Topics in Digital and Robust Control of Linear Systems

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To the memory of Maj-Britt Bernhardsson

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Abstract

This thesis deals with several problems in the theory of linear systems. It consists of an introduction and six papers.

Paper I solves the problem of obtaining a finite dimensional sampled representation of a continuous time, state-space system with several time delays. Necessary and sufficient conditions for existence of a finite dimensional sampled system are given. A short algorithm for computing the sampled system is given. Paper II presents existence conditions for so called pure-mixed saddle equilibria. These conditions give insight into the properties of min-max controllers used in H_∞-control theory and stochastic differential games. Paper III analyzes a mixed H_2/H_∞-control problem. It is shown how recent results in this field can be obtained using standard differential game theory and a recently presented separation theorem due to Pierre Bernhard. The problem is solved by completion of squares. The time-varying, finite time horizon problem is solved. An explicit formula for the value of the game is obtained. New formulas for the discrete time case are given. A simple but rich example illustrates the equations. Paper IV presents a dual relationship between two special problem classes; a question that has been discussed previously by other authors. It is shown that duality can be obtained if generalized problem formulations including dynamic weighting functions are used. The result is illustrated by a derivation of a polynomial solution to the frequency-weighted discrete-time multivariable LQG feedforward control problem. Paper V discusses the concept of strong stabilization. The paper points out several weaknesses of the present concept. Paper VI describes how information from standard system identification can be used to find robust performance controllers. The expected value of the H_2-norm of the closed loop system is minimized by rewriting the problems as an LQG-problem for an extended problem.

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Contents

Preface

The thesis consists of an introduction and the following six papers:

I. Sampling of State-Space Systems With Several Time Delays, submitted to the *12th IFAC World Congress.*

II. Existence of Pure-Mixed Nash Equilibria for Continuous Partly Convex Games.

III. Min-Mix Control – A Classical Stochastic Differential Games Approach.

IV. Feedforward Control is Dual to Deconvolution, accepted for publication in *International Journal of Control.* Joint work with M. Sternad.

V. On the Notion of Strong Stabilizability. Published in *IEEE Transactions on Automatic Control,* Vol. AC-35, No. 8, pp. 927–929, August 1990. Joint work with P. Hagander. With addendum.

VI. Robust Performance Optimization of Open Loop Type Problems Using Models From Standard Identification, submitted to the *12th IFAC World Congress.*

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severely pressed for time. He has also let me explain my work to him on several occasions along the road, and has for instance checked the counterexample in paper III. His great knowledge of people and broad overviews over the status of different research areas have been of great help. I also thank him for a lot of good advice, some of which I did not take.

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I have had constant inspiring discussions with Anders Rantzer. He has also given me a lot of moral support. He, for instance, convinced me that the results of paper II should be written up. I would also like to thank Mikael Sternad who has read and commented a preliminary version of the manuscript. He has been a stimulating coauthor of paper IV. I also want to thank him for providing preliminary versions of several articles which have helped me to get the final ideas of paper VI. My interest in game theory was aroused by a course held by Professor Tamer Basar. He has also responded to some questions during the preparation of this work. I am also indebted to Michael Green for giving an interesting course in H_∞-control. Among my friends at the mathematical department in Lund I would like to mention Gunnar Sparr and Sven Spanne, with whom I have had many pleasant talks about mathematics.

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Finally but most importantly I thank my dear wife Helen for her generous support during occasional times of hard work.

Introduction

This chapter gives a brief summary of the contents of the different papers. The main contributions are presented and they are related to other work. The goal of the work is to investigate linear systems with time delays (Paper I), model uncertainty (Papers III and VI) and controller design with the restriction on using a stable controller (Paper V). Part of the work is motivated by contacts with colleagues in the signal processing field (Paper IV,VI) and in the area of game theory (Paper II,III). All papers concern Linear Systems.

I. Sampling of State-Space Systems with Several Time Delays

This paper solves a basic problem in digital control. It is a natural generalization of results given in courses in digital control. The problem concerns digital control of systems with several internal time delays. It is well known that a linear system preceded by a time delay can be sampled and that the sampled system is finite dimensional. Most software packages for control system design contain routines doing this. Since a time delay is an infinite dimensional system, it is a surprising coincidence that the sampled system becomes finite dimensional. It has also been noted before that a system consisting of a time delay between two linear systems becomes finite dimensional when sampled.

Natural questions arising are: 'What happens if there are several time delays that are arbitrarily connected to other finite dimensional linear systems? When is the sampled system finite dimensional ? How should the sampled system be computed?' The paper describes how to obtain a zero-order hold sampled version of a state space system containing several time delays at arbitrary positions. It is not assumed that the ratios of the time delays are rational numbers. It is also described when the sampled system is finite dimensional for all sampling periods. The condition for obtaining a finite dimensional sampled system is shown to be that there are no loops around time delays.

A short and constructive algorithm for sampling such a system is presented. The key idea is a handy description of the solution of a class of difference-differential systems, obtainable by semigroup theory. The sampled system requires an augmented state vector. The state variables of the sampled system contain the continuous time state variables at the sampling instances as a subset. The physical interpretations of states are therefore retained. All calculations can be performed using standard software for sampling systems. There are several possibilities for further work in this area. It would be nice with an improved algorithm guaranteeing a minimal order representation. Using the same techniques as in the paper it should also be possible to find sampled representations of systems with stochastic noise. The goal is then to find a discrete noise sequence giving the same first two moments as the continuous noise at sampling points. It should also be possible to obtain formulas for sampling a continuous time quadratic loss function.

II. Existence of Pure-Mixed Nash Equilibria for Continuous Partly Convex Games

This paper describes a simple, but useful, result concerning the structure of saddle equilibria in differential games. The result says that if the loss function is convex in some of the players variables, these players can use pure strategies. The rest of the players must generally use mixed strategies. This simple result is useful for understanding the saddle point properties of H_∞-control for different γ-levels and the min-mix controllers in Paper III . The proof is straightforward, using basic functional analysis. I find it pedagogical to see that existence theorems for all three kinds of equilibria: pure, mixed, and pure-mixed equilibria really follows from the same basic theorem, Kakutani's fixed point theorem.

III. Min-Mix Control Using Classical Stochastic Game Theory

In this paper we will present a *completion of squares method* for a mixed H_2 and H_∞ problem that will be called "the min-mix problem". Both the continuous time and discrete time case are treated. A conjectured generalization of a dynamic programming separation principle by Bernhard is used to obtain the controller. For the infinite time horizon case the formulas reduce to those in [Doyle *et al.*, 1992], however there obtained for a problem without stochastic disturbances. That paper also uses different methods to obtain the results. The full finite time horizon, time varying problem is treated and new formulas are given. We also obtain new, explicit, formulas for the value of the game. New discrete time formulas also follows from an analogous treatment, which illustrates the close connection. Relationships to earlier results on game theory are also presented. This gives insight into the importance of the information structure. A simple, but rich, example illustrates the theory and the equations obtained. When working on this part, we also found several problems worthy of further investigation.

IV. Feedforward Control is Dual to Deconvolution

The duality between two special problem classes has been discussed in a previous paper [Sternad-Ahlén, 1988]. In this paper it was pointed out that there are close correspondences between feedforward based on disturbance measurement and deconvolution. By using loop transformations on scalar systems they showed how one problem could be transformed into the other. No dual relationships were, however, obtained. This paper demonstrates that the problems are dual if and only if a generalized problem formulation, with frequency-shaped weighting in the criteria, are used. From one of the problems, the dual problem can then be obtained immediately form the block diagram, by reversing the directions of arrows, interchanging summation points and node points and transposing all transfer function matrices. This result applies to continuous and discrete time problems, as well as for minimization of $J = \|G\|$, for any transfer function norms such that $\|G^T\| = \|G\|$. A derivation of a new polynomial solution to the frequency-weighted discrete-time multivariable linear quadratic Gaussian feedforward control problem illustrates how the duality can be used.

V. On the Notion of Strong Stabilizability

The goal of this paper is to understand a result in multivariable control which says that it is almost always possible to stabilize a MIMO control system with a *stable* controller. This has been described as a "proof" of the superiority of MIMO control compared to decentralized control. The contribution of the paper is to reveal, in several ways, the practical weakness of the present concept of strong stabilization. An example also illustrates that unstable controllers can be desperately required for good closed-loop performance, although they are not required for stability.

Insight is derived from an explicit solution of an H_∞-control problem with a side condition on controller stability. We show that examples exist that give arbitrarily bad performance with stable controllers, but acceptable performance with an unstable controller.

VI. Robust Performance Optimization of Open Loop Type Problems Using Models From Standard Identification

A natural approach to robust performance design is to use the information about model quality obtained from standard identification. The idea is to take the likelihood of different parameter variations into account. The controller design is then made by taking the most probable parameter variations into account, instead of very rare worst cases based on hard bounds on uncertainty. This has been done in an interesting paper which investigates some scalar robust estimation and robust feedforward control problems, [Sternad and Ahlén, 1992]. Optimal controllers are there obtained using polynomial calculations. Paper VI extends these results to a more general class of problems, and develops a short and instructive algorithm for the solution. Some restrictions on where the uncertain parameters enter are made. If these restrictions are not met, other identification methods and/or other measures of closed loop performance have to be used. Examples of different open loop type problems that satisfy the assumptions are given.

Paper I

Sampling of State Space Systems with Several Time Delays

Bo Bernhardsson

Abstract: The article discusses zero-order hold sampling of a state space system containing several time delays at arbitrary positions. It is described when the sampled system is finite dimensional for all sampling periods. The condition for obtaining a finite dimensional sampled system is shown to be that there is no signal loop around any of the time delays. A short and constructive state space algorithm is presented for sampling such systems. No assumption is made on commensurability of the time delays. The critical idea is a useful description of the solution for a class of difference-differential systems. The sampled system requires an augmented state vector. The states of the sampled system contain the continuous time states at the sampling instances as a subset. The physical interpretation of states is therefore retained. All calculations can be performed using standard programs for sampling systems.

1. Introduction

Many industrial processes contain several time delays. This is, for instance, common in chemical engineering processes, where time delays results from piping between units. Problems with several time delays also occur frequently in manufacturing processes, due to transportation delay and in signal processing applications, due to calculation delay or information delay. The time behavior of such systems can often be adequately described by linear, continuous time, differential-difference equations (DDEs). Such equations has been the subject of much research and there exists a rich literature on different aspects on DDE:s going far back [Choksy, 1960], [Bellman and Cooke, 1963], [Marshall, 1979].

The control of systems containing time-delays is generally difficult both in theory and in practice. Often time delays put severe restrictions on achievable feedback performance. It is therefore important to have good methods for analysis and design of such systems. One possibility is to sample the system and use digital control. As we will see, it can then happen that the sampled system becomes finite dimensional. Further analysis is then much simplified. This also opens up the way for many standard design techniques such as pole placement, linear quadratic control or H_∞-methods.

Control issues for differential-difference equations have received considerable attention in recent years. Stability questions have been studied in a number of papers, see e.g. [Kamen, 1982], [Mori, 1985], [Kamen *et al.*, 1985], [Lee and Radovic, 1988] and [Mori and Kokame, 1989]. Criteria for controllability, observability and stabilizability that parallels the delay-free case have been obtained, see e.g. [Salamon, 1984], [Emre, 1984] and [Fiagbedzi and Pearson, 1986]. Stochastic control of systems with time delays is studied in [Lindquist, 1969], [Lindquist, 1972], [Milman and Schumitzky, 1991]. Robustness of time delay systems is studied in, e.g., [Barmish and Shi, 1989]. A variety of methods to define and compute optimal control laws have also been suggested, [Lindquist, 1969], [Krasovskii, 1963], [Ross, 1971], and [Kwon and Lee, 1988]. Special identification methods which estimate unknown time delays also exist, [Pearson and Wuu, 1984], [Gawthrop and Nihtilä, 1985]. Simulation of DDE:s are treated in [Hairer *et al.*, 1987].

As soon as computers were being used to implement control systems in the 1950s it was found how to describe the discrete time equivalent of a continuous-time, linear, finite-dimensional system connected to a computer via A-D and D-A converters. Using a zero order hold function the input will be piecewise constant and the relationship between the input sequence $\{u(kh)\}$ and the output sequence $\{y(kh)\}$ can be described

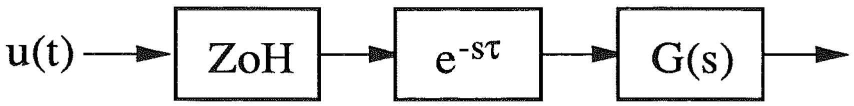

Figure 1. Hold circuit, time delay and linear system.

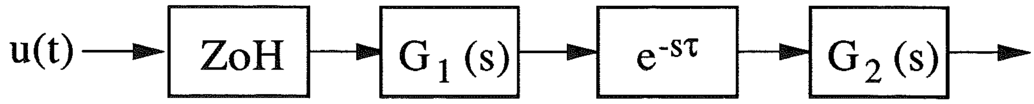

Figure 2. The case with a single inner time delay.

with a discrete time system. This was called "sampling the system". It is hard to determine the first reference, [Ragazzini and Franklin, 1958], [Mason, 1956], [Jury, 1958], [Ragazzini and Zadeh, 1952]. The method is by now classical and formulas appear in every book in computer-controlled systems, see e. g. [Åström and Wittenmark, 1990]. Algorithms are included in most packages for controller design.

A continuous time linear system with a time delay is an infinite dimensional system. To model the delay one must store a function of time over a time interval equal the length of the time delay. It was therefore a surprise when it was found that the sampled version of the system in Figure 1 is finite-dimensional. Formulas for the sampled system are easily obtained, see e. g. [Franklin and Powell, 1980] or [Åström and Wittenmark, 1990]. It is straightforward using these results to sample a multivariable linear system with time delays in control variables only. The solution consists of storing state variables and delayed input signals in a finite number of sampling points and showing that this information suffices to update the system equations. Notice that state variables often have physical interpretations. To keep the engineering intuition from the continuous time model, it is preferable to obtain a sampled system from which the state variables at the sampling points can be obtained.

Sampling of systems with internal time delays has received much less attention in the literature and few results have been obtained. The problem has only been solved for simple systems. The setup in Figure 1 was slightly generalized in [Araki *et al.*, 1984], [Wittenmark, 1985], [Fujinaka and Araki, 1987]. As a result the system in Figure 2 can also be sampled. Here the time delay is situated between two linear systems. The sampled system is finite dimensional in this case also. Notice that the problem of sampling the system in Figure 2 can not be trivially solved by changing the order of $G_1(s)$ and the time delay and reducing the problem to the system in Figure 1. The pulse transfer function between input and output will of course be the same, but the transformation changes the states of G_1 from $x_1(kh)$ to $x_1(kh - \tau)$ and one will hence not obtain a state space representation with the values of all state variables at the sampling points. This delay can be a problem if, for instance, the state variables should be use for state feedback. See [Wittenmark, 1985] for further motivation and discussion.

The problem with several time delays at arbitrary positions in a multivariable linear system arises naturally. When is the sampled system finite dimensional? This problem has not previously been solved. One reason might be that the answer, as we will see, is that the sampled system is not always finite dimensional. This was discussed, e.g., in [Koepcke, 1965]. Sampling of general time delay systems can therefore be very hard and the success will depend on where the time delays are situated. In this paper we will describe what systems become finite dimensional when sampled and we will present a short and constructive algorithm for sampling such systems.

Since the problem with several inner time delays has not been solved before, it has been circumvented in different ways in computer controlled systems. A standard method is to approximate delays with finite dimensional systems. Different methods exist. A popular method is to use Padé-approximations and other Taylor-series expansions. This were also common when analogue techniques were used for implementing time delays. The conclusion is normally that a very high order approximation has to be used if the time-delays are long and an accurate approximation is required. Notice, e.g., that an Nth order system can give at most $N\pi$ radians phase lag, whereas a time delay gives arbitrarily large phase lag. The approximation can therefore only be used in a limited frequency range. A high order approximation will of course both increase computation time in simulations and make analysis more difficult. If the system contains several time delays, this will often not be a satisfactory method.

Another method commonly used is to neglect the time delays. This is not recommended as a general method. As mentioned above delays can severely restrict achievable performance of the system, and it is hence important to model these correctly. The success of all approximate methods will of course depend on the situation. The problem is generally harder the longer the time delays are. A comparison of some approximation methods used on an industrial example is made in [Hammarstrom and Gros, 1980].

Some results about differential-difference equations require a commensurability condition between all time delays, see e.g. [Morse, 1976], [Kamen, 1982]. This means that all time delays should be an integer multiple of the same real number r. One idea is then to sample the system with the sampling rate r. All time delays are then integer multiples of the sampling rate, which could allow for simplifications. If, for instance, the time delays have lengths $h/7$, $h/11$ and $h/12$, where h is the sampling period, one must sample the system at the rate $h/924$. Having done this, one can then calculate the sampled system at the rate h assuming u constant over 924 time units. This method will often lead to very high dimensional systems.

From both a theoretical and practical viewpoint it is preferable with an exact representation of the sampled system. This is the aim of the current paper. In Section 2 we define notation and introduce two examples to illustrate ideas. In Section 3 we show how a finite dimensional sampled system can be obtained for systems having no feedback loop around any time delay. We also show that this is not possible for all systems. In Section 4 we present necessary conditions for a system to be finite dimensionally samplable for all values of time delays. Section 5 discusses how to reduce the order of the obtained realization. Conclusions and open questions are presented in Section 6.

For a deeper understanding of differential-difference systems it is often useful to use ring theory, 2-D theory, and the theory of semi-groups, [Sontag, 1976], [Morse, 1976], [Šebek, 1987], [Goldstein, 1985], [Hale, 1971]. We have, however, tried to avoid such theory to make the article as elementary and self contained as possible. The appendix, Section 7, contains a brief review of some results needed.

2. Problem Formulation

Consider the problem in Figure 3. The system consists of a general linear system with two types of inputs and two types of outputs. There is also an upper feedback loop consisting of time delays $\tau_1, \ldots, \tau_p$. This is a useful representation which covers many linear control problems, see e.g. [Pernebo, 1981] and [Boyd and Barratt, 1991]. Here w represents generalized external signals, such as disturbances and reference signals which can not be affected by the controller, u represents controlled signals, z are outputs that should be controlled and y are measured outputs that can be used by the controller.

We will in this article only be interested in the relationship between u and y. The reason we want to present the problem in the framework of Figure 3 is that the method presented in this paper can, with small additions, be extended to cover also sampling of systems including stochastic signals $w(t)$ and sampling of quadratic loss functions described in terms of $z(t)$, see Section 6. We will in this paper not use w and z.

All time delays have been collected in the upper loop. This is done

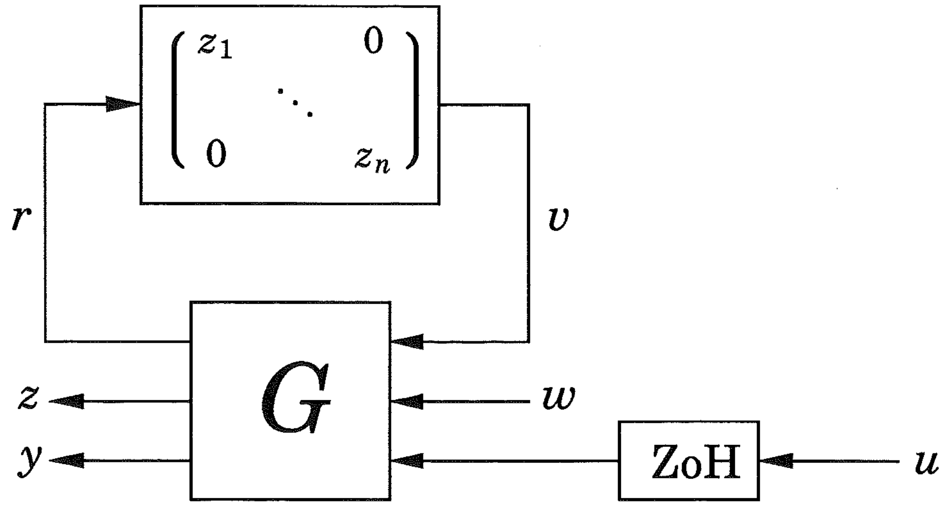

Figure 3. General linear problem with several time delays. All time delays are collected in the diagonal matrix Z.

by collecting all signals going into a time delay in the vector r, and all signals going out from delays in the vector v. The matrix Z will then be diagonal.

$$Z = \begin{pmatrix} z_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & z_n \end{pmatrix}$$

Here time delay number i is denoted z_i which will be interpreted either as a complex number, $z_i = e^{-s\tau_i}$, or in operator form, with z_i operating on functions of time via

$$v(t) = z_i f(t) = f(t - \tau_i)$$

In Figure 3, G denotes a linear, time invariant, causal, continuous time system that will be described in state-space form

$$\begin{aligned} \dot{x} &= Ax + B \begin{pmatrix} v \\ u \end{pmatrix} \\ \begin{pmatrix} r \\ y \end{pmatrix} &= Cx + D \begin{pmatrix} v \\ u \end{pmatrix} \end{aligned}$$

We will assume that zero-order hold circuits are used. This means that control signals are held constant between sampling points:

$$\bar{u}(t) = u(kh) \qquad t \in [kh, kh + h)$$

Here h is the sampling period. Uniform sampling is assumed. Rewriting Figure 3 into "state space" form we assume that the continuous time open loop system is given in the following form, which is standard in the differential-difference literature, [Bellman and Cooke, 1963]:

$$
\begin{aligned}
\dot{x}(t) &= A_0 x(t) + \sum_{i=1}^{p} A_i x(t-\tau_i) + B_0 u(t) + \sum_{i=p+1}^{r} B_i u(t-\tau_i) \\
&= A_0 x(t) + \sum_{i=1}^{p} A_i z_i x(t) + B_0 u(t) + \sum_{i=p+1}^{r} B_i z_i u(t) \\
&= A(z)x(t) + B(z)u(t)
\end{aligned} \tag{1}
$$

where $z = (z_1, \ldots, z_r)$ and

$$
A(z) = A_0 + \sum_{i=1}^{p} A_i z_i \qquad B(z) = B_0 + \sum_{i=p+1}^{r} B_i z_i
$$

EXAMPLE 1
The system in Figure 2 with one inner time delay will be used in the following as a test example. Written in the form (1), it becomes:

$$
\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ A_{12} & 0 \end{pmatrix}\begin{pmatrix} zx_1 \\ zx_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ 0 \end{pmatrix} u \tag{2}
$$

Here x_i is a state vector corresponding to system i, $zx_i = x_i(t-\tau)$ and $A_{12} = B_1 C_2$. From the results of [Wittenmark, 1985] we know that the sampled system can be written

$$
\begin{aligned}
x_1(kh+h) &= \Phi_1(h)x_1(kh) + \Gamma_1(h)u(kh) \\
x_2(kh+h) &= \Phi_{21}^{-}x_1(kh-h) + \Phi_2(h)x_2(kh) \\
&\quad + \Gamma_2^{-}u(kh-h) + \Gamma_2'(h-\tau)u(kh)
\end{aligned} \tag{3}
$$

where

$$
\begin{aligned}
\Phi_i(t) &= e^{A_i t}; \quad \Gamma_i(t) = \int_0^t e^{A_i s} B_i \, ds; \quad i = 1,2 \\
\Phi_{21}(t) &= \int_0^t e^{A_2 s} A_{21} e^{A_1(t-s)} \, ds \\
\Gamma_2'(t) &= \int_0^t e^{A_2 s} A_{21} \Gamma_1(t-s) \, ds \\
\Phi_{21}^{-} &= \Phi_{21}(h)\Phi_1(h-\tau) \\
\Gamma_2^{-} &= \Phi_{21}(h)\Gamma_1(h-\tau) + \Phi_{21}(h-\tau)\Gamma_1(\tau) + \Phi_2(h-\tau)\Gamma_2'(\tau)
\end{aligned}
$$

□

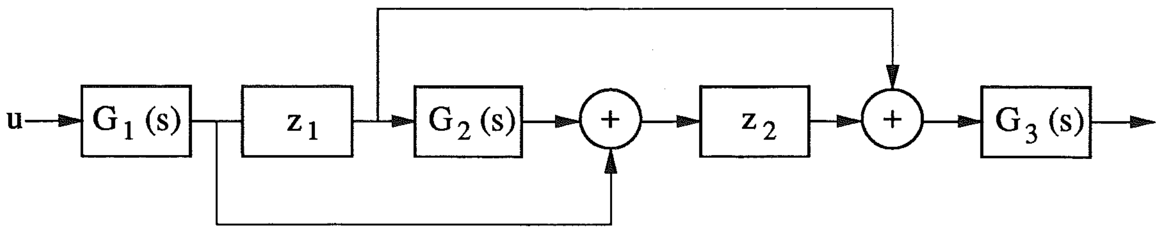

Figure 4. A simple problem with several time delays. The system consists of three mixing tanks described by first order systems and have two transportation delays.

EXAMPLE 2
As a slightly more challenging example we use the setup of Figure 4. This system describes a chemical processes with two transport delays and three mixing tanks, described by first order systems. The system can be written as:

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & 0 & 0 \end{pmatrix} z_1 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{4}$$

$$+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_{31} & a_{32} & 0 \end{pmatrix} z_2 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ 0 \\ 0 \end{pmatrix} u$$

□

3. Sampling Systems with Several Time Delays

It is easy to see that not all systems containing time delays become finite dimensional when sampled.

EXAMPLE 3
Consider the system in Figure 5, with transfer function $1/(1-e^{-s\tau})$. (This is not a system of the form (1).) This system gives an infinite dimensional sampled system unless τ is a rational multiple of the sampling rate h. In fact the system has poles, see appendix A, at the zeros of $1-e^{-s\tau}$, i.e. at

$$s_k = \frac{2k\pi i}{\tau} \qquad k = 0, \pm 1, \ldots$$

After sampling, the poles are transformed to

$$s_k \mapsto e^{s_k h} = e^{2k\pi i h/\tau}, \qquad k = 0, \pm 1, \ldots$$

and if h/τ is irrational there are infinitely many discrete time poles (the spectrum is in fact given by the entire unit circle). The sampled system can therefore not be finite dimensional.

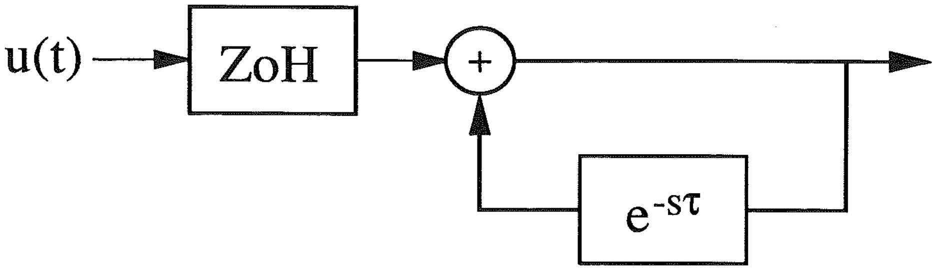

Figure 5. This system becomes finite dimensional when sampled if τ is a rational multiple of the sampling period h. If τ and h are incommensurable the sampled system is infinite dimensional.

On the other hand if h/τ is rational it is easy to see that the sampled system is finite dimensional. In fact, if $\tau = mh/n$ we can oversample the system at rate h/n and use that u is constant over m samples. It therefore suffices to consider the case when τ is an integer multiple, mh, of the sampling interval. We then, however, have

$$x(kh) = x(kh - mh) + u(kh)$$

which clearly is finite dimensional. □

To formalize we introduce the following definition:

DEFINITION 1
We say that a differential-difference system of the form (1) is a finite dimensionably samplable (FDS) system, if it's zero order hold sampled version can be represented with a finite dimensional discrete time system.

□

DEFINITION 2
A function $x(t)$ belongs to the set $\mathcal{N}$ if it is real-analytic and of exponential type α. This means that the Taylor-series expansion of x converges:

$$x(s+t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} x^{(k)}(s)$$

and that there exists constants C, α such that $|x(t)| \leq Ce^{\alpha|t|}, \ \forall t$.

□

It is easy to see that the following operator is well defined on $\mathcal{N}$ for all t:

$$\phi(t,z) = \sum_{k=0}^{\infty} \frac{t^k}{k!} (A(z))^k \tag{5}$$

which is defined by the sum

$$\phi(t,z)x(s) = \sum_{k=0}^{\infty} \frac{t^k}{k!} v_k(s)$$

where $v_0(s) = x(s)$ and

$$v_k(s) = A_0 v_{k-1}(s) + A_1 v_{k-1}(s-\tau_1) + \cdots + A_p v_{k-1}(s-\tau_p)$$

Using the restriction on exponential growth of x one easily proves that there exists a real number D such that for fixed s $\|v_k(s)\| \le D^k$ for all k. The sum will therefore converge (absolutely) for all t, s.

The following lemma is a description of the solution to (1). It can be seen as a result about the semigroup $\phi(t,z)$ generated by the infinitesimal generator $A(z)$ (see appendix for more details):

LEMMA 1
Assume that $x, u \in \mathcal{N}$ satisfy (1) for all t. We then also have

$$x(s+t) = \phi(t,z)x(s) + \int_0^t \phi(t-r,z)B(z)u(s+r)\,dr \qquad \forall s,t \tag{6}$$

Proof: From $\dot{x}(s) = A(z)x(s) + B(z)u(s)$ we obtain by successive differentiation that

$$x^{(k)}(s) = A^k(z)x(s) + \sum_{j=0}^{k-1} A^{k-1-j}(z)B(z)u^{(j)}(s)$$

This gives

$$\begin{aligned}
x(s+t) &= \sum_{k=0}^{\infty} \frac{t^k}{k!} x^{(k)}(s) = \\
&= \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k(z)x(s) + \sum_{k=1}^{\infty}\sum_{j=0}^{k-1} \frac{t^k}{k!} A^{k-1-j}(z)B(z)u^{(j)}(s) = \\
&= \phi(t,z)x(s) + \sum_{j=0}^{\infty}\sum_{l=0}^{\infty} \frac{t^{l+j+1}}{(l+j+1)!} A^l(z)B(z)u^{(j)}(s) \\
&= \phi(t,z)x(s) + \sum_{l=0}^{\infty}\sum_{j=0}^{\infty} \int_0^t \frac{(t-r)^l r^j}{l!\, j!}\,dr\, A^l(z)B(z)u^{(j)}(s) = \\
&= \phi(t,z)x(s) + \int_0^t \phi(t-r,z)B(z)u(s+r)\,dr
\end{aligned}$$

Notice that all sums converge according to the discussion above. □

Remark. Notice that in the delay-free case Lemma 1 reduces to the classical 'Variations of Constants Lemma'.

We noticed in Example 1 that not all systems are FDS. The following condition will be the key to categorizing FDS-systems.

DEFINITION 3
A system of the form (1) is said to be feedback free (FBF), if and only if, for all sets of indices $\{i_1, \dots i_p\}$ for which some of the positive indices are equal ($i_r = i_s > 0$), we have

$$\prod_j A_{i_j} = 0 \tag{7}$$

For instance, we must have $A_1 A_0 A_1 = 0$ and $A_2^2 = 0$ but there is no restriction on e.g. $A_1 A_0^2 A_2 A_0$, since the index 0 is not positive. □

The FBF-condition can be checked in a finite number of operations for a given system using the definition directly. This follows from Cayley-Hamilton's theorem. The condition is, however, more easily checked in a block diagram of the system. This becomes easy if the system equations are written in a form where the matrices A_i all are of rank one. This means that time delays of equal length, situated at different positions in a block diagram, are treated as separate. We will assume this in what follows. Condition FBF can be seen to be satisfied if there are no feedback loops in the system around any of the delays.

Remark. The condition above is given for the open loop system (1). without regulator. There can, of course, be feedback control present around the sampled system.

LEMMA 2
If (7) is satisfied there exist continuous functions $F_0(t), F_1(t), \dots, F_{1 \dots p}(t)$, where p is the number of internal time delays, such that

$$\begin{aligned}\phi(t,z) = {} & F_0(t) + F_1(t) z_1 + \cdots + F_p(t) z_p + \\ & + F_{12}(t) z_1 z_2 + \cdots + F_{p-1,p}(t) z_{p-1} z_p \\ & + F_{123}(t) z_1 z_2 z_3 + \cdots + F_{p-2,p-1,p}(t) z_{p-2} z_{p-1} z_p \\ & + \cdots \\ & + F_{1 \dots p}(t) z_1 z_2 \cdot \ldots \cdot z_p \qquad \forall t, z \end{aligned} \tag{8}$$

Proof: Treat $z_1, \dots, z_p$ as algebraic variables. The infinite sum

$$\phi(t,z) = \sum_{k=0}^{\infty} \frac{t^k}{k!} (A_0 + A_1 z_1 + \cdots + A_p z_p)^k$$

then converges absolutely for all t and z. When we expand the terms in the sum, all terms where some z_i is multiplied by itself become zero. This is exactly (7). This leaves us with the terms stated in the theorem. The manipulations involved in collecting terms in this way are allowed because of absolute convergence. □

Remark. Notice that for complex numbers $z_1, \dots, z_p$ we have

$$\phi(t,z) = e^{tA(z)} \tag{9}$$

This is just a formal identity. When $z_1, \dots, z_p$ are interpreted as operators it must be handled with care. Compare with the formulation of Taylor's formula as

$$e^{hD} f(t) = f(t+h) \qquad D = \frac{d}{dt}$$

which is valid only under special conditions on f. The operator $\phi(t,z)$ is a priori defined only on the class $\mathcal{N}$ above. Using expression (8) we, however, now see that the definition can be extended to all piecewise continuous u and differentiable x satisfying (1). Lemma 1 will remain valid since the right hand side of (6) still is well defined. We also notice that the right hand side of (6) will only depend on a finite number of old values of x. The results above could also have been obtained using semigroup theory, working with other classes of functions than $\mathcal{N}$, see the appendix.

Remark. The easiest way to find the F:s given a certain system satisfying condition FBF is to treat the z_i:s as complex numbers and to identify the left and right hand sides in

$$\begin{aligned} e^{t(A_0+A_1z_1+\dots A_pz_p)} = {} & F_0(t) + F_1(t)z_1 + \cdots + F_p(t)z_p + \\ & + F_{12}(t)z_1z_2 + \cdots + F_{p-1,p}(t)z_{p-1}z_p \\ & + \dots \\ & + F_{1\dots p}(t)z_1z_2 \cdot \dots \cdot z_p \end{aligned} \tag{10}$$

for some different choices of z:s. This gives a number of linear equations to determine the F:s. Since $\{1, z_1; z_2; \dots; z_p; z_1z_2; \dots; z_1z_2 \cdot \dots \cdot z_p\}$

are linearly independent functions it is possible to construct a nonsingular linear system of equations in this way. By a good choice of the z:s we can in fact write down explicit formulas. If we successively put $(z_1, \ldots, z_p)$ equal to all possible vectors consisting of zeros and ones we get a triangular system of equations from which the F:s can be determined recursively. This gives:

LEMMA 3
The functions in Lemma 2 are given by

$$\begin{aligned}
F_0(t) &= e^{A_0 t} \\
F_1(t) &= e^{(A_0+A_1)t} - F_0(t) \\
&\vdots \\
F_p(t) &= e^{(A_0+A_p)t} - F_p(t) \\
F_{12}(t) &= e^{(A_0+A_1+A_2)t} - F_0(t) - F_1(t) - F_2(t) \\
&\vdots \\
F_{1\ldots p}(t) &= e^{(A_0+A_1+\ldots A_p)t} - F_0(t) - F_1(t) - \cdots - F_{2\ldots p}(t)
\end{aligned} \tag{11}$$

□

Notice that all calculations can be performed using standard numerical software.

EXAMPLE 1, revisited
Let us use Lemma 2 on the system with the inner time delay in Figure 2. Since

$$A_0 + A_1 z = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ a_{21} & 0 \end{pmatrix} z$$

we get, identifying terms using $z = 0$ and $z = 1$, that

$$\phi(t,z) = e^{A_0 t} + (e^{(A_0+A_1)t} - e^{A_0 t})z = \begin{pmatrix} e^{a_1 t} & 0 \\ \alpha(t)z & e^{a_2 t} \end{pmatrix}$$

with $\alpha(t) = a_{21}(e^{a_1 t} + e^{a_2 t})/(a_1 + a_2)$. This can of course also easily be verified directly. □

EXAMPLE 2, revisited
Using (8) on the chemical engineering example in Figure 5 gives

$$\phi(t,z) = e^{(A_0+z_1A_1+z_2A_2)t} = F_0(t) + F_1(t)z_1 + F_2(t)z_2 + F_{12}(t)z_1z_2 \tag{12}$$

Putting $z = (1,0)$, $(0,1)$ and $(1,1)$ and identifying left and right hand sides of (12) we get

$$F_0(t) = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}; \qquad F_1(t) = \begin{pmatrix} 0 & 0 & 0 \\ \delta_{21} & 0 & 0 \\ \delta_{31} & 0 & 0 \end{pmatrix}$$

$$F_2(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \delta_{31} & \delta_{32} & 0 \end{pmatrix}; \qquad F_{12}(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \gamma & 0 & 0 \end{pmatrix}$$

where $\alpha_i = \exp(a_i t)$, $i = 1,2,3$, $\delta_{ij} = a_{ij}(\alpha_i - \alpha_j)/(a_i - a_j)$ and $\gamma = a_{32}\, a_{21}(\alpha_1(a_2 - a_3) - \alpha_2(a_1 - a_3) + \alpha_3(a_1 - a_2))/((a_1 - a_2)(a_1 - a_3)(a_2 - a_3))$

□

The following theorem gives a constructive solution to the sampling problem for FBF systems:

THEOREM 1
If condition FBF is satisfied, the system (1) is FDS. One finite dimensional sampled representation is given by

$$X_e(kh + h) = \Phi_e X_e(kh) + \Gamma_e U(kh)$$

Here X_e is an extended state space vector of the form

$$X_e(t) = \begin{pmatrix} x(t) \\ x(t - \tau_1) \\ x(t - \tau_2) \\ \vdots \\ x(t - \tau_1 - \cdots - \tau_p) \end{pmatrix}$$

The matrix Φ_e is constructed using (8) and (11):

$$\Phi_e = \begin{pmatrix} F_0(h) & F_1(h) & \cdots & F_{1\ldots p}(h) \\ F_0(h - \tau_1) & F_1(h - \tau_1) & \cdots & F_{1\ldots p}(h - \tau_1) \\ \vdots & \vdots & & \vdots \\ F_0(h - \tau_1 - \cdots - \tau_p) & F_1(h - \tau_1 - \cdots - \tau_p) & \cdots & F_{1\cdots p}(h - \tau_1 - \cdots - \tau_p) \end{pmatrix}$$

The matrix U_e contains the $d + 1$, where $(d - 1)h < \sum \tau_i \leq dh$, last values of the control signals:

$$U_e = \begin{pmatrix} u(kh) \\ u(kh - h) \\ \vdots \\ u(kh - dh) \end{pmatrix}$$

The matrix Γ_e is determined by using that $u(t)$ is constant between samples in the following relation

$$\begin{pmatrix} \int_0^h \phi(h-r,z)B(z)u(kh+r)\,dr \\ \int_0^{h-\tau_1} \phi(h-\tau_1-r,z)B(z)u(kh+r)\,dr \\ \vdots \\ \int_0^{h-\tau_1-\cdots-\tau_p} \phi(h-\tau_1-\cdots-\tau_p-r,z)B(z)u(kh+r)\,dr \end{pmatrix} = \Gamma_e U_e$$

Proof: The theorem follows directly by using Lemmas 1 and 2 with $s = kh$ and $t = h$, $t = h - \tau_1, \ldots, t = h - \tau_1 - \cdots - \tau_p$, since we can use these lemmas to update the full state vector $X_e(kh+h)$ from the value of $X_e(kh)$ and the $d+1$ last values of $u(kh)$. □

EXAMPLE 2, continued
We have that

$$x(kh+t) = \begin{pmatrix} F_0(t) & F_1(t) & F_2(t) & F_{12}(t) \end{pmatrix} \begin{pmatrix} x(kh) \\ x(kh-\tau_1) \\ x(kh-\tau_2) \\ x(kh-\tau_1-\tau_2) \end{pmatrix} + \Gamma_0(t)u(kh) + \Gamma_1(t)u(kh-h) \tag{13}$$

where the F:s are given above. If we assume for simplicity in notation that $\tau_1 + \tau_2 \le h$ we get for $t \le h$:

$$\begin{aligned} \Gamma_0(t) &= \int_0^t F_0(t-s)B\,ds + \int_{\min(\tau_1,t)}^t F_1(t-s)B\,ds + \\ &+ \int_{\min(\tau_2,t)}^t F_2(t-s)B\,ds + \int_{\min(\tau_1+\tau_2,t)}^t F_{12}(t-s)B\,ds \\ \Gamma_1(t) &= \int_0^{\min(\tau_1,t)} F_1(t-s)B\,ds + \int_0^{\min(\tau_2,t)} F_2(t-s)B\,ds + \\ &+ \int_0^{\min(\tau_1+\tau_2,t)} F_{12}(t-s)B\,ds \end{aligned}$$

With $a_1 = -1, a_2 = -2, a_3 = -3$ and $a_{21} = a_{31} = a_{32} = b_1 = 1, \tau_1 = 1/5, \tau_2 = 1/4, h = 1$ we obtain for $t \ge \tau_1 + \tau_2$:

$$\Gamma_0(t) = \begin{pmatrix} 1 - e^{-t} \\ 0.5 - 1.2214\,e^{-t} + 0.7459\,e^{-2t} \\ 0.8333 - 2.0369\,e^{-t} + 1.2298\,e^{-2t} + 0.0136\,e^{-3t} \end{pmatrix}$$

$$\Gamma_1(t) = \begin{pmatrix} 0 \\ 0.2214\,e^{-t} - 0.2459\,e^{-2t} \\ 0.5369\,e^{-t} - 0.7298\,e^{-2t} + 0.1531\,e^{-3t} \end{pmatrix}$$

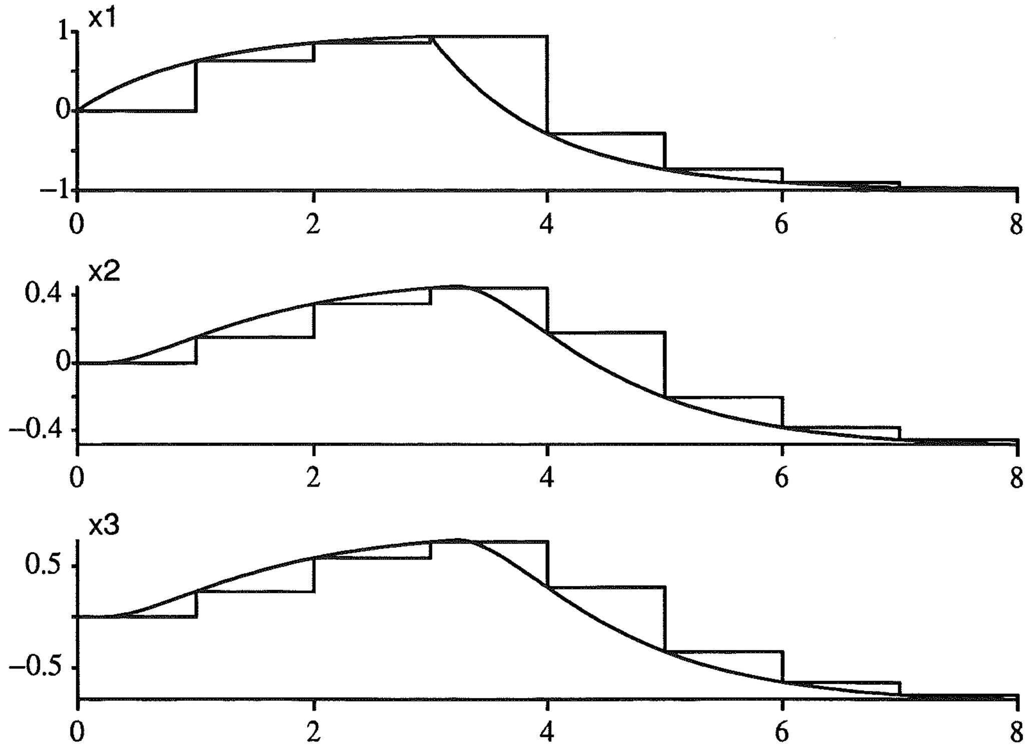

Figure 6. Simulation verifying the equations for the sampled version of Example 2.

Equation (13) can now be used for $t = h, h - \tau_1, h - \tau_2$ and $h - \tau_1 - \tau_2$ to update the full X_e-vector. We will not present the full discrete time system since Φ_e is a 12 by 12-matrix. We will discuss how the order can be reduced in Section 5.

Figure 6 shows a simulation of the continuous time system in Example 3 and of the sampled data representation that results from Theorem 1. The system is started from zero initial condition at $t = 0$. The input is 1 until $t = 3$ and then -1. The plots confirm the calculations.

4. What Systems Have a Finite Dimensional Sampled Counterpart?

Theorem 1 shows that condition FBF is sufficient for a system of the form (1) to be FDS. We also have the following result of necessity:

THEOREM 2
Assume the system is given by (1). Then if the open loop system contains a feedback loop around any of the delays, the system can not be FDS for all sampling rates h and delays τ_i.

Proof: Assume that the transmission from the output of delay z_1 to the input of the same delay is $n_2(s,z)/d_2(s,z) \neq 0$. We then have poles at $1 - z_1 n_2/d_2 = 0$, so

$$d_2(s, z_2, \ldots, z_p) - z_1 n_2(s, z_2, \ldots, z_p) = 0 \tag{14}$$

Notice that for special values of time delays we might have, e.g., $z_1 = z_2 z_3$, but since the condition is required to be satisfied for all τ_i, z_i are independent variables. Since $n_2 \neq 0$ and $d_2 \neq 0$, (14) is a quasi-polynomial of the form in Lemma 3 in the appendix which is not free of z_1. The continuous time system will therefore have infinitely many poles. Since poles are mapped as $s_k \mapsto e^{s_k h}$ when sampled, see appendix, we conclude that the discrete system can not be finite dimensional. Notice that the mapping $s_k \mapsto e^{s_k h}$ is not injective for a fixed h, but since the condition is required to be satisfied for all h we obtain a contradiction.

□

Remark. Example 1 shows that a system with feedback loop around a time delay can be FDS for some special values of h. It is also easy to construct an example with a feedback loop that is FDS for special values of the length of the time delays. Theorems 1 and 2 are therefore the best possible.

With Theorems 1 and 2 we have both necessary and sufficient conditions for a system to be FDS for all values of time delays τ_i and sampling periods h. Other systems will require an infinite dimensional state vector to be represented exactly.

5. Obtaining a Minimal Realization

Theorem 1 gives a realization that provides the values of all states for more time instances than the sampling points. If only the states are needed at the sampling points, which is a natural situation for digital controller design, the order of the representation can be further reduced. Consider, for instance, Example 1. The formulas in (5) give a third order representation if $0 < \tau \leq h$, but Theorem 1 gives a fifth order system. The input-output behavior is, however, the same. The system obtained from Theorem 1 is therefore not minimal. The same comments are true for Example 2.

Of course standard realization techniques, such as Kalman decomposition or balanced realization can now be used to obtain a minimal realization. However, it would be nicer to have a direct way to reduce the order of the sampled representation in Theorem 1. Looking closer at Example 1, we see that we calculated zx_2 although this delayed state was not needed in the other equations. One idea is therefore to figure out exactly what states in the extended state vector X_e that are needed to update the other states. This can be found using the block diagram. A part of the X_e-vector, say $z_4 z_3 z_1 x_2$ is needed only if a signal starting in system 2 can go through time delays number 1, 3 and then 4. This idea will normally reduce the order of the sampled representation considerably for dense systems. It is, however, not enough to guarantee minimality.

EXAMPLE 2, continued
In Example 2, only the state variables x_1, x_2, x_3, $z_1 x_1$, $z_2 x_1$, $z_2 x_2$ and $z_1 z_2 x_1$ are needed. This gives a representation with seven states (+ one delayed control signal). More careful analysis, however, shows that the minimal order is four (if $0 < \tau_1 + \tau_2 < h$).

6. Conclusions and Open Questions

We have discussed how to obtain, when possible, a finite dimensional system when sampling a general state-space system containing several time delays. This result generalizes the results of [Araki *et al.*, 1984, Wittenmark, 1985, Fujinaka and Araki, 1987]. We have shown that a necessary and sufficient condition for obtaining a finite dimensional system is that the system has no feedback loop around any time delay. The algorithm has been used on two examples. The formulas for the sampled representation have been verified by simulation.

A representation of a system with several time delays as a finite dimensional sampled system makes state space analysis and design easier. It is, however, important to understand that the inherent restrictions on achievable control performance due to time delays are still present. Also notice that the behavior between sampling points is not described by the discrete time system.

It would be nice with an improved algorithm that guarantees a minimal order sampled representation of the input-output behavior. The minimal order is an open question. The results in this article could possibly also be generalized to higher order hold circuits, as used in, e.g., [Kabamba, 1987] and to multirate sampling. This could be a goal for further research.

Using, e.g., Matlab or Maple, it is straightforward to implement the algorithm presented in the paper. The calculations parallels the standard case, so existing software can be used.

Using Lemma 1 and Lemma 2 it should also be possible to generalize the results of this paper to sampling of stochastic, continuous time systems, i.e. when the signal w in Figure 3 is white noise. The continuous noise is then represented with sampled noise with the same first two moments. To do this, one must calculate integrals of the form

$$\int_0^h \phi(h-t,z)R\phi^T(h-t,z)\,dt$$

where R is a covariance matrix and $\phi(t,z)$ is given by (8). Sampling of quadratic loss functions

$$\int_0^h \begin{pmatrix} x \\ u \end{pmatrix}^T Q \begin{pmatrix} x \\ u \end{pmatrix} dt$$

also give rise to similar integrals. Lemma 2 is here the key.

7. Appendix

We will here describe some results from the theory of differential difference equations that are needed in the article. Proofs of the following results can be found in [Bellman and Cooke, 1963], unless stated otherwise.

Consider the following system of differential-difference equations (DDEs).

$$\dot{x}(t) - \sum_{i=0}^{p} A_i x(t-\tau_i) = f(t) \tag{15}$$

where we assume that τ_i are constant and

$$0 = \tau_0 \leq \tau_1 \leq \tau_2 \leq \cdots \leq \tau_p$$

Remark. Differential-difference equations were studied in the 18th century in connection with geometrical problems, for an early reference see [Lacroix, 1819]. Since we assume all $\tau_i \geq 0$, the equations above are of the so called retarded type. In older literature such equations were also designated 'hystero-differential' equations.

Existence Theorem

In principle existence theorems for retarded type DDEs is an easy matter. Suppose that $x(t)$ is known for a period of time equal to the longest delay time. Then all delayed states are known functions of time and the equation becomes an ordinary differential equation and can be treated by known existence theories. We have the following result:

THEOREM 3—Existence Theorem
Consider the system (15) with initial conditions given by

$$x(t) = g(t), \qquad 0 \le t \le \tau_p$$

Suppose that f is of class C^0 on $[0, \infty)$ and that g is continuous on $[0, \tau_p)$. Then there exists one and only one function $x(t)$ which is continuous for $t \ge 0$, satisfies the initial conditions, and (15) for $t > \tau_p$. Moreover, $x(t)$ is of class C^1 on (τ_p, ∞). This function will be called the solution. The existence theorem can be generalized to piecewise continuous functions f. This is needed when a zero order hold-circuit is used for control signals. □

Spectrum

DEFINITION 4
The *characteristic equation* connected with (15) is given by

$$p(s, z_1, \ldots, z_p) := \det\left(sI - \sum A_i z_i\right) = 0 \qquad z_i = e^{-s\tau_i} \tag{16}$$

A function p which is a polynomial in $(s, e^{-s\tau_1}, \ldots, e^{-s\tau_p})$ is called an exponential-polynomial or quasi-polynomial. □

Suppose that g is of class C^1 and let $x(t)$ be the unique solution to (3) and (15) with $f(t) = 0$. Let

$$p(s) = \det\left(sI - \sum_{i=1}^{p} A_i e^{-s\tau_i}\right)$$

$$r(s) = g(0) + \int_0^{\tau_p} g'(t)\,dt - \sum_{i=1}^{p} e^{-s\tau_i} \int_0^{\tau_p - \tau_i} A_i g(t) e^{-st}\,dt$$

Then for t large we have

$$x(t) = \lim_{l\to\infty} \sum_{C_l} e^{s_k t} q_k(t)$$

where $e^{s_k t} q_k(t)$ denotes the residues of $e^{ts} p^{-1}(s) r(s)$ at a zero s_k of $\det(p(s))$. The limit is uniform in any finite interval $t_0 \le t \le t_1$ if t_0 is large. The function $q_k(t)$ is a vector polynomial of degree less than the multiplicity of s_k. Here C_l denotes an increasing sequence of regions in the complex plane as defined in [Bellman and Cooke, 1963]. The order of selecting partial sums is important as to obtain convergence.

Zeros of Quasipolynomial

A system of the form (15) can have infinitely many poles. The following result is useful:

LEMMA 4
Let

$$p(s) = p_0(s) + \sum_{i=1}^{N} e^{-s\tau_i} p_i(s) \tag{17}$$

where $p_i(s), i = 0, \ldots, N,\ N \geq 1$, are nonzero polynomials and $\tau_i \neq 0$, then $p(s)$ has infinitely many zeros.

Proof: Suppose p has a finite number of zeros with multiplicities $r_0, r_1, \ldots, r_M$. It then follows from Hadamard's Factorization theorem, see e.g. [Boas, 1954], that

$$p(s) = s^{r_0} e^{h(s)} \prod_{k=1}^{M} (1 - s/s_k)^{r_k}$$

where $h(s)$ is a first order polynomial. It is easy to see that this is in contradiction with (17). □

The Sampled Spectrum—Basic Semigroup Theory

We need a description of what happens with the singularities of $\dot{x} = A_0 x(t) + \sum_i A_i x(t - \tau_i)$ when the system is sampled. In what sense is it true that, the possibly infinitely many, poles are mapped as $s \mapsto e^{sh}$? This is best discussed using the framework of semigroup theory. For a good introduction to the area see [Goldstein, 1985].

Let $\mathcal{X}$ be a real or complex Banach space. In our application it will be the space of all continuous functions $C([-\tau_p, 0])$, where τ_p is the longest internal time delay. Let $\mathcal{A}$ be a linear operator from a domain of definition $D(\mathcal{A})$ dense in $\mathcal{X}$ to $\mathcal{X}$. A goal of semigroup theory is to give meaning to equations like

$$\dot{x} = \mathcal{A}x, \qquad x \in \mathcal{X}$$

and analyze them using notations like $e^{t\mathcal{A}}$. Notice that x might be infinite dimensional. We will use this framework to study (1).

A family $T = \{T(t) : 0 \leq t < \infty\}$ of linear operators from $\mathcal{X}$ to $\mathcal{X}$ is called a (C_0) *semigroup* if

1. $\|T(t)\| < \infty \quad \forall t$
2. $T(t+s)f = T(t)T(s)f \quad \forall f \in \mathcal{X}$ and all $t, s \geq 0$
3. $T(0) = I$
4. $t \mapsto T(t)f$ is continuous for $t \geq 0$ for each $f \in \mathcal{X}$.

The infinitesimal generator $\mathcal{A}$ is then defined by the equation

$$\mathcal{A}f = \lim_{t\to 0^+} \frac{T(t)f - f}{t}$$

where f is in $D(\mathcal{A})$ iff this limit exists. Convergence is defined in the topology of $\mathcal{X}$. Formally the semigroup property suggests that $T(t) =$ "$e^{t\mathcal{A}}$" where $\mathcal{A} = (d/dt)T(t)|_{t=0}$.

The spectrum σ of a closed operator $\mathcal{A}$ is defined as the set of complex values s such that the operator $(sI - \mathcal{A})^{-1}$ is not bijective and bounded.

THEOREM 4—The Spectral Mapping Theorem
Let $\mathcal{A}$ generate a (C_0) semigroup T on $\mathcal{X}$. Then

$$\exp[t\sigma_p(\mathcal{A})] \subset \sigma_p[T(t)] \subset \exp[t\sigma_p(\mathcal{A})] \cup \{0\} \tag{18}$$

for $t \geq 0$, where σ_p denotes point spectrum.

Proof: See Theorem 9.5 [Goldstein, 1985]. □

To apply this we have to show how to interpret (1) as a strongly continuous semigroup of linear bounded operators. Define $\mathcal{X} = C[-\tau_p, 0]$ and the group action

$$[T(t)x](s) = \begin{cases} x(s+t) & s \in [-\tau_p, -t] \\ x(0) + \int_0^{s+t} \sum_i A_i x(r - \tau_i)\, dr & s \in [-t, 0] \end{cases}$$

It is easy to see that $T(t)$ is closed and satisfies all properties stated above when the sup-norm topology is used on C. The generator is given by

$$\mathcal{A}x(s) = \begin{cases} \dot{x}(s) & \text{if } -\tau_p \leq s < 0 \\ \sum_i A_i x(-\tau_i) & \text{if } s = 0 \end{cases} \tag{19}$$

and the domain of definition is given by

$$D(\mathcal{A}) = \left\{ x \in C^1([-\tau_p, 0]) \ : \ \dot{x}(0) = \sum_i A_i x(-\tau_i) \right\}$$

which is dense in C. We can now use the following result:

THEOREM 5
The spectrum $\sigma(\mathcal{A})$ of the infinitesimal generator $\mathcal{A}$ is a pure point spectrum given by

$$s \in \sigma_p(A) \quad \Longleftrightarrow \quad \det\left(sI - \sum_i A_i e^{-s\tau_i}\right) = 0$$

Proof: This follows from Lemma 20.1 in [Hale, 1971]. □

From the discussion above and since $x(kh + h) = T(h)x(kh)$ we can conclude that the discrete time spectrum contains the values $e^{s_k h}$ where s_k are the continuous time poles and h is the sampling period. This is needed in the article in Section 4.

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PAPER II

Existence of Pure-Mixed Nash Equilibria for Continuous Partly Convex Games

Bo Bernhardsson

Abstract: An N-person nonzero-sum game is considered for which the cost functionals are convex for some, but not all, players. It is shown that a Nash-equilibrium exists with pure strategies for the convex players and mixed strategies for other players. Such equilibria will be called 'pure-mixed'. The theorem is useful for understanding the saddle-point properties of the H_∞-controllers. An example from linear quadratic differential games occurring in H_∞-control theory, see [Başar and Bernhard, 1991], is sketched. A simple proof of the classical theorem of existence of mixed equilibria to continuous games also follows.

1. Introduction

The existence of different types of saddle equilibria is at the heart of game theory. Let $J_i(u_1, \ldots, u_N)$ be real-valued continuous functionals to be minimized by the noncooperative players $i = 1, \ldots, N$ respectively. We remind the reader of the definition of Nash-equilibria.

DEFINITION 1
$(u_1, \ldots, u_N)$ is a Nash equilibrium to the set of cost functionals $J_1, \ldots, J_N$ iff $\forall i$

$$J_i(u_1, \ldots, u_i, \ldots, u_N) \leq J_i(u_1, \ldots, v_i, \ldots, u_N) \qquad \forall v_i$$

This means that a single player can not improve upon his cost functional if the other players strategies are kept fixed. The following existence result is classical:

THEOREM 1
Assume that J_i is convex in $u_i \in \Omega_i$, where Ω_i are compact, convex sets. Then the associated N-person nonzero-sum game admits a Nash equilibrium in pure strategies.

Proof: The theorem follows directly from Kakutani's fixed point theorem, which is a slight generalization of Brouwer's fixed point theorem. For details, see [Kakutani, 1941] and [Stoer and Witzgall, 1970]. □

Remark 1. If all J_i:s are strictly convex the equilibrium is unique.

Remark 2. Continuity can be relaxed to semi-continuity, see [Başar and Olsder, 1982] and the references therein.

Remark 3. For a two player zero-sum game $J_2 = -J_1$, hence J_1 is convex in u_1 and concave in u_2.

Remark 4. A short combinatorical proof of Brouwer's theorem can be found in, e.g., [Stoer and Witzgall, 1970].

If we do not have convexity in J_i and the spaces Ω_i, we can not in general hope for pure strategy equilibria. However in the enlarged class of mixed strategies an equilibrium exists. In the following E denotes the expectation operator and $d\mu_i, d\nu_i$ are positive Borel measures with unit mass (i.e., probability measures).

DEFINITION 2
$(d\mu_1, \ldots, d\mu_N)$ is a Nash-equilibrium in mixed strategies iff $\forall i$

$$E\{J_i(d\mu_1, \ldots, d\mu_i, \ldots, d\mu_N)\} \le E\{J_i(d\mu_1, \ldots, dv_i, \ldots, d\mu_N)\} \qquad \forall dv_i$$

This means that each player chooses a probability distribution for his argument and that no better distribution exists if the other players distributions are kept fixed.

THEOREM 2
An N-person nonzero-sum game, in which the action spaces Ω_i are compact and the cost functionals J_i are continuous, admits a Nash equilibrium in mixed strategies.

Proof: A proof of this theorem can be found in e.g. [Owen, 1974]. The underlying idea is there to approximate the J_i:s with discrete version matrix games. The compactness of the action spaces is shown to ensure that the limit of the sequences of solutions obtained for the approximating finite matrix games exists. Other proofs also exist, see e.g. [Glicksberg, 1952]. See also the proof of Theorem 3 below. □

It is natural to ask what happens if the convexity assumption is kept for some, but not all, players. This situation arises frequently in control theory, see e.g. the example below. The answer is given by the following presumably new theorem:

THEOREM 3
Assume for $i = 1, \ldots, N$ that $J_i(u_1, \ldots, u_N)$ are continuous in the compact sets $\Omega_1, \ldots, \Omega_N$. Assume also for $i = 1, \ldots, M$ that each J_i is convex in u_i (keeping other arguments fixed), and that Ω_i are convex sets. The associated N-person nonzero-sum game then admits a pure-mixed solution. By this we mean that players $1, \ldots, M$ use pure strategies and players $M+1, \ldots, N$ mixed strategies.

Proof: Let

$$P_i = \{d\mu_i \mid \int_{\Omega_i} d\mu_i = 1 \;\&\; d\mu_i \ge 0\} \qquad i = M+1, \ldots, N$$

be the set of all probability measures on Ω_i. Consider

$$\tilde{J}_i(u_1, \ldots, u_M, d\mu_{M+1}, \ldots, d\mu_N) := E\{J_i(u_1, \ldots, u_N)\}$$

where expectation is performed over variables $(u_{M+1}, \dots, u_N)$. This is a function from $\Omega_1, \dots, \Omega_M, P_{M+1}, \dots, P_N$ to R. We note that $\tilde{J}_i$ is convex in it's ith argument (linear for $i = M+1, \dots, N$). We introduce the weak* topology on P_i. It is then easy to see that $\tilde{J}_i$ is jointly continuous in all arguments. In fact it suffices to check that if $d\mu \to d\nu$ and $u \to v$ in respective topologies, then

$$\int J(u,x)\, d\mu(x) \to \int J(v,x)\, d\nu(x)$$

However, this follows from

$$\int J(u,x)\, d\mu(x) - \int J(v,x)\, d\nu(x) =$$

$$\int (J(u,x) - J(v,x))\, d\mu(x) + \int J(v,x)\, d\mu(x) - \int J(v,x)\, d\nu(x) \to 0 + 0$$

The first limit follows from uniform continuity, the second from the definition of convergence in the weak*-topology.

The sets P_i are clearly convex. They are also compact. This follows from Riesz's representation theorem and the fact that the unit ball in the dual space to a Banach space is compact in the weak*-topology. The elementary facts from functional analysis used above can be found in, e.g., [Rudin, 1987]. The theorem now follows from Theorem 1 above. □

Remark. Putting $M = 0$ we see that Theorem 2 follows from Theorem 3.

Remark. For a two persons zero-sum game the theorem states that convexity in u alone is enough to guarantee the existence of a pure-mixed saddle equilibrium. This means that

$$\min_u \max_{d\nu(w)} EJ(u,w) = \max_{d\nu(w)} \min_u EJ(u,w)$$

2. An Example from H_∞-Control Theory

Differential game theory has been used lately to give new insight into the area of H_∞-control, see e.g. [Başar and Bernhard, 1991]. The problem is to find the control law that minimizes the worst case effect of an L_2-bounded disturbance. The control laws were previously found using more complicated methods from operator theory. It was later realised that if the controller and disturbance were viewed as opposing players in a (zero-sum) game theory context, the control laws already existed in the game theory literature. In fact [Medanic, 1967], [Mageirou, 1976] gives the control laws for H_∞-control in the case of full state information. This was published several years before the H_∞-problem was formulated in the control literature, see [Zames and Francis, 1983].

To be more concrete let the process be given in state-space form by

$$\begin{aligned} x_{k+1} &= \Phi x_k + \Gamma_1 w_k + \Gamma_2 u_k, \quad k = 0, \ldots, K \\ z_k &= C_1 x_k + D_{12} u_k \\ y_k &= C_2 x_k + D_{21} w_k \\ x_0 &\text{ given} \end{aligned} \tag{1}$$

Here $u \in R^{m_1}$ is the control signal, $w \in R^{m_2}$ the disturbance and $z \in R^{p_1}$ the signal the controller aims to minimize. The control signal u is restricted to be a function of the measurement signal $y \in R^{p_2}$. The H_∞-control problem is to find a controller, $u = Ky$, that solves

$$\min_K \max_{\|w\|=1} \|z\|^2 =: \gamma_0^2 \tag{2}$$

Here the L_2-norm is used, i.e. $\|z\|^2 = \sum_{k=0}^{K} |z_k|^2$. No explicit formula for the optimal controller is known for the general case. However the suboptimal control problem has been solved consisting of determining if (with $x_0 = 0$)

$$\min_u \max_{\|w\|=1} \|z\|^2 = \min_u \max_w \frac{\|z\|^2}{\|w\|^2} \leq \gamma^2 \tag{3}$$

where γ is some given real number. For γ below the lower limit γ_0, no solution to (3) exists. The limit is normally found by iteration on γ. Explicit formulas for the optimal γ has only been found for special classes of problems.

In most control literature (3) is replaced by the seemingly equivalent problem

$$\min_u \max_w \{\|z\|^2 - \gamma^2 \|w\|^2\} \tag{4}$$

This new cost functional is convex in u and (if γ is sufficiently large) concave in w. Standard dynamical game theory can then be employed to find a pure strategy saddle point. The difference between problems (3) and (14) can now be understood in the light of Theorem 3.

Pure Feedback Equilibrium

The following are recursive formulas for a pure feedback saddle point (u^*, w^*) to (4) when $\gamma > \gamma_0$. They are a slight generalization of the results in [Başar and Bernhard, 1991] (where $Q_{12} = 0$ and $Q_{22} = I$) for the full information case ($y_k = x_k$). Note that the Riccati equation (9) progresses in backward time. Note also that the standard Linear Quadratic Regulator is obtained by letting $\gamma \to \infty$.

$$
\begin{aligned}
w_k^* &= -L_1(k)x_k && (5)\\
u_k^* &= -L_2(k)x_k && (6)\\
L_1(k) &= -\gamma^{-2}\Gamma_1^T(\Gamma_2 Q_{22}^{-1}\Gamma_2^T - \gamma^{-2}\Gamma_1\Gamma_1^T + P_k^{-1})^{-1}\widetilde{\Phi} && (7)\\
L_2(k) &= Q_{22}^{-1}Q_{12}^T + Q_{22}^{-1}\Gamma_2^T(\Gamma_2 Q_{22}^{-1}\Gamma_2^T - \gamma^{-2}\Gamma_1\Gamma_1^T + P_k^{-1})^{-1}\widetilde{\Phi} && (8)\\
P_k &= Q_{11} - Q_{12}Q_{22}^{-1}Q_{12}^T + \widetilde{\Phi}^T(\Gamma_2 Q_{22}^{-1}\Gamma_2^T - \gamma^{-2}\Gamma_1\Gamma_1^T + P_{k+1}^{-1})^{-1}\widetilde{\Phi} && (9)\\
\widetilde{\Phi} &= \Phi - \Gamma_2 Q_{22}^{-1}Q_{12}^T && (10)\\
P_{K+1} &= 0 && (11)
\end{aligned}
$$

where we have used the notation

$$
\begin{pmatrix} C_1^T \\ D_{12}^T \end{pmatrix} \begin{pmatrix} C_1 & D_{12} \end{pmatrix} =: \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{pmatrix}
$$

and assumed D_{12} left invertible so that Q_{22}^{-1} exists. The value of the game is $x_0^T P_0 x_0$.

Remark. Corresponding equations for the case of output feedback and infinite time horizon K, are given in [Başar and Bernhard, 1991].

Pure-Mixed Feedback Equilibrium

Existence of a saddle point equilibrium to (3) is however not guaranteed by existence for (4). Actually the control problem (3) can not be expected to have a pure saddle point solution, because the problem is not concave in w. In fact it can be shown that (3) need not have any pure-strategy saddle point. One can however obtain formulas for a pure-mixed saddle point to (3). This means that

$$
\min_u \max_{\substack{dv(w) \\ \|w\|=1}} E\|z\|^2 = \max_{\substack{dv(w) \\ \|w\|=1}} \min_u E\|z\|^2 \qquad (12)
$$

Therefore the controller does not have to use a mixed strategy which, perhaps, is a practical advantage.

The pure-mixed saddle point can be described as follows. Let γ_0 be the optimal γ defined in (2) and let η be an eigenvector corresponding to a maximal eigenvalue of $\Gamma_1^T N \Gamma_1$ where

$$N = P_1 + \gamma_0^{-2} \sum_{k=1}^{N} \Phi_1^{*T} \ldots \Phi_{k-1}^{*T} \Phi_k^{*T} L_1^T(k) L_1(k) \Phi_k^* \Phi_{k-1}^* \ldots \Phi_1^*$$

and where

$$\Phi_k^* = (I + (\Gamma_2 Q_{22}^{-1} \Gamma_2^T - \gamma^{-2} \Gamma_1 \Gamma_1^T) P_{k+1}) \widetilde{\Phi}$$

Let $\xi_0 = \pm\eta$, where the positive sign is chosen with probability 1/2, and put $\xi_{k+1} = \Phi_k^* \xi_k$ and $\xi_1 = \Gamma_1 \xi_0$. Then

$$w_k^* = c L_1(k) \xi_k, \qquad k = 0, \ldots, N \tag{13}$$

and u_k^* given by (6), is a pure-mixed saddle equilibrium to (2). Here c is chosen so that $\|w^*\| = 1$. This result is a slight generalization of Prop. 3.1 in [Başar and Bernhard, 1991]. The proof follows the same lines. We refer the reader to [Başar and Bernhard, 1991] for details. Note that w_k^* are highly correlated for different k. The randomness in w_0^* is however enough to put w^* in pure-mixed saddle equilibrium with u^*.

Remark. For a control theorist it is interesting to know if pure-mixed saddle points exist for other problems of the form (3) that use other norms than the L_2-norm. If u can be shown to be bounded, this follows from Theorem 3.

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PAPER III

Min-Mix Control – A Classical Stochastic Differential Games Approach

Bo Bernhardsson

Abstract: In this paper it is attempted to formulate and solve a central problem in control, which captures the idea that disturbances may be both deterministic and stochastic. The deterministic behavior models worst case situations and stochastic disturbances model average properties. The name min-mix is coined for the problem because the operations of minimizing, maximizing and taking expected values appear in the problem. The problem is very rich. It has connections both to recent results in H_∞-control and also to classical results in stochastic control and differential games. The analysis makes it possible to unify several results and to gain insight. A method for solving the problem is presented which is based on a recent dynamic programming separation principle due to Bernhard, which is a generalization of Isaacs' equation.

The solution is obtained by *completion of squares* and is given by three coupled Riccati equations. These equations generalize the equations obtained, however for a slightly different problem, in [Doyle *et al.*, 1992]. The time-varying, finite time horizon version of the problem is treated here, both in continuous and discrete time. This will probably be the key for solving the open issues still present in the infinite time horizon case. We also obtain explicit formulas for the value of the game. We show by an example that this information is useful for obtaining the correct controller in the infinite time horizon case.

1. Introduction

A number of recent papers have addressed different so called mixed H_2/H_∞ control problems[1]. The setup in these papers differ, but they all concern mixing H_2 and H_∞ norms and/or using different models of disturbance signals. See e.g. [Doyle *et al.*, 1989] for details and references.

Mixing different norms, in particular H_2 and H_∞ norms, can be motivated in several ways. When designing control systems it is necessary to understand what limits achievable performance. It is then a good idea to trade off different performance measures. A typical question is "Can a controller simultaneously achieve performance and robustness?". Different norms are good for measuring different things. This motivates the use of mixed norms.

Another motivation comes from modeling of disturbance signals. In H_∞-control the system is

$$\dot{x} = Ax + B_1 d + B_2 u \tag{1}$$

$$y = Cx + De \tag{2}$$

Here d and e denote process and measurement disturbances, u control signals and y measurements. The H_∞ control problem optimizes system performance against worst disturbances d and e in the class of bounded L_2-signals d, e. This means that it solves the following constrained optimization problems

$$\min_K \max_{d,e} \|z\|_2^2, \qquad \text{when } \|d\|_2^2 + \|e\|_2^2 \le 1$$

where z describes some signals to be minimized. The constraint means that process disturbances and measurement noise can not both be large. It is unnatural to couple process disturbance and measurement noise in this way. It often turns out that the worst case signals in H_∞-control contain only one frequency. This is however not a natural assumption for measurement noise. It is better to assume that some signals are stochastic.

It has also been found that *optimal* H_∞-controllers tend to have undesirable properties, e.g. infinite sensitivity to unmodeled nonzero initial values. Some of the problems arise because the solution is expressed in equations that are poorly conditioned other are due to poor software, yet other are due to the problem formulation. The equations are obtained under the assumption that there are no stochastic process disturbances or measurements errors. There is folklore among the

1 See Section 11 for notations

H_∞community that says: '*slightly suboptimal controllers are better when analyzed with respect to all the additional considerations that arise in practice*'. Lately the situation has however improved due to increased understanding of the quite complicated optimal case, see e. g. [Glover *et al.*, 1991], [Safonov and Limebeer, 1988]. See also [Hagander and Bernhardsson, 1992] for an illustration of what can happen in a simple second order case.

Another motivation comes from the fact that suboptimal H_∞controllers are not unique. Even the optimal H_∞-controller can be non-unique. Different choices of which controller to use have been suggested. The most common choice is the central controller, which is obtained by putting $Q = 0$ in the formula for all suboptimal controllers, see [Doyle *et al.*, 1989]. This choice has several motivations. It minimizes a certain entropy norm, see [Mustafa, 1989] and it is equivalent to the risk-sensitive controller, see [Whittle, 1990a, Whittle, 1990b]. So called superoptimal controllers, which minimize also the second largest singular value etc, have been suggested, see e. g. [Tsai *et al.*, 1988]. Equalizing controllers, i. e. controllers giving a constant magnitude of the performance measure at all frequencies, are used in the polynomial version of H_∞-control, [Kwakernaak, 1985]. In the sense of the H_2-norm these are the worst of all H_∞-controllers The most natural way to reduce the lack of uniqueness is to introduce stochastic disturbances.

It is now well understood that there are several close correspondences between H_2 (LQG) and H_∞-control, see e. g. [Doyle *et al.*, 1989], [Whittle, 1990a]. The H_2-results can often be recovered by letting $\gamma \to \infty$ in the H_∞-results. The motivation for the present author has been to increase the understanding of how to unify the H_2 and H_∞-theories. The problem formulation is so natural that it is also motivated from a pure system theory point of view.

In this paper we will present a completion of squares method for the min-mix problem. Both continuous time and discrete time problems will be treated. A conjectured generalization of a dynamic programming separation principle by Bernhard is used to obtain the controller. For the infinite time horizon case the formulas obtained are the same as in [Doyle *et al.*, 1992], however there obtained for a problem without stochastic disturbances. That paper also uses different methods to obtain the results. The full finite time horizon, time varying problem is treated and new formulas are given. We also obtain new, explicit, formulas for the value of the game. New discrete time formulas follows from an analogous treatment, which illustrates the close connection.

Relationships to earlier results on game theory are presented. This gives insight into the importance of the information structure. A simple, but rich, example illustrates the theory and the equations obtained.

The paper is organized as follows. Sections 2 and 3 present the problem and relate it to other setups that also go under the name of mixed H_2/H_∞problems. Some connections between different problems are mentioned. In Section 4 we relate the problem to previous results in the last 25 years of game theory and discuss the importance of information structure. Section 5 gives a very short introduction to some basic results in game theory needed in the following. Different separation theorems and dynamic programming techniques are discussed in Section 6. An example illustrates that a better certainty equivalence principle than the present H_∞-separation principle in [Başar and Bernhard, 1991] is needed. A weakness in the formulation of their principle is in fact pointed out. We summarize a recent dynamic programming separation principle by Bernhard, that will be an important tool for obtaining the solution. Two generalizations of his dynamic programming principle are conjectured. The proof of these conjectures are open problems at the moment. In Sections 7 and 8 we obtain a completion of squares solution of the min-mix problem for continuous and discrete time respectively. These are the main results. We also give an illustrative example. Section 9 presents suggestions for future work and Section 10 gives conclusions. Section 11 gives details in the notation.

2. Mixed H_2/H_∞ Problems

Several different problems go under the name of mixed H_2/H_∞controller design. It is important to separate between them even though they are all more or less related.

H_∞–Constrained H_2–Optimization

The following problem is formulated in e. g. [Rotea and Khargonekar, 1990], see Figure 1. Find a controller K, which gives internal stability and solves the optimization problem

$$\min_K \|G_{z_0,w_0}\|_2 \quad \text{under the restriction} \quad \|G_{z_1,w_1}\|_\infty \leq \gamma \tag{3}$$

Here system performance is expressed by the H_2-norm. Notice that performance is only evaluated for the nominal system. The H_∞ norm can be used to guarantee robust stability under unstructured uncertainty. The uncertainty can be introduced in many different ways, e.g. in additive, multiplicative, feedback or normalized coprime form, as explained in e. g. [McFarlane and Glover, 1990].

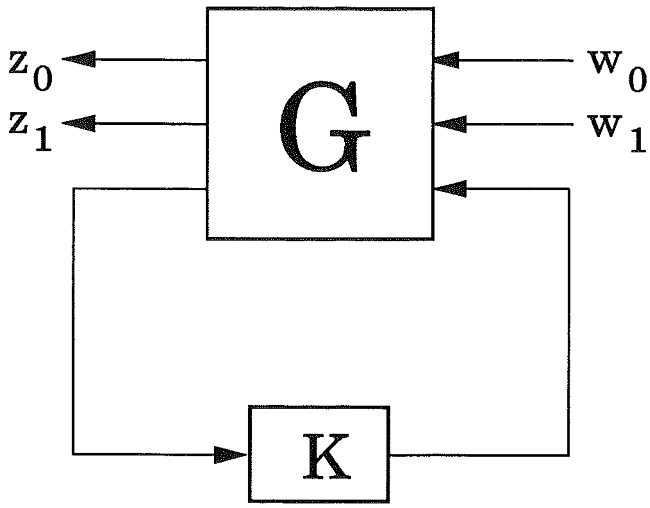

Figure 1. A general linear system with two types of disturbances, w_0 and w_1. The figure captures some mixed H_2/H_∞-control problems.

Remark 1. It has, rightly or wrongly, become generally accepted that the H_∞-norm is good for formulating robustness problems. This is because the norm is submultiplicative, i. e. $\|AB\| \leq \|A\|\|B\|$. This property is needed to use the small-gain theorem, see e. g. [Francis, 1987]. It is important to understand that minimizing an H_∞-norm will not automatically give a robust closed loop. It of course depends on what you take the norm of. Actually it is much more important what you take the norm of, than which norm you use. Although obvious, several misconceptions on this point can be found in the literature. One can also discuss how relevant the H_∞-constrained H_2-minimization problem is in practice. Notice that the H_2-performance is only evaluated for the nominal system. The problem does therefore not measure *robust performance*, only nominal performance and robust stability. A simple thought experiment is illuminating: Assume that a known disturbance w enters a system with additive stable uncertainty Δ, e. g. a system of the form $y = G_1 w + (G_2 + \Delta)u$ and that we want to construct a robust performance controller $u = K_1 w + K_2 y$. It is possible to construct examples were the optimal nominal H_2 controller uses pure feedforward from the disturbance and no feedback. This will then also be the H_∞-constrained H_2-optimal controller, if the H_∞ norm measures stability against the additive uncertainty Δ. The H_∞-norm will in this case be zero and there is no risk that the system becomes unstable for any stable perturbation Δ. The performance of such a controller can however be sensitive to nonzero Δ, and in practice it would often be preferable with some feedback.

The problem (3) is quite hard. Few results are known even if many good researchers have worked with it during recent years. Notice that the optimal H_2-controller often solves (3). All that is needed is that this controller also satisfies the H_∞ bound. The interesting situation is when the two criteria are competing. It is then easy to see that there must be equality in the H_∞-bound. The reason for this is that we then have a convex functional to be minimized over a convex subset of controllers where the optimal unconstrained controller is outside the set. A simple convexity argument then says that the minimum is achieved on the boundary. This case is studied in a number of recent papers, see [Rotea and Khargonekar, 1991] for results on the state feedback case. In [Khargonekar and Rotea, 1991] it is shown that the full measurement case can be written as a convex optimization problem. No numerical results are presented. In [Ridgely *et al.*, 1992] seven necessary, coupled, nonlinear, matrix equations are presented for the optimal controller using Lagrange techniques. The problem of sufficiency is still open. Numerical results are presented in [Ridgely and Valavani, 1992]. The optimal controller order is not necessarily bounded by the system order n, as is shown in [Wells and Ridgely, 1992]. For related work see also [Belcastor *et al.*, 1991],[Peters and Stoorvogel, 1992], [Kraffer, 1992].

An interesting result on simultaneous H_2 and H_∞-minimization is given in [Foias and Frazho, 1992]. There, a commutant lifting approach is used to show, in the scalar case, that it is possible for any $\delta > 1$ to find stable $q(s)$ such that simultaneously

$$\|f+q\|_\infty \le \delta d_\infty \quad \text{and} \quad \|f+q\|_2 \le \frac{\delta d_2}{\sqrt{\delta^2-1}}$$

Some important contributions on slightly different mixed H_2 and H_∞ problems will now be reviewed.

Auxiliary Cost Minimization

In a number of recent papers the H_2-norm is replaced by an upper bound on the H_2-norm, the so called auxiliary cost:

$$J_{aux}(G,\gamma) = \text{Trace}\left[Q_s \tilde{C}^T \tilde{C}\right] \tag{4}$$

Here Q_s is the solution of a certain Riccati equation and $\tilde{C}$ is the observation matrix for the closed loop system, see [Bernstein and Haddad, 1989, Haddad *et al.*, 1991] for further details. The idea is that minimizing the auxiliary cost instead of the H_2-norm, under an H_∞-constraint, gives a solvable problem with similar performance. It seems unlikely that the optimal controller for the auxiliary cost minimization problem and the H_∞-constrained H_2-problem are the same, although the titles

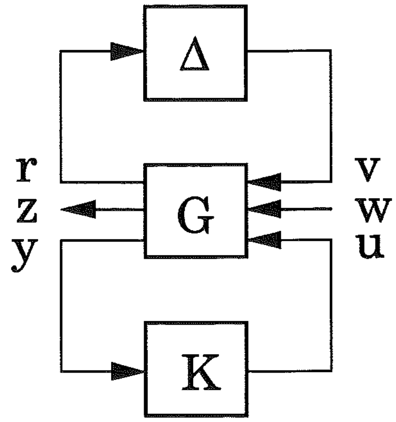

Figure 2. A general formulation of many robust performance problems.

of the early papers suggested so. Counterexamples can be given. It is known that if w_0 and w_1 enter in the same way and $z_0 = z_1$, the central and entropy minimizing H_∞-controller is optimal also for the auxiliary H_2/H_∞-problem, see [Mustafa, 1989]. A duality with the work of [Zhou *et al.*, 1990] described below was indicated in [Yeh *et al.*, 1992].

Other Mixed Problems

Several robust performance results can be illustrated by Figures 1 and 2. The following problems are also called mixed H_2/H_∞-problems:

$$\begin{aligned}
J_1 &= \min_K \max_{\|\Delta\|_\infty \le 1} \mathop{E}_{w_0} \|z\|^2 \\
J_2 &= \min_K \max_{\|w_1\| \le 1} \mathop{E}_{w_0} \|z\|^2 \\
J_3 &= \min_K \max_{w_1} \mathop{E}_{w_0} \|z\|^2 - \gamma^2 \|w_1\|^2 \qquad \text{(min-mix)} \\
J_4 &= \min_K \max_{w_1 \in BP} \max_{w_0 \in BS} \|z\|^2 - \gamma^2 \|w_1\|^2
\end{aligned}$$

See Section 11 for details about notation. Both finite time and infinite time horizons problems can be formulated. If the time horizon is infinite the minimization should be performed over all controllers K giving a stable closed loop system. In this paper we will deal with problem J_3.

In H_∞-theory it is common practice to replace

$$\max_{\|w_1\| \le 1} \|z\| \qquad \text{by} \qquad \max_{w_1 \ne 0} \frac{\|z\|}{\|w_1\|}$$

Notice that this does not make sense in the mixed case where there is also stochastic noise w_0, because $w_1 = 0$ does not imply that $z = 0$. The problems J_3 and J_4 are closely related. We have the following result:

LEMMA 1
The following relation holds between the problems above:

$$J_2 \leq J_3 + \gamma^2$$

Suppose that $z = G_0 w_0 + G_1 w_1$ where $w_1 = \Delta z$ and that $\|G_1\|_\infty \leq \gamma < 1$, then

$$J_1 \leq \frac{J_3}{1-\gamma^2}$$

Proof: The first inequality follows directly from the formulation. The second inequality is proved in [Zhou *et al.*, 1992]. We present the idea here also. Notice that for any $w_1 \in \mathcal{P}$, we have by the definition

$$\|z\|_P^2 - \gamma^2 \|w_1\|_P^2 \leq J_3$$

Now $w_1 = \Delta z$, hence

$$\|z\|_P^2 \leq \gamma^2 \|w_1\|_P^2 + J_3 = \gamma^2 \|\Delta z\|_P^2 + J_3$$

Therefore, for any $\Delta \in \mathcal{R}H_\infty$ with $\|\Delta\| \leq 1$ we have

$$\|z\|_P^2 \leq \frac{J_3}{1-\gamma^2}.$$

Both bounds are conservative. □

The information structure is important in the problems above. Different information structures give different optimal controllers and different existence conditions. There has been a lot of confusion on this issue over the years. It is non-trivial to find a clean notation for the exact problem formulation. This will be discussed more in section 4.

3. The Min-Mix Control Problem

The J_3-problem concerns controller design for a system with mixed deterministic and stochastic signals. To separate it from other mixed H_2/H_∞problems, it will in what follows be called "The Min-Mix problem". Notice that if $w_1 = 0$, the problem reduces to the standard H_2-problem, and if $w_0 = 0$ it reduces to the H_∞-problem. It is assumed the system is given by

$$\begin{aligned}
\dot{x} &= Ax + B_0 w_0 + B_1 w_1 + B_2 u \\
z &= C_1 x + D_{12} u \\
y &= C_2 x + D_{20} w_0 + D_{21} w_1
\end{aligned}$$

where w_0 is formal white noise. Stochastic differential equations will be interpreted and analyzed using Ito-calculus, see e. g. [Åström, 1970]. We will also treat the discrete time case obtained by the obvious change to difference equations.

Notice that with a $D_{10}w_0$ term there is a white noise component in z which gives infinite loss (unless there is also a direct term in the controller cancelling this). We can also assume that $D_{22}u = 0$ since this can be achieved by a loop transformation. It is also assumed that $D_{11}w_1 = 0$ in what follows. Compare with [Safonov and Limebeer, 1988]. It will also be assumed that certain matrices are of full rank. These conditions correspond to stabilizability and detectability, [Doyle *et al.*, 1989]. This will not be further elaborated. The problem of determining the weakest possible conditions for solution of the min-mix problem is still open.

The problem

$$J_3 = \min_K \max_{w_1} \mathop{E}_{w_0} \left\{ \|z\|^2 - \gamma^2 \|w_1\|^2 \right\} \tag{5}$$

in words means that the controller should be found, which minimizes the worst influence of a causal disturbance w_1 that knows the controller strategy. The motivation for modeling a disturbance in this way is often to cover a "worst-case" situation. Different information structures on u and w_1 are possible. As described further below we will assume that w_1 is a causal function of w_0, x and u. We also assume that u is a causal function of y. This is called the semicomplete information case. In the discrete time case $u(k)$ is a function of $y(k-1)$ and older values. The case with a direct term in the controller gives more complicated formulas and is still an open problem.

4. Historical Remarks

The min-mix control problem has a long history in the game theory literature. Stochastic differential game theory has been an active research area for over 25 years. It is therefore not surprising that several authors have worked on special versions of this problem before. Most papers have dealt with finite horizon problems, with the focus being on information structures (e. g. open loop control, full state feedback, measurement delays, noisy measurements) and existence and characterization of saddle points.

Existence of saddle point strategies, is not so important for controller design. From a controller point of view it is only the upper value

Table 1. A collection of early game theory references treating versions of the min-mix problem with different informations structures. The present paper is concerned with the first row, i. e. w_1 has perfect (causal) information

w_1 \ u	Perfect Measurement	Noisy Measurement	No Measurement
Perfect Measurement	[Bryson+Ho 1969]	[Behn+Ho 1968] [Rhodes+Luenberger 1969a]	[Bryson+Ho 1969]
Noisy Measurement	[Behn1968]	[Willman 1969] [Rhodes+Luenberger 1969b] [Bley+Stear 1969]	[Rhodes+Luenberger 1969a]
No Measurement	[Bryson+Ho 1969]	[Rhodes+Luenberger 1969a]	[Bryson+Ho 1969]

of the game that is of interest. The saddle point theory is however useful both for the understanding and for obtaining the solution. The information structure is always a central issue.

Different Information Structures

For the min-mix control problem there is an ambiguity as to what information structure u and w_1 have. The reference [Ho, 1970] gives a nice review of the situation in the field of finite horizon stochastic games in the late sixties. This paper includes discussion of different assumptions on the information available to u and w_1. Different information structures lead to different existence conditions for saddle point equilibria. Different solution techniques are also required. The most difficult case is the one where both players have different, noisy information. It is then hard even to find a good problem formulation, see [Witsenhausen, 1968]. We will now briefly review some of the references. The list is far from complete.

Behn–Ho

The work [Behn and Ho, 1968],[Behn, 1968] solves a pursuit and evasion game with two players with energy restrictions. In the notation of that paper the game is defined by the dynamical system

$$\dot{y} = G_p u(t) - G_e v(t), \quad y(t_0) = y_0 \tag{6}$$

and the criterion

$$J = \frac{a^2}{2}\|y(t_f)\|^2 + \frac{1}{2}\int_{t_0}^{t_f} \|u(t)\|_{R_p}^2 - \|v(t)\|_{R_e}^2 \, dt$$

The problem structure is special. The pursuer is assumed to have perfect knowledge of the state while the evader makes noisy measurements of the form

$$z = y(t) + w(t)$$

where $w(t)$ is a Gaussian white noise with zero mean and covariance $Q(t)$. Controller formulas are obtained using a Lagrange multiplier method, varying controller parameters around a nominal optimal controller. The following necessary conditions are obtained:

$$\begin{aligned}
\dot{\Gamma}_1 &= \Gamma_1(G_pR_p^{-1}G_p^T - G_eR_e^{-1}G_e^T)\Gamma_1, \quad \Gamma_1(t_f) = a^2I \\
\dot{\Gamma}_2 &= \Gamma_2G_pR_p^{-1}G_p^T\Gamma_2 + \Gamma_2(G_pR_p^{-1}G_p^TK^{-1} + PH^TQ^{-1}H), \\
&+ \quad (\Gamma_1G_pR_p^{-1}G_p + H^TQ^{-1}HP)\Gamma_2 + \Gamma_1G_eR_e^{-1}G_e\Gamma_1, \quad \Gamma_2(t_f) = 0 \\
u^0(t) &= -R_p^{-1}(t)G_p^T(t)\Gamma_1(t)y(t) - R_p^{-1}G_p^T(t)\Gamma_2(t)(y(t) - \hat{y}(t)) \\
v^0(t) &= -R_e^{-1}(t)G_e^T(t)\Gamma_1(t)\hat{y}(t) \\
\dot{\hat{y}}(t) &= (G_pC_p - G_eC_e)\hat{y}(t) + PH^TQ^{-1}(z - H\hat{y}(t)), \quad \hat{y}(t_0) = \hat{y}_0
\end{aligned}$$

To obtain these equations, it is assumed that the controller is of a certain structure, using feedback from a state estimate. No proof is actually given that better controllers of other structures can not be found.

Rhodes–Luenberger

[Rhodes and Luenberger, 1969a] solves a similar finite horizon problem as Behn-Ho using different techniques. The problem is slightly more general since (6) is replaced by

$$\dot{x} = F(t)x + G_1(t)u(t) + G_2(t)v(t), \quad x(t_0) = x_0 \tag{7}$$

The solution is given by three coupled equations and three matrices N, M and P, similar to the equations above. The case where one player has open loop information and the other noisy measurements is also treated.

In both [Behn and Ho, 1968] and [Rhodes and Luenberger, 1969a] the problem is treated formally. Several technicalities are overseen, as noted in later papers. Nothing is for example said about what happens when the Riccati equations fail to have solutions.

Witsenhausen

If both players have different noisy measurements available then the problem is much harder. Many problems arise. It is no longer true that

[Başar and Olsder, 1982] and [Başar and Bernhard, 1991] for more details.

Let $J(u, w)$ be a real valued function defined on the spaces U and W. The problem of minimizing J for u and maximizing J for w is called a static zero-sum game. The pair (u_0, w_0) is called a pure saddle point if the following two inequalities hold for all $u \in U$ and $w \in W$.

$$J(u^*, w) \leq J(u^*, w^*) \leq J(u, w^*) \tag{20}$$

We always have

$$J_* := \max_w \min_u J(u, w) \leq \min_w \max_u J(u, w) =: J^* \tag{21}$$

Here J_* is called the lower value and J^* the upper value of the game. If the spaces U and W are infinite, min and max may have to be replaced by inf and sup. The lower and upper values coincide if the problem has a pure saddle point. It is also easy to see that different pure saddle equilibria all give the same value of the game. The following theorem, which is a direct consequence of Kakutani's fixed point theorem, guarantees the existence of pure saddle points for so called convex-concave problems

THEOREM 2
Let J be continuous and convex in u on the convex compact set U and concave in w on the convex compact set W then J has a pure saddle point (u_0, w_0). If the problem is strictly convex-concave the saddle point is unique.

Proof: A direct application of Kakutani's fixed point theorem and standard functional analysis, see e. g. [Başar and Olsder, 1982]. □

If the conditions on J are relaxed we may have to settle for so called mixed saddle points. A mixed strategy means that a player chooses a probability distribution, i. e. a positive Borel measure $d\mu$ with unit mass, for his choice and plays accordingly. The expected value of the game is then evaluated as

$$\overline{J} = \int J(u, w)\, d\mu(u)\, d\nu(w) \tag{22}$$

We have the following existence theorem:

THEOREM 3
If J is continuous on the compact sets U and W then the problem has a mixed strategy solution. □

Moreover, when these conditions hold, one such controller is

$$K(s) := \left[\begin{array}{c|c} A_{ml} + B_2 F & -L \\ \hline F & 0 \end{array}\right] \tag{19}$$

□

Warning: In the published papers [Doyle *et al.*, 1989] and [Zhou *et al.*, 1990] these equations contained algebraic mistakes. These mistakes are corrected in the unpublished manuscript [Doyle *et al.*, 1992]. In earlier papers both necessity and sufficiency were also claimed. In [Doyle *et al.*, 1992] there are more conditions for necessity. As we will see later there is problem even with sufficiency.

Nikoukhah–Delebecque

Lagrange multiplier techniques are used in [Nikoukhah and Delebecque, 1991, Nikoukhah and Delebecque, 1992] to obtain necessary equations. These conditions are slightly different from the ones presented in [Doyle *et al.*, 1992]. The difference seems to be due to the fact that optimum can be obtained at the boundary of all possible controller parameters, see [Doyle *et al.*, 1992]. This possibility is overlooked in [Nikoukhah and Delebecque, 1992]. There are no references to the work of Zhou et.al. or to early work in game theory literature. The paper contains numerical results on a randomized first order example.

Bernhard

Two recent papers, [Bernhard and Colomb, 1988, Bernhard, 1992] contain a useful dynamic programming separation principle, which is proved rigorously. The separation theorem is further described in Section 7.

Other papers that deal with related problems include [Limebeer *et al.*, 1991a, Limebeer *et al.*, 1991b], [Leondes and Mons, 1979], [Sun and Ho, 1976], [Ho, 1974], [Willman, 1969],[Schömig and Ly, 1992], [Bagchi and Basar, 1981], [Başar, 1981, Başar, 1985], [Başar, 1989], [Uchida and Fujita, 1992]. These papers also contain further references.

5. Game Theoretical Background

To fully understand the problem it is helpful to have a background in game theory. Here we will summarize the most important facts and describe them in a simple setting. We refer the reader to [Isaacs, 1965], optimal control rules are linear in observables. This was shown in classical examples of [Witsenhausen, 1968]. The new feature is that the zero-sum nature of the game is lost and that players' actions can achieve not only control but also communication, and non-linearities are needed for the optimal conveyance of information.

Structure of the Riccati Equation

During the sixties and seventies a number of papers further analyzed the structure of the Riccati equations with indefinite matrices. This was found to have applications to game theory, see [Medanic, 1967, Willems, 1971, Willems, 1974, Mageirou, 1976, Pachter and Bullock, 1977]

Kumar–van Schuppen

The paper [Kumar and Schuppen, 1980] treats a case where both players have incomplete measurements. It is assumed that one player knows both players observations. This reduces the complexity of the problem. The solution is given in the form of seven coupled Riccati equations. Three with initial conditions, four with final conditions. It is not analyzed what happens when the equations fail to have a solution.

Zhou, Doyle, Glover, Bodenheimer

A number of interesting recent contributions [Doyle *et al.*, 1989, Zhou *et al.*, 1990, Zhou *et al.*, 1992, Doyle *et al.*, 1992] treat the problem J_4. The problem is formulated without introducing stochastics. Instead induced norms are used, as is explained in Section 11. The papers are closely related to the min-mix problem J_3, where w_0 is assumed to be white noise. For special cases, the solution obtained can be found in older game theory literature, see [Behn and Ho, 1968], [Rhodes and Luenberger, 1969a], [Kumar and Schuppen, 1980]. The paper [Doyle *et al.*, 1992] does however not contain any references to such papers. To describe the results we assume

$$D_{12}^T \begin{pmatrix} C_1 & D_{12} \end{pmatrix} = \begin{pmatrix} 0 & I \end{pmatrix}$$

for ease of notation. The formulas (without this normalization) are given in [Zhou *et al.*, 1990]. The system is given by

$$G = \left[\begin{array}{c|ccc} A & B_0 & B_1 & B_2 \\ \hline C_1 & 0 & 0 & D_{12} \\ C_2 & D_{20} & D_{21} & 0 \end{array}\right], \tag{8}$$

The following assumptions are made in [Zhou *et al.*, 1990]

- (A, B_2) is stabilizable and (C_2, A) is detectable.
- D_{12} has full column rank.
- $D_{20}D_{20}^T > 0$
- $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all $\omega \in R$.
- $\begin{bmatrix} A - j\omega I & B_0 \\ C_2 & D_{20} \end{bmatrix}$ has full row rank for all $\omega \in R$.

The problem is formulated as

Problem (G) Given the plant G, a constant γ, exogenous signals w_0, with $S_{w_0,w_0} = I$ and $w_1 \in \mathcal{P}$ depending causally on w_0. Find a proper controller K such that

$$\min_K \sup_{w_1 \in \mathcal{P}} \{\|z\|_P^2 - \gamma^2 \|w_1\|_P^2\} \tag{9}$$

where the minimization is constrained to those K providing internal stability.

THEOREM 1—J_4, infinite time horizon [Doyle *et al.*, 1992]
Given $\gamma > 0$ and a plant G, there exists a controller $K(s)$ which solves Problem (G) if the following conditions hold:

- There exists a real definite matrix such that

$$AX + XA^T + C_1^T C_1 + X(\gamma^{-2} B_1 B_1^T - B_2 B_2^T)X = 0 \tag{10}$$
$$X \geq 0 \text{ and } A_c := A + (\gamma^{-2} B_1 B_1^T - B_2 B_2^T)X \quad \text{is stable} \tag{11}$$

- There exist L, Y and P which satisfy

$$Y(LD_{20}D_{20}^T + B_0 D_{20}^T + PC_2^T + \tag{12}$$
$$+\gamma^{-2} PXB_1 D_{21}^T) + \gamma^{-2} PY(B_1 + LD_{21})D_{21}^T = 0 \tag{13}$$
$$YA_{\mathrm{ml}} + A_{\mathrm{ml}}^T Y + Y\tilde{R}Y + F^T F = 0 \tag{14}$$
$$Y \geq 0 \text{ and } A_{\mathrm{ml}} + \tilde{R}Y \text{ is stable} \tag{15}$$
$$(A_{\mathrm{ml}} + \tilde{R}Y)P + P(A_{\mathrm{ml}} + \tilde{R}Y)^T + (B_0 + LD_{20})(B_0 + LD_{20})^T = 0 \tag{16}$$

where

$$\tilde{R} = \gamma^{-2}(B_1 + LD_{21})(B_1 + LD_{21})^T$$
$$A_{\mathrm{ml}} = A + \gamma^{-2} B_1 B_1^T X + L(C_2 + \gamma^{-2} D_{21} B_1^T X) \tag{17}$$
$$F = -B_2^T X \tag{18}$$

where $V(t, x(t_f)) = q(x(t_f))$. This equation gives a sufficient condition for optimal control. The result is is however useful only for the full information case. It gives pure state feedback equilibria $u^* = u(x)$ and $w^* = w(x)$. Isaacs' equation should be combined with some kind of separation principle. For linear quadratic games one can normally conclude both sufficiency and necessity by careful study of the Riccati equations resulting from Isaacs' equation. The theory of conjugate points is often useful.

6. Separation Theorems

Several different separation theorems exist for H_2 and H_∞control, see e.g. [Doyle *et al.*, 1989] and [Başar and Bernhard, 1991]. There is also an elegant risksensitive certainty equivalence principle, see [Whittle, 1990b]. The theory requires great care in problem formulation especially regarding the information structure. It is easy to make mistakes as illustrated in the example to follow.

Başar and Bernhard

We cite the following certainty equivalence principle (Theorem 6.1) from [Başar and Bernhard, 1991] using their notation. Consider the system

$$x_{k+1} = f_k(x_k, u_k, w_k) \tag{26}$$
$$y_k = h_k(x_k, w_k) \tag{27}$$

which may be nonlinear. Let the criterion by

$$J(u, w) = M(x_{K+1}) + \sum_{k=1}^{K} g_k(x_k, u_k, w_k) + N(x_1) \tag{28}$$

For a given pair $\overline{u}^{\tau-1} = (\overline{u}_1, \overline{u}_2, \ldots \overline{u}_{\tau-1})$ and $\overline{y}^\tau = (\overline{y}_1, \overline{y}_2, \ldots, \overline{y}_\tau)$ introduce the constraint set

$$\Omega_\tau(\overline{u}^{\tau-1}, \overline{y}^\tau) = \{w^\tau \mid y_k = \overline{y}_k, k = 1, \ldots, \tau\}$$

This is the set of all w which are consistent with past observations. Let the value of the full state-feedback game from time k be denoted by $V_k(x_k)$ and let the corresponding state feedback controller be $\mu_k^*(x_k)$. It is assumed that $\mu_k^*(x_k)$ is unique. Introduce the auxiliary performance index

$$G^{\tau-1}(u^{\tau-1}, w^{\tau-1}) = V_\tau(x_\tau) + \sum_{k=1}^{\tau-1} g_k(x_k, u_k, w_k) + N(x_1)$$

on γ and the information. $\overline{J} = J^*$ can always be obtained by a mixed $w = (w_1, w_2)$. The situation is illustrated in Figure 4. □

Remark 3. In the general H_∞-case (24) the situation is similar. We have $J_* = x^T(0)X(0)x(0)$, where $X(0)$ is given by a certain Riccati equation, but

$$J^* = \begin{cases} x^T(0)X(0)x(0) & \text{if } \gamma > \gamma^* \\ \infty & \text{if } \gamma < \gamma^* \end{cases}$$

Here γ^* depends on information structure. The more information u has the lower γ^* will be. The value of the game for $\gamma > \gamma^*$, $x^T(0)X(0)x(0)$, does however not depend on information structure. This is a somewhat singular situation. Typically there are many controllers that give concavity in w. All these give the same value of the game. This e. g. shows up in the following way in H_∞control: If the state estimator is

$$\dot{\check{x}} = A\check{x} + \gamma^{-2}B_1B_1^TX\check{x} + B_2u - L(y - C_2\check{x} - \gamma^{-2}D_{21}B_1^TX\check{x}) \tag{25}$$

then L can be chosen in many different ways. Even $L = 0$, corresponding to open loop control, gives a suboptimal H_∞-controller for the measurement feedback case, if only the value of γ is sufficiently large. The value must be larger than the γ corresponding to open loop control. These examples show that it is difficult to find "the correct" separation principle for H_∞-control, since the controller is non-unique. It is further elaborated in Section 6.

Isaacs' Equation

The following generalization of the classical Hamilton Jacobi Bellmann equation in control is called Isaacs' equation after one of the founders of dynamical game theory, see [Isaacs, 1965]. See also [Zachrisson, 1964]. It is used to find a recursion for the future loss $V(t, x)$ of the game. Consider a system described by

$$\dot{x} = f(t, x, u, w)$$

Let the loss function be

$$J = q(x(t_f)) + \int_0^{t_f} g(t, x, u, w)\, dt \; ; \qquad x(0) = x_0$$

Isaacs' equation is given by

$$0 = \min_u \max_w \left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f + g \right) = \max_w \min_u \left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f + g \right)$$

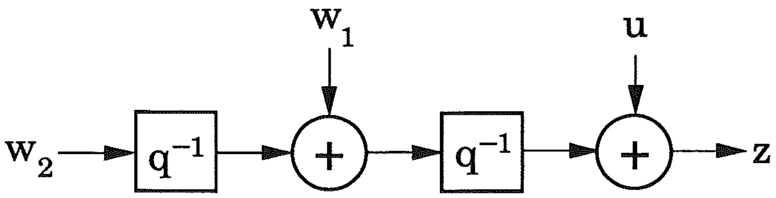

Figure 3. A simple example to illustrate that the pure-pure saddle strategy turns to a pure-mixed saddle equilibria when γ is decreased.

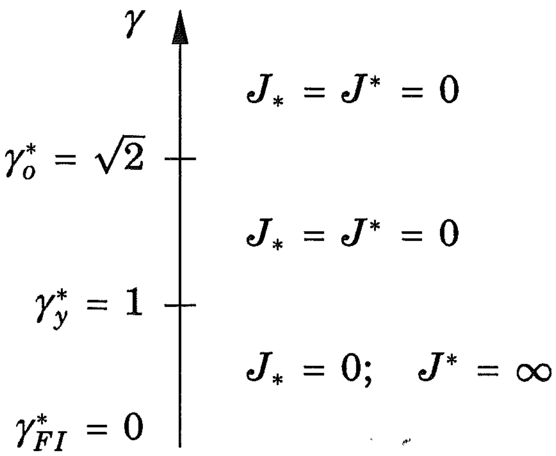

Figure 4. Different information structures give different γ^* for the problem (24) in Example 2. The lower and upper values of the game are shown for different levels of γ for the measurement, feedback case i.e. $u = u(y)$

For $\gamma > 1$ we find that $J_* = J^* = 0$ and that $(u^*, w^*) = (0, 0)$ is the pure saddle point. For $\gamma < 1$ concavity in w is lost. The lower value of the game is still $J_* = 0$, but the upper value is unbounded. For any u, it is possible to choose w such that J becomes arbitrarily large. Using a mixed w, for instance $w = \pm A$ with A large, gives $\overline{J} = J^* = \infty$. $\square$

It is easy to see that for (24) we always have $\overline{J} = J^*$, which means that the pure-mixed solution attains the upper value of the game. This is a consequence of convexity in u.

EXAMPLE 5—Information Structure

Consider the dynamic game in Figure 3, where w_1 and w_2 are disturbances, u is the control signal and y is a measurement. Assume the loss function is given by

$$J := \{u(2) + w_1(1) + w_2(0)\}^2 - \gamma^2\{w_1^2(1) + w_2^2(0)\}$$

We consider three different information structures. In the full information case both w_1 and w_2 are known by u. In the measurement case u knows $y = w_2$. In the open loop case he has no information about w_1 or w_2. It is easy to see that the lower values in all three cases are $J_* = 0$, independently of γ. The upper value is $J^* = 0$ or $J^* = \infty$ depending

In several instances one has convexity in one player, e.g. the controller, but no concavity in the other player, e.g. the disturbance. We then have the following intuitive and useful result proved in [Bernhardsson, 1992]:

THEOREM 4
Let $J : U \times W \to R$ be continuous on the compact sets U, W. Further assume that J is convex in u (keeping w fixed), and that U is a convex set. The associated two player zero-sum game then admits a pure-mixed solution, which means that u uses a pure strategy and w a mixed strategy. □

Remark 2. The compactness assumption is often violated in practice. One can however often restrict the game to a compact set by careful study of the loss function.

The H_∞ Problem

In H_∞-control the induced 2-norm problem

$$\min_K \max_{w \neq 0} \frac{||z||_2}{||w||_2} < \gamma \tag{23}$$

for all L_2-disturbances w, is rewritten as

$$\min_K \max_w ||z||_2^2 - \gamma^2 ||w||_2^2 < 0 \tag{24}$$

Assume that z depends linearly on u and w. Then both problems are convex in u. The problem (23) is not concave in w, and can hence not be expected to have a pure-saddle point. Problem (24) is concave in w if the value of γ is sufficiently large, i.e. $\gamma > \gamma^*$, it will then have a pure saddle point. If $\gamma < \gamma^*$ the value of the game can be made arbitrarily large with pure-mixed strategies.

DEFINITION 1
The number γ^* is the infimum over all γ such that the upper value of the game (24) is bounded.

The problem in (24) is illustrated by the following simple example.

EXAMPLE 4—Concavity
Consider the (static) game. Let u and w be real numbers and let $J(u, w)$ be given by

$$J = z^2 - \gamma^2 w^2 = (u + w)^2 - \gamma^2 w^2 = -(\gamma^2 - 1)\left(w - \frac{u}{\gamma^2 - 1}\right)^2 + \frac{\gamma^2}{\gamma^2 - 1} u^2$$

and consider the auxiliary problem

$$Q^\tau(\overline{u}^{\tau-1},\overline{y}^{\tau-1}) \;:\; \max_{w^{\tau-1}\in\Omega_{\tau-1}(\overline{u}^{\tau-2},\overline{y}^{\tau-1})} G^{\tau-1}\left(\overline{u}^{\tau-1},w^{\tau-1}\right)$$

Let $\hat{w}^\tau$ be a solution of the auxiliary problem Q^τ, and $\hat{x}^\tau$ be the trajectory generated by $\overline{u}^{\tau-1}$ and $\hat{w}^\tau$. Notice that in $\hat{w}^\tau$ the first part $\hat{w}^{\tau-1}$ fulfills Q^τ while $\hat{w}_\tau$ is arbitrary. The controller is now

$$u_\tau = \mu^*(\hat{x}_\tau) = \hat{\mu}_\tau(\overline{u}^{\tau-1},\overline{y}^{\tau-1}) \tag{29}$$

In this way the full information strategy μ^* defines a strictly causal measurement feedback controller. Intuitively one can argue as follows: Consider the situation at time τ. Knowing y_k up to $k = \tau - 1$, one should look for the worst possible disturbance w compatible with the available information and "play" as if the current state were actually the most unfavorable. The future loss should then be evaluated using the assumption that u may use full state-feedback! This is formulated it the following theorem

THEOREM 5—[Başar and Bernhard, 1991]–Theorem 6.1
If, for every y and every $\tau \in [1,k]$, the auxiliary problem $Q^\tau(\hat{\mu}(y),y)$ has a unique maximum in w, generating a state trajectory $\hat{x}^\tau$, then (29) defines a inf sup controller for $\inf_u \sup_w J(u,w)$ and the min – max cost is

$$\min_u \max_w J = \max_{x_1}(V_1(x_1)+N(x_1)) = V_1(\hat{x}_1^1)+N(\hat{x}_1^1)$$

□

It is claimed in Remark 6.1 [Başar and Bernhard, 1991] and [Bernhard, 1991] that this H_∞-separation principle is valid also for the nonlinear nonquadratic case since the same proof applies. This is not true. Formally the flaw in the proof as presented in [Başar and Bernhard, 1991] is that the cited representation theorem (Theorem 2.5) is also wrong or wrongly formulated. Just consider the case with $J \equiv 0$ independently of u and w.

EXAMPLE 6
Consider the following nonquadratic game. Here $x(k)$ denotes the value of the state x at time k.

$$\begin{aligned} x(2) &= w \qquad w \ge 0 \\ x(3) &= x(2)+u \\ J &= x^2(3)-w^4 \end{aligned}$$

Where w and u are short for $w(1)$ and $u(2)$ respectively. This non-quadratic game satisfies the conditions of the theorem above. The initial condition $x(1)$ is uninteresting and there is no term $N(x_1)$. There are no measurements so the problem is actually open loop for u. For this problem the unique full-information law is $u = \mu_2^*(x(2)) = -x(2)$. Furthermore we have $V_2(x(2)) = 0$. The auxiliary problem Q^2 becomes

$$\hat{w} = \operatorname{argmax}\{V_2(x(2)) + g_1(x_1, u_1, w_1)\} = \left(0 - w(1)^4\right) = 0$$

The state trajectory generated by this unique $\hat{w}$ is $\hat{x}(2) = 0$. According to Theorem 6.1 of [Başar and Bernhard, 1991] (notice that the remarks after the theorem emphasize that it holds in the nonlinear nonquadratic situation) we should therefore have the open loop law:

$$u = \hat{\mu}_2 = \mu_2^*(\hat{x}(2)) = -\hat{x}(2) = 0$$

This controller however gives $\sup_w J = \max_{w \geq 0} w^2 - w^4 = 1/4$. Which is clearly different from zero so this is actually not the the correct value of the game. Notice that w knows that u is playing open loop. To find the correct min-sup controller we solve

$$\min_u \sup_{w \geq 0} \left((w+u)^2 - w^4\right)$$

"by hand". The solution, see Figure 5, is given by $u = -1/(3\sqrt{3})$ and the value of the game is $J = 1/27$. The mixed disturbance

$$w = \begin{cases} 0 & \text{with prob. } 2/3 \\ 1/\sqrt{3} & \text{with prob. } 1/3 \end{cases}$$

is in pure-mixed saddle equilibria with $u^* = -1/(3\sqrt{3})$. We then have $\min \sup E\, J = \sup \min E\, J = 1/27$.

Remark 4. Notice that the value of the game was $J = 0$ if u had full state information. Not only existence of an upper value, but also the actual value of this game therefore depends on the information player u has. This is a significant difference between the linear quadratic and the non-linear nonquadratic cases. It is hence vital that correct assumptions are made on the future game (e.g. open loop versus closed loop etc). This is a similar issue to the early discussion for LQG-problems in the nonlinear, nonquadratic case, see [Lindquist, 1973].

Remark 5. The theorem is true for linear quadratic problems. This should however be proven in another way.

A better formulation is possible as we will see. The key will be that the worst disturbance w should also include a term $(x - \hat{x})$, i. e. the error in u:s estimate. This will then capture the difference in information structures between u and w and the possibility for w to mess things up.

Bernhard's Dynamic Programming Separation Principle

A rigorous separation principle for a special class of stochastic difference games was given in [Bernhard and Colomb, 1988]. The theorem gives sufficient conditions. An extension in [Bernhard, 1992] gives both necessary and sufficient conditions. The problems studied in those papers are discrete time, nonlinear, stochastic, two player problems, defined on discrete (=finite) spaces. One player has noise corrupted measurements, the other perfect causal state information. The theorem can be used to construct mixed saddle point strategies using dynamic programming. We will now *try* to describe the main ideas.

The following is a brief summary of the results in [Bernhard, 1992], We use the notation of that paper and refer to it for further details. Let the system be given by

$$x_{t+1} = f_t(x_t, u_t, v_t, w_t) \tag{30}$$

$$y_t = h_t(x_t, w_{t-1}) \tag{31}$$

The initial value x_1 is known to both player u and v. The sequence $(y_1, \dots y_t)$ is denoted y^t. The player u_t knows y^{t-1}, and player v_t has full causal information at time t, i.e., x^t, y^t and also u^{t-1}. The white stochastic disturbance w_t is described by the probability distribution $W_t(w_t)$ known to both players. Both players and stochastic disturbance are defined on finite spaces. The criterion is given by

$$G = E\left[\sum_{t=1}^{T-1} L_t(x_t, u_t, v_t) + K(x_T)\right] \tag{32}$$

The game is stopped at time T. Let ν_t be a probability distribution over $X^t \times V^{t-1}$ describing player v:s conditional probability on (x^t, v^{t-1}). Assume u^t given as well as a behavioral strategy ψ^t of player v. The definition of behavioral strategies is given in [Bernhard, 1992]. Using (30) and (31) we can propagate ν_t into a probability $\overline{\nu}_{t+1}$ over $X^{t+1} \times V^t$ in the following way. Let $a = (a^t, a_{t+1}) \in X^{t+1}$ and $b \in V^t$

$$\overline{\nu}_{t+1}(a, b) = \nu_t(a^t, b^{t-1})\psi_t\left[a^t, u^{t-1}, b^{t-1}\right](b_t) \cdot \tag{33}$$

$$\sum_{w_t} \delta(a_{t+1} - f_t(a_t, u_t, b_t, w_t))W_t(w_t) \tag{34}$$

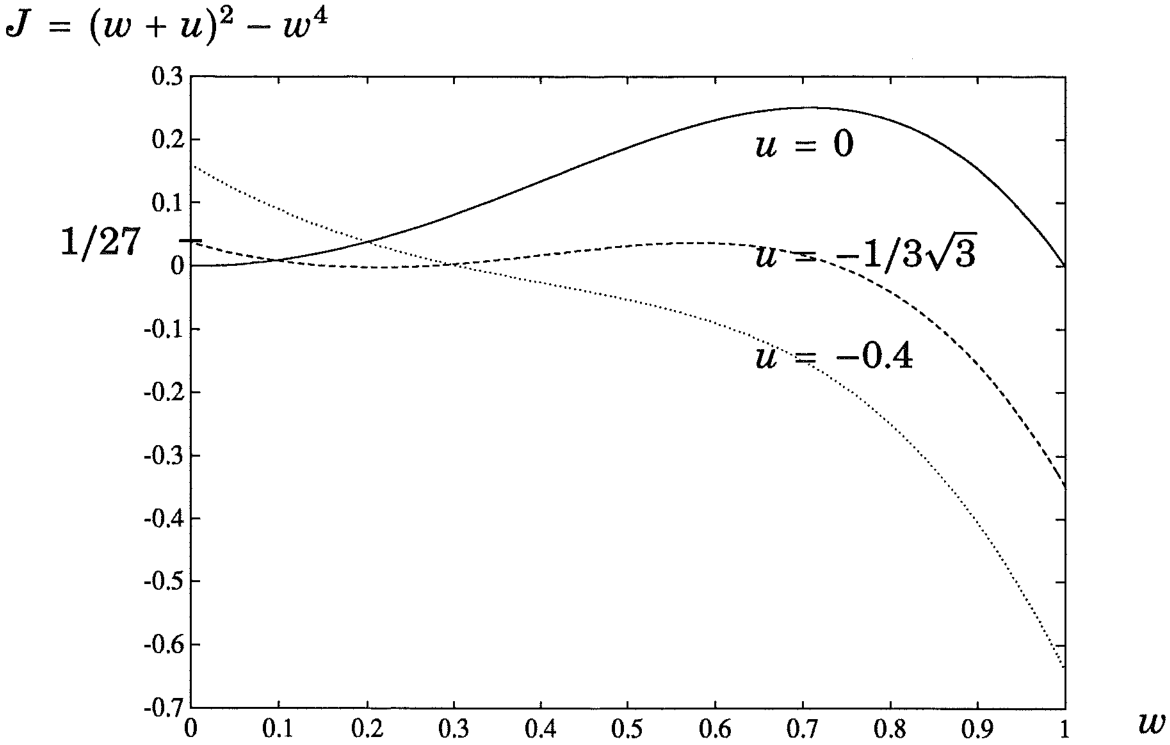

Figure 5. The cost function $J(u, w)$ in Example 3 for $u = 0$ (full), $u = -1/3\sqrt{3}$ (dashed) and $u = -0.4$ (dotted). The pure-mixed saddle point is given by $u = -1/3\sqrt{3}$ and the mixed disturbance $w = 0$ (w.p. 2/3), $w = 1/\sqrt{3}$ (w.p. 1/3), the value of the game is 1/27

Remark 6. The continuous time case, Theorem 5.1 in [Başar and Bernhard, 1991], is treated more carefully. In an appendix containing Danskin's theorem, more restrictions like uniqueness, rules out counterexamples analogous to the example given above. Therefore the continuous time separation theorem as formulated in [Başar and Bernhard, 1991] is correct, also for the nonquadratic case.

The discrete time H_∞-control problem is in principle therefore not solved in [Başar and Bernhard, 1991]. One can, however prove by other means, e. g. using Bernhard's newer separation principle, see below, that the corresponding theorem is correct in the linear quadratic case, at least for the suboptimal case ($\gamma > \gamma^*$).

The conclusion we want to draw from the previous discussion is that there is a need for a separation/certainty equivalence principle that is more easy to work with. The fact that there is a minor omission in previous work is of much less importance. What is more important is to realize that it will be hard to generalize this type of certainty equivalence principle to the min-mix case involving stochastic disturbances, since the value of the game will then depend on the information structure. The optimization over all possible past disturbances w is hard to generalize.

where $\delta(x) = 1$ if $x = 0$ and 0 otherwise. When measurement y_{t+1} becomes available, the a posteriori probability ν_{t+1} on (x^{t+1}, v^t) can be computed. This defines a filter

$$\nu_{t+1} = F_t\left(v_t, u^t, y_{t+1}, \psi_t\right) \tag{35}$$

and a function

$$\nu_t = N_t\left(u^{t-1}, y^t, \psi^{t-1}\right) \tag{36}$$

By summation over the component subspaces we project ν_t on X^t alone. Let ρ^t be that law on x^t. Further project on component X_t yielding a law

$$\rho_t = R_t\left(u^{t-1}, y^t, \psi^{t-1}\right) \tag{37}$$

Bernhard's theorem now states the following:

THEOREM 6
Let a stochastic dynamic game be given by (30)-(32) and let (φ^*, ψ^*) be a saddle point in behavioral strategies. Let $\rho_t^* = R_t(u^{t-1}, y^t, \psi^*)$. The game admits a saddle point in behavioral strategies of the form $\varphi_t^*[u^{t-1}, y^t] = \hat{\varphi}[\rho_t^*]$, $\psi_t^*[x^t, u^{t-1}, v^{t-1}] = \hat{\psi}_t[x_t, \rho_t^*]$. There exists a filter

$$\rho_{t+1} = g_t(\rho_t, u_t, y_{t+1}, \psi_t) \tag{38}$$

and a sequence of functions $V_t(x_t, \rho_t)$ such that for all (x_t, ρ_t) that are reached with a nonzero probability while playing optimally we have

$$\begin{aligned} V_t(x_t,\rho_t) &= \max_{v\in V_t} \sum_{w\in W_t} \sum_{u\in U_t} [V_{t+1}(f_t(x_t,u,v,w,), g_t(\rho_t,u,y_{t+1},\hat{q}_t)) \\ &\qquad + L_t(x_t,u,v)]\,\hat{p}_t(u) W_t(w) \\ &= \sum_{v\in V_t} \sum_{w\in W_t} \sum_{u\in U_t} [V_{t+1} + L_t]\,\hat{p}_t(u) W_t(w) \hat{q}_t(v) \end{aligned} \tag{39}$$

and

$$\begin{aligned} \sum_{x_t} V_t(x_t,\rho_t)\rho_t(x) &= \min_{u\in U_t} \sum_{x_t} \sum_{w\in W_t} \sum_{v\in V_t} [v_{t+1}(f_t(x_t,u,v,w,), g_t(\rho_t,u,y_{t+1},\hat{q}_t)) \\ &\qquad + L_t(x_t,u,v)]\,\hat{q}_t(v) W_t(w) \rho_t(x_t) = \\ = \sum_{u\in U_t} \sum_{x_t} \sum_{w\in W_t} \sum_{v\in V_t} & [V_{t+1} + L_t]\,\hat{q}_t(v) W_t(w) \rho_t(x_t) \hat{p}(u) \end{aligned} \tag{40}$$

Here $\hat{p}_t$ denotes $\hat{\varphi}_t[\rho_t]$, $\hat{q}_t$ for $\hat{\psi}_t[x_t, \rho_t]$, and y_{t+1} for $h_{t+1}(f_t, w)$, and

$$\forall\, x_T, \forall\, \rho, \quad V_T(x_T, \rho) = K(x_T)$$

Conversely, if a sequence of functions $\hat{\varphi}_t$, $\hat{\psi}_t$, and V_t satisfy these equations, they provide a saddle point of the game. □

Remark 7. Notice that the theorem gives *both necessary and sufficient conditions*.

The theorem can be interpreted as follows: The state of the system is described by (x_t, ρ_t). The function $V_t(x_t, \rho_t)$ describes the expected future loss. Two optimization problems must be solved, one for the maximizer v, another for the minimizer u. In the maximization, it can be assumed that $u = u^*$. The filter ρ will then not necessarily describe u:s conditional distribution of x, but this is not needed in the proof. One should not attempt now to interpret ρ as a conditional distribution. Instead it should be regarded as a cleverly introduced function. In the minimization, it can be assumed that $v = v^*$. The function ρ then equals the conditional distribution of x given past observations. Furthermore R_t is the filter giving this distribution.

The theorem basically answers the question "How should the state estimate $\hat{x}$ be interpreted and under what assumptions should the conditional expectations be performed?" The most severe restrictions is the assumption on finite spaces. This restriction will guarantee that minima and maxima exist. It is natural to conjecture that a similar theorem as Theorem 6 is true for more general spaces in the linear dynamics, quadratic loss function case. Let the system be given by

$$\begin{aligned} qx &= Ax + B_0 w_0 + B_1 w_1 + B_2 u && (41)\\ x(t_0) &= x_0 && (42)\\ y &= C_2 x + D_{20} w_0 + D_{21} w_1 && (43)\\ L_t &= x^T C_1^T C_1 x + u^T u - \gamma^2 w_1^T w_1 && (44)\\ K_{t_f} &= x(t_f)^T Q_f x(t_f) && (45) \end{aligned}$$

where w_0 is Gaussian white noise and let $\rho = \{\check{x}, P\}$ describe the mean value and covariance matrix of the Gaussian probability distribution obtained from the filter g described in (38) and let $\tilde{x} = x - \check{x}$. In the following $\tilde{x}$ is shorthand for the full probability distribution of x, described via $\check{x}$ and P.

CONJECTURE 1

For the discrete time, linear quadratic stochastic differential game there is a pure saddle point $(u^*(y), w_1^*)$ if and only if there exists a function $V_t(x, \tilde{x})$ such that

$$V_t(x_t, \tilde{x}_t) = \max_{w_1} \mathop{E}_{w_0} \{ V_{t+1}(Ax_t + B_0 w_0 + B_1 w_1 + B_2 u^*, \tilde{x}_{t+1}) + L_t(x_t, u^*, w_1) \}$$

where $\tilde{x}_{t+1}$ is calculated using y^t and $u = u^*$, and such that

$$\underset{x_t}{E}\, V_t(x_t, \tilde{x}_t) =$$
$$= \min_u \underset{x_t}{E}\,\underset{w_0}{E}\left\{V_{t+1}(Ax_t + B_0 w_0 + B_1 w_1^* + B_2 u, \tilde{x}_{t+1}) + L_t(x_t, u, w_1^*)\right\}$$

Here E_{x_t} is calculated using $x_t = \check{x}_t - \tilde{x}_t$ and the probability distribution given by $\rho_t(x_t)$. The distribution of x_{t+1} is recursively calculated using the assumption $w_1 = w_1^*$ and using measurements up to y_t. When calculating the extrema, there should be equality for $w_1 = w_1^*$ and $u = u^*$. □

We also present the corresponding conjecture for continuous time problems:

CONJECTURE 2
In continuous time the above theorem is replaced with

$$\max_{w_1} \underset{w_0}{E} \left(\frac{dV_t}{dt}(x, \tilde{x}) + L_t(x, u, w_1) \right)\Bigg|_{u=u^*} = 0 \tag{46}$$

$$\min_u \underset{x}{E}\,\underset{w_0}{E} \left(\frac{dV_t}{dt}(x, \tilde{x}) + L_t(x, u, w_1) \right)\Bigg|_{w_1=w_1^*} = 0 \tag{47}$$

The extrema above should be achieved for $w_1 = w_1^*$ and $u = u^*$. The arguments to the functions and the interpretations are analogous to the previous conjecture.

The value ∞ is considered as possible for a game in the discussion above. The theorems do a priori only give mixed strategies for u and w_1. The result in [Bernhardsson, 1992] indicates however that the control signal u can in fact use a pure strategy. Moreover the player w_1 can also use a pure strategy if the problem is concave in w_1. The upper value of the game is otherwise unbounded. We have not yet been able prove Conjectures 1 and 2 rigorously. It means that at this point the necessity and sufficiency of the three coupled Riccati equations obtained in sections 7 and 8 are really only conjectures.

7. Continuous Time Min-Mix Controllers

In this section we obtain formulas for the finite time, time varying case of the min-mix problem J_3 by completion of squares.

An LQ Lemma

To develop the controllers for the min-mix case we use the following lemma which is "semi-classical" LQ control, see e.g. [Willems, 1971]:

LEMMA 2
Let the system be given by

$$\begin{aligned} \dot{x} &= Ax + B_0 w_0 + B_1 w_1 \\ x(t_0) &= x_0 \end{aligned}$$

The signal w_1 with full causal information of x that maximizes

$$J = \underset{w_0}{E}\left\{\int_{t_0}^{t_f} \left[x^T Q x - \gamma^2 w_1^T w_1\right] dt + x^T(t_f) Q_f x(t_f)\right\}$$

is given by

$$w_1^* = \gamma^{-2} B_1^T X x \tag{48}$$

where

$$-\dot{X} = A^T X + XA + Q + \gamma^{-2} X B_1 B_1^T X, \qquad X(t_f) = Q_f \tag{49}$$

If the Riccati equation (49) fails to exist for all t, or said in other words, has a conjugate point, then the value can be made arbitrarily large by w_1, i.e. $\max J = \infty$.

Proof: A proof can be found in the [Willems, 1971]. We will indicate how to use dynamic programming to motivate the equations. We show that

$$V(t,x) = \max_{w_1} \underset{w_0}{E}\left\{\int_t^{t_f} x^T Q x - \gamma^2 w_1^T w_1 \, d\tau\right\} = x^T(t) X(t) x(t) + f(t) \tag{50}$$

where $X(t)$ is given by Riccati equation (49) and $f(t)$ will be determined later. We use dynamic programming:

$$\begin{aligned} &\underset{w_0}{E}\left(\frac{dV}{dt} + x^T Q x - \gamma^2 w_1^T w_1\right) = \\ &\quad = \underset{w_0}{E}\left\{x^T(-A^T X - XA - Q - \gamma^{-2} X B_1 B_1^T X)x\right\} + \mathrm{Tr}\,(B_0^T X B_0) + \dot{f}(t) + \\ &\qquad + \underset{w_0}{E}\, 2x^T X(Ax + B_0 w_0 + B_1 w_1) + x^T Q x - \gamma^2 w_1^T w_1 = \\ &\quad = -\gamma^2 (w_1 - \gamma^{-2} B_1^T X x)^T (w_1 - \gamma^{-2} B_1^T X x) \;\leq\; 0 \end{aligned}$$

The term $\mathrm{Tr}(B_0^T X B_0)$ comes form Ito's differentiation rule. See Section 11. We have also used

$$f(t) = \int_t^{t_f} \mathrm{Tr}\, B_0^T X B_0 \, dt$$

Notice that equality is obtained if and only if $w_1 = w_1^*$. Once the completion of squares is done, the rest of the theorem follows by application of standard results using the theory of conjugate points. □

The Min-Mix Case

To treat the full min-mix case we first motivate the form of V for this problem as a specialization of (50). Let the system be given by

$$\begin{aligned} \dot{x} &= Ax + B_2 u + B_0 w_0 + B_1 w_1 \\ x(t_0) &= x_0 \\ y &= C_2 x + D_{20} w_0 + D_{21} w_1 \end{aligned} \tag{51}$$

and the criterion by

$$J = \underset{w_0}{E} \int_{t_0}^{t_f} \left(x^T C_1^T C_1 x + u^T u - \gamma^2 w_1^T w_1 \right) dt + x(t_f)^T Q_f x(t_f) \tag{52}$$

Introduce the following signal

$$\frac{d}{dt}\check{x} = A\check{x} + B_2 u + \gamma^{-2} B_1 B_1^T X \check{x} - L\left(y - C_2 \check{x} - \gamma^{-2} D_{21} B_1^T X \check{x} \right) \tag{53}$$

where L will be chosen later. The motivation for this signal will also be clear later. Introduce also $\tilde{x} = x - \check{x}$. Then

$$\begin{aligned} \frac{d}{dt}\tilde{x} = A\tilde{x} + B_0 w_o + B_1 \left(w_1 - \gamma^{-2} B_1^T X (x - \tilde{x}) \right) + \\ + L\left(C_2 \tilde{x} + D_{20} w_0 + D_{21}(w_1 - \gamma^{-2} B_1^T X (x - \tilde{x})) \right) \end{aligned} \tag{54}$$

We hence obtain with the definition $u^* := -B_2^T X \check{x} = -B_2^T X (x - \tilde{x})$

$$\begin{aligned} \frac{d}{dt}\begin{pmatrix} x \\ \tilde{x} \end{pmatrix} = \begin{pmatrix} A - B_2 B_2^T X & B_2 B_2^T X \\ -\gamma^{-2} B_{1d} B_1^T X & A + LC_2 + \gamma^{-2} B_{1d} B_1^T X \end{pmatrix} \begin{pmatrix} x \\ \tilde{x} \end{pmatrix} \\ + \begin{pmatrix} B_0 \\ B_{0d} \end{pmatrix} w_0 + \begin{pmatrix} B_1 \\ B_{1d} \end{pmatrix} w_1 + \begin{pmatrix} B_2 \\ 0 \end{pmatrix} (u - u^*) \end{aligned} \tag{55}$$

where

$$B_{0d} := B_0 + LD_{20} \tag{56}$$
$$B_{1d} := B_1 + LD_{21} \tag{57}$$

When $u = u^*$ this can be considered as a problem of the form in Lemma 2

$$\begin{aligned}\dot{\overline{x}} &= \overline{A}\overline{x} + \overline{B_0}w_0 + \overline{B_1}w_1\\ \overline{x}(0) &= \overline{x}_0\end{aligned}$$

where $\overline{x} = \begin{pmatrix} x^T & \tilde{x}^T \end{pmatrix}^T$ and

$$\begin{aligned}\overline{A} &= \begin{pmatrix} A - B_2B_2^TX & B_2B_2^TX \\ -\gamma^{-2}B_{1d}B_1^TX & A + LC_2 + \gamma^{-2}B_{1d}B_1^TX \end{pmatrix}\\ \overline{B_0} &= \begin{pmatrix} B_0 \\ B_{0d} \end{pmatrix}\\ \overline{B_1} &= \begin{pmatrix} B_1 \\ B_{1d} \end{pmatrix}\\ \overline{Q} &= \begin{pmatrix} C_1^TC_1 + XB_2B_2^TX & -XB_2B_2^TX \\ -XB_2B_2^TX & XB_2B_2^TX \end{pmatrix}\end{aligned}$$

Here $\overline{Q}$ follows from $u = u^* = -B^TX(x - \tilde{x})$ inserted into (48). If we guess that the corresponding $\overline{X}$ will be block diagonal, say

$$\overline{X} := \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$$

it follows from Lemma 2 that

$$\begin{aligned}V(t, x, \tilde{x}) = {} & \begin{pmatrix} x \\ \tilde{x} \end{pmatrix}^T \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \begin{pmatrix} x \\ \tilde{x} \end{pmatrix} + \\ & \int_t^{t_f} \mathrm{Tr} \begin{pmatrix} B_0 \\ B_{0d} \end{pmatrix}^T \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \begin{pmatrix} B_0 \\ B_{0d} \end{pmatrix} dt\end{aligned} \tag{58}$$

The Riccati equation (49) for the elements of the symmetric $\overline{X}$ yields

$$-\dot{X} = XA + A^TX + C_1C_1^T - XB_2B_2X + \gamma^{-2}XB_1B_1^TX \tag{59}$$
$$0 = \gamma^{-2}XB_1B_{1d}^TY + XB_2B_2^TX - XB_2B_2^TX + \gamma^{-2}XB_1B_{1d}Y \tag{60}$$
$$-\dot{Y} = YA_e + A_e^TY + XB_2B_2^TX - \gamma^{-2}YB_{1d}B_{1d}^TY \tag{61}$$

where

$$A_e := A + LC_2 + \gamma^{-2} B_{1d}(B_1^T X + B_{1d}^T Y)$$

Notice that (59) is the standard control Riccati equation from H_∞control. Equation (60) is obviously satisfied and (61) is a new equation. We also recognize the stationary equations obtained by setting derivatives to zero as two of the equations obtained in [Doyle *et al.*, 1992] for the problem J_4. We also obtain

$$w_1^* = \gamma^{-2} \overline{B_1} \, \overline{X} \begin{pmatrix} x \\ \tilde{x} \end{pmatrix} = \gamma^{-2}(B_1^T X x + B_{1d}^T Y \tilde{x}) \qquad (62)$$

Now when we have found a guess on V and w_1^* it is easy to complete the full min-mix case.

CONJECTURE 3
Optimal min-mix controllers satisfy

$$\begin{aligned}
\dot{\check{x}} &= A\check{x} + B_2 u + \gamma^{-2} B_1 B_1^T X \check{x} - L(y - C_2 \check{x} - \gamma^{-2} D_{21} B_1^T X \check{x}) && (63)\\
u^* &= -B_2^T X \check{x} && (64)\\
w_1^* &= \gamma^{-2}(B_1^T X x + B_{1d}^T Y \tilde{x}) && (65)
\end{aligned}$$

where $\tilde{x} = x - \check{x}$, and where

$$\begin{aligned}
-\dot{X} &= XA + A^T X + C_1^T C_1 - X(B_2 B_2^T - \gamma^{-2} B_1 B_1^T)X && (66)\\
X(t_f) &= Q_f && (67)\\
-\dot{Y} &= YA_e + A_e^T Y + X B_2 B_2^T X - \gamma^{-2} Y B_{1d} B_{1d}^T Y && (68)\\
Y(t_f) &= 0 && (69)\\
\dot{P} &= A_e P + P A_e^T + B_{0d} B_{0d}^T && (70)\\
P(t_0) &= 0 && (71)\\
(B_0 + LD_{20}) D_{20}^T &= -P\left[C_2 + \gamma^{-2} D_{21}(B_1^T X + (B_1 + LD_{21})^T Y)\right]^T && (72)\\
A_e &= A + LC_2 + \gamma^{-2} B_{1d}(B_1^T X + B_{1d}^T Y) && (73)
\end{aligned}$$

Proof: Let

$$V(t, x, \tilde{x}) = x^T X x + \tilde{x}^T Y \tilde{x} + \int_t^{t_f} \mathrm{Tr}\left(B_0^T X B_0 + B_{0d}^T Y B_{0d}\right) d\tau \qquad (74)$$

We now follow Conjecture 2 and define

$$F := \underset{w_0}{E}\left\{\frac{dV}{dt} + x^T C_1^T C_1 x + u^T u - \gamma^2 w_1^T w_1\right\} \tag{75}$$

where formally

$$\underset{w_0}{E}\frac{dV}{dt} = \underset{w_0}{E}\left[\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial \tilde{x}}\dot{\tilde{x}}\right] + \text{ Ito}$$

From (63)-(65) together with (52) it follows that

$$\dot{\tilde{x}} = A_e\tilde{x} + B_{1d}(w_1 - w_1^*) + B_{0d}w_0 \tag{76}$$

so we have

$$\begin{aligned}
\frac{\partial V}{\partial t} &= x^T\dot{X}x + \tilde{x}^T\dot{Y}_f\tilde{x} - \text{Tr}\,(B_0^T X B_0 + B_{0d}^T Y B_{0d}) && (77)\\
\frac{\partial V}{\partial x}\dot{x} &= 2x^T X(Ax + B_2 u + B_0 w_0 + B_1 w_1) && (78)\\
\frac{\partial V}{\partial \tilde{x}}\dot{\tilde{x}} &= 2\tilde{x}^T Y(A_e\tilde{x} + B_{1d}(w_1 - w_1^*) + B_{0d}w_0) && (79)\\
\text{Ito} &= \text{Tr}\,(B_0^T X B_0 + B_{0d}^T Y B_{0d}) && (80)
\end{aligned}$$

Collecting terms and completing squares gives

$$\begin{aligned}
F &= x^T\{\dot{X} + XA + A^TX + C_1^TC_1\}x + \tilde{x}^T\{\dot{Y}_f + YA_e + A_e^TY\}\tilde{x}\\
&\quad + 2x^TXB_2u + 2x^TXB_1w_1 + 2\tilde{x}^TYB_{1d}(w_1 - w_1^*) + u^Tu - \gamma^2 w_1^Tw_1\\
&= (u + B_2^TXx)^T(u + B_2^TXx) - (B_2^TX\tilde{x})^TB_2^TX\tilde{x} - \gamma^{-2}a^Ta + \gamma^{-2}b^Tb\\
&\quad + 2a^Tw_1 + 2b^T(w_1 - w_1^*) - \gamma^2 w_1^Tw_1\\
&= (u - u^* + c)^T(u - u^* + c) - c^Tc - \gamma^2(w_1 - w_1^*)^T(w_1 - w_1^*)\\
&\quad + \gamma^2 w_1^{*T}w_1^* - \gamma^{-2}a^Ta + \gamma^{-2}b^Tb - 2b^Tw_1^*\\
&= |u - u^* + c|^2 - c^Tc - \gamma^2|w_1 - w_1^*|^2 && (81)
\end{aligned}$$

where $a := B_1^TXx$, $b := B_{1d}^TY\tilde{x}$, and $c := B_2^TX\tilde{x}$. Until this point we have only introduced some new notation and completed squares. Equation (81) is revealing and is the key for understanding saddle point properties and obtaining min-mix controllers.

Now notice that if $u = u^*$ the maximum of F over w_1 is zero and is achieved if and only if $w_1 = w_1^*$. This proves half of the theorem.

What then needs to be shown is that by a special choice of L we can use Bernhard's certainty equivalence. The trick is that we can choose L such that $(u-u^*)$ becomes orthogonal to c. This is achieved by (70)-(73), which can be seen using standard Kalman filter results in the following way. With $w_1 = w_1^*$ we have

$$\dot{\tilde{x}} = A_e\tilde{x} + B_{0d}w_0 \tag{82}$$

so the matrix Riccati equation in (70) gives $P(t) = E[\tilde{x}_t^2 \mid w_1 = w_1^*, y^t]$. According to Bernhard's separation theorem $(\check{x}_t, P)$ should describe the conditional distribution of x_t given $w_1 = w_1^*$ and given past observations. We have for $w_1 = w_1^*$ that

$$\dot{x} = (A + \gamma^{-2}B_1B_1^TX)x + B_2u + B_0w_0 + \gamma^{-2}B_1B_1^TY\tilde{x} \tag{83}$$

$$y = C_2x + D_{20}w_0 + \gamma^{-2}D_{21}(B_1^TXx + B_{1d}^TY\tilde{x}) \tag{84}$$

It is a standard LQG-result, see e. g. , [Åström, 1970], that if L and P satisfy

$$-LD_{20}D_{20}^T = B_0D_{20}^T + P\left[C_2 + \gamma^{-2}D_{21}(B_1^TX + (B_1 + LD_{21})^TY)\right]^T \tag{85}$$

then

$$\dot{\check{x}} = (A + \gamma^{-2}B_1B_1^TX)\check{x} + B_2u - L(y - C_2\check{x} - \gamma^{-2}D_{21}B_1^TX\check{x}) \tag{86}$$

gives $E[x^2 \mid w_1 = w_1^*, y^t] = \check{x}$, so that $\check{x} \perp \tilde{x}$, i. e. $E[\check{x}\tilde{x}^T \mid w_1 = w_1^*, y^t] = 0$, and thus

$$(u - u^*) \perp c$$

To see this it is valuable to form

$$\tilde{y} = y - C_2\check{x} - \gamma^{-2}D_{21}B_1^TX\check{x} = D_{20}w_0 + (C_2 + \gamma^{-2}D_{21}(B_1^TX + B_{1d}^TY))\tilde{x}$$

The theorem now follows by application of the generalization of Bernhard's dynamic programming principle (Conjecture 2). □

Remark 8. If derivatives are put to zero, these equations by [Doyle *et al.*, 1992] for the infinite time horizon case for problem J_4 follows.

Remark 9. Formula (72) can be used to calculate L if P and Y are known. The equation is linear in L and can be rewritten using Kroenecker products to a linear equation that is singular if and only if

$$I \otimes I + (D_{20}D_{20}^T)^{-1}D_{21}D_{21}^T \otimes \gamma^{-2}PY \tag{87}$$

is singular. If $P \geq 0$ and $Y \geq 0$ it is easy to see that this matrix is nonsingular. L is therefore uniquely given if P and Y are known.

The Closed Loop

By using $u = u^*$ and $w_1 = w_1^*$ we get the following equations for the resulting system

$$\frac{d}{dt}\begin{pmatrix} x \\ \tilde{x} \end{pmatrix} = \begin{pmatrix} A_c & B_2B_2^TX + \gamma^{-2}B_{1d}^TY \\ 0 & A_e \end{pmatrix}\begin{pmatrix} x \\ \tilde{x} \end{pmatrix} + \begin{pmatrix} B_0 \\ B_{0d} \end{pmatrix} w_0 \tag{88}$$

where $A_c = A + (\gamma^{-2}B_1B_1^T - B_2B_2^T)X$ and A_e is given by (73). This describes the transfer from stochastic noise w_0 to x and $\tilde{x}$. We hence understand the conditions that A_c and A_e should be stable matrices in the infinite time horizon case.

It is rewarding for the understanding to study how the calculations reduce to (partly) known results in the H_2 and H_∞ cases:

Summary of H_∞-Controllers—Continuous Time

The standard H_∞-controller formulas exist in several different forms. Two different forms are obtained using $\check{x}$ or $\hat{x}$ as states in the controller, see [Rhee and Speyer, 1991]. Introduce

$$0 = X_\infty A + A^TX_\infty + C_1^TC_1 - X_\infty\left(B_2B_2^T - \gamma^{-2}B_1B_1^T\right)X_\infty \tag{89}$$
$$0 = AY_\infty + Y_\infty A^T + B_1B_1^T - Y_\infty\left(C_2^TC_2 - \gamma^{-2}C_1^TC_1\right)Y_\infty \tag{90}$$
$$X_\infty \geq 0 \tag{91}$$
$$\gamma^2Y_\infty^{-1} - X_\infty \geq 0 \tag{92}$$
$$Z = (I - \gamma^{-2}Y_\infty X_\infty)^{-1} \tag{93}$$

More conditions also have to be satisfied. These are stability of A_c and A_e described above, which is now seen as very natural, see (88).

For simplicity in the description below we assume $D_{21}B_1^T = 0$ and $D_{21}D_{21}^T = I$. The first form of the controller is given by

$$\dot{\hat{x}} = A\hat{x} + B_2u + \gamma^{-2}Y_\infty C_1^TC_1\hat{x} + Y_\infty C_2^T(y - C_2\hat{x}) \tag{94}$$
$$u^* = -B_2^TX_\infty Z_\infty\hat{x} \tag{95}$$

An equivalent form is given by

$$\dot{\check{x}} = A\check{x} + B_2u + \gamma^{-2}B_1B_1^TX_\infty\check{x} - L(y - C_2\check{x}) \tag{96}$$
$$u^* = -B_2^TX_\infty\check{x} \tag{97}$$
$$L = -ZY_\infty C_2^T = -(I - \gamma^{-2}Y_\infty X_\infty)^{-1}Y_\infty C_2^T \tag{98}$$

The forms are connected through $\check{x} = Z\hat{x}$, but only if $u = u^*$.

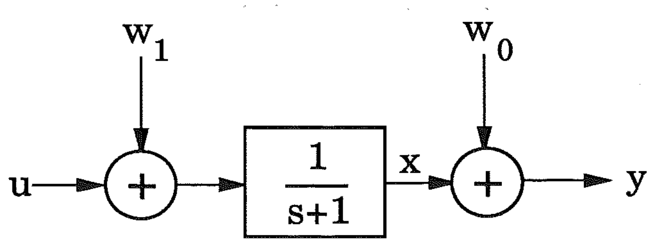

Figure 6. The system in Example 4

H_2 and H_∞ as Special Cases

The connection between Y and Y_∞ in H_∞control is the following

$$Y = \gamma^2 Y_\infty^{-1} + X_\infty \tag{99}$$

Using (72) with $D_{21}B_1^T = 0$, $D_{21}D_{21}^T = I$ we obtain

$$-LD_{20}D_{20}^T = B_0 D_{20}^T + P(C_2^T + \gamma^{-2} Y L) \tag{100}$$

setting $D_{20} = 0$ and assuming P nonsingular we obtain the H_∞formulas above.

We now also have an equation for the worst w_1^*. Notice that the equation obtained above for w_1^* reduces to

$$w_1^* = \gamma^{-2} B_1^T X x + \gamma^{-2} B_{1d} Y \tilde{x} \tag{101}$$

and hence contains a term proportional to $x - \check{x}$. This term is invisible if both players play optimally, but is the key for understanding how w can take advantage of player u:s lack of information, i. e. when $\check{x} \neq x$ and for obtaining the controller formulas by simple completion of squares!

To obtain the LQG formulas we put $B_1 = 0$ and $D_{21} = 0$. We then get $A_e = A + LC_2$ and (72) becomes

$$-LD_{20}D_{20}^T = B_0 D_{20}^T + PC_2^T \tag{102}$$

which we recognize from the Kalman filter.

EXAMPLE 7

The feedback problem in Figure 6 gives a lot of insight into the behavior of min-mix controllers. It will also illuminate some flaws in the infinite time horizon theorem in [Doyle *et al.*, 1992]. The problem is given by

$$\dot{x} = -x + w_1 + u \tag{103}$$

$$x(0) = 0 \tag{104}$$

$$z = \begin{pmatrix} x \\ u \end{pmatrix} \tag{105}$$

$$y = x + \sigma w_0 \tag{106}$$

that is

$$G : \left[\begin{array}{c|ccc} -1 & 0 & 1 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & \sigma & 0 & 0 \end{array}\right] \tag{107}$$

Notice first that if $\gamma > \gamma_o$, where $\gamma_o = 1$ corresponds to open-loop, then the best min-mix controller is obviously $K(s) = 0$ and the worst disturbance is $w_1 = 0$. The value of the game is then zero, since this controller does not introduce any measurement noise into the process. If $\gamma_\infty < \gamma < \gamma_o$, where $\gamma_\infty = 1/\sqrt{2}$ is the full-information γ, there is a trade off between counteracting w_1 and introducing measurement noise w_0 into the process.

The solution to the min-mix equations is now obtained "by hand". We first study the infinite time-horizon problem. For this process Equations (63)-(73) reduce to

$$\begin{aligned} 0 &= -2X + 1 - X^2(1 - \gamma^{-2}) && (108)\\ 0 &= 2A_eY + X^2 - \gamma^{-2}Y^2 && (109)\\ 0 &= 2A_eP + L^2\sigma^2 && (110)\\ L\sigma^2 &= -P && (111)\\ A_e &= -1 + L + \gamma^{-2}(X + Y) && (112)\\ A_c &= -1 + (\gamma^{-2} - 1)X && (113) \end{aligned}$$

From (110) and (111) we get two solutions:

$$(L, P) = (0, 0) \qquad \text{or} \qquad (L, P) = (2A_e, -2A_e/\sigma^2) \tag{114}$$

The solution with $L = 0$ can be used if $\gamma > \gamma_o = 1$. For $\gamma \searrow 1$ we get for $L = 0$ that

$$\begin{aligned} X &\to 1/2 \\ A_c &\to -1 \\ Y &\nearrow 1/2 \\ A_e &= -1/2 + Y \nearrow 0 \\ L &= 0 \\ P &= 0 \end{aligned}$$

The other solution, with $L = 2A_e$, is valid for all $\gamma > \gamma_\infty = 1/\sqrt{2}$. For $\gamma \searrow \gamma_\infty$ we get

$$\begin{aligned} X &\to 1 \\ A_c &\nearrow 0 \\ Y &\to (\sqrt{7}-1)/6 \\ A_e &\to -(2+\sqrt{7})/3 \\ L &= 2A_e \\ P &\to 2\left(2+\sqrt{7}\right)\sigma^2/3 \end{aligned}$$

Notice that L is independent of σ, which is surprising. The controller is given by

$$\begin{aligned} \dot{\check{x}} &= A_c\check{x} - L(y-\check{x}) \\ u &= -X\check{x} \end{aligned}$$

with $A_c = -1 - X(1-\gamma^{-2})$. The value of the game can now be calculated using the new formula

$$V(t,x,\tilde{x}) = x^T X x + \tilde{x}^T Y \tilde{x} + \int_t^{t_f} \operatorname{Tr} \sigma^2 L^2 Y \, d\tau \tag{115}$$

Notice that the value is strictly positive when $L \neq 0$ and $Y > 0$.

The situation is therefore the following: For $\gamma_o < \gamma < \gamma_\infty$ there is only one solution satisfying the equations and stability conditions. For large γ there are however two solutions to the equations and stability conditions given in [Doyle *et al.*, 1992] and in this paper. Only the choice with $L = 0$ gives the optimal min-mix controller. The formulation of the theorem in [Zhou *et al.*, 1990, Doyle *et al.*, 1992] is therefore not satisfactory. That the equations are satisfied and all stability conditions are met *are not enough to guarantee that the optimal min-mix controller is found*. One could make the mistake of using the controller with $L = 2A_e$ also for the case with large γ, which would not be optimal.

The formulas therefore have to be complemented with more information, like our expression for the value of the game. We strongly believe that intuition from the finite-time horizon results is necessary to fully understand the infinite horizon case, and that further study of the finite time horizon case will be rewarding for finding these conditions. This is an open issue for future research. The present paper contributes by presenting the solution (modulo Conjecture 2) to the finite horizon case. □

We now obtain analytical expressions for $\gamma^{-2} = 7/4$. This value satisfies $\gamma_o < \gamma < \gamma_\infty$ and the solution $L = 0$, $P = 0$ will hence not give a stable system. This shows up in that the equation for Y gets a conjugated point. We will use the stationary value $X(t) \equiv 2/3$ in what follows. The dynamic equations are given by

$$
\begin{aligned}
A_c &= -1 - (1 - \gamma^{-2})X \\
A_e &= 1 + L + 2Y \\
L &= -P/\sigma^2 \\
\dot{P} &= 2A_e P + L^2\sigma^2 \\
\dot{Y} &= -X^2 + (2 + 2P/\sigma^2 - 2\gamma^{-2}X)Y - \gamma^{-2}Y^2
\end{aligned}
$$

The solution for $P(t) \equiv 0$ and $L(t) \equiv 0$ was obtained using Maple. It is given by

$$
Y_i(t) = 9\left(\sqrt{3}\tan\left(\frac{(t - C_1)\sqrt{3}}{2}\right) - \frac{1}{3}\right)/8
$$

Notice that Y_i explodes if the time interval is sufficiently long.

For this γ value the stationary values when $L = 2A_e$ are given by

$$
\begin{aligned}
X &= 2/3 \\
P &= \left(2 + \sqrt{85}\right)\sigma^2/9 \\
Y &= (-2 + 2\sqrt{85})/63 = 0.2609
\end{aligned}
$$

We have not been able to obtain explicit solutions for the dynamic P and Y Riccati equations There are several open questions concerning the dynamical equations, which should be rewarding to investigate.

8. Discrete Time Min-Mix Controllers

The calculations in discrete time are analogous to continuous time. The algebra is however more complicated. The calculations have hopefully been reduced to a minimum in what follows. Let the, possibly time-varying, system be given by

$$
\begin{aligned}
qx &= Ax + B_0 w_0 + B_1 w_1 + B_2 u && (116) \\
x(0) &= x_0 && (117) \\
y &= C_2 x + D_{20} w_0 + D_{21} w_1 && (118)
\end{aligned}
$$

where q is the forward shift operator, i.e. $qx = x(k+1)$. Let the criterion be to find saddle equilibria for

$$\mathop{E}_{w_0}\left\{\sum_{k=0}^{t_f-1} x^T(k)C_1^TC_1x(k) + u^T(k)u(k) - \gamma^2 w_1^T(k)w_1(k) + x^T(t_f)Q_f x(t_f)\right\}$$

where the disturbance consists of $w = (w_1(0), \ldots, w_1(t_{f-1}))$ and the controller is $u = (u(0), u(1), \ldots, u(t_{f-1}))$. The controller is assumed to be of the form $u(k) = f(y^{k-1})$. In what follows the time index is assumed to be k unless explicitly stated otherwise.

We first mention the following LQ-lemma, which is the counterpart of Lemma 2:

LEMMA 3
Specialize the system to

$$\begin{aligned} qx &= Ax + B_0w_0 + B_1w_1 &&(119)\\ x(0) &= x_0 &&(120)\end{aligned}$$

and the criterion to

$$J = \mathop{E}_{w_0}\sum_{k=0}^{t_f-1}(x^TQx - \gamma^2 w_1^T w_1) + x^T(t_f)Q_f x(t_f) \tag{121}$$

If $S_k \geq 0$ for all k, where

$$\begin{aligned} S_k &= \gamma^2 I - B_1^T X_{k+1} B_1 &&(122)\\ X_k &= Q + A^T V_k A = Q + A^T X_{k+1} A - A^T X_{k+1} B_1 S_k^{-1} B_1^T X_{k+1} A\\ &= Q + A^T X_{k+1} A - F_k^T S_k F_k &&(123)\\ X_{t_f} &= Q_f &&(124)\\ V_k &= (X^{-1} - \gamma^{-2} B_1 B_1^T)^{-1} &&(125)\\ S_k F_k &= B_1^T X_{k+1} A &&(126)\end{aligned}$$

then the worst case w_1 is given by

$$w_1^*(k) = \gamma^{-2} B_1^T V_k A x(k) = S_k^{-1} B_1^T X_{k+1} A x(k) = F_k x_k \tag{127}$$

If S_k is singular, there are many solutions F_k. The value of the game is

$$\mathcal{V}_k = x^T(k) X_k x(k) + \sum_{j=k+1}^{t_f} \operatorname{Tr} B_0^T X_j B_0 \tag{128}$$

If S_k fails to be positive semidefinite for some k, then J can be made arbitrarily large by w_1.

Proof: The lemma is a special case of the results in e.g. [Başar and Bernhard, 1991]. It can be proved by the following calculation

$$\begin{aligned}\mathcal{V}_k &= \underset{w_0}{E}\Big\{(Ax + B_0w_0 + B_1w_1)^T X_{k+1}(Ax + B_0w_0 + B_1w_1) + x^T Qx \\ &\qquad\qquad -\gamma^2 w_1^T w_1 + \sum_{j=k+2}^{t_f} B_0^T X_j B_0\Big\} \\ &= \sum_{j=k+1}^{t_f} B_0^T X_j B_0 + (Ax + B_1w_1)^T X_{k+1}(Ax + B_1w_1) + x^T Qx - \gamma^2 w_1^T w_1 \\ &= x^T X_k x + \sum_{j=k+1}^{t_f} B_0^T X_j B_0 - (w_1 - F_k x)^T S_k (w_1 - F_k x)\end{aligned}$$

from which the theorem follows immediately. Notice how the continuous time problem with conjugated point theory is much simplified in discrete time. □

We now motivate the form of $\mathcal{V}$ for the full min-mix case in the same way as in continuous time. Introduce the signals

$$u^* = F_2\check{x} \tag{129}$$

$$q\check{x} = A\check{x} + B_2u_k + B_1F_1\check{x} - L(y - C_2\check{x} - D_{21}F_1\check{x}) \tag{130}$$

$$\tilde{x} = x - \check{x} \tag{131}$$

where L is determined later and

$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} \gamma^{-2}B_1^T \\ -B_2^T \end{pmatrix} VA; \qquad V = (X_{k+1}^{-1} - \gamma^{-2}B_1B_1^T + B_2B_2^T)^{-1} \tag{132}$$

The form of $\check{x}$ will be motivated later. Consider it for the moment as a suitably chosen signal. We then have

$$q\tilde{x} = (A + LC_2)\tilde{x} + B_{0d}w_0 + B_{1d}w_1 - B_{1d}F_1(x - \tilde{x}) \tag{133}$$

where $B_{0d} := B_0 + LD_{20}$ and $B_{1d} := B_1 + LD_{21}$. The system can thus be written as

$$\begin{aligned} q\begin{pmatrix} x \\ \tilde{x} \end{pmatrix} &= \begin{pmatrix} A + B_2F_2 & -B_2F_2 \\ -B_{1d}F_1 & A + LC_2 + B_{1d}F_1 \end{pmatrix}\begin{pmatrix} x \\ \tilde{x} \end{pmatrix} + \begin{pmatrix} B_0 \\ B_{0d} \end{pmatrix} w_0 \\ &\qquad + \begin{pmatrix} B_1 \\ B_{1d} \end{pmatrix} w_1 + \begin{pmatrix} B_2 \\ 0 \end{pmatrix}(u - u^*) \end{aligned} \tag{134}$$

Inspired by Lemma 3 and the continuous time case we define

$$\mathcal{V}_k = x^T X x + \tilde{x}^T Y \tilde{x}^T + \sum_{k+1}^{t_f} \mathrm{Tr}\left(B_0^T X B_0 + B_{0d} Y B_{0d}^T\right) \tag{135}$$

To find the new Riccati equations we use Lemma 3 on the extended system. Let

$$\overline{V} = \left(\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}^{-1} - \gamma^{-2}\begin{pmatrix} B_1 \\ B_{1d} \end{pmatrix}\begin{pmatrix} B_1^T & B_{1d}^T \end{pmatrix}\right)^{-1} \tag{136}$$

and

$$\overline{A} = \begin{pmatrix} A + B_2F_2 & -B_2F_2 \\ -B_{1d}F_1 & A + LC_2 + B_{1d}F_1 \end{pmatrix} \tag{137}$$

A straightforward calculation shows that

$$\overline{V}\,\overline{A} = \overline{V}\begin{pmatrix} A \\ A + LC_2 \end{pmatrix}\begin{pmatrix} 0 & I \end{pmatrix} + \begin{pmatrix} I \\ 0 \end{pmatrix} VA \begin{pmatrix} I & -I \end{pmatrix} = \begin{pmatrix} VA & * \\ 0 & * \end{pmatrix} \tag{138}$$

Therefore the right hand side of the extended Riccati equation (123) becomes

$$\begin{aligned}\overline{Q} + \overline{A}^T\overline{V}\,\overline{A} = {} & \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} F_2^T \\ -F_2^T \end{pmatrix}\begin{pmatrix} F_2 & -F_2 \end{pmatrix} \\ & + \begin{pmatrix} A + B_2F_2 & -B_2F_2 \\ -B_{1d}F_1 & A + LC_2 + B_{1d}F_1 \end{pmatrix}^T \overline{V}\,\overline{A}\end{aligned} \tag{139}$$

Using (138) again we obtain the following three equations

$$q^{-1}X = Q + F_2^TF_2 + F_2^TB_2^TVA + A^TVA = Q + A^TVA \tag{140}$$

$$0 = -F_2^TF_2 - F_2^TB_2^TVA \tag{141}$$

$$q^{-1}Y = \begin{pmatrix} -B_2F_2 \\ A + LC_2 + B_{1d}F_1 + F_2^TF_2 \end{pmatrix}^T \overline{V}\begin{pmatrix} -B_2F_2 \\ A + LC_2 + B_{1d}F_1 \end{pmatrix} \tag{142}$$

The first Riccati equation is the standard full-information H_∞equation, the second equation is obviously satisfied and the third equation reduces after simplification using (138) to

$$q^{-1}Y = \begin{pmatrix} A \\ A + LC_2 \end{pmatrix}^T \overline{V}\begin{pmatrix} A \\ A + LC_2 \end{pmatrix} - A^TVA \tag{143}$$

Inspired by Lemma 3 we also define

$$w_1^* = \gamma^{-2} \begin{pmatrix} B_1^T & B_{1d}^T \end{pmatrix} \overline{V}\,\overline{A} \begin{pmatrix} x \\ \tilde{x} \end{pmatrix} = F_1 x + (N - F_1)\tilde{x} \tag{144}$$

with

$$N := \gamma^{-2} \begin{pmatrix} B_1^T & B_{1d}^T \end{pmatrix} \overline{V} \begin{pmatrix} A \\ A + LC_2 \end{pmatrix} \tag{145}$$

We now obtain

$$q\tilde{x} = A_e \tilde{x} + B_{0d} w_0 + B_{1d}(w_1 - w_1^*) \tag{146}$$

where

$$A_e = A + LC_2 + B_{1d} N \tag{147}$$

Notice the resemblance to continuos time. All that is needed now is to determine L using Bernhard's separation theorem. With $w_1 = w_1^*$ the covariance matrix $E[\tilde{x}^2]$ is given by

$$qP = A_e P A_e^T + B_{0d} B_{0d}^T \tag{148}$$

and the system equations become

$$\begin{aligned} x(k+1) &= [A + B_1 F_1(k)]x(k) + B_0 w_0(k) + B_2 u(k) \\ &\quad + B_1[N(k) - F_1(k)]\tilde{x}(k) \\ y(k) &= [C_2 + D_{21} F_1(k)]x(k) + D_{20} w_0(k) + D_{21}[N(k) - F_1(k)]\tilde{x}(k) \end{aligned}$$

We have

$$\begin{aligned} &\operatorname{cov} \begin{pmatrix} qx - (A + B_1 F_1)\check{x} - B_2 u \\ y - (C_2 + D_{21} F_1)\check{x} \end{pmatrix} = \\ &\operatorname{cov} \begin{pmatrix} (A + B_1 N)\tilde{x} + B_0 w_0 \\ (C_2 + D_{21} N)\tilde{x} + D_{20} w_0 \end{pmatrix} =: \begin{pmatrix} V_a & V_b \\ V_b^T & V_c \end{pmatrix} \end{aligned}$$

where

$$V_a = (A + B_1 N)P(A + B_1 N)^T + B_0 B_0^T \tag{149}$$
$$V_b = (A + B_1 N)P(C_2 + D_{21} N)^T + B_0 D_{20}^T \tag{150}$$
$$V_c = (C_2 + D_{21} N)P(C_2 + D_{21} N)^T + D_{20} D_{20}^T \tag{151}$$

Standard estimation theory now says that if L satisfies

$$L(k)V_c + V_b = 0 \tag{152}$$

that is

$$-L\left\{[C_2 + D_{21}N]P[C_2 + D_{21}N]^T + D_{20}D_{20}^T\right\} = \\ = [A + B_1N(k)]P(k)[C_2 + D_{21}N(k)]^T + B_0D_{20}^T \tag{153}$$

giving

$$-LD_{20}D_{20}^T = A_eP(C_2 + D_{21}N)^T + B_0D_{20}^T \tag{154}$$

and if

$$\check{x}(k+1) - (A + B_1F_1)\check{x}(k) - B_2u(k) = -L(k)[y(k) - C_2 - D_{21}F_1(k)\check{x}(k)] \tag{155}$$

then

$$\tilde{x}(k+1) \perp \mathcal{Y}_k \tag{156}$$

where $\mathcal{Y}_k$ denotes all past information. This is the classical orthogonality principle. It is now possible to finish the calculations in the discrete time case with a completion of squares similar to the one in continuous time. We will not present these details here. It parallels the continuous time case, but with more algebraic problems. The details will be given in a future paper.

In summary the discrete time min-max problem boils down to the following equations:

CONJECTURE 4

$$\begin{aligned}
q\check{x} &= A\check{x} + B_2u_k + B_1F_1\check{x} - L(y - C_2\check{x} - D_{21}F_1\check{x}) && (157)\\
u^* &= F_2\check{x} && (158)\\
w^* &= F_1x + (N - F_1)\tilde{x} && (159)\\
q^{-1}X &= Q + A^TVA && (160)\\
q^{-1}Y &= \begin{pmatrix} A \\ A + LC_2 \end{pmatrix}^T \overline{V} \begin{pmatrix} A \\ A + LC_2 \end{pmatrix} - A^TVA && (161)\\
qP &= A_ePA_e^T + B_{0d}B_{0d}^T && (162)\\
-LD_{20}D_{20}^T &= A_eP(C_2 + D_{21}N)^T + B_0D_{20}^T && (163)\\
A_e &= A + LC_2 + B_{1d}N && (164)
\end{aligned}$$

The definitions of $F_1, F_2, \overline{V}$ and V are given above. □

As a side result we also obtain the following for H_∞control:

Relations to Discrete Time H_∞-Controllers

Different formulas for discrete time H_∞-controllers exist. The connections between some different forms are discussed in [Walker, 1990]. The following results concern the delayed case, e. g. when $u(k)$ is a function of $y(k-1)$ which means that there is a delay of one sample in the calculation of control signal. The case with a direct term in the controller is algebraically more complicated, see [Limebeer *et al.*, 1989], [Gu *et al.*, 1989], [Başar and Bernhard, 1991] or [Stoorvogel, 1992]. We will not discuss this case here. Assume for ease of notation in the following two theorems that $B_1 D_{21}^T = 0$ and $D_{21} D_{21}^T = I$

THEOREM 7
If $\gamma > \gamma^*$ then a pure saddle point is represented by

$$
\begin{aligned}
u^*(k) &= -B_2^T V_k A Z_k \hat{x}(k) && (165)\\
w_1^*(k) &= \gamma^{-2} B_1 W_k A Z_k \hat{x}(k) && (166)\\
\hat{x}(k+1) &= A\hat{x}(k) + B_2 u(k) + AW_k(\gamma^{-2} C_1^T C_1 \hat{x}(k) + \\
&\qquad C_2^T (y(k) - C_2 \hat{x}(k))) && (167)\\
\hat{x}(0) &= Z_0 x_0 && (168)\\
X_k &= A^T V_k A + B_1 B_1^T; \qquad X_{t_f} = Q_f && (169)\\
Y_{k+1} &= AW_k A^T + C_1^T C_1; \qquad Y_0 = Q_0^{-1} && (170)\\
Z_k &= (I - \gamma^{-2} Y_k X_k)^{-1} && (171)\\
V_k &= (X_{k+1}^{-1} + B_2 B_2^T - \gamma^{-2} B_1 B_1^T)^{-1} && (172)\\
W_k &= (Y_k^{-1} + C_2^T C_2 - \gamma^{-2} C_1^T C_1)^{-1} && (173)
\end{aligned}
$$

The concavity conditions are, according to [Başar and Bernhard, 1991] in the finite time horizon case that

$$
\begin{aligned}
X_{k+1}^{-1} - \gamma^{-2} B_1 B_1^T &> 0\\
Y_k^{-1} + C_2^T C_2 - \gamma^{-2} C_1^T C_1 &> 0\\
\gamma^2 Y_k^{-1} - X_k &> 0
\end{aligned}
$$

For the infinite time horizon case there are other conditions as well.

Proof: See e.g. [Başar and Bernhard, 1991] or [Whittle, 1990b]. Many other references also exist. □

An Alternative Form

The min-mix formulas suggest another, but equivalent form of the suboptimal discrete time H_∞controllers. The alternative form uses a different state in the controller, $\check{x}$. This form is so natural that it is probably not new, although we have not been able to find any references:

question of when the equations are necessary and sufficient is still not satisfactorily solved. It is also not clear what are the weakest possible existence conditions for obtaining an optimal controller. The singular cases require special techniques. The techniques in [Stoorvogel, 1990, Stoorvogel, 1992] can probably be applied. This is however non trivial.

It would be interesting to investigate what happens as $\gamma \searrow \gamma_{opt}$, the optimal case. How does the controller obtained in this way compare with the equalizing controller or the controller minimizing the H_2-norm with an H_∞ constraint? Example 4 should be useful when doing this. Another simple result in this direction would be:

CONJECTURE 5
The optimal γ-values are the same for J_3 and H_∞.

Proof: Take any $\gamma > \gamma_\infty^*$ and a controller achieving this H_∞-bound. The quadratic term in w_1 is then negative definite and the system is internally stable. The extra w_0-terms can not give infinite extra loss. J_3 is hence bounded. □

It should be possible to see this from the equations directly. Another open problem is to determine when the min-mix controller is unique. The intuition is here that if there is process noise "on all state variables" the controller should be unique. It is also important to investigate further connections between the different mixed H_2/H_∞ results.

The relation with the work of Whittle, see [Whittle, 1990b] is also open right now. Is there a way to connect the results to the risk-sensitive approach? The beautiful symmetric formulation of two-point boundary problems using the path formalism of Whittle should be aimed for. It should also be possible to show what kind of operator factorizations the Riccati equations really are doing, [Hagander, 1973].

If the motivation for mixing norms is to obtain robust performance controllers, one should probably tie the robust controller design closer to how the knowledge of the system is obtained. One such approach for optimizing system performance for uncertain systems is given in [Bernhardsson, 1992], where covariance information on uncertainty is used to find the expected H_2-performance. If the uncertain parameters enter in a feedforward fashion, explicit formulas for the controller optimizing the expected H_2-performance can be obtained.

THEOREM 8
If $\gamma > \gamma^*$ then the following represents a pure saddle point

$$\begin{aligned} \check{x}(k+1) &= A\check{x}(k) + B_1 w_1^*(k) + B_2 u(k) + Z_{k+1} A W C_2^T (y - C_2 \check{x}_k) && (174) \\ \check{x}(0) &= x_0 && (175) \\ u^*(k) &= -B_2^T V_k A \check{x}(k) && (176) \\ w_1^*(k) &= \gamma^{-2} B_1^T V_k A \check{x}(k) && (177) \end{aligned}$$

where V, W and Z are given by the same equations as above.

Proof: The proof is obtained from Theorem 7 by straightforward algebraic manipulations. □

The states are related in the following way. If $u = u^*$ then $\check{x} = Z\hat{x}$. An advantage with the second form of the controller is that $\check{x}$ has the interpretation of a state estimate for u using $w = w^*$. It is also easier to see the second form of the controller as a special case of the min-mix case.

Relation to Discrete Time LQG

At least for the case $D_{21} = 0$ the complicated L-equation (163) simplifies considerably. Some matrix algebra gives

$$A_e = A + LC_2 + B_{1d}N = [I + W(X+Y)]A + [I + WY]LC_2$$

where

$$W = B_1[B_1^T(X+Y)B_1 - \gamma^2 I]^{-1} B_1^T$$

so that

$$-LD_{20}D_{20} - [I + WY]LC_2PC_2^T = [I + W(X+Y)]APC_2 + B_0 D_{20}^T \quad (178)$$

This equation is now linear in L, and it's solvability can be discussed as in the continuous time case. If also $B_1 = 0$, i. e. in the LQG case, then $W = 0$, and we recognize the well known formula for the Kalman filter gain L. The P-equation with $A_e = A + LC_2$ gives the covariance.

9. Some Ideas for Future Work

The paper presents several problems worthy of further investigation. The first goal should be to prove the two separation theorems given in the text. This should not be difficult for the linear quadratic case. The

10. Conclusions

The paper presents a short completion of squares argument for the solution of the min-mix controller problem both in continuous and discrete time. A conjectured generalization of a dynamic programming separation principle by Bernhard was used to obtain the controller. The full finite time horizon, time varying problem was treated and new formulas were given. We also obtained new, explicit, formulas for the value of the game. Relationships to earlier results on game theory were presented. An example was also presented that illustrated the theory and the equations obtained.

11. Notation

If $G(s)$ is a stable, continuous time transfer function, then

$$||G||_\infty = \sup_w \overline{\sigma}(G(iw))$$

where $\overline{\sigma}$ is the largest singular value and

$$||G||_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{Tr} G^*(iw)G(iw)dw$$

In discrete time iw is replaced by e^{iw} and the integration is over the unit circle. We also need Ito's differentiation rule. Let

$$dx = f(x,t)dt + \sigma(x,t)dw$$

where $w(t)$ is a Wiener process. Then if $y = y(x,t)$ we have

$$dy = \left(y_t + y_x^T + \frac{1}{2}\operatorname{Tr}\left(y_{xx}\sigma\sigma^T\right)\right)dt + y_x^T\sigma dw$$

In special if

$$dx = Axdt + Bdw$$

then

$$\begin{aligned} d\left(x^T(t)X(t)x(t)\right) &= \left(x^T\dot{X}x + x^T(XA + A^TX)x + \operatorname{Tr}\left(BXB^T\right)\right)dt \\ &\quad + dw^TB^TXx + x^TXBdw \end{aligned}$$

Notice the term $\operatorname{Tr}(BXB^T)$, it is denoted "Ito" in the calculations in the paper. See [Åström, 1970] for further details.

Bounded Power Signals (BP)

The paper [Zhou *et al.*, 1990, Zhou *et al.*, 1992] avoids introducing stochastic noise. They instead use the spaces (BP) and (BS) and induced norms. The definitions used in these papers are as follows:

The autocorrelation matrix for a given signal is defined as

$$R_{uu}(\tau) := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} u(t+\tau)u^T(t)\, d\tau \tag{179}$$

if the limits exists for all τ. Further assume the Fourier transform of the signal's autocorrelation matrix function exists (but may contain impulses). This Fourier transform is called the *spectral density* of u, denoted $S_u(j\omega)$

$$S_{uu}(j\omega) := \int_{-\infty}^{\infty} R_{uu}(\tau)e^{-j\omega\tau}{}'d\tau \tag{180}$$

A signal $u(t)$ is called a *power signal* if $u(t)$ satisfies the following conditions:

(BP1) $u(t) \in L_\infty$

(BP2) $R_{uu}(\tau)$ exists for all τ

(BP3) $S_{uu}(j\omega)$ exists (it need not be bounded and may include impulses)

A power signal is said to have bounded power if it is a power signal and the following (semi-)norm is bounded

$$\|u\|_P^2 = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|u\|^{2\prime} dt = \mathrm{Tr}\,[R_{uu}(0)] = \int_{-\infty}^{\infty} \mathrm{Tr}\,[S_{uu}(j\omega)\; d\omega/2\pi \tag{181}$$

Bounded Spectrum Signals

We call a signal $u(t)$ a *spectral signal* if the following conditions are satisfied:

(BS1) $u(t) \in L_\infty$

(BS2) $R_{uu}(\tau)$ exists for all τ

(BS3) $S_{uu}(j\omega)$ exists

A signal is said to have bounded spectrum if these conditions are satisfied and the seminorm defined by

$$\|u\|_S^2 := \|S_{uu}(j\omega)\|_\infty \tag{182}$$

is bounded. Both H_2 and H_∞ can be seen as induced norms using these two seminorms.

Some other notation

The following notation is also used in the paper

$$
\begin{aligned}
B_{0d} &= B_0 + LD_{20} \\
B_{1d} &= B_1 + LD_{21} \\
A_c &= A + (\gamma^{-2} B_1 B_1^T - B_2 B_2^T) X \\
A_o &= A + LC_2
\end{aligned}
$$

The notation $A := B$ means that A is defined as expression B.

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PAPER IV

Feedforward Control is Dual to Deconvolution

Bo Bernhardsson and **Mikael Sternad**[1]

Abstract: A duality is demonstrated between optimal feedforward control and optimal deconvolution, or input estimation. These two problems are normally discussed separately in the literature, but have close similarities. Duality between them can be demonstrated if and only if one uses general problem formulations, with frequency-shaped weighting in the criteria. From one of the problems, the dual problem can then be obtained immediately form the block diagram, by reversing the directions of arrows, interchanging summation points and node points and transposing all transfer function matrices. This result applies for continuous and discrete time problems, as well as for minimization of $J = ||G||$, for any transfer function norms for which $||G^T|| = ||G||$. A derivation of a polynomial solution to the frequency-weighted discrete-time MIMO LQG feedforward control problem illustrates the use of the duality.

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1. Introduction

Duality relations have a long history as fruitful tools in control and estimation theory. All control engineers are well aware of the dualities between LQ state feedback and Kalman state estimation, see [Kalman, 1960] or, e.g., [Kwakernaak and Sivan, 1972]. Similar duality results are of use in the study of H_∞-control and state estimation, see [Doyle *et al.*, 1989] and [Shaked, 1990]. For a linear time-invariant system in state space form, the dual system is obtained by reversing the role of inputs and outputs and by transposing all matrices.

For a problem in block diagram form, there is however no need to compute a state space realization to obtain the dual problem. In Sections 2 and 3, we will present an elegant way of obtaining the dual of a linear time-invariant control problem directly, from its block diagram. Although we have not found Theorem 1 in Section 2 stated explicitly in the literature, we doubt that it is novel. However, we include it because it is important for the following discussion. Also the elegant algorithm in Section 3 is believed to be a part of the folklore of control theorists, but is hard to find in the literature. The result is, however, very useful e.g when doing polynomial calculations and has certainly not greatly penetrated the literature.

This method is used in Sections 4–6 to clarify the question of what kind of estimation problems are dual to feedforward control problems. This question has been discussed, e.g., in [Sternad and Ahlén, 1988], where close correspondences were pointed out between disturbance measurement feedforward control and deconvolution, also called input estimation. In fact, by using loop transformations on scalar problems, it was shown how one problem could be transformed into the other. No dual relationship could, however, be obtained. As will be clarified below, it is possible to demonstrate a dual relationship, if the formulation of both problems are made more general than the ones discussed in [Sternad and Ahlén, 1988]. The general formulations include *dynamic cost weighting* in the criteria.

Our interest has been mainly in LQG (or H_2)-solutions based on polynomial equations, a method pioneered by [Kučera, 1979]. However, the duality holds for any criterion $J = \min \|G\|$ based on a norm of a rational matrix G, for which $\|G^T\| = \|G\|$. It does, e.g., hold also for an H_∞ norm but not in the MIMO case for the L_1 norm [Dahleh and Pearson, 1987], defined by

$$\|G\|_{L_1} = \max_i \sum_{j=1}^{n} \|g_{ij}\|_1$$

$$J_1 = \min_{K_1} \|G^1_{zw}\| \qquad J_2 = \min_{K_2} \|G^2_{zw}\|$$

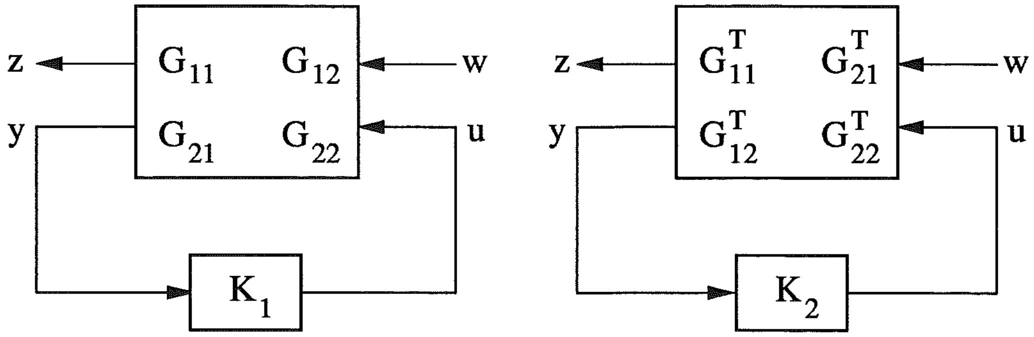

Figure 1. Dual problems. The left hand figure represents the standard problem G^1. The dual problem G^2 is given to the right.

where $g_{ij}(t)$ is the impulse-response of element (i,j) of $G(s)$.

By clarifying the duality relation between the two types of problems, we achieve two goals. Firstly, the many correspondences between them are explained, and the understanding of both problems is enhanced, see Sections 6 and 8. Intuition from the feedforward problem can be used in the formulation and solution of input estimation problems and vice versa. Secondly, the construction of algorithms for computer-aided design is simplified. Only one algorithm, which solves both kinds of problems, needs to be implemented. We illustrate this in Section 7 by deriving a polynomial solution to the discrete time LQG feedforward control problem from the corresponding input estimator design equations.

2. Duality

We begin our discussion by establishing a duality relation between the two problems described by the block diagrams in Figure 1. The left-hand diagram in Figure 1 represents the "standard problem." It was introduced around 1980 as a standard way of representing a large collection of control and signal estimation problems. See e.g. [Pernebo, 1981], or [Doyle *et al.*, 1989]. Polynomial optimization of LQG-controllers for the standard problem is described in [Grimble, 1991] and [Hunt *et al.*, 1991].

In Figure 1, y represents the measured variables, z are signals to be controlled, w are exogenous signals and u are the control inputs. Many control and filtering problems are formulated as design of K_1, to minimize the influence of w on z. All G_{ij}'s are here linear time-invariant transfer functions, in continuous or discrete time. (Time-arguments of signals, and arguments of transfer functions, are suppressed in the following.) Duality between the two block diagrams in Figure 1 can now be stated as follows.

THEOREM 1—Problem Duality
For all norms satisfying $||G^T|| = ||G||$, the two problems

$$J_1 = \min_{K_1} ||G^1_{zw}||, \qquad J_2 = \min_{K_2} ||G^2_{zw}||$$

in Figure 1 are dual, in the sense that the two optima are equal $J_1 = J_2$, and the optimal controllers are related by $K_1^T = K_2$. A necessary condition for the problems to be dual is that the minimal values of the norms are invariant under transposition.

Proof: The closed loop from w to z in the first problem is given by

$$G^1_{zw} = G_{11} + G_{12}(I - K_1 G_{22})^{-1} K_1 G_{21}$$

Transposing gives

$$\begin{aligned}(G^1_{zw})^T &= G_{11}^T + G_{21}^T K_1^T (I - G_{22}^T K_1^T)^{-1} G_{12}^T = \\ &= G_{11}^T + G_{21}^T (I - K_1^T G_{22}^T)^{-1} K_1^T G_{12}^T\end{aligned}$$

which is exactly the closed loop from w to z in the second case if $K_2 = K_1^T$. The sufficiency of the assumption $||G^T|| = ||G||$ follows from

$$||G^1_{zw}|| = ||(G^1_{zw})^T|| = ||G^2_{zw}||$$

If $||G^1_{zw}|| \neq ||(G^1_{zw})^T||$ at the minimum, then J_1 and J_2 will differ. Thus, it is necessary that $||G^1_{zw}|| = ||(G^1_{zw})^T||$ at the minimum. □

Remark 1. The dual system can be obtained by transposing the following matrix, where the state space representation of the transfer functions G_{ij} is $[A, B_j, C_i, D_{ij}]$, $i,j = 1,2$:

$$\begin{pmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{pmatrix}$$

Remark 2. Depending on the specific type of problem set-up and norm, restrictions may have to be imposed on the properness and stability of some or all of the blocks G_{ij}.

Remark 3. Fundamentally, duality is a relation between two *systems*: the role of their inputs and outputs are interchanged, and the time is reversed. For a time varying, continuous time system, the transformation can be seen as obtaining the adjoint system, followed by a time reversal. With finite final time t_f, the transformation of the state-space description is

$$\begin{pmatrix} A(t) & B(t) \\ C(t) & D(t) \end{pmatrix} \xrightarrow{\text{Adjoint system}} \begin{pmatrix} -A^T(t) & -C^T(t) \\ B^T(t) & D^T(t) \end{pmatrix} \xrightarrow{\text{Time reversal}} \begin{pmatrix} A^T(t_f-t) & C^T(t_f-t) \\ B^T(t_f-t) & D^T(t_f-t) \end{pmatrix} \quad (1)$$

The last transformation follows because the solution to $\dot{v} = -f(t_f - t, v(t))$ is the time reverse of the solution to $\dot{x} = f(t, x(t))$, see e.g. [Kwakernaak and Sivan, 1972], lemma 4.1. (The dual system is identical to the so called *modified adjoint system*, see [Kailath, 1980]). With time-invariant systems, the transformation (1) reduces to a transposition.

Duality between *systems* can be used for obtaining correspondences, or dualities, between *optimization problems*. The original example is LQ state feedback and Kalman filtering [Kalman, 1960]. If the solution is time-invariant, we no longer have to think of the dual problem as defined in reversed time. Theorem 1 states that duality essentially involves only transposition for a large class of problems with time-invariant solutions.

3. Block Diagram Version of Duality

For time invariant systems given in block diagram form, the dual to an optimization problem, in the sense of Theorem 1 can, if it exists, be obtained directly from the block diagram. The idea is old, but has to the authors knowledge not been published for the general setup of Theorem 1. The correctness of the following algorithm is easily proved and is left as a nice exercise.

Algorithm: The following block diagram transformations give the dual block diagram:

- Exchange w:s and z:s
- Exchange u:s and y:s
- Reverse directions of arrows (2)
- Interchange summation points and node points
- Transpose the transfer function blocks

An example is given in Figure 2 and Figure 4 below.

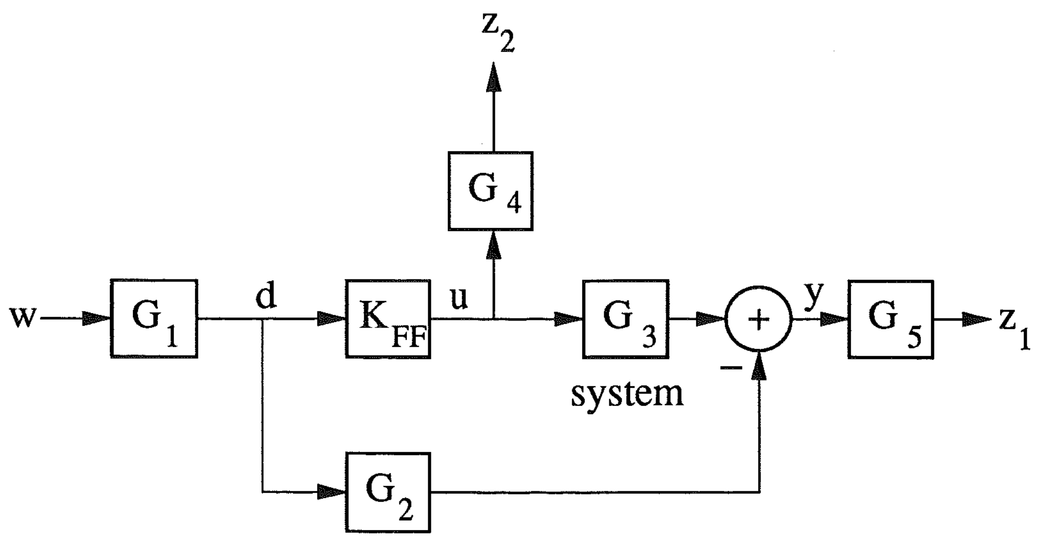

Figure 2. Feedforward control problems.

4. Feedforward Control

The design of feedforward links from measurable disturbances and from command signals is an important complement to a feedback design. We will here, in particular, consider the design of LQG (or H_2)-controllers.

The feedforward problem to be considered is shown in Figure 2. The system output is described by

$$y = G_3 u - G_2 d$$

where G_3 represents the system, including a possible fixed feedback controller. Here, d is a measurable signal, which is modeled as filtered white noise

$$d = G_1 w$$

The problem is to calculate the optimal causal, stable and linear feedforward regulator

$$u = K_{FF}\, d$$

that minimizes a mean square of the sum of filtered outputs and filtered control signals:

$$\begin{aligned} &\min \; E(\text{tr } z_1 z_1^T + \text{tr } z_2 z_2^T) \\ &z_1 = G_5 y \\ &z_2 = G_4 u \end{aligned}$$

All transfer functions are assumed known, stable, proper (in continuous time) or causal (in discrete time). In discrete-time problems, both G_2 and

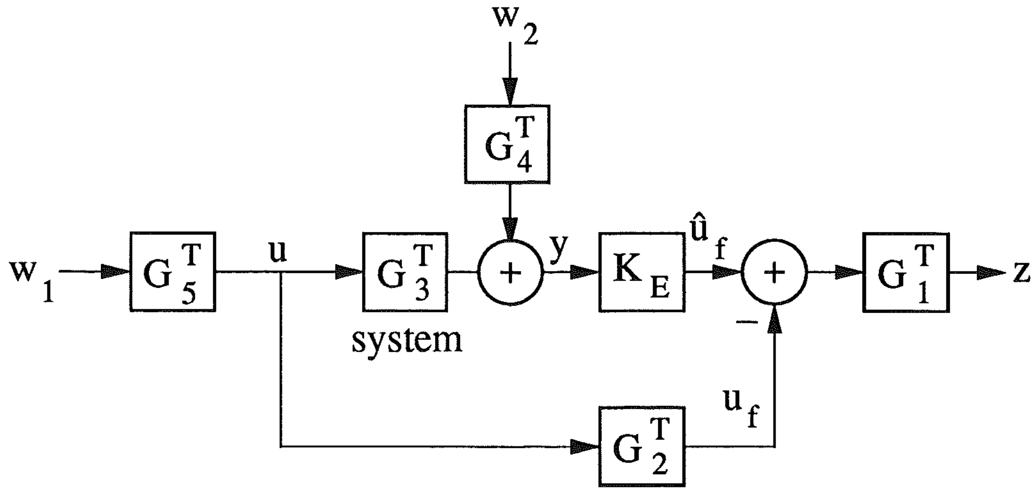

Figure 3. Signal estimation. The filter K_E is sought which estimates (a filtered version of) the input u.

G_3 may include delays. In a *disturbance measurement feedforward* problem, d represents measurable disturbances. They are eliminated in frequency regions of interest (defined by G_5), if $z_1 = G_5(G_3K_{FF} - G_2)G_1w$ is small. In command feedforward problems, d represents command signals, and G_1w are stochastic models describing their second order properties. Servo filters K_{FF} are then to be designed, based on a response model G_2. Good model following is achieved, in frequency regions of interest, if z_1 is small. (In a multivariable setting, d can of course include both measurable disturbances and command signals.)

For a discussion of scalar discrete-time LQG feedforward design, see e.g. [Sternad and Söderström, 1988] or [Hunt, 1989]. Multivariable problems are discussed in [Hunt and Šebek, 1989] and [Sternad and Ahlén, 1992], using the polynomial equations approach. A solution to MIMO discrete time problems is discussed in Section 7.

5. Estimation of the Input to a Dynamic System

Many filtering, prediction and smoothing problems are special cases of a set-up presented in Figure 3. The signal u is the input to a linear system G_3^T. A possibly filtered version of it, $u_f = G_2^T u$, is to be estimated, based on noisy measurements y of the system output. With white w_1 and w_2, $G_5^T w_1$ and $G_4^T w_2$ represent stochastic models of signal and noise. The transfer function G_1^T is a frequency shaping weighting filter.

When G_3^T contains dynamic elements, the problem is an input estimation or *deconvolution* problem. Otherwise, we have an output or state estimation problem. A dynamic element G_3^T may represent an analog or digital communication channel. The filter K_E is then a linear recursive equalizer. It's task is to reconstruct the transmitted signal u. In process control and supervision, G_3^T can represent a transducer, with slow dynamics. The task of the filter K_E is then to estimate, and possibly predict, the input u to the transducer.

The filter G_2^T in the lower (fictitious) signal path can be of use in several ways. In discrete time, it may include an advance or delay Iq^{-m}, i.e. $u_f(t) = u(t-m)$. Depending on m, $\hat{u}(t-m \mid t)$ is then a prediction ($m < 0$), filtering ($m = 0$) or a fixed lag smoothing ($m > 0$) estimate. The block G_2^T may also contain filters, to emphasize the estimation accuracy in certain frequency regions. Filters in either G_1^T or G_2^T can be used for affecting the relative accuracy, in different frequency regions, in the estimation of u. For a discussion of advantages and disadvantages of these two methods, see [Ahlén and Sternad, 1989]. Thus, the measured output is described by

$$y = G_3^T u + G_4^T w_2; \qquad u = G_5^T w_1$$

All systems are assumed to be known and stable and the white noise signals w_i are stationary, zero mean and mutually uncorrelated. We consider the problem of finding the best causal, stable and linear estimator of a filtered version $G_2^T u$ of the input

$$\hat{u}_f = K_E\, y$$

which minimizes a frequency weighted version of the mean square estimation error

$$\min E(\mathrm{tr}\ zz^T); \qquad z = G_1^T(\hat{u}_f - G_2^T u)$$

Such an estimator is a Wiener or stationary Kalman filter. All blocks, except G_2^T, are assumed proper in continuous time and causal in discrete time.

A solution to the discrete-time version of the MIMO H_2 estimation problem introduced above will be discussed in Section 7.

6. Duality Between Feedforward Control and Input Estimation

Using the setup in Figure 1, the feedforward problem in Figure 2 is represented within the standard problem by the rational matrix

$$\begin{pmatrix} \begin{pmatrix} -G_5 G_2 G_1 \\ 0 \end{pmatrix} & \begin{pmatrix} G_5 G_3 \\ G_4 \end{pmatrix} \\ G_1 & 0 \end{pmatrix}$$

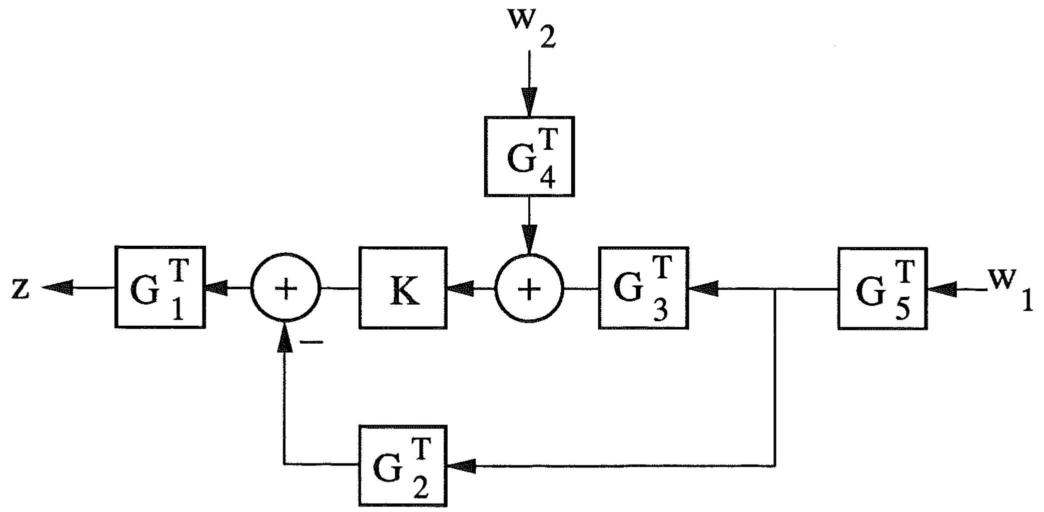

Figure 4. Result of block diagram transformations on the feedforward problem in Fig 2. The result equals the mirror image of the block diagram of the estimation problem in Fig 3.

with z_1, z_2 as controlled outputs, d as measured output, w as exogenous input and u as control input. Transposing this matrix gives the dual problem

$$\begin{pmatrix} \begin{pmatrix} -G_1^T G_2^T G_5^T & 0 \end{pmatrix} & G_1^T \\ \begin{pmatrix} G_3^T G_5^T & G_4^T \end{pmatrix} & 0 \end{pmatrix}$$

This is exactly the estimation problem of Figure 3, with z as output to be minimized, y as measured output, w_1, w_2 as exogenous inputs, $\widehat{u}_f$ as "control input", and $K_E = K_{FF}^T$. Alternatively, we may use the block diagram transformations in Section 3 on Figure 2 directly, which gives Figure 4.

Thus, the model $G_1$of the signal d corresponds to the weighting function of the estimation problem. The systems G_3 correspond to each other. The control weighting G_4 corresponds to the measurement noise, while the output weighting G_5 corresponds to the signal model of the estimation problem. As in other dual problems, minimum variance control ($G_4 = 0$) corresponds to estimation with noise-free measurements. In continuous time, both are singular problems. Some of the consequences of these correspondences will be discussed in Section 8.

7. An Illustration: Polynomial Solutions to Discrete-Time Input Estimation and Feedforward Control Problems

Input Estimation/Deconvolution

In a discrete-time estimation problem described by Figure 3, let the noise-corrupted measurement vector $y(t)$, of dimension p, and the input $u(t)$, of dimension s, be given by

$$\begin{aligned} y(t) &= A^{-1}Bu(t) + N^{-1}Mw_2(t) \\ u(t) &= D^{-1}Cw_1(t) \end{aligned} \tag{3}$$

Here, (A, B, N, M, D, C) are polynomial matrices in the backward shift operator q^{-1}, of dimension $p|p$, $p|s$, $p|p$, $p|r$, $s|s$ and $s|k$, respectively. The noise signals $\{w_1(t)\}$ and $\{w_2(t)\}$ are assumed white and stationary, with zero means and covariance matrices normalized to unit matrices. An optimal linear estimator

$$\widehat{u}_f(t) = K_E(q^{-1})y(t) \tag{4}$$

of a filtered version of the input, of dimension l

$$u_f(t) = T^{-1}Su(t-m) \tag{5}$$

is sought, such that the frequency weighted quadratic criterion

$$J = \text{tr } E\{z(t)z^T(t)\}; \quad z(t) = U^{-1}V(\widehat{u}_f(t) - u_f(t)) \tag{6}$$

is minimized. In (5) and (6), T, S, U, V are polynomial matrices of dimensions $l|l$, $l|s$, $l|l$ and $l|l$.

Comparing with Figures 3 or 4, we have

$$\begin{aligned} G_1^T &= U^{-1}V \\ G_2^T &= T^{-1}Sq^{-m} \\ G_3^T &= A^{-1}B \\ G_4^T &= N^{-1}M \\ G_5^T &= D^{-1}C \end{aligned}$$

We make the following two assumptions:

Assumption 1. The polynomial matrices $A(q^{-1}), N(q^{-1}), D(q^{-1}), T(q^{-1})$, $U(q^{-1})$, and $V(q^{-1})$ all have stable determinants and non-singular leading coefficient matrices. (Thus, they have stable and causal inverses.)

Assumption 2. The spectral density matrix $\Phi_y(e^{i\omega})$ of the measurement $y(t)$ is nonsingular for all ω.

Define the following coprime factorizations

$$BD^{-1} = \tilde{D}^{-1}\tilde{B} \tag{7}$$
$$\tilde{D}AN^{-1} = \tilde{N}^{-1}\tilde{A} \tag{8}$$
$$VT^{-1}SD^{-1} = \tilde{T}^{-1}\tilde{S} \tag{9}$$

Stability of det T and det D and coprimeness of $\tilde{T}^{-1}\tilde{S}$ implies that det $\tilde{T}$ will be stable. Causality of T^{-1} and D^{-1} implies that $\tilde{T}^{-1}$ will be causal. Let P_* denote the conjugate transpose $P^T(q)$ of a polynomial matrix $P(q^{-1})$. Define the following left polynomial spectral factorization,

$$\beta\beta_* = \tilde{N}\tilde{B}CC_*\tilde{B}_*\tilde{N}_* + \tilde{A}MM_*\tilde{A}_* \tag{10}$$

Under assumption 2, (10) will always have a solution $\beta(q^{-1})$, of dimension $p|p$, with stable determinant and nonsingular leading coefficient matrix, see e.g. [Anderson and Moore, 1979], [Kučera, 1979], [Kučera, 1980], [Ježek and Kučera, 1985]. [2] The following result now holds:

THEOREM 2—The Wiener Estimator
Let the system and input model be described by (3), see Figure 3. Introduce the coprime factorizations (7)–(9) and the spectral factorization (10). Under assumptions 1 and 2, a stable and causal H_2-optimal estimator (4), minimizing (6), is then given by

$$\hat{u}_f(t) = V^{-1}\tilde{T}^{-1}Q_E\beta^{-1}\tilde{N}\tilde{D}A\, y(t) \tag{11}$$

where $Q_E(q^{-1})$, together with $L_*(q)$, both of dimension $l|p$, are given by the unique solution to the bilateral Diophantine equation

$$q^{-m}\tilde{S}CC_*\tilde{B}_*\tilde{N}_* = Q_E\beta_* + q\tilde{T}UL_* \tag{12}$$

2 Two conditions on the polynomial matrices appearing in (10) do, together, guarantee that A2 holds. (1): The matrix $[\tilde{N}\tilde{B}C \quad \tilde{A}M]$ should have full (normal) row rank p and (2): The greatest common left divisor of $\tilde{N}\tilde{B}C$ and $\tilde{A}M$ should have nonzero determinant on $|z| = 1$. While (1) is a condition for existence of spectral factors, (2) provides a spectral factor $\beta(z^{-1})$ such that det $\beta(z^{-1}) \neq 0$ on $|z| = 1$.

Proof: In [Ahlén and Sternad, 1991], this result is derived for the case $U = V = I$. Only small modifications, leading to the equations (9), (11), and (12) above, while (7), (8), (10) remain unchanged, are required to extend that result to filtered criteria $z(t) = U^{-1}V(\hat{u}_f(t) - u_f(t))$. Note that since $\tilde{T}^{-1}$ and β^{-1} are stable and causal, (11) will be stable and causal. □

For a more detailed discussion of Wiener filter design using polynomial equations, see [Ahlén and Sternad, 1991].

Feedforward Control

Let us, in the same way, express a discrete time feedforward structure, described by Figure 2, by right matrix fraction descriptions:

$$\begin{aligned}
\text{Disturbance/reference dynamics: } G_1 &= G_c H_c^{-1} \\
\text{Disturbance transfer/desired response model: } G_2 &= q^{-m} D_c F_c^{-1} \\
\text{System: } G_3 &= B_c A_c^{-1} \\
\text{Input weighting function: } G_4 &= W_c N_c^{-1} \\
\text{Output weighting function: } G_5 &= V_c U_c^{-1}
\end{aligned} \tag{13}$$

Assume A_c, N_c, U_c, F_c, H_c and G_c to have stable determinants and nonsingular leading coefficient matrices. Introduce the coprime factorizations

$$U_c^{-1} B_c = \tilde{B}_c \tilde{U}_c^{-1} \tag{14}$$

$$N_c^{-1} A_c \tilde{U}_c = \tilde{A}_c \tilde{N}_c^{-1} \tag{15}$$

$$U_c^{-1} D_c F_c^{-1} G_c = \tilde{G}_2 \tilde{F}_2^{-1} \tag{16}$$

Stability of det U_c and det F_c and coprimeness of $\tilde{G}_2 \tilde{F}_2^{-1}$ implies that det $\tilde{F}_2$ will be stable. Causality of U_c^{-1} and F_c^{-1} implies that $\tilde{F}_2^{-1}$ will be causal. Define the following criterion-related right polynomial spectral factorization

$$\beta_{c*}\beta_c = \tilde{N}_{c*}\tilde{B}_{c*}V_{c*}V_c\tilde{B}_c\tilde{N}_c + \tilde{A}_{c*}W_{c*}W_c\tilde{A}_c \tag{17}$$

Assume that the right-hand side of (17) is nonsingular on the unit circle. Then, (17) will have a solution β_c with stable and causal inverse.

Now, the polynomial solution to the LQG feedforward design problem can be stated as follows.

THEOREM 3—The LQG Feedforward Regulator
Let the system and weighting functions in Figure 3 be given by the right MFD's (13). Introduce the coprime factorizations (14)–(16) and the spectral factorization (17), nonsingular on $|z| = 1$. Then, a stable and causal H_2-optimal feedforward regulator, minimizing

$$\operatorname{tr} E\left\{(G_5 y(t))(G_5 y(t))^T + (G_4 u(t))(G_4 u(t))^T\right\}$$

is

$$u(t) = A_c U_c \widetilde{N}_c \beta_c^{-1} Q_{FF} \widetilde{F}_2^{-1} G_c^{-1} d(t) \tag{18}$$

where $Q_{FF}(q^{-1})$, together with $L_{1*}(q)$ are given by the unique solution to the bilateral Diophantine equation

$$q^{-m} N_{c*} \widetilde{B}_{c*} V_{c*} V_c \widetilde{G}_2 = \beta_{c*} Q_{FF} + q L_{1*} H_c \widetilde{F}_2 \tag{19}$$

Proof: The solution to this problem will be derived by duality with (3)–(6). Use of the duality relations give (with P^{-T} denoting transpose and inverse of P)

$$\begin{array}{rcl}
G_1^T = H_c^{-T} G_c^T & \longleftrightarrow & U^{-1}V \\
G_2^T = F_c^{-T} D_c^T q^{-m} & \longleftrightarrow & T^{-1}Sq^{-m} \\
G_3^T = A_c^{-T} B_c^T & \longleftrightarrow & A^{-1}B \\
G_4^T = N_c^{-T} W_c^T & \longleftrightarrow & N^{-1}M \\
G_5^T = U_c^{-T} V_c^T & \longleftrightarrow & D^{-1}C
\end{array}$$

By making the substitutions above, and transposing the equations (7)–(12), design equations are obtained for the LQG feedforward regulator. Substitution into (7)–(9) gives

$$\begin{aligned}
B_c^T U_c^{-T} &= \widetilde{D}^{-1}\widetilde{B} \\
\widetilde{D} A_c^T N_c^{-T} &= \widetilde{N}^{-1}\widetilde{A} \\
G_c^T F_c^{-T} D_c^T U_c^{-T} &= \widetilde{T}^{-1}\widetilde{S}
\end{aligned}$$

By transposing these factorizations, and defining $\widetilde{B}_c \triangleq \widetilde{B}^T$, $\widetilde{U}_c \triangleq \widetilde{D}^T$, $\widetilde{A}_c \triangleq \widetilde{A}^T$, $\widetilde{N}_c \triangleq \widetilde{N}^T$, $\widetilde{G}_2 \triangleq \widetilde{S}^T$, $\widetilde{F}_2 \triangleq \widetilde{T}^T$, we obtain (14)–(16).

Use of the substitutions and of (14),(15) in the spectral factorization (10) gives

$$\beta\beta_* = \widetilde{N}_c^T \widetilde{B}_c^T V_c^T V_{c*}^T \widetilde{B}_{c*}^T \widetilde{N}_{c*}^T + \widetilde{A}_c^T W_c^T W_{c*}^T \widetilde{A}_{c*}^T$$

By transposing and defining $\beta_c \triangleq \beta^T$, we obtain the criterion-related right spectral factorization (17).

The feedforward filter (18) is obtained by substitution into (11) and transposition:

$$u(t) = (G_c^{-T}\widetilde{F}_2^{-T}Q_E\beta^{-1}\widetilde{N}_c^T U_c^T A_c^T)^T d(t)$$

By defining $Q_{FF} \triangleq Q_E^T$, (18) is obtained. The filter will, of course, be stable and causal, since $\widetilde{F}_2^{-1}$ and β^{-1} are stable and causal. Substitution into the Diophantine equation (12) gives

$$q^{-m}\widetilde{G}_2^T V_c^T V_{c*}^T \widetilde{B}_{c*}^T \widetilde{N}_{c*}^T = Q_E\beta_{c*}^T + q\widetilde{F}_2^T H_c^T L_* \tag{20}$$

By transposing this equation and using $Q_{FF} = Q_E^T$ and $L_{1*} \triangleq L_*^T$, we obtain the Diophantine equation (19). □

The minimal criterion value will, of course, be equal in the two dual problems.

The feedforward design equations (14)–(19) constitute an extension of earlier known results. In [Sternad and Ahlén, 1992], only the special case of polynomial penalties in the criterion, $N_c = I$ and $U_c = I$, were considered. (The two coprime factorizations (14), (15) are then superfluous, with $\widetilde{B}_c = B_c$, $\widetilde{A}_c = A_c$.) [Hunt and Šebek, 1989] consider a different combined feedback and feedforward problem, without dynamic cost weighting.

Remark. The extension to dynamic cost weighting clearly shows how the weights influence the controller. This can help the user in the choice of weights. It should however be noted that another possibility is to include the weights in an extended system description, see e.g. [Hunt, 1989].

Remark. In both the estimation and the feedforward control problems, one can derive a second Diophantine equation. For unstable systems, it would sometimes have to be used in combination with (12) and (19) to determine the filter uniquely. This is never necessary for systems with poles on or inside the stability limit. Since strictly unstable systems are of little relevance in the open-loop design problems considered here, we have not introduced this second equation, which would just complicate the solution. However, the duality relations do of course hold for that equation as well.

8. Concluding Discussion

It has been demonstrated that *feedforward control* problems are dual to a special type of estimation problems: *deconvolution* or *input estimation* problems, where the input to a dynamic system G_3^T is sought. (Output or state estimation problems, without transducer dynamics G_3^T, would correspond to rather trivial feedforward control problems, with no dynamics between control input u and the output y.)

In [Sternad and Ahlén, 1988], several close correspondences were noted between scalar Wiener-input estimation and LQG feedforward control problems. These correspondences could not be interpreted as duality relations. The reason for this can now be seen in the too restrictive problem formulations used in [Sternad and Ahlén, 1988]: $G_4 = I$ and $G_5 = I$ in the control problem and $G_2^T = I$ and $G_1^T = I$ in the input estimation problem. With duality established between the more general problems discussed in this paper, the correspondences between (MIMO) LQG feedforward controllers and Wiener input estimators can now be placed into their correct perspective. Some design guidelines also follow:

- When the system G_3 is of low-pass-type, both feedforward controllers and Wiener input estimators tend to be high-pass. In the control problem, an input penalty G_4, penalizing high-frequency components of the input, will reduce the high frequency gain of the controller K_{FF}. The introduction of measurement noise $G_4^T w_2$ with significant high-frequency content has the same effect on the estimator K_E. For scalar problems, a resonance peak in G_4 introduces a notch in both K_{FF} and K_E. Note the presence of $\widetilde{N}$ in (11) and of $\widetilde{N}_c$ in (18). They equal N and N_c, respectively in scalar problems.
- Use of a *positive* smoothing lag m in the estimation problem (with $q^{-m}I$ in G_2^T) corresponds to a delay in the disturbance path ($q^{-m}I$ in G_2) of a regulator problem. A larger smoothing lag/delay will improve the filtering/control performance.
- A *negative* m (prediction) would correspond to a noncausal block G_2, containing $q^{-m}I$, in the control problem of Figure 2. This is equivalent to forcing a delay $q^m I$, i. e. a computational delay, into the controller. (If $G_2 = q^{-m}IG_2'$, the block $q^{-m}I$ can be moved up to G_1 in Figure 2, while its inverse, $q^m I$, is included in the controller.) With everything else being equal, the achievable performance would deteriorate as the prediction horizon/computation delay $-m$ increases.
- There are two ways of reducing the static feedforward control error: either one of G_1 or the output penalty G_5 should have high gain at low frequencies. Likewise, there are two ways to obtain an estimator

with small error at low frequencies: either one of the frequency weighting G_1^T or the input model G_5^T should have high gain at low frequencies.

- The polynomial solutions to the two dual discrete-time problems, discussed in Section 7, involve a spectral factorization, a Diophantine equation and up to three coprime factorizations. (The same is true for the solutions to the corresponding continuous-time problems.) The transpositions used in going from one problem to the other explains why a *left* spectral factorization (10), where β appears to the left, is involved in the filtering solution, while a *right* spectral factorization (17) appears in the control solution. Also, note that while it is natural to start from a left MFD model (3) in the estimation problem, the dual control problem is expressed in right MFD form (13). See also [Kučera, 1991], where use is made of duality relations to investigate several other types of LQ problems, using the polynomial equations approach.

We have compared properties of the LQ (or H_2) solutions to the dual control and signal processing problems above. Very similar remarks apply to all criteria for which the duality holds, i. e. all norms for which $||G^T|| = ||G||$. In particular, this applies to H_∞-optimal solutions.

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PAPER V

On the Notion of Strong Stabilizability

Per Hagander and **Bo Bernhardsson**

Abstract: A system is called strongly stabilizable (SS) if it can be stabilized using a stable controller. Systems that are non-SS are a real world problem. However the concept is not strong enough to guarantee that SS-systems can be controlled well by stable controllers. Especially for multivariable and sampled systems, strange controllers might be required. The weakness of the SS-concept is illustrated by several examples, and it is shown how unstable controllers might be required for good closed-loop performance, even if they are not required for stability.

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PER HAGANDER AND BO BERNHARDSSON

***Abstract*—A system is called strongly stabilizable (SS) if it can be stabilized using a stable controller. Systems that are non-SS are a real world problem. However the concept is not strong enough to guarantee that SS-systems can be controlled well by stable controllers. Especially for multivariable and sampled systems, strange controllers might be required. The weakness of the SS-concept is illustrated by several examples, and it is shown how unstable controllers might be required for good closed-loop performance, even if they are not required for stability.**

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I. Introduction

The structure of controllers that stabilize a linear system is a topic that recently has received renewed interest. Observer state feedback can be used to design a controller of the same order as the system, but the designs might very well lead to unstable controllers.

The implementation of an unstable controller should be done with care and a well-designed safety net. One way is to use a stable observer fed with actual input in addition to the measured output. In case of actuator saturation, no windup occurs if the observer sees the saturated input signal. Similarly, the observer should provide a test against sensor failure.

Stable controllers are easier to work with, although integrators are standard elements in most process control designs. As demonstrated in [6], nonlinearities like friction might drastically deteriorate the behavior when an unstable regulator is used to achieve a high-performance servo.

If the plant is unstable, you need a safety net and a special logic for startup, even in case of stable controllers. Of course, an unstable controller might add a little to their complexity. Note further that an unstable system with a saturated actuator can be made stable only in a limited region of the state space.

II. Criterion for Strong Stabilizability

For single-input single-output systems, it has long been known, e.g., [5], that certain plants $P(s)$ require unstable controllers $C(s)$ for stabilization. An odd number of unstable poles p_i of $P(s)$ between two unstable zeros z_1 and z_2, i.e.,

$$0 \leq z_1 < p_i < z_2 \leq \infty \tag{1}$$

makes a $P(s)$ require an unstable $C(s)$. Note that $z_2 = \infty$ means that $P(s)$ is strictly proper. An equivalent criterion is that for $P(s) = n(s)/d(s)$, there exist two real zeros $z_1 \geq 0$ and $z_2 > 0$ such that

$$\operatorname{sign} d(z_1) \neq \operatorname{sign} d(z_2). \tag{2}$$

A simple root-locus argument shows that the root locus stays on the unstable real axis unless the controller $C(s)$ adds one unstable real pole between the two zeros.

It is more difficult to prove that all other systems are actually strongly stabilizable (SS), i.e., can be stabilized using a stable controller. The proofs, e.g., [8] or [7], use interpolation theory, and they do not give an explicit lower bound on the order of such a stable stabilizing controller.

III. Physical Examples

There is a significant class of real world systems that requires unstable controllers. Two simple examples are given to provide some physical insight.

Example 1: Regard a mixed culture of growing cells. The total number of cells is measured, and cells are harvested by removing cells in pairs, one slow growing cell attached to each fast growing cell:

$$\dot{x}_1 = ax_1 - u$$
$$\dot{x}_2 = bx_2 - u$$
$$y = x_1 + x_2.$$

The transfer function for $a = 1$ and $b = 3$ would be

$$P(s) = \frac{2(2-s)}{(s-1)(s-3)}$$

and any stabilizing harvest policy would have to be an unstable dynamical system. □

Example 2: Horizontal acceleration u of the pivot point is used to stabilize an inverted pendulum based on measurement of the angular velocity x_2. The linearized model would be

$$\dot{x}_1 = x_2$$
$$\ddot{x}_2 = \omega^2 x_1 + u$$
$$y = kx_2$$

giving the transfer function $P(s) = ks/(s-\omega)(s+\omega)$. An unstable controller is required also in this case. □

A Class of Non-SS Systems: Both of these examples belong to a quite general class of systems that are not strongly stabilizable. Consider scalar transfer functions that can be written as

$$G(s) = \sum_{i=1}^{n} \frac{A_i}{s - a_i} \tag{3}$$

with real distinct poles a_i and real positive residues A_i. Such systems have a single real zero between each adjacent pair of real poles, and they are not strongly stabilizable if they have any unstable finite zero.

IV. The Multiinput Multioutput Case

Almost any multivariable system would be strongly stabilizable. For a system $P(s)$ not to be strongly stabilizable, it has to have unstable *blocking* zeros z_1 and z_2 and an odd number of poles p_i that fulfill (1). A blocking zero z zeros *all* the elements of the transfer-function matrix, i.e., $P_{ij}(z) = 0$ for all i and j.

One could argue that this shows the strength of interacting controllers for multivariable plants. An opposite viewpoint would be that it shows a weakness in the SS concept for multivariable systems. The intuition available from the SISO case has to be used with great care. One might assume, that for, e.g., a two-by-two system P with no interaction whatsoever, it would be sufficient to look at the individual links. This is not true unless it is also required that the controller C is restricted to be diagonal.

Surprisingly enough, it is thus, for instance, possible to find a stable controller to stabilize the cell cultures in Example 1 by utilizing an arbitrary, totally unrelated part of the plant, like a loop for the temperature control of the room. A stable interacting controller can be designed by utilizing the temperature loop to form an unstable net compensator for the first loop, still stabilizing the overall system. Of course, such a controller would require exactly the same precautions as an unstable compensator. One might argue that whether the temperature dynamics is part of the plant or part of the compensator should be irrelevant, but it is not.

Example 3: Extend Example 2 with $k = 2$ and $\omega = 1$ to

$$y = Pu, \qquad P(s) = \begin{pmatrix} \frac{1}{s-1} + \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s+1} \end{pmatrix}.$$

A simple proportional controller

$$u = v + Cy, \qquad C(s) = \begin{pmatrix} -2 \\ 2 \end{pmatrix} (1 \quad 1)$$

would give a stable closed-loop system

$$y = (I - PC)^{-1} Pv$$

with all three poles in $s = -1$. The resulting controller around P_{11} would, however, be unstable:

$$u_1 = v_1 + (C_{11}y_1 + C_{12}P_{22}(1 - C_{22}P_{22})^{-1}(C_{21}y_1 + v_2))$$
$$= -2\frac{s+1}{s-1}y_1 + v_1 - \frac{2}{s-1}v_2$$

but the overall system would certainly be internally stable. A block diagram of the system is given in Fig. 1. □

V. Sampled Systems

A corresponding parity property of the interlacing of real poles and zeros is required for strong stabilization of discrete-time systems [7]. The basic properties of zeros obtained when sampling were given in [1]. In [3], it was shown that a sampled system obtained from an SS continuous-time system would be SS for fast enough sampling. It was also shown that the sampled system could be SS for continuous-time systems that

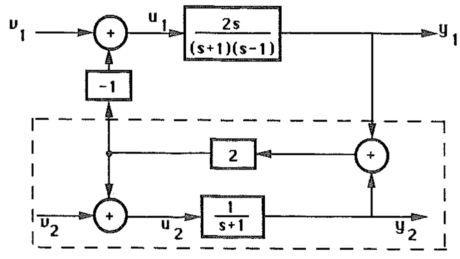

Fig. 1. Stabilizing controller for Example 3.

are non-SS, and most remarkably, that this could occur for infinitely fast sampling in special cases.

On the other hand, for scalar transfer functions as in (3), zero-order hold sampling maintains the pole-zero interlacing:

$$H(z) = \sum_{i=1}^{n} \frac{A_i \beta_i}{z - \alpha_i}, \qquad \alpha_i = e^{a_i T}, \quad \beta_i = \begin{cases} (\alpha_i - 1)/a_i, & a_i \neq 0 \\ T, & a_i = 0 \end{cases} \tag{4}$$

and positive residues are maintained for any sampling period T both for stable and unstable poles a_i. This simple result seems to have been found and elaborated by [4]. So here is a common class of systems for which the property of not being strongly stabilizable is preserved for any sampling period.

In the multivariable case, it would be extremely rare that an unstable blocking zero would be maintained after sampling.

Example 4: Regard, for instance,

$$P(s) = \begin{pmatrix} \dfrac{s-1}{(s+1)(s-2)} & 0 \\ 0 & \dfrac{s-1}{(s+1.0001)(s-2)} \end{pmatrix}$$

with $P(1) = 0$. The corresponding sampled system would have no finite blocking zero for any sampling interval, and it would thus be strongly stabilizable. □

VI. Concluding Remarks

Systems that are non-SS are a real-world problem. Systems in the class described by (3) with some $z_i > 0$ are actually non-SS, even if slightly perturbed. However, the concept is not strong enough to guarantee that SS systems can be controlled well by stable controllers.

Strong stabilizability is in the multivariable case a concept to be used with greatest care, and if you insist on stable controllers, you may engage a whole plant in the control of a single isolated loop. You may tend to absorb dynamic excursions after a disturbance in innocent parts of the plant instead of in your controller of the loop.

The single-input single-output system

$$P(s) = \frac{(s-1)^2 + \epsilon}{(s+1)(s+2)(s-2)} \tag{5}$$

is strongly stabilizable unless $\epsilon \leq 0$, and the stable controllers possible for small $\epsilon > 0$ are drastically different from unstable controllers required for $\epsilon \leq 0$. The robustness of the loop would be quite bad for such a stable controller. It was also noted in [8] that the stable controllers required for many SS systems were quite sensitive to the controller parameters.

Similarly, when the system (5) with $\epsilon = 0$ is sampled, it becomes SS for any sampling interval T. The optimal H_∞-norm from an output disturbance to the output would, however, increase dramatically if you insist on stable controllers. Using the method in [2] and Macsyma, it can be shown that the optimal loss increases from $V = 9$ to $V = 3 \exp(\pi\sqrt{2}(1/T + 1) - 4/3) \to \infty$ for small T. This quantifies how bad the performance of the best stable controller would be, compared to what is achievable using an unstable controller.

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1. Addendum – Calculations Exposed

In the article "On the Notion of Strong Stabilizability" some comments at the end of the article deserve a more thorough explanation. The idea is to quantify the increase in H_∞-norm of the sensitivity function, that is the transfer function from output disturbance w to output z, see Figure 1, required when insisting on using a stable controller $K(s)$. We will see that the sampled system can be controlled quite well with unstable controllers, but if one requires the controller to be stable the system will for small h have arbitrarily large H_∞-loss. This is another example of the practical weakness of the SS-concept.

The sampled version of the system

$$G(s) = \frac{(s-1)^2}{(s+1)(s+2)(s-2)}$$

is

$$H(z) = -4/3\,\frac{1-\exp(-h)}{z-\exp(-h)} + 9/8\,\frac{1-\exp(-2h)}{z-\exp(-2h)} - 1/24\,\frac{1-\exp(2h)}{z-\exp(2h)}$$

with pole-zero digram as shown in Figure 2. For any sampling period $h > 0$ the two zeros are off the positive real axis. The system is therefore strongly stabilizable, i.e. can be stabilized with a stable controller. We will now study the two problems

$$J_1 = \min_K \|T_{zw}\|_\infty$$

and

$$J_2 = \min_{K\ stable} \|T_{zw}\|_\infty$$

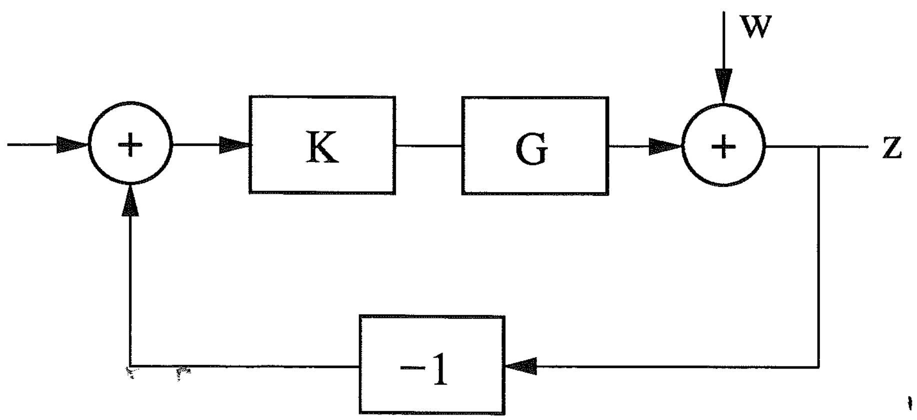

Figure 1. Block diagram for sensitivity minimization. The influence of disturbance w should be minimized on output z.

Optimal Non-Stable H_∞-Controller

The sensitivity minimization problem mentioned above is of one-block type and an explicit solution can hence be obtained using Nevanlinna-Pick interpolation. The result in continuous time is that proper controllers close to

$$K(s) = -10/9\frac{(s+1)(s+2)}{(s-1/2)}$$

give a stable closed loop with transfer function close to

$$T_{zw} = -9\frac{(s-2)(s-1/2)}{(s+2)(s+1/2)}$$

The minimal loss is hence 9. For the sampled system it is easy to see that the minimal loss for small h is

$$J_1 = 9 + 18h + O(h^2)$$

There is hence no dramatic problem controlling the system for small sampling periods if one is allowed to use unstable controllers. We will not give any details of these calculations. The amplitude margin of a resulting controller is 1.1 and the phase margin is only 5 degrees. The design will also be a little noise sensitive. Such restrictions on achievable performance are however inevitable. One can not expect better for an unstable and non-minimum phase process.

Optimal stable H_∞-controller

To calculate the optimal H_∞-performance with a stable controller we briefly review the results of [Boyd, 1987]. Notice when reading [Boyd, 1987] that the stability domain is defined to be the exterior of the unit circle. We will use $D = \{|z| < 1\}$ as the stability domain in what follows.

If there are no zeros outside D, including infinity, the optimal sensitivity is $J_1 = J_2 = 1$, and it can approximately be achieved using high gain feedback. Let otherwise $z_1, \ldots, z_k$, denote the zeros outside D, including infinity if the system is strictly proper. Assume for simplicity that the zeros are simple. Introduce the Blaschke product of unstable poles, i. e.

$$B_p(\lambda) = \prod_{1\le i\le n} \frac{\lambda - p_i}{\lambda \overline{p}_i - 1}\frac{p_i}{|p_i|}$$

The following theorem is proved in [Boyd, 1987]:

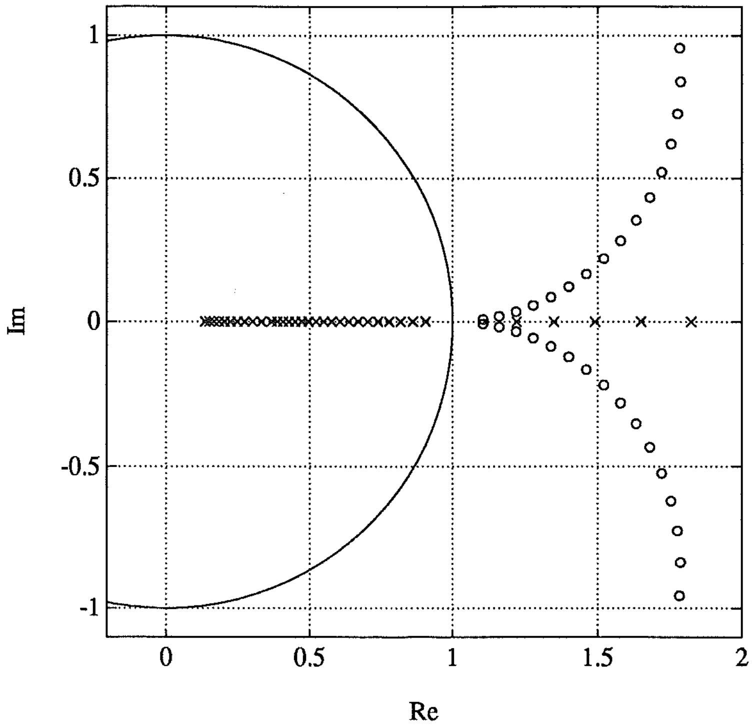

Figure 2. Pole-zero diagram of the sampled system $H(z)$ for h varying from 0.1 to 1. Notice that the system is SS for all sampling periods $h > 0$.

where in both cases the controller $K(s)$ should be proper and give a stable closed loop system. The difference $J_2 - J_1$ indicates the extra cost associated with insisting on a stable controller. Notice that the sensitivity is evaluated for the discrete time systems. Another possibility is to view the continuous time system and discrete time controller as a hybrid system and calculate the continuous time sensitivity function. This however leads to a much more difficult problem, and the difference is probably very small for small sampling periods h.

There are a number of recent papers discussing both H_2 and H_∞ optimization with side condition on stable controllers. In [Boyd, 1987] the scalar weighted sensitivity H_∞-problem is solved using classical complex theory arguments, i. e. Nevanlinna-Pick interpolation and complex logarithms of H_∞-units, see also [Ganesh and Pearson, 1986]. In [Ganesh and Pearson, 1989] the optimal H_2-problem with stable controllers is rewritten using the Q-parametrization to a general, nonconvex optimization problem. No analytical solution is to the author's knowledge known for any H_2-problem. Upper bounds on the controller degree required for SS are given in, e. g., [Dorato *et al.*, 1989].

THEOREM 1
The optimal H_∞-norm is given by $J_2 = \exp(-\delta)$ where δ is the smallest solution of the eigenvalue problem

$$\det(T^*R + RT - 2\delta R) = 0 \tag{1}$$

where

$$T = \operatorname{diag}\left(g_1 + 2\pi j n_1 \quad \ldots \quad g_k + 2\pi j n_k \right)$$

with $g_i = -\log B_p^{-1}(z_i)$ (the principle branch of logarithms) and n_i are positive integers satisfying $n_i = -n_j$ whenever $z_i = \overline{z}_j$ and R is the positive definite Hermitian matrix given by

$$R_{ij} = \frac{1}{1 - \overline{z}_i^{-1} z_j^{-1}}$$

The numbers n_i should be chosen in such a way that the smallest solution δ is maximized. The search over n_i:s can be reduced to a finite set.

□

The Example

For the system $H(z)$ above the zeros are for small sampling periods given by

$$\begin{aligned}
z_1(h) &= 1 + h + \left(1 - i\sqrt{2}\right) h^2/2 + O\left(h^3\right) \\
z_2(h) &= 1 + h + \left(1 + i\sqrt{2}\right) h^2/2 + O\left(h^3\right) \\
z_3(h) &= \infty
\end{aligned}$$

and using Maple we have

$$R = \begin{pmatrix} \frac{1}{2h} + 1/2 + \frac{13}{48} h & \frac{1}{2h} + 1/2 - \frac{i\sqrt{2}}{4} + \frac{h}{48} + & 1 \\ \frac{1}{2h} + 1/2 + \frac{i\sqrt{2}}{4} + \frac{h}{48} & \frac{1}{2h} + 1/2 + \frac{13}{48} h & 1 \\ 1 & 1 & 1 \end{pmatrix} + O\left(h^2\right)$$

It is easy to check that $n_1 = n_2 = 0$ gives the optimal solution. We then have, up to $O\left(h^2\right)$-terms, that

$$T = \begin{pmatrix} -\ln(3) - \pi i + 2\sqrt{2}/3hi & 0 & 0 \\ 0 & -\ln(3) + \pi i - 2\sqrt{2}/3hi & 0 \\ 0 & 0 & -2h \end{pmatrix}$$

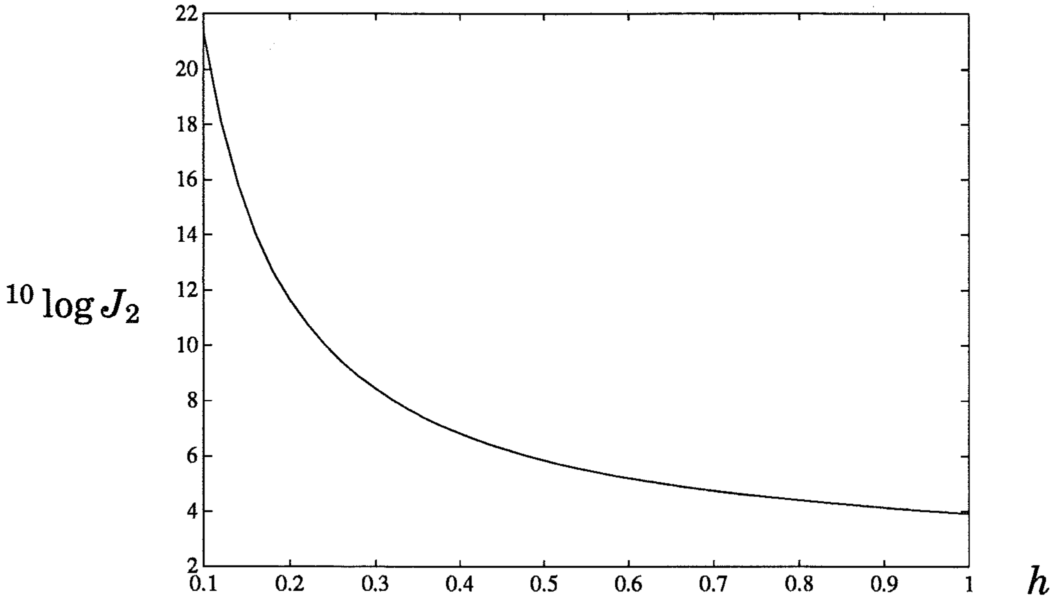

Figure 3. Optimal H_∞-norm of the sensitivity, J_2, as a function of sampling period, when requiring a stable controller. Notice the logarithmic scale. Compare this plot with the value $J_1 = 9 + 18h + O(h^2)$ achievable with unstable controllers, this value is below the lower value of the figure. There is a large extra cost associated with using stable controllers for this problem.

There are three solutions to the eigenvalue equation (1), one with $\delta > 0$, one with $\delta = O(h)$, and the interesting one:

$$\delta = -\sqrt{2}\pi/h + -\sqrt{2}\pi + 4/3 - \ln(3) + O(h)$$

Therefore the H_∞-loss is for small sampling periods approximately given by

$$J_2(h) = \exp(-\delta) = 3\exp(\sqrt{2}\pi(h^{-1} + 1) - 4/3) \to \infty \ \ as \ \ h \to 0$$

This function is plotted in Figure 3.

According to the rule of thumb in [Åström and Wittenmark, 1990], a reasonable sampling period for this process is $h \in [0.1, 0.3]$ Notice the very large extra cost associated with requiring a stable controller even for reasonable sampling periods. For $h = 0.5$ we e.g. have $J_2 \approx 10^6$. Using a standard approximation argument we can also conclude that rational stable controllers exist giving an H_∞-norm arbitrarily close to the optimum. The order of such controllers is often quite large. If we were to put a restriction also on controller order, say twice the plant order, then it is probable that no stable stabilizing controller exists for reasonable sampling periods.

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PAPER VI

Robust Performance Optimization of Open Loop Type Problems Using Models From Standard Identification

Bo Bernhardsson

Abstract: This paper discusses the problem of finding the controller that optimizes the expected H_2-norm for an uncertain system. The approach is connected to the so called stochastic embedding approach. A closed form solution is given using a minimum of calculations for a class of problems including interesting signal processing applications such as feedforward design, channel equalization, noise cancellation and signal filtering. The method uses covariance information on model uncertainty and can therefore be used together with standard identification methods. By using the probability distribution of model error we avoid the conservativeness related to designing for worst cases. We then obtain robust designs with soft bounds. The paper gives a unification of many similar results. It is shown how the optimal controller can be found by rewriting the problem as a standard H_2-problem for an extended system. The solution can hence be obtained using standard methods and software. The paper uses restrictions on where uncertain parameters enter into the system. Such restrictions are inevitable if hard bounds on parameters are to be avoided. The method has direct applications in adaptive signal processing and adaptive feedforward control.

1. Introduction and Motivation

No design method can tackle all the issues that have to be studied when making a real world design. Typically a synthesis method will focus on one or two issues and leave others to the designer's common sense as often hidden conditions.

The goal of robust controller design is to achieve robust performance, that is good performance in face of plant uncertainty. This is a much harder problem than the robust stability problem which has been studied intensively during the last decade. One difference is that it requires good engineering intuition to define what is meant by good performance, while robust stability is closer to pure mathematics. Interesting historical remarks on the dominating focus on stability issues are given in [Boyd and Barratt, 1991]. Most control systems can be rendered useless for much smaller system variations than are needed for rendering the system unstable. This is of course well realized by people working in the robust stability area and it is a major goal of recent research to find results for robust performance.

The formulation of uncertainty models is a fundamental issue. The most common assumption is hard bounds on the uncertainty, and design for worst cases. Since standard identification methods do not give hard bounds, new identification methods are being developed, e.g. [Norton, 1987b],[Norton, 1987a], [Kosut, 1988] or [Wahlberg and Ljung, 1991]. It is hard to develop such methods that are not too conservative, that is, give too large upper bounds on the model uncertainty. Too conservative estimates will lead to conservative regulator designs resulting in unnecessarily low performance. A natural approach is instead to try to use information obtained from standard identification and to take the likelihood of different parameter variations into account in the controller design. The controller is then chosen taking the probability of parameter variations into account, instead of designing for very rare worst cases. This has been suggested several times before. An interesting recent article solves the case of scalar robust filtering and feedforward design, see [Sternad and Ahlén, 1993].

The present paper will describe results for a certain general class of problems, where we put restrictions on how uncertain elements enter, see Assumptions 1 and 2 below. We then present a short and instructive algorithm for obtaining the optimal controller. If the assumptions on how the uncertain parameters enter are not met, we strongly believe that new identification methods giving hard bounds on parameters and/or new definitions of closed loop performance have to be used. The current paper should be seen as describing a class of problems where standard identification methods suffice for designing robust performance controllers.

2. Design Methods For Robust Performance

Several approaches have been suggested before for obtaining robust performance controllers, see e. g. [Maciejowski, 1989]

QFT

One early design method for obtaining robust performance is the Quantitative Feedback Theory (QFT), developed by Horowitz and others. The idea is to present the uncertain model in a Nichols chart where amplitude and phase uncertainty can be separated. The method also puts emphasis on the closed loop specifications to be met. The engineer is then left more or less on his own to find the controller parameters. A positive effect is that the designer's understanding of the process is increased by this activity. The calculation of uncertainty templates helps the designer in understanding achievable performance. The method is best suited for single input single output (SISO) and is also rather involved when the number of unstable open loop poles varies. It can then be hard to understand which part of a template that should go where. For details on the QFT-method see, e.g., [Horowitz and Sidi, 1972] or [Maciejowski, 1989].

The LQG/LTR Method

The Linear Quadratic Gaussian Loop Transfer Recovery (LQG/LTR) method is a procedure that aims at recovering the amplitude and phase margins of the state feedback controller (LQ). The recovery is made at the expense of a somewhat increased LQG loss. The margins of the LQ controller can not be obtained in general and should not be aimed for. The robustness can however normally be improved using LTR. We refer to the discussions in [Stein and Athans, 1987], [Maciejowski, 1989] or [Bourles and Irving, 1991] for further information and discussion.

The μ Method

Another idea is to write an H_∞-performance problem for an uncertain system as a μ-design with an extended Δ matrix, see Figure 1. The drawback is that there will be structure in the Δ-matrix, so that a full μ-problem has to be solved. Today there is no good software for the synthesis problem. Some preliminary algorithms are given in the so called robust toolbox available in Matlab. It is unclear if there will ever

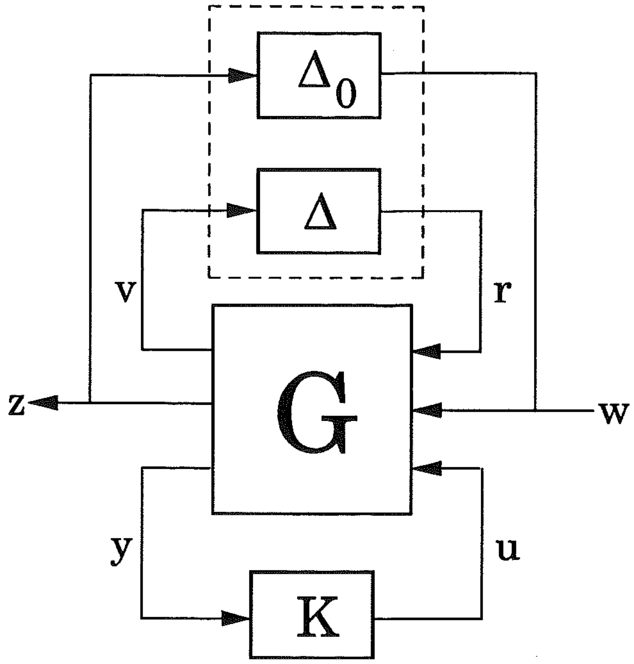

Figure 1. The small gain theorem can be used to write a robust performance problem as a robust stability problem using the μ-framework. Notice that the extended Δ matrix will be structured.

be good software for the problem or if people are going to turn to other problem formulations before that happens. For details on the μ-design see [Maciejowski, 1989], or [Doyle *et al.*, 1982]. The method often leads to high order controllers and model reduction is then applied.

Mixed H_2/H_∞ Control

Since robustness under nonparametric uncertainty can be formulated with H_∞-norms, using the small gain theorem, a natural idea is to try to minimize some performance measure under an H_∞-norm restriction. The case with mixed H_2 and H_∞ norms has been studied in a number of recent papers, see e.g. [Khargonekar and Rotea, 1991], [Rotea and Khargonekar, 1991] and [Ridgely, 1991]. No analytical solutions for both necessity and sufficiency has so far been found. It is known how to write the problem as a convex optimization problem, which might help in obtaining numerical solutions. This is a recent popular area and more evaluation is needed to judge the merits of the method. Notice that this approach only evaluates performance for the nominal design. The problem does not formulate robust performance. Connected work has also been done by [Doyle *et al.*, 1989, Zhou *et al.*, 1990, Zhou *et al.*, 1992, Doyle *et al.*, 1992].

The (Convex) Optimization Approach

Many optimization problems can be formulated where some type of robustness condition is included. Numerical solutions can then often be found, see e.g. [Polak, 1973] and [Polak *et al.*, 1984]. This is made more easy if the problem can be written as a convex optimization problem, [Boyd and Barratt, 1991] since there is then no problem with local minima in the optimization procedure. One drawback is that numerical solutions give too little insight into the problem and how changing the specifications changes the solution. Much numerical work is needed to get such insight. The merits of the method is that many different aspects on controller design can be merged. One can sometimes at least draw useful conclusions on limits of achievable performance. The methods often lead to high order controllers and model reduction is then applied.

Adaptivity

Adaptive control is a good method to design controllers for very uncertain processes. The method leads to nonlinear controllers and a lot of research effort is still being put in to understand their behavior. One trend in adaptive control today is to combine adaptivity with results from robust control theory. One can then use a robust but low performance controller when parameters are uncertain and increase performance when the uncertainty in parameters has been reduced by identification. To combine results in robust control with adaptivity it is crucial that the information needed for controller design matches the information obtained from identification. This is not the case using todays standard methods. Several recent articles have discussed different ways for marrying identification and robust controller design, see e.g. [Goodwin *et al.*, 1990, Goodwin *et al.*, 1989, Goodwin and Salgado, 1989, Krause *et al.*, 1992, Iglesias, 1990, Zhang *et al.*, 1991] just to mention a few.

The Shifted Stability Domain Method

Results from the area of robust stability can sometimes be generalized to so called Ω-stability. This means that closed loop poles should lie in a certain set Ω. In this way some special aspects of performance can be guaranteed, e.g. sufficiently good damping. Notice that the closed loop zeros are considered unimportant using such an approach. The method has restricted value in robust control.

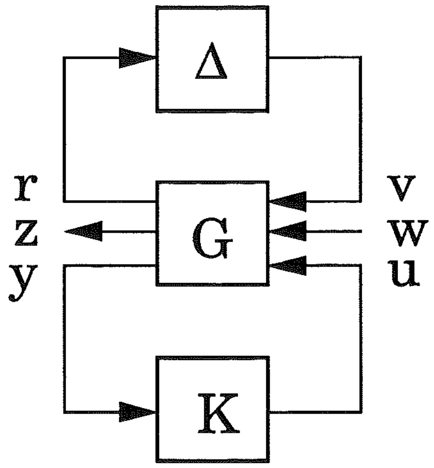

Figure 2. The uncertainty model used in this paper.

3. Problem Formulation

Uncertainty Models

A useful way to capture many robust performance problems is given in Figure 2. Weighting functions, signal models etc. are all included in the extended system matrix G and all uncertain elements have been collected in an upper loop in a matrix Δ. This matrix can be structured in several different ways. The uncertainty can, e.g., be parametric and/or nonparametric. It is common to assume that Δ is block diagonal with

$$\Delta = \mathrm{diag}(\Delta_R, \Delta_C, \Delta_{\mathrm{full}}) \tag{1}$$

For a description of the different blocks see [Doyle *et al.*, 1982]. More general structures also exist.

As a typical example consider a state space system with

$$\begin{aligned} \dot{x} &= (A + \Delta_A)x + (B + \Delta_B)u \\ y &= (C + \Delta_C)x + (D + \Delta_D)u \end{aligned} \tag{2}$$

This can easily be written in the form of Figure 2.

A natural question is how general Figure 2 is. If Δ is represented by a matrix with elements δ_{ij} one easily sees that the closed loop transfer function is rational in the elements δ_{ij}. One can, e.g., not write the system with transfer function $(s + e^{\delta})/(s + \delta)$ as Figure 2. It is conjectured that a state space system can be written as in Figure 2 with Δ being a matrix of independent parameters, if and only if, all elements in the system matrices are rational in the uncertain elements in Δ. Another conjecture is that a transfer function $n(s, \theta)/d(s, \theta)$ depending on a vector of independent parameters θ can be realised as in Figure 2 if and only if n/d is rational in s and θ.

Covariance Information

For an introduction to the area of identification we refer to, e.g., [Ljung, 1987]. What can be said easily is that standard identification often gives covariance information on parameters. This can be represented by a matrix

$$P = E(\tilde{\theta}\tilde{\theta}^T) \tag{3}$$

where E denotes mathematical expectation given observations of the system and where $\tilde{\theta}$ denotes the parameter error.

Bias Error and Variance Error

If the system is under-modeled, there is no parameter θ that gives a perfect model match. Let θ^* denote the best parameter fit in the given model class. Under reasonable assumptions we have as the number of data, N, tends to infinity that

$$\hat{\theta}_N \to \theta^* \tag{4}$$

The error is then normally split into two parts

$$G_T - G(\hat{\theta}_N) = \underbrace{G_T - G(\theta^*)}_{\text{Bias Error}} + \underbrace{G(\theta^*) - G(\hat{\theta}_N)}_{\text{Variance Error}}$$

where G_T denotes "the true" system. Many results in robust controller design using models from identification have been criticized for not taking the effect of undermodeling into account. We however quote [Ljung *et al.*, 1991] for a recent opinion:

> *It has often been said, also by the authors of this paper, that traditional identification has neglected the systematic error: the bias contribution. In light of priors and "accepted models" /.../, it is natural to get this impression. Nevertheless the statement is not quite true.*

and later they continue with:

> *In other words, we could say that for an unfalsified model, the bias error has not been found to be significantly larger than the random error. The traditional variance bound /.../ is thus relevant also to describe the total mean square error for an unfalsified model – we might just like to reflect the presence of the bias term when determining the width of the confidence interval in terms of number of standard deviations.*

See [Ljung *et al.*, 1991] for a further discussion, also on some philosophical aspects of the problem.

Hard or Soft Bounds

It is often argued that robust controller design needs hard bounds on parameters. Several suggestions have been made on how to reject the traditional disturbance description and develop "hard bounds" for the models deviation from "the real" system. These methods are known as set membership, ellipsoidal or unknown but bounded noise methods. See [Gutman, 1988, Kosut *et al.*, 1990, Younce and Rohrs, 1990, Wahlberg and Ljung, 1991] for how to use these ideas to deal with undermodeling. We again quote [Ljung *et al.*, 1991]:

The current state of affairs in this area is somewhat confused

See [Ljung *et al.*, 1991] for further information. Also notice the well known fact that hard bound estimates are extremely sensitive to outliers, i. e. underestimating the disturbance bounds.

The paper [Bertsekas and Rhodes, 1971] presents a result, which can be used to obtain hard bounds from the standard P-matrix if L_2-bounds are known for measurement errors and time-variability of the parameters. Unfortunately such bounds are seldom valid.

4. Optimal Controller Design

The goal is now to optimize expected controller performance measured from external signals w to outputs z, see Figure 2. We use the H_2-norm to measure performance:

DEFINITION 1
For a stable system with transfer function G, the H_2-norm is defined as

$$\|G\|_2 := \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{tr}\, G(j\omega)^* G(j\omega)\, d\omega\right)^{\frac{1}{2}} \tag{5}$$

□

This norm has several useful interpretations both in frequency domain, as in (5), or in time domain using Parseval's theorem. It can also be interpreted, like in standard LQG, as the stationary variance of the output in the case of white noise system input. There is also the following less known worst case interpretation

$$\|G\|_2 = \max_{\|w\|_2=1} \sup_t |z(t)| \tag{6}$$

where $z = Gw$.

The optimal robust performance controller is defined as the linear time invariant controller $K(s)$ that stabilizes the loop under all uncertainties Δ of a certain structure and minimizes the expected H_2-norm:

$$\min_{K} \mathop{E}_{\Delta} \|T_{zw}(K)\|_2 \tag{7}$$

where E denote mathematical expectation and $T_{zw}(K)$ denotes the closed loop system from w to z. Notice that with this interpretation the expectation is calculated using a fixed a priori covariance information on parameters. Another possibility is to update the a posteriori information of parameters when more data is made available.

Problems like (7) have been proposed previously and are generally quite hard. This is because the expectation is normally difficult to evaluate and results in a function that depends in a complicated way on the controller. Notice that (7) is related to the stochastic embedding method used by [Goodwin and Salgado, 1989]. Different criteria that also take the likelihood of modeling errors into account have been used by, e.g., [Chung and Belanger, 1976], [Grimble, 1984]. See also [Nahi and Knobbe, 1976], [Kassam and Poor, 1985] and [Stengel and Ray, 1991]. A recent article uses the problem formulation above for a robust input estimation of scalar systems using polynomial calculations, see [Sternad and Ahlén, 1993]. This paper and the concept of stochastic embedding have been an important source of inspiration for the present study. The results of [Sternad and Ahlén, 1993] can be obtained as a special case of Theorem 1 below. Our results also covers the MIMO case with little extra work. We also refer the reader to that paper for a more extensive overview of related work.

Assume that the system is given by :

$$\begin{aligned} r &= G_{00}v + G_{01}w + G_{02}u \\ z &= G_{10}v + G_{11}w + G_{12}u \\ y &= G_{20}v + G_{21}w + G_{22}u \\ u &= Ky \\ v &= \Delta r \end{aligned} \tag{8}$$

where Δ is a real matrix of random variables. We assume that $E(\Delta) = 0$ and that we are given covariance information on the elements of Δ. We leave the issue of parametrization of Δ to later.

Although Figure 2 represents a quite general way of representing robust stability and robust performance problems we will now see that one actually can solve problem (7) in closed form for a class of interesting problems:

Assumption 1

$$G_{00} = 0$$

Assumption 2

$$G_{02} = 0 \qquad \text{or} \qquad G_{20} = 0$$

See Section 6 for a discussion on relaxation of Assumption 1. Examples of problems satisfying Assumptions 1 and 2 are, e.g., feedforward problems, deconvolution or channel equalization problems, robust filtering and noise cancellation.

It is easy to see that if Assumptions 1 and 2 are satisfied, the stability of the "closed loop" system will not depend on the actual value of Δ. If we do not want to assume hard bounds on parameters this is crucial. If there is feedback around an uncertain element for which there is no hard bound, there is a risk for the system to become unstable for large parameter errors. A positive probability for an unstable system will dominate in the calculation of expected H_2-norm and render the problem formulation useless.

THEOREM 1
Assume that $G_{00} = 0$ and $G_{02} = 0$. Introduce the spectral factorization

$$P(s)P^*(s) = \underset{\Delta}{E}\,(\Delta G_{01}(s)G_{01}^*(s)\Delta^*) \tag{9}$$

Here Δ is the matrix of uncertain parameters in Figure 2 and G_{01} is given in (9). The optimal controller solving (7) is then given by the solution of the following standard H_2-problem.

$$\min_K \left\| \begin{pmatrix} G_{10}P & G_{11} \end{pmatrix} + G_{12}K(I - G_{22}K)^{-1} \begin{pmatrix} G_{20}P & G_{21} \end{pmatrix} \right\|_2^2 \tag{10}$$

Proof: Closing the lower loop, using $u = Ky$ gives

$$\begin{aligned} r &= [G_{00} + G_{02}K(I - G_{22}K)^{-1}G_{20}]v + [G_{01} + G_{02}K(I - G_{22}K)^{-1}G_{21}]w \\ z &= [G_{10} + G_{12}K(I - G_{22}K)^{-1}G_{20}]v + [G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}]w \end{aligned}$$

If we now were to close the upper loop, we would in general get linear fractional transformations of linear fractional transformations and the closed loop would be a complicated function of the controller. However, assuming $G_{00} = 0$ and $G_{02} = 0$ we can use $v = \Delta r = \Delta G_{01} w$ to reduce the problem of minimizing (7), to

$$\begin{aligned} T_{zw}(K) = {} & G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21} + \\ & + [G_{10} + G_{12}K(I - G_{22}K)^{-1}G_{20}]\Delta G_{01} \end{aligned} \tag{11}$$

Notice that the information needed to perform the expectation in (7) is exactly the covariance between different elements in Δ. Performing the expectations in (7) and noting that the terms linear in Δ disappear due to $E(\Delta) = 0$ we get, by forming $T_{zw}T_{zw}^*$, that

$$\|T_{zw}\|_2^2 = \left\| \begin{pmatrix} G_{10}P & G_{11} \end{pmatrix} + G_{12}K(I - G_{22}K)^{-1} \begin{pmatrix} G_{20}P & G_{21} \end{pmatrix} \right\|_2^2$$

This is a standard LQG-problem, and it can be solved by any favorite method. □

Remark 1. For $P = 0$ the design of course reduces to the nominal case. The larger P is, the more influence will come from the uncertainty loop. The number of columns in P is given by $\min(n_\Delta, n_w)$, where n_Δ is the size of Δ and n_w is the number of external signals w.

Remark 2. We are implicitly assuming that the LQG-problem above is nonsingular and solvable. This includes assumptions on stabilizability and detectability etc. Problems not satisfying these assumptions are often badly formulated.

The corresponding result when $G_{20} = 0$ instead of $G_{02} = 0$ follows by the following dual theorem:

THEOREM 2
Assume that $G_{00} = 0$ and $G_{20} = 0$. Introduce the spectral factorization

$$P^*(s)P(s) = \mathop{E}_{\Delta} (\Delta^* G_{10}^*(s) G_{10}(s)\Delta) \tag{12}$$

The optimal controller solving (7) is then given by the solution of the following standard H_2-problem.

$$\min_K \left\| \begin{pmatrix} PG_{01} \\ G_{11} \end{pmatrix} + \begin{pmatrix} PG_{02} \\ G_{12} \end{pmatrix} K(I - G_{22}K)^{-1} G_{21} \right\|_2^2 \tag{13}$$

Proof: Similar to above, see Theorem 3 for another possibility. □

We also mention the following duality result which is helpful for establishing correspondences between different problems:

THEOREM 3
For all norms satisfying $\|G^T\| = \|G\|$ the two problems

$$J_1 = \min_{K_1} \mathop{E}_{\Delta_1} \|T_{zw}^1\|, \qquad J_2 = \min_{K_2} \mathop{E}_{\Delta_2} \|T_{zw}^2\| \tag{14}$$

where the probability distribution of Δ_2 equals the probability distribution of Δ_1^T, see Figure 3, are dual, in the sense that the two optima are equal $J_1 = J_2$, and the optimal controllers are related by $K_1^T = K_2$, □

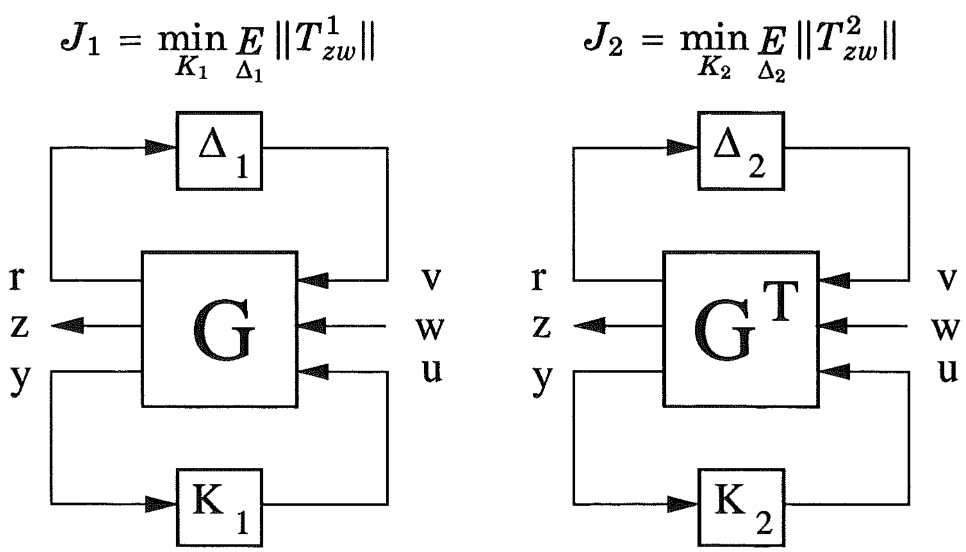

Figure 3. Duality between robust performance problems.

Proof: Follows directly by calculating the closed loop for the two problems and taking transposes. □

An Example. The following result on how to compute (9) is a slight generalization of results in [Sternad and Ahlén, 1993]. The result is useful for several popular parametrizations of models, like finite impulse response, Laguerre or Kautz functions or other orthogonal basis functions. Assume

$$\Delta_1(s) = \sum_{0 \le i \le n} \tilde{\theta}_i \varphi_i(s) = \varphi(s)\tilde{\theta} \tag{15}$$

and that $E(\tilde{\theta}\tilde{\theta}^*) =: P$, then

$$E(\Delta_1(s)\Delta_1(s)^*) = \varphi P \varphi^* = \sum_{i,j} P_{i,j}\varphi_i(s)\varphi_j^*(s) \tag{16}$$

so the averaged term $\Delta_1 = \Delta G_{01}$ in (9) is in this case readily obtainable. This result is given in [Sternad and Ahlén, 1993] for the discrete time case with $\varphi_i(q) = q^{-1}$.

5. An Example

The following example is made very simple to illustrate ideas, it can be solved by many other methods. We know that successful feedforward control crucially depends on a good model of the system. To exemplify

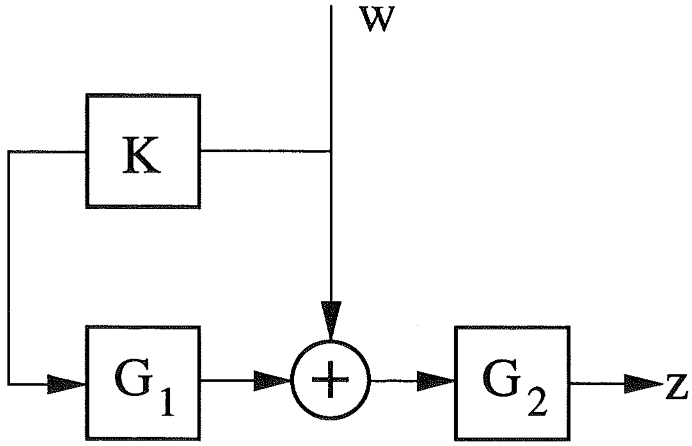

Figure 4. Block diagram for the feedforward example.

this we consider the feedforward problem in Figure 4, where

$$G_1 = \frac{s^2 + \theta s + 1}{(s+1)^2} \qquad G_2 = \frac{1}{s+1} \tag{17}$$

Notice that there is a substantial risk that G_1 is nonminimum phase. The system is described by the following state space equations:

$$\begin{aligned}
\dot{x}_1 &= -2x_1 - x_2 + u \\
\dot{x}_2 &= x_1 \\
\dot{x}_3 &= (\theta_0 + \tilde{\theta} - 2)x_1 - x_3 + v + w + u \\
z &= x_3 \\
y &= w \\
r &= x_1
\end{aligned}$$

We assume that $\theta_0 = 0.1$, $E(\tilde{\theta}) = 0$, and $E(\tilde{\theta}^2) = \sigma^2$. Written in the form (9) the system is

$$G = \begin{pmatrix} 0 & 0 & \frac{\cancel{s}}{(s+1)^2} \\ \frac{1}{s+1} & \frac{1}{s+1} & \frac{s^2+\theta_0 s+1}{(s+1)^3} \\ 0 & 1 & 0 \end{pmatrix}$$

Notice that $G_{00} = G_{20} = 0$, we can therefore use Theorem 2. Since $G_{10} = 1/(s+1)$, the spectral factorization (12) will in fact be scalar and

$$P(s) = \sigma G_{10}(s) = \frac{\sigma}{s+1}$$

follows immediately. With more than one uncertain element a method for doing multivariable spectral factorization should be used, see [Anderson and Moore, 1990].

Using Theorem 2 the problem can be solved as H_2 minimization, with respect to stable $K(s)$, of the transfer function

$$T_{zw}(s) = \begin{pmatrix} 0 \\ \frac{1}{s+1} \end{pmatrix} + \begin{pmatrix} \frac{\sigma s}{(s+1)^3} \\ \frac{s^2+\theta_0 s+1}{(s+1)^3} \end{pmatrix} K(s) =: \begin{pmatrix} a(s)K(s) \\ b(s)+c(s)K(s) \end{pmatrix} \tag{18}$$

Introducing $P^*P = a^*a + c^*c$ and multiplying from the left with an inner matrix we see that the optimal controller is given by the equation

$$PK = -\left(\frac{c^*b}{P^*}\right)_+$$

where $(\cdot)_+$ denotes the stable part. The resulting controller is given by

$$u = K(s)w = -\frac{S(s)}{R(s)}w$$

where

$$S(s) = \frac{2+\theta_0}{2+\sqrt{(\theta_0^2+\sigma_0^2)}}(s+1)^2$$

$$R(s) = s^2 + \sqrt{(\theta_0^2+\sigma_0^2)}s + 1$$

The amplitude of $K(i\omega)$ for $\sigma = 0, 0.2, 0.4$ is plotted in Figure 5. Notice that $\sigma = 0$ corresponds to certainty equivalence and that when the uncertainty in θ is increased the robust controller decreases the maximum amplitude. This seems intuitively correct.

As a concluding remark we mention that feedforward control could be combined with adaptivity if the models are very bad. An idea is of course to combine the robust performance results described here and in [Sternad and Ahlén, 1993] with adaptivity. Several suggestions for improving the certainty equivalence principle used in adaptive control has been made, e. g. the cautious controllers minimizing the variance of the one-step ahead control error, see [Wittenmark, 1975]. Notice that for many systems there is a trade off between good identification and good control. This is however not so for the problems studied in this paper. This trade off is studied further in the area of dual control, see e. g. [Sternby, 1977].

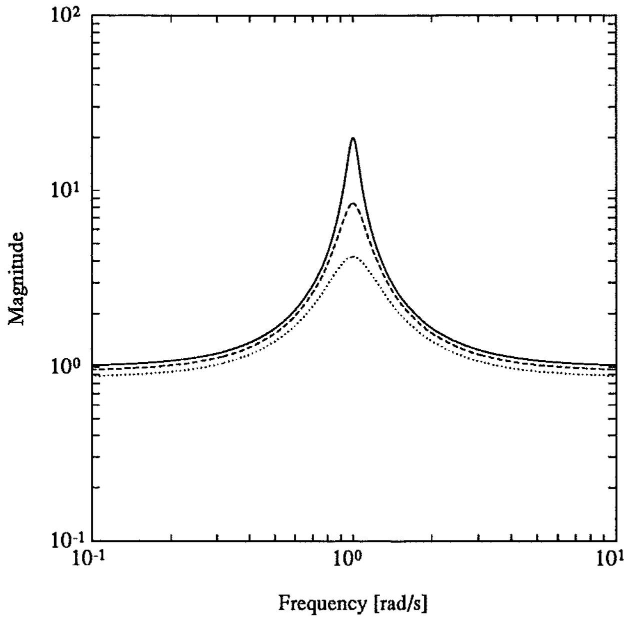

Figure 5. Solution for the Feedforward Example, $\sigma = 0$ (solid), $\sigma = 0.2$ (dashed) and $\sigma = 0.4$ (dotted).

6. Conclusions and Open Problems

A natural performance robustness problem has been formulated. The formulation uses information obtainable from standard identification in the form of covariances of parameters. In this way we avoid finding hard bounds on parameters and by using expected values we avoid the conservativeness related to designing for worst cases. We have shown how the problem can be solved for a class of problems that includes interesting applications in signal processing such as feedforward design, channel equalization, noise cancellation, optimal differentiation[Carlsson *et al.*, 1991] and robust filtering. The optimal "controller" can then be found using standard LQG-software with a modest increase in computational complexity, one extra Riccati equation or matrix spectral factorization. It is also easy to see how uncertainty in the elements influence the resulting controller, see e. g. (9) and (10). We have assumed that the resulting H_2-control problem is nonsingular and solvable. This is the case for most applications.

The setup used in the present paper relates to the setup made be Doyle and others in the μ-synthesis analysis. The theorems presented solve the problem in a very simple way and only standard numerical software is needed here.

The assumption $G_{00} = 0$ can be slightly relaxed, e. g. to nilpotency of $G_{00}\Delta$. This means that there is no signal loop around an uncertain parameter in the open loop system but the signals may pass Δ several times. Higher order moments of the probability distribution of elements of Δ are needed to calculate the expected value of the H_2-norm. The higher order moments can be calculated from covariance information if parameters are assumed Gaussian. The problem can then be solved analogously to above.

The problem has been described in continuous time. The corresponding results can be obtained for discrete time systems. The only change needed is a to change the definition of the H_2 norm to integration over the unit circle.

The proposed method has interesting applications in adaptive control and adaptive signal processing of open loop type problems. We now have a situation where identification results match the "controller" design. Applications of this are left for future research. It is also an important project to compare all existing techniques for robust performance design. See [Sternad and Ahlén, 1993] for some further interpretations of the result and for simulations and comparison with other methods.

The case with feedback around uncertain parameters are much harder. Feedback however often automatically decreases the sensitivity to model errors. There are at least two possibilities to formulate sensible robust performance problems. One is to change the identification methods to obtain hard bounds on parameters. This has been tried by several authors. Another interesting possibility is to use other definitions of robust performance. It is easy to formulate reasonable problems, harder to solve them. One possibility could, e.g., be to maximize the probability that performance $\|T\|$ is sufficiently good, i. e. $\|T\| \leq \gamma$. Another idea is to optimize some criteria that can accept a rare risk for instability without becoming infinite, e. g. find the controller that maximizes $E\{\exp -(\|T\|)\}$. Explicit solution can not be expected, but numerical results can perhaps lead to useful insight. One should also keep in mind what is the intended use of the design and how knowledge and understanding of the system can be improved. A healthy attitude is to keep a holistic view but to solve concrete problems.

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