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MATHEMATICAL METHODS OF A PULP AND PAPER MILL SCHEDULING PROBLEM [x]

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ABSTRACT

Mathematical methods for production scheduling of a complex integrated pulp and paper mill are discussed. The scheduling problem is formulated as an optimal control problem for a multivariable deterministic system. A method of solution for the problem, using optimal control theory, is derived. The method developed is suitable for an on-line process control computer.

x This work has been carried out as part of a process computer project at the Gruvön mill of Billeruds AB.

Contents

1. INTRODUCTION

A production co-ordination problem of the Gruvön pulp and paper mill of Billeruds AB has been described in [7] and in further detail in [8] and [10]. In this report, the mathematical methods developed to solve the problem are discussed.

A mathematical model of the mill is presented in ch 2. In ch 3, scheduling objectives are discussed and formulated mathematically. When formulating the problem, computational aspects are discussed, bearing in mind that the problem must be solved on a process computer (IMB 1800) with a computing time not exceeding 1-2 hours.

The scheduling problem has been attacked using simulation and optimization methods. The simulation and its results are described in ch 4. The simulation technique is not really useful during operating conditions, since the manual work required is too great. However, it is a very valuable aid when studying the problem.

The first optimization attempt was linear programming, as described in ch 5. The size of the problem turned out to be far beyond the capacity of a process computer. Thus, the execution time for a typical problem was about 40 minutes on an IBM 7044. Looking into other methods, a formulation based upon the Pontryagin maximum principle turned out to be successful. In ch 6, the maximum principle is applied to the planning problem. In ch 7, a method of solution for the problem is derived. The work is carried out using a simple two-dimensional model. The results are then generalized to the model of Gruvön and the final formulation of the problem is presented, ch. 8.

The solution technique developed, as described in ch 8, can be characterized as a successive solution of a number of small linear programming problems (about 50 rows and 40 columns) defined and linked together by means of the maximum principle. The result can also be interpreted as a decomposition algorithm of a linear programming problem.

To carry out the calculations, FORTRAN programs are written for the IBM 1800. The programs cover about 15,000 words and the execution time during time-sharing is about one hour, assuming 50% load of priority programs.

In order to illustrate the structure of the solutions obtained by the optimization method developed, some planning examples are given in ch 9.

2. MATHEMATICAL MODEL

In order to handle the co-ordination problem, a mathematical model of the mill has been developed. The modeling and the approximations made are discussed in [7] and in further detail in [8] .

The model is illustrated in figure 2.1. It consists of

3 paper machines,	the production of which is assumed to be known as a function of time during a period of 2-3 days (the planning period)
9 process units,	the production of which during the period is to be determined
10 storage tanks,	the contents of which are assumed to be known at the beginning of the planning period.

The model flows are of pulp, liquors and steam. Concentrations of chemicals are assumed to be constant. The dynamics of individual processes is neglected. The ratio between flows around a process unit is assumed to be constant.

The model is described mathematically in the following way (cf fig. 2.1). The state of the system is described by the storage tank levels $x_1,\dots,x_{10}$, components of the state vector x(t). As controls $u_1,\dots,u_9$ (components of the control vector u(t) the productions of the processes are chosen. One of the flows entering or leaving a process is regarded as a measure of the production of that process. Since all flows around a process are assumed to be proportional, the choice is arbitrary from a mathematical point of view. The actual choice of controls (cf fig. 2.1) is motivated by practical reasons (agreement with mill practice, possibility of measurement).

The given paper production is regarded as a disturbance to the system, denoted by the disturbance vector v(t) with 3 components. Since the steam flow of the system has no storage tank, a scalar, S, is introduced in order to describe the steam balance of the system.

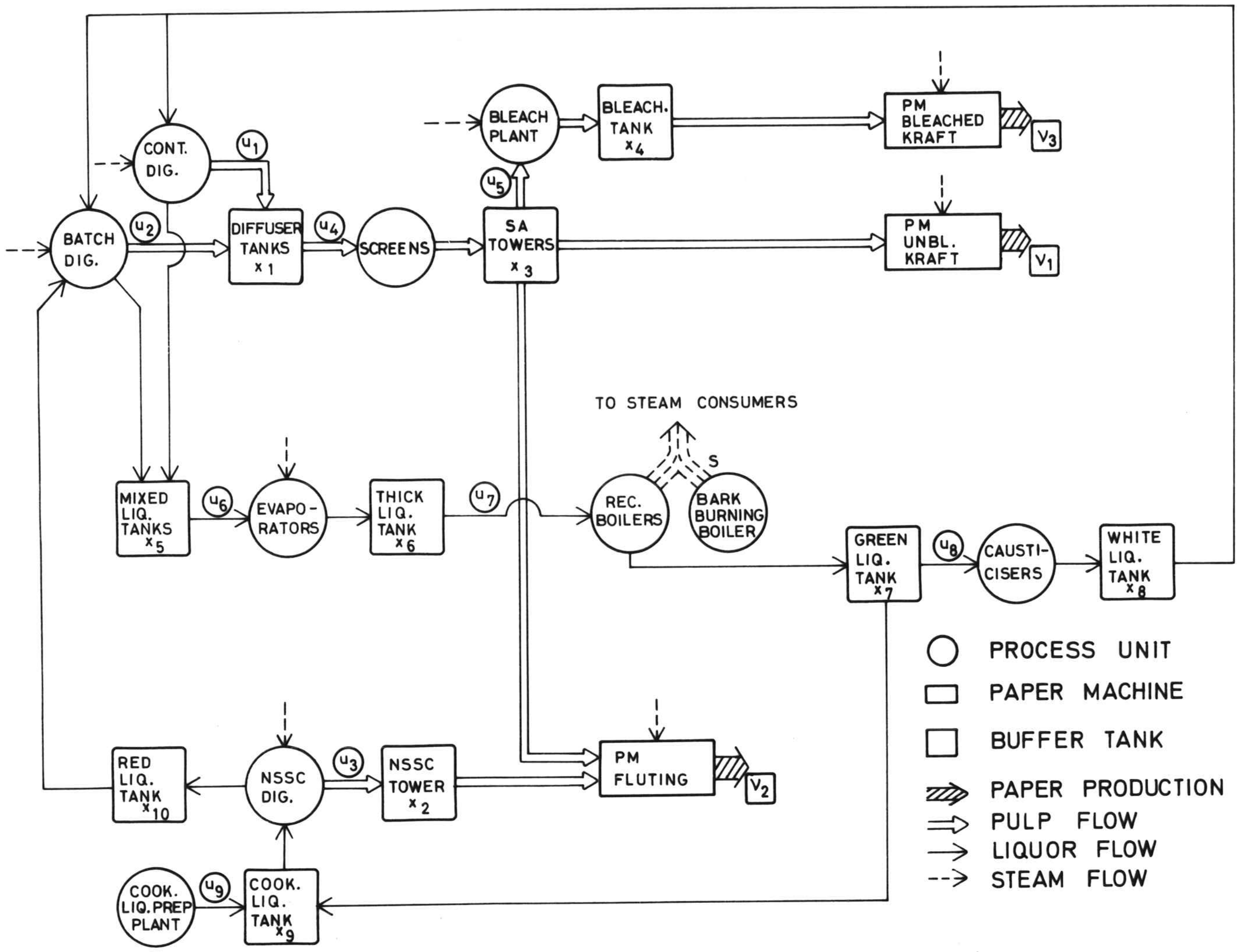

Fig *2.1* Model of the Gruvön mill. The model consists of 3 paper machines, 9 process units and 10 storage tanks, interconnected by flows of pulp, liquors and steam. The ratio between flows around a process is assumed to be constant.

The relations between the state vector x(t), the control vector u(t), the disturbance vector v(t) and the steam variable S(t) are described by

$$\frac{dx(t)}{dt} = B \cdot u(t) + C \cdot v(t) \qquad (2.1)$$

$$S(t) = D \cdot u(t) + E \cdot v(t) \qquad (2.2)$$

B, C, D and E are coefficient matrices describing relations between flows of the model. The matrix sizes are 10 x 9, 10 x 3, 1 x 9 and 1 x 3 respectively.

Since most tanks have one input and one output only, most elements of B and C are equal to zero. The structure of the B- and C-matrices is illustrated in figure 2.2.

$$B = \begin{bmatrix} x & x & o & x & o & o & o & o & o \\ o & o & x & o & o & o & o & o & o \\ o & o & o & x & x & o & o & o & o \\ o & o & o & o & x & o & o & o & o \\ x & x & o & o & o & x & o & o & o \\ o & o & o & o & o & x & x & o & o \\ o & o & o & o & o & o & x & x & x \\ x & x & o & o & o & o & o & x & o \\ o & o & x & o & o & o & o & o & x \\ o & x & x & o & o & o & o & o & o \end{bmatrix} \qquad C = \begin{bmatrix} o & o & o \\ o & x & o \\ x & x & o \\ o & o & x \\ o & o & o \\ o & o & o \\ o & o & o \\ o & o & o \\ o & o & o \\ o & o & o \end{bmatrix}$$

Fig. 2.2 Structure of B- and C- matrices. Elements not identically zero are marked with x.

The variables of the model are constrained by capacity limits, described by

$$x_i^{min} \le x_i(t) \le x_i^{max} \qquad i=1,\ldots,10 \tag{2.3}$$

$$u_i^{min} \le u_i(t) \le u_i^{max} \qquad i=1,\ldots,9 \tag{2.4}$$

$$S^{min} \le S(t) \le S^{max} \tag{2.5}$$

Numerical values of relations between flows of the system are derived in [8] . Numerical values of matrix elements and capacity limits are given in [10] .

From a control point of view the system as described by (2.1)-(2.5) has the following properties:

- the system is not controllable
- the disturbance is deterministic
- the right member of (2.1) does not depend on the state vector x
- the control space is constrained to a convex hyperpolyhedron
- the state space is constrained to a hyperparallelepiped

The model is a rather rough approximation of the real plant. The accuracy of the model has been tested by simulations, based upon measurements performed during normal operating conditions. This is reported in [7] . Despite the approximations made, the model describes the plant in a satisfactory manner [7] .

3. FORMULATION OF THE PROBLEM

Scheduling objectives

The planning must obviously fulfil the following requirements:

- pulp production must satisfy the demand from the planned paper production
- the storage tanks of the system must not be empty or overflowing
- production and consumption of steam must balance

In addition to this, however, it is desirable [7] , [10] to make production schemes involving

(i) few changes in the production rates of the processes
(ii) possibility of indirect storage of steam
(iii) acceptable tank levels at the end of the planning period

Since any change in production rate introduces a disturbance in the system, it is desirable to keep each process production as even as possible. The steam supply of the system is barely sufficient and there is no real possibility of storing steam directly [7] . It would, therefore, be desirable to store steam indirectly. This can be done by filling up pulp buffers and the thick liquor tank [10] during a period of low steam consumption. After this period, the high levels of these tanks (No. 2,3,6) enable us to run the paper machines harder because there will be less demand for steam to digesters and evaporators. Condition (iii) implies that tank levels at the end of the planning period must give a good position before the next period commences. This means, that final tank levels should usually be about 50 %.

As initial data, planned paper production $v(t)$ during the planning period $0 \leq t \leq T$ and initial tank levels $x(0)$ are assumed to be given.

The scheduling problem can now be formulated as follows: given $x(0)$ and $v(t)$, $0 \leq t \leq T$, determine $u(t)$, $0 \leq t \leq T$ without violating the restrictions (2.3)-(2.5) and while satisfying conditions (i) - (iii) as well as possible.

Computational aspects

From practical reasons, once a production planning problem is stated, a solution must be obtained within one or at the most two hours [11]. This fact, together with the computer capacity available (IBM 1800), naturally has considerable influence on the choice of solution methods.

In principle, there are two ways of attacking the problem:

1. "Standard" planning problems are defined and solved on a large computer. These solutions are then stored on the process computer disk. When a planning problem arises a similar standard case is looked for on the disk and used as a solution of the current problem.
2. Each planning problem is regarded as unique and solved when it arises.

The first method is attractive since we can use a large computer for the calculations. However, the number of standard cases which must be solved turns out to be enormous. Assume that there are 100 different planning problems. However, how to run the processes depends also on buffer tank levels at the beginning of the planning period. Consider five different initial levels in each tank, i.e. 0, 25, 50, 75 and 100 %. Since we have 10 tanks the number of standard cases to be solved is

$$100 \cdot 5^{10} \approx 10^{9}$$

Even if the time needed to solve each problem is as short as 1 sec. the total time required would be at least 30 years. Hence, this method is impossible.

Now, let us regard each problem as unique. In this case the most attractive method would be the following. Collect initial data and define the problem by means of a process control computer. Then solve the problem on a large computer using a teleprocessing terminal or by manual delivery of the problem. Because of the geographical position of the mill, the latter method was not possible and nor was the teleprocessing technique available. Hence, the problem had to be solved using the process computer. This fact implies great demands on the solution technique, which has to be chosen with due allowance for short execution time and limited core storage capacity.

Optimality criteria

In this section the planning problem is formulated as an optimization problem and the mathematical formulation of the scheduling objectives is discussed.

The scheduling objective (i) (few production changes) is expressed by the performance functional. Objectives (ii) and (iii) (indirect storage of steam and acceptable final tank levels) are expressed by establishing a suitable boundary value x(T).

The performance functional is of the form

$$J(u) = \int_0^T \|u(t) - a(t)\| \, dt$$

where a(t) is a vector that can be physically interpreted as a desired mean value of the production vector u(t) during the planning period. The loss function has the following properties:

- it depends only on the control u(t)
- it is non-linear
- it is a continuous approximation to a minimization of the number of production rate changes
- different a(t) can give the same number of production rate changes

The production scheduling problem can be formulated as an optimization problem in the following way:

Given the system eq.

$$\frac{dx}{dt} = B \cdot u(t) + C \cdot v(t) \qquad (3.1)$$

$$S(t) = D \cdot u(t) + E \cdot v(t) \qquad (3.2)$$

the constraints

$$x_i^{min} \leq x_i(t) \leq x_i^{max} \qquad i = 1,\ldots,10 \qquad (3.3)$$

$$u_i^{min} \leq u_i(t) \leq u_i^{max} \qquad i = 1,\ldots,9 \qquad (3.4)$$

$$S^{min} \leq S(t) \leq S^{max} \qquad (3.5)$$

the planned paper production

$v(t),\ 0 \leq t \leq T$

and the initial tank levels

$x(0)$,

calculate a control strategy $u(t),\ 0 \leq t \leq T$, minimizing the performance functional

$$J(u) = \int_0^T G(x,u,s)ds + G_0(x(T)) \tag{3.6}$$

Proper mathematical formulation of the functions G and G_0 is very important, since the structure of the optimal solution is determined by the optimality criteria. However, the formulation of an objective function is not obvious. Usually, an optimization problem is formulated as a cost-minimizing or profit-maximizing problem. In this case, however, there is no basis for a specification in dollars and cents of the cost of a production change or of a divergence from the desired final tank level. Moreover, criteria (i) and (iii) are contradictory. Thus, it is often possible to obtain fewer production rate changes if the requirement of acceptable final tank levels is waived. It would be very difficult, however, to weigh a production rate change against a certain divergence from the desired end point.

When formulating the scheduling objectives mathematically we must also take the computational aspects into account. Since the problem has to be solved on a process computer, the optimization problem must be rather small and the objective function must be simple.

The state variables (tank levels) of the system are constrained. When a tank is run empty, one of the surrounding processes must be shut down. This will probably result in production drop-out. Tank overflow is also out of the question. Hence, the restrictions of $x(t)$ are essential from a practical point of view. However, constrained state variables are difficult to handle in most optimization methods and it would be desirable to express them via the criteria. This can be done in the following way:

Assume that the constraints of the state variable x are expressed by

$0 \leq x \leq 1$

This can be formulated in the objective function as

$$G(x) = \begin{cases} 0 \text{ if } 0 \leq x \leq 1 \\ \infty \quad \text{otherwise} \end{cases}$$

A continuous approximation to this function is

$$G(x) = (2x - 1)^{2n} \qquad n \text{ integer}$$

A good approximation requires $n \gg 1$. This gives a complicated objective function and difficulties when solving the problem numerically will occur. For n=1 (linear-quadratic optimization) numerical methods are available [6] . However, n=1 will give great disadvantages:

- the function is not sharp enough $\Rightarrow$ the tanks can run empty or flow over
- the control algorithm will try to keep the tanks half-full, i.e. the storage capacities are level controlled
- the solution will be
  $u(t) = L(x(t))$
  where L is a linear function, i.e. u(t) will be changed permanently in opposition to condition (i)

Hence, no attempts have been made to include the constraints of the state variables in the objective function.

Scheduling objective (i)

Criterion (i) requires as few production rate changes as possible. This can be expressed mathematically by the following formulation of G(x,u,t) in eq. (3.6)

$$G(x,u,t) = \sum_{i} \left| \operatorname{sgn}\left(\frac{du_i(t)}{dt}\right)\right| \qquad (3.7)$$

i.e. the objective function is given the value 0 if no production rate has changed and is increased by 1 each time a change occurs.

Formulation of (3.7) in discrete time gives

$$G(x,u,t) = \sum_i |\text{sgn}\ (u_i(t) - u_i(t - \Delta t))| \quad (3.8)$$

where Δt is a suitably chosen time interval.

For the one-dimensional case we will get the following function:

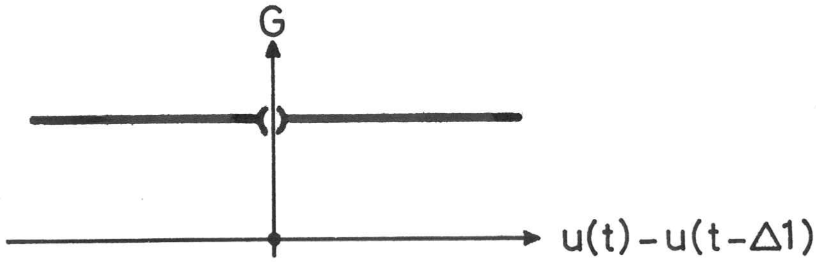

Fig 3.1 Objective function minimizing the number of production rate changes. The function is discontinuous and integervalued.

However, (3.8) is discontinuous and integer-valued. A continuous, real-valued approximation is

$$G(x,u,t) = \sum_i |u_i(t) - u_i(t - \Delta t)| \quad (3.9)$$

For the one-dimensional case we get:

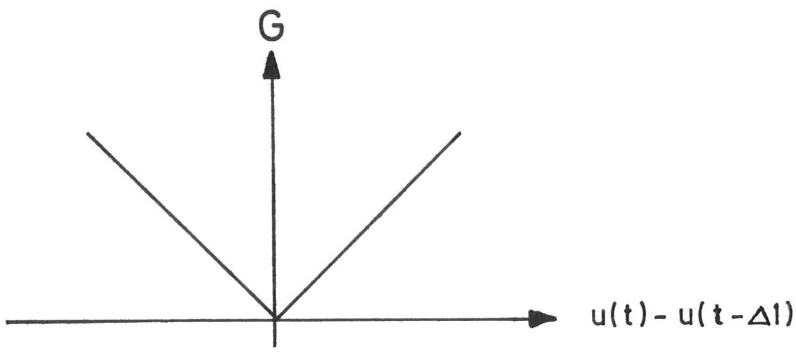

Fig 3.2 A continuous, real-valued approximation to the objective function of figure 3.1.

Put $\Delta t = 1$ (the time unit). Hence

$$G(x,u,t) = \sum_i |u_i(t) - u_i(t-1)| \tag{3.10}$$

This objective function can be linearized (by introducing auxiliary variables) and has been utilized in the linear programming attempts reported in ch. 5. However, the formulation (3.10) has a disadvantage. Once a production rate change has been made, eq. (3.10) tends to keep the production at the new value. This will often result in unacceptable tank levels at the end of the planning period. The following formulation of (i) has been found to be better in this respect:

$$G(x,u,t) = \sum_i |u_i(t) - a_i(t)| \tag{3.11}$$

where a_i are components of a vector a that can be physically interpreted as a desired average of u(t) during the time T.

As an illustration of the calculation of a, consider the model of fig. 3.3. Assume that the output v(t) of the storage tank in figure 3.3 is given, as well as the initial tank level x(0) and the final tank level x(T).

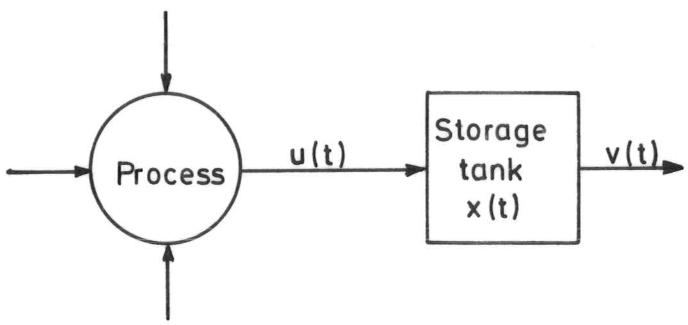

Fig 3.3 A process unit and a storage tank. A value of the vector a can be calculated from a material balance over the storage tank.

A material balance over the tank gives:

$$x(T) = x(0) + \int_0^T u(t)dt - \int_0^T v(t)dt \tag{3.12}$$

Thus

$$\overline{u} = \frac{1}{T}\int_0^T u(t)dt = \frac{x(T) - x(0) + \int_0^T v(t)dt}{T} \tag{3.13}$$

is determined. Now, assume that the storage capacity of x is very great. Then a good control strategy (i.e. a strategy giving few production changes) whould be to run the process u at the rate

$$u(t) = \overline{u}$$

during the whole planning period. This strategy will give no production change and we will reach the desired final levels. In the actual problem, the tank capacities are limited. Besides, the demand for steam balance further complicates the problem, However, in ch. 7 we will show that by putting

$$a = \overline{u}$$

and minimizing

$$|u - a|$$

i.e. by minimizing the deviations from an "ideal" trajectory we will obtain production schedules with few production rate changes. The solution technique developed implies an iteration over the vector a.

Scheduling objectives (ii) and (iii)

The criteria (ii) and (iii) (indirect storage of steam and acceptable final tank levels) can be expressed either as a specified boundary value

$$x(T) \text{ given} \tag{3.14}$$

or via $G_0(x(T))$, for example as

$$G_0(x(T)) = |x(T) - x^*(T)| \tag{3.15}$$

where $x^*(T)$ is a fixed value. We have chosen the formulation (3.14) implying

$$G_0(x(T)) \equiv 0$$

in order to simplify the loss function.

However, it is not essential from a practical point of view to reach the fixed final levels exactly (cf the scheduling objectives). Besides, a fixed boundary value x(T) implies a risk that the problem will be too rigidly structured. In order to avoid this, the following property of the solution technique has been developed: if the restrictions do not permit the tanks to reach the desired values, we will obtain tank levels as close to the fixed ones as possible.

4. SIMULATION

Two different approaches have been used when trying to solve the planning problem

- simulation
- optimization

The simulations have been performed in the following way. Given the planned paper production $v(t), 0 \leq t \leq T$, and the initial tank levels, $x(0)$, guess how to run the processes during the planning period, i.e. guess the control $u(t)$, $0 \leq t \leq T$. Integrate the system equation (2.1) and calculate the steam demand as described by (2.2). If all restrictions are fulfilled, the problem is solved. If not, try new functions $u(t)$ until the restrictions are fulfilled.

This technique does of course, guarantee short solution times on a computer. Even on an IBM 1800 the total time required for integrating the system equations and checking the restrictions will be some few seconds. However, the manual work for problem specification, stating a suitable control function $u(t)$ and changing it is so much more. Thus this method is not really useful during operating conditions. Besides, even if we have found a solution satisfying the restrictions, it is probably not the best one in respect of criteria (i) - (iii). To be able to choose between two or more permitted solutions, a quantitative formulation of the criteria is necessary. However, the simulation technique has been found to be a very valuable aid when studying the behaviour of the model and in order to gain insight into the problem.

As an illustration of the results obtained by the simulation technique, consider the following planning problem. Assume the planning period to be 48 hours, divided into 6 intervals of 8 hours each. The initial tank levels are fixed at 50% and we wish to return to approximately the same levels. Planned paper production is shown by fig. 4.1 A, illustrating a wire change on a large kraft paper machine during interval No. 2 and a wire change on the fluting machine during interval No. 4. The task is to determine a production schedule for all the processes.

The schedule obtained by simulations is shown in figure 4.1.B and the

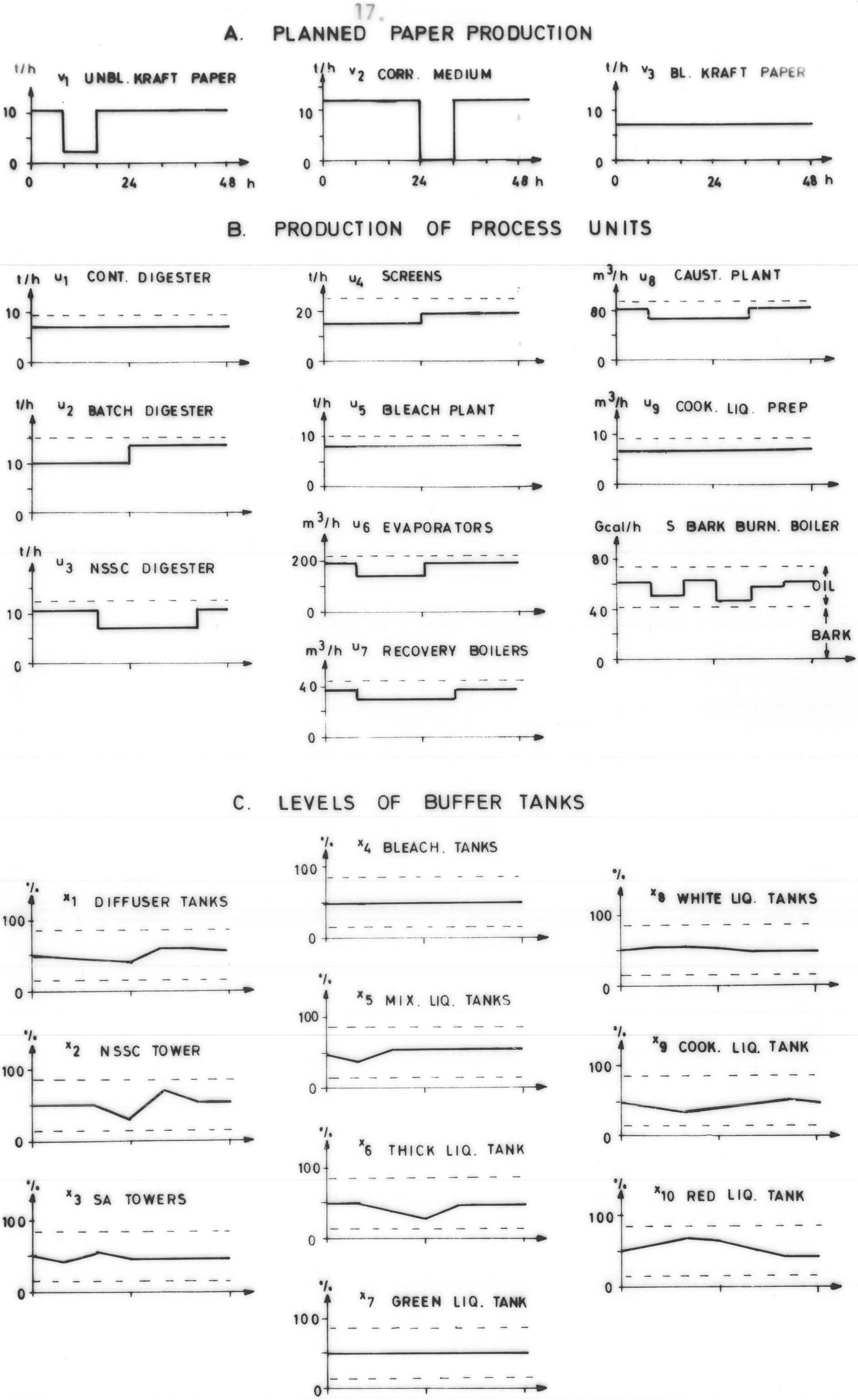

Fig 4.1 Solution of a planning example, obtained by the simulation technique. Figures A show the planned paper production, figures B the production schedule as obtained by simulation and figures C the resultant storage tank levels. Dotted lines are capacity restrictions.

resultant tank levels in figure 4.1.C. The total number of production rate changes during the planning period is 10.

Attempts have been made to systemize the simulations by defining "standard" functions $u_i(t)$ for each process and then combining the functions by combinatorial means, checking if the restrictions are fulfilled [10]. Computer programs have been written in order to carry out the combinations and calculations. For the example above, 288 different combinations were tested by the computer. 96 turned out to fulfil the restrictions. The schedule illustrated in figure 4.1 is one of these.

5. LINEAR PROGRAMMING

In this chapter, the scheduling problem is formulated as a linear programming problem [5] and the results of some tests are discussed.

The magnitude of the LP problems obtained is about 750 rows and 400 columns. The density of the LP matrix (i.e. the relative number of elements $\neq 0$) is very low ($< 0.5\%$). The LP programs have been run on an IBM 7044 and the execution time for a typical problem was about 40 minutes. Thus, the size of the LP problems is far beyond the capacity of the 1800 computer with regard to both execution time and memory capacity.

The optimization problem is given by eq. (3.1)-(3.6). Divide the planning period into N intervals of length τ_k:

$$T = \sum_{k=1}^{N} \tau_k$$

During each interval the production rates $u_i(k)$, $i=1,\ldots,9$ $k=1,\ldots,N$ are assumed to be constant. The sequence

$$\left\{u_i(k)\right\}_{k=1}^{N} \qquad i = 1,\ldots,9$$

is to be calculated. The initial state

$$x_i(0) \quad i=1,\ldots,10$$

and the sequence

$$\left\{v_i(k)\right\}_{k=1}^{N} \qquad i=1,\ldots,3$$

are considered to be known.

We have used two different objective functions. The first one minimizes the production rate changes:

$$V = \sum_{j=1}^{9} \sum_{k=1}^{N} \left| u_j(k) - u_j(k-1) \right| \tag{5.1}$$

The other objective function minimizes the deviation from specified final tank levels:

$$V = \sum_{i=1}^{10} \left| x_i(N) - x_i^*(N) \right| \tag{5.2}$$

$u_i(0)$ of eq. (5.1) and $x_i^*(N)$ of eq. (5.2) are given.

(5.1) and (5.2) are non-linear, but can be linearized by introducing auxiliary variables. To linearize (5.1), introduce the variables

$$g_i(k),\ h_i(k) \quad i=1,..,9 \ k=1,...,N \quad g,\ h \geq 0$$

defined by

$$u_i(k) = u_i(k-1) + g_i(k) - h_i(k)$$

and minimize

$$V = \sum_{i,k} (g_i(k) + h_i(k))$$

Of course, (5.2) is linearized in a similar way.

If we choose (5.1) as an objective function we get the following linear programming problem:

Minimize

$$V = \sum_{i=1}^{9} \sum_{k=1}^{N} \left\{ g_i(k) + h_i(k) \right\}$$

subject to

$$u_i^{min} \leq u_i(k) \leq u_i^{max} \qquad i=1,...,9 \ k=1,..,N$$

$$S^{min} \leq \sum_{j=1}^{9} d_j u_j(k) + \sum_{j=1}^{3} e_j v_j(k) \leq S^{max} \qquad k=1,..,N$$

$$x_i^{min} \leq x_i(k-1) + \tau_k \left\{ \sum_{j=1}^{9} b_{ij} u_j(k) + \sum_{j=1}^{3} c_{ij} v_j(k) \right\} \leq x_i^{max} \quad i=1,..,10 \quad k=1,..,N$$

$$u_i(k) = u_i(k-1) + g_i(k) - h_i(k) \qquad i=1,..,9 \quad k=1,..,N$$

$$u_i(k),\ g_i(k),\ h_i(k) \geq 0 \qquad i=1,..,9 \quad k=1,..,N$$

$u_i(0)$, $i=1,..,9$, $x_i(0)$ $i=1,..,10$ and $\left\{v_j(k)\right\}_{k=1}^{N}$ $j=1,..,3$ are given.

b_{ij}, c_{ij}, d_j and e_j are elements of the matrices B, C, D and E, respectively.

The size of this LP problem is:

49·N restrictions
27·N variables (not including slack variables and artificial variables)

For a realistic problem we must have

$N \approx 15$

giving an LP problem for about 400 variables and 750 restrictions.

An LP problem of this size is relatively large. Analysis of the problem shows, however, that the density (i.e. the relative number of elements not identically zero) of the LP matrix is very low and that the matrix has a particular structure, illustrated by figure 5.1. Due to the low density and simple structure of the LP matrix, rather short execution times for the LP program would be expected.

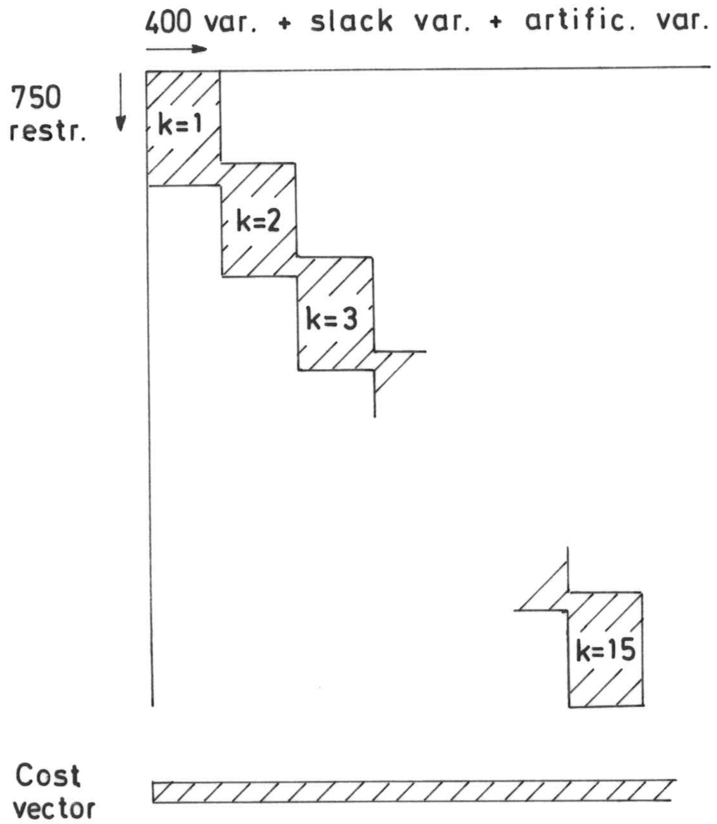

Fig 5.1 The LP matrix. Shaded areas contain elements not identically zero.

Tests of the LP technique have been performed, utilizing an IBM 7044 and the IBM standard program LP 40. Despite the low density (< 0.5%) a problem consisting of about 600 rows gave an execution time of about 40 minutes. It is not possible to solve problems of this size in a process computer.

A detailed description of the LP tests is given in [9]. To illustrate the structure of the solutions obtained and to show the difficulty in formulating a proper objective function, two of these examples are presented here.

LP example 1

The planning problem is identical to the problem discussed in the simulation chapter. The planning period is 44 hours, divided into 11 intervals of 4 hours each. Wire changes are to be made on PN UNBL during 8-16 h and on PM 6 during 24-32 h (cf fig 5.2 A). The initial tank levels are assumed to be 50%. As an objective function, eq. (5.2) has been used. x_i^* (N) are fixed to different numbers, about 50% for most of the tanks.

The production schedule as calculated by the LP program is illustrated in fig. 5.2.B and the resultant tank levels in fig 5.2.C. The target defined has been reached for all tanks except for No. 6 and No. 8. No target had been defined for tank No. 3.

As illustrated by figure 5.2B the production rates have been changed very often. Further, the production often changes from the lower limit to the upper and vice versa. One reason for this is the absence of terms punishing production changes. Another reason is that the solution of an LP problem always lies on the boundary of the permitted region. From a practical point of view it is very disadvantageous to stop and start processes frequently. Hence, a solution as illustrated by this example has no practical value.

LP example 2

In this example, the objective function (5.1) has been used. The planning problem is identical to the problem above. However, in order to reduce the computational time required, the planning period of 48 hours has been divided into 4 intervals of 12 hours each. For this reason, adjustments of the wire changes have been necessary (cf fig. 5.3 A)

The calculated production schedule is illustrated by figure 5.3.B and the resultant tank levels by figure 5.3.C. The production rates for all the processes except one (the recovery boiler) have remained unchanged during the period. However, the storage tanks are made full use of and at the end of the period most of them are empty

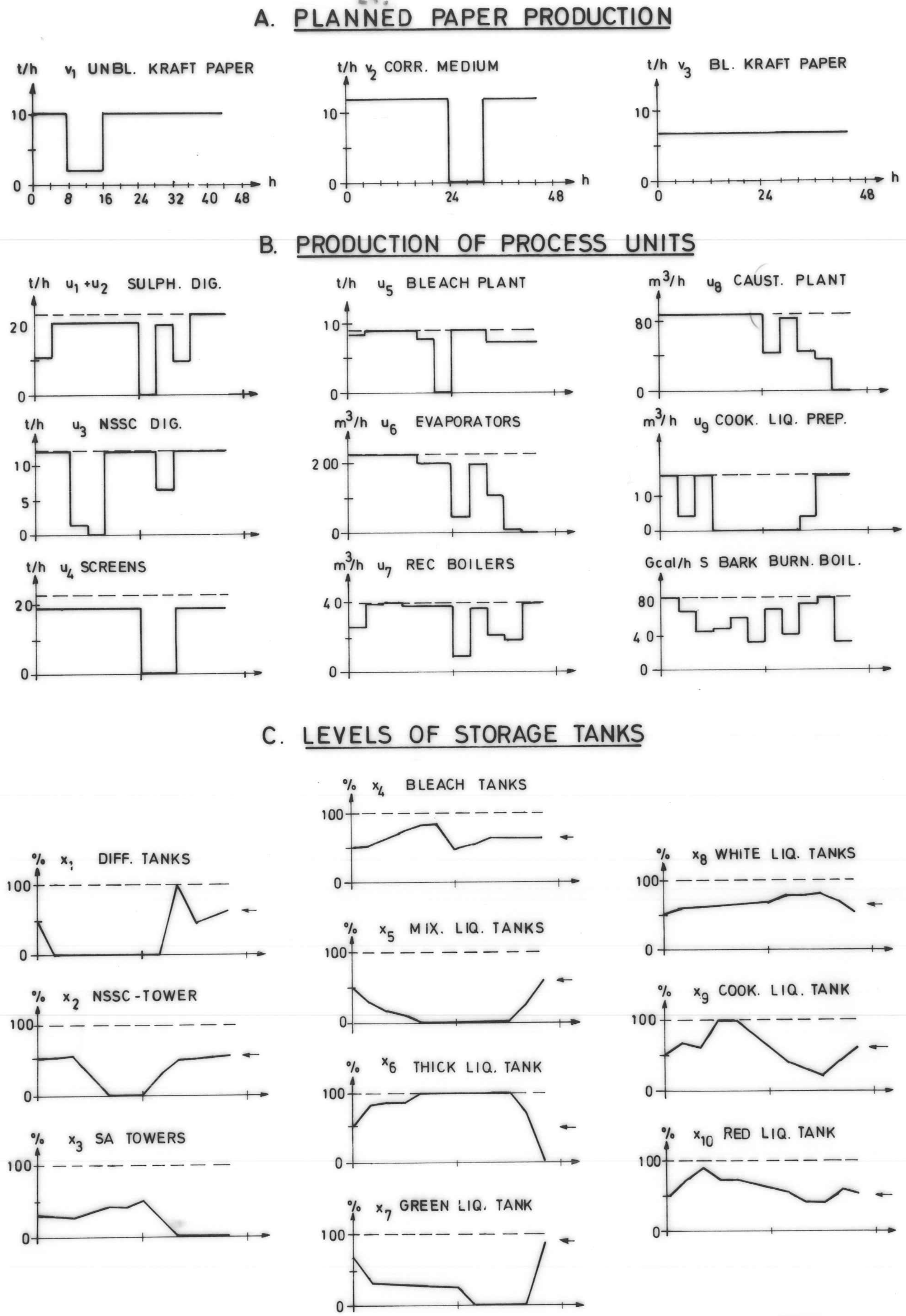

Fig 5.2 Optimal solution of LP example 1 (minimization of the deviation from specified final tank levels). Figures A show the planned paper production, figures B the production schedule as obtained by linear programming and figures C the resultant tank levels. Dotted lines are capacity restrictions. The desired final tank levels are marked by arrows.

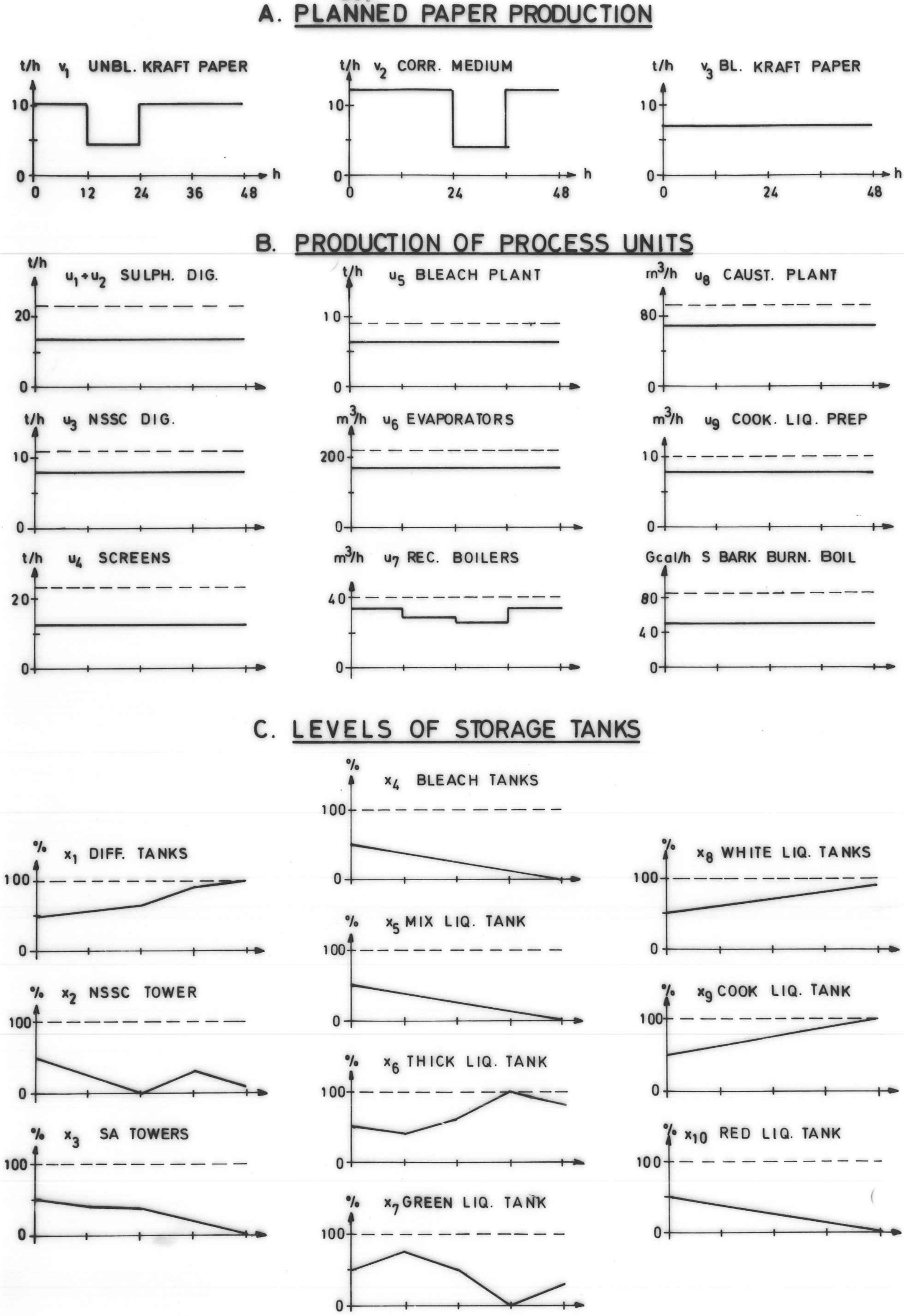

Fig 5.3 Optimal solution of LP example 2 (minimization of the production rate changes). Figures A show the planned paper production, figures B the production schedule as obtained by linear programming and figures C the resultant tank levels. Dotted lines are capacity restrictions.

or completely filled up. This means that the position before the next planning period is very bad. Hence, the solution is not applicable in practice.

Theoretically, the two objective functions (5.1) and (5.2) can be combined. This implies that it must be possible to weigh the cost of a production change against a divergence from the desired end point. As mentioned above, no data were available for such a specification.

Another possible way of solving this problem is to define the wish for acceptable final tank levels as a restriction:

$$x_i^-(N) \leq x_i(N) \leq x_i^+(N) \qquad i = 1,..,10$$

However, the LP demand of computational speed and core storage capacity is far beyond the capacity of a process computer. Hence, the technique described in this section was judged to be impractical for the Gruvön problem.

6. CALCULUS OF VARIATIONS - THE MAXIMUM PRINCIPLE

Neither simulation technique nor linear programming satisfies the requirements on a suitable solution method. Hence, the possibilities of applying the methods developed by Bellman and Pontryagin and based upon calculus of variations were investigated [1].

Dynamic programming [3] , [4] was judged to be unrealistic owing to the requirements of core storage capacity. However, a formulation based upon the theory of the maximum principle [2], [12] turned out to be successful.

In this chapter we will apply the Pontryagin maximum principle to the scheduling problem. A physical interpretation of the adjoint variables will also be given.

We will find that the adjoint vector p(t) of the maximum principle is constant as long as no tank limit is reached. When reaching a tank limit, the corresponding component of p makes a jump to a new constant value, while the other components remain unchanged. Physically, this implies that an input or output flow to the tank must be changed.

It is shown that only a finite number of p-values influence the optimal solution. A vector A, defined by

$$A = p^* B$$

where B is the system matrix, is introduced. It is shown that in principle only 3 values of each component A_i influence the optimal solution, namely

$$|A_i| < 1, \quad A_i > 1, \quad A_i < -1$$

Physically, A has the following properties

$\lvert A_i \rvert < 1 \ \forall i$	the productions of all processes remain unchanged (if the restrictions permit)
$A_i > 1, \lvert A_j \rvert < 1 \quad j \neq i$	the production of the process u_i is reduced, the other productions remain unchanged.
$A_i < -1, \lvert A_j \rvert < 1 \quad j \neq i$	the production of u_i is increased

Application of the maximum principle to the scheduling problem

Our system is described by (cf ch. 2)

$$\frac{dx(t)}{dt} = B \cdot u(t) + C \cdot v(t) \qquad (6.1)$$

where

$x(t)$ is an n-vector of state variables (n=10)

$u(t)$ is an m-vector of control variables (m=9)

$v(t)$ is a given vector function

t is the time

B and C are time-independent matrices

Both the control space and the state space are constrained

$$u(t) \in \Omega_u \subset E^m, \quad x(t) \in \Omega_x \subset E^n$$

Ω_u is a convex hyperpolyhedron described by

$$u_i^{min} \leq u_i(t) \leq u_i^{max} \qquad i=1.,,,9$$

$$S^{min} \leq D \cdot u(t) + E \cdot v(t) \leq S^{max}$$

where D and E are time-independent row vectors

Ω_x is a hyperparallelepiped, described by

$$x_i^{min} \leq x_i(t) \leq x_i^{max} \qquad i=1,,..,10$$

The initial state $x(0)$ is known.

Consider the system during a fixed time-interval $0 \leq t \leq T$ and assume that $x(T)$ is fixed. Thus our problem is a fixed-time, fixed-endpoint problem.

Problem: find a control strategy $u(t)$, $0 \leq t \leq T$, $u(t) \in \Omega_u$ minimizing the performance functional (cf chapter 3)

$$J(u) = \int_0^T \| u(t) - a(t) \| \, dt \qquad (6.2)$$

where $\| y \|$ is the norm

$$\| y \| = \sum_i | y_i |$$

and a(t) is a vector function, that can be calculated from x(0), x(T) and v(t), $0 \leq t \leq T$, and physically interpreted as a desired average of the production vector u(t).

Solution: Introduce the Hamiltonian function

$$H(x,u,p,t) = \| u(t) - a(t) \| + \langle p(t), Bu(t) + Cv(t) \rangle \qquad (6.3)$$

where $\langle a,b \rangle$ denotes the scalar product of the vectors a and b and p(t), the adjoint vector, is a new vector function, not identically zero and satisfying the ordinary differential equation

$$\frac{dp(t)}{dt} = - \frac{\partial H(x,u,p,t)}{\partial x} \qquad (6.4)$$

if the optimal trajectory of the system lies in the interior of Ω_x, and the ordinary differential equation

$$\frac{dp(t)}{dt} = - \frac{\partial H}{\partial x} + \lambda(t) \cdot \frac{\partial \varphi(x,u)}{\partial x} \qquad (6.5)$$

if the optimal trajectory lies on the boundary of Ω_x.

The boundary condition of the differential equations is:

$$x(T) \text{ given} \qquad (6.6)$$

In (6.5) $\lambda(t)$ are certain Lagrange multipliers [12] . φ is defined by

$$\varphi(x,u) = \langle \text{grad } g(x), Bu(t) + Cv(t) \rangle \qquad (6.7)$$

where

$$g(x) = 0 \qquad (6.8)$$

describes the boundary of Ω_x. g(x) must have continuous second partial derivatives [12] . This condition is not fulfilled on the edges and corners of the box Ω_x. However, this difficulty can be avoided by smoothing the edges and corners.

When reaching a boundary of Ω_x, the vector p(t) makes a jump [12] . Assume that the optimal trajectory of the system during the time $t < t_0$ lies in the interior of Ω_x and during $t > t_0$ on the boundary of Ω_x. Introduce

$p^-(t)$ = p-vector for $t < t_0$

$p^+(t)$ = p-vector for $t > t_0$

The following relationship between p^- and p^+ is valid (the "jump condition", [12])

$$p^+(t_0) = p^-(t_0) + \mu \cdot \text{grad } g(x(t_0)) \tag{6.9}$$

where μ is real and not identically zero.

Since Ω_x is a hyperparallelepiped with edges parallel to the co-ordinate axes, the gradient of g is parallel to the unit vector e_i if x_i is the state variable that has reached its upper or lower limit:

$$\text{grad } g(x(t_0)) \sim (0,\ldots 0,\underset{\substack{\uparrow \\ \text{component No. } i}}{1},0,\ldots,0) \tag{6.10}$$

Hence

$$p_j^+(t_0) = p_j^-(t_0) \quad \text{if } j \neq i \tag{6.11}$$

$$p_i^+(t_0) = p_i^-(t_0) + \mu \qquad \mu \neq 0 \tag{6.12}$$

This means the following. When a storage tank has reached its limit, the corresponding element of p makes a jump to a new value, while the other components remain unchanged. Physically, this implies that an input or output flow of the tank must be changed.

From (6.7) and (6.10) we find that $\varphi(x,u)$ is a piecewise constant function of x. Hence

$$\frac{\partial \varphi}{\partial x} = 0 \tag{6.13}$$

(6.5) together with (6.13) now give that the relation (6.4) remains unchanged, i.e.

$$\frac{dp(t)}{dt} = - \frac{\partial H}{\partial x} \tag{6.14}$$

is valid also on the boundary of Ω_x.

Thus we find from (6.3) and (6.14)

$$\frac{dp(t)}{dt} = 0 \tag{6.15}$$

in every point where it is defined. (6.15) together with (2.11) and (2.12) now imply

$$p(t) = \text{piecewise constant} \tag{6.16}$$

during the planning period $0 \leq t \leq T$. The discontinuities appear when a tank reaches its limit.

The Pontryagin maximum principle (MP) states that a necessary condition for minimum of the performance functional $J(u)$ is that

$H(x,u,p,t)$ is minimized as a function of u

The maximum principle is valid if the system is controllable and if certain regularity conditions (continuity, derivability), [2] are fulfilled. Except for the conditions on $g(x)$ discussed above, the regularity conditions are fulfilled here.
However, our system as described by (2.1) is not controllable. (This fact is due to the approximations made when building the model. The model can be made controllable by introducing more variables (cf the planning example of Ref [7]). However, we have chosen to introduce as few variables as possible in order to reduce the size of the model).

Since $v(t)$, $0 \leq t \leq T$, is given, the maximum principle now dictates that a necessary condition for minimum of the performance functional (2.2) is that

$$\|u(t)-a(t)\| + \langle p(t), B \cdot u(t) \rangle \tag{6.17}$$

is minimized as a function of u. The vector p(t) is a piece-wise constant function of t, satisfying the boundary condition

x(T) given

We will now give a physical interpretation of the adjoint vector p(t).

Physical interpretation of adjoint variables

It follows from the maximum principle that the function

$$||u(t) - a(t)|| + p_0^* \cdot B \cdot u(t) \qquad (6.18)$$

is to be minimized. p_0^* is the transpose of the piecewise constant adjoint vector p_0. B, the system matrix, is given. Introduce a vector A, defined by

$$A = p_0^* \cdot B \qquad (6.19)$$

To illustrate the influence of the vector A on the optimal solution, consider the following 2-dimensional problem.

Minimize

$$V = |u_1| + |u_2| + A_1 u_1 + A_2 u_2 \qquad (6.20)$$

subject to

$$|u_1| \leq 1, \quad |u_2| \leq 1 \qquad (6.21)$$

We find that if

$|A_2| < 1$

the optimal solution is

$u_1 = u_2 = 0 \quad \text{if} \ |A_1| < 1$

$u_1 = 1 \quad u_2 = 0 \quad \text{if} \ A_1 < -1$

$u_1 = -1 \quad u_2 = 0 \quad \text{if} \ A_1 > 1$

$u_1 \in [-1, 0] \quad u_2 = 0 \ \text{if} \ A_1 = 1$

$u_1 \in [0, 1] \quad u_2 = 0 \ \text{if} \ A_1 = -1$

Since the problem is symmetric in u_1 and u_2 similar conditions are valid for A_2.

Curves of equal V for some values of A_1 and A_2 are shown in figures 6.1 and 6.2. The optimal point (or line) minimizing V is also marked in the figures.

However, from a numerical point of view, the degenerated cases

$$A_1, A_2 = \pm 1$$

are not of interest. If we disregard these cases, we find that the two-dimensional problem, defined by (6.20) and (6.21) has only 9 distinct optimal points, namely

- the corners of the square (correspond to the result of an LP optimization)
- the intersections between the square and the co-ordinate axes
- the origin

depending on the nine combinations of

$$|A_i| < 1, A_i > 1 \text{ and } A_i < -1, \ i=1,2$$

This is illustrated in figure 6.3

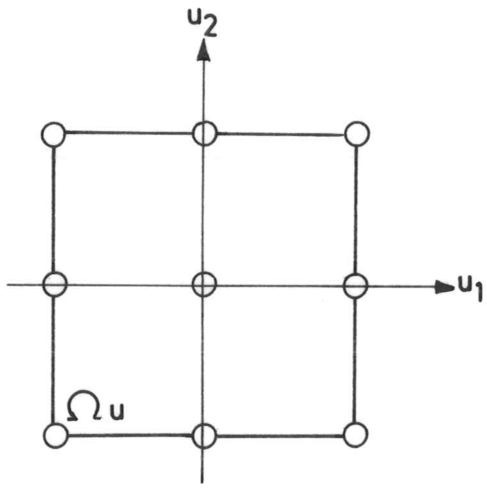

Fig 6.3 **Distinct optimal points (encircled) to the objective function**
$$V = |u_1| + |u_2| + A_1u_1 + A_2u_2$$

Generalizing this result to an n-dimensional problem with Ω_u as a convex hyperpolyhedron, we find that in principle only 3^m A-vectors

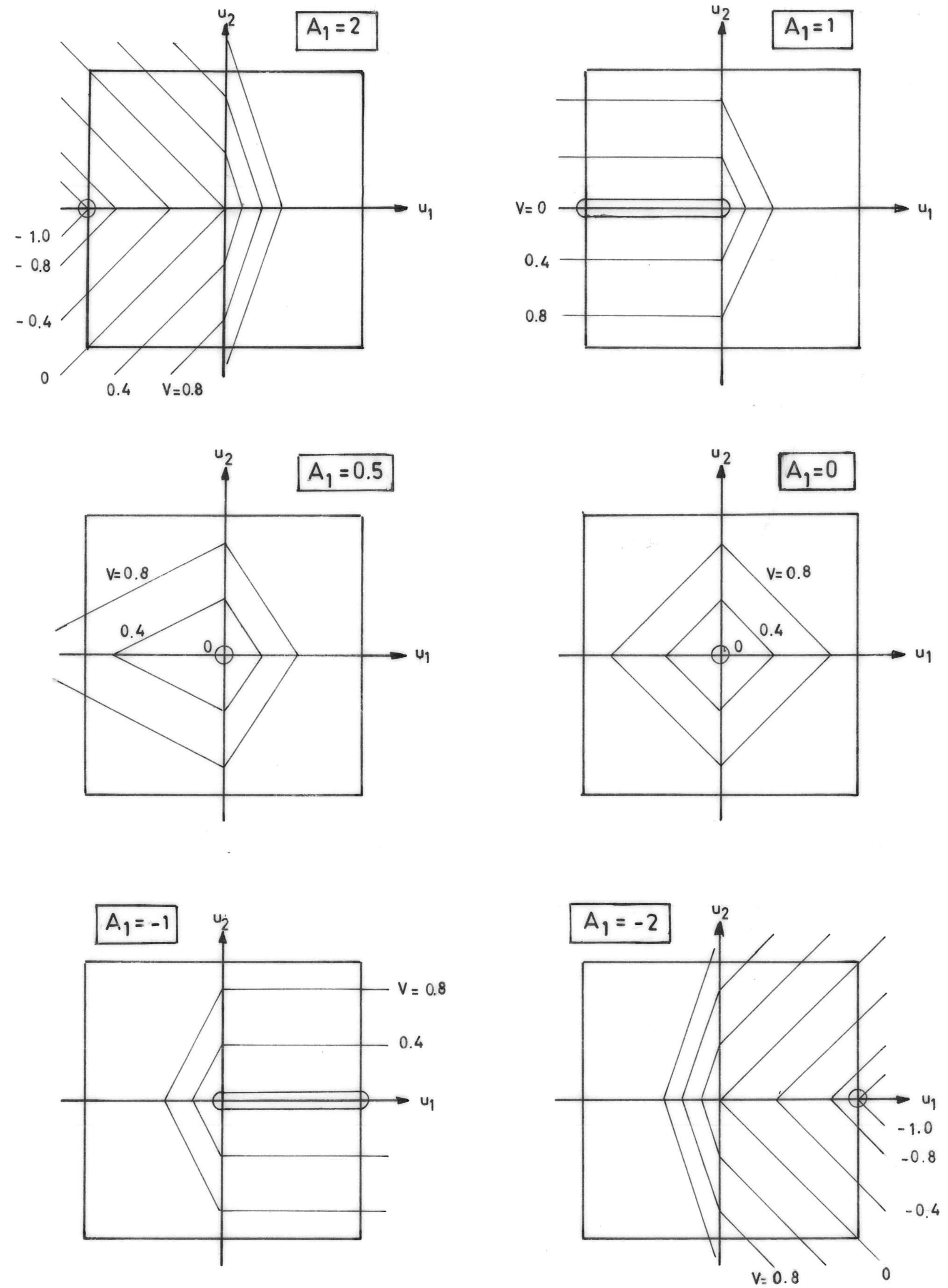

Fig 6.1 Curves of equal V using the objective function

$$V = |u_1| + |u_2| + A_1 u_1 + A_2 u_2$$

with $A_2 = 0$ and varying A_1. Permitted control region:

$$|u_1| \leq 1, \quad |u_2| \leq 1$$

The optimal point (or line) is encircled.

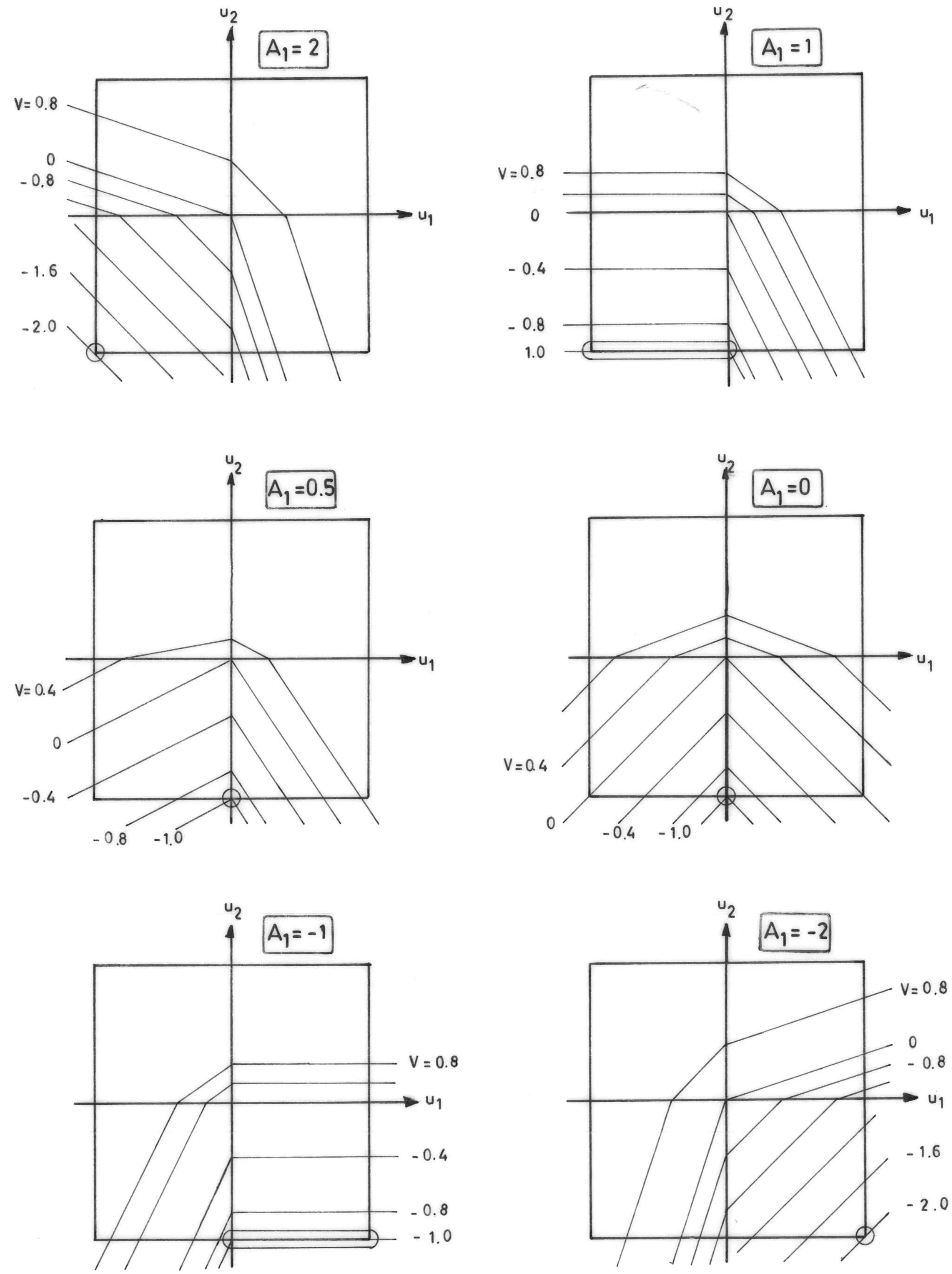

Fig 6.2 Curves of equal V using the objective function

$$V = |u_1| + |u_2| + A_1u_1 + A_2u_2$$

with $A_2 = 2$ and varying A_1. Permitted control region:

$$|u_1| \leq 1, \quad |u_2| \leq 1$$

The optimal point (or line) is encircled.

influence the solution. This means that only a finite number, 3^m, of values of the constant vector p_0 must be taken into account as long as no tank has reached its limits. Each time a tank reaches its limit, new 3^m values must be considered.

If Ω_u is a hyperpolyhedron the only possible points minimizing

$$V = \sum_{i=1}^{m} (|u_i - a_i| + A_i u_i)$$

are

- the corners of the hyperpolyhedron
- the intersection between a boundary surface of Ω_u and a line through the point a, parallel to one of the co-ordinate axes
- the point a

Physically, the different A-values and the corresponding points of Ω_u mean the following:

If $\max_i |A_i| \ll 1$ and $a \in \Omega_u$	we try to keep the productions of all processes u_i at the level a_i.
if $A_j \ll -1$ and $|A_i| \ll 1$, $i \neq j$	the production of process u_j will be increased until some constraint is reached. The productions of the other processes are kept at the level a_i, if possible.
if $A_j \gg 1$ and $|A_i| \ll 1$, $i \neq j$	the production of u_j will be reduced.

These properties of the A-vector will be utilized when solving the problem numerically in the next chapter.

7. SOLUTION METHODS ILLUSTRATED BY A TWO-DIMENSIONAL EXAMPLE

In the preceding chapter, the planning problem has been given a formulation based upon the maximum principle and consequences of the formulation have been discussed. In this chapter we will discuss the solution of the problem, especially

- determination of the a-vector
- utilization of the physical interpretation of the A-vector
- numerical solution of the problem

There is no direct way of solving the optimization problem. Thus, some iterative technique must be used. An immediate method is the following: guess the A-vector, solve the optimization problem and iterate until the dėsired boundary value is reached. It will be shown that this method leads to excessive computations. Instead, a technique using iteration over the a-vector and utilizing the physical interpretation of the A-vector is derived. This method does not ensure convergence to the true minimum (in terms of production rate changes). However, it has been demonstrated to give solutions close to the optimal one, and the computational time is drastically reduced.

The iteration technique is developed in two numerical examples, based upon a simple two-dimensional model. In the first example, a systematic way of finding an initial a-value is derived and two different iteration techniques are illustrated. The ability of $|A| < 1$ to keep the production at the level a is utilized. It is shown that the minimum number of production rate changes in this example is 2 and that there is an infinite number of solutions with two production rate changes. The influence of the sampling interval on the solution is also discussed. In the second example, the ability of $|A| > 1$ to reduce or increase the production is illustrated.

The optimization problems of this chapter are easily solved graphically or by manual calculations. This, of course, is not possible for the real 9-dimensional planning problem. However, the calculations described can be performed using a sequence of LP programs. Since the planning problem can be solved sequentially, each LP problem will be relatively small. This is further discussed in the next chapter.

Model

The two-dimensional model, illustrated in figure 7.1, consists of

1 paper machine (v)
2 process units (u_1,u_2)
3 storage tanks (x_1, x_2, x_3)

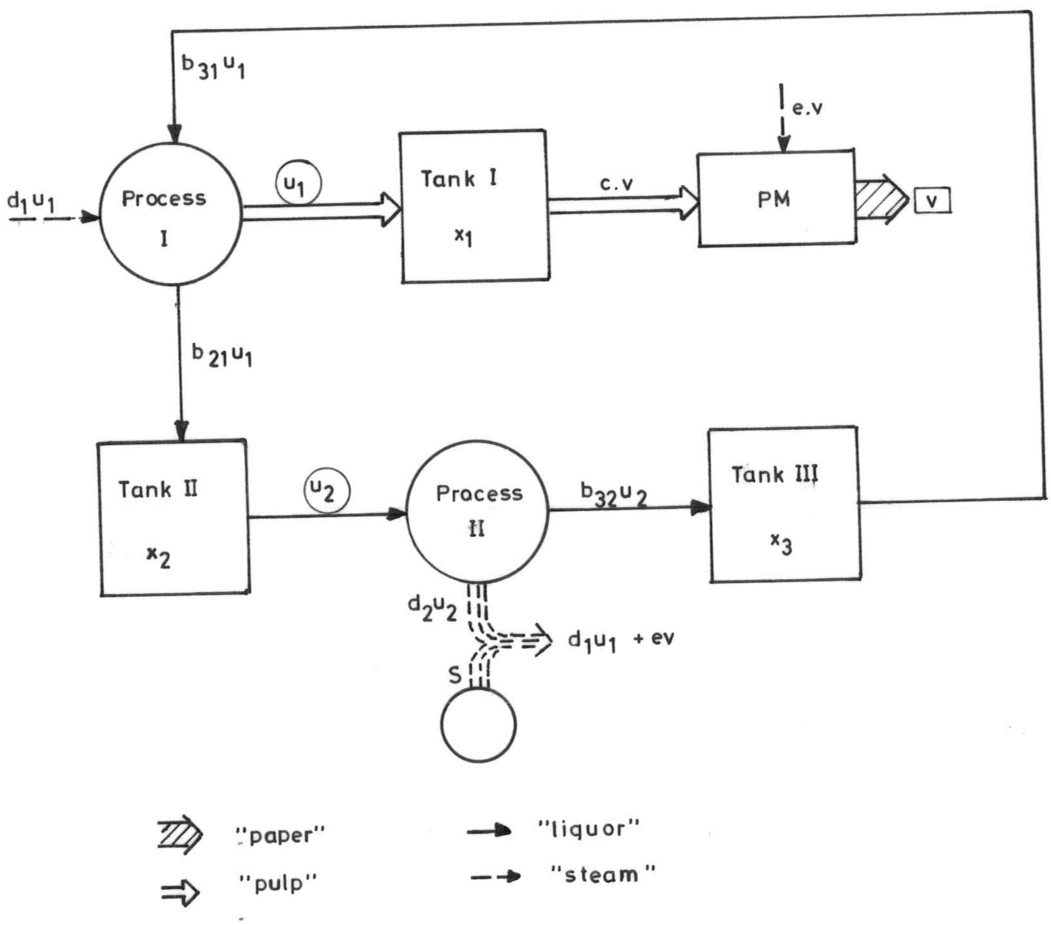

Fig 7.1 A two-dimensional model consisting of 1 paper machine, 2 process units and 3 storage tanks.

One flow of the system has no storage tank (the "steam") and the steam balance of the system is maintained by an extra steam producer, S. In agreement with the model of the Gruvön mill, this simplified model is not controllable.

The model is described by the following equations, expressing material balances over the tanks and the demand for steam balance of the system:

$$\dot{x}_1 = u_1 - c \cdot v \tag{7.01}$$

$$\dot{x}_2 = b_{21}u_1 - u_2 \tag{7.02}$$

$$\dot{x}_3 = - b_{31}u_1 + b_{32}u_2 \tag{7.03}$$

$$S = d_1u_1 - d_2u_2 + e \cdot v \tag{7.04}$$

In matrix form we get

$$\dot{x} = B \cdot u + C \cdot v \tag{7.05}$$

$$S = D \cdot u + E \cdot v \tag{7.06}$$

Put

$$b_{21} = b_{31} = b_{32} = d_1 = 1 , \quad d_2 = e = 2 , \quad c = 1$$

and we get the system matrices

$$B = \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & -2 \end{bmatrix}$$

$$E = \begin{bmatrix} 2 \end{bmatrix}$$

The capacity restrictions of the system are assumed to be given by

$$0.5 \leq u_1 \leq 1 \tag{7.7}$$

$$0.2 \leq u_2 \leq 1 \tag{7.8}$$

$$0 \leq S \leq 2 \tag{7.9}$$

$$0 \leq x_i \leq 1 \qquad i=1,2,3 \tag{7.10}$$

The planned paper production and the initial tank levels are given. The length of the planning period is assumed to be 5 time units. When solving the problem numerically this period is divided into 5 intervals of equal length. The production of the processes, i.e.

$$u_i(1),\ldots,u_i(5) \quad i = 1, 2$$

is to be calculated.

For each time interval k we can now define the following optimization problem:

Minimize

$$V_k = |u_1(k) - a_1| + |u_2(k) - a_2| + A_1u_1(k) + A_2u_2(k) \tag{7.11}$$

subject to

$$0.5 \leq u_1(k) \leq 1 \tag{7.12}$$

$$0.2 \leq u_2(k) \leq 1 \tag{7.13}$$

$$0 \leq D\cdot u(k) + E\cdot v(k) \leq 2 \tag{7.14}$$

$$0 \leq x_i(k-1) + (Bu(k) + Cv(k))_i \leq 1 \tag{7.15}$$

where

$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = p^* B = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix} \cdot B = \begin{bmatrix} p_1 + p_2 - p_3 \\ - p_2 + p_3 \end{bmatrix}$$

$x_i(0)$ i=1,2,3 and $v(k)$ k = 1,...,5 are given.

The vector p is to be chosen in a manner satisfying the boundary condition

$x(5)$ given

In chapter 6 it was shown that only a finite number of p-vectors influences the optimal solutions. In this case, only $3^2 = 9$ different values of $A = p^* B$ must be checked if the consequences of the constrained state variables are disregarded. However, each time a tank reaches a limit, the p-vector makes a jump and a new A-vector has to be taken into account. For the real case (m=9) this implies the following: for each of the $3^9 = 19683$ original A-vectors, $3^9 = 19683$ new ones must be checked each time a tank reaches its limits. Hence, an iteration technique, using iteration over the p-vector, is not realistic. Instead, we will derive an iteration over the a-vector utilizing the ability of A to keep the production at the level a ($|A| < 1$), to increase the production ($A < -1$) or to reduce it ($A > 1$) (cf the physical interpretation of the A-vector of the preceding chapter). We will derive the iteration in two numerical examples.

Example 1

In this example we will use the ability of $|A| < 1$ to keep the production at a certain level a. A systematic manner of finding an initial a-value is derived. It is shown that the planning problem of this example has no solution with 0 or 1 production rate change and that there is an infinite number of solutions with 2 changes. Two different techniques, using iteration over the a-vector and giving two of these solutions, are derived. The influence of the choice of sampling interval is illustrated.

Consider the following planning problem.
Planned paper production:

$$v(1) = v(2) = v(4) = v(5) = 0.9 \quad v(3) = 0$$

$v(3) = 0$ can be interpreted as a "wire change".

Initial state:

$$x_i(0) = 0.5 \quad i=1,2,3$$

Desired final state:

$$x_1(5) = x_2(5) = 0.5$$

(Since the system is not controllable, all final tank levels cannot be fixed).

Calculate $u_i(1),\ldots,u_i(5)$ $i=1,2$

Calculation of an initial a-value

An initial a-value can be calculated from the planned paper production, the initial tank levels and the final tank levels. The a-value can be physically interpreted as a desired mean value of the productions of the processes during the planning period (cf chapter 3).

The total paper production is given:

$$\int_0^T v(t)dt = \sum_{i=1}^{5} v(i) \qquad (7.16)$$

The initial and the final levels of tank No. 1 are known, i.e.

$x_1(0)$ and $x_1(5)$ given

Thus, the total flow

$$\int_0^T u_1(t)dt = \sum_{i=1}^{5} u_1(i)$$

is determined by the material balance

$$\int_0^T u_1(t)dt = x_1(T) - x_1(0) + c \cdot \int_0^T v(t)dt \qquad (7.17)$$

An average production

$$\overline{u_1} = \frac{1}{T} \cdot \int_0^T u_1(t)dt \qquad (7.18)$$

can thus be calculated. Now, assume that the storage capacity of x_1 is very great. Then a good control strategy would be to run process I at the rate of

$u_1(t) = \overline{u_1}$

during the whole planning period. Thus, by putting $A_1=0$ and choosing $a_1=\overline{u_1}$ we would reach the desired end point and no production rate change would be required. However, even when the capacity of x_1 is limited, we have found that

$a_1 = \overline{u_1}$

is a useful initial value.

In the same way, put

$$a_2 = \overline{u_2}$$

where

$$\overline{u_2} = \frac{1}{T} \cdot \int_0^T u_2(t)dt \qquad (7.19)$$

is calculated from

$$x_2(0),\ x_2(5) \text{ and } \int_0^T u_1(t)dt$$

Now, we have no possibility of controlling the final level of x_3.

Numerically we get in this example

$$\left.\begin{array}{l} \int_0^T v(t)dt = 4 \cdot 0.9 = 3.6 \\ x(0) = x(5) = 0.5 \end{array}\right\} \Rightarrow a_1 = a_2 = 0.72$$

Putting

$$A_1 = A_2 = 0$$

our optimization problem is the following:

For k=1,2,..,5 minimize

$$V_k = |u_1(k) - 0.72| + |u_2(k) - 0.72|$$

subject to

$$0.5 \leq u_1(k) \leq 1$$

$$0.2 \leq u_2(k) \leq 1$$

$$0 \leq S(k) \leq 2$$

$$0 \leq x_i(k) \leq 1$$

The result of the optimization is illustrated in figure 7.2.

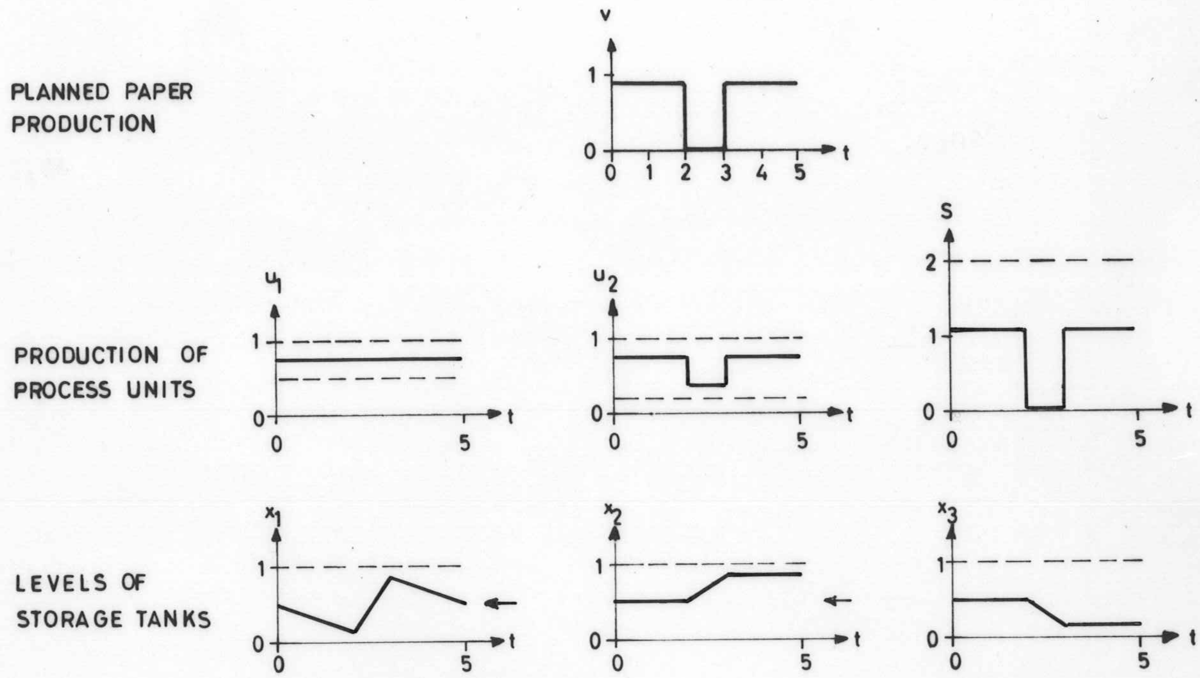

Fig 7.2 Optimal solution of the two-dimensional example using the objective function

$$V = |u_1-0.72| + |u_2-0.72|$$

Planning period: 5 time units divided into 5 equal intervals. Dotted lines are capacity restrictions. Desired final tank levels are marked by arrows. Since the production of process II has been reduced during interval No. 3, the desired level of tank II has not been reached.

No production change of u_1 has been necessary and x_1 has reached the desired value. However, the production of u_2 has been reduced during interval No. 3, since the S-variable has reached its lower limit. In consequence of the production change, the final level of x_2 is incorrect.

Iteration over the a-vector, method I

An iteration over the a-vector can be performed in the following way. The problem is first solved using the above values of a_1 and a_2. No correction of a_1 is necessary, since the desired final level of x_1 is reached. A better value of a_2 can be calculated if the production reduction of u_2 during interval 3 is taken into account. Thus, put

$$a_2 = \frac{T \cdot \overline{u_2} - u_2(3)}{T - 1} = \frac{3.6 - 0.36}{4} = 0.81$$

Thus, the new objective function is

$$V = |u_1 - 0.72| + |u_2 - 0.81|$$

This criterion gives a production schedule as illustrated in figure 7.3. The correct final levels are now reached.

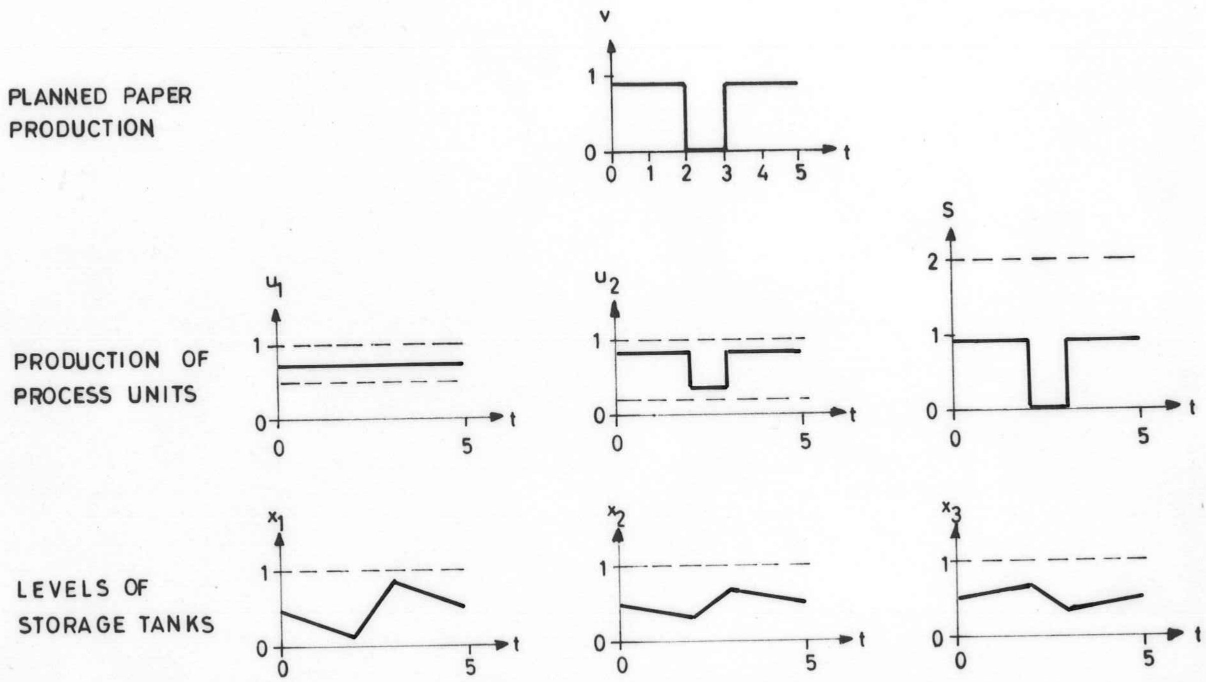

Fig 7.3 **Optimal solution of the two-dimensional example using the objective function**

$$V = |u_1-0.72| + |u_2-0.81|$$

The desired final tank levels have been reached.

The iteration technique implies, however, that the complete optimization problem must be solved many times. This is disadvantageous if the computational time is critical. Besides, this method is an iteration "by inspection" that will be very difficult to systemize for the real case with 9 processes and 10 storage tanks.

The following technique will utilize the production and storage capacities more and will sometimes increase the number of production changes. However, it is systematic and the saving of computational time is considerable since the problem need be solved only once.

Iteration over the a-vector, method II

The problem is solved with the original a-values until, during a certain interval k, some production change is obtained. Then, for intervals > k, the a-vector is recalculated with regard to the production change during interval k. Thus, the change is compensated for during the remaining intervals only.

In our example, $a_1 = a_2 = 0.72$ is used during the intervals No. 1, 2,3. During interval No 3,

$$u_2 = 0.36$$

i.e. we have got a production drop-out of

$$0.72 - 0.36 = 0.36$$

units. To compensate for this drop-out, the production during intervals No 4 and 5 must be

$$\frac{2 \cdot 0.72 + 0.36}{2} = 0.90$$

If the problem is solved using this technique we will get results according to figure 7.4. The desired tank levels have been reached.

Discussion of uniqueness of optimal solution

By different choices of the vector a(t), we have found two different solutions to the problem, both with two production rate changes. We will now prove that the minimum number of production rate changes in this example is 2 and that there is an infinite number of solutions with two changes.

First we will show that there is no solution with 0 or 1 production change.

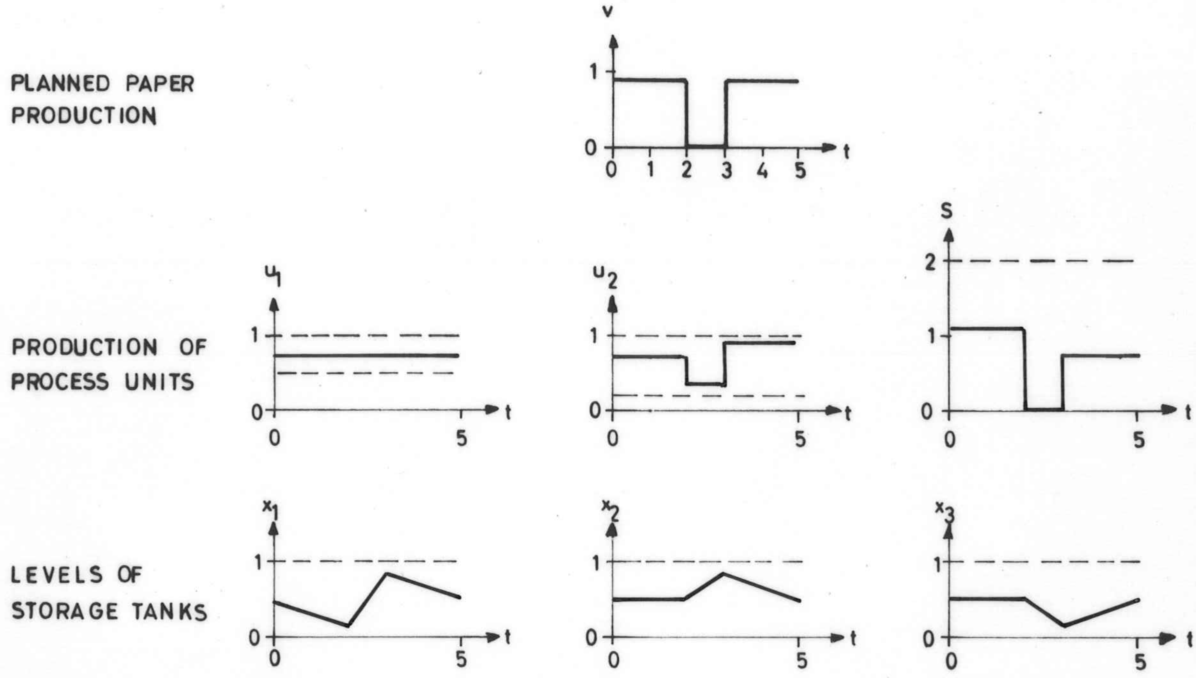

Fig 7.4 Optimal solution of the two-dimensional example using the objective function

$V = |u_1-0.72| + |u_2-0.72|$ during intervals No. 1,2,3

$V = |u_1-0.72| + |u_2-0.90|$ during intervals No. 4,5

The desired final tank levels have been reached. Compared to figure 7.3, the production and storage capacities are utilized harder.

Zero production changes require that the productions of both process I and process II are constant during the whole period. If $u_1(t)$ is assumed to be constant, a material balance over tank I gives (eq. 7.17 and 7.18):

$$u_1(t) = 0.72 \quad 0 \leq t \leq T$$

Now, assuming that $u_2(t)$ is constant, a material balance over tank II gives

$$u_2(t) = 0.72 \quad 0 \leq t \leq T$$

Eq. (7.4) gives during interval 3

$$S(3) = -0.72$$

i.e. S is out of the limits given by (7.9).

This means, in mill terms, that we are "blowing steam on the roof", a very uneconomical way of running a mill. Thus the problem has no solution with zero production changes.

Now, let us consider one change. First, assume zero changes of $u_1(t)$ and one of $u_2(t)$. The steam balance (7.4) requires

$$u_2 = 0.36$$

during interval 3. One production change of process II implies one of the following shapes of $u_2(t)$ (Fig. 7.5).

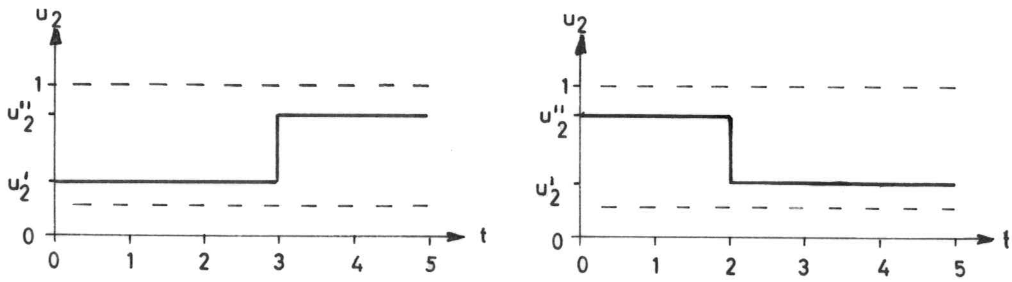

Fig 7.5 **Shape of $u_2(t)$ if only one production rate change is permitted and steam balance during interval No. 3 is required.**

Since

$$x_2(0) = x_2(5) = 0.5 \quad \text{and} \quad \sum_{i=1}^{5} u_1(i) = 3.6$$

a material balance over tank II gives

$$\left.\begin{array}{l} 3u_2' + 2u_2'' = 3.6 \\ u_2' = 0.36 \end{array}\right\} \Rightarrow u_2'' = 1.26$$

i.e. u_2'' is out of the limits.

Now, assume one change on u_1 and zero on u_2. Zero changes on u_2 implies

$$u_2(t) = 0.72 \quad 0 \leq t \leq T$$

Steam balance during interval 3 gives

$$S(3) = u_1(3) - 1.44$$

The constraint (7.09) now gives

$$1.44 \leq u_1(3) \leq 3.44$$

i.e. u_1 is beyond the limits.

Thus, the problem has no solution with zero or one production change.

We will now derive an infinite number of changes of a form, illustrated by figure 7.6.

We require zero changes on $u_1(t)$, implying

$$u_1(t) = 0.72 \quad 0 \leq t \leq T$$

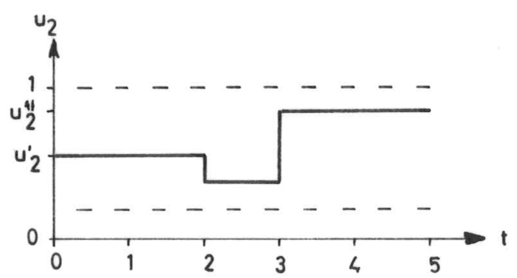

Fig 7.6 **Shape of $u_2(t)$ with two production rate changes and requiring steam balance during interval No. 3.**

Put

$$u_2(1) = u_2(2) = u_2'$$

$$u_2(3) = 0.36$$

$$u_2(4) = u_2(5) = u_2''$$

After some calculations we find that each pair of numbers u_2', u_2'', satisfying

$$u_2' + u_2'' = 1.62$$

$$0.47 \leq u_2', u_2'' \leq 0.97$$

will fulfil all system constraints and will give the desired final tank levels.

$$u_2' = 0.81 \quad u_2'' = 0.81 \text{ (fig. 7.3) and } u_2' = 0.72, \ u_2'' = 0.90 \text{ (fig. 7.4)}$$

are two of these solutions.

The solution region is illustrated in figure 7.7.

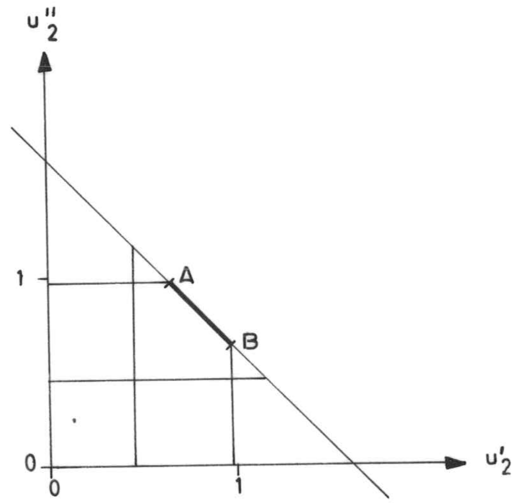

Fig 7.7 The two-dimensional example has an infinite number of solutions, all giving two production rate changes. Each pair of numbers u_2' and u_2'' lying on the line segment A-B gives two production rate changes.

The extreme cases

$$u_2' = 0.65,\ u_2'' = 0.97$$

and

$$u_2' = 0.97,\ u''_2 = 0.65$$

are illustrated in figure 7.8.

Subcriteria

We have now got an infinite number of solutions, all optimal in respect of the criteria. In order to choose between these solutions we can define subcriteria.

From a practical point of view, the solution of fig. 7.3 is better than the extreme cases of fig. 7.8. The reason is that the schedule of fig. 7.3 gives a certain safety margin against unforeseen events, since the production and storage capacities are not utilized so hard. Thus, one possible subcriterion is to look for solutions that do not utilize the capacities too hard.

Another possibility is to consider the production changes at $t = 0$. It is not probable that the calculated production during the beginning of the planning period, i.e. $u(t= + 0)$ will coincide with the production $u(t=-0)$ at the end of the preceding planning period. Thus we will as a general rule get production changes at $t = 0$. This gives us the possibility of choosing the conditions

$$u(t=+0) = u(t=-0)$$

as a subcriterion. However, this criterion will often utilize the capacities rather hard, thus contradicting the subcriterion discussed above.

For the real problem with 9 process units, the computational time is critical. Hence, no attempts have been made to improve the solutions by defining subcriteria.

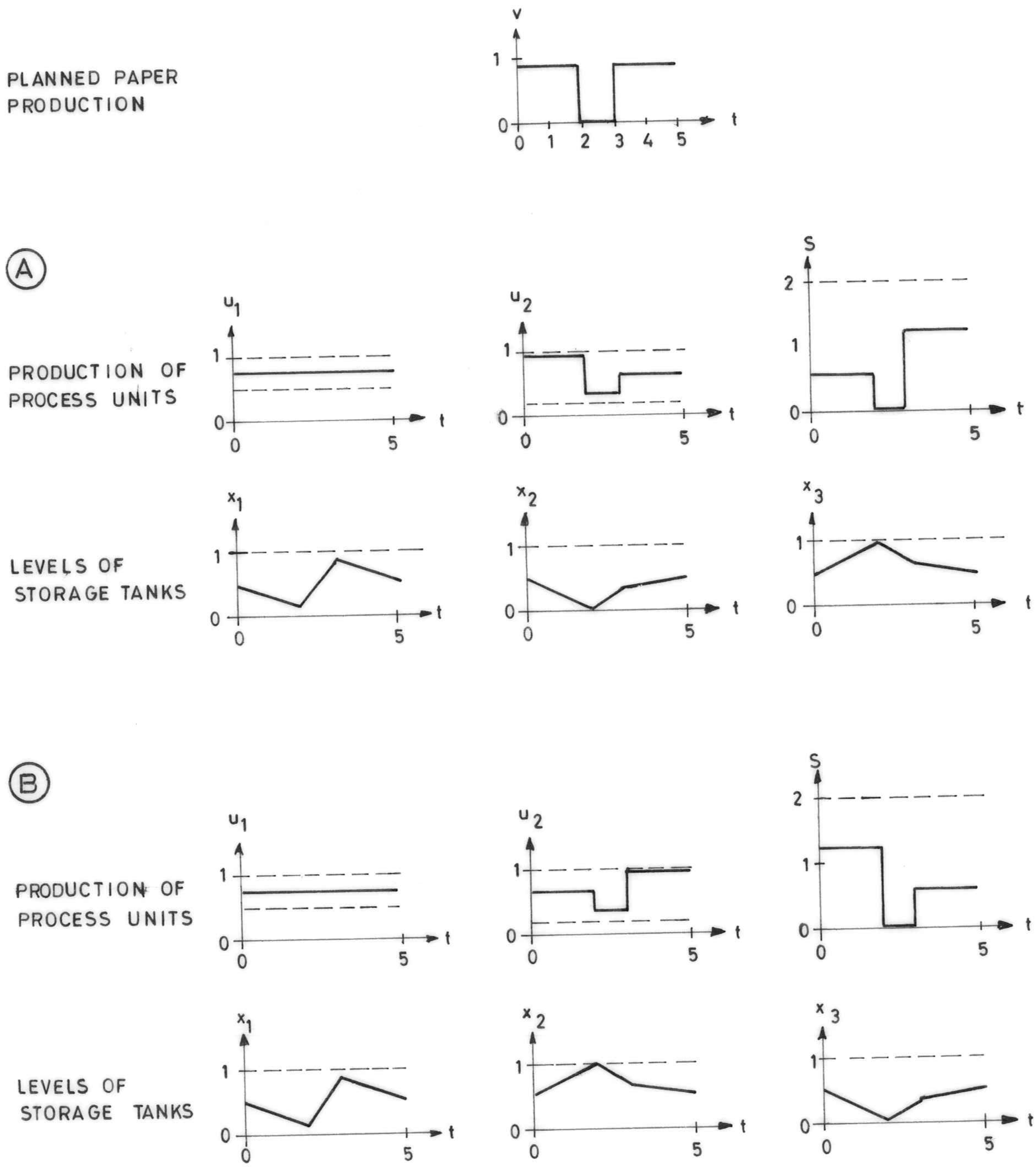

Fig 7.8 A) Optimal solution of the two-dimensional example, using the objective function

$V = |u_1-0.72| + |u_2-0.65|$ during intervals 1,2,3

$V = |u_1-0.72| + |u_2-0.97|$ during intervals 4,5

The solution corresponds to point A in fig 7.7.

B) Optimal solution of the two-dimensional example, using the objective function

$V = |u_1-0.72| + |u_2-0.97|$ during intervals 1,2,3

$V = |u_1-0.72| + |u_2-0.65|$ during intervals 4,5

The solution corresponds to point B in fig 7.7.

Influence of sampling intervals

Consider one of the solutions obtained for our 2-dimensional example. When solving the problem numerically we divided the planning period into 5 intervals of equal length. However, we will get exactly the same solution with every time quantization that does not have t=2 and t=3 in the interior of an interval. The reason is that the steam balance is the critical constraint of this example. Thus for every quantization of the time interval

$$2 \leq t \leq 3$$

the optimal solution lies on the steam constraint. For

$$0 \leq t \leq 2$$

and

$$3 \leq t \leq 5$$

the optimal point lies in the interior of the permitted region, independent of the time quantization. However, if a tank restriction is the critical constraint of the problem, the solution is not independant of the sampling of the planning period. As an illustration of this, assume that the capacity limits of tank I are given by

$$0.2 \leq x_1(t) \leq 0.8$$

i.e. the capacity of x_1 is reduced by 40 %.

We have solved the planning problem for 3 different time quantizations (using recalculation of a-values according to method II):

A) $t_1 = t_2 = .. = t_5 = 1$

B) $t_1 = 2 \quad t_2 = 1 \quad t_3 = 2$

C) $t_1 = \frac{5}{3} \quad t_2 = \frac{1}{3} \quad t_3 = \frac{1}{2} \quad t_4 = \frac{1}{2} \quad t_5 = \frac{3}{2} \quad t_6 = \frac{1}{2}$

The results are illustrated in figure 7.9.

As can be seen from the figures, the choice of sampling intervals is important in this case. Thus, an ideal iteration method should iterate also on the sampling intervals. However, since the computational time is critical no attempts have been made to iterate on the quantization of the planning period.

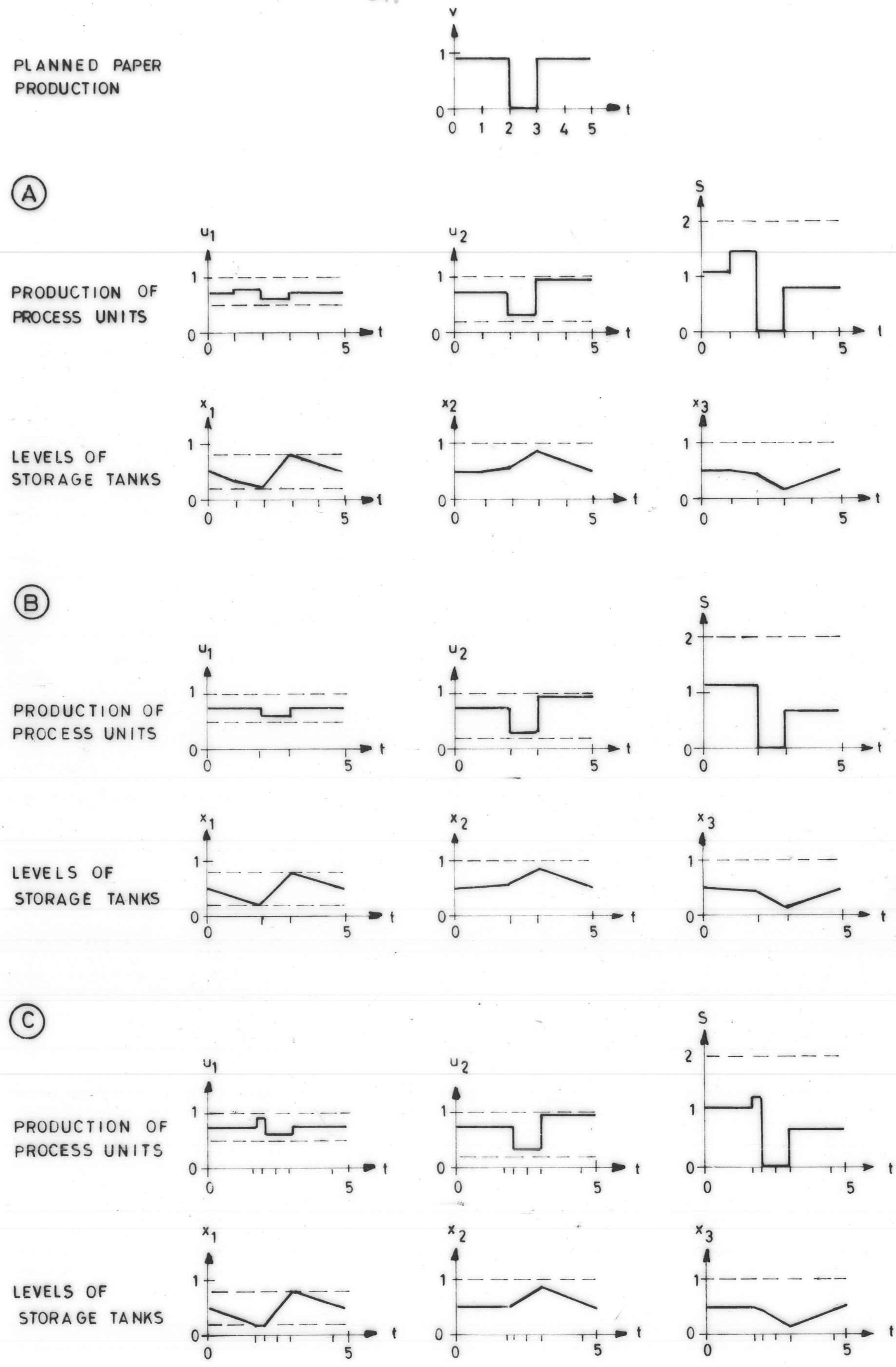

Fig 7.9 Optimal solution of the two-dimensional example (reduced size of tank II) for three different time quantizations of the planning period:

A) 5 intervals, 1 time unit each

B) 3 intervals, 2, 1, and 2 time units respectively

C) 6 intervals, 5/3, 1/3, 1/2, 1/2, 3/2, and 1/2 time units respectively.

Comparison between ideal and actual objective function

The objective function used

$$V = \sum_i |u_i - a_i| \tag{7.20}$$

is an approximation to the mathematically ideal one

$$V = \sum_i |\text{sgn}(u_i - a_i)| \tag{7.21}$$

To illustrate the accuracy of the approximation, figure 7.10 shows curves of equal V and permitted control regions Ω_u for example 1 using the criterion

$$V = |u_1 - 0.72| + |u_2 - 0.72| \tag{7.22}$$

It is immediately seen from the figures, that the criterion

$$V = |\text{sgn}(u_1 - 0.72)| + |\text{sgn}(u_2 - 0.72)| \tag{7.23}$$

will give the same optimal points for intervals No, 1, 2, 4 and 5.

During interval No. 3, there will be a difference as illustrated by figure 7.11.

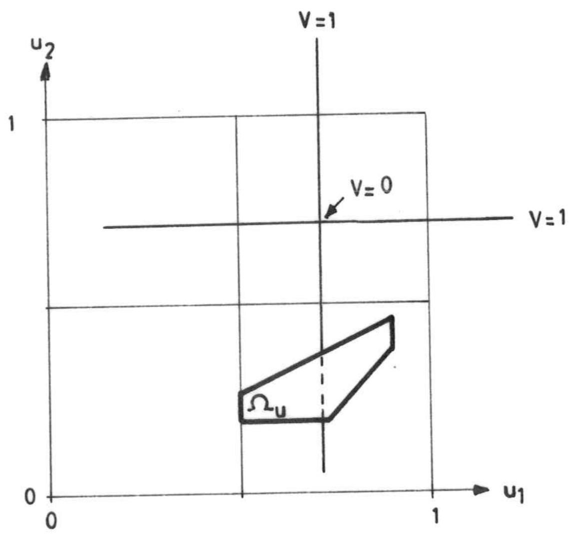

Fig 7.11 Permitted control region Ω_u and curves of equal V during interval No. 3, using the objective function $V = |\text{sgn}(u_1-0.72)| + |\text{sgn}(u_2-0.72)|$ All points on the dotted line minimize V.

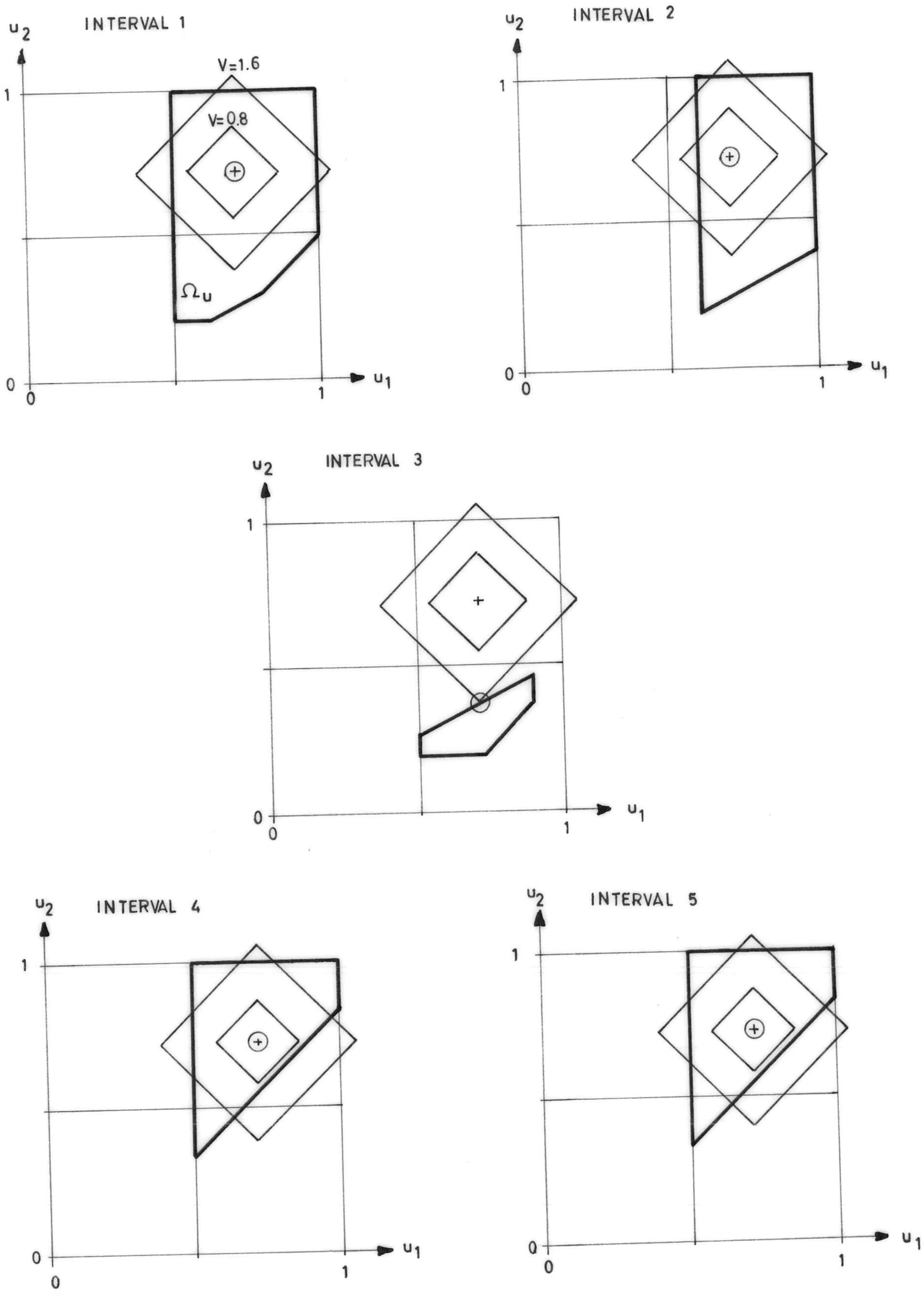

Fig 7.10 Permitted control region Ω_u and curves of equal V using the objective function

$$V = |u_1 - 0.72| + |u_2 - 0.72|$$

Planning period: 5 time units, divided into 5 equal intervals. The optimal points are encirled.

For this interval, all points on the dotted line are optimal points to the criterion (7.23). However, in this specific case, the solutions in the interior of Ω_u are inferior to the boundary solution, since they will utilize the capacity of x_2 harder (cf the discussion of subcriteria).

Example 2

In this example, the ability of $|A|>1$ to reduce or increase the production is utilized.

The planning problem is identical to the problem of example 1, but in addition we want to run process II at the lowest possible rate during interval No. 3 ("maintenance"). This can be formulated as a constraint

$$u_2(3) = 0.2$$

However, this formulation implies a risk that the problem is so rigidly structured that no feasible solution exists. To avoid this, the production reducing properties of $A_2 > 1$ can be utilized. If the restrictions permit, u_2 will be reduced to its lower limit. Otherwise, we will get the smallest possible production.

When calculating a_2, the planned reduction of u_2 must be taken into account. In figure 7.12 the result, using

$$V = |u_1-0.72| + |u_2-0.85| + A_1u_1 + A_2u_2$$

where

$$A_1 = A_2 = 0 \text{ during intervals 1, 2, 4, 5}$$

$$A_1 = 0 \quad A_2 = 2 \text{ during interval No 3}$$

is illustrated.

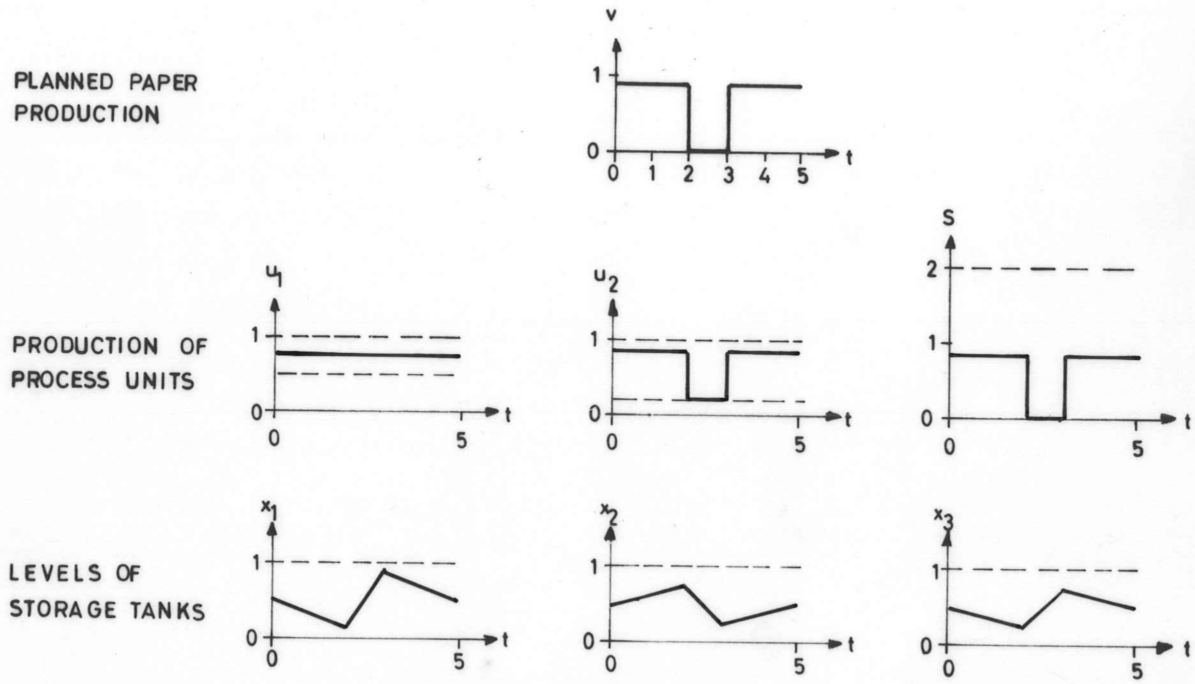

Fig 7.12 Optimal solution of the two-dimensional example using the objective function

$$V = |u_1-0.72| + |u_2-0.85| + A_1u_1 + A_2u_2$$

with $A_1 = A_2 = 0$ during intervals No. 1,2,4,5 and $A_1 = 0$, $A_2 = 2$ during interval No. 3.

8. FINAL PROBLEM FORMULATION AND SOLUTION TECHNIQUE

In this chapter, the final formulation of the Gruvön scheduling problem is given.

The mathematical model of the mill (ch 2) has been obtained as a combination of analysis, simulation and experimental verification [7] . The number of state variables of the model is 10 and the number of control variables 9. The scheduling problem has been formulated as an optimal control problem for a multivariable deterministic system (ch. 6). The scheduling objectives: few production rate changes, indirect storage of steam and acceptable final tank levels, have been formulated mathematically (ch. 3). The solution technique developed (ch. 7) is a fusion of methods and ideas taken from the maximum principle, linear programming,heuristic argumentation, and physical interpretation of the mathematical relations. Thus, the formulation of the scheduling problem and the solution method developed is no strict application of any existing theory.

The solution technique implies that the planning period is divided into a number of time intervals, not necessarily of equal length. During each interval, the productionsof the processes are assumed to be constant. For each LP problem, the objective function has the form

$$\sum_i |u_i-a_i| + A_i u_i$$

A_i are parameters, related to the adjoint variables of the Pontryagin theory. a_i are components of a vector a, that can be physically interpreted as a desired average of the production vector u(t). An initial a-value can be calculated from the planned paper production, initial tank levels, and final tank levels. The solution technique implies an iteration over a. The LP problems are solved sequentially. The transfer of information between the LP programs is handled partly by the vector a(t), partly by the adjoint vector. The execution time on an IBM 1800 of a typical problem is expected to be about one hour on-line and during time-sharing (50% load of priority programs).

Problem formulation

Given the system equations

$$\frac{dx(t)}{dt} = B\cdot u(t) + C\cdot v(t)$$

$$S(t) = D\cdot u(t) + E\cdot v(t)$$

the constraints

$$u_j^{min} \leq u_j(t) \leq u_j^{max} \qquad j = 1,\ldots,9$$

$$x_i^{min} \leq x_i(t) \leq x_i^{max} \qquad i = 1,\ldots,10$$

$$S^{min} \leq S(t) \leq S^{max}$$

the initial state

$$x(0)$$

and the function

$$v(t), \quad 0 \leq t \leq T$$

Calculate

$$u(t), \quad 0 \leq t \leq T$$

satisfying the restrictions and the scheduling objectives of ch. 3.

Problem solution

Suitable boundary values $x(T)$ are fixed. The planning period T is divided into a number of intervals τ_k (not necessarily of equal length):

$$T = \sum_{k=1}^{N} \tau_k$$

For $k=1,2,\ldots,N$, the following optimization problems are solved successively:

Minimize

$$V_k = \sum_{j=1}^{9} \left\{ g_j(k) + h_j(k) + A_j u_j(k) \right\}$$

subject to

$$u_j^{min} \le u_j(k) \le u_j^{max} \qquad j=1,..,9 \quad k=1,..,N$$

$$S^{min} \le \sum_{j=1}^{9} d_j u_j(k) + \sum_{j=1}^{3} e_j v_j(k) \le S^{max} \qquad k=1,..,N$$

$$x_i^{min} \le x_i(k-1) + \tau_k \cdot \left\{ \sum_{j=1}^{9} b_{ij} u_j(k) + \sum_{j=1}^{3} c_{ij} v_j(k) \right\} \le x_i^{max}$$

$$i=1,..,10 \quad k=1,..,N$$

$$u_j(k) = a_j(k) + g_j(k) - h_j(k) \qquad j=1,..,9 \quad k=1,..,N$$

$$u_j(k),\ g_j(k),\ h_j(k) \ge 0$$

b_{ij}, c_{ij}, d_j and e_j are elements of the matrices B, C, D and E respectively.

$a_j(1)$ are calculated from x(0), x(T) and v(t), $0 \le t \le T$

$a_j(k)$ k=2,...,N are calculated from the result of the optimization for interval No. k-1.

A_j are components of a vector A, related to the adjoint vector p of the Pontryagin theory by

$$A = p^* \cdot B$$

Usually, A = 0. The numbers A_i influence the optimal solution in the following manner:

$\lvert A_i \rvert \ll 1$	the production of u_i is kept at the rate a_i (if possible)
$A_i \gg 1$	the production of u_i is reduced
$A_i \ll -1$	the production of u_i is increased

Each time interval gives rise to a linear programming problem with

27 variables (excluding slack variables and artificial variables) and 49 restrictions.

A computer program written in Basic FORTRAN IV has been developed to carry out the scheduling calculations. Fig. 8.1 shows a simplified flow chart of the program.

The program size is about 15,000 words on an IBM 1800 (software floating point, single precision) and the execution time for a problem with 15 time intervals is about 30 minutes (off-line execution 4 μs cycle time).

In the practical implementation, the program will be executed as a non-process program during time-sharing. The total execution time is expected to be about 1 hour assuming 50% load of priority programs. Initial data to the program will be received partly from analog inputs to the computer (tank levels), partly from a card reader (planned production).

Comparison with simulation and linear programming

As can be seen from the examples in the next chapter, the optimization technique developed gives production schedules which are superior to the results of simulation and linear programming. Compared to simulations the solutions are qualitatively better, the method is systematic and the manual work required is insignificant.

In comparison with linear programming, the solutions are superior and the demand on computational speed and core storage capacity is much less. The execution time of an LP problem is approximately proportional to the cube of the number of restrictions. Thus, compared to LP as described in ch. 5 the execution time has been reduced by a factor

$$15\left(\frac{50}{750}\right)^3 \approx 0.005$$

The core storage required for an LP problem is approximately proportional to the square of the restrictions. This implies a reduction by a factor

$$\left(\frac{50}{750}\right)^2 \approx 0.005$$

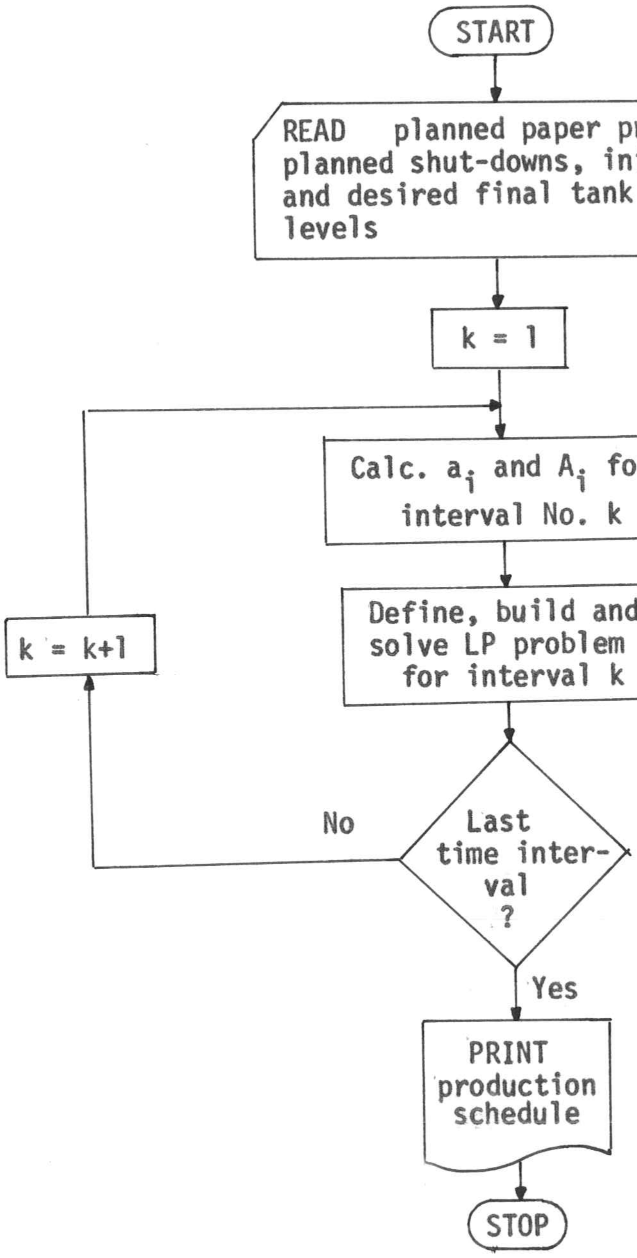

Fig 8.1 Simplified flow chart of the computer program.

The solution technique has been achieved on the basis of the maximum principle. However, it can also be interpreted as a decomposition of the LP problem defined in ch. 5. The solution technique implies that the interconnection between the time intervals has been broken (cf fig. 5.1) and the small LP problems are solved sequentially, as illustrated in fig. 8.2.

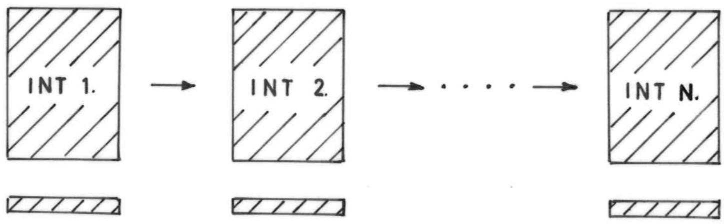

Fig 8.2 Interpretation of the solution method as a decomposition of the large LP problem illustrated in fig 5.1. The small LP problems are solved sequentially.

The transfer of information between the separated problems is handled partly by the vector a, partly by the adjoint vector (i.e. the vector A).

The solution technique has been achieved on the basis of the maximum principle. However, it can also be interpreted as a decomposition of the LP problem defined in ch. 5. The solution technique implies that the interconnection between the time intervals has been broken (cf fig. 5.1) and the small LP problems are solved sequentially, as illustrated in fig. 8.2.

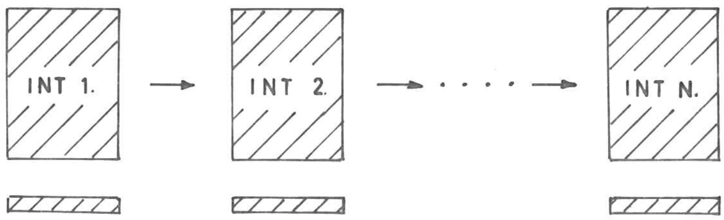

Fig 8.2 Interpretation of the solution method as a decomposition of the large LP problem illustrated in fig 5.1. The small LP problems are solved sequentially.

The transfer of information between the separated problems is handled partly by the vector a, partly by the adjoint vector (i.e. the vector Λ).

9. SOME PLANNING EXAMPLES

The solution technique has been tested on a number of planning examples. Three of these examples are discussed in this section. Further examples are given in [11].

Planning example 1

This example is identical to the simulation example of ch. 4 and the LP examples of ch. 5. The planning period is 48 hours, divided into 6 intervals of 8 hours each. Planned paper production according to fig. 9.1A. Initial tank levels: 50%, desired final tank levels:50%.

The schedule calculated by the optimization program is shown in figure 9.1B (dotted lines are capacity restrictions). Only two changes of production rate during the planning period have been necessary (recovery boilers, interval 4). The corresponding number obtained by simulation is 10 (cf fig. 4.1).

The variations of the steam consumption are firstly compensated for by changing the oil feed to the bark burning boiler. As this variable, S, has reached its lower limit during interval No. 4 the steam production of the recovery boilers has also been reduced.

The resultant tank levels are shown in figure 9.1C. Upper (85%) and lower (15%) limits are marked by dotted lines. Most of the tanks have reached the desired final values, 50%. However, since the system is not controllable, fixed end points for all ten tanks cannot be handled (cf ch. 7). This is the reason why x_{10} differs from 50% at the end of the period.

As can been seen in figure 9.1C, the solution technique has the intuitively correct ability to reduce the pulp buffers before the shut-down of the paper machines.

Planning example 2

The problem is the same as in example 1. However, we will reach the more offensive goal of storing steam indirectly, i.e. the levels of tanks No. 2, 3 and 6 shall be high at the end of the period. The following desired final tank levels are chosen:

80% for tanks No. 2, 3 and 6
50% for the other tanks

The schedule obtained is illustrated by fig. 9.2B and 9.2C. The production of the recovery boiler is changed during interval No. 4 to obtain steam balance in the system. The NSSC digester is changed during the same interval to prevent x_2 from flowing over. The desired final levels have been reached.

Planning example 3.

No wire changes are planned in this example (fig. 9.3A). However, the evaporators must be stopped for cleaning during interval No. 3. Initial levels: 50%, desired final levels:50%.

The result of the optimization is shown in fig. 9.3B and 9.4C. The shut-down of the evaporators causes changes in the surrounding processes (batch digesters, recovery boilers). The final levels were set at 50%. However, if the restrictions do not permit the tanks to reach the desired values, we obtain levels as close to the fixed ones as possible. Thus, the upper limitation of the evaporators has caused final levels of x_5 and x_6 that differ from 50%.

An additional planning example (a combination of ex. 1 and 3) is given in [7].

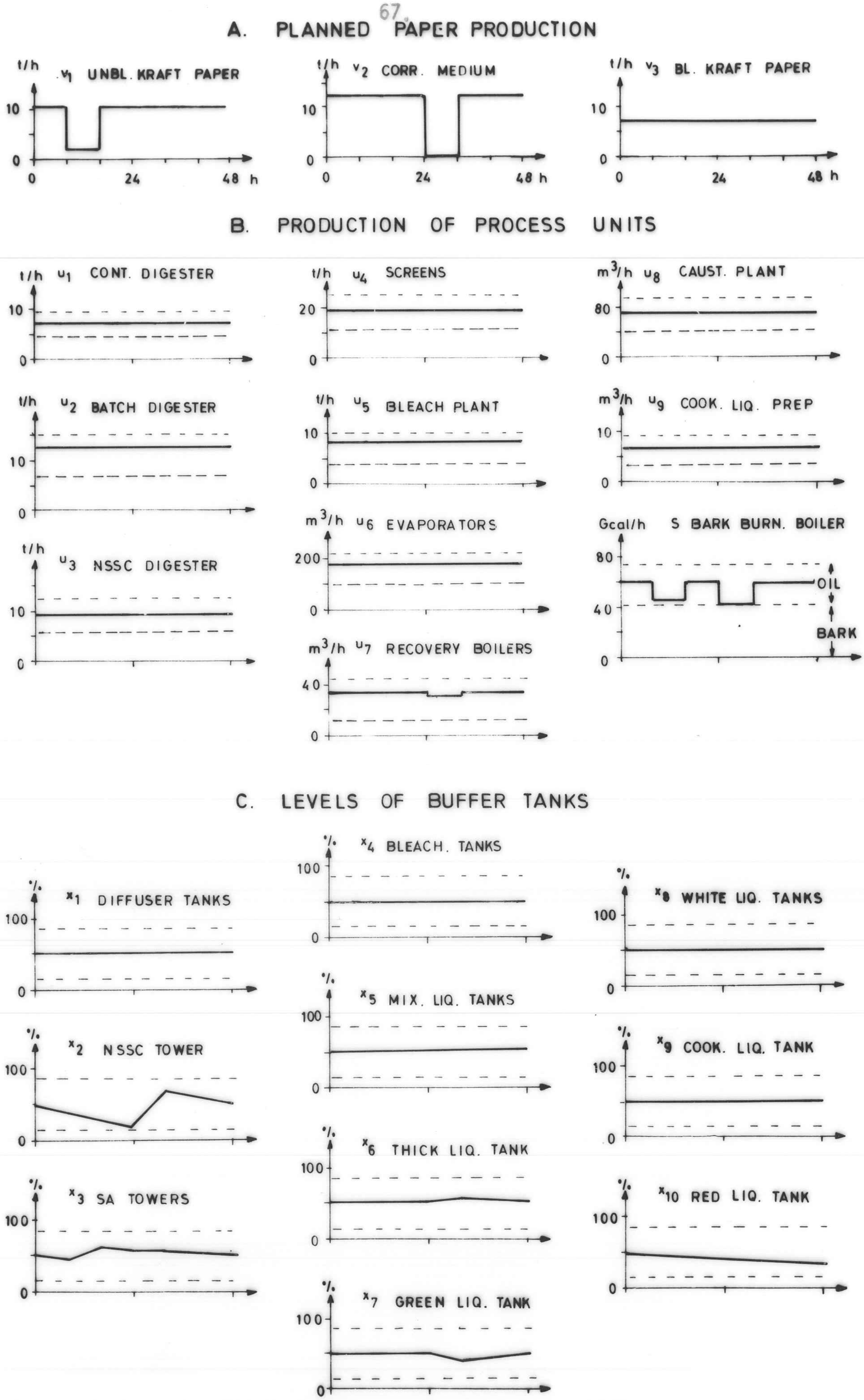

Fig 9.1 Planning example 1. Figures A show the planned paper production, figures B the production schedule as calculated by the optimization program and figures C the resultant tank levels. Dotted lines are capacity restrictions.

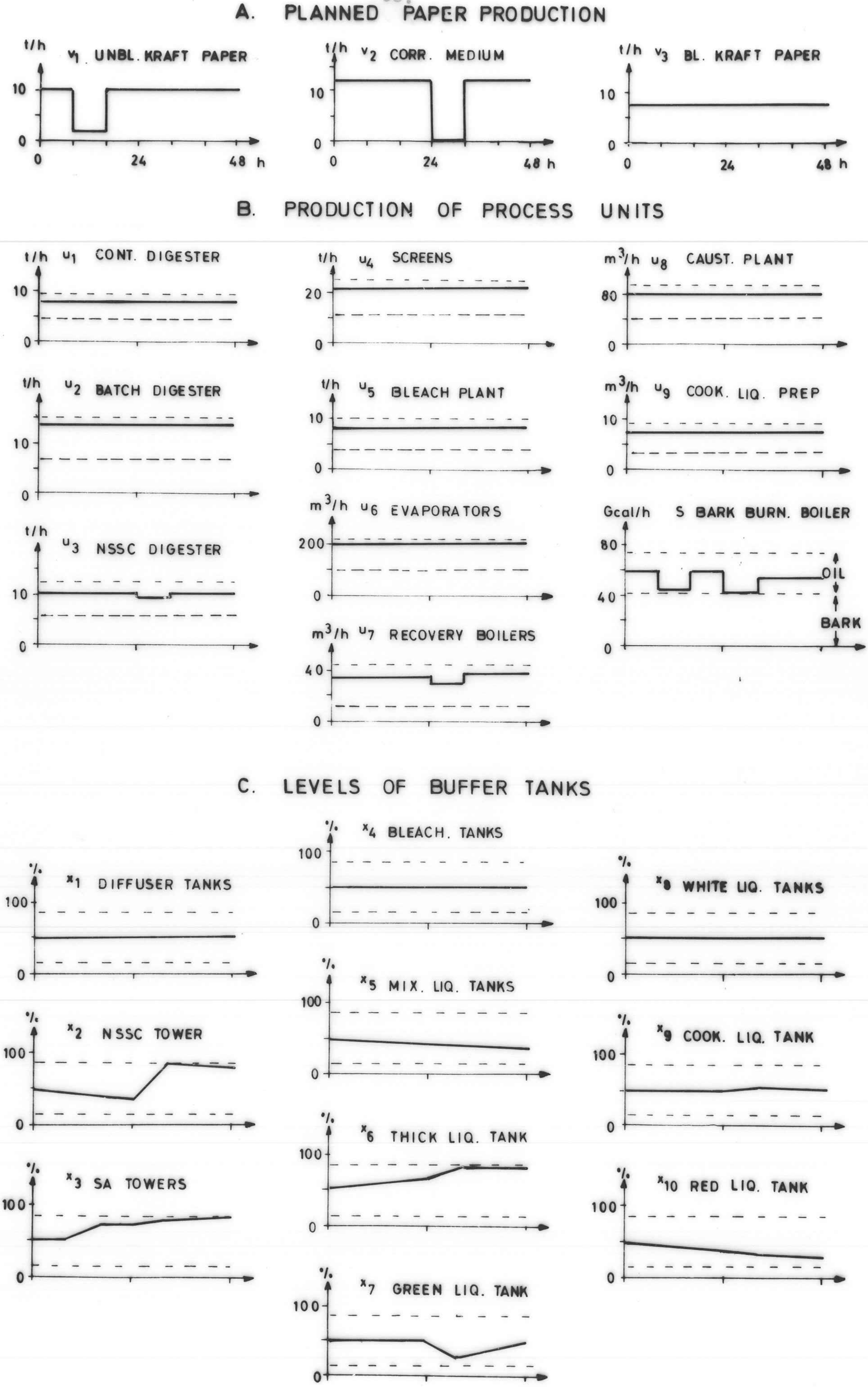

Fig 9.2 Planning example 2. Figures A show the planned paper production, figures B the production schedule as calculated by the optimization program and figures C the resultant tank levels. Dotted lines are capacity restrictions.

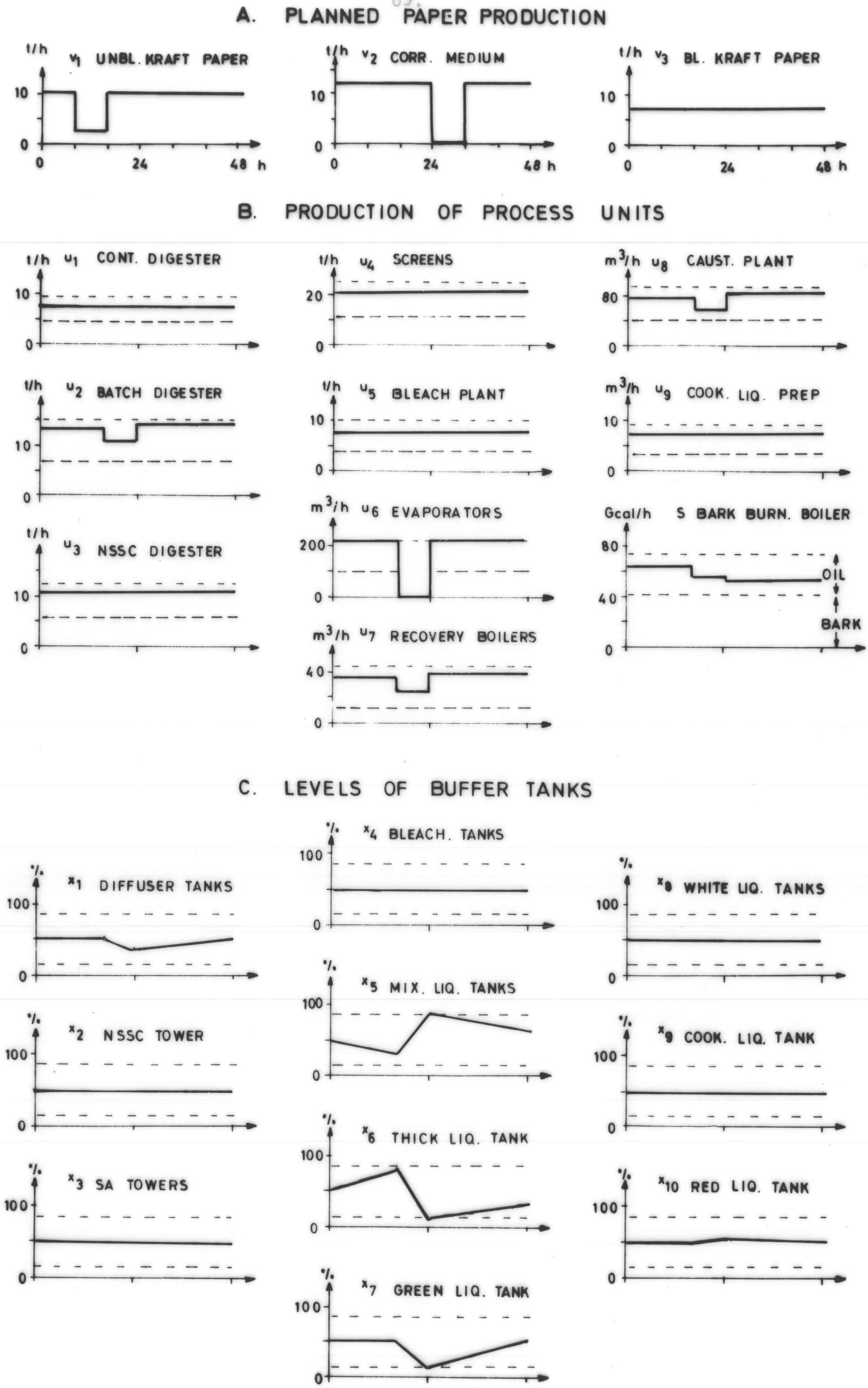

Fig 9.3 Planning example 3. Figures A show the planned paper production, figures B the production schedule as calculated by the optimization program and figures C the resultant tank levels. Dotted lines are capacity restrictions.

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