a positive value, computable by convex optimization, such that the simplification error is always bounded by the sum of these values corresponding to the simplified components. Copyright © 1999 John Wiley & Sons, Ltd.

Key words: model simplification; error bounds; uncertain models; integral quadratic constraints

1. INTRODUCTION

In modern robust control design it is common to model both the dominant system dynamics as well as system uncertainty, for example, noise, disturbances, parameter variations, non-dominant non-linearities and uncertain dynamics. This often results in models that have high state order and complicated uncertainty descriptions, thus these models may be difficult to analyse and the subsequent controller design may be both difficult and computationally expensive. The resulting controller is likely to be even more complex and may therefore be expensive and difficult to implement as well. For these reasons there is a need to develop systematic methods to simplify the descriptions of the non-linearity and uncertainty in the model as well as the state order.

For linear time-invariant models without uncertainty there are well-known order reduction methods. Two of these methods are balanced truncation, see References 1, 2, and singular perturbation approximation, see References 3–5, which have associated error bounds, see References 6–9. The balanced truncation method has been generalized to models with norm-bounded uncertainty, see References 10–12.

In this paper we show that similar bounds can be obtained for comparison and simplification of a general class of dynamical models. The proposed method includes simplification by truncation and singular perturbation approximation as special cases. The models may contain both non-linearities and uncertainty, and are described using integral quadratic constraints. This

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latter framework was outlined in Reference 13 and further developed in References 14, 15, and leads to noticeably simpler proofs.

The paper is organized as follows. We start, in Section 2, by describing the modelling framework and stating both the comparison problem and the closely related simplification problem. In Section 3 some previously obtained results are reviewed, including balanced truncation and singular perturbation approximation of standard state-space models, and truncation of uncertain models. These results are generalized in Section 4 based on the use of Integral Quadratic Constraints (IQCs). In Section 5 the connection between the preceding results and those obtained in this paper are examined; non-linear systems are discussed separately in Section 6. Section 7 contains some concluding remarks.

2. PRELIMINARIES

In this section we describe the modelling framework and state both the model comparison and model simplification problem. The results hold both in discrete and continuous time, but will, for convenience of presentation, in general be stated for continuous time models.

2.1. Notation

We let $\mathbf{L}_2^p[0, \infty)$ denote the space of square integrable vector signals of dimension p, with corresponding inner product defined as

$$\langle v, w \rangle = \int_0^\infty v(t)^T w(t)\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} v(\mathrm{i}\omega)^* w(\mathrm{i}\omega)\,\mathrm{d}\omega$$

where the last equality is the well-known Parseval formula. Note that the same notation is used for time and frequency signals. The signal norm is defined as $\|v\| = \sqrt{\langle v, w \rangle}$.

An operator on $\mathbf{L}_2^n[0, \infty)$ is a map $\Delta : \mathbf{L}_2^n[0, \infty) \to \mathbf{L}_2^n[0, \infty)$; the operator gain is given by the induced norm

$$\|\Delta\| = \sup_{\substack{z \in \mathbf{L}_2^n[0,\,\infty) \\ z \neq 0}} \frac{\|\Delta z\|}{\|z\|}$$

When Δ is linear, we use Δ^* to denote the adjoint operator of Δ defined by

$$\langle \Delta z_1, z_2 \rangle = \langle z_1, \Delta^* z_2 \rangle$$

Although most results are given in continuous time, similar results also hold in discrete time using the signal space $l_2^p[0, \infty)$ with inner product

$$\langle v, w \rangle = \sum_{k=0}^{\infty} v_k^T w_k$$

2.2. Model description

The modelling framework considered in this paper includes a number of well-known model structures such as state-space models and Linear Fractional Transformations (LFTs).

The model is defined by the interconnection of a pair (Δ, M) according to the relations

$$x = \Delta z$$

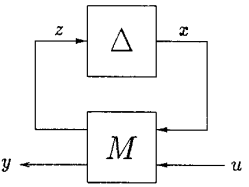

Figure 1. Feedback interconnection representing a model with non-linearities and uncertainty

and

$$\begin{bmatrix} z \\ y \end{bmatrix} = M \begin{bmatrix} x \\ u \end{bmatrix}$$

as illustrated in Figure 1. We let Δ represent non-linearities, uncertainty and dynamic elements, while M represents linear time-invariant dynamics.

Typically, M is the nominal system model, which is often assumed to consist of a transfer matrix including model weighting functions. However, as will be seen in Section 3, it is often advantageous to let M be a constant matrix. The delay or integral operator is in this case included in Δ, and M will no longer be a nominal model. Throughout the paper, we will use M to denote both constant matrices and transfer function matrices, where the interpretation should be clear from the context.

The operator Δ is assumed to have a block diagonal structure

$$\Delta = \text{diag}(\Delta_1, \ldots, \Delta_r)$$

where each of the blocks satisfies a constraint, e.g. a norm bound or more generally an integral quadratic constraint, but otherwise, the specific operator Δ need not be known. Thus Δ is particularly useful for describing uncertainty.

The signals in the interconnection are the input $u \in \mathbf{L}_2^m[0, \infty)$, the output $y \in \mathbf{L}_2^p[0, \infty)$ and the internal signals $z \in \mathbf{L}_2^n[0, \infty)$ and $x \in \mathbf{L}_2^n[0, \infty)$.

The transfer matrix M is partitioned consistently with the signal dimensions as

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

The input–output mapping of the interconnection is then defined by the Redheffer star product

$$y = (\Delta \bigstar M)u = [D + C\Delta(I - A\Delta)^{-1}B]u$$

For simplicity we assume that the model is causal, well-defined and stable. A model (Δ, M) is *well-posed* if the operator $I - A\Delta$ is causally invertible on $\mathbf{L}_2^n[0, \infty)$. If in addition $(I - A\Delta)^{-1}$ is bounded then the model is *stable*.

Example 1 (Uncertain spring constant)

A mass and spring system can be described using the equation

$$m\ddot{\xi} + d\dot{\xi} + k\xi = F$$

where ξ is the position, F an external force, m the mass, d the damping and k the spring constant. Assume that the spring constant is known with only 10 per cent accuracy. This may be written as

$$k = k_0(1 + 0{\cdot}1\delta)$$

where k_0 is the nominal spring constant and δ a real unknown constant satisfying $|\delta| \leqslant 1$. The equation then becomes

$$m\ddot{\xi} + d\dot{\xi} + k_0\xi = F - 0{\cdot}1k_0\delta\xi$$

The described model, with input F and output ξ, can be written on standard form using

$$M(s) = \frac{1}{ms^2 + ds + k_0}\begin{bmatrix} -0{\cdot}1k_0 & 1 \\ -0{\cdot}1k_0 & 1 \end{bmatrix}$$

$$\Delta = \delta$$

2.3. *Problem formulation*

In this paper we study model comparison and simplification. The comparison problem defined below is interesting, since it can be viewed as the basic problem in well-known reduction methods.

Consider comparison of two models (Δ, M) and $(\hat{\Delta}, M)$ where

$$M = \begin{bmatrix} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ C_1 & C_2 & D \end{bmatrix}, \qquad \Delta = \begin{bmatrix} \Delta_{\mathrm{U}} & 0 \\ 0 & \Delta_{\mathrm{L}} \end{bmatrix}, \qquad \hat{\Delta} = \begin{bmatrix} \Delta_{\mathrm{U}} & 0 \\ 0 & \hat{\Delta}_{\mathrm{L}} \end{bmatrix},$$

as illustrated in Figure 2. We assume that the upper blocks in Δ and $\hat{\Delta}$, denoted $\Delta_{\mathrm{U}} = \mathrm{diag}(\Delta_1, \ldots, \Delta_{\hat{r}})$, are identical while the lower blocks denoted $\Delta_{\mathrm{L}} = \mathrm{diag}(\Delta_{\hat{r}+1}, \ldots, \Delta_r)$ and $\hat{\Delta}_{\mathrm{L}} = \mathrm{diag}(\hat{\Delta}_{\hat{r}+1}, \ldots, \hat{\Delta}_r)$, respectively, are different.

To compare the two models we consider the output difference

$$e = y - \hat{y} = (\Delta \bigstar M)u - (\hat{\Delta} \bigstar M)u$$

or more precisely the norm

$$\| \Delta \bigstar M - \hat{\Delta} \bigstar M \|$$

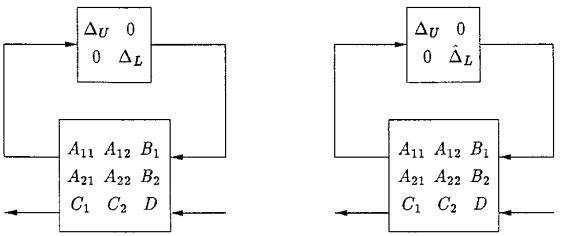

Figure 2. Two models for comparison

The purpose of this paper is to find upper bounds on this difference between models in the described framework.

The implications of such bounds are that they can be used to analyse the importance of model components, such as uncertainties and non-linearities. Both closed-loop models and open-loop models can be analysed. The bounds can be used to indicate less important components of a model. Note, however, that less important components of an open-loop process model may have significant importance in closed loop, see e.g. Reference 16.

The above-described comparison of models may be used for model simplification. The second model, in this case, is considered to be a simplification of the first model and is obtained by replacing Δ_L with a fixed transfer matrix $\hat{\Delta}_L$. One result of this simplification is that the matrix dimension of both Δ and M may be reduced. This follows by observing that

$$\begin{bmatrix} \Delta_U & 0 \\ 0 & \hat{\Delta}_L \end{bmatrix} \star M = \Delta_U \star \hat{M} \tag{1}$$

where

$$\hat{M} = \begin{pmatrix} A_{11} & B_1 \\ C_1 & D \end{pmatrix} + \begin{pmatrix} A_{12} \\ C_2 \end{pmatrix} \hat{\Delta}_L (I - A_{22}\hat{\Delta}_L)^{-1} (A_{21} \quad B_2)$$

It is crucial for this reduction that $\hat{\Delta}_L$ be a fixed operator, and not a set of operators, so that $\hat{M}$ is a fixed operator. In some cases it is also useful to restrict M to be constant; to maintain this property after reduction, $\hat{\Delta}_L$ must also be a constant matrix.

Note that in general, we will not give any guidance among the possible selections for fixed $\hat{\Delta}_L$.

3. RELATED RESULTS

In this section we review known results that are relevant to the new results presented in this paper.

3.1. Balanced model reduction

Continuous time state-space models

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{2}$$

with signals in $\mathbf{L}_2[0, \infty)$ can be modelled in the aforementioned framework using

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad \Delta = s^{-1} I_n$$

where s^{-1} is the transfer function of the integral operator. Two well-known reduction methods for stable-space models are balanced truncation, see References 1, 2, 6, 7, and singular perturbation approximation, see References 3–5.

The elimination of states via truncation is accomplished by setting the fixed matrix $\hat{\Delta}_L = 0$ in formula (1) to obtain

$$\hat{M} = \begin{bmatrix} A_{11} & B_1 \\ C_1 & D \end{bmatrix}, \qquad \Delta_U = s^{-1} I_{\hat{n}}$$

and via singular perturbation approximation by letting $\hat{\Delta}_{\mathrm{L}} \to \infty \cdot I$ to obtain

$$\hat{M} = \begin{bmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & B_1 - A_{12}A_{22}^{-1}B_2 \\ C_1 - C_2A_{22}^{-1}A_{21} & D - C_2A_{22}^{-1}B_2 \end{bmatrix}, \qquad \Delta_{\mathrm{U}} = s^{-1}I_{\hat{n}}$$

The reduced order models obtained by truncation have no error at very high frequencies, while the singular perturbation approximations have no error at zero frequency (stationarity).

In both of these methods it is assumed that the realization is balanced, i.e. satisfies the two equalities

$$A\Sigma + \Sigma A^{\mathrm{T}} + BB^{\mathrm{T}} = 0$$

$$\Sigma A + A^{\mathrm{T}}\Sigma + C^{\mathrm{T}}C = 0$$

for some real diagonal matrix

$$\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n) > 0$$

This is not a major restriction since every stable minimal state-space model can be transformed into a balanced realization.

The most noteworthy property of balanced realizations is that the states are equally controllable and observable from an energy perspective. The values of the elements in Σ give a measure of the controllability and observability of the states. Small-valued elements in Σ indicate states with little importance and vice versa. The Σ-matrix is therefore an important guide for selecting which of the states to eliminate. If we let $G(s)$ be the transfer function of the original model and $\hat{G}(s)$ be the transfer of the reduced order model then an upper bound on the error is given by

$$\| \Delta \bigstar M - \hat{\Delta} \bigstar M \| = \| G(s) - \hat{G}(s) \|_{\infty} \leqslant \sum 2\sigma_k$$

where the sum is taken over the eliminated states.

Similar results exist for discrete time models, see References 8, 9 in addition to the previous references. A discrete time model can be described as

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad \text{and} \quad \Delta = z^{-1}I_n$$

where z^{-1} is the transfer function of the delay operator, and the Lyapunov equations are

$$A\Sigma A^{\mathrm{T}} - \Sigma + BB^{\mathrm{T}} = 0$$

$$A^{\mathrm{T}}\Sigma A - \Sigma + C^{\mathrm{T}}C = 0$$

Balanced truncation is obtained using $\hat{\Delta}_{\mathrm{L}} = 0$, as discussed before, while the singular perturbation approximation,

$$\hat{M} = \begin{bmatrix} A_{11} + A_{12}(I - A_{22})^{-1}A_{21} & B_1 + A_{12}(I - A_{22})^{-1}B_2 \\ C_1 + C_2(I - A_{22})^{-1}A_{21} & D + C_2(I - A_{22})^{-1}B_2 \end{bmatrix}, \qquad \Delta_{\mathrm{U}} = z^{-1}I_{\hat{n}}$$

is obtained using $\hat{\Delta}_{\mathrm{L}} = I$.

Finally, we remind the reader that there exist many other results for simplification of linear time-invariant models in addition to those discussed above.

3.2. Model reduction of uncertain models

For models where M is a constant matrix and Δ consists of repeated[†] linear scalar operators,

$$\Delta = \mathrm{diag}(\delta_1 I_{n_1}, \ldots, \delta_r I_{n_r})$$

satisfying $\|\delta_k\| \leqslant 1$ for $k = 1, \ldots, r$, a truncation method similar to balanced truncation has been developed, see References 10–12. This type of model may typically be used to describe discrete-time systems with norm-bounded uncertainty. The linear time-invariant dynamics are modelled by letting one of the δ_k's be the delay operator.

The Lyapunov equalities are replaced by Lyapunov inequalities

$$\begin{aligned} A\Sigma A^{\mathrm{T}} - \Sigma + BB^{\mathrm{T}} &< 0 \\ A^{\mathrm{T}}\Sigma A - \Sigma + C^{\mathrm{T}}C &< 0 \end{aligned} \tag{3}$$

There are many realizations of a given model for which there exist a $\Sigma > 0$ such that these inequalities hold. The Σ-matrix for a given realization is also non-unique. The error bounds are calculated as before but with the sum taken over all truncated elements in Δ instead of just eliminated states. To obtain low error bounds we have to find an equivalent realization as well as a Σ so that Σ has small elements. This is not a convex optimization problem, but a co-ordinate transformation which results in a realization with a suboptimal Σ will be described below.

Note that when we replace the Lyapunov equalities by inequalities, we can no longer make direct controllability or observability interpretations.

3.3. Co-ordinate transformations

Both the described methods depend strongly on co-ordinate transformations. The error bounds for balanced truncation only hold if the realization is balanced. For uncertain models error bounds may, in some cases, be obtained without co-ordinate transformations but are then, in general, conservative. Note that in the main result of this paper, we will not utilize allowable co-ordinate transformations. This will be discussed later.

Co-ordinate transformations from internal signals $\bar{x}$ and $\bar{z}$ to internal signals x and z are described by

$$x = T\bar{x}, \qquad z = T\bar{z}$$

where T is invertible and commutes with Δ, i.e. $T\Delta T^{-1} = \Delta$. This type of transformations only changes the internal signals and not the input–output relation. The transformation also transforms

$$\bar{M} = \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & D \end{bmatrix} \text{ into } M = \begin{bmatrix} T\bar{A}T^{-1} & T\bar{B} \\ \bar{C}T^{-1} & D \end{bmatrix}$$

but does not affect Δ. If we consider a model

$$\bar{M} = \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & D \end{bmatrix}, \qquad \Delta = z^{-1} I_n$$

[†] Each of the operators in a repeated operator $\delta_k I_{n_k}$ have the same input–output behaviour.

which is not balanced, then a balanced realization can be obtained using a co-ordinate transformation. The matrix T is found in the following way. Solve for $P = P^{\mathrm{T}} > 0$ and $Q = Q^{\mathrm{T}} > 0$ satisfying the Lyapunov equations

$$\bar{A}P\bar{A}^{\mathrm{T}} - P + \bar{B}\bar{B}^{\mathrm{T}} = 0$$

$$\bar{A}^{\mathrm{T}}Q\bar{A} - Q + \bar{C}^{\mathrm{T}}\bar{C} = 0$$

Then use the Cholesky factorization $Q = Q_1^{\mathrm{T}}Q_1$ and the singular value decomposition $Q_1PQ_1^{\mathrm{T}} = U\Sigma^2U^{\mathrm{T}}$ here $U^{\mathrm{T}}U = I$. The transformation matrix is then given by $T = \Sigma^{-1/2}U^{\mathrm{T}}Q_1$.

If we assume that we have a stable uncertain model

$$\bar{M} = \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & D \end{bmatrix}, \qquad \Delta = \mathrm{diag}(\delta_1 I_{n_1}, \ \dots, \delta_r I_{n_r})$$

then a transformation matrix resulting in a suboptimal matrix Σ is obtained in the following way, see Reference 11. Solve for block diagonal matrices

$$P = \mathrm{diag}(P_1, \ \dots, P_r) \quad \text{where } P_k \in \mathbf{R}^{n_k \times n_k}$$

$$Q = \mathrm{diag}(Q_1, \ \dots, Q_r) \quad \text{where } Q_k \in \mathbf{R}^{n_k \times n_k}$$

with minimal traces[‡] such that the Lyapunov inequalities

$$\bar{A}P\bar{A}^{\mathrm{T}} - P + \bar{B}\bar{B}^{\mathrm{T}} < 0$$

$$\bar{A}^{\mathrm{T}}Q\bar{A} - Q + \bar{C}^{\mathrm{T}}\bar{C} < 0$$

hold. Each minimization is a convex problem involving Linear Matrix Inequalities (LMIs). Efficient algorithms for solving such problems have been developed,[17] and reliable software packages exist, see for example Reference 18. After finding P and Q the co-ordinate transformation

$$T = \mathrm{diag}(T_1, \ \dots, T_r), \qquad T_k \in \mathbf{R}^{n_k \times n_k}$$

is obtained using the Cholesky factorizations $Q_k = Q_{k(1)}^{\mathrm{T}}Q_{k(1)}$, the singular value decompositions $Q_{k(1)}P_kQ_{k(1)}^{\mathrm{T}} = U_k\Sigma_k^2U_k^{\mathrm{T}}$ where $U_k^{\mathrm{T}}U_k = I_{n_k}$, and the formula $T_k = \Sigma_k^{-1/2}U_k^{\mathrm{T}}Q_{k(1)}$. It then holds that $\Sigma = \mathrm{diag}(\Sigma_1, \ \dots, \Sigma_r)$ satisfies inequalities (3) for the new realization. Note that the block structure of the co-ordinate transformation is consistent with the structure of Δ.

The example below illustrates the model reduction procedure, including a co-ordinate transformation, of an uncertain model.

[‡] The trace of a matrix is the sum of its diagonal elements.

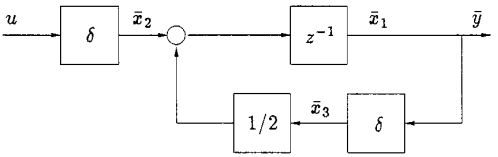

Figure 3. The model in Example 2

Example 2 (Minimal realizations)

Consider the model in Figure 3 with norm bounded uncertainty $\|\delta\| \leqslant 1$. This model is written on the form $(\Delta, \bar{M})$ using

$$\bar{M} = \begin{bmatrix} 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \qquad \Delta = \mathrm{diag}(z^{-1}, \delta, \delta)$$

A co-ordinate transformation obtained using the algorithm described previously is given by

$$T = \begin{bmatrix} 1{\cdot}0000 & 0 & 0 \\ 0 & 1{\cdot}0000 & 0{\cdot}5000 \\ 0 & -0{\cdot}0097 & 0{\cdot}0146 \end{bmatrix}$$

This transformation does not change the input–output properties of the model since the latter two elements in Δ are equivalent. We obtain

$$M = \begin{bmatrix} 0 & 1{\cdot}0000 & 0{\cdot}0000 & 0 \\ 0{\cdot}5000 & 0 & 0 & 1{\cdot}0000 \\ 0{\cdot}0146 & 0 & 0 & -0{\cdot}0097 \\ 1{\cdot}0000 & 0 & 0 & 0 \end{bmatrix}$$

and Δ as before. This new model satisfies inequalities (3) with

$$\Sigma = \mathrm{diag}(1{\cdot}3333, 1{\cdot}3333, 0{\cdot}0004)$$

This shows that truncation of the last entry in δ will result in an error less than $2\sigma_3 = 0{\cdot}0008$, which is negligible.

The exact values of the elements in the matrix $\hat{M}$ and the corresponding value of σ_3 depends on the numerical precision of the solver. The value of σ_3 is smaller for solvers of higher precision. A zero value cannot be obtained unless the inequalities are replaced by non-strict ones.

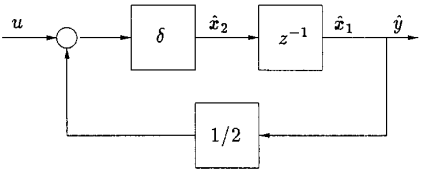

Figure 4. A minimal description of the model in Example 2

The reduced dimension model is, with the five digit precision used in this example, identical to the one obtained for $\sigma_3 = 0$,

$$\hat{M} = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \qquad \Delta_U = \begin{bmatrix} z^{-1} & 0 \\ 0 & \delta \end{bmatrix}$$

This model shown in Figure 4 has the same input–output properties as the original model in Figure 3. The original model may therefore be called non-minimal. Minimality for models with uncertainty is studied in References 12, 19–21.

4. COMPARISON AND SIMPLIFICATION USING IQCs

The results discussed in the previous section provided us with error bounds for simplification of some commonly used models. However, error bound results for uncertain models of continuous time systems are missing, as well as error bounds for singular perturbation approximation of uncertain models. Also, the error bounds for uncertain models may be unnecessarily conservative if, e.g. we have uncertain constant parameters since the bounds are constructed for any linear operator, including time-varying operators.

In this section we provide the main results needed to solve these remaining problems. We first introduce Integral Quadratic Constraints (IQCs) as a general framework to describe the operator Δ. This framework includes norm-bounds and restrictions on the time-variation as special cases. Based on these constraints we then obtain bounds on the difference between two models, useful for comparison and simplification of a large variety of models. The obtained bounds have a similar form to those obtained in the previous section and hold for a general class of simplification methods that includes truncation and singular perturbation approximation of uncertain models.

At the end of the section we describe numerical computations and some properties of the method.

4.1. Integral quadratic constraints

As a general framework for describing non-linearities, and system uncertainty, we use integral quadratic constraints, see References 13, 14. This framework includes a number of well-known constraints such as passivity and norm bounds.

Let $\Pi : i\mathbf{R} \to \mathbf{C}^{2n \times 2n}$ be a bounded measurable[§] function, taking Hermitian values. The operator Δ is said to *satisfy the integral quadratic constraint defined by the multiplier* Π if

$$\int_{-\infty}^{\infty} \begin{bmatrix} z(i\omega) \\ x(i\omega) \end{bmatrix}^* \Pi(i\omega) \begin{bmatrix} z(i\omega) \\ x(i\omega) \end{bmatrix} d\omega \geqslant 0 \tag{4}$$

for all vectors z, $x = \Delta z \in \mathbf{L}_2^n[0, \infty)$.

It will be shown later that the multipliers Π play a fundamental role in computing upper bounds on the difference between two models.

The same Δ may satisfy integral quadratic constraints defined by many different multipliers. The particular choice of multiplier will influence the bound. In order to obtain less conservative bounds it is therefore important to choose the multiplier properly. Restricting the set of multipliers beforehand may therefore result in more conservative bounds than necessary.

The following example illustrates that the more information we have about the operator Δ the more freedom we have in the choice of multiplier.

Example 3 (Multipliers)

- Let Δ be any operator with gain (induced two-norm) less than one. Then Δ satisfies all integral quadratic constraints defined by

$$\begin{bmatrix} xI & 0 \\ 0 & -xI \end{bmatrix}$$

  where $x \geqslant 0$ is a constant number.
- Let Δ be any linear time-invariant operator with gain (H_∞ norm) less than one. Then Δ satisfies all integral quadratic constrains defined by

$$\begin{bmatrix} x(i\omega)I & 0 \\ 0 & -x(i\omega)I \end{bmatrix}$$

  where $x(i\omega) \geqslant 0$ is a bounded measurable function.
- Let Δ be defined as multiplication with a time-varying real scalar with absolute value less than one. Then Δ satisfies all integral quadratic constraints defined by

$$\begin{bmatrix} X & Y \\ Y^{\mathrm{T}} & -X \end{bmatrix}$$

  where $X = X^{\mathrm{T}} \geqslant 0$ and $Y = -Y^{\mathrm{T}}$ are real matrices.
- Let Δ be defined as multiplication with a constant real scalar with absolute value less than one. Then Δ satisfies all integral quadratic constraints defined by

$$\begin{bmatrix} X(i\omega) & Y(i\omega) \\ Y(i\omega)^* & -X(i\omega) \end{bmatrix}$$

  where $X(i\omega) = X(i\omega)^* \geqslant 0$ and $Y(i\omega) = -Y(i\omega)^*$ are bounded measurable matrix functions.

[§] For a definition and review of measurable functions, see Reference 22.

Below are two useful properties of integral quadratic constraints.

P1. Assume that Δ satisfies the integral quadratic constraints defined by $\Pi_1, \ldots, \Pi_r$ then Δ also satisfies the integral quadratic constraint defined by

$$\sum_{k=1}^{r} \alpha_k \Pi_k$$

for every $\alpha_k \geqslant 0$, $k = 1, \ldots, r$.

P2. Assume that Δ has a block diagonal structure,

$$\Delta = \mathrm{diag}(\Delta_1, \ldots, \Delta_r)$$

and that Δ_k satisfies the integral quadratic constraint defined by Π_k for $k = 1, \ldots, r$. Then Δ satisfies the integral quadratic constraint defined by

$$\Pi = \mathrm{daug}(\Pi_1, \ldots, \Pi_r) = \left[\begin{array}{ccc|ccc} \Pi_{1(11)} & & 0 & \Pi_{1(12)} & & 0 \\ & \ddots & & & \ddots & \\ 0 & & \Pi_{r(11)} & 0 & & \Pi_{r(12)} \\ \hline \Pi_{1(21)} & & 0 & \Pi_{1(22)} & & 0 \\ & \ddots & & & \ddots & \\ 0 & & \Pi_{r(21)} & 0 & & \Pi_{r(22)} \end{array}\right]$$

where

$$\Pi_k = \begin{bmatrix} \Pi_{k(11)} & \Pi_{k(12)} \\ \Pi_{k(21)} & \Pi_{k(22)} \end{bmatrix}$$

4.2. Bounds on the difference

Upper bounds on the difference between two models with Δ described by integral quadratic constraints are given in the following theorem. This theorem is the main result of the paper.

Theorem 1

Assume that both (Δ, M) and $(\hat{\Delta}, M)$ are stable and that the operators

$$\Delta = \mathrm{diag}(\Delta_1, \ldots, \Delta_r)$$

$$\Delta = \mathrm{diag}(\Delta_1, \ldots, \Delta_{\hat{r}}, \hat{\Delta}_{\hat{r}+1}, \ldots, \hat{\Delta}_r)$$

are linear. Also assume that Δ_k satisfies the integral quadratic constraint defined by Π_k for $k = 1, \ldots, r$, and that $\hat{\Delta}_k$ satisfies the integral quadratic constraint defined by Π_k for $k = \hat{r} + 1, \ldots, r$. Collect the Π_k matrices in $\Pi = \mathrm{daug}(\Pi_1, \ldots, \Pi_r)$ and assume that

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \Pi \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} \leqslant \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \quad \text{for all } \omega \in [0, \infty] \tag{5}$$

If there exists a real matrix $\Sigma = \mathrm{diag}(\sigma_1 I_{n_1}, \ldots, \sigma_r I_{n_r}) > 0$ such that

$$\begin{bmatrix} A \\ I \end{bmatrix}^* \begin{bmatrix} \Sigma^2 & 0 \\ 0 & \Sigma^2 \end{bmatrix} \Pi \begin{bmatrix} A \\ I \end{bmatrix} + C^*C \leqslant 0 \quad \text{for all } \omega \in [0, \infty] \tag{6}$$

then

$$\| \Delta \bigstar M - \hat{\Delta} \bigstar M \| \leqslant 2\sigma_{\hat{r}+1} + \cdots + 2\sigma_r$$

Proof. See Appendix. □

Note that the inequalities are symmetric since Π commutes with the matrix $\mathrm{diag}(\Sigma^2, \Sigma^2)$.

The interpretation of this theorem is that if a subset of the operators Δ_k are replaced by different, possibly fixed, operators, denoted by $\hat{\Delta}_k$ and satisfying the same integral quadratic constraints, then the difference is bounded by the sum of the positive σ_k-values corresponding to the replaced operators. The non-unique Σ matrix is found as a solution to inequalities (5) and (6).

The theorem is applicable to uncertain models since it is sufficient that Δ and $\hat{\Delta}$ belong to the same set described by integral quadratic constraints. It is, thus, not necessary to know the specific operators Δ and $\hat{\Delta}$. Possible selections for the replacing operators $\hat{\Delta}_k$ are $\hat{\Delta}_k = 0$, corresponding to truncation and $\hat{\Delta}_k \to \infty I$, corresponding to singular perturbation approximation.

For practical reasons the non-strict inequalities (5) and (6) are usually replaced by strict inequalities. One reason for this is that many numerical routines are developed for solving strict inequalities. Another reason is that stability of the original and simplified model essentially follows if either of the strict inequalities holds. We refer to References 14, 15 for details on proving stability using integral quadratic constraints.

Equal σ_k-values in the bound need only be added once, as shown by the following arguments: If there are equal σ_k-values then the corresponding blocks can be collected in one block and the corresponding Π_k matrices collected in one larger Π_k matrix. The σ_k-value corresponding to this larger Δ_k is the same as for the smaller ones.

4.3. Remarks

The blocks in Δ play a fundamental role for the bounds obtained using Theorem 1. It is important that we capture as much as possible the behaviour of each block Δ_k. This is done by considering as many candidates for the corresponding multiplier Π_k as possible and selecting the one resulting in the smallest Σ.

Each block is described separately. This does not allow us to describe interdependence between blocks, e.g. a physical parameter appearing in more than one block. We therefore expect the bounds to be more conservative if such interdependence exists. Also, whole blocks must be eliminated. No bounds are available for elimination of parts of a block.

If there are related blocks in Δ, then we may collect these blocks in one larger block and describe the interdependence using integral quadratic constraints. This will most likely result is less conservative bounds, but we then only obtain bounds for elimination of all of these blocks at the same time, and not for elimination of separate blocks.

Another possibility is to apply co-ordinate transformations a priori, such that the interdependence between the blocks in the new model has little effect on the bounds. Since co-ordinate transformations only change the internal signals and leave the input–output relation

unchanged, this seems to be a good idea. However, finding such co-ordinate transformations are in general difficult and may also have to be avoided if the internal description is important. An example of the later is when the blocks in Δ are components with a physical interpretation. Transformations must then be avoided if we would like to maintain the physical interpretation, or if we would like to know the importance of a physical component.

4.4. Numerical computations

The problem of finding upper bounds on the difference has been rewritten in Theorem 1 as a problem of finding values σ_k that satisfy inequalities (5) and (6). The scalars σ_k satisfying these inequalities are not unique, thus it is useful to introduce an optimization criterion. Since the values of σ_k should be small, in order to obtain low bounds on the differences, one may choose to minimize the trace of Σ.

There are many matrices Π defining integral quadratic constraints satisfied by both Δ and $\hat{\Delta}$, and which thus could be used. Assume that we select one of these a priori, then the inequalities become linear matrix inequalities in Σ^2. A matrix Σ^2 with minimal trace can then be found numerically using for example the LMI control toolbox.[18]

The obtainable values on σ_k, and thus the bound, depend strongly on the choice of Π. The above-described approach does therefore in general not seem reasonable. Instead we must consider as many candidates for Π as possible. These candidates are typically described using linear matrix inequalities, as seen in Example 3. The numerical algorithm should minimize the trace of Σ not only *w.r.t.* $\Sigma > 0$ but also *w.r.t.* Π. The resulting problem is in general non-convex and difficult to solve. The following suboptimal two-step algorithm is therefore proposed. The inequalities are strict for numerical reasons and should be satisfied for $\omega \in [0, \infty]$.

Step 1: Let Σ_0 be an initial guess for Σ. Find an allowable $\Pi(i\omega)$ that minimizes γ under the constraints

$$\gamma^2 \begin{bmatrix} A \\ I \end{bmatrix}^* \begin{bmatrix} \Sigma_0^2 & 0 \\ 0 & \Sigma_0^2 \end{bmatrix} \Pi \begin{bmatrix} A \\ I \end{bmatrix}^* + C^*C < 0$$

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \Pi \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} < \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$$

Step 2: Let Π be the solution obtained in Step 1. Find Σ that minimizes $\operatorname{tr}\Sigma^2$ under the constraint

$$\begin{bmatrix} A \\ I \end{bmatrix}^* \begin{bmatrix} \Sigma^2 & 0 \\ 0 & \Sigma^2 \end{bmatrix} \Pi \begin{bmatrix} A \\ I \end{bmatrix} + C^*C < 0$$

Note that this algorithm may end up in local minima. The initial choice Σ_0 is therefore important. In the case where Δ only contains one block, it is easy to verify that the algorithm will result in the best possible value on σ, satisfying the inequalities.

The algorithm may be modified. It is, e.g. possible to use different weights for different blocks instead of minimizing the trace of Σ^2. It is also possible to use the algorithm iteratively, with Σ obtained in Step 2 replacing Σ_0 in Step 1 when a new iteration begins.

Note that the inequalities in general are frequency dependent. It is then possible to apply the Kalman–Yakubovich–Popov lemma to obtain frequency-independent inequalities. Further

complications are due to the optimization over the allowable Π-matrices. These are in general transfer functions in an infinite dimensional space. A parameterization of a subspace of these transfer functions is therefore needed. Numerical solutions to similar problems have been studied in Reference 15.

5. SPECIAL CASES

In this section we first consider models where Δ represents norm-based uncertainty, and has a repeated scalar structure. Error bounds for truncation of such models were presented in Section 3 and we will show that the same error bound follows from Theorem 1, but then also is proven to hold for additional simplification methods. One of these methods may be thought of as singular perturbation approximation. Similar results are also obtained for models where Δ represents passive uncertainty.

5.1. Uncertain norm bounded components

Consider the uncertain model described in Section 3. Based on Theorem 1 we prove the error bound presented there, and at the same time show that this bound holds for simplification methods other than truncation. This is summarized in the following corollary.

Corollary 1

Let M be a constant matrix and let

$$\Delta = \mathrm{diag}(\delta_1 I_{n_1}, \ldots, \delta_r I_{n_r})$$

$$\Delta = \mathrm{diag}(\delta_1 I_{n_1}, \ldots, \delta_{\hat{r}} I_{n_{\hat{r}}}, \hat{\delta}_{\hat{r}+1} I_{n_{\hat{r}+1}}, \ldots, \hat{\delta}_r I_{n_r})$$

be linear operators such that $\|\delta_k\| \leqslant 1$ for $k = 1, \ldots, r$ and $\|\hat{\delta}_k\| \leqslant 1$ for $k = \hat{r}+1, \ldots, r$.

If there exists a real matrix $\Sigma = \mathrm{diag}(\sigma_1 I_{n_1}, \ldots, \sigma_r I_{n_r}) > 0$, such that

$$\begin{aligned} A\Sigma A^* - \Sigma + BB^* &< 0 \\ A^*\Sigma A - \Sigma + C^*C &< 0 \end{aligned} \tag{7}$$

then

$$\|\Delta \star M - \hat{\Delta} \star M\| \leqslant 2\sigma_{\hat{r}+1} + \cdots + 2\sigma_r$$

Proof. The multiplier

$$\Pi = \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & -\Sigma^{-1} \end{bmatrix}$$

defines an integral quadratic constraint that is satisfied for any Δ with the above structure. Using this multiplier we find that inequality (6) simplifies to $A^*\Sigma A - \Sigma + C^*C \leqslant 0$ and inequality (5) can be simplified in the following way:

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & -\Sigma^{-1} \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} \leqslant \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} A^* \\ B^* \end{bmatrix} \Sigma^{-1} [A \quad B] \leqslant \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & I \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} \Sigma^{1/2} A^* \Sigma^{-1/2} \\ B^* \Sigma^{-1/2} \end{bmatrix} [\Sigma^{-1/2} A \Sigma^{1/2} \quad \Sigma^{-1/2} B] \leqslant \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\Leftrightarrow [\Sigma^{-1/2} A \Sigma^{1/2} \quad \Sigma^{-1/2} B] \begin{bmatrix} \Sigma^{1/2} A^* \Sigma^{-1/2} \\ B^* \Sigma^{-1/2} \end{bmatrix} \leqslant I$$

$$\Leftrightarrow A\Sigma A^* - \Sigma + BB^* \leqslant 0$$

The stability assumption in Theorem 1 is obtained by replacing the non-strict inequalities with strict inequalities. This may be shown using stability theory based on integral quadratic constraints but has also been proven elsewhere, see e.g. Reference 11. This completes the proof. □

The positive number σ_k gives a measure on the importance of the repeated scalar block $\delta_k I_{n_k}$. The difference between the original model and the model where this block is replaced by $\hat{\delta}_k I_{n_k}$ is bounded by $2\sigma_k$.

The bounds hold for a number of different simplified models. These models are obtained using different fixed operators, satisfying $\|\hat{\delta}_k\| \leqslant 1$. To obtain models where $\hat{M}$, obtained using formula (1), is a real constant matrix, we restrict the fixed operators $\hat{\delta}_{\hat{r}+1}, \ldots, \hat{\delta}_r$ to be constant scalars in the interval $[-1, 1]$. Note that the choice $\hat{\delta}_k = 0$ corresponds to truncation, i.e. $\hat{x}_k = 0$ and the choice $\hat{\delta}_k = 1$ corresponds to singular perturbation approximation, i.e. $\hat{x}_{k+1} = \hat{x}_k$.

The next example will illustrate that the actual difference depends on the fixed operator.

Example 4 (Reduction methods)

Consider a discrete time state-space model where $\Delta = z^{-1} I_3$ and

$$A = \begin{bmatrix} 0{\cdot}9638 & 0{\cdot}0784 & 0{\cdot}0082 \\ -0{\cdot}0784 & 0{\cdot}5261 & -0{\cdot}1824 \\ 0{\cdot}0082 & 0{\cdot}1824 & 0{\cdot}1602 \end{bmatrix}$$

$$B = \begin{bmatrix} 1{\cdot}3639 \\ 1{\cdot}3762 \\ -0{\cdot}1835 \end{bmatrix}, \quad C = [1{\cdot}3639 \quad -1{\cdot}3762 \quad -0{\cdot}1835]$$

This model has a zero at $z = 0$ and poles at $z = 0{\cdot}95$, $z = 0{\cdot}4$ and $z = 0{\cdot}3$.

Assume that we would like to find a first-order approximation of this model together with error bounds. We may then choose $\Delta = \text{diag}(z^{-1}, \hat{\delta}_2, \hat{\delta}_3)$ with $\hat{\delta}_2, \hat{\delta}_3 \in [-1, 1]$, and note that inequalities (7) are satisfied for

$$\Sigma = \text{diag}(26{\cdot}4, 2{\cdot}85, 0{\cdot}134)$$

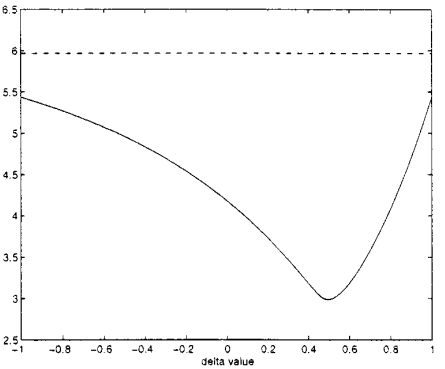

Figure 5. Actual error as a function of δ. The upper bound is shown in a dashed line

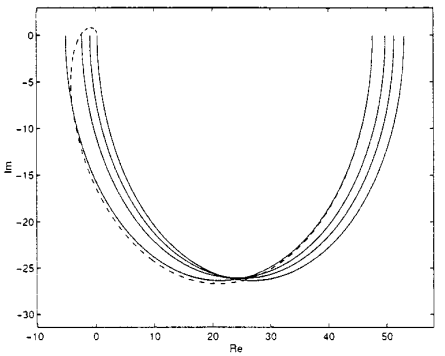

Figure 6. Nyquist plots for the simplified models obtained using $\delta = 1, 0{\cdot}5, 0, -1$ (shown in that order in solid lines), and the original model (dashed line)

so that the upper-bound is given by

$$\| \Delta \bigstar M - \hat{\Delta} \bigstar M \| \leqslant 2(\sigma_2 + \sigma_3) = 5{\cdot}97$$

Figure 5 shows the actual error in the case where $\hat{\delta}_2 = \hat{\delta}_3 = \delta$. It is seen that we obtain a smaller error by chosing $\delta = 0{\cdot}5$ than we do with truncation, $\delta = 0$, and singular perturbation approximation $\delta = 1$. This shows that the proposed general method may result in smaller error than that for either truncation or singular perturbation approximation. The choice for δ_k resulting in the smallest error, however, is not directly given by the method.

To see why the error is smallest for $\delta = 0{\cdot}5$, we consider the Nyquist plots in Figure 6. We see that singular perturbation approximation, i.e. $\delta = 1$, has no error at low frequencies (on the right side of the plot) but has a large error as $\omega \to \infty$. We also see that $\delta = 0{\cdot}5$ has a medium error for most frequencies, and thus not a large error at any frequency.

5.2. *Uncertain passive components*

Consider the case where the δ_k are passive operators. This case includes continuous time models with passive uncertainties. The following corollary shows that bounds on the differences are found easily for such models using Theorem 1.

Corollary 2

Let M be a constant matrix and let

$$\Delta = \mathrm{diag}(\delta_1 I_{n_1}, \ldots, \delta_r I_{n_r})$$
$$\Delta = \mathrm{diag}(\delta_1 I_{n_1}, \ldots, \delta_{\hat{r}} I_{n_{\hat{r}}}, \hat{\delta}_{\hat{r}+1} I_{n_{\hat{r}+1}}, \ldots, \hat{\delta}_r I_{n_r})$$

where $\delta_k + \delta_k^* \geqslant 0$ for $k = 1, \ldots, r$ and $\hat{\delta}_k + \hat{\delta}_k^* \geqslant 0$ for $k = \hat{r} + 1, \ldots, r$.

If there exists a real matrix $\Sigma = \mathrm{diag}(\sigma_1 I_{n_1}, \ldots, \sigma_r I_{n_r}) > 0$, such that

$$\begin{aligned} A\Sigma + \Sigma A^* + BB^* &< 0 \\ A^*\Sigma + \Sigma A + C^*C &< 0 \end{aligned} \tag{8}$$

then

$$\| \Delta \bigstar M - \hat{\Delta} \bigstar M \| \leqslant 2\sigma_{\hat{r}+1} + \cdots + 2\sigma_r$$

Remark

In this case we choose $\hat{\delta}_k \in [0, \infty)$ in order that $\hat{M}$ obtained using formula (1) be a constant matrix.

Proof. The multiplier

$$\Pi = \begin{bmatrix} 0 & \Sigma^{-1} \\ \Sigma^{-1} & 0 \end{bmatrix}$$

defines an integral quadratic constraint that is satisfied for every Δ with the above structure. Using this multiplier we find that inequality (5) becomes

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \begin{bmatrix} 0 & \Sigma^{-1} \\ \Sigma^{-1} & 0 \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -I \end{bmatrix} \leqslant 0$$

which is equivalent to

$$\Sigma A^* + A\Sigma + BB^* \leqslant 0$$

Furthermore, (6) becomes

$$A^*\Sigma + \Sigma A + C^*C \leqslant 0$$

It can be shown that stability follows if we use strict inequalities. Applying Theorem 1 completes the proof. □

5.3. *Remarks. Bilinear transformation*

The results in Sections 5.1 and 5.2 are related by a bilinear transformation. The bilinear transformation takes a norm-bounded operator Δ_n to a passive operator $\Delta_p = (I - \Delta_n)^{-1}(I + \Delta_n)$.

Since here $\Delta_p + \Delta_p^* \geqslant 0 \Leftrightarrow \|\Delta_n\| \leqslant 1$ we have been able to prove the passive case results using the norm-bounded results and vise versa.

It is interesting to note that the result for truncation of norm-bounded operators combined with the bilinear transformation are not sufficient to obtain the truncation result for passive operators. The reason is that truncation, corresponding to $\hat{\Delta} = 0$, for norm-bounded operators, does not correspond to truncation for passive operators.

5.4. Remarks. Co-ordinate transformations

The results in this section may be of little use without an a priori applied coordinate transformation, as illustrated by the following example.

Example 5

Consider a discrete time state space model with $\Delta = z^{-1} I_3$ and

$$A = \begin{bmatrix} 0{\cdot}3 & 0 & 0 \\ 1 & 0{\cdot}4 & 0 \\ 0 & 1 & 0{\cdot}95 \end{bmatrix}$$

$$B = \begin{bmatrix} 0{\cdot}3 \\ 1 \\ 0 \end{bmatrix}, \quad C = [0 \quad 0 \quad 1]$$

This model has a zero at $z = 0$ and poles at $z = 0{\cdot}95$, $z = 0{\cdot}4$ and $z = 0{\cdot}3$.

Applying Corollary 1 fails, since there does not exist any real matrix $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \sigma_3) > 0$ satisfying inequalities (7).

If the particular realization is not important then we can transform the model using the co-ordinate transformation described in Section 3. We then obtain the model in Example 4 which satisfies inequalities (7) for both

$$\Sigma = \mathrm{diag}(26{\cdot}4, 2{\cdot}85, 0{\cdot}134) \quad \text{and} \quad \Sigma = 41{\cdot}9 I_3$$

The first case formally corresponds to the case where Δ is assumed to consist of different operators and the second case when Δ is assumed to consist of one repeated scalar operator.

Note, that the error bound for reduction of all states are

$$2(26{\cdot}4 + 2{\cdot}85 + 0{\cdot}134) = 58{\cdot}7 \quad \text{and} \quad 2 \cdot 41{\cdot}9 = 83{\cdot}8$$

respectively. We therefore obtain a lower bound using $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \sigma_3)$ than when we use $\Sigma = \sigma I_3$. □

This example indicates that solutions to inequalities (7) may not exist or may result in very conservative bounds. This does not imply that we must use a priori applied co-ordinate transformations. The major problem instead is how Theorem 1 was applied to obtain Corollary 1. Considering the proof of the corollary we see that we have used the assumption that Δ satisfies the integral quadratic constraint defined by the multiplier

$$\Pi = \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & -\Sigma^{-1} \end{bmatrix}, \qquad \Sigma = \mathrm{diag}(\sigma_1 I_{n_1}, \ldots, \sigma_r I_{n_r}) > 0 \tag{9}$$

This means that we have selected one specific multiplier among a large number of possibilities. It does in fact hold that Δ satisfies any integral quadratic constraint defined by

$$\Pi = \begin{bmatrix} X & 0 \\ 0 & -X \end{bmatrix}, \qquad X = X^{\mathrm{T}} = \mathrm{diag}(X_1, \ldots, X_r) > 0 \tag{10}$$

where X is a real matrix. Note that $X = \Sigma^{-1}$ gives us the previous multiplier (9). One expects that the extra freedom can be exploited to improve the error bounds.

The additional freedom in multiplier (10) compared to the multiplier (9) has an interpretation. Consider the following proposition.

Proposition 1

Let Π have the form (10) and factor $X = T^*\Sigma^{-1}T$. The inequalities in Theorem 1 are then equivalent to

$$\bar{A}\Sigma\bar{A}^* - \Sigma + \bar{B}\bar{B}^* \leqslant 0$$
$$\bar{A}^*\Sigma\bar{A} - \Sigma + \bar{C}^*\bar{C} \leqslant 0$$

where

$$\bar{A} = TAT^{-1}, \qquad \bar{B} = TB, \qquad \bar{C} = CT^{-1}$$

and T is any invertible matrix of the form

$$T = \mathrm{diag}(T_1, \ldots, T_r)$$

Proof. Factor $X = T^*\Sigma^{-1}T$ and use the steps in the proof of Corollary 1. □

The proposition shows that the additional freedom consists of the allowable co-ordinate transformations. Thus, the given realization has no importance since co-ordinate transformations are captured by the inequalities. This statement is, however, only true if we have a method that finds an optimal solution to the inequalities. If we only can find suboptimal solutions, then these solutions may depend on the given realization.

Example 6

Return to Example 5 and consider simplification of the only repeated scalar block, $z^{-1}I_3$, in that model. Applying Theorem 1 using multiplier (10) to either of the two realizations in Example 5 we find $\Sigma = 41{\cdot}9 \cdot I_3$ as the optimal value. The corresponding error-bound is given by $2\sigma = 83{\cdot}8$. This is the same error bound as the one obtained in Example 5 after an a priori applied co-ordinate transformation.

Now assume that we only would like to consider elimination of some of the states in the model. Then we must consider the repeated delay operators as different operators. Then multiplier (10) must have $X = \mathrm{diag}(x_1, x_2, x_3)$. Applying Theorem 1 we find

$$\Sigma = \mathrm{diag}(22{\cdot}9, 45{\cdot}7, 34{\cdot}0)$$

The corresponding error bounds are expected to be conservative since the existing dependence between the blocks cannot be used. The error bound for reduction of all the states in this case will

be $2(\sigma_1 + \sigma_2 + \sigma_3) = 205$, which is much larger than when we considered the whole block at the same time.

Note, however, that the individual σ_k-values measure the importance of the states (delay operators) in the given realization. This should be compared to the σ_k-values obtained in Example 5 which measured the importance of the states in the transformed realization.

6. NON-LINEAR MODELS

The results presented in this paper, so far, assume that the operator Δ is linear. In this section we show how the results may be generalized to include non-linearities.

To extend the results to the non-linear case we require the following definitions. An operator Δ is *odd* if

$$\Delta(-z) = -\Delta z$$

An integral quadratic constraint is said to be *satisfied incrementally* if

$$\int_{-\infty}^{\infty} \begin{bmatrix} z_2 - z_1 \\ x_2 - x_1 \end{bmatrix}^* \Pi(\mathrm{i}\omega) \begin{bmatrix} z_2 - z_1 \\ x_2 - x_1 \end{bmatrix} \mathrm{d}\omega \geqslant 0$$

holds $\forall z_1, z_2, x_1 = \Delta z_1, x_2 = \Delta z_2 \in \mathbf{L}_2^n[0, \infty)$. These two properties are sufficient for the results to hold also in the non-linear case.

Theorem 2

Assume that all the assumptions in Theorem 1, except the linearity assumption, hold. Also assume that Δ_k is odd and satisfies the integral quadratic constraints incrementally for $k < r$. Then

$$\| \Delta \bigstar M - \hat{\Delta} \bigstar M \| \leqslant 2(\sigma_{\hat{r}+1} + \cdots + \sigma_r)$$

Proof. See Appendix.

Remark

Similar results also hold if Δ is non-causal.

The following example will illustrate what the incrementally condition means for specific static non-linearities.

Example 7 (Incrementality conditions)

Consider a static non-linearity in the sector $|\delta(z)| \leqslant |z|$, see Figure 7(a). This operator satisfies the integral quadratic constraint defined by

$$\Pi = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

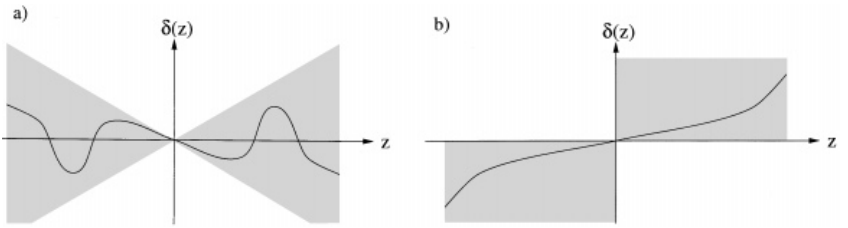

Figure 7. Two static non-linearities

The incrementality condition is satisfied if the *Lipschitz condition*

$$|\delta(z_2) - \delta(z_1)| \leqslant |z_2 - z_1|$$

is satisfied. A static non-linearity in the first and third quadrant

$$z \cdot \delta(z) \geqslant 0$$

see Figure 7(b), satisfies the integral quadratic constraint defined by

$$\Pi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The incrementality condition is satisfied if the *monotonicity condition*

$$[\delta(z_2) - \delta(z_1)] \cdot [z_2 - z_1] \geqslant 0$$

is satisfied.

The last example illustrates how the results can be applied to a non-linear example.

Example 8 (Dead zone non-linearity)

Consider the feedback connection in Figure 8, where the dead zone non-linearity $x = \delta(z)$ is defined by

$$\delta(z) = \begin{cases} 0, & |z| \leqslant 1 \\ z - 1, & z \geqslant 1 \\ z + 1, & z \leqslant -1 \end{cases}$$

Assume that the peak value of the output signal, $\|y\|_\infty$, is bounded,[¶] then the maximum gain of δ will be restricted to

$$K = \sup_{|y| \leqslant \|y\|_\infty} \frac{\delta(y)}{y} = \begin{cases} 0 & \|y\|_\infty \leqslant 1 \\ \dfrac{\|y\|_\infty - 1}{\|y\|_\infty}, & \text{otherwise} \end{cases}$$

[¶] A bounded value on $\|y\|_\infty$ exists if, e.g. the peak value on the input signal, $\|u\|_\infty$, is bounded and $\|G\|_{L_1} < 1$.

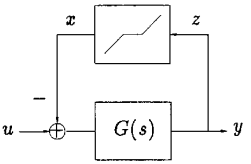

Figure 8. Feedback connection with a dead zone non-linearity

In the following derivation we assume, for simplicity, that $K \neq 0$ and $G(i\omega) \not\equiv 0$, but the final result will hold also in these cases. It holds that

$$[Ky - \delta(y)]\delta(y) \geqslant 0$$

or equivalently

$$\begin{bmatrix} y \\ \delta(y) \end{bmatrix}^{\mathrm{T}} \Pi \begin{bmatrix} y \\ \delta(y) \end{bmatrix} \geqslant 0$$

for any multiplier

$$\Pi = \gamma \begin{bmatrix} 0 & 1 \\ 1 & -2/K \end{bmatrix}, \qquad \gamma \in \mathbf{R}^+$$

Assume that the non-linear feedback is disconnected. An upper bound on the difference between this model and the original model can be obtained using Theorem 2. We let

$$M(s) = \begin{bmatrix} -G(s) & G(s) \\ -G(s) & G(s) \end{bmatrix}, \qquad \Delta = \delta$$

Note that δ need not satisfy any incrementality condition, since we truncate the only non-linear block appearing in Δ. Inequality (6) becomes equivalent to

$$\sigma^2 \geqslant \frac{1}{\gamma} \sup_{\omega} \frac{|G|^2}{1 + K\,Re\,G}$$

and inequality (5) becomes equivalent to the inequalities

$$\gamma \leqslant \inf_{\omega} 4\,\frac{1 + K\,Re\,G}{|G|^2 K^2}$$

and

$$1 + K\,Re\,G \geqslant 0$$

The last inequality replaced by a strict inequality is the stability condition for a static non-linearity in a sector $[0, K]$, obtained using the circle criterion, see, e.g. Reference 23. Combining the other two inequalities, with the largest value on γ and smallest value on σ, gives the bound

$$2\sigma = \sup_{\omega} \frac{|G|^2 K}{1 + K\,Re\,G}$$

This upper bound is smaller for smaller K or equivalently for smaller $\|y\|_\infty$ and is large close to equality in the stability inequality. Note that a more conservative bound on the difference is obtained if a smaller value on γ is used.

7. CONCLUSIONS

In this paper we have shown that error bounds can be found, using convex optimization, for a general class of dynamical models. The proposed method includes simplification by truncation and singular perturbation approximation. The models may contain both non-linearities and uncertainty, and are described using integral quadratic constraints. Special attention has been paid to compare the results with those in Reference 10 which are a special case of the results in this paper.

APPENDIX: PROOFS OF THEOREM 1 AND THEOREM 2

We start by proving the case $\hat{r}+1=r=2$. The proof in the case $\hat{r}=r=1$ is almost identical; ignore all parts containing index 1 and then replace index 2 with index 1. We partition the matrix M as

$$M=\begin{bmatrix} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ C_1 & C_2 & D \end{bmatrix}$$

Since we have two models (Δ, M) and $(\hat{\Delta}, M)$ we also have two feedback inter-connections and also two sets of signals. We introduce the following notation for the (frequency) signals:

$$\begin{bmatrix} z_1 \\ z_2 \\ y \end{bmatrix} = M \begin{bmatrix} x_1 \\ x_2 \\ u \end{bmatrix}, \qquad \begin{bmatrix} \hat{z}_1 \\ \hat{z}_2 \\ \hat{y} \end{bmatrix} = M \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ u \end{bmatrix}$$

We assume that the initial conditions are zero, which is realistic since the two models are assumed to be stable in a strict input–output sense. Note that the input signal u is the same for both models.

Multiplying inequality (5) with the vector $(x_1+\hat{x}_1, x_2+\hat{x}_2, 2u)$ from the right and its complex conjugate from the left, and adding this to the inequality obtained by multiplying inequality (6) with the column vector $\sigma_2^{-1}(x_1-\hat{x}_1, x_2-\hat{x}_2)$ from the right and its complex conjugate from the left, we then get the following inequality by noting that $z-\hat{z}=A(x-\hat{x})$, $y-\hat{y}=C(x-\hat{x})$ and using some matrix manipulations:

$$\begin{aligned} 0 \geqslant & \begin{bmatrix} z_1+\hat{z}_1 \\ x_1+\hat{x}_1 \end{bmatrix}^* \Pi_1 \begin{bmatrix} z_1+\hat{z}_1 \\ x_1+\hat{x}_1 \end{bmatrix} + \sigma_1^2\sigma_2^{-2} \begin{bmatrix} z_1-\hat{z}_1 \\ x_1-\hat{x}_1 \end{bmatrix}^* \Pi_1 \begin{bmatrix} z_1-\hat{z}_1 \\ x_1-\hat{x}_1 \end{bmatrix} \\ & + 2\begin{bmatrix} z_2 \\ x_2 \end{bmatrix}^* \Pi_2 \begin{bmatrix} z_2 \\ x_2 \end{bmatrix} + 2\begin{bmatrix} \hat{z}_2 \\ \hat{x}_2 \end{bmatrix}^* \Pi_2 \begin{bmatrix} \hat{z}_2 \\ \hat{x}_2 \end{bmatrix} + \sigma_2^{-2}|y-\hat{y}|^2 - 4|u|^2 \end{aligned} \tag{11}$$

Integrating the expression on the right-hand side of the inequality along the frequency axis causes the first four terms to become non-negative. In the linear case, the first and second term become non-negative since $\Delta_1=\hat{\Delta}_1$ satisfies the integral quadratic constraints defined by Π_1 and are linear. In the non-linear case this follows from the additional assumptions that the integral quadratic constraint is satisfied incrementally and that Δ_1, $\hat{\Delta}_1$ are odd. The third and fourth term become non-negative since Δ_2, $\hat{\Delta}_2$ satisfy the integral quadratic constraint defined by Π_2. This completes the proof for the case $\hat{r}+1=r=2$.

More generally one can generate a sequence of new models by replacing $\Delta_r, \ldots, \Delta_{\hat{r}+1}$ one at a time. At each step, the above argument can be used and the total difference becomes

$$\| y - \hat{y} \| \leqslant 2(\sigma_r + \cdots + \sigma_{\hat{r}+1}) \| u \|$$

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Paper 3

Frequency Dependent Error Bounds for Uncertain Linear Models

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Abstract

In this paper we study frequency dependent error bounds for approximation and truncation of linear dynamic models with uncertainty. A possible application is in model simplification relevant for control design. The uncertainty is described by quadratic constraints and the error bounds are calculated based on solutions to linear matrix inequalities.

1. Introduction

Mathematical models describing the behavior of physical systems are important in many areas. The required accuracy of the model largely depends on the purpose for which the model is intended.

In control engineering there is usually a trade off between model simplicity and model accuracy. Simulation and controller design based on complex models may result in time consuming computations as well as numerical difficulties. The designed controller often also becomes complex and may therefore be expensive and difficult to implement. Models with low accuracy, on the other hand, may give rise to incorrect simulations, as well as control loops with low performance or even instability.

For these reasons there is a need to use a hierarchy of models, each equipped with a quality measure. The most appropriate should then be used for each task. Computation of such quality measures is the objective of this paper. We analyze the importance of different components in

a model. We then simplify the description of the less important components to a suitable level of accuracy. In some cases it is desirable to do this analysis frequency by frequency. For example, in controller design, the accuracy of the process model usually needs to be high close to the cross-over frequency, seeAnderson and Liu (1989). The resulting model is obtained together with a quality measure.

For linear time-invariant models without uncertainty there are well-known order reduction methods. One of these methods is balanced truncation, see Moore (1981); Enns (1984); Glover (1984). This method was adapted to the needs in controller design, by the introduction of frequency weights, see Enns (1984).

Generalizations of the balanced truncation method to models with norm-bounded uncertainty is given in Beck *et al.* (1996); Beck (1996) and to a more general class of uncertainty descriptions as well as nonlinearities in Andersson *et al.* (1999).

In this paper we use the ideas in Andersson *et al.* (1999) to develop frequency dependent error bounds for uncertain linear time-invariant models. Further results and details are available in the report Andersson and Rantzer (1998). The paper is organized as follows. The modeling framework is described in Section 2. The model truncation problem is defined in Section 3 and the solution in terms of error bounds is given in Section 4. In Section 5 an example is given and in Section 6 a generalization and proof of the results is given. The paper is concluded in Section 7.

2. Model description

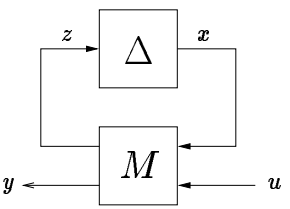

Figure 1. Feedback interconnection representing a model with uncertainty.

The model is described, which is common in modern robustness analysis and control design, by the interconnection of a pair (Δ, M) according to

the relations

$$x = \Delta z, \qquad \text{and} \qquad \begin{bmatrix} z \\ y \end{bmatrix} = M \begin{bmatrix} x \\ u \end{bmatrix}, \tag{1}$$

as illustrated in Figure 1. The signals in the interconnection are the input $u \in \mathbf{L}_{2e}^m[0,\infty)$, the output $y \in \mathbf{L}_{2e}^p[0,\infty)$ and the internal signals $z \in \mathbf{L}_{2e}^n[0,\infty)$ and $x \in \mathbf{L}_{2e}^n[0,\infty)$. The space $\mathbf{L}_{2e}[0,\infty)$ is the space of functions that are square integrable over any finite time interval.

We will usually assume, for stability and well-posedness reasons, that Δ and M are stable proper transfer matrices and that at least one of them is strictly proper. The transfer matrix Δ is used to represent uncertain dynamics as well as known dynamics considered for simplification while M represents the remaining part of the model.

The transfer matrix M is partitioned consistently with the signal dimensions as

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \tag{2}$$

The input-output mapping of the interconnection in Figure 1 is then defined by

$$y = (\Delta \star M)u = (D + C\Delta(I - A\Delta)^{-1}B)u. \tag{3}$$

The transfer matrix Δ is assumed to have a block diagonal structure

$$\Delta = \operatorname{diag}(\Delta_1, \ldots, \Delta_r), \tag{4}$$

where each of the blocks satisfies a quadratic constraint, for example a norm bound. If Δ_k is uncertain we use a quadratic constraint that is satisfied for all Δ_k in the uncertainty set, for example $(\Delta_k - \hat{\Delta}_k)^*(\Delta_k - \hat{\Delta}_k) < \varepsilon_k^2$, where $\hat{\Delta}_k$ is fixed.

We say that the matrix Δ satisfies the quadratic constraint defined by the hermitian matrix Π if

$$\begin{bmatrix} I \\ \Delta \end{bmatrix}^* \Pi \begin{bmatrix} I \\ \Delta \end{bmatrix} \geq 0. \tag{5}$$

If we assume that Δ_k satisfies the quadratic constraints defined by Π_k, for $k = 1, \ldots, r$, then the block diagonal matrix $\Delta = \operatorname{diag}(\Delta_1, \ldots, \Delta_r)$ satisfies

the quadratic constraint defined by $\Pi = \text{daug}(\Pi_1, \ldots, \Pi_r) =$

$$\left[\begin{array}{ccc|ccc} \Pi_{1(11)} & & 0 & \Pi_{1(12)} & & 0 \\ & \ddots & & & \ddots & \\ 0 & & \Pi_{r(11)} & 0 & & \Pi_{r(12)} \\ \hline \Pi_{1(21)} & & 0 & \Pi_{1(22)} & & 0 \\ & \ddots & & & \ddots & \\ 0 & & \Pi_{r(21)} & 0 & & \Pi_{r(22)} \end{array}\right]. \tag{6}$$

3. Problem formulation

The problem considered is to analyze the importance of the different blocks in Δ for different frequencies. We would in particular like to compare the original model with one where some of the blocks in Δ have been truncated.

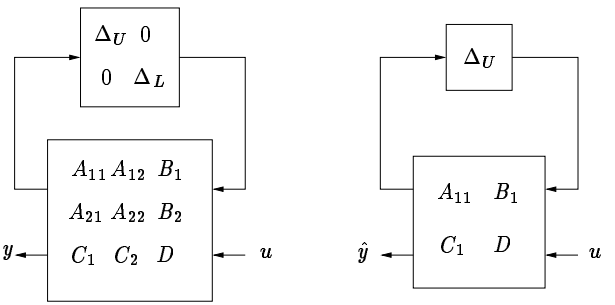

Figure 2. Model truncation.

Model truncation is illustrated in Figure 2. We assume, without loss of generality, that the model is partitioned in such a way that the lower blocks of Δ and the corresponding part of M are considered for truncation. Partitioning the model as

$$M = \left[\begin{array}{ccc} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ C_1 & C_2 & D \end{array}\right], \qquad \Delta = \left[\begin{array}{cc} \Delta_U & 0 \\ 0 & \Delta_L \end{array}\right], \tag{7}$$

where

$$\Delta_U = \mathrm{diag}(\Delta_1, \ldots, \Delta_{\hat{r}})$$
$$\Delta_L = \mathrm{diag}(\Delta_{\hat{r}+1}, \ldots, \Delta_r),$$

then the truncated model is given by

$$\hat{M} = \begin{bmatrix} A_{11} & B_1 \\ C_1 & D \end{bmatrix}, \qquad \hat{\Delta} = \Delta_U. \tag{8}$$

To compare the two models we would like to find upper bounds on the error,

$$e = y - \hat{y} = (\Delta \star M - \hat{\Delta} \star \hat{M})u, \tag{9}$$

or more precisely the norm

$$\|(\Delta \star M - \hat{\Delta} \star \hat{M})(i\omega)\|, \tag{10}$$

between the models. The norm (gain) of the transfer matrix, for each frequency, is defined by the maximum singular value as

$$\|G(i\omega)\| = \bar{\sigma}(G(i\omega)). \tag{11}$$

4. Error bounds

To analyze what parts of the model to truncate we assign positive real-valued functions $\sigma_k(\omega)$ to each of the blocks in Δ. Each function $\sigma_k(\omega)$ gives a measure on the importance of the corresponding block Δ_k at each frequency. When the model is truncated then the error at each frequency, is bounded by two times the sum of the σ_k-functions corresponding to the truncated blocks. This is stated in the following theorem.

THEOREM 1
Assume that Δ and M are stable proper transfer matrices and at least one of them is strictly proper. If there exist bounded measurable functions $\Pi_1(i\omega), \ldots, \Pi_r(i\omega)$ taking hermitian values, and real valued functions $\sigma_1(\omega), \ldots, \sigma_r(\omega) > 0$ such that $\forall \omega \in [0, \infty]$

A. Π_k defines a constraint satisfied by $\tau\Delta_k$,
for $k = 1, \ldots, r$, and $\tau \in [0, 1]$,

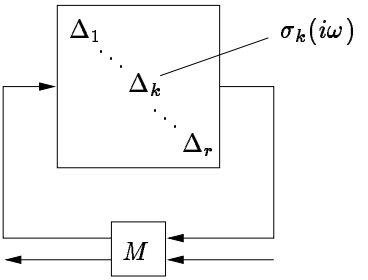

Figure 3. Each block Δ_k has an associated function $\sigma_k(\omega)$ that measure the importance of the block at each frequency.

B. $\quad \Pi_{k(11)} \geq 0$, for $k = \hat{r}+1, \ldots, r$

C. $$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \mathrm{daug}(\Pi_1, \ldots, \Pi_r) \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} < \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix},$$

D. $$\begin{bmatrix} A \\ I \end{bmatrix}^* \mathrm{daug}(\sigma_1^2\Pi_1, \ldots, \sigma_r^2\Pi_r) \begin{bmatrix} A \\ I \end{bmatrix} + C^*C < 0,$$

then both the original and truncated models are stable and

$$\|(\Delta \star M - \hat{\Delta} \star \hat{M})(i\omega)\| \leq 2\sigma_{\hat{r}+1}(\omega) + \cdots + 2\sigma_r(\omega).$$

REMARK 1
The homotopy parameter τ is only needed for stability. If the original model is stable or if $\Pi_{k(11)} \geq 0$ and $\Pi_{k(22)} \leq 0$ then it is sufficient to only consider $\tau = 1$. □

REMARK 2
If stability of the model can be shown, using the results in Megretski and Rantzer (1997), then there also exist functions σ_k satisfying the conditions above. □

Proof See Section 6. □

The scalar functions $\sigma_k(\omega)$ and matrix functions $\Pi_k(i\omega)$ are not unique. The global minimization of the function values of $\sigma_k(\omega) > 0$ is in general non-convex and difficult to solve. However, a simple suboptimal algorithm, based on two steps is as follows. The steps must be repeated for each frequency of interest.

First fix values of $\hat{\sigma}_k$, and let $\sigma_k = \gamma\hat{\sigma}_k$. Then find Π_k that minimizes γ under the conditions A-D. When all the conditions are formulated as linear matrix inequalities in Π_k then the obtained problem is a generalized eigenvalue minimization problem, that can be solved by convex optimization.

In the second step use the Π_k-matrices obtained in the previous step and minimize a linear combination of the σ_k-values, for example the sum, under condition *D*. This problem can also be solved by convex optimization.

In the case where Δ only contains one block, it is easy to verify that the algorithm will result in the best possible value on σ, independent of the initial choice $\hat{\sigma}$. This is the situation in the example in the next section. If there are more than one block, improved values, at a given frequency, can usually be obtained by iterating the two step algorithm. The obtained values of σ_k are then used as initial values $\hat{\sigma}_k$ when a new iteration begins.

The theorem may also be used to obtain error bound formulas in special cases. In the closely related paper Andersson *et al.* (1999) it was shown that the error bounds presented there could be used to derive the error bounds for truncation of the generalized balanced realization given in Beck *et al.* (1996); Beck (1996).

Properties of the obtainable error bounds can be found in Andersson and Rantzer (1998).

5. Flexible servo example

In this section we consider the flexible servo in Figure 4. We investigate the influence of uncertainty in one of the spring constants.

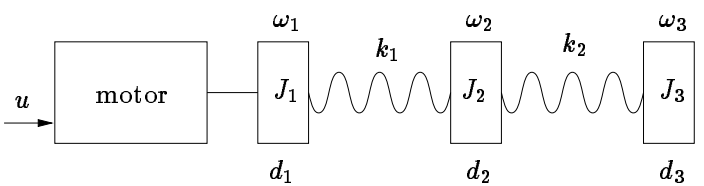

Figure 4. A flexible servo.

A simple model of the servo is given by

$$\begin{aligned}
J_1\dot{\omega}_1 &= -k_1(\phi_1 - \phi_2) - d_1\omega_1 + k_u u \\
J_2\dot{\omega}_2 &= k_1(\phi_1 - \phi_2) - k_2(\phi_2 - \phi_3) - d_2\omega_2 \\
J_3\dot{\omega}_3 &= k_2(\phi_2 - \phi_3) - d_3\omega_3 \\
\dot{\phi}_1 &= \omega_1 \\
\dot{\phi}_2 &= \omega_2 \\
\dot{\phi}_3 &= \omega_3 \\
y &= k_\omega\omega_1
\end{aligned}$$

where ω_k denotes angular velocity and ϕ_k the corresponding angle. The system output is the voltage y representing the angular velocity of the first mass. We assume that the spring constant k_1 is known with only 10% accuracy. This uncertain spring constant k_1 is modeled as

$$k_1 = \bar{k}_1(1 + 0.1\delta)$$

where $|\delta| \leq 1$ and $\bar{k}_1$ is the nominal spring constant. The parameter values are

$$\begin{aligned}
J_1 &= 50 \cdot 10^{-6}\ \text{kgm}^2 \\
J_2 &= 20 \cdot 10^{-6}\ \text{kgm}^2 \\
J_3 &= 55 \cdot 10^{-6}\ \text{kgm}^2 \\
d_1 &= d_2 = d_3 = 30 \cdot 10^{-6}\ \text{Nm/rad/s} \\
\bar{k}_1 &= k_2 = 400 \cdot 10^{-6}\ \text{Nm/rad} \\
k_u &= 25 \cdot 10^{-3}\ \text{Nm/V} \\
k_\omega &= 0.1\ \text{V/rad/s}.
\end{aligned}$$

The Bode diagram for the model when $k_1 = \bar{k}_1$ is given in Figure 5. We see that there are resonances at 9 rad/s and 22 rad/s and notches at 6 rad/s and 21 rad/s.

We are interested in analyzing in what frequency ranges the nominal model is a sufficient description of the uncertain model. We will therefore calculate error bounds that measure the difference between the nominal model and the uncertain model.

The uncertainty is represented by a unity norm bounded real scalar δ. Such a scalar satisfies any quadratic constraint defined by

$$\Pi(i\omega) = \begin{bmatrix} x(\omega) & y(i\omega) \\ y(i\omega)^* & -x(\omega) \end{bmatrix}$$

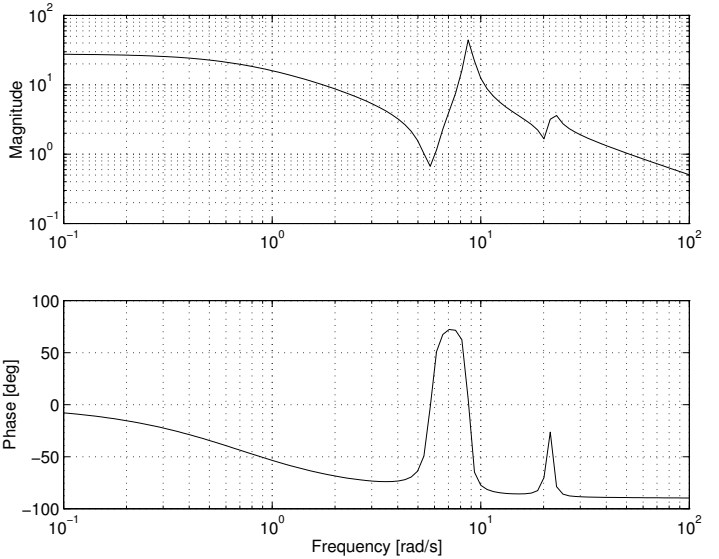

Figure 5. Bode diagram for the flexible servo. There is one resonance at 9 rad/s and one at 22 rad/s.

where $x(\omega) \geq 0$ is real and $\operatorname{Re} y(i\omega) = 0$.

We find, using for example the given numerical algorithm, the upper bound $2\sigma(\omega)$, shown in Figure 6 with a solid line. The true worst case error is shown with a dashed line. We see that the obtained upper bound is close to the true worst case error.

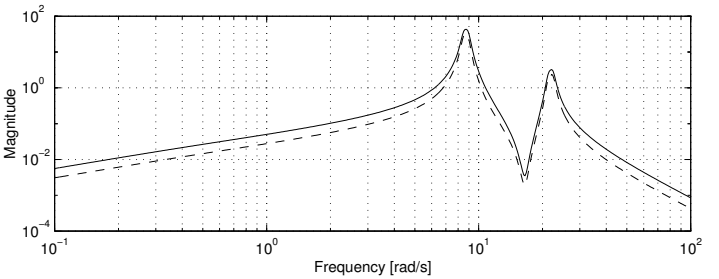

Figure 6. This figure shows upper bounds on the error when the uncertainty in the spring constant k_1 is neglected. The solid line shows the error bound obtained using our method while the dashed line shows the true worst case error.

The upper bounds show that the error is large close to the resonance and notch frequencies. This is intuitively reasonable since the resonances and notches move when the spring constant is changed, resulting in large differences in amplification near the resonance and notch frequencies.

The accuracy needed in the process model is different at different frequencies due to the effects of the closed-loop interconnection. Such considerations for stability and performance are discussed in Anderson and Liu (1989), where it is shown that high accuracy usually is needed close to the cross over frequency. Thus, if we, in this example, will design a system with low bandwidth, then the uncertainty in the spring constant may be neglected during the controller design.

6. Generalization and proof of Theorem 1

We now consider comparison of two models that are identical except for some of the blocks in Δ. We assume that the upper blocks denoted $\Delta_U = \mathrm{diag}(\Delta_1, \ldots, \Delta_{\hat{r}})$, are identical while the lower blocks denoted $\Delta_L = \mathrm{diag}(\Delta_{\hat{r}+1}, \ldots, \Delta_r)$ and $\hat{\Delta}_L = \mathrm{diag}(\hat{\Delta}_{\hat{r}+1}, \ldots, \hat{\Delta}_r)$, respectively, are different. We partition the two models consistently as

$$M = \begin{bmatrix} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ C_1 & C_2 & D \end{bmatrix},$$

$$\Delta = \begin{bmatrix} \Delta_U & 0 \\ 0 & \Delta_L \end{bmatrix}, \quad \hat{\Delta} = \begin{bmatrix} \Delta_U & 0 \\ 0 & \hat{\Delta}_L \end{bmatrix}.$$

The described comparison of models may be used for model simplification. The simplification is done by replacing Δ_L with a fixed transfer matrix $\hat{\Delta}_L$, with the same block structure as Δ_L. The spatial dimension of the simplified model may be reduced. This follows by observing that

$$\begin{bmatrix} \Delta_U & 0 \\ 0 & \hat{\Delta}_L \end{bmatrix} \star M = \Delta_U \star \hat{M},$$

where

$$\hat{M} = \begin{bmatrix} A_{11} & B_1 \\ C_1 & D \end{bmatrix} + \begin{bmatrix} A_{12} \\ C_2 \end{bmatrix} \hat{\Delta}_L (I - A_{22}\hat{\Delta}_L)^{-1} \begin{bmatrix} A_{21} & B_2 \end{bmatrix}.$$

Truncation considered earlier corresponds to choosing $\hat{\Delta}_L = 0$.

THEOREM 2
Assume that M is a stable proper (strictly proper) transfer matrix and that we have two stable strictly proper (proper) block diagonal transfer function matrices

$$\begin{aligned}\Delta &= \operatorname{diag}(\Delta_1,\dots,\Delta_r),\\ \hat{\Delta} &= \operatorname{diag}(\Delta_1,\dots,\Delta_{\hat{r}},\hat{\Delta}_{\hat{r}+1},\dots,\hat{\Delta}_r).\end{aligned}$$

If there exist bounded measurable functions $\Pi_k(i\omega)$ where $k = 1\dots r$, taking hermitian values, and real valued functions $\sigma_1(\omega),\dots,\sigma_r(\omega) > 0$ such that $\forall\omega \in [0,\infty]$

A. Π_k defines a constraint satisfied by $\tau\Delta_k$, for $k = 1,\dots,r$, and $\tau \in [0,1]$.

B. Π_k defines a constraint satisfied by $\tau\hat{\Delta}_k$, for $k = \hat{r}+1,\dots,r$, and all $\tau \in [0,1]$.

C. $$\begin{bmatrix} A & B\\ I & 0\end{bmatrix}^* \operatorname{daug}(\Pi_1,\dots,\Pi_r)\begin{bmatrix} A & B\\ I & 0\end{bmatrix} < \begin{bmatrix} 0 & 0\\ 0 & I\end{bmatrix},$$

D. $$\begin{bmatrix} A\\ I\end{bmatrix}^* \operatorname{daug}(\sigma_1^2\Pi_1,\dots,\sigma_r^2\Pi_r)\begin{bmatrix} A\\ I\end{bmatrix} + C^*C < 0,$$

then both (Δ, M) and $(\hat{\Delta}, M)$ are stable and

$$\|(\Delta \star M - \hat{\Delta} \star M)(i\omega)\| \le 2\sigma_{\hat{r}+1}(\omega) + \cdots + 2\sigma_r(\omega).$$

Proof Stability can be shown using the general stability theorem given in Megretski and Rantzer (1997). A direct proof is given in Andersson and Rantzer (1998).

We now prove the error bounds for the case with $\hat{r} = 1$ and $r = 2$. The case with $\hat{r} = r = 1$ is almost identical, but less pedagogical. We partition the matrix M as

$$M = \begin{bmatrix} A_{11} & A_{12} & B_1\\ A_{21} & A_{22} & B_2\\ C_1 & C_2 & D\end{bmatrix}.$$

Since we have two models (Δ, M) and $(\hat{\Delta}, M)$ we also have two feedback interconnections and also two sets of signals. We introduce the following notation for the Laplace transformed signals:

$$\begin{bmatrix} z_1\\ z_2\\ y\end{bmatrix} = M\begin{bmatrix} x_1\\ x_2\\ u\end{bmatrix}, \qquad \begin{bmatrix} \hat{z}_1\\ \hat{z}_2\\ \hat{y}\end{bmatrix} = M\begin{bmatrix} \hat{x}_1\\ \hat{x}_2\\ u\end{bmatrix}.$$

We assume that the initial conditions are zero. This is realistic since the two models are stable. Note that the input signal u is the same for both models.

Multiplying the inequality (C) with the vector $(x_1+\hat{x}_1, x_2+\hat{x}_2, 2u)$ from the right and its complex conjugate from the left, and adding this to the inequality obtained by multiplying the inequality (D) with the column vector $\sigma_2^{-1}(x_1-\hat{x}_1, x_2-\hat{x}_2)$ from the right and its complex conjugate from the left, we then get by noting that $z-\hat{z}=A(x-\hat{x})$, $y-\hat{y}=C(x-\hat{x})$ and using some matrix manipulations, the following inequality

$$
\begin{aligned}
0 \geq & \begin{bmatrix} z_1+\hat{z}_1 \\ x_1+\hat{x}_1 \end{bmatrix}^* \Pi_1 \begin{bmatrix} z_1+\hat{z}_1 \\ x_1+\hat{x}_1 \end{bmatrix} \\
& + \sigma_1^2\sigma_2^{-2} \begin{bmatrix} z_1-\hat{z}_1 \\ x_1-\hat{x}_1 \end{bmatrix}^* \Pi_1 \begin{bmatrix} z_1-\hat{z}_1 \\ x_1-\hat{x}_1 \end{bmatrix} \\
& + 2 \begin{bmatrix} z_2 \\ x_2 \end{bmatrix}^* \Pi_2 \begin{bmatrix} z_2 \\ x_2 \end{bmatrix} + 2 \begin{bmatrix} \hat{z}_2 \\ \hat{x}_2 \end{bmatrix}^* \Pi_2 \begin{bmatrix} \hat{z}_2 \\ \hat{x}_2 \end{bmatrix} \\
& + \sigma_2^{-2}|y-\hat{y}|^2 - 4|u|^2.
\end{aligned}
$$

The first and second term are nonnegative since $\Delta_1 = \hat{\Delta}_1$ satisfies the quadratic constraint defined by Π_1 and $x_1+\hat{x}_1 = \Delta_1(z_1+\hat{z}_1)$. The third and fourth term are nonnegative since Δ_2 and $\hat{\Delta}_2$ both satisfy the quadratic constraint defined by Π_2. This completes the proof for the case $\hat{r}+1 = r = 2$.

More generally one can generate a sequence of new models by replacing the blocks $\Delta_r, \ldots, \Delta_{\hat{r}+1}$ one at a time. At each step, the above argument can be used and the total error bound becomes

$$
|(y-\hat{y})(i\omega)| < 2(\sigma_r(\omega) + \cdots + \sigma_{\hat{r}+1}(\omega))|u(i\omega)|.
$$

□

7. Conclusions

In this paper we have derived upper bounds on the error between two uncertain linear time-invariant models. The results can be used to analyze the importance of different components of a model and for model reduction. The obtained error bounds are frequency dependent which is an important feature in many applications such as controller design where different frequencies have different importance.

Further research is needed for practical application of the results. In particular the properties of the proposed numerical algorithm need further investigation. Also the conservativeness of the theoretically obtainable error bounds needs to be investigated. Some results on the conservativeness are given in Andersson and Rantzer (1998).

Acknowledgment

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Extension of Paper 3

Properties of the Error Bounds

Lennart Andersson and Anders Rantzer

Abstract

This is an extension of the paper "Frequency dependent error bounds for uncertain linear models". In this extension properties of the error bounds are investigated. First we discuss some properties related to the choice of simplified model. Then situations in which the error bounds are expected to be conservative will be given and discussed. At the end it will be shown that the error bounds are optimal in the one-component case.

8. Choosing a simplified model

To choose which of the blocks in Δ to simplify, we look at the σ_k-values since they indicate for which of the blocks in Δ simplification is cheapest. The level of accuracy required is then used to determine the number of blocks simplified. A noteworthy property is that equal σ_k-values in the bound need only be added once.

PROPOSITION 3—IMPROVED ERROR BOUNDS
Equal σ_k in the error bound need only be added once.

Proof If there are equal σ_k-values then the corresponding blocks can be collected in one block and the corresponding Π_k matrices collected in one larger Π_k matrix. The σ_k-value corresponding to this larger Δ_k is the same as for the smaller ones. □

If one decides to simplify a block corresponding to a repeated σ_k-value, one might as well simplify all of the blocks corresponding to this σ_k-value. At least from an upper bound point of view.

The simplification is done by replacing the selected blocks by fixed transfer matrices. By selecting frequency independent matrices we avoid unnecessary dynamics, and by choosing matrices close to the original matrices we obtain a simplified model close to the original model. The last

statement is motivated by the following result, which shows that the error between the two models is small if Δ and $\hat{\Delta}$ are close to each other.

PROPOSITION 4—CONTINUITY OF ERROR
Assume that (Δ, M) is stable. Then for every $\varepsilon > 0$ there exist a $\delta > 0$ such that

$$\|\Delta \star M - \hat{\Delta} \star M\| < \varepsilon \qquad \text{when} \qquad \|\Delta - \hat{\Delta}\| < \delta.$$

Proof This follows since

$$\Delta \star M - \hat{\Delta} \star M = C\Delta(I - A\Delta)^{-1}B - C\hat{\Delta}(I - A\hat{\Delta})^{-1}B$$

is a continuous function of the matrix elements, and no singularities, on the imaginary axis, occur thanks to the stability assumption. □

In fact not only the error but also the error bound will be small if Δ and $\hat{\Delta}$ are close to each other. This follows by the following proposition showing that we for any stable model (Δ, M) can make Σ, and thus the error bound, arbitrary small by restricting $\hat{\Delta}$ to be sufficiently close to Δ.

PROPOSITION 5—CONTINUITY OF ERROR BOUND
Consider a fixed frequency. Given any

$$\begin{aligned} M &= \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \\ \Delta &= \operatorname{diag}(\Delta_1, \ldots, \Delta_r), \\ \Sigma &= \operatorname{diag}(\sigma_1 I_{n_1}, \ldots, \sigma_r I_{n_r}) > 0, \end{aligned}$$

such that $I - \Delta A$ is invertible[1], there exist $\delta > 0$ and $\lambda > 0$ such that

$$\Pi = -\lambda \begin{bmatrix} \Delta^* \\ -I \end{bmatrix} \begin{bmatrix} \Delta & -I \end{bmatrix} + \lambda\delta^2 \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},$$

satisfies the conditions

I.
$$\begin{bmatrix} I \\ \hat{\Delta} \end{bmatrix}^* \Pi \begin{bmatrix} I \\ \hat{\Delta} \end{bmatrix} \geq 0, \quad \text{when} \quad \|\Delta - \hat{\Delta}\| \leq \delta$$

II.
$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \Pi \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -I \end{bmatrix} < 0 \qquad (C)$$

$$\begin{bmatrix} \Sigma A \\ \Sigma \end{bmatrix}^* \Pi \begin{bmatrix} \Sigma A \\ \Sigma \end{bmatrix} + C^* C < 0. \qquad (D)$$

[1]The invert-ability condition holds if (Δ, M) is stable.

Proof Consider the multiplier Π defined in the proposition. The first condition holds since

$$\frac{1}{\lambda}\begin{bmatrix} I \\ \hat{\Delta} \end{bmatrix}^* \Pi \begin{bmatrix} I \\ \hat{\Delta} \end{bmatrix} = -(\Delta - \hat{\Delta})^*(\Delta - \hat{\Delta}) + \delta^2.$$

The second condition is first shown for $\delta = 0$. The multiplier Π does in this case satisfy inequality (C) for any $\lambda > 0$ since

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \Pi \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -I \end{bmatrix}$$
$$= \begin{bmatrix} -\lambda(I-\Delta A)^*(I-\Delta A) & \lambda(I-\Delta A)^*\Delta B \\ \lambda(\Delta B)^*(I-\Delta A) & -\lambda(\Delta B)^*\Delta B - I \end{bmatrix} < 0$$
$$\Longleftrightarrow$$
$$\begin{bmatrix} -\lambda(I-\Delta A)^*(I-\Delta A) & 0 \\ 0 & -I \end{bmatrix} < 0,$$

where the last equivalence follows using the Schur complement

$$-\lambda(\Delta B)^*\Delta B - I + \lambda(\Delta B)^*(I-\Delta A) \cdot \{\lambda(I-\Delta A)^*(I-\Delta A)\}^{-1}\lambda(I-\Delta A)^*\Delta B = -I.$$

The inequality (D) for any given $\Sigma > 0$ is satisfied by the multiplier Π (when $\delta = 0$) if λ is chosen sufficiently large. This follows by the following arguments.

$$\begin{bmatrix} \Sigma A \\ \Sigma \end{bmatrix}^* \Pi \begin{bmatrix} \Sigma A \\ \Sigma \end{bmatrix} + C^*C$$
$$= -(\Delta A - I)^*\Sigma\lambda\Sigma(\Delta A - I) + C^*C.$$

Since $\Sigma > 0$ and $\Delta A - I$ has full rank we see that we can make the expression negative by choosing λ sufficiently large.

We have so far shown that for any Σ, no matter how small, our Π, with $\delta = 0$ and λ chosen sufficiently large, defines a quadratic constraint satisfied by $\hat{\Delta} = \Delta$ and it also satisfies inequalities (C) and (D). This shows that the error bound can be made arbitrary small when the compared models are equal.

Finally, since the expressions in the two inequalities (C) and (D) are continuous in the matrix elements it follows that sufficiently small changes in Π do not affect the negative definiteness of the inequalities. Thus, the inequalities will be satisfied for a sufficiently small $\delta > 0$. □

Another guide in the selection of simplified blocks is the Π_k-matrices corresponding to the σ_k-values. They define sets for which the error bounds will hold. Apart from these basic ideas not much guidance among the possible choices of replacing matrices will be given. The choice will therefore in general be based on intuition and physical knowledge.

9. Some indications of conservatism

The blocks in Δ play a fundamental role for the bounds obtained using Theorem 1. It is important that we capture as much as possible the behavior of each block Δ_k. This is done by considering as many candidates for the corresponding multiplier Π_k as possible and selecting the one resulting in the smallest Σ.

Each block is described separately. This does not allow us to describe interdependence between blocks, for example a physical parameter appearing in more than one block. We therefore expect the bounds to be more conservative if such interdependence exists. Also, whole blocks must be eliminated. No bounds are available for elimination of parts of a block.

If there are related blocks in Δ, then we may collect these blocks in one larger block and describe the interdependence using quadratic constraints. This will most likely result is less conservative bounds, but we then only obtain bounds for elimination of all of these blocks at the same time, and not for elimination of separate blocks.

Another possibility is to apply coordinate transformations a priori, such that the interdependence between the blocks in the new model has little effect on the bounds. Since coordinate transformations only change the internal signals and leave the input-output relation unchanged, this seems to be a good idea. However, finding such coordinate transformations are in general difficult and may also have to be avoided if the internal description is important. An example of the later is when the blocks in Δ are components with a physical interpretation. Transformations must then be avoided if we would like to maintain the physical interpretation, or if we would like to know the importance of a physical component.

Even if there is no interdependence between the blocks and each block is described accurately using quadratic constraints, we still expect the error bounds to be conservative, when there are more than one block. The reason is that the σ_k-values are expected to increase with the number of blocks in Δ, since each σ_k-value should be used to calculate more than one upper bound. If we simplify one block then σ_k is used in the error bound $2\sigma_k$, which must hold independent of how many blocks there are in Δ. We therefore do not expect σ_k to decrease with the number of blocks in Δ. On the other hand if we increase the number of blocks in Δ then σ_k should be

used to calculate more and more upper bounds. The σ_k-values may then have to be increased. The following example illustrates the idea.

EXAMPLE 1
Consider the function

$$f(\delta_1, \delta_2, \delta_3) = (\delta_1 - \delta_2)(\delta_1 - \delta_3),$$

defined for $|\delta_k| \leq 1$. Think of each δ_k as a transfer function and the function f as the input-output relation of a model. Assume that the nominal model is given by $G(\delta) = f(\delta, \delta, \delta) = 0$ and consider the following three simplified models,

$$\begin{aligned}
\hat{G}_2(\delta) &= f(\delta, \hat{\delta}_2, \delta) = 0, \\
\hat{G}_3(\delta) &= f(\delta, \delta, \hat{\delta}_3) = 0, \\
\hat{G}_{23}(\delta) &= f(\delta, \hat{\delta}_2, \hat{\delta}_3) = (\delta - \hat{\delta}_2)(\delta - \hat{\delta}_3).
\end{aligned}$$

We see that replacing only one of the transfer functions δ_2 and δ_3 does not result in any input-output error. If we on the other hand replace both δ_2 and δ_3 then the error satisfies

$$|G - \hat{G}_{23}| = |\hat{G}_{23}| \leq 4,$$

where the bound is reached for $\hat{\delta}_2 = \hat{\delta}_3 = -\delta = \pm 1$. This shows that the error when we replace more than one component of a model may be larger than the sum of the errors for replacing one component. This indicates that it may be necessary to increase the value on σ_k when the number of blocks considered for replacement is increased. □

The conservatism illustrated in the example above is not a problem specific to the theorem analyzed here. It is an inherent property of all methods trying to measure the importance of individual components.

10. The one block case is non-conservative

It was shown in a previous section that the error bounds are small when the error is small. It was also shown that the bound may be conservative in some cases. In this section we show that the bound is non-conservative if only one block is considered.

THEOREM 6—NON-CONSERVATIVE BOUND
The upper bound is non-conservative if the original model contains one block.

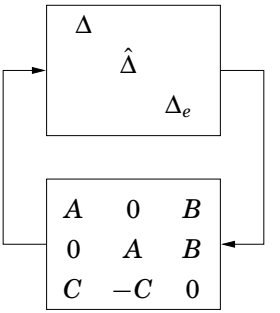

Figure 7. The stability problem associated with the error bound.

Proof Consider the two models (Δ, M) and $(\hat{\Delta}, M)$, where both Δ and $\hat{\Delta}$ satisfy the two quadratic constraints defined by Π and $\hat{\Pi}$. Assume that $\|\Delta_e\| \leq 1/\gamma$, or equivalently, Δ_e satisfies the quadratic constraint defined by

$$\Pi_e = \begin{bmatrix} \frac{1}{\gamma} I & 0 \\ 0 & -I \end{bmatrix}.$$

Then the error between the two models is less than γ if the model in Figure 7 is stable, that is, see Megretski and Rantzer (1997), when

$$\left[\begin{array}{ccc} A & 0 & B \\ 0 & A & B \\ C & -C & 0 \\ \hline I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{array}\right]^* \left[\begin{array}{ccc|ccc} \Pi_{11} & & & \Pi_{12} & & \\ & \hat{\Pi}_{11} & & & \hat{\Pi}_{12} & \\ & & \frac{1}{\gamma} I & & & 0 \\ \hline \Pi_{21} & & & \Pi_{22} & & \\ & \hat{\Pi}_{21} & & & \hat{\Pi}_{22} & \\ & & 0 & & & -I \end{array}\right] \left[\begin{array}{ccc} A & 0 & B \\ 0 & A & B \\ C & -C & 0 \\ \hline I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{array}\right] < 0.$$

Rewriting this expression results in the new equivalent expression

$$\left[\begin{array}{ccc} A & 0 & B \\ I & 0 & 0 \\ \hline 0 & A & B \\ 0 & I & 0 \\ \hline C & -C & 0 \\ \hline 0 & 0 & I \end{array}\right]^* \left[\begin{array}{cccccc} \Pi_{11} & \Pi_{12} & & & & \\ \Pi_{21} & \Pi_{22} & & & & \\ & & \hat{\Pi}_{11}\hat{\Pi}_{12} & & & \\ & & \hat{\Pi}_{21}\hat{\Pi}_{22} & & & \\ & & & & \frac{1}{\gamma} I & \\ & & & & & -I \end{array}\right] \left[\begin{array}{ccc} A & 0 & B \\ I & 0 & 0 \\ \hline 0 & A & B \\ 0 & I & 0 \\ \hline C & -C & 0 \\ \hline 0 & 0 & I \end{array}\right] < 0.$$

Rewriting again gives

$$\begin{bmatrix} A & 0 & B \\ I & 0 & 0 \end{bmatrix}^* \Pi \begin{bmatrix} A & 0 & B \\ I & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & A & B \\ 0 & I & 0 \end{bmatrix}^* \hat{\Pi} \begin{bmatrix} 0 & A & B \\ 0 & I & 0 \end{bmatrix} + \frac{1}{\gamma^2} [\, C \quad -C \quad 0\,]^* [\, C \quad -C \quad 0\,] - [\, 0 \quad 0 \quad I\,]^* [\, 0 \quad 0 \quad I\,].$$

Multiplying with the vector

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

from the right and its transpose from the left gives

$$\begin{bmatrix} A \\ I \end{bmatrix}^* (\Pi + \hat{\Pi}) \begin{bmatrix} A \\ I \end{bmatrix} + \frac{4}{\gamma^2} C^* C < 0.$$

Multiplying with the matrix

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

from the right and its transpose from the left that

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* (\Pi + \hat{\Pi}) \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -I \end{bmatrix} < 0.$$

This shows that if the error is less than γ then, the inequalities in Theorem 1 are satisfied using $\tilde{\Pi} = \Pi + \hat{\Pi}$ and $\sigma^2 = \frac{\gamma^2}{4}$. It then follows that the error is less that $2\sigma = \gamma$. □

Paper 4

Robustness of Equilibria in Nonlinear Systems

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Abstract

Robust stability of equilibria of nonlinear models is considered. The nonlinear model is assumed to have a parametric uncertainty. Stability is investigated using the linearized uncertain model. Particular attention is paid to the fact that the location of the equilibria depends on the uncertain parameters. It is shown that structured singular values in several cases can be used to analyze also the effects of perturbed equilibria.

1. Introduction

There are many methods for investigation of stability in nonlinear control systems, see for example Khalil (1996). Methods for global stability analysis of control systems include the describing function method, Lyapunov methods and methods related to the small gain theorem.

Local stability in a neighborhood of an equilibrium is usually shown based on Lyapunov methods. One of the results of Lyapunov theory is that local exponential stability of an equilibrium of a nonlinear model can be analyzed by studying the linearized model.

The methods referred to above as global, have shown to be well suited for analyzing robust stability. A lot of attention has also been paid to robust stability analysis of linear models with uncertainty, see for example Zhou *et al.* (1996). However, less attention has been paid to nonlinear models with perturbed equilibria. Some simple examples are considered

in text books, see for example Morari and Zafiriou (1989), but the problem if often neglected and deserves more attention. Some such work was reported in the paper Michel and Wang (1993) for models with unstructured perturbations and in the paper Wada *et al.* (1998) for Lur'e systems with parametric uncertainty. In this paper we consider parametric uncertainty for a general class of nonlinear models.

The paper is organized as follows. In Section 2 properties of nonlinear models are discussed resulting in a useful problem formulation. Examples illustrating how the problem may be solved using a change of variables is presented in Section 3. In Section 4 approximation methods are discussed. The paper is concluded in Section 5.

2. Problem formulation

In this section we first consider some selected properties of nonlinear models with uncertainty; stability and perturbed equilibria. Based on these properties we state the problem.

Consider the nonlinear model

$$\frac{dx}{dt} = f(x, \delta), \tag{1}$$

where $f : (D, \Omega) \to R^n$ is continuously differentiable, and $\delta \in \Omega \subset R^r$ is an uncertain parameter vector. The equilibria, x_0, are given by

$$f(x^0, \delta) = 0. \tag{2}$$

Some of these equilibria may move as δ changes. The rate of change with respect to changes in the parameter δ is given by

$$\frac{dx^0}{d\delta} = -\left[\frac{\partial f}{\partial x^0}\right]^{-1} \frac{\partial f}{\partial \delta}. \tag{3}$$

At points where

$$\det\left(\frac{\partial f}{\partial x^0}(x^0, \delta)\right) = 0$$

it may happen that the number of equilibria changes, that is, a bifurcation.

EXAMPLE 1
Consider the simple nonlinear model

$$\dot{x} = \delta - x^2.$$

This model has no equilibria when $\delta < 0$, one equilibrium at the origin when $\delta = 0$ and two equilibria $x^0 = \pm\sqrt{\delta}$ when $\delta > 0$. The location of the equilibria are different for different values of $\delta > 0$, and the number of equilibria changes at $\delta = 0$. □

Asymptotic stability of an equilibrium, x^0, of the nonlinear model may be analyzed by considering stability of the linearized model,

$$\frac{d(\tilde{x})}{dt} = \frac{\partial f}{\partial x}(x^0, \delta)\tilde{x}, \tag{4}$$

where $\tilde{x} = x - x^0$.

THEOREM 1—LYAPUNOV'S INDIRECT METHOD

Consider the nonlinear model (1) for a given value of $\delta \in \Omega$. Assume that x^0 is an equilibrium, that is $f(x^0, \delta) = 0$. Let

$$A = \frac{\partial f}{\partial x}(x^0, \delta)$$

then

- x^0 is asymptotically stable if $\mathrm{Re}(\lambda_i) < 0$ for all eigenvalues, λ_i, of A
- x^0 is unstable if $\mathrm{Re}(\lambda_i) > 0$ for one or more eigenvalues, λ_i, of A

Proof See for example Khalil (1996). □

EXAMPLE 2
Consider the nonlinear model

$$\frac{dx}{dt} = f(x, \delta) = \delta(\delta^4 - x^2) = \delta(\delta^2 + x)(\delta^2 - x).$$

It holds that

$$A = \frac{\partial f}{\partial x}(x^0, \delta) = -2\delta x^0.$$

The equilibrium $x^0 = \delta^2 > 0$ is stable when $\delta > 0$ and unstable when $\delta < 0$, see Figure 1. This shows that the stability properties of this equilibrium

changes at $\delta = 0$. Similarly, the equilibrium $x^0 = -\delta^2 < 0$ is unstable when $\delta > 0$ and stable when $\delta < 0$. This means that there for the same value of δ are both stable and unstable equilibria; *e.g.* for $\delta = 1$ the equilibrium $x^0 = 1$ is stable and $x^0 = -1$ is unstable. Note also that the same equilibrium location may be stable for one value of δ and unstable for an other value; for example the point $x = 1$ is a stable equilibrium for $\delta = 1$ and an unstable equilibrium for $\delta = -1$. □

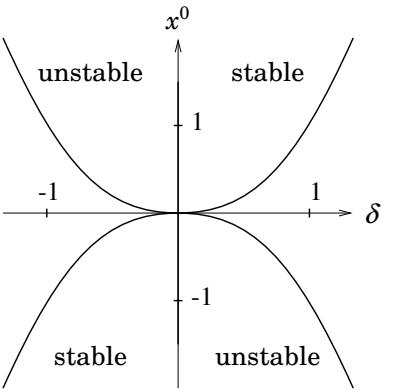

Figure 1. The location of the equilibria x^0 vary with the parameter δ. It is seen that there are two equilibria for each value of $\delta \neq 0$, and that the same location x^0 is an equilibrium for more than one value of δ, in Example 2.

A robust stability problem

The examples above point out two important considerations for investigation of robust local stability. The first is to take into account that x^0 changes if δ is changed. The second is to define, for each value of δ, which of the equilibria we are interested in. Both these considerations are captured by assuming that the equilibrium of interest is a function of the uncertain parameter

$$x^0 = g(\delta), \qquad \delta \in \Omega. \tag{5}$$

The robust stability problem can then be formulated in the following way.

PROBLEM 1
Determine if for all $\delta \in \Omega$ the eigenvalues of

$$J(\delta) = \frac{\partial f}{\partial x}(g(\delta), \delta) \tag{6}$$

stay in the open left half plane. □

If g is continuous and $f \in C^1$ then an equivalent problem formulation is as follows.

PROBLEM 2
Assume that there exists $\delta_{nom} \in \Omega$ such that the eigenvalues of

$$\frac{\partial f}{\partial x}(g(\delta_{nom}), \delta_{nom})$$

are in the open left half plane. Determine if

$$\det\left(i\omega I - \frac{\partial f}{\partial x}(g(\delta), \delta)\right) \neq 0$$

for all $\delta \in \Omega$ and all $\omega \in R$. □

Problem reformulation

The robust stability problem can be solved in different ways. One approach is to make a grid over possible values of the vector δ and for each of these values calculate the equilibrium of interest and the corresponding eigenvalues. This will result in long computation times if the number of uncertainties is high.

A similar problem also occurs for linear models with uncertainty. For such models the method of structured singular values (μ-analysis) have been developed, see for example Zhou *et al.* (1996); Skogestad and Postlethwaite (1996); Balas *et al.* (1993). These methods can be used to determine stability of a linear model with uncertainty

$$\frac{dz}{dt} = A(\delta)z$$

where $A(\delta)$ is rational in the elements of the uncertain parameter vector δ. These methods can therefore be directly applied to solve our problem if the Jacobian in Equation (6) is rational in the elements of δ. The remaining part of the paper will therefore focus on finding rational Jacobians.

Heat exchanger example

The example in this section illustrates the importance of taking into account that the location of the equilibrium depends on the uncertain parameter. Also when doing so, the resulting Jacobian is shown to be rational, which makes stability analysis easy using available methods.

Consider the heat exchanger in Figure 2. Two liquid flows q_1 and q_2 with temperatures $T_{1,in}$ and $T_{2,in}$, respectively, are entering the heat exchanger. We assume that the liquids in the two containers are well mixed, and that the outflows are as large as the inflows. The temperature difference between the two containers, $T_2 - T_1$, results in a heat transfer.

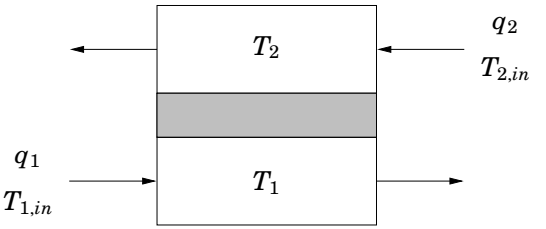

Figure 2. Heat exchanger

We will consider the flow q_2 as the control signal, T_1 as the measured output signal and the flow q_1 as well and the inflow temperatures $T_{1,in}$ and $T_{2,in}$ as disturbances.

A very simple nonlinear model for the heat exchanger is

$$\begin{aligned}\frac{dT_1}{dt} &= \frac{q_1}{V_1}(T_{1,in} - T_1) - \frac{\alpha_1}{V_1}(T_1 - T_2)\\ \frac{dT_2}{dt} &= \frac{q_2}{V_2}(T_{2,in} - T_2) - \frac{\alpha_2}{V_2}(T_2 - T_1),\end{aligned}$$

where $\{V_i\}$ are the container volumes and

$$\alpha_1 = \frac{kA}{\rho_1 C_{p_1}}, \qquad \alpha_2 = \frac{kA}{\rho_2 C_{p_2}},$$

where A is the heat transfer area, k a heat transfer constant, $\{C_{p_i}\}$ heat capacities and $\{\rho_i\}$ the liquid densities. The model is nonlinear since q_1 and q_2 enter multiplicatively.

The nonlinear model has one equilibrium described by

$$q_1^0 = \alpha_1 \frac{T_1^0 - T_2^0}{T_{1,in}^0 - T_1^0}, \qquad q_2^0 = \alpha_2 \frac{T_2^0 - T_1^0}{T_{2,in}^0 - T_2^0}.$$

Linearization around this equilibrium gives the linear model

$$\begin{aligned}\frac{dx}{dt} &= \begin{bmatrix} -\frac{q_1^0+\alpha_1}{V_1} & \frac{\alpha_1}{V_1} \\ \frac{\alpha_2}{V_2} & -\frac{q_2^0+\alpha_2}{V_2} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{T_{2,in}^0 - T_2^0}{V_2} \end{bmatrix} u + w \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x, \end{aligned} \tag{7}$$

where the states, control signal and output of the linearized model are

$$x = \begin{bmatrix} T_1 - T_1^0 \\ T_2 - T_2^0 \end{bmatrix}, \quad u = q_2 - q_2^0, \quad y = T_1 - T_1^0$$

and the disturbances

$$w = \begin{bmatrix} \frac{T_{1,in}^0 - T_1^0}{V_1} & \frac{q_1^0}{V_1} & 0 \\ 0 & 0 & \frac{q_2^0}{V_2} \end{bmatrix} \begin{bmatrix} q_1 - q_1^0 \\ T_{1,in} - T_{1,in}^0 \\ T_{2,in} - T_{2,in}^0 \end{bmatrix}.$$

The temperature T_1 is controlled using a PID-controller,

$$G_{pid}(s) = K(1 + \frac{1}{sT_i} + sT_d),$$

which manipulates the flow q_2, see Figure 3.

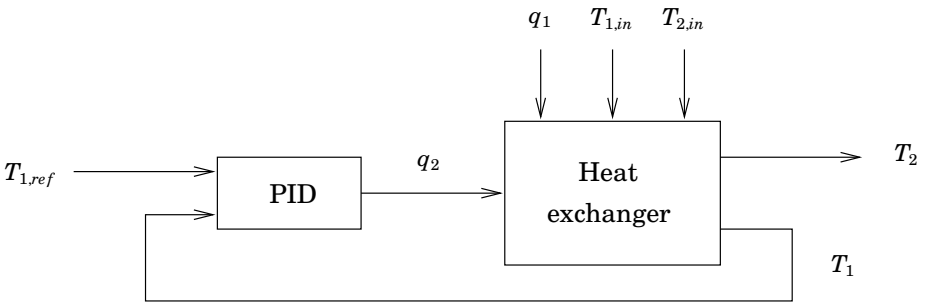

Figure 3. PID-control of the heat exchanger

The integral action in the controller implies that the temperature T_1, for a stable closed loop model, will be correct in stationarity. Assuming

that all the inputs to the closed loop model are constant we obtain the following equations specifying the equilibrium:

$$\begin{aligned} T_1^0 &= T_{1,ref}^0 \\ T_2^0 &= T_1^0 + \frac{q_1^0}{\alpha_1}(T_1^0 - T_{1,in}^0) \\ q_2^0 &= \frac{q_1^0\alpha_2(T_1^0 - T_{1,in}^0)}{q_1^0(T_{1,in}^0 - T_1^0) + \alpha_1(T_{2,in}^0 - T_1^0)}. \end{aligned} \tag{8}$$

These equations show in a direct way how the location of the equilibrium is changed when one of the stationary inputs to the closed loop model is changed.

Assume that the controller parameters are chosen as $K = 2$, $T_i = 0.2$ and $T_d = 0.05$ and that the heat exchanger parameters are $\alpha_1 = \alpha_2 = 1$ and $V_1 = V_2 = 1$. Also assume that the stationary input temperatures are

$$T_{1,in}^0 = 0, \quad T_{2,in}^0 = 30, \quad T_{1,ref}^0 = 10.$$

If the stationary input flow $q_1^0 = 1$, then

$$q_2^0 = 1, \quad T_1^0 = 10, \quad T_2^0 = 20,$$

and the linearized model (7) becomes

$$\begin{aligned} \frac{dx}{dt} &= \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 10 \end{bmatrix} u + w \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x. \end{aligned}$$

The eigenvalues of the closed loop model are -4.7 and $-0.17 \pm 4.6\mathrm{i}$. The PID-design is on purpose very poor.

Now assume that the flow q_1^0 is uncertain. For what values of q_1^0 is the closed loop model stable? To answer this question we must take into account that the other variables, in stationarity, are affected by q_1^0, as described by the equations (8), and illustrated in Figure 4. In this example it is easy to express the equilibrium variables in the uncertain parameter q_1^0. The linearized uncertain model is

$$\begin{aligned} \frac{dx}{dt} &= \begin{bmatrix} -1 - q_1^0 & 1 \\ 1 & \frac{-2}{2-q_1^0} \end{bmatrix} x + \begin{bmatrix} 0 \\ 10(2 - q_1^0) \end{bmatrix} u + w \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x. \end{aligned}$$

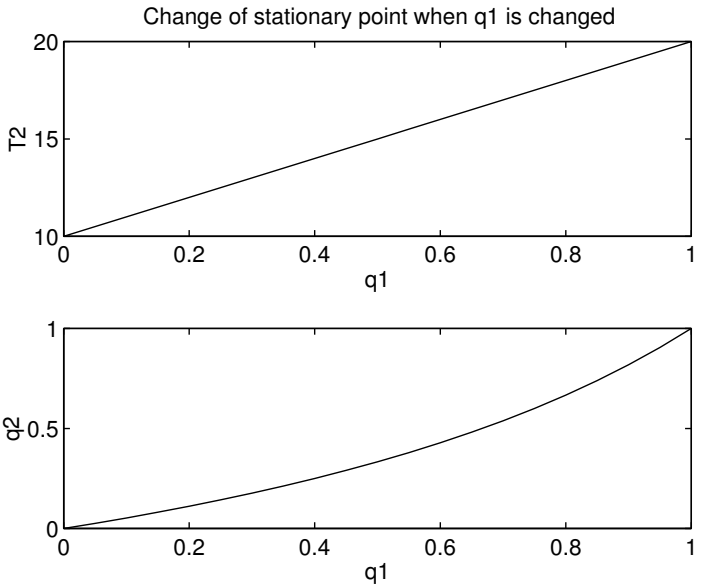

Figure 4. The stationary point moves as q_1 is changed.

The elements of the linearized model are rational functions of q_1^0 and the model can therefore be rewritten in a way suitable for robustness analysis, see Skogestad and Postlethwaite (1996). The closed loop linear uncertain model is stable for $0.78 < q_1^0 < 2.0$. We then also draw the conclusion that the equilibrium of the nonlinear uncertain model is stable for $0.78 < q_1^0 < 2.0$.

It is important to take into account that the location of the equilibrium is changed when the uncertain flow q_1^0 is changed. If we do not, we would obtain the *incorrect* linear uncertain model

$$\frac{dx}{dt} = \begin{bmatrix} -1 - q_1^0 & 1 \\ 1 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 10 \end{bmatrix} u + w$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x,$$

which would lead us to believe that the closed loop model is stable for $0.53 < q_1^0$.

Note that the equilibrium also changes location if the reference value $T_{1,ref}$ or any of the other disturbances to the closed-loop are changed.

3. Change of variables

Linearizing the heat exchanger, with an uncertain flow, resulted in a linear model where the state-space matrices were rational functions of the uncertain parameter. Stability analysis of such models can be done using available methods, see Skogestad and Postlethwaite (1996). If, however, the state-space matrices are not rational functions of the uncertain parameters these methods do not work. One way to cope with this problem is to use a change of variables.

Assume that the Jacobian given in Equation (6) is not rational in the elements of δ, then it might be possible to use a change of variables

$$\delta = T(\varepsilon)$$

such that

$$\frac{\partial f}{\partial x}(g(T(\varepsilon)), T(\varepsilon))$$

is rational in the elements of the new uncertain variable ε.

A common situation is that the Jacobian is a rational function of the elements in x^0 and δ. The main difficulty is then the function $x^0 = g(\delta)$. This function is usually not rational in the elements of δ and it may also be difficult to find an analytical expression. Ideas of how these problems can be avoided using a change of variables is demonstrated by two examples.

Double integrator example

In this example the function $g(\delta)$ is easy to find, but it is not rational. A change of variables is used to cope with this problem.

Consider the process

$$\begin{aligned}\frac{dx_1}{dt} &= x_2\\ \frac{dx_2}{dt} &= f(u) + v,\end{aligned}$$

which is a double integrator with an input saturation. The saturated input signal $f(u)$ together with a load disturbance v, enters the double integrator. The load disturbance is either constant $v(t) = v^0$ or a step. The process is controlled by a PID-controller. The process and its connection to the PID-controller is shown in Figure 5. The nonlinear closed loop model has one equilibrium. The stationary value of the output $x_1 = x_1^0$ is determined by the constant reference value $y_{ref} = y_{ref}^0$, as the controller contains an integrator. The input to the process integrators must be zero

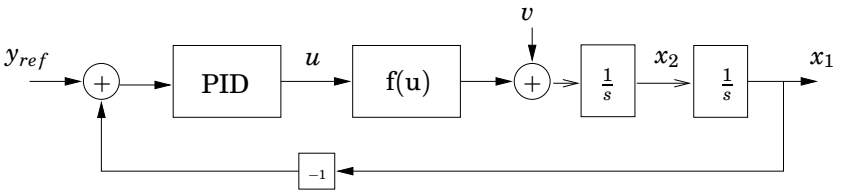

Figure 5. PID-control of a double integrator with an input saturation.

in stationarity; $v = v^0$ must therefore be compensated by a stationary control signal. The location of the equilibrium for different values of y^0_{ref} and small enough v^0 is therefore

$$\begin{aligned} x_1^0 &= y^0_{ref} \\ x_2^0 &= 0 \\ u^0 &= f^{-1}(-v^0). \end{aligned}$$

Linearizing the process model at the equilibrium results in the linearized model

$$\begin{aligned} \frac{d(\tilde{x}_1)}{dt} &= \tilde{x}_2 \\ \frac{d(\tilde{x}_2)}{dt} &= \frac{\partial f}{\partial u}(u^0)\tilde{u} + \tilde{v}, \end{aligned}$$

where we have introduced $\tilde{x}_1 = x_1 - x_1^0$, $\tilde{x}_2 = x_2 - x_2^0$, $\tilde{u} = u - u^0$, and $\tilde{v} = v - v^0$.

Now assume that we would like to investigate for what constant values of the disturbance $|v^0| < \pi/2$ the closed-loop model is stable when

$$f(u) = \arctan(u).$$

The matrix

$$\frac{\partial f}{\partial u}(u^0) = \frac{1}{1 + (u^0)^2} = \frac{1}{1 + \tan^2(v^0)} = \cos^2(v^0)$$

is not rational in v^0. We would therefore like to introduce a new uncertain parameter, ε, instead of v^0, in such a way that the Jacobian is rational in ε. There are different possibilities to do this.

The most immediate idea would be to let $\varepsilon = u^0$ so that

$$\frac{\partial f}{\partial u}(u^0) = \frac{1}{1+\varepsilon^2},$$

which is rational in ε. When we have analyzed the values of ε for which the closed-loop model is stable we find the corresponding values of v^0 using the equation

$$v^0 = -\arctan(\varepsilon).$$

Another possible change of uncertain variable is to introduce

$$\varepsilon = \cos^2(v^0)$$

so that

$$\frac{\partial f}{\partial u}(u^0) = \varepsilon.$$

The best choice among different possibilities depends on how difficult the relation between ε and δ (in this case v^0) will be, as well as on the complexity of the obtained stability problem.

Another example

In this example it is illustrated how the need for explicit equilibrium expressions, that is explicit expressions for $g(\delta)$, can be avoided using a change of variables.

Consider the closed-loop nonlinear system

$$\frac{dx}{dt} = -x + (\delta + x)^2,$$

where δ is an uncertain parameter. The equilibria are the solutions to the equality

$$x^0 = (\delta + x^0)^2.$$

Let $\tilde{x} = x - x^0$, then the linearized model becomes

$$\frac{d(\tilde{x})}{dt} = [-1 + 2(\delta + x^0)]\tilde{x}.$$

If we use the same approach as in the previous example we replace x^0 by

$$x^0 = \frac{1}{2} - \delta \pm \sqrt{\frac{1}{4} - \delta}, \quad \delta \leq \frac{1}{4}$$

resulting in the linearized model

$$\frac{d(\tilde{x})}{dt} = \pm\sqrt{1-4\delta}\cdot\tilde{x}.$$

This expression is not rational in the uncertainty δ and we may proceed with a change of variables as in the previous example to see that the equilibrium

$$x^0 = \frac{1}{2} - \delta - \sqrt{\frac{1}{4}-\delta}, \qquad \delta < \frac{1}{4},$$

is stable while the equilibrium

$$x^0 = \frac{1}{2} - \delta + \sqrt{\frac{1}{4}-\delta}, \qquad \delta < \frac{1}{4},$$

is unstable.

A disadvantage with this approach is that we have to find an analytical expression for the equilibrium of interest, which may be difficult or impossible. This can sometimes be avoided if we use a change of variables without explicit expressions for $x^0 = g(\delta)$.

Consider the change of variables

$$\varepsilon = \delta + x^0$$

then

$$\begin{aligned} x^0 &= (\delta + x^0)^2 = \varepsilon^2 \\ \delta &= \varepsilon - x^0 = \varepsilon(1-\varepsilon). \end{aligned}$$

The interpretation of this change of variables is that we instead of considering δ as uncertain consider $\sqrt{x^0}$ as the uncertain parameter. The resulting linearized model is

$$\frac{d(\tilde{x})}{dt} = (2\varepsilon - 1)\tilde{x},$$

which is rational in the uncertain parameter ε. The uncertain model is stable for $\varepsilon < \frac{1}{2}$.

As a result the stable equilibria and the corresponding values on δ are parameterized as

$$x^0 = \varepsilon^2, \quad \delta = \varepsilon(1-\varepsilon), \qquad \varepsilon < \frac{1}{2},$$

and the unstable as

$$x^0 = \varepsilon^2, \quad \delta = \varepsilon(1-\varepsilon), \qquad \varepsilon > \frac{1}{2}.$$

To obtain the location and type of equilibria as a function of δ numerical methods may be used.

4. Approximation

The idea so far has been to write the Jacobian as a rational function of some uncertain parameters

$$J(\delta) = \frac{\partial f}{\partial x}(g(\delta), \delta) = Q(\varepsilon),$$

where Q is a matrix rational in the elements of ε, $\delta = T_\delta(\varepsilon)$, and $x^0 = T_x(\varepsilon)$.

Not all models have a Jacobian that can be rewritten as a rational function of the uncertainties. In such cases a possible approach would be to approximate it by such a matrix function and capture the approximation error by an additional uncertainty. The ideas can be outlined as follows.

Assume that the Jacobian $\frac{\partial f}{\partial x}(x^0, \delta)$ is a rational function of the scalars x^0 and $|\delta| \leq 1$, while $x^0 = g(\delta)$ is not. Consider Maclaurins formula

$$x^0 = g(\delta) = g(0) + \frac{dg}{d\delta}(0)\delta + \underbrace{\frac{1}{2}\frac{d^2g}{d\delta^2}(\theta\delta)\cdot\delta^2}_{\varepsilon},$$

where $0 \leq \theta \leq 1$. Replacing the last term by an uncertainty, ε, results in a Jacobian rational in δ and ε. The first and second derivatives of $g(\delta)$ can be obtained by differentiating the expression $f(g(\delta), \delta) = 0$ two times. A bound on ε is then obtained using the supremum

$$|\varepsilon| \leq \sup_{|\delta|\leq 1, 0\leq\theta\leq 1} \frac{1}{2}\frac{d^2g}{d\delta^2}(\theta\delta)\cdot\delta^2.$$

5. Conclusions

In this paper we investigated robust stability of equilibria in nonlinear models having parametric uncertainty. The main difficulty was that the

location of the equilibria depends on the uncertain parameters. A problem formulation was given to cope with perturbed equilibria. Examples were given in which the problem could be solved based on a change of variables. When this is possible needs to be further investigated. At the end of the paper approximation methods were discussed as a possible resort when a change of variables does not work.

Acknowledgement

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Paper 5

Robustness Analysis of Large Differential-Algebraic Models with Application to Power Systems

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Abstract

A general computational methodology is introduced for robustness analysis of large differential-algebraic models with parametric uncertainty. The objective is to convert the analysis problem into computation of structured singular values, while at the same time keeping the matrix dimensions down.

The proposed methodology has been implemented using the *OmSim* package for object-oriented modeling and *Matlab* for matrix computations. It is applied to a model of the Nordel power system, including 16 generators, 16 power loads and 20 transmission lines. There are 63 states in total and more than 500 parameters.

1. Introduction

Methods for robustness analysis with respect to parametric uncertainty have during the last decades been developed based on computations of the structured singular value μ. It has been verified that every rational transfer matrix with rational dependence on some uncertain parameters can in principle be analyzed with respect to stability and L_2-gain performance using such computations.

However, even if the algorithms for structured singular value computations can handle matrices of dimension as high as 50-100, not many

applications of this size have been reported. One of the reasons is that proper generation of input data for large problems is a non-trivial task. A typical scenario is as follows. First simulate the large and complicated differential-algebraic model to find a stable equilibrium. The objective is to compute the maximal range of parameter variations for which this equilibrium remains locally stable. To do so, linearize the model around the equilibrium point and then rewrite the resulting parameter dependent transfer matrix as a feedback interconnection involving a constant transfer matrix $M(s)$ and a diagonal parameter matrix $\Delta = \text{diag}\{\delta_1, \ldots, \delta_r\}$. The desired parameter range is obtained from a μ-computation for $M(i\omega)$.

Our focus is on the transition from a nonlinear differential-algebraic model to a transfer matrix $M(s)$. This problem is far from trivial and earlier publications on this topic Lambrechts *et al.* (1993); Steinbuch *et al.* (1991); Skogestad and Postlethwaite (1996); Zhou *et al.* (1996), leave room for many improvements.

As an application of the methodology, we analyze a model of the Nordel power system network. The model is built component-wise in the object oriented modeling tool OmSim described in Andersson (1993); Andersson (1994). The tool is able to generate Matlab m-files with the model matrices on symbolic form. A Matlab script then does the conversion to the input format of the μ-toolbox along the lines described below.

The paper is organized as follows. In the first section we review μ-analysis as a general tool for robustness analysis. Object-oriented modeling using Omola and OmSim is then shortly described in Section 3. Transformation of linear models with uncertainty to a form suitable for μ-analysis is then presented Section 4. The generalization of the ideas to nonlinear models is discussed in Section 5. The derived methodology is applied to a model of the Nordel power system in Section 6. The paper is concluded in Section 7.

2. Preliminaries

This section will present a short review of the structured singular value μ and the concepts associated with it. This short presentation is based on Zhou *et al.* (1996) and Doyle *et al.* (1991).

Definition of μ

The structured singular value μ has been an important concept in robustness analysis since it was introduced in the beginning of the 80s, as a possibility to analyze stability of models with structured uncertainty. One important example of such uncertainties is parametric uncertainty, which will be considered in this paper.

DEFINITION 1
The *Structured Singular Value* is defined as

$$\mu_\Delta(M) = \frac{1}{\min\{\bar{\sigma}(\Delta) : \det(I - M\Delta) = 0\}},$$

unless no Δ makes $I - M\Delta$ singular, in which case $\mu_\Delta = 0$. □

In words; the μ-value is the inverse of the smallest perturbation that causes $I - M\Delta$ to become singular. Note that the μ-value usually is frequency dependent since M usually is frequency dependent.

It is often difficult to calculate μ exactly, in particular when the dimensions of Δ and M are large. Fortunately there exist good and easily computable upper and lower bounds. This is discussed in Fan *et al.* (1991), Doyle *et al.* (1991) and Zhou *et al.* (1996).

Robust stability

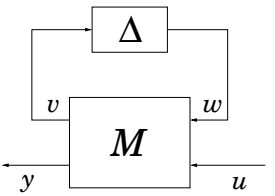

Figure 1. Uncertainty feedback form.

The μ-value as defined in Definition 1 can be used to determine if a dynamic model fulfills robust stability, for a given uncertainty set. In order to use μ-analysis, we need to write the model on *uncertainty feedback form* defined by the relations

$$w = \Delta v$$

and

$$\begin{bmatrix} v \\ y \end{bmatrix} = M \begin{bmatrix} w \\ u \end{bmatrix},$$

as illustrated in Figure 1. Partitioning the matrix M as

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \in \mathbf{C}^{(n+p)\times(n+m)},$$

robust stability can be investigated based on the following theorem.

THEOREM 1
The model in Figure 1 is robustly stable for all Δ such that $\bar{\sigma}(\Delta) < 1$ if and only if

$$\mu_{\Delta}(M_{11}(i\omega)) < 1, \qquad \forall \omega \in [0, \infty].$$

□

Robust performance

In practice we need the system to fulfill performance requirements, and it is not sufficient that the system be robustly stable. Robust performance can be examined by including a feedback Δ_F between the input and output. The resulting model can be seen in Figure 2. The fictitious uncertainty

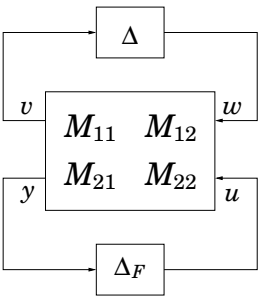

Figure 2. Model for analyzing robust performance.

block, Δ_F, is a full complex uncertainty block. This makes this problem similar to the robust stability problem. The μ-value can be calculated in the same way as before, using Δ_P,

$$\Delta_P = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_F \end{bmatrix},$$

and the full M matrix.

THEOREM 2
The following expressions are equivalent:

$$\mu_{\Delta_P(M)} < 1 \Longleftrightarrow \begin{cases} \mu_{\Delta}(M_{11}) < 1 \\ \| M_{22} + M_{21}\Delta(I - M_{11}\Delta)M_{12} \|_{\infty} < 1 \end{cases}$$

□

Thus, if the μ-value obtained using the performance test is less than one then the model will be stable and fulfill the performance criterion $\|y\|_2 < \|u\|_2$. The model will have robust performance.

3. Object-oriented modeling in Omola

In this section we discuss object-oriented modeling in the modeling language Omola, and how the modeling tool OmSim can be used to provide a set of model equations which later can be used to obtain a model on uncertainty feedback form.

Only a very brief description of the possibilities of Omola is presented here. The interested reader should consult Andersson (1994) for a detailed description of Omola and OmSim and Mattsson (1992) for a description of power system modeling in Omola. A tutorial on Omola and OmSim is presented in Andersson (1993).

Using Omola as a modeling tool

Omola is based on object-oriented modeling, and each type of model is described by different classes. The actual models are instantiations of the class descriptions. The concept of inheritance allows for a natural way of building a class library with re-usage of model-code and good interpretation of the model structure. Modeling in Omola is basically done in two ways; coding directly in the Omola language and graphically using the tool OmSim. The graphical tool is particularly useful for creating large models by the interconnection of sub-models. The tool OmSim also contains a good simulator.

The modeling environment OmSim and the modeling language Omola are often used to investigate dynamical models with algebraic constraints. Nominal stability for a given trajectory of such models can be investigated in OmSim using simulations, whereas there is little support for robust stability analysis. This is the main motivation behind the work described in this paper.

Matrix generation from OmSim

The OmSim tool can export Omola models to Matlab. The matrices of the exported model are expressed in the parameter names used in Omola, and values of these parameters are provided. This symbolic representation is useful in the analyses made in this paper. To be able to use this feature the model has to have the differential-algebraic form

$$\begin{aligned} E(\delta)\dot{x} &= A(\delta)x + B(\delta)u + F(\delta)z \\ G(\delta)z &= H(\delta)x + J(\delta)u \\ y &= C(\delta)x + D(\delta)u + K(\delta)\dot{x} + L(\delta)z. \end{aligned} \tag{1}$$

This is a restriction, as it excludes nonlinear models.

4. Transformation to uncertainty feedback form

The aim of this section is to present a general way of transforming a model described by differential-algebraic equations with parametric uncertainties to a form suitable for robustness analysis using the μ-toolbox.

Transformation without inputs and outputs

Physical models are often stated in terms of algebraic and differential equations. If we exclude inputs and outputs, they can be written, after linearization, on the form

$$\begin{aligned} E\dot{x} &= Ax + Fz \\ Gz &= Hx, \end{aligned}$$

where x is the vector of states and z the vector of algebraic variables. Provided that E and G are invertible, the z-variable can be eliminated and the equations can be rewritten on state space form as

$$\dot{x} = E^{-1}(A + FG^{-1}H)x$$

However, it is often advisable to avoid inverting the matrices and keep the equation on the original form, as it often allows the matrix coefficients to be expressed linearly in terms of physical parameters.

With uncertain parameters appearing linearly in all the matrices A, E, F, G and H, the equations become

$$\begin{aligned} \left(E_0 + \sum_k \delta_k E_k\right)\dot{x} &= \left(A_0 + \sum_k \delta_k A_k\right)x + \left(F_0 + \sum_k \delta_k F_k\right)z \\ \left(G_0 + \sum_k \delta_k G_k\right)z &= \left(H_0 + \sum_k \delta_k H_k\right)x, \end{aligned} \tag{2}$$

where $\delta_1, \ldots, \delta_r \in [-1, 1]$. If E_0 and G_0 are invertible these equations can be rewritten as

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ z \end{bmatrix} &= (R + \sum_{k=1}^{r} \delta_k R_k) \begin{bmatrix} \dot{x} \\ x \\ z \end{bmatrix} \\ &= R \begin{bmatrix} \dot{x} \\ x \\ z \end{bmatrix} + \sum_{k=1}^{r} S_k \delta_k T_k \begin{bmatrix} \dot{x} \\ x \\ z \end{bmatrix}. \end{aligned} \tag{3}$$

The rank factorization $R_k = S_k \delta_k T_k$ should be done with a minimal number n_k of columns in S_k. Define

$$\Delta = \operatorname{diag}\{\delta_1 I_{n_1}, \ldots, \delta_r I_{n_r}\} \qquad S = [\, S_1, \quad \ldots \quad S_r,]$$
$$T = \begin{bmatrix} T_1 \\ \vdots \\ T_r \end{bmatrix} \qquad K(s) = \begin{bmatrix} I & 0 \\ s^{-1} I & 0 \\ 0 & I \end{bmatrix}.$$

Then the equations can be rewritten in the frequency domain as

$$\begin{bmatrix} \begin{bmatrix} sx \\ z \end{bmatrix} \\ v \end{bmatrix} = \begin{bmatrix} R & S \\ T & 0 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} sx \\ x \\ z \end{bmatrix} \\ \Delta v \end{bmatrix} = \begin{bmatrix} R & S \\ T & 0 \end{bmatrix} \begin{bmatrix} K(s) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} sx \\ z \end{bmatrix} \\ \Delta v \end{bmatrix},$$

where s is the Laplace variable. Closing the $\left[\begin{smallmatrix} \dot{x} \\ z \end{smallmatrix}\right]$-loop gives the equation $v = M(s)\Delta v$ where

$$M(s) = TK(s)[I - RK(s)]^{-1} S. \tag{4}$$

Hence the stability condition in Theorem 1 generalizes as follows.

PROPOSITION 3
Suppose that the model (3) is Hurwitz stable for $\delta_1 = \ldots = \delta_r = 0$. Then it is stable for all $\delta_1, \ldots, \delta_r \in [-1, 1]$ if and only if

$$\mu_\Delta \left(TK(i\omega)[I - RK(i\omega)]^{-1} S \right) < 1, \qquad \forall \omega \in [0, \infty]$$

where Δ is described by

$$\Delta = \operatorname{diag}(\delta_1 I_{n_1}, \ldots, \delta_r I_{n_r}).$$

□

Transformations with inputs and outputs

Consider the differential-algebraic model

$$\begin{aligned} E\dot{x} &= Ax + Bu + Fz \\ Gz &= Hx + Ju \\ y &= Cx + Du + K\dot{x} + Lz \end{aligned}$$

where u are the inputs and y the outputs. Performance analysis of this model can be done similar to the approach in the previous section. First write the model on the form

$$\begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} = R \begin{bmatrix} \dot{x} \\ x \\ z \\ u \end{bmatrix} + \sum_{k=1}^{r} S_k \delta_k T_k \begin{bmatrix} \dot{x} \\ x \\ z \\ u \end{bmatrix},$$

where $\delta_1, \ldots, \delta_r \in [-1, 1]$. Define S and T as before to get

$$\begin{bmatrix} \begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} \\ v \end{bmatrix} = \begin{bmatrix} R & S \\ T & 0 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \dot{x} \\ x \\ z \\ u \end{bmatrix} \\ \Delta v \end{bmatrix}.$$

By changing the block partitioning and by introducing $K(s)$ as before we get

$$\begin{bmatrix} \begin{bmatrix} sx \\ z \end{bmatrix} \\ \begin{bmatrix} y \\ v \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \hat{R} & \hat{S} \\ \hat{T} & \hat{U} \end{bmatrix} \begin{bmatrix} K(s) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} sx \\ z \end{bmatrix} \\ \begin{bmatrix} u \\ \Delta v \end{bmatrix} \end{bmatrix}.$$

Closing the $\left[\begin{smallmatrix} \dot{x} \\ z \end{smallmatrix}\right]$-loop gives an equation of the form

$$\begin{bmatrix} y \\ v \end{bmatrix} = M(i\omega) \begin{bmatrix} u \\ \Delta v \end{bmatrix},$$

where

$$M(i\omega) = \hat{T} K(s) [I - \hat{R} K(s)]^{-1} \hat{S} + \hat{U}.$$

Hence applying Theorem 2, the performance theorem, we obtain the following proposition.

PROPOSITION 4
The performance specification $\|y\|_2 \leq \|u\|_2$ holds for all u if and only if

$$\mu_{\Delta_P}(\hat{T} K(i\omega) [I - \hat{R} K(i\omega)]^{-1} \hat{S} + \hat{U}) < 1, \qquad \forall \omega \in [0, \infty]$$

where

$$\Delta_P = \{\mathrm{diag}\{\Delta_F, \delta_1 I_{n_1}, \ldots, \delta_r I_{n_r}\} : \ \bar{\sigma}(\Delta_F) \leq 1, \ \delta_1, \ldots, \delta_r \in [-1, 1]\}.$$

□

Model reduction

When the model with uncertainty is transformed into uncertainty feedback form we need repeated entries of each uncertain parameter. The number of repeated entries of each uncertainty is determined by the rank of a matrix. If the resulting dimension of Δ is large, the model will be time-consuming to analyze. We therefore need to reduce the dimension of Δ, while keeping the dominant behavior of the model.

Reduction algorithms that considers this problem exist, see for example Beck *et al.* (1996); Andersson *et al.* (1999). These methods are related to Balanced truncation and provide error bounds based on solutions to linear matrix inequalities. These methods are, however, time-consuming when the number of uncertainties or the number of repeated entries is large. We therefore present another method based on singular value decomposition.

When doing the rank factorization numerically we consider a singular value decomposition. The rank of a matrix is equal to the number of non-zero singular values. If some singular values are small the impact of considering these as zero will often also be small. This procedure will reduce the rank and thus the dimension of the resulting matrices M and Δ.

The proposed reduction method reduces the number of repeated entries in an efficient manner. However, there are no bounds on the resulting error. This is shown by the following example.

EXAMPLE 1
Consider the following model with uncertainty

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} -a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + \begin{bmatrix} \delta & 0 \\ 0 & \delta \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix},$$

where u is the input, y the output, x the state and $|\delta| \leq 1$ an uncertain real constant. Note that the model contains a two times repeated uncertainty δ. The method suggests that the number of repeated uncertainties could be reduced if either σ_1 or σ_2 is small. The approximation is then done by letting the corresponding σ_k be replaced by zero, implying that one of the repeated entries of δ may be truncated.

The transfer function of the model is

$$G(s) = \frac{cb}{s + a - \delta\sigma_1} + d\delta\sigma_2,$$

so putting $\sigma_1 = 0$ or $\sigma_2 = 0$ will result in the errors

$$e_1 = \frac{cb}{s+a-\delta\sigma_1} - \frac{cb}{s+a} = \frac{cb\delta\sigma_1}{(s+a-\delta\sigma_1)(s+a)},$$
$$e_2 = d\delta\sigma_2 - 0 = d\delta\sigma_2,$$

respectively. Whether σ_1 can be regarded as small or not depends strongly on the values of a, b and c. A predefined level for which σ_1 can be regarded as small can not be given. A similar argument holds for σ_2 (importance depends on d). Also, the values of σ_1 and σ_2 cannot be used directly to compare the importance of the repeated entries of δ. □

The described method has weaknesses. However, it is presently the only fast reduction algorithm available for large models with uncertainty.

5. Nonlinear models

In this section we consider robust stability analysis of nonlinear models. We linearize the model and then apply the results in the previous sections. Special attention has to be payed to the fact that the equilibria change location when a parameter is changed.

Linearization

Linearization of the dynamics is one of the most well-known techniques to simplify nonlinear models. The resulting model is linear, which makes it easier to analyze. A nice property of linearization is that the local stability properties mainly are left unchanged, see Khalil (1996). The discussion below considers an ordinary differential equation but the ideas are easily generalized to differential-algebraic models.

Consider the following uncertain model

$$\frac{dx}{dt} = f(x, \delta). \tag{5}$$

The equilibria of this model are the solutions of the equation

$$f(x^0, \delta) = 0.$$

The location of the equilibria therefore depend on the value of δ. The deviation of the variables from an equilibrium is denoted

$$\tilde{x} = x - x^0(\delta).$$

The dynamics close to an equilibrium is described by

$$\frac{d\tilde{x}}{dt} = \frac{\partial f}{\partial x}\,\tilde{x}.$$

The partial derivative depends on x^0 and δ. As x^0 depends of δ we can rewrite the partial derivative as a functions of δ

$$A(\delta) = \frac{\partial f}{\partial x}(x^0(\delta), \delta). \tag{6}$$

The linearized uncertain model then becomes

$$\frac{d\tilde{x}}{dt} = A(\delta)\tilde{x}. \tag{7}$$

To analyze local stability of the nonlinear model we can therefore analyze stability of this linear model for each possible value of δ.

Series expansion of the uncertainty description

The location of the equilibrium usually depends on the parameters in a complex way and it is often impossible to obtain explicit analytical expressions for $x^0(\delta)$, see Andersson and Rantzer (1999). The elements of the matrix in the linearized model therefore also often depend on the parameters in a complex way. To cope with this problem different types of approximations have been used and suggested, see Andersson and Rantzer (1999).

One way is to simply assume that

$$x^0(\delta) \approx x^0(0),$$

and investigate stability of the Jacobian

$$\frac{\partial f}{\partial x}(x^0(0), \delta).$$

This approach is useful when the location of the equilibrium is almost independent of the uncertain parameters. It also requires that the Jacobian be rational in the elements of the uncertain parameter δ, which is often the case.

If, however, the equilibrium location depends on the parameters, a more accurate model will be obtained if we add another term in the series expansion,

$$x^0(\delta) \approx x^0(0) + \frac{\partial x^0}{\partial \delta}(0)\delta,$$

and investigate stability of the Jacobian

$$\frac{\partial f}{\partial x}(x^0(0) + \frac{\partial x^0}{\partial \delta}(0)\delta, \delta).$$

The Jacobian must in this case be rational in both the elements of δ and x^0. This is often the case.

A third possibility is to approximate the Jacobian directly

$$A(\delta) \approx A_0 + \sum_k \delta_k A_k. \tag{8}$$

This approach does not require that the Jacobian be rational. The above form of the model is also the result of the other two methods if the Jacobian is linear in the parameters instead of rational.

Note that the stability properties of the approximate models only approximate the stability properties of the original model.

The complexity of models of the form (8) may be compared by considering the rank of the matrices A_k. The reason for this is that the rank determines the number of repeated entries we need to use on uncertainty feedback form. The first of the three approaches will in general be the least complex, as changes in equilibrium location usually influences many of the elements in A_k, and therefore often increases the rank dramatically.

Using OmSim

From OmSim it is possible to export an analytical model of the form (1), under the assumption that the Omola model is linear. By linearizing nonlinear components it will be possible to use OmSim to generate a model with the equilibrium variables added to the set of parameters. OmSim may for example generate a model of the form

$$\frac{d\tilde{x}}{dt} = A(x^0, \delta)\tilde{x}.$$

The value of x^0 for different values of δ can be obtained by simulating the original nonlinear model. We can based on such simulations compute approximations of x^0 or A.

It is worth noting that there are many simulation programs that provide features for automatic linearization (as well as exportation of linearized models). Examples are the general purpose simulation program Dymola, Elmqvist *et al.* (1996) and the power system simulation program described in EUROSTAG (1995). Using such programs it would be possible to avoid the manual linearization step.

6. Robustness analysis of the Nordel power system

In this section we apply the methodology described in the previous sections to a power system model.

A Nordel power system model

The considered model is described in, Akke (1989); Eliasson (1990), and models the Nordel power system. The loads and generating units are described using 16 nodes each consisting of one load and one generator. The interconnections are described using 20 lines. The aggregated model is illustrated in Figure 3. The considered model consists of 63 states and hundreds of parameters.

Generating units The generating units are described as synchronous generators. These generators, in the Nordel power system, generate sinusoidal voltages and currents with a frequency close to 50Hz, which is the reference frequency.

Each generator is described by the electrical dynamics

$$T_{d0}^{'}\frac{dE_q^{'}}{dt} = -E_q^{'} + E_f + (X_d - X_d^{'})i_{sd}$$

$$\begin{aligned} u_{sd} &= -X_q^{'} i_{sq} \\ u_{sq} &= X_d^{'} i_{sd} + E_q^{'}, \end{aligned}$$

the generated terminal voltage and electrical power

$$V_t = \sqrt{u_{sd}^2 + u_{sq}^2}$$

$$P_{el} = E_q^{'} i_{sq} + (X_d^{'} - X_q) i_{sq} i_{sd},$$

and the mechanical dynamics

$$\frac{2H}{\omega_{el}^0}\frac{d\omega_{el}}{dt} + \frac{D}{S_n}\frac{1}{\omega_{el}^0}\omega_{el} = \frac{P_{mek}}{S_n} - \frac{P_{el}}{S_n}$$

$$\frac{d\theta}{dt} = \omega_{el} - \omega_{el}^0.$$

The generator variables can be found in Table 1 and the parameters in Table 2.

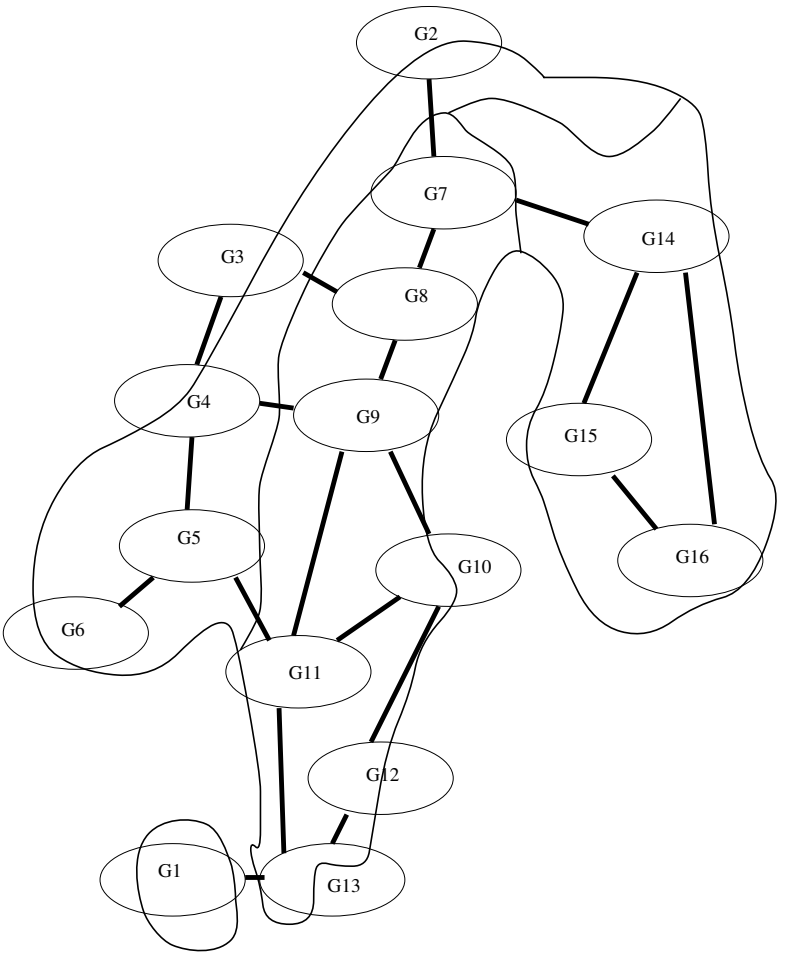

Figure 3. A model of the Nordel power system.

As different generators only rotate with approximately the same angular velocity it is common to use a reference frame rotating with angular velocity ω_{el}^0. The angular deviation from the reference is denoted θ. To express the local currents and voltages in the global reference frame we use the coordinate transformations

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} u_{sd} \\ u_{sq} \end{bmatrix}$$
$$\begin{bmatrix} i_x \\ i_y \end{bmatrix} = \begin{bmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} i_{sd} \\ i_{sq} \end{bmatrix}.$$

i_d	current, d-axis component
i_q	current, q-axis component
u_d	voltage, d-axis component
u_q	voltage, q-axis component
$E_q^{'}$	q-axis transient emf
E_f	Field voltage
V_t	terminal voltage
P_{el}	Generated active power
P_{mek}	Mechanical power
ω_{el}	rotor angular velocity, electrical radians/sec
θ	rotor angle, electrical radians
i_x	current, global d-axis component
i_y	current, global q-axis component
u_x	voltage, global d-axis component
u_y	voltage, global q-axis component

Table 1. Generator variables.

ω_{el}^0	reference angular velocity, $2\pi \cdot 50$ rad/s
$T_{d0}^{'}$	transient open-circuit time constant, d-axis
X_d	synchronous reactance, d-axis
$X_d^{'}$	transient reactance, d-axis
$X_q^{'}$	transient reactance, q-axis
H	Inertia constant
D	Damping coefficient
S_n	Base machine power

Table 2. Generator parameters.

The different coordinates are illustrated in Figure 4.

Controllers It is important to keep the frequency and voltage at their reference values. In case of the frequency this is done by controlling the turbine that drives the generator. The turbine speed was earlier controlled by a Watt centrifugal mechanism using fly-balls but is now done electro-hydraulicly. The voltage is controlled using a so called automatic voltage regulator (AVR). This controller measures the generator voltage and manipulates the field voltage, E_f, delivered by the exciter. In some cases the

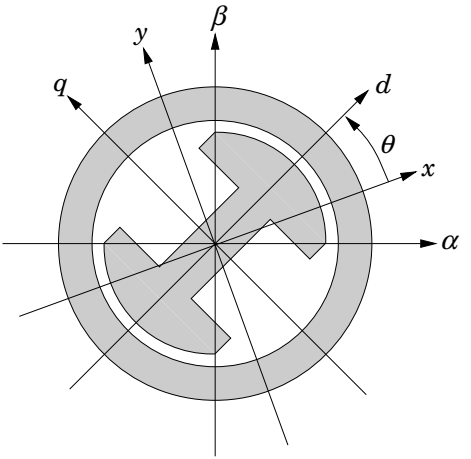

Figure 4. Coordinate systems. The stator is fixed in $\alpha\beta$-coordinates and the rotor in dq-coordinates. The angle of the rotor is θ radians in the reference xy-coordinates. The dq and xy vectors are rotating with the angular velocity ω_{el} relative to the $\alpha\beta$ coordinates.

performance of the power system needs to be improved. It is in such cases common to add a power system stabilizer (PSS) to the AVR. This controller often measures the generated power, P_e, and adds a contribution to the field voltage. For further details see Machowski *et al.* (1997).

In this paper we use an AVR of the form

$$E_f = E_{f0} + \frac{K_v}{sT_v + 1}(V_{ref} - V_t),$$

where $T_v = 0.5$. We assume that the mechanical power P_m is constant and do not use any PSS.

Load and lines Individual power system loads are of various types and are switched on and off at different times. It is therefore difficult to make accurate models of the aggregated load. Commonly used approximations assume that the power consumption depends on voltage and perhaps frequency, but not on time. In case of voltage dependence it is common to use a series expansion

$$P = P_0 + k_1 V + k_2 V^2.$$

The first term corresponds to loads with constant power, the second to loads with constant current and the last to loads with constant impedance.

In this paper we assume that the load impedance is constant. Assuming that also the lines can be modeled as impedances then the grid can be described by the complex admittance based equation $I = YV$, resulting in

$$\begin{bmatrix} I_x \\ I_y \end{bmatrix} = \begin{bmatrix} \text{Re } Y & -\text{Im } Y \\ \text{Im } Y & \text{Re } Y \end{bmatrix} \begin{bmatrix} V_x \\ V_y \end{bmatrix}, \quad I_x = \begin{bmatrix} i_{x1} \\ i_{x2} \\ \vdots \end{bmatrix}, \quad \text{etc.}$$

The complex admittance matrix has the following structure. Off-diagonal elements represent lines, while diagonal elements represent both loads and lines.

$$\mathrm{Y} = \begin{bmatrix}
* & & & & & & & & & * & & & \\
 & * & & & & & & & & & & & \\
 & & * & * & & & & & & & & & \\
 & & * & * & & & & & & & & & \\
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 & & & & & & & & & & & & * & * & * \\
 & & & & & & & & & & & & * & * & *
\end{bmatrix}.$$

Further details on power system modeling can be found in Machowski *et al.* (1997); Kundur (1994); Anderson and Fouad (1977); Bergen (1986).

Robustness analysis

Robustness analysis of power systems is a difficult problem. First of all, power system models are usually nonlinear with many states and parameters. Secondly, the parameter values and the model structure varies with time. The parameter variations are often caused by variations in power

demand and power consumption and a possible cause of model structure changes is failures.

It is of course not possible to investigate all possible situations. Instead some interesting cases are studied. For each of these cases it is interesting to vary some of the more important parameters. Typical examples of interesting parameters to vary are reference values, power demand, power generation, and controller parameters. If we assume that the parameter changes appear abruptly and seldom then it is possible to assume that the parameters are constant but uncertain.

In this paper, as mentioned earlier, we consider a model of the Nordel power system, consisting of 63 states and hundreds of parameters. The eigenvalues of this model, in the nominal case considered, are shown in Figure 5. The plot shows that many of the eigenvalues have low damping.

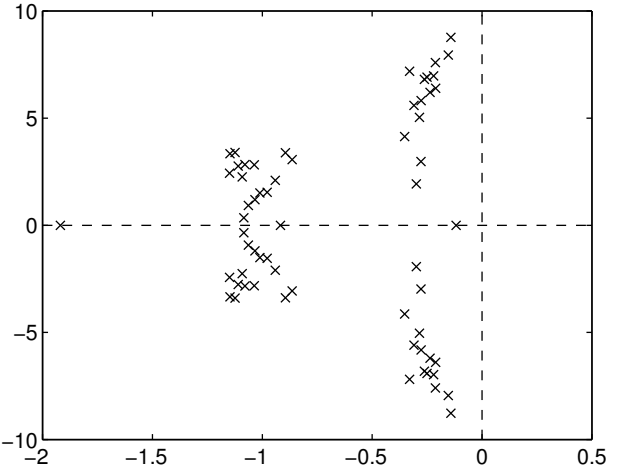

Figure 5. Nominal eigenvalues.

We therefore expect that relatively small load changes could make the model unstable. We will assume that each aggregated load varies with at most 10% and analyze robust stability using the methodology presented in previous sections.

Model with uncertainty The loads are modeled by impedances and we will assume that the real part of each load-impedance (corresponding to active power) is uncertain. Two different models of the form

$$\frac{d\tilde{x}}{dt} = \left(A_0 + \sum_{k=1}^{16} A_k \delta_k \right) \tilde{x}, \tag{9}$$

are used. In the first model we assume that the location of the considered equilibrium does not depend on load changes, whereas in the second model a first order approximation of such effects are included. The rank of the matrices $\{A_k\}$ is two in the first case and 32 in the second. This shows that including such effects dramatically increases complexity.

It is interesting to investigate the change of equilibrium. The change is more significant locally than globally, which means that changes in the load at for example Node 10 has more impact on signals at Node 10 than at other nodes. This is illustrated in Figure 6, where the generator angle, θ, is shown before and after increasing the load by 10%. The local changes, at

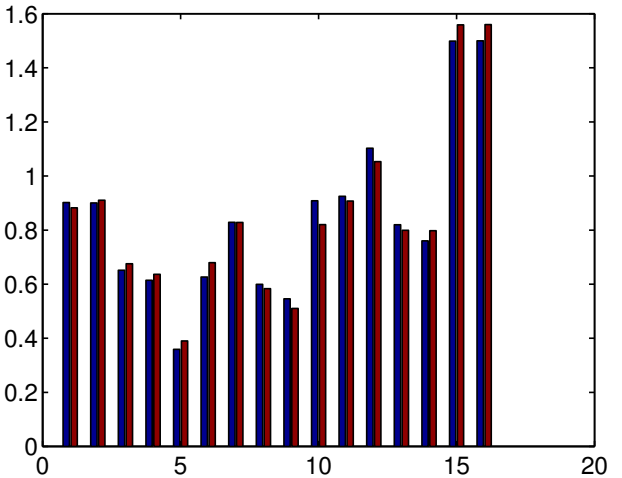

Figure 6. The normalized angle θ at the 16 different nodes before and after increasing the load at Node 10 by 10%.

Node 10, are shown in Figure 7. Among the different variables it seems that the angle is the most sensitive one. This is a general observation for all nodes. The changes in equilibrium location are large enough to influence the robustness analysis. This is shown below.

μ-Analysis To investigate robust stability we apply μ-analysis, see for example Zhou *et al.* (1996). This can be done without any problems for the first model, where the dimension of Δ is $16 \cdot 2 = 32$. If, however, the changes of equilibrium location is considered, the dimension of Δ will instead be $16 \cdot 32 = 512$. Then the computation time, for a reasonable frequency resolution, will be many days on a Sun Ultra 10. It is then valuable to reduce the dimension of Δ.

The singular values of $\{A_k\}$ are shown in Figure 8 and Figure 9. The

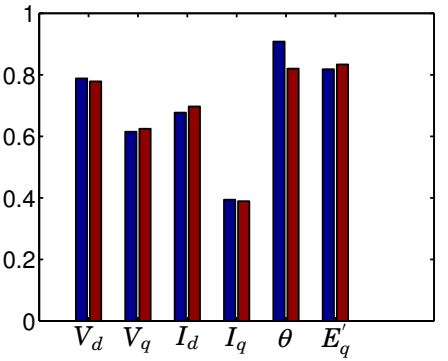

Figure 7. The normalized local variables at Node 10 before and after increasing the load at Node 10 by 10%.

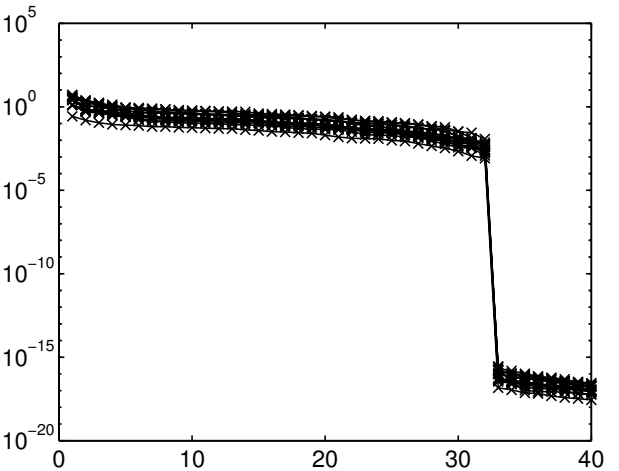

Figure 8. The first 40 singular values of $\{A_k\}$.

first figure confirms that each matrix has rank 32. The second shows that there is no obvious way to choose a level σ_{min} such that all smaller singular values can be neglected. Some levels and the corresponding dimensions of Δ are given in Table 3. Computations show that numerically feasible models are obtained if σ_{min} is about 0.3 or larger. Upper bounds on μ for some of the models in Table 3 are shown in Figure 10. It is seen that the frequencies 2.1 rad/s and 2.9 rad/s are the most critical ones for all models, and that we can allow about 10% changes in the loads without loosing stability. Decreasing σ_{min} below 0.5 affects the magnitude at the

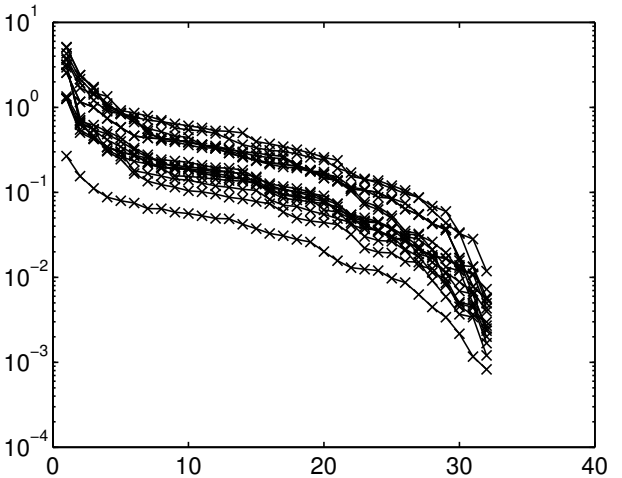

Figure 9. The first 32 singular values of $\{A_k\}$.

σ_{min}	$\dim(\Delta)$
0	512
0.1	296
0.2	189
0.3	136
0.5	76

Table 3. Level for truncation of singular values and the corresponding dimensions of Δ.

most critical frequency but not at other frequencies.

We also investigate the influence on μ caused by neglecting changes in equilibrium location. Figure 11 compares the upper bounds obtain with and without considering that the equilibrium is perturbed. The figure shows that there is a noticeable difference between the μ-plots. Note for example that the critical frequencies are shifted.

It is possible to use μ-analysis to compute critical values of the uncertain parameters. Figure 12 shows the critical changes of the different loads. Some of the loads should be increased while others should be decreased in order to make the model unstable. The resulting eigenvalues are shown in Figure 13. Note that two of the eigenvalues are in the right half plane and that the resulting model therefore is unstable.

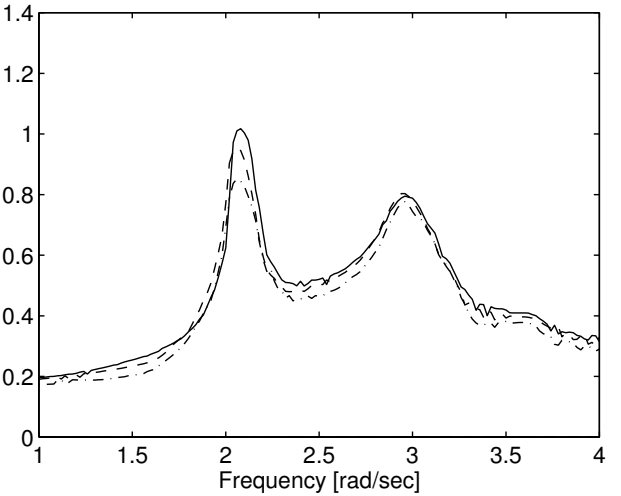

Figure 10. Upper bounds on μ. The solid line shows μ when $\sigma_{min} = 0.2$, the dashed line when $\sigma_{min} = 0.3$ and the dash-dotted when $\sigma_{min} = 0.5$.

We conclude that there is a trade-off between the computation-time and the accuracy of the μ-plot. Neglecting that the location of the analyzed equilibrium depends on the uncertain parameters or using large reductions result in fast computations but a less accurate μ-plot.

7. Conclusions

We have presented a general methodology for robustness analysis of large nonlinear differential-algebraic models with uncertain parameters. The methodology is based on model simplification followed by μ-analysis, and has been successfully applied to a large power system model.

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We would like to thank Olof Samuelsson and Mats Larsson at the Department of Industrial Electrical Engineering and Automation, Lund Institute of Technology, for valuable comments and discussions.

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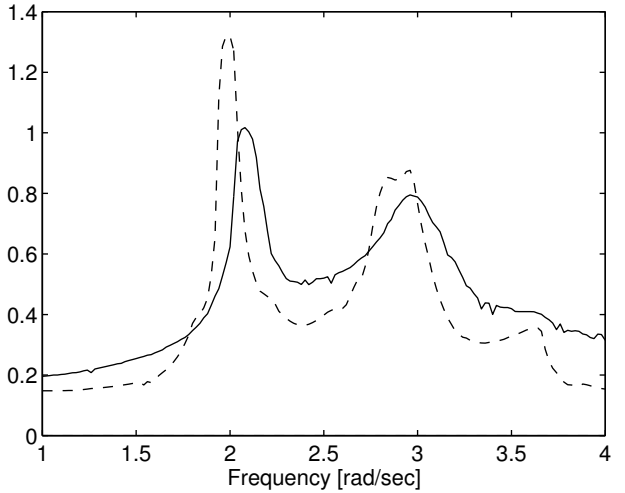

Figure 11. Upper bounds on μ. The solid line shows μ when changes in equilibrium location are considered, while the dashed line shows μ when these effects are neglected.

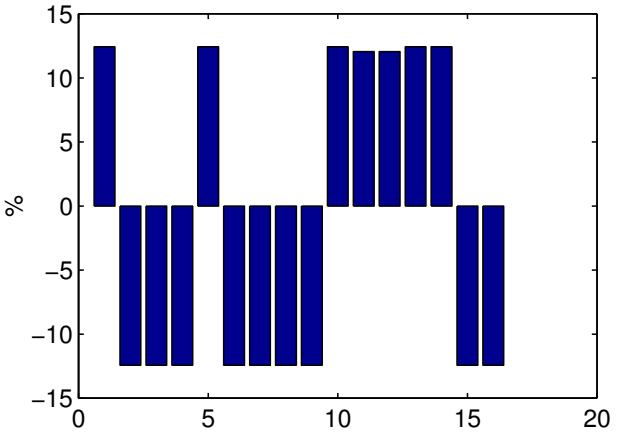

Figure 12. Critical change of the different loads, which will make the model become unstable.

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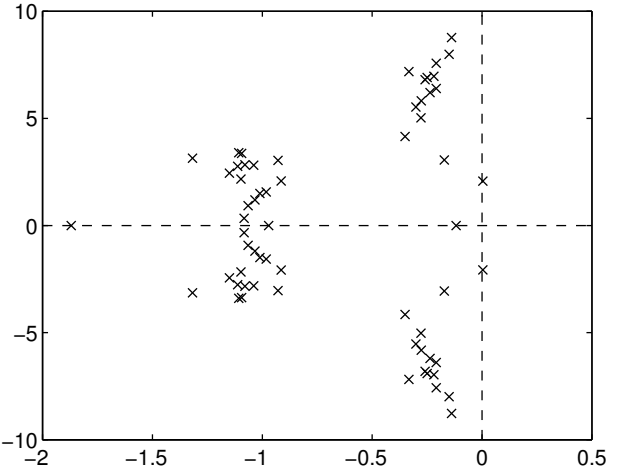

Figure 13. Eigenvalues for the constructed load changes.

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