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IDENTIFICATION OF LINEAR, MULTIVARIABLE PROCESS DYNAMICS USING CLOSED LOOP EXPERIMENTS

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I. Gustavsson, L. Ljung and T. Söderström

ABSTRACT

The identifiability of linear, multivariable, time invariant systems under output feedback control is analysed. Vector difference equation models are used. It is shown that the open loop characteristics can often be determined, even if data from closed loop experiments are used. One approach is to treat the closed loop data as in the open loop case using e.g. the maximum likelihood method. Another technique would be to start by estimating the closed loop transfer function and then try to compute the open loop characteristics using the knowledge of the regulator. It is shown that the two approaches have the same identifiability properties. One advantage of the straightforward method is that no special algorithms are necessary. If the regulator is time varying, or non-linear, or if disturbances act in the feedback loop, or if an external input is applied, then the identifiability is most often secured. In the case of a linear, time invariant, noise free regulator without any extra input signal, the identifiability conditions will involve the structures of the system, the regulator and the model. For the single-input single-output case explicit necessary and sufficient conditions are given. One practical way to achieve identifiability is to alternate between several different linear feedback laws. It is shown that the required number of regulators depends only on the number of inputs and outputs of the system.

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1. INTRODUCTION

Mathematical models of dynamical systems are often desired, e.g. for design of control strategies. There are essentially two ways to construct such models. One way is to use basic physical laws and the other to use system identification, i.e. to determine the model from process measurements. The field of identification has developed rapidly during the past decade. A great number of applications have now been reported, see e.g. the survey papers Baeyens and Jacquet (1973), Bekey (1973) and Gustavsson (1973).

It is most often desired for control design purposes to know the open loop characteristics of the process. Therefore the identification techniques have mostly been developed to handle processes operating in open loop during the experiments. However, in practice it is frequently desirable and even necessary to perform the experiments in closed loop. One reason may be that the process behaviour itself is unstable or very poor without the control. Risks of process damages, loss of production or reduction of process efficiency may prevent open loop experiments. Another reason may be that a linear model, valid around a certain point, is desired. The process should thus be kept near this point during the experiments. In all these cases a regulator or manual control must keep the process within the desired limits. Another very important reason is that there are systems that are inherently in closed loop and for which it is impossible to break up the feedback loop. Many economical and biological systems are of this kind.

It turns out to be very important to be able to identify processes operating in closed loop during the experiments. It has been realized since long that identification of such processes causes extra difficulties. In fact there are simple, but yet realistic cases, when identification is impossible under closed loop. Consider for example the system (y(t) denotes the output, u(t) the input and e(t) white noise)

$$y(t+1) + a\, y(t) = b\, u(t) + e(t+1)$$

with a proportional linear regulator

$$u(t) = g\, y(t)$$

An attempt to estimate the parameters a and b, e.g. by the least squares method, shows that all parameter estimates

$$\hat{a} = a + \gamma\, g$$

$$\hat{b} = b + \gamma$$

γ arbitrary, give the same value for the identification criterion. For $\gamma \neq 0$ an erroneous description of the open loop system is obtained. Notice in particular that it is of no help to know the regulator parameter, g.

Many of the analyses of identification schemes assume explicitly that the experiments are carried out in open loop. Several methods will actually give wrong estimates when applied to data from closed loop systems. Other methods may or may not produce correct results depending on the structure of the process, the model and the regulator. However, the possibilities of identifying the open loop characteristics from closed loop experiments and the necessary conditions that must be fulfilled in order to obtain reasonable estimates seem to be largely unknown. This is particularly the case for multivariable systems and for the case with a time varying regulator. There is no paper indicating these problems clearly and giving the solutions systematically. This has resulted in a somewhat confusing situation concerning what can actually be done.

An attempt is made in this report to give a unified approach to the problem of identification of linear, multivariable systems operating in closed loop. Previous contributions in the area are reviewed in Section 2, where also the basic problem formulation is given. Formal definitions of the models and the identification methods used are presented in Section 3. The identifiability concepts are introduced in Section 4. Identification of the open loop dynamics from measurements of the inputs and the outputs is treated in Section 5. In Section 6 it is shown that this approach has the same identifiability properties as the indirect identification approach, in which the closed loop transfer function is estimated first. Then it is solved for the open loop characteristics. In this case the regulator has to be known. One way to achieve identifiability is to alternate between several different feedback laws. In Section 7 the number of feedback laws that is sufficient for identifiability is determined. It is never possible to test identifiability by evaluation of the experimental data only. This is demonstrated in Section 8. In Section 9 conditions, which are both necessary and sufficient for identifiability, are given explicitly for the single-input single-output case. In Section 10 some numerical illustrations are presented. The main conclusions are summarized in Section 11.

2. SURVEY OF PREVIOUS RESULTS

Various results on identification of systems operating in closed loop have been presented in a number of papers. In this section some of them will be reviewed. The basic problem formulation will first be given in order to obtain a unified and systematic notation in the following. Some properties of identification techniques will also be discussed.

2.1 Problem formulation

In the following the closed loop system configuration shown in Fig. 2.1 will be considered.

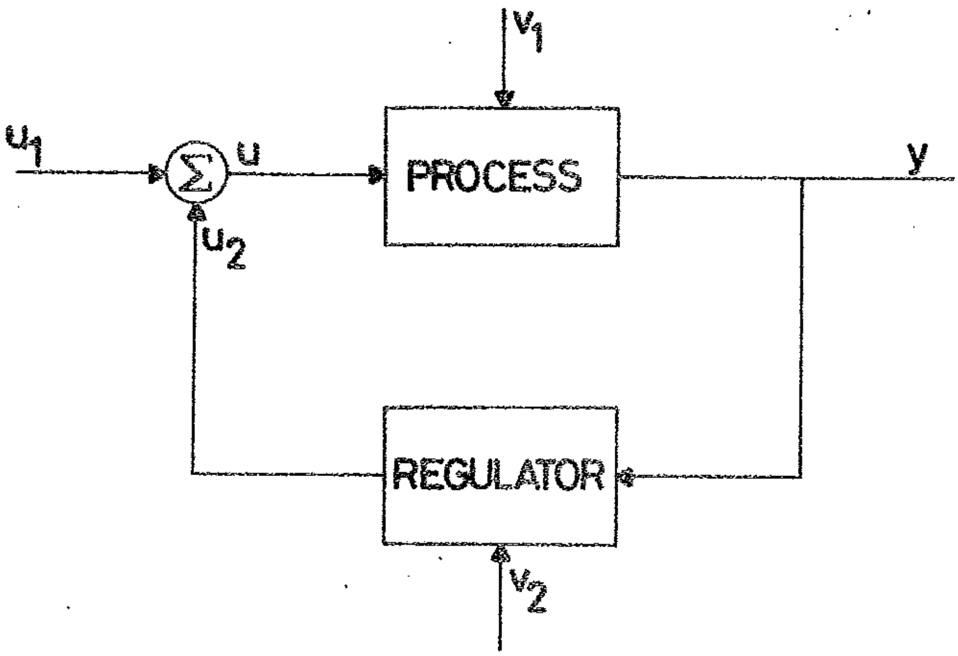

Fig. 2.1 Block diagram of a closed loop system

u	input signal to the process (measurable)
y	output signal
u_1	extra perturbation signal (measurable)
u_2	feedback signal
v_1, v_2	disturbances (unmeasurable)

The main problem is to investigate under what conditions on the process, the regulator and the signals it is possible to determine the open loop characteristics, i.e. the transfer function of the process and the characteristics of the disturbances on the output, from measurements on the closed loop system.

In the following it is assumed that the process† itself is linear and time invariant. Disturbances are acting on the process. They are denoted by v_1. This signal includes the measurement noise on the output signal and is usually non zero. The input u is assumed to be measured without noise. The signal u_1 is a deliberately applied input signal. It is natural to assume that it is independent of v_1. A disturbance v_2, that is not directly measurable, may act in the feedback loop. The disturbances v_1 and v_2 are also assumed to be independent. Notice that only output feedback is allowed.

It turns out that it is convenient to distinguish between some different cases:

1) $u_2 = 0$

2) $u_1 \not\equiv 0$ and/or $v_2 \not\equiv 0$

3) $u_1 = 0$, $v_2 = 0$

   a) nonlinear regulator
   b) time varying regulator
   c) linear, time invariant regulator

The first case is the open loop case, which will be considered only briefly. The second case is when there is a deliberately applied input signal or when there is a disturbance source in the feedback loop. The third case covers the situations when the feedback is noise free and when no extra perturbation u_1

† "Process" and "system" will be used synonymously.

is injected. Identifiability problems arise in particular in the third case, when the regulator is linear and time invariant (see Section 5).

2.2 Identification techniques

There are several ways to identify the open loop characteristics. The straightforward technique is to use measurements of the input u and the output y without assuming that the process is in closed loop. This approach will be called direct identification. By indirect identification is meant identification of the closed loop behaviour and solution for the open loop dynamics. The regulator has to be known exactly in this case. The closed loop behaviour from u_1 to y is determined, or if $u_1 = 0$, time series analysis of the output y is performed. Notice that indirect identification is not suitable, e.g. if the regulator is time varying or if there is noise in the feedback loop. Indirect identification includes also all methods that require an exact knowledge of the regulator. Such identification techniques can e.g. estimate the system parameters directly but use the closed loop transfer function as model structure with the regulator parameters inserted.

Many different identification methods are available, see e.g. the survey paper Åström and Eykhoff (1971). For some of them physical realizability (causality) of the model is not assumed. Such methods will be called noncausal methods. Cross spectral analysis is an example of such a method. For other methods, on the other hand, parametric models whose structure implies causality are postulated. Many of these methods can be considered as prediction error methods which minimize some prediction error criterion in order to get the parameter estimates. Examples are the least squares method and the maximum likelihood method. Most of the results in this report are derived for prediction error methods applied to vector difference equation models.

2.3 Review of previous contributions

In a short review like this it is not possible to give full details. Notions like identifiability etc. will therefore be used without giving the definitions that were used in each particular paper. Nevertheless it is most often intuitively clear what is meant. For details it is referred to the papers themselves. In Table 2.1 an attempt is made to systemize the contributions according to the cases discussed and the methods and model structures used. The papers dealing with the adaptive situation are not included in the table. In the following the notations from Sections 2.1 and 2.2 will be used. The papers are reviewed in chronological order.

Fisher (1965) discusses identifiability of continuous, deterministic, linear state space systems with no extra perturbation and with a time invariant, noise free feedback. A least squares approach to the parameter estimation is taken. Necessary and sufficient conditions for identifiability are in this case that the control law is not linear in the states, and that the system is completely controllable. This result corresponds to the result in this report that identifiability is secured if the regulator has high enough order or is nonlinear. Linear state feedback does not fulfil this condition.

The problem of identification of closed loop systems has been treated in several papers by Akaike (1967, 1968). There are also several applications using his ideas, e.g. Otomo, Nakagawa and Akaike (1969) and Itoh, Saito, Takumi and Shimizu (1971). Akaike points out that cross spectral analysis requires, that the input is measured without noise and that the noise and the input are independent. This last condition is violated for closed loop systems. If there is an extra perturbation, i.e. $u_1 \neq 0$, cross spectral analysis can, however, be used in a special way. Using the signals u_1, u and y it is possible to estimate the open loop transfer function without knowing the regulator. This approach cannot be directly re-

Author(s)	Feedback cases					Methods				Systems		Model structures			
	$u_1 \neq 0$	$v_2 \neq 0$	$u_1=0; v_2=0$												
			Time invariant reg.		Time varying reg.										
			Linear	Nonlinear		Direct	Indirect	Prediction error	Noncausal	SISO	MIMO	State space equation	Difference equation	Impulse response	Frequency response
Fisher (1965)			(x)	x		x		x		x		x			
Akaike (1967, 1968)	x	x				x	(x)	x	x	x	x			x	x
Priestley (1969)		x	x			x		x	x	x				x	x
Åström and Eykhoff (1971)			(x)			(x)			(x)	(x)					(x)
Bohlin (1971 a, b)	x		x			x	(x)	x	(x)	x			x		
Schultze (1971)	x	x				x		x		x			x		
Box and Mac Gregor (1972)	x	x	x				x		x	x			x	x	
Caines and Wall (1972)		x					x	x		x	x		x		
Leonhard (1972)	x	x	x			(x)	x	x		x				x	
Eykhoff (1973)	x		(x)			x	x	x		x			x	(x)	
Glover (1973)	(x)		x				x	x		x	x	x			
Goodwin, Payne and Murdoch (1973)	x						x	x		x			x		
Lindberger (1973 a, b)		x	(x)				x	x		x			x		
Gustavsson, Ljung and Söderström (this report)	x	x	x	(x)	x	x	x	x	(x)	x	x		x		

Table 2.1. The contents of some important papers dealing with identification of systems operating in closed loop. Parentheses indicate that only a brief discussion is given.

SISO - single-input single-output

MIMO - multiple-input multiple-output

ferred to be either direct or indirect identification as defined in Section 2.1, since besides u and y also the signal u_1 is used. Akaike treats particularly the case when there is an unmeasurable disturbance in the feedback loop. Then ordinary cross spectral analysis fails. Instead the problem is solved by introducing a causal time domain model. Direct identification is used with an impulse response model with a finite number of parameters. The condition of identifiability is that there is a delay in the system or in the regulator, so that a change in the input signal to the system does not instantaneously influences on the feedback signal.

Priestley (1969) treats the problem of estimating the process transfer function from data consisting of records of the input and the output from a closed loop system with additive disturbances. No known perturbation is available and the feedback is linear. Direct identification is used. The least squares method is compared with the weighted least squares method. It is claimed that the two methods give identical results for the open loop case and for the closed loop case with noise free feedback. It is also remarked that if the models are not restricted to be causal, e.g. by using cross spectral analysis, the estimated models may be quite incorrect. Nevertheless cross spectral analysis is recommended as an initial step for identification of closed loop systems.

In the survey paper on identification by Åström and Eykhoff (1971) identification of closed loop systems is only discussed very briefly. An example is given for the case of noise free, linear, time invariant regulator without extra perturbation. In this case a noncausal method will give the inverse of the regulator as the model. However, a method using a causal model may or may not give the correct model according to the results in e.g. Section 9 of this report.

The most general discussion is given by Bohlin (1971a). This paper is concerned with the basic limitations of identification and the practical implications of the mathematical assumptions involved. Thus also identification of processes operating in closed loop during the experiments is treated. Bohlin remarks that there are very simple but realistic cases when identification is impossible under closed loop conditions. However, a closed loop during the experiments does not necessarily prevent identification of the open loop characteristics. In the case of noise free, linear feedback without any extra perturbation the process is not identifiable without a priori knowledge of the structure. A chosen structure cannot be validated by the data, cf Section 8 of this report. In general, feedback is allowed only from the output and from no other sources. Otherwise identification is not possible. This condition is violated e.g. if a human operator intervenes in order to adjust for unfavourable behaviour of other variables than the input and the output. Bohlin also points out that it is necessary to use methods for which open loop is not postulated. Otherwise an incorrect model is obtained. Even methods allowing for feedback may result in uncertain, ambiguous or undeterminable solutions. Some examples of identification of industrial processes operating in closed loop during the experiments are given. Direct identification with the maximum likelihood method is used.

In Bohlin (1971b) the ambiguity of the maximum likelihood method is treated. It is demonstrated that if the regulator is noise free, linear, time invariant and if no extra perturbation is injected only the noise transfer function for the closed loop system is identifiable. When the structure is known, identifiability of the open loop transfer function is secured if there is a one to one correspondence between the parameters of the open loop transfer function and the closed loop noise transfer function. No explicit conditions for the identifiability of the open loop dynamics are given. Bohlin also treats the special case, when the regulator is known to be a minimum variance regulator. If the transport delay is known, it is possible in this case to obtain a model of the open loop dynamics which is equivalent to the true process in the sense that they have the same minimum variance control law. Finally Bohlin remarks that introduction of a small persistently exciting perturbation generally removes the ambiguity problems under closed loop conditions, cf Section 5 of this report.

Schultze (1971) treats the case with an extra perturbation u_1 or with an unknown disturbance v_2. In fact he states that extra perturbation or noise is necessary for identifiability but that it need not be directly measurable, cf however Sections 5 and 9 in this report. The choice of model structure and identification criterion is also discussed. It is demonstrated that this choice is of crucial importance. For the closed loop case the difference equation model for the process must include possibilities to handle correlated disturbances. Direct identification is proposed. An application to a ball mill is presented.

Box and MacGregor (1972) consider identification of closed loop single-input single-output systems using impulse response and difference equation models. Indirect identification with noncausal methods is used. It is shown that crosscorrelation of the input and output sequences for identifying the process transfer function is invalid in closed loop situations. Under noise free, linear feedback without extra perturbation this crosscorrelation procedure gives the inverse of the regulator transfer function. If the regulator is known and if an independent noise sequence is added to the input, then the transfer function can be identified by crosscorrelating this added noise with the output. This method is analogous to the one used by Akaike for the same case. It is also shown that tests on the residuals in order to verify the obtained model are not reliable in many closed loop situations. Several different models may yield the same residual sequence. It is stated that the regulator must be known in order to perform identification of the process transfer function, cf however Section 5 of this report. Under certain conditions the regulator can be identified if it is not known a priori. Some applications to real data are given.

Caines and Wall (1972) discuss parameter estimation of closed loop systems and particularly the case with an unknown disturbance in the feedback loop. They state that direct identification using the loss function for the open loop maximum likelihood estimation will not give the maximum likelihood estimates for the closed loop case. Instead it is proposed to estimate the parameters in the model consisting of the inputs and outputs expressed as time series of the noise sources. This idea is now being studied by Chan (1973).

Leonhard (1972) discusses the identification of systems operating in closed loop. Indirect identification with the least squares method is used. It is stated that the open loop transfer function can be obtained if an extra perturbation u_1 is injected or when there is a disturbance v_2 acting in the feedback loop. An example using direct identification is also given. From the results it seems, however, that a noncausal model structure has been used in this example.

Eykhoff (1973) discusses briefly identification of closed loop systems in connection with least squares estimation. The case with noise free feedback and with a perturbation signal is treated. Indirect and direct identification are proposed as the two existing possibilities. It is mentioned that difficulties can arise in indirect identification because it is not always trivial to solve for the open loop dynamics from the closed loop transfer function. For direct identification on the other hand the problem with dependence between input and noise arises. A simple example of direct identification is given. An impulse response model is shown to be non identifiable if the output depends instantaneously on the input.

Glover (1973) is mainly concerned with parameter identifiability of continuous and discrete time state space systems. The example from Åström and Eykhoff (1971) is used to illustrate that identification in the presence of feedback can cause significant problems. A recommended way to model a system with feedback is to write down the state space equations for the open loop system and then modify these equations with the feedback law. The closed loop system is then expressed in terms of the open loop parameters and the parameters of the feedback law. The identifiability questions for the unknown parameters of this closed loop system can then be answered as for the open loop case.

Goodwin, Payne and Murdoch (1973) are mainly concerned with the synthesis of optimal test signals for closed loop identification. A variant of the case with an extra perturbation is treated. According to our notion indirect identification is used. A prediction error method is one of the proposed estimation techniques. Identifiability problems are not discussed explicitly. However, the criterion for the synthesis of optimal test signals is a function of the uncertainty of the estimated parameters. The two problems are thus closely related. A few simulations are presented.

Lindberger (1973a, b) has discussed problems related to the identification of closed loop systems. In principle the case with unmeasurable disturbances in the feedback loop is treated. Lindberger uses indirect identification according to our notion. He also proposes a strategy in order to try to achieve identifiability by successively increasing the complexity of the regulator and performing one experiment with each regulator until a reasonable model is obtained. This strategy, however, need not necessarily lead to the correct model. Lindberger gives some necessary conditions for identifiability but they are not complete.

The papers reviewed so far have mainly been concerned with the identification of the open loop dynamics from closed loop experiments with constant regulators. Identifiability problems for closed loop systems appear also naturally in many adaptive control situations. Adaptive techniques often consist of simultaneous estimation and control, see Fig. 2.2. The current estimate and the regulator setting depend on the measurements. This implies that the feedback is time varying in a very subtle way. It thus becomes very difficult to verify the identifiability of the system. A few references are given below.

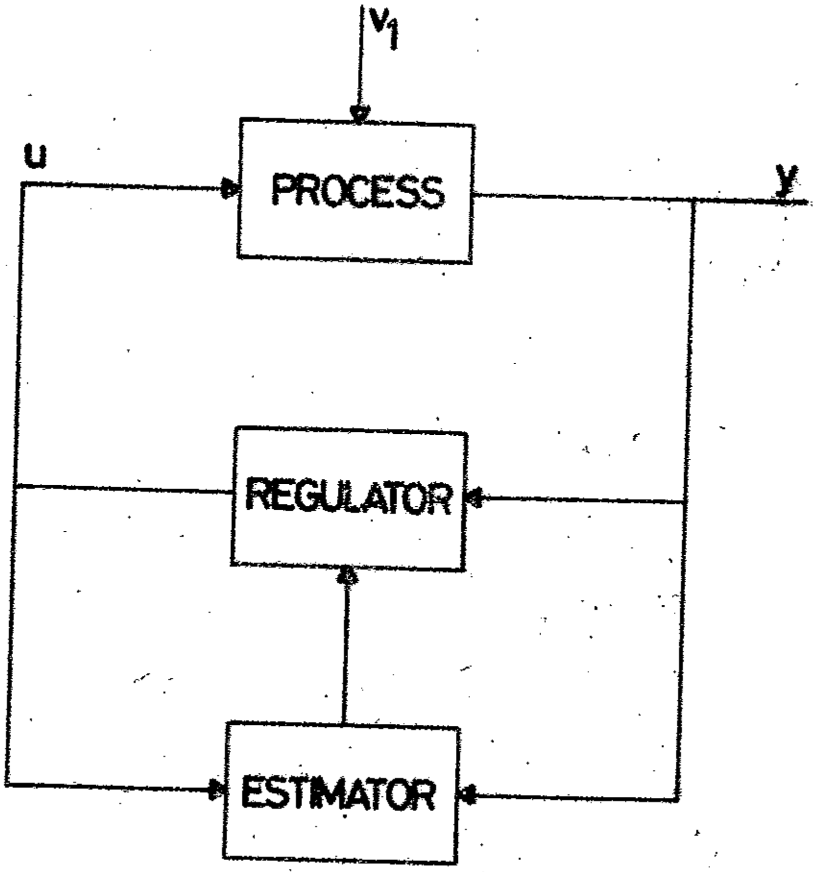

Fig. 2.2 Typical configuration for adaptive control

Turtle and Phillipson (1971) investigate identification and simultaneous control under a feedback control system of Box and Jenkins type and without any extra perturbations. The case when the estimates and thus also the regulator parameters are updated only every n-th measurement is discussed.

The open loop transfer function will be identifiable if the process has transport delays enough. Furthermore sufficient conditions for identifiability are given for a more general structure. The linear system of equations in the unknowns which is obtained in indirect identification should have a unique solution. However, no explicit, complete conditions on the process and the regulator are given. The accuracy of the estimates obtained by indirect identification is also briefly discussed. The paper deals exclusively with indirect identification and it is remarked that a feedback of a mathematically undefined nature, e.g. manual control, will give data records virtually useless for reliable parameter estimation. According to the results in Section 5 of this report it may however be possible to obtain a correct model if direct identification is used instead of indirect identification.

Saridis and Lobbia (1972) and Lobbia and Saridis (1972) consider the problem of consistent parameter identification of stochastic, linear, discrete time processes driven by feedback controllers with different structures. In the first paper single-input single-output systems are treated. Direct identification is performed by stochastic approximation algorithms. The model is in state space form. The identifiability condition for the case of linear state feedback without extra perturbation was established. It turns out that the order of the regulator has to be sufficiently large. This condition is considered to be a too restrictive condition in practice. The case with an arbitrary feedback structure and with an extra perturbation signal is also treated in this paper. The perturbation signal is assumed to be white noise. The same case is also treated in the second paper for the multivariable case. Because of the extra perturbation no restrictions on the feedback are necessary, cf Section 5 of this report.

Balakrishnan (1973) presents an approach to adaptive control of linear, continuous systems in state space form. It turns out that a crucial assumption for the algorithm for optimal adaptive control is that a certain identifiability condition is fulfilled. A sufficient condition for identifiability in closed loop is given. It can be formulated as nonsingularity of a certain matrix. It is claimed that the condition can be checked straightforwardly. If the condition is satisfied the asymptotic consistency of the maximum likelihood estimates is assured in the case the input to the system is an (adaptive) feedback control. An extra perturbation may be present but is not necessary.

From the given review it is clear that there exists no paper treating all the feedback cases for the discrete time case. Very few papers deal with the multivariable case. In the following an attempt is made to systemize the identifiability properties for different feedback configurations.

3. MODELS AND IDENTIFICATION METHODS

The models and the identification methods, that are considered in this report, and the concept of persistently exciting signals are defined in this section.

3.1 Models

Vector difference equations are frequently used as models of multivariable systems. They constitute quite a general class of linear models. If so desired, they can be transformed to various state space realizations. Most of the results in this report refer to vector difference equations.

Consider a system S. Assume that the input output relationship and the noise characteristics can be described by

$$A_S(q^{-1})\ y(t) = B_S(q^{-1})\ u(t) + C_S(q^{-1})\ e(t) \qquad (3.1)$$

where $A_S(z)$ etc. are matrix polynomials in z:

$$A_S(z) = I + z\ A_{S,1} + \dots + z^{na}\ A_{S,n_a}$$

$$B_S(z) = z\ B_{S,1} + \dots + z^{n_b}\ B_{S,n_b}$$

$$C_S(z) = I + z\ C_{S,1} + \dots + z^{n_c}\ C_{S,n_c}$$

The operator q^{-1} is the backward shift operator:

$$q^{-1}\ y(t) = y(t-1)$$

The variables e(t), t = 0, 1,... form a sequence of independent random variables with zero mean value and covariance $E\ e(t)\ e^T(t) = \Lambda$. This matrix is assumed to be nonsingular. The output y(t) is a vector of dimension n_y and the input u(t) has dimension n_u. It has been assumed that e(t) has the same dimension as y(t), which can be shown to be no loss of generality. It is also assumed that $\det\{C_S(z)\}$ has all zeroes strictly outside the unit circle. Then the inverse $C_S^{-1}(q^{-1})$ is an exponentially stable filter. $C_S(z)$ is a constant, square matrix for each given z. The inverse $C_S^{-1}(z)$ is therefore straightforwardly defined.

$A_{S,i}$, $B_{S,i}$ and $C_{S,i}$ are matrices of proper dimensions. The elements of these matrices may be partly known and partly unknown. The identification problem is to determine the unknown elements. To do so, these elements or functions thereof are collected into a parameter vector θ. The model of the system then is

$$A_M(q^{-1},\theta)\ y(t) = B_M(q^{-1},\theta)\ u(t) + C_M(q^{-1},\theta)\ \varepsilon(t) \tag{3.2}$$

where, as before

$$A_M(z,\theta) = I + z\ A_{M,1}(\theta) + \ldots + z^{\hat{n}_a}\ A_{M,\hat{n}_a}(\theta)$$

etc., and $\{\varepsilon(t)\}$ is a sequence of independent random variables with zero mean value and covariance $\hat{\Lambda}$. The model (3.2) will be denoted by $M(\theta)$. The parameter vector θ is to be chosen so that $M(\theta)$ in some sense describes the system S. This is further discussed in Section 4.

The matrices $A_{M,i}(\theta)$ etc. can depend on θ in several ways. As discussed above, some elements in $A_{S,i}$ etc. may be known, and some may have to be assigned certain values. When the parameter vector θ varies over a set of feasible values, $M(\theta)$, given by eq. (3.2), defines a set of models that will be denoted by M. This set M can be described by the way the parameter vector θ enters in the matrices $A_{M,i}(\theta)$, $B_{M,i}(\theta)$ and $C_{M,i}(\theta)$, as well as by the variables $\hat{n}_a$, $\hat{n}_b$ and $\hat{n}_c$. The terms <u>model structure</u> and <u>model parametrization</u> will therefore also be used for the set M. This is an important concept, especially for multivariable systems. If a canonical representation related to the observability indices is chosen, M can be described by these indices, see e.g. Rissanen (1972). For single input single output (SISO) systems it is accustomed to use a structure, where the $\hat{k}$ first B-matrices (scalars) are assumed to be zero and all the other matrices unknown. This means that θ consists of $\hat{a}_1, \ldots, \hat{a}_{\hat{n}_a}, \hat{b}_{\hat{k}+1}, \ldots, \hat{b}_{\hat{k}+\hat{n}_b}, \hat{c}_1, \ldots, \hat{c}_{\hat{n}_c}$, and M can be described by the set $\{\hat{n}_a, \hat{n}_b, \hat{n}_c, \hat{k}\}$. Notice that apart from the orders n_a, n_b and n_c no concept of structure or parametrization has been introduced for the system S.

For notational convenience it will be written

$$A_S(q^{-1}) = A, \quad B_S(q^{-1}) = B, \quad C_S(q^{-1}) = C$$

$$A_M(q^{-1},\theta) = \hat{A}, \quad B_M(q^{-1},\theta) = \hat{B}, \quad C_M(q^{-1},\theta) = \hat{C}$$

when no ambiguity can occur. Also the time argument in y(t) etc. will sometimes be omitted.

3.2 Prediction

Knowing the inputs u(t) and the outputs y(t) up to time t, it is possible to predict y(t+1) as follows. From eq. (3.1):

$$C_S^{-1}(q^{-1})\, A_S(q^{-1})\, y(t) = C_S^{-1}(q^{-1})\, B_S(q^{-1})\, u(t) + e(t)$$

and

$$y(t+1) = \left[I - C_S^{-1}(q^{-1})\, A_S(q^{-1})\right] y(t+1) +$$

$$+ C_S^{-1}(q^{-1})\, B_S(q^{-1})\, u(t+1) + e(t+1)$$

The right hand side of the above expression contains only y(s) and u(s) up to time t. The term e(t+1) is independent of these variables, also in the case when u is determined from output feedback. Hence the best prediction of y(t+1) is

$$\hat{y}(t+1|t;S) = \left[I - C_S^{-1}(q^{-1})\, A_S(q^{-1})\right] y(t+1) + C_S^{-1}(q^{-1})\, B_S(q^{-1})\, u(t+1) \tag{3.3}$$

Analogously, the best prediction of y(t+1), assuming the model (3.2) is true, is

$$\hat{y}(t+1|t;M(\theta)) = \left[I - C_M^{-1}(q^{-1},\theta)\, A_M(q^{-1},\theta)\right] y(t+1) + C_M^{-1}(q^{-1},\theta)\, B_M(q^{-1},\theta)\, u(t+1) \tag{3.4}$$

The optimal prediction error is

$$y(t+1) - \hat{y}(t+1|t;S) = e(t+1) \tag{3.5}$$

Analogously, the prediction error for the prediction given by eq. (3.4) is

$$y(t+1) - \hat{y}(t+1|t;M(\theta)) = \varepsilon(t+1,M(\theta))$$

3.3 Identification methods

Mainly, identification methods that minimize some function of the prediction error will be discussed in this report. The maximum likelihood method is of this type.

Some scalar function of the matrix

$$Q_N(\theta;S,M) = \frac{1}{N}\sum_{t=0}^{N-1} \varepsilon(t+1,M(\theta))\, \varepsilon^T(t+1,M(\theta)) = \frac{1}{N}\sum_{t=0}^{N-1} [y(t+1) - \hat{y}(t+1|t;M(\theta))][y(t+1) - \hat{y}(t+1|t;M(\theta))]^T$$

is minimized over the model set M, giving the estimate θ_N and the model $M(\theta_N)$. The covariance matrix $\hat{\Lambda}$ is then estimated as

$\hat{\Lambda}_N = Q_N(\theta_N; S, M)$

Usual choices for scalar loss functions are

$$V_N(\theta; S, M) = \det Q_N(\theta; S, M) \tag{3.6}$$

or

$$V_N(\theta; S, M) = \operatorname{tr} Q_N(\theta; S, M) \tag{3.7}$$

The function (3.6) corresponds to the maximum likelihood method in case $\{e(t)\}$ are jointly normal, Eaton (1967). Clearly the two loss functions are the same in the single-output case.

3.4 Persistently exciting signals

In order to achieve identifiability the input signals must satisfy certain conditions. Trivially, it is impossible to estimate $B_M(q^{-1}, \theta)$ if $u(t) \equiv 0$. Caines (1970) has extended the definition of persistently exciting signals given in Åström and Bohlin (1965) as follows:

The signal u(t) is said to be persistently exciting of order n if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} u(t)\, u^T(t+k) = R(k) \text{ exists, } 0 \leqslant k \leqslant n,$$

where R(k) is an $n_u | n_u$ matrix, and the Toeplitz block matrix R of dimension $n | n_u$, generated by R(k), is positive definite.

If u(t) is persistently exciting of order n and $K(q^{-1})$ is a linear filter of order n-1

$$K(q^{-1}) = k_1 + k_2 q^{-1} + \dots + k_n q^{-(n-1)}$$

where k_i are row vectors of dimension n_u, then

$$K(q^{-1})\ u(t) \equiv 0 \Rightarrow k_i = 0 \qquad i = 1,2,\ldots,n$$

In particular, let u(t) be persistently exciting of sufficiently high order. Assume that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} v^T(t)\ v(t) = 0$$

where

$$v(t) = \left[C_S^{-1}(q^{-1})\ B_S(q^{-1}) - C_M^{-1}(q^{-1},\theta)\ B_M(q^{-1},\theta)\right] u(t)$$

Then it follows that

$$C_S^{-1}(z)\ B_S(z) = C_M^{-1}(z,\theta)\ B_M(z,\theta) \qquad \text{a.e.}$$

4. IDENTIFIABILITY CONCEPTS

The question of identifiability concerns the principal possibility to determine the characteristics of a system using input output data. The data may have been collected during closed loop operation.

It is important to distinguish between two concepts. For control purposes it is sufficient to have a model that describes the input output relationship as well as the noise characteristics of the output. On the other hand, the objective of the identification may be to estimate certain parameters in the model. These may, for instance, correspond to physical constants. Notions of identifiability, that correspond to these two situations, are introduced in this section.

The identification result and hence the identifiability properties depend on the following factors:

- S: The system, described by eq. (3.1)
- M: The set of models (model structure), $M(\theta)$, described by eq. (3.2). (In this report all models are described by vector difference equations).
- I: The identification method
- X: The experiment conditions, e.g. the choice of (extra) input signal, the feedback structure and the regulator parameters.

Of these factors, M and I can usually be chosen freely. In many cases also some of the experiment conditions X are disposable. However, as pointed out in the introduction, they cannot always be chosen in the most favourable way, i.e. to perform the experiments in open loop with a persistently exciting input signal.

Consider for given S,M,I, and X the sequence of estimates θ_N. The smallest set into which the estimates converge w.p. 1

will be denoted by

$D_I(S,M,I,X)$

i.e.

$\theta_N \rightarrow D_I(S,M,I,X)$ w.p. 1 as $N \rightarrow \infty$[†]

$D_I(S,M,I,X)$ therefore defines the obtained models.

When only a description of the open loop system is required, the subset of desired models is defined by

$$D_T(S,M) = \{\theta \,|\, A_S^{-1}(z)\, B_S(z) = A_M^{-1}(z,\theta)\, B_M(z,\theta) \text{ and}$$
$$A_S^{-1}(z)\, C_S(z) = A_M^{-1}(z,\theta)\, C_M(z,\theta)\}$$

The first equality in this definition means that the model $M(\theta)$ has the same input output relation as the system S. The second equality means that the noise characteristics of the output are the same for the model and the system. If S,M,I and X are such that all obtained models belong to this set of desired models, the true input output relation and the true noise characteristics of the system S, eq. (3.1), have been identified. The notion System Identifiability will be used for this situation. Clearly, it is possible to obtain system identifiability only for such models that $D_T(S,M)$ is non-empty. If this is the only requirement on M to obtain system identifiability, the system is said to be Strongly System Identifiable. The concept Parameter Identifiable will be used when θ_N actually converges to a value that corresponds to a "true parameter value". These concepts are formally defined as follows:

[†]The notion $\theta_N \rightarrow D_I$ as $N \rightarrow \infty$ means that $\inf_{\theta \in D_I} |\theta - \theta_N| \rightarrow 0$ as $N \rightarrow \infty$. It does not imply that $\{\theta_N\}$ converges to a certain value.

Definitions

For a given identification method, I, and a given set of experiment conditions, X,

a) The system S is said to be System Identifiable for the model structure M, $SI(M,I,X)$, if $\theta_N \to D_T(S,M)$ w.p. 1 as $N \to \infty$, i.e. $D_I(S,M,I,X) \subset D_T(S,M)$.

b) The system S is said to be Strongly System Identifiable, $SSI(I,X)$, if it is $SI(M,I,X)$ for all model structures M such that $D_T(S,M)$ is non-empty.

c) The system S is said to be Parameter Identifiable for the model structure M, $PI(M,I,X)$, if it is $SI(M,I,X)$ and $D_T(S,M)$ consists of only one element.

Remark: Notice that $D_I \subset D_T$ in particular implies that D_T is non-empty.

The arguments S,M,I and X in $SI(M,I,X)$, $SSI(I,X)$ and $PI(M,I,X)$, as well as in $D_I(S,M,I,X)$ and $D_T(S,M)$ will sometimes be suppressed.

Clearly the concepts SI, SSI and PI depend on the choices of M, I and X. It is the purpose of the following analysis to clarify the effects of different M, I and X. The following general comments about the choices of M, I and X apply. More detailed analysis is given in Sections 5-9.

M: A necessary condition on M to achieve $SI(M,I,X)$ clearly is that $D_T(S,M)$ is non-empty. If the system is $SSI(I,X)$, this condition is also a sufficient condition for $SI(M,I,X)$. In that case the fact that the system may operate in closed loop does not add any extra difficulties when choosing appropriate model structures.

I: Some different methods will be considered. It will be shown that a prediction error method as described in Section 3.3 can be applied to closed loop data exactly as for open loop data. If SI(*M*,*I*,*X*) is not obtained by this method, any other method will also fail, (see Section 6).

X: Naturally it is most desirable if the experiments can be performed in open loop and with a persistently exciting input, u_1. If this is not possible, similar identifiability properties are obtained if an extra perturbation signal is added to the input of the process, if noise is added in the regulator loop, or if the regulator is time varying, (see Sections 5 and 7). The only really unfavourable situation is when the feedback is linear, time invariant and noise free, and when no extra input can be added, (see Sections 5 and 9). Then the system is not SSI(*I*,*X*), and it cannot be tested from input output data whether it is SI(*M*,*I*,*X*), or not, (see Section 8).

Notice that PI(*M*,*I*,*X*) is always implied by SI(*M*,*I*,*X*) if $D_T(S,M)$ consists of only one value. This condition does not involve neither *I* nor *X*. Therefore it is most suitable to study the effects of different experiment conditions on the identifiability properties by considering SI(*M*,*I*,*X*). If PI(*M*,*I*,*X*) is desired, this property follows from SI(*M*,*I*,*X*) exactly as in the open loop case. As mentioned below, the latter problem is treated by other authors.

The concepts can be illustrated by the following simple examples:

EXAMPLE 4.1

Consider the single-input single-output system

$$y(t+1) = b\,u(t) + e(t+1) \qquad (4.1)$$

Let the experiment conditions be that the system is governed by the feedback law, X_1,

$$u(t) = g\, y(t) \tag{4.2}$$

Assume that the parameter $\hat{b}$ in the model structure M_1

$$y(t+1) = \hat{b}\, u(t) + \varepsilon(t+1) \tag{4.3}$$

shall be estimated using the method of least squares (I_1). It is easy to see that $\hat{b}$ will converge to b. Hence the system S is $SI(M_1,I_1,X_1)$, as well as $PI(M_1,I_1,X_1)$.

Consider now a different model structure, M_2:

$$y(t+1) + \hat{a}\, y(t) = \hat{b}\, u(t) + \varepsilon(t+1) \tag{4.4}$$

Insert the feedback law (4.2) in eq. (4.4):

$$y(t+1) + (\hat{a}-g\hat{b})\, y(t) = \varepsilon(t+1)$$

Hence all $\hat{a}$ and $\hat{b}$ such that

$$\hat{a} - g\hat{b} = -gb$$

will give a correct description of the closed loop system (4.1) with the feedback law (4.2). The set $D_I(S,M_2,I_1,X_1)$ therefore consists of all $\hat{a}$ and $\hat{b}$ satisfying this equation. However, obviously e.g.

$$\hat{a} = 1$$
$$\hat{b} = b + 1/g$$

in eq. (4.4) does not give the correct transfer function of the system S. The system S is thus neither $SI(M_2,I_1,X_1)$ nor $PI(M_2,I_1,X_1)$. In particular S is not $SSI(I_1,X_1)$ for the experimental configuration (feedback law) under consideration.

EXAMPLE 4.2

Consider again the system (4.1) and suppose that the input is white noise, uncorrelated with $\{e(t)\}$. This means that the system is operating in open loop. Let these experiment conditions be denoted by X_2. Let the model structure M_3 be given by the set of models

$$y(t+1) + \hat{a}\, y(t) = \hat{b}_1\, u(t) + \hat{b}_2\, u(t-1) + \varepsilon(t+1) + \hat{c}\, \varepsilon(t) \quad (4.5)$$

The maximum likelihood method, I_2, is used to estimate the parameters $\hat{a}$, $\hat{b}_1$, $\hat{b}_2$ and $\hat{c}$. It can be shown that the method gives estimates that converge to values satisfying

$$\begin{aligned} \hat{a} &= \mu \\ \hat{b}_1 &= b \\ \hat{b}_2 &= \mu b \\ \hat{c} &= \mu \end{aligned}$$

where μ is any real number. S is thus not $PI(M_3,I_2,X_2)$. However, the transfer function of eq. (4.5) for these parameter values is the correct one for any value of μ. Consequently, S is $SI(M_3,I_2,X_2)$. In fact, for this experiment configuration, X_2, (i.e. choice of input signal), the system S is $SSI(I_2,X_2)$

Remark: Parameter identifiability for systems operating in open loop has been discussed by a number of authors, see e.g. Tse and Anton (1972), Glover and Willems (1973). Usually the system matrices are assumed to correspond to a certain parameter value θ^o for the given model parametrization. In such a case the parameter θ^o is said to be identifiable w.p. 1 (in probability) if there exists a sequence of estimates that tends to θ^o w.p. 1 (in probability). This definition is given by Tse and Anton (1972). They use convergence in probability as the convergence concept. Now, the sequence of estimates converges to θ^o w.p. 1 if and only if the set $D_I(S,M,I,X) = \{\theta^o\}$.

Therefore the definition just cited is a special case of the definition c) above.

Clearly, a system S can be $PI(M,I,X)$ only if $D_T(S,M) = \{\theta^o\}$. This means that there exists a one to one correspondence between the transfer function and the parameter vector θ^o. This one to one correspondence can hold globally or locally around a given value. The terms global and local identifiability have been used for the two cases, see e.g. Bellman and Åström (1970). Definition c) clearly corresponds to global parameter identifiability.

The problem to obtain a one to one correspondence falls in the field of canonical representation of transfer functions. This is a field that has received much attention. The special questions related to canonical forms for identification have been treated by e.g. Caines (1971), Mayne (1972) and Rissanen (1973).

5. DIRECT IDENTIFICATION USING PREDICTION ERROR METHODS

When direct identification is applied, the feedback is completely disregarded. The input and the output from the system are measured and treated as in the open loop case. In this section the identifiability properties of direct identification are considered for various feedback cases.

It is assumed that a prediction error identification method is used, and that either the loss function (3.6) or (3.7) is minimized with respect to θ. In Ljung (1974) it is shown that for this identification method, I,

$$\theta_N \to D_I(S,M,I,X) = \{\theta \mid \lim_{N\to\infty} \frac{1}{N} \sum_0^{N-1} E\,||\hat{y}(t+1|t;S) - \hat{y}(t+1|t;M(\theta))||^2 = 0\} \qquad (5.1)^\dagger$$

It is straightforward to show that D_T also can be written

$$D_T(S,M) = \{\theta \mid C_S^{-1}(z)\, A_S(z) = C_M^{-1}(z,\theta)\, A_M(z,\theta) \text{ and } C_S^{-1}(z)\, B_S(z) = C_M^{-1}(z,\theta)\, B_M(z,\theta)\}$$

It follows from (5.1), (3.3) and (3.4) that $D_T \subset D_I$. Now, if $D_T \neq D_I$, the system is not $SI(M,I,X)$. Assume that $\theta^* \in D_I$. Denote

$$\begin{aligned} K(q^{-1}) &= K = C_S^{-1}(q^{-1})\, A_S(q^{-1}) - C_M^{-1}(q^{-1},\theta^*)\, A_M(q^{-1},\theta^*) \\ L(q^{-1}) &= L = C_S^{-1}(q^{-1})\, B_S(q^{-1}) - C_M^{-1}(q^{-1},\theta^*)\, B_M(q^{-1},\theta^*) \end{aligned} \qquad (5.2)$$

---

† It is here assumed that the processes are basically stationary so that the limit exists. For very special feedback laws (linear ones that converge to low order regulators) the set D_I given by (5.1) may actually not be the smallest set into which θ_N converges. Such laws, however, will not be considered and these problems are overlooked.

Then K and L are linear time invariant operators. Since $\theta^* \in D_I$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{0}^{N-1} E\{||Ky - Lu||^2\} = 0 \qquad (5.3)$$

If it is possible to show that (5.3) cannot be satisfied unless $K = 0$ and $L = 0$, then it follows from (5.2) that $\theta^* \in D_T(S,M)$. This means that $D_T \supset D_I$, and consequently that the system is $SI(M,I,X)$.

Consider the various feedback cases discussed in Section 2. Clearly, in all cases K must be zero if L is zero. Otherwise (5.3) would state that y(t) is not persistently exciting of a certain order. This contradicts the fact that y(t) contains filtered white noise.

The input signal can be decomposed as

$u(t) = u_1(t) + u_2(t)$ (see Fig. 2.1)

where u_1 is independent of e. If u_1 is persistently exciting of sufficiently high order, the signal Lu_1 will be non zero for non zero L. Then eq. (5.3) implies that also the signal $Ky - Lu_2$ is non zero. This signal contains a component that is formed from e, and consequently cannot be completely correlated with Lu_1. Hence, $L = 0$ and $K = 0$ follows as soon as u_1 is persistently exciting.

Consider now the different experiment conditions X: (cf Fig.2.1)

- <u>There is an extra perturbation signal u_1 at the experiment designer's disposal.</u> Then this signal can always be chosen to be persistently exciting of sufficiently high order, which implies that $K = 0$ and $L = 0$.

- <u>There is extra noise v_2 added in the feedback loop.</u> Suppose that this noise, independent of v_1 (and u_1), is added to each component of u_2 and is persistently exciting. Such noise will be called non degenerate. Then u(t) can be decomposed as described above and K = 0, L = 0 follows.

- o <u>Non linear regulator.</u> The signal $u = u_2$ may or may not be persistently exciting, depending on the number of inputs and how the non linearities enter. If it is persistently exciting, both K and L are either zero or non zero. In the latter case (5.3) would imply a linear, time invariant and noise free relationship between the input and output. This cannot be the case for a non linear feedback, which is non linear in every component. Such a non linear feedback yielding a persistently exciting signal u_2 will be called non degenerate.

- o <u>Time varying regulator.</u> A time invariant relationship (5.3) imposes important restrictions on how the regulator may vary in time. These restrictions can always be violated if the variation of the regulator is at the experiment designer's disposal. Such a variation will be called persistent. Then K = 0 and L = 0 follows. In Section 7 it is discussed how the regulator can be varied in order to achieve SSI.

In all these cases it follows that K = 0 and L = 0, which implies System Identifiability for the chosen model structure M. This is true regardless of the model structure, as long as D_T is non empty and the signals are persistently exciting of proper order. Hence the results can be summarized as follows:

IDENTIFIABILITY RESULT 5.1

Let the identification method I be a straightforwardly applied prediction error method. Let the experiment conditions X be such that either

- o there is a persistently exciting perturbation signal u_1
- o there is a non degenerate noise source in the feedback loop
- o the regulator is a non degenerate non linear feedback or
- o the regulator is persistently time varying

Then, for this I and X, the system is Strongly System Identifiable.

Suppose that the regulator is linear, time invariant and noise free. Then (5.3) can be satisfied with non zero K and L. It must then be examined, whether such K and L can be consistent with the model structure and the system according to (5.3). Thus the conditions for SI(M,I,X) will contain relationships involving the chosen model structure, as well as orders and other characteristics of the feedback law and the system. It is quite clear that SSI cannot be obtained in this case. For example, a model structure M can be chosen, which allows both the true system and the system which is obtained when the input is eliminated, cf also Example 4.1. In Section 9 the case with linear, time invariant feedback is discussed at length for the single-input single-output case.

6. COMPARISON BETWEEN DIRECT AND INDIRECT IDENTIFICATION

In Section 2 two principally different identification methods were described: direct and indirect identification. In this section these methods are compared. Applicability, as well as identifiability properties, are considered.

By Indirect identification is meant a variety of different identification schemes. The open loop system parameters are determined in one or another way from the closed loop system characteristics. In this section the following technique is considered. The output of the closed loop system is modelled as an autoregressive-moving average process. The parameters of the open loop system are then solved for from the parameters of this closed loop system using the knowledge of the feedback law. It is thus required that the feedback law is known exactly. This means that when there is noise in the feedback loop this scheme cannot be straightforwardly applied. If the regulator is time varying, it must essentially be switching between different constant regulators. Then the data can be processed separately for each regulator. Special indirect identification schemes can be designed for other feedback structures. These techniques are not considered here.

Direct identification can be used without any restrictions on the regulators.

It is shown in this section that the two identification methods are equivalent from the identifiability point of view. The results hold for the asymptotic case. That is, it is assumed that sufficiently many data are supplied so that closed loop characteristics, as well as open loop system parameters, can be estimated with arbitrary accuracy.

The case when there is no extra perturbation signal is treated in the following:

RESULT 6.1

Consider the multiple input, multiple output system (3.1)

$$A_S(q^{-1})\ y(t) = B_S(q^{-1})\ u(t) + C_S(q^{-1})\ e(t) \qquad (6.1)$$

with the feedback law(s)

$$F^{(i)}(q^{-1})\ u(t) = G^{(i)}(q^{-1})\ y(t) \qquad i = 1,\dots,\nu$$

where

$$F^{(i)}(z) = I + zF_1^{(i)} + \dots + z^{n_{fi}}\ F_{n_{fi}}^{(i)}$$

$$G^{(i)}(z) = G_0^{(i)} + zG_1^{(i)} + \dots + z^{n_{gi}} F_{n_{gi}}^{(i)}$$

The model is given by eq. (3.2)

$$A_M(q^{-1},\theta)\ y(t) = B_M(q^{-1},\theta)\ u(t) + C_M(q^{-1},\theta)\ \varepsilon(t)$$

In the case of direct identification (I_1) the input output data of the system (6.1), obtained with the regulator switching between the ν values, are treated as data from an open loop system. A prediction error method is used to determine the parameter vector θ.

In the case of indirect identification (I_2) the resulting closed loop characteristics for each regulator is determined (with arbitrary accuracy) by any kind of time series analysis of the output. The model parameter vector θ is then determined so that the model plus feedback i, $i = 1,\dots,\nu$ (which must be known) accounts for the estimated closed loop transfer function with regulator i, $i = 1,\dots,\nu$.

Suppose that the number of input output data obtained for each regulator tends to infinity. Let regulator i be used $\gamma_i > 0$ part of the total time. Assume also that the regulator switches so seldom that initial effects can be neglected. Then the limits of the estimates that may result from the two identification methods are the same, i.e.

$$D_I(S,M,I_1,X) = D_I(S,M,I_2,X)$$

Furthermore

$$D_I(S,M,I_1,X) = \{\theta \mid \hat{C}^{-1}[\hat{A}-\hat{B}(F^{(i)})^{-1}\, G^{(i)}] = $$
$$= C^{-1}[A-B(F^{(i)})^{-1}\cdot G^{(i)}]\,; i = 1,\ldots,\nu\} \qquad (6.2)$$

PROOF

Consider first direct identification. It follows from (5.1) that $D_I(S,M,I_1,X)$ is defined by

$$\sum_{i=1}^{r} \gamma_i E\{[(C^{-1}A-C^{-1}B(F^{(i)})^{-1}G^{(i)})-(\hat{C}^{-1}\hat{A}-\hat{C}^{-1}\hat{B}(F^{(i)})^{-1}G^{(i)})]\; y\}^T$$
$$\{[(C^{-1}A-C^{-1}B(F^{(i)})^{-1}G^{(i)})-(\hat{C}^{-1}\hat{A}-\hat{C}^{-1}\hat{B}(F^{(i)})^{-1}G^{(i)})]\; y\} = 0$$

The set $D_I(S,M,I_1,X)$ for direct identification is thus characterized by (6.2)

In the case of indirect identification the closed loop transfer functions

$$y(t) = H^{(i)}(q^{-1})e(t)$$

where

$$H^{(i)}(q^{-1}) = [A_S(q^{-1})-B_S(q^{-1})(F^{(i)}(q^{-1}))^{-1}G^{(i)}(q^{-1})]^{-1}\cdot C_S(q^{-1})$$

corresponding to the various regulators are determined. To obtain the open loop system parameters knowing the feedback laws, the following equation must be solved for θ

$$[\hat{A} - \hat{B}(F^{(i)})^{-1}G^{(i)}]^{-1}\hat{C} = H^{(i)} \qquad i = 1,\ldots,\nu \qquad (6.3)$$

Here $H^{(i)}$, $F^{(i)}$ and $G^{(i)}$ are known filters. Inserting the expression for $H^{(i)}$ it is found that this equation is nothing but (6.2).

The possible estimates for direct and indirect identification are consequently identical. □

The set $D_I(S,M,I,X)$ clearly depends on $F^{(i)}$, $G^{(i)}$ (corresponding to X), the true system S and the model structure M. The system is $SI(M,I,X)$ if $D_I \subset D_T$, and $PI(M,I,X)$ if $D_T = \{\theta^o\}$. In the next section the solution set D_I of (6.2) is investigated. Necessary and sufficient conditions for SSI are given there.

When there is an extra (persistently exciting) perturbation signal u_1, it can be shown, in the same way as in Result 6.1, that SI is obtained also for indirect methods, if the feedback is linear and noise free.

Since direct and indirect identification are equivalent from the identifiability point of view, there is no reason for using indirect identification. Indirect identification can be used only when certain restrictions on the regulator are satisfied. Also it requires a priori knowledge about the feedback laws.

7. SYSTEM IDENTIFIABILITY IN THE CASE WITH SEVERAL LINEAR REGULATORS

The possible system parameter estimates, obtained from experiments where the regulator switches between several linear feedback laws, are studied in this section. These estimates are given by (6.2), which can be written

$$\hat{C}^{-1}\hat{A} - C^{-1}A - (\hat{C}^{-1}\hat{B}-C^{-1}B)(F^{(i)})^{-1}G^{(i)} = 0 \qquad i = 1,\ldots,\nu$$

or

$$[\hat{C}^{-1}\hat{A} - C^{-1}A;\ C^{-1}B-\hat{C}^{-1}\hat{B}]R_\nu = [0\ \ldots\ 0] \tag{7.1}$$

where

$$R_\nu = \begin{bmatrix} I & I & \ldots & I \\ (F^{(1)})^{-1}G^{(1)} & (F^{(2)})^{-1}G^{(2)} & \ldots & (F^{(\nu)})^{-1}G^{(\nu)} \end{bmatrix}$$

The dimensions of the matrices in (7.1) are $n_y|(n_y+n_u)$, $(n_y+n_u)|(n_y\cdot\nu)$ and $n_y|(n_y\cdot\nu)$ respectively. Notice that each element in the matrices is a function of z.

Suppose now that

$$\text{rank } R_\nu = n_y + n_u \qquad \text{a.e.} \quad z \tag{7.2}$$

Then (7.1) implies that

$$\hat{C}^{-1}\hat{A} - C^{-1}A = 0$$

$$C^{-1}B - \hat{C}^{-1}\hat{B} = 0 \tag{7.3}$$

i.e. that $D_I(S,M,I,X) = D_T(S,M)$. This means that SI is achieved if (7.2) is satisfied. It is important to notice that this condition can be met by conditions on the regulators only. Thus (7.2) is a sufficient condition for SSI(I,X). A necessary condition for (7.2) to hold is obviously

$$\nu n_y \geq n_y + n_u$$

Introduce ν_o as the smallest integer $\geq 1 + n_u/n_y$. The condition can then be written

$$\nu \geq \nu_o \qquad (7.4)$$

Consequently, if the number of regulators is at least ν_o, it is always possible to choose them so that SSI is secured. That is, the true open loop characteristics can be estimated regardless of model structure and the true system (as long as $D_T(S,M)$ is non-empty).

In case $n_u = n_y$, it is sufficient to use two regulators. These shall be chosen so that

$$\det[(F^{(1)}(z))^{-1}G^{(1)}(z) - (F^{(2)}(z))^{-1}G^{(2)}(z)] \not\equiv 0 \qquad (7.5)$$

This is quite a mild condition. It is e.g. satisfied with two "proportional" regulators

$$u(t) = G_0^{(i)}y(t) \qquad i = 1,2$$

for which $G_0^{(1)} - G_0^{(2)}$ is a non singular matrix.

Now, (7.2) is a sufficient but not necessary condition for SI(M,I,X). If it is not satisfied, the null space of R_ν^T is non-empty. However, the null space may not contain any non zero vector of the form

$$[\hat{C}^{-1}\hat{A} - C^{-1}A,\ C^{-1}B - \hat{C}^{-1}\hat{B}]$$

for the assumed structure M,so (7.3) may still follow from (7.1). Notice that R_ν, as well as the vector above, are functions of z. The relationships discussed must hold for almost all z.

The condition that vectors of this form do not belong to the null space of R_ν is clearly quite complex and involves relationships between the regulators, the model structure and the true system. For sufficiently flexible M they can always be violated. The conditions are derived for the case $n_u = n_y = \nu = 1$ in Section 9.

To summarize, it has been shown that in the case where at least ν_o regulators are used it is possible to guarantee Strongly System Identifiability from conditions on the regulators only. Each regulator must be used a non negligable part of the total time. If fewer regulators are used, then conditions for SI(M,I,X) contain also characteristics of the true system,and the system is not Strongly System Identifiable.

8. A POSTERIORI TEST OF IDENTIFIABILITY

In the previous sections criteria for a priori test of identifiability have been given. That is, from such criteria it may be known that the system is Strongly System Identifiable from conditions on the experiment configuration only. For example, if it is known that the feedback is such that Result 5.1 is applicable or that the condition (7.2) is satisfied, then it follows that the system is SSI. In these cases the conditions for SSI do not contain the characteristics of the system. When the number of linear regulators is less than $1 + n_u/n_y$, the criteria are of a different kind. Then the criteria for SI(M,I,X) involve both the model structure and the true system. The conditions for SI(M,I,X) can be tested only if some characteristics of the system is known. In case such a priori knowledge about the system is available, SI(M,I,X) can be tested before the experiment is performed. Otherwise the question of identifiability cannot be resolved a priori. It is then relevant to ask whether it is possible to test identifiability after the identification experiment has been evaluated; a posteriori. That is, is it possible to decide if the system actually was SI(M,I,X) e.g. from the number and character of the solutions of (6.2) or from the estimated structure of the system?

The answer is no. This is seen as follows:

Consider the multivariable system S_1

$$A_{S_1}(q^{-1})\ y(t) = B_{S_1}(q^{-1})u(t) + C_{S_1}(q^{-1})\ e(t) \tag{8.1}$$

with the feedback law R

$$F(q^{-1})\ u(t) = G(q^{-1})\ y(t)$$

Suppose that the system S_1 is SI(M,I,X) for a given model structure M, but that this is not known a priori.

Consider the system S_2

$$\left[A_{S_1}(q^{-1}) + L(q^{-1})G(q^{-1})\right]y(t) = \left[B_{S_1}(q^{-1}) + L(q^{-1})F(q^{-1})\right]u(t) +$$

$$+ C_{S_1}(q^{-1})\ e(t) \qquad (8.2)$$

where L(z) is an arbitrary matrix polynomial.

If this system S_2 is governed by the same feedback law R as S_1, the two closed loop systems have the same input output relationship and noise characteristics, since

$$LG\ y = LF\ u$$

can be eliminated in (8.2). Hence from input output experiments with the feedback R, the systems S_1 and S_2 cannot be distinguished. Thus even if the system (8.1) is found as a unique solution of (6.2) (for the given M) it is impossible to decide whether the true system actually is S_1 or S_2. This means that it cannot a posteriori be established that the obtained system is SI(M,I,X), and hence that the estimated model has the same transfer function as the system. It also means that reliable order tests cannot be performed in such cases.

Consequently, if no a priori information about the structure of the system is available, the feedback case with a linear, time invariant, noise free regulator is not identifiable, in the sense that the result of the identification is not reliable. More experiments have to be performed to check the validity of the model. However, as shown in Section 7, using v_o different regulators it is not only possible to test identifiability but also actually to find the true system.

The conclusions of this section and Section 7 can be summarized as follows:

IDENTIFIABILITY RESULT 8.1

Suppose that linear, noise free feedback with no extra perturbation signal is used in the closed loop configuration. Assume further that no other a priori knowledge about the system than that it is linear is available.

To obtain estimates that can be relied upon it is necessary to use ν_o different linear regulators in the feedback loop. They must be chosen so that (7.2) is satisfied. The number ν_o is the smallest integer that is greater or equal to $1 + n_u/n_y$.

With such regulators Strong System Identifiability is guaranteed.

With fewer regulators Strong System Identifiability cannot be achieved.

9. NECESSARY AND SUFFICIENT CONDITIONS FOR IDENTIFIABILITY OF SINGLE-INPUT SINGLE-OUTPUT SYSTEMS OPERATING WITH A LINEAR FEEDBACK

It was found in Section 6 that direct and indirect identification are equivalent from the identifiability point of view and that the set $D_I(S,M,I,X)$ is given by (assuming that a time invariant regulator is used) eq. (6.2)

$$\hat{C}^{-1}(\hat{A}-\hat{B}F^{-1}G) = C^{-1}(A-BF^{-1}G)$$

The purpose of this section is to analyse this equation in detail for the special case of single-input single-output systems. As before it is generally assumed that a prediction error method, which in this case means the maximum likelihood method, is used. In the foregoing analysis of multiple-input multiple-output systems an unspecified model structure has been assumed, i.e. it is not specified how the polynomials depend on the vector θ. In the examination of single-input single-output systems in this section the following commonly used structure will be considered.

The system S is assumed to be given by, cf eq. (3.1)

$$A(q^{-1})\; y(t) = q^{-k}\; B(q^{-1})\; u(t) + C(q^{-1})e(t) \tag{9.1}$$

where

$$A(z^{-1}) = A = 1 + a_1 z^{-1} + \ldots + a_{n_a} z^{-n_a} \qquad a_{n_a} \neq 0$$

$$B(z^{-1}) = B = \qquad b_1 z^{-1} + \ldots + b_{n_b} z^{-n_b} \qquad b_1 \neq 0,\; b_{n_b} \neq 0$$

$$C(z^{-1}) = C = 1 + c_1 z^{-1} + \ldots + c_{n_c} z^{-n_c} \qquad c_{n_c} \neq 0$$

$$k \geq 0 \quad n_a \geq 0 \quad n_b \geq 1 \quad n_c \geq 0$$

There is no common factor to the three polynomials A, B and C.

The feedback is given by

$$F(q^{-1})\ u(t) = G(q^{-1})\ y(t) \qquad (9.2)$$

where

$$F(z^{-1}) = F = 1 + f_1 z^{-1} + \ldots + f_{n_f} z^{-n_f} \qquad f_{n_f} \neq 0$$

$$G(z^{-1}) = G = g_o + g_1 z^{-1} + \ldots + g_{n_g} z^{-n_g} \qquad g_{n_g} \neq 0$$

Some of the first coefficients of G may be zero. F and G have no common factor.

The model structure M is assumed to be given by

$$\hat{A}(q^{-1})\ y(t) = q^{-\hat{k}}\ \hat{B}(q^{-1})\ u(t) + \hat{C}(q^{-1})\ \varepsilon(t) \qquad (9.3)$$

where

$$\hat{A}(z^{-1}) = \hat{A} = 1 + \hat{a}_1 z^{-1} + \ldots + \hat{a}_{\hat{n}_a} z^{-\hat{n}_a}$$

$$\hat{B}(z^{-1}) = \hat{B} = \hat{b}_1 z^{-1} + \ldots + \hat{b}_{\hat{n}_b} z^{-\hat{n}_b}$$

$$\hat{C}(z^{-1}) = \hat{C} = 1 + \hat{c}_1 z^{-1} + \ldots + \hat{c}_{\hat{n}_c} z^{-\hat{n}_c}$$

$$\hat{n}_a \geq 0,\ \hat{n}_b \geq 1,\ \hat{n}_c \geq 0,\ \hat{k} \geq 0$$

The parameter vector θ is given by

$$\theta = [\hat{a}_1 \ \ldots\ \hat{a}_{\hat{n}_a}\ \hat{b}_1\ \ldots\ \hat{b}_{\hat{n}_b}\ \hat{c}_1\ \ldots\ \hat{c}_{\hat{n}_c}]$$

In order to guarantee that $D_T(S,M)$ is non-empty it is necessary and sufficient to assume that

$$\hat{k} \le k$$

$$n^* = \min(\hat{n}_a - n_a,\ \hat{n}_b + \hat{k} - n_b - k,\ \hat{n}_c - n_c) \ge 0 \tag{9.4}$$

In order to treat the case $\hat{k} < k$ in a nice way in the analysis a new polynomial $\tilde{B}(z^{-1})$ is introduced through

$$z^{-k}B(z^{-1}) = z^{-\hat{k}}\tilde{B}(z^{-1})$$

Clearly, $\deg\tilde{B} = \tilde{n}_b = n_b + k - \hat{k}$

Before going into details in the analysis two more integers must be defined. The polynomial $AF - z^{-k}BG$ may not be of degree $\max(n_a+n_f,\ k+n_b+n_g)$, since some of the last coefficients can be zero.

Assume that the true degree is $\max(n_a+n_f,\ k+n_b+n_g)-r$. Note that $r > 0$ implies that $n_a+n_f = k+n_b+n_g$. Assume further that the polynomials C and $AF - z^{-k}BG$ have exactly n_p common factors. Introduce the three polynomials D, H and P by

$$C = DP$$

$$AF - z^{-k}BG = HP \tag{9.5}$$

where P is the common factor of degree n_p. Clearly, the degree of H is

$$n_h = \max(n_a+n_f,\ k+n_b+n_g) - n_p - r$$

The equation (6.2) can be rewritten as

$$C[\hat{A}F - z^{-\hat{k}}\hat{B}G] - \hat{C}[AF - z^{-\hat{k}}\tilde{B}G] = 0 \tag{9.6}$$

or after use of (9.5)

$$D[\hat{A}F - z^{-\hat{k}}\hat{B}G] - \hat{C}H = 0 \tag{9.7}$$

The solutions of equation (9.7) describe the set $D_I(S,M,I,X)$. The remaining part of this section is organized as follows. First, necessary conditions for $SI(M,I,X)$ and $PI(M,I,X)$ are considered. Then sufficient conditions for the same concepts are derived. It is proved that the necessary and the sufficient conditions are equivalent. In the end of the section the obtained results are discussed and compared. The concept of $SSI(I,X)$ is also analysed. Finally, as an illustration, the obtained results are applied to a system with a minimum variance controller.

9.1 Necessary conditions for identifiability

Now, necessary conditions for identifiability for the model structure (9.3) will be derived. The definition of $SI(M,I,X)$ is $D_I(S,M,I,X) \subset D_T(S,M)$. It is thus necessary to express the latter set. It is given by the solutions w. r. t. θ of

$$\hat{C}^{-1}\hat{A} = C^{-1}A$$
$$\hat{C}^{-1}\hat{B} = C^{-1}\tilde{B} \qquad (9.8)$$

It is shown in Söderström (1973) that the general solution of (9.8) is given by

$$\hat{A} = AL$$
$$\hat{B} = \tilde{B}L \qquad (9.9)$$
$$\hat{C} = CL$$

where

$$L = L(z^{-1}) = 1 + \ell_1 z^{-1} + \ldots + \ell_{n^*} z^{-n^*}$$

and the coefficients $\{\ell_i\}$ are arbitrary. By a direct insertion of (9.9) into (9.6) it is easy to see that

$D_T(S,M) \subset D_I(S,M,I,X)$. If the system S is $SI(M,I,X)$ then $D_T(S,M) = D_I(S,M,I,X)$.

A necessary condition for $SI(M,I,X)$ is derived as follows. Collect the coefficients of the polynomials $\hat{A} - A$, $\hat{B} - \tilde{B}$ and $\hat{C} - C$ in the vector x of dimension $(\hat{n}_a+\hat{n}_b+\hat{n}_c)$. Then (9.9) describes a n^*-dimensional subspace in the x-space. Moreover, the equation (9.7), which describes the set $D_I(S,M,I,X)$ is in fact a homogeneous system of linear equations in x. Then a necessary and sufficient condition for $SI(M,I,X)$ is that the corresponding linear transformation has a null space of dimension equal to n^*. Since $D_T \subset D_I$, the dimension of the null space is always at least n^*. The dimension is equal to the number of unknowns minus the number of linear independent equations. Denote the latter number by ρ. Then the condition can be expressed as

$$\hat{n}_a + \hat{n}_b + \hat{n}_c - \rho = n^*$$

But in view of (9.7) there is an upper limit of ρ, namely

$$\rho \leq \max[n_c-n_p+\hat{n}_a+n_f,\ n_c-n_p+\hat{k}+\hat{n}_b+n_g,\ \hat{n}_c+n_h]$$

which after inserting n_h gives the following <u>necessary condition</u> for $SI(M,I,X)$

$$[\hat{n}_a+\hat{n}_b+\hat{n}_c] - \max[n_f+\max(\hat{n}_a+n_c,\ n_a+\hat{n}_c-r),$$

$$n_g + \max(\hat{k}+\hat{n}_b+n_c,\ k+n_b+\hat{n}_c-r)] + n_p$$

$$\leq \min[\hat{n}_a-n_a,\ \hat{n}_b+\hat{k}-n_b-k,\ \hat{n}_c-n_c] \qquad (9.10)$$

From this calculation it is easy to get the necessary condition for $PI(M,I,X)$. Clearly (9.10) must be fulfilled. Moreover it is obvious that also $n^* = 0$ is a necessary condition for $PI(M,I,X)$ since (9.9) gives solutions of (9.8). Thus a necessary condition for $PI(M,I,X)$ can be written as

$$n^* = 0$$

$$\hat{n}_a + \hat{n}_b + \hat{n}_c \leq \max[n_f + \max(\hat{n}_a + n_c,\ n_a + \hat{n}_c - r),$$

$$n_g + \max(\hat{k} + \hat{n}_b + n_c,\ k + n_b + \hat{n}_c - r)] - n_p \tag{9.11}$$

Note that (9.11) just means that in the system of equations (9.7) the number of unknowns is less or equal the number of equations.

9.2 Sufficient conditions for identifiability

Sufficient conditions for identifiability will now be considered. The solutions of the equations (9.7) will be considered. Conditions, ensuring that the solutions fulfil (9.9) will be stated. Since D(z) and H(z) by construction are relatively prime, all solutions of (9.7) must fulfil

$$\hat{C} = DQ$$

$$\hat{A}F - z^{-\hat{k}}\hat{B}G = HQ \tag{9.12}$$

$$Q = Q(z^{-1}) = 1 + q_1 z^{-1} + \ldots + q_{n_q} z^{-n_q}$$

where

$$n_q = \min[\hat{n}_c - n_c + n_p,\ \max(\hat{n}_a + n_f,\ \hat{k} + \hat{n}_b + n_g) - n_h]$$

The second equation in (9.12) gives after multiplication with P(z)

$$P[\hat{A}F - z^{-\hat{k}}\hat{B}G] = Q[AF - z^{-k}BG]$$

or rewritten

$$F[\hat{A}P - AQ] = z^{-\hat{k}}G[\hat{B}P - \tilde{B}Q] \tag{9.13}$$

Assume for a moment that it follows from (9.13) that the expressions in the brackets are zero. After multiplication with D the following equations are obtained

$$[\hat{A}P-AQ]D = 0$$

$$[\hat{B}P-\tilde{B}Q]D = 0$$

With use of (9.5) and (9.12) these equations become

$$\hat{A}C - A\hat{C} = 0$$
$$\hat{B}C - \tilde{B}\hat{C} = 0 \qquad (9.14)$$

which is nothing but (9.8) rewritten into a new form. Thus if (9.12) implies (9.14), then the system is SI(*M*,*I*,*X*). It remains to give conditions such that (9.13) implies (9.14). Since F and G have no common factors, all solutions of (9.13) are such that $\hat{A}P-AQ$ is a factor of $z^{-\hat{k}}G$ and $\hat{B}P-\tilde{B}Q$ is a factor of F. Thus a sufficient condition for SI(*M*,*I*,*X*) is that at least one of the following two inequalities is satisfied

$$n_f \geq \deg[\hat{B}P-\tilde{B}Q]$$

$$\hat{k} + n_g \geq \deg[\hat{A}P-AQ]$$

or expressed in a more compact way

$$n_r \hat{=} \max[n_f-\max(\hat{n}_b+n_p,\ n_b+k-\hat{k}+n_q),$$
$$\hat{k} + n_g - \max(\hat{n}_a+n_p,\ n_a+n_q)] \geq 0 \qquad (9.15)$$

Sufficient conditions for PI(*M*,*I*,*X*) can easily be derived from the discussion above. Suppose that the sufficient condition for SI(*M*,*I*,*X*), (9.15), is fulfilled. Then, as mentioned above, the solutions of (9.14) are given by (9.9). Thus a sufficient condition for PI(*M*,*I*,*X*) is

$$n_r \geq 0 \qquad n^* = 0 \qquad (9.16)$$

9.3 Equivalence of necessary and sufficient conditions

The foregoing analysis is summed up and extended in the following theorem.

Theorem 9.1. Consider the system S given by (9.1) with the feedback (9.2) and the model structure M given by (9.3). Assume that the condition (9.4) holds. Let the identification method I be the maximum likelihood method.

i) If the condition

$$\max(n_f - n_b, n_g + k - n_a) - n_p \geq 0 \tag{9.17}$$

is not fulfilled, then there is no model structure M of the considered form, such that the system S is System Identifiable in this model structure M with the identification method I and the experimental configuration X. Moreover, if (9.17) is fulfilled, then there is at least one model structure of the considered form, such that the system S is System Identifiable, as well as Parameter Identifiable, in this model structure M with the identification method I and the experimental configuration X, SI(M,I,X) and PI(M,I,X). One model structure M of this kind is given by

$$\hat{n}_a = n_a \quad \hat{n}_b = n_b \quad \hat{n}_c = n_c \quad \hat{k} = k$$

ii) A necessary and sufficient condition for System Identifiability in the model structure M with the identification method I and the experimental configuration X, SI(M,I,X), is (9.10)

$$[\hat{n}_a + \hat{n}_b + \hat{n}_c] - \max[n_f + \max(\hat{n}_a + n_c,\ n_a + \hat{n}_c - r),$$

$$n_g + \max(\hat{k} + \hat{n}_b + n_c,\ k + n_b + \hat{n}_c - r)] + n_p$$

$$\leq \min[\hat{n}_a - n_a,\ \hat{n}_b + \hat{k} - n_b - k,\ \hat{n}_c - n_c]$$

iii) A necessary and sufficient condition for Parameter Identifiability in the model structure M, with the identification method I and the experimental configuration X, PI(M,I,X), is (9.11)

$$\hat{n}_a + \hat{n}_b + \hat{n}_c - \max[n_f + \max(\hat{n}_a + n_c,\ n_a + \hat{n}_c - r),$$

$$n_g + \max(\hat{k} + \hat{n}_b + n_c,\ k + n_b + \hat{n}_c - r)] + n_p \leq 0$$

$$\min[\hat{n}_a - n_a,\ \hat{n}_b + \hat{k} - n_b - k,\ \hat{n}_c - n_c] = 0$$

Proof: The proof of part ii) consists of tedious calculations and comparisons showing that (9.10) and (9.15) are equivalent. This part of the proof is given in the Appendix.

Part iii) follows from part ii) since the necessary as well as the sufficient condition for PI(M,I,X) are both obtained from the corresponding kind of condition for SI(M,I,X) by adding $n^* = 0$.

Part i) is proved as follows. It follows from part ii) that it is a necessary and sufficient condition for SI(M,I,X) that $n_r \geq 0$, eq. (9.15).

It can be seen after some consideration that n_r is a decreasing function of $\hat{n}_a$, $\hat{n}_b$, $\hat{n}_c$ and $-\hat{k}$. Thus the most favourable choice of these integers is n_a, n_b, n_c and $-k$. Then PI(M,I,X) and SI(M,I,X) are equivalent concepts. For this specific choice the integer n_q simplifies to n_p and n_r becomes

$$n_r = \max[n_f - n_b - n_p,\ k + n_g - n_a - n_p]$$

Thus if (9.17) is not fulfilled, then there is no model structure M such that the system S becomes SI(M,I,X). On the other hand, if it is fulfilled, the model structure given by $\hat{n}_a = n_a$, $\hat{n}_b = n_b$, $\hat{n}_c = n_c$ and $\hat{k} = k$ will give SI(M,I,X), as well as PI(M,I,X). □

Remark 1. The condition (9.17) is both necessary and sufficient for existence of a model structure M of the considered type (9.3) such that the system S is SI(M,I,X). It is clearly not dependent on M.

Remark 2. As shown in Section 5 the system S cannot be strongly system identifiable for the given identification method and experimental conditions (i.e. SSI(I,X)). This can be shown explicitly using the result of the theorem. Choose especially $\hat{n}_a = \hat{n}_b = \hat{n}_c = \hat{n}$ and $\hat{k} = k$, which satisfies (9.4) provided only that

$$\hat{n} \geq \max(n_a, n_b, n_c)$$

However, straightforward calculation simplifies (9.10) to

$$\hat{n} \leq \max[n_f + \max(n_c, n_a - r), n_g + k + \max(n_c, n_b - r)] -$$

$$- \max[n_a, n_b, n_c]$$

which clearly cannot be true if $\hat{n}$ is chosen large. This means that (9.4) is true and that (9.10) is violated. Thus the system is not SI(M,I,X) for such a model structure, i.e. it is not SSI(I,X).

9.4 Application to systems with minimal variance control.

Now the obtained results are applied to a system controlled with a minimal variance strategy. Assume that B(z) has all zeros outside the unit circle. Then the feedback law is given by, see Åström (1970),

$$F(z^{-1}) = zB(z^{-1})/b_1\tilde{F}(z^{-1})$$

$$C(z^{-1}) = A(z^{-1})\tilde{F}(z^{-1}) - b_1 z^{-k-1}G(z^{-1}) \qquad (9.18)$$

$$\tilde{F}(z^{-1}) = 1 + f_1 z^{-1} + \ldots + f_k z^{-k}$$

For this feedback it can be found by trivial calculations

$$n_f = n_b - 1 + k$$

$$n_g = \max(n_c - k - 1,\ n_a - 1)$$

$$n_p = n_c \qquad (9.19)$$

$$r = \max(n_a - n_c + k,\ 0)$$

$$n_h = n_b - 1$$

$$H(z) = B(z) z^{-1}/b_1$$

The necessary condition (9.17) for identifiability becomes for this system

$$n_c \leq \max(k-1,\ n_c - n_a - 1)$$

Since $n_c > n_c - n_a - 1$, the following condition is in fact necessary

$$k \geq n_c + 1 \qquad (9.20)$$

Assume that (9.20) holds. Then two of the expressions in (9.19) simplify to $n_g = n_a - 1$ and $r = n_a - n_c + k$.

The necessary and sufficient condition for $SI(M,I,X)$ becomes

$$\hat{n}_a + \hat{n}_b + \hat{n}_c \leq \max[\hat{n}_a + n_b - 1 + k,\ \hat{k} + \hat{n}_b + n_a - 1,\ \hat{n}_c + n_b - 1] + \min[\hat{n}_a - n_a,\ \hat{n}_b + \hat{k} - n_b - k,\ \hat{n}_c - n_c] \qquad (9.21)$$

A necessary and sufficient condition for $PI(M,I,X)$ is obtained by adding the condition

$$n* = 0$$

to (9.21)

10. NUMERICAL ILLUSTRATION

In order to illustrate the obtained theoretical results some systems were simulated and then identified. The calculations have been carried out on a PDP/15. An interactive identification program, IDPAC, see Gustavsson, Selander and Wieslander (1973) was applied and the maximum likelihood method was used in all the examples. The given accuracies of the parameter estimates are estimated standard deviations. The number of data was for all the examples 1000 and the noise e(t) was generated as gaussian distributed with zero mean and variance 1.0. In all the examples with exclusion of **Example 10.4 direct identification has been used.**

Example 10.1

The system is

$$(1+a_1q^{-1}+a_2q^{-2})y(t) = q^{-1}(b_1q^{-1}+b_2q^{-2})u(t) + e(t) \qquad (10.1)$$

$a_1 = -1.5$, $a_2 = 0.7$, $b_1 = 1.0$, $b_2 = -0.45$

Clearly $n_a = 2$, $n_b = 2$, $n_c = 0$ and $k = 1$. The feedback law is

$$(1+f_1q^{-1}+f_2q^{-2})u(t) = (g_o+g_1q^{-1})y(t) \qquad (10.2)$$

$f_1 = 1.05$, $f_2 = -0.675$, $g_o = -1.55$, $g_1 = 1.05$

Consider the condition for identifiability (9.17). Clearly, for this case $\max(n_f-n_b,\ n_g+k-n_a) = \max(2-2,\ 1+1-2) = 0$. Moreover, n_p must be 0 since the C-polynomial is degenerated to 1. Thus the condition (9.17) is satisfied. The model structure was chosen as $\hat{n}_a = n_a$, $\hat{n}_b = n_b$, $\hat{n}_c = n_c$ and $\hat{k} = k$, which is the most favourable case. The result of the identification is given in Table 10.1.

Parameter	Theoretical value	Obtained results
a_1	-1.5	-1.502 ± 0.032
a_2	0.7	0.706 ± 0.062
b_1	1.0	1.014 ± 0.046
b_2	-0.45	-0.433 ± 0.065
V	500.0	525.0154

Table 10.1. Identification results of Example 10.1

Example 10.2

The system and the feedback are the same as in Example 10.1. However, it is now assumed that k = 1 is not known a priori. For this reason the model structure given by $\hat{n}_a = 2$, $\hat{n}_b = 3$, $\hat{n}_c = 0$ and $\hat{k} = 0$ was tried. Consider the condition (9.15). For this example

$$n_r = \max[2-\max(3+0,\ 2+1-0+n_q),\quad 0+1-\max(2+0,\ 2+n_q)]$$

$$= \max[2-\max(3,\ 3+n_q),\ 1-\max(2,\ 2+n_q)] =$$

$$= \max[-1-\max(0,\ n_q),\ -1-\max(0,\ n_q)] = -1-n_q$$

which is strictly negative irrespective of n_q. Actually, the value is -1. The process is thus not SI(M,I,X). It is possible to obtain an expression for the global minimum points of the loss function by finding the general solution of (9.6) for this case. After inserting the numerical values it can be written as

$$\begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ 0 \\ b_1 \\ b_2 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ -0.69 \\ -0.65 \\ -0.73 \\ 0.47 \end{bmatrix}$$

The result from the identification was

$$\begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{bmatrix} = \begin{bmatrix} 0.33 \\ -0.54 \\ -1.18 \\ -0.23 \\ 0.37 \end{bmatrix}$$

and the loss function has for these parameter values the value 525.0154. A comparison with Table 10.1 indicates that the loss function has a valley in which it changes very little. It was found from the identification that the matrix of second order derivatives has one very small eigenvalue. Also this fact demonstrates the existence of the valley. By comparing the identification result with Table 10.1 the following equation for the valley can be found.

$$\begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{bmatrix} = \begin{bmatrix} -1.50 \\ 0.71 \\ 0 \\ 1.01 \\ -0.43 \end{bmatrix} + \mu \begin{bmatrix} 1.0 \\ -0.68 \\ -0.65 \\ -0.71 \\ 0.44 \end{bmatrix}$$

which is in good accordance with the expected result.

Example 10.3

The system considered is still the one given by (10.1). The feedback, however, is simplified to

$$u(t) = g_o y(t) \tag{10.3}$$

with $g_o = -0.2$. For this feedback $n_f = n_g = 0$, and it is easy to see that the necessary condition (9.17) for identifiability is <u>not</u> satisfied. The model orders were chosen as $\hat{n}_a = 2$, $\hat{n}_b = 2$, $\hat{n}_c = 0$ and $\hat{k} = 1$. It is possible to calculate the general solution of (9.6). It is given by

$$\hat{a}_1 = a_1$$
$$\hat{a}_2 = a_2 + \mu g_o$$
$$\hat{b}_1 = b_1 + \mu$$
$$\hat{b}_2 = b_2$$

The result of the identification was that the loss function has a valley of global minimum points. The loss function changes less than 10^{-4} in the valley. The computer program stopped in the point

$$\hat{a}_1 = -1.44 \pm 0.02$$
$$\hat{a}_2 = 0.76 \pm 0.02$$
$$\hat{b}_1 = 0.34 \pm 0.21$$
$$\hat{b}_2 = -0.32 \pm 0.21$$

which shows that the system is not $SI(M,I,X)$ for this combination of M, I and X.

<u>Example 10.4</u>

For this example indirect identification has been used. Two regulators were used as discussed in Section 7. The system is still given by (10.1). The feedback is given by (10.3) with $g_o = -0.2$ for the first 500 samples and with $g_o = -0.1$ for the last 500 samples. The closed loop system is given by

$$M(q^{-1})y(t) = (1+m_1q^{-1}+m_2q^{-2}+m_3q^{-3})y(t) = e(t)$$

with

$$m_1 = a_1$$
$$m_2 = a_2 - g_o b_1$$
$$m_3 = \quad -g_o b_2$$

The identification results are given in Table 10.2

Parameter	$g_o = -0.2$	$g_o = -0.1$
m_1	-1.45 ± 0.04	-1.47 ± 0.04
m_2	0.84 ± 0.07	0.79 ± 0.07
m_3	-0.065 ± 0.04	-0.041 ± 0.04

Table 10.2. Primary identification results for Example 10.4.

In order to find estimates of (a_1, a_2, b_1, b_2) the following overdetermined system of equations must be solved. The superscript defines the part of the total experiment.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -g_o^1 & 0 \\ 0 & 0 & 0 & -g_o^1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & -g_o^2 & 0 \\ 0 & 0 & 0 & -g_o^2 \end{bmatrix} \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{b}_1 \\ \hat{b}_2 \end{bmatrix} = \begin{bmatrix} m_1^1 \\ m_2^1 \\ m_3^1 \\ m_1^2 \\ m_2^2 \\ m_3^2 \end{bmatrix}$$

This system can be treated in several ways. A natural way is to compute the least squares solution. After simple calculations it is found to be

$$\hat{a}_1 = \frac{m_1^1 + m_1^2}{2}$$

$$\hat{a}_2 = \frac{-g_o^2 m_2^1 + g_o^1 m_2^2}{g_o^1 - g_o^2}$$

$$\hat{b}_1 = \frac{-m_2^1 + m_2^2}{g_o^1 - g_o^2}$$

$$\hat{b}_2 = -\frac{g_o^1 m_3^1 + g_o^2 m_3^2}{(g_o^1)^2 + (g_o^2)^2}$$

From these expressions it is easy to compute the estimates of the accuracies as well. The results are the following:

$$\sigma_{\hat{a}_1} = \sigma_{m_1} / \sqrt{2}$$

$$\sigma_{\hat{a}_2} = \sigma_{m_2} [(g_o^1)^2 + (g_o^2)^2]^{1/2} / |g_o^1 - g_o^2|$$

$$\sigma_{\hat{b}_1} = \sigma_{m_2} \sqrt{2} / |g_o^1 - g_o^2|$$

$$\sigma_{\hat{b}_2} = \sigma_{m_3} / [(g_o^1)^2 + (g_o^2)^2]^{1/2}$$

Inserting the numerical values from Table 10.2 the result given in Table 10.3 is obtained.

Parameter	Theoretical value	Obtained estimate
a_1	-1.5	-1.46 ± 0.03
a_2	0.7	0.74 ± 0.16
b_1	1.0	0.5 ± 0.5
b_2	-0.45	-0.34 ± 0.18

Table 10.3: Final identification results from Example 10.4.

Example 10.5

The same system and the same time varying feedback as in Example 10.4 were considered, but direct identification was applied using all the measurements simultaneously. The result is given in Table 10.4.

Parameter	Theoretical value	Obtained estimate
a_1	-1.5	-1.47 ± 0.03
a_2	0.7	0.70 ± 0.05
b_1	1.0	0.82 ± 0.28
b_2	-0.45	-0.40 ± 0.19

Table 10.4. Identification results for Example 10.5.

Example 10.6

The system is given by

$$(1+aq^{-1})\ y(t) = bq^{-4}\ u(t) + (1+cq^{-1})\ e(t) \tag{10.4}$$

$a = -0.8$, $b = 1.0$, $c = 0.7$

and the feedback is the minimum variance control of (10.4), which can be calculated to be

$$(1+f_1q^{-1}+f_2q^{-2}+f_3q^{-3})\ u(t) = g_o y(t)$$

$f_1 = 1.5$, $f_2 = 1.2$, $f_3 = 0.96$, $g_o = -0.768$

It is easy to see that the necessary condition for identifiability (9.20) is fulfilled. The model orders were chosen in the most favourable way, i.e. $\hat{n}_a = \hat{n}_b = \hat{n}_c = 1$ and $\hat{k} = 3$. This means that $PI(\mathcal{M}, \mathcal{I}, \mathcal{X})$ is expected. The result of the identification is given in Table 10.5. It confirms the theory.

Parameter	Theoretical value	Obtained estimate
a_1	-0.8	-0.77 ± 0.02
b_1	1.0	1.01 ± 0.04
c_1	0.7	0.71 ± 0.03

Table 10.5. Identification results for Example 10.6.

Example 10.7

The system and the feedback law are the same as in Example 10.6. However, in this example the model structure given by $\hat{n}_a = \hat{n}_b = \hat{n}_c = 2$ and $\hat{k} = 3$ was used. It can be seen from (9.21) that for this model structure the system is not PI(M,I,X) but SI(M,I,X). The theoretical estimates can be found from (9.9). They are given by

$$1 + a_1 q^{-1} + a_2 q^{-2} = (1+aq^{-1})\ (1+\ell q^{-1})$$

$$b_1 q^{-1} + b_2 q^{-2} = bq^{-1}\ (1+\ell q^{-1})$$

$$1 + c_1 q^{-1} + c_2 q^{-2} = (1+cq^{-1})\ (1+\ell q^{-1})$$

which after insertion of the values of a, b and c gives the theoretical values given in Table 10.6. The parameter ℓ is quite arbitrary. The obtained estimates differ very little from the theoretical ones if the parameter ℓ is chosen equal to -1.0. This means that the obtained model gives an almost correct transfer function and that the system in fact is SI(M,I,X).

Parameter	Theoretical estimate	Obtained estimate
a_1	$-0.8 + \ell$	-1.78 ± 0.03
a_2	-0.8ℓ	0.80 ± 0.05
b_1	1.0	0.96 ± 0.06
b_2	ℓ	-1.07 ± 0.09
c_1	$0.7 + \ell$	-0.28 ± 0.03
c_2	0.7ℓ	-0.70 ± 0.03

Table 10.6. Identification results for Example 10.7.

11. CONCLUSIONS

In this section the main results are summarized. The assumptions made and the definitions of some of the concepts used are repeated in order to reduce the need of references to earlier sections and to make the section as self-contained as possible.

The identification of multivariable, linear and time invariant systems under output feedback control has been considered. It has been assumed that the system can be described by a vector difference equation. The feedback law is allowed to be fairly general. In Fig. 11.1, which is identical to Fig. 2.1, a block diagram of the closed loop system configuration considered is shown. The signal u_1 is a known injected perturbation, while v_1 and v_2 are unknown disturbances acting on the system and in the feedback loop respectively. Several principally different cases can be distinguished depending on the structure of

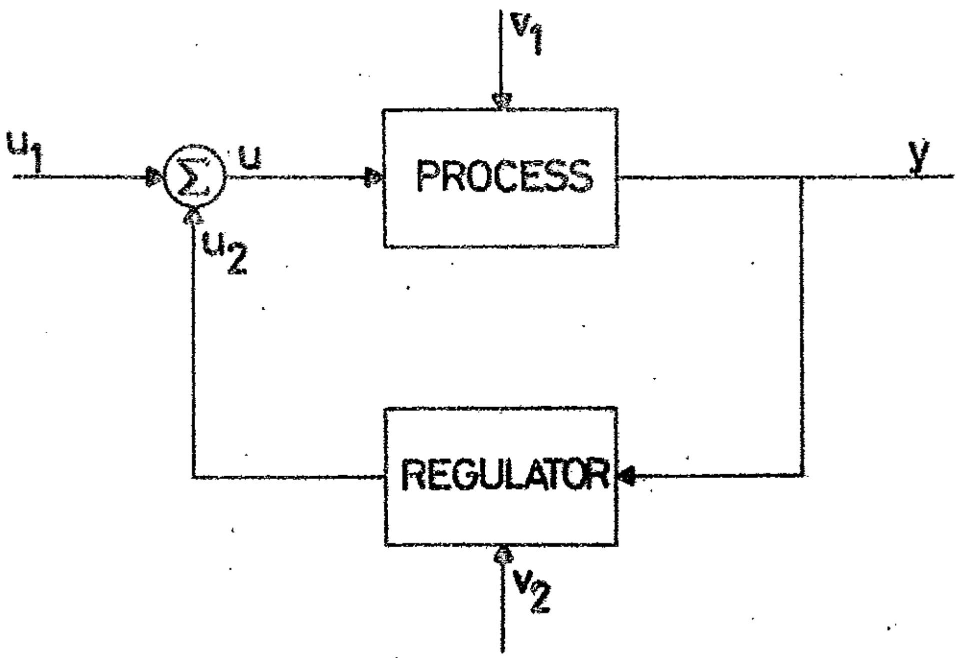

Fig. 11.1. Block diagram of a closed loop system.

the regulator and on the existence of the signals u_1 and v_2, (cf. Section 2).

Two approaches to the identification of a system operating under closed loop are possible. One way is to start by identifying the closed loop behaviour. Assuming that the regulator is exactly known, it may then be possible to determine the open loop characteristics. This approach is called indirect identification. There are several feedback cases for which indirect identification cannot be used. In direct identification, on the other hand, the measurements of the input u and the output y are used straightforwardly as in the open loop case without assuming that the system is in closed loop.

Most of the available identification methods can be divided into two principally different classes. Methods for which causality of the model is not assumed are called noncausal methods. Such methods often fail, e.g. if the feedback law is linear, time invariant and noise free and if no extra perturbation is injected. The other class consists of prediction error methods using a model structure restricted to models that can be physically realizable. In many cases such methods can lead to a correct estimation of the open loop characteristics also in cases where noncausal methods would fail.

Identifiability concepts have been introduced in order to be able to characterize the possibilities to obtain reasonable models by identification. Identifiability depends on the system itself, the set of models considered, M, the identification method used, I, and the experiment conditions, X.

A system is said to be System Identifiable, SI(M,I,X), if the estimated model and the system have the same input output relationship (transfer function) and the same noise characteristics. In some cases it is desirable to estimate certain parameters in a given model structure. If the parameters can be estimated consistently, the system is said to be Parameter Identifiable, PI(M,I,X). It is known that a system in open loop is in general SI(M,I,X) provided the experiment conditions are such that the input is persistently exciting. An open loop system is sometimes not PI(M,I,X) e.g. due to redundancy in the chosen parametrization of the model structure. For systems operating in closed loop the property of SI(M,I,X) may be lost (Example 4.1) regardless of how the model structure is chosen. Therefore it would be desirable to be able to characterize identifiability of systems by a concept that is independent of the choice of model structure. If the only requirement for SI(M,I,X) on the class of models considered is that it includes the true transfer function and noise characteristics, then the system is said to be Strongly System Identifiable, SSI(I,X). If a closed loop system is SSI(I,X), the identifiability problem is of the same kind as for an open loop system.

In many cases the identifiability of a system depends explicitly on the system configuration. This means that a detailed knowledge of the system must be available in order to test identifiability a priori. Such knowledge may, however, be lacking. It is then not possible to determine if the system is identifiable by using experimental data either. The assumed structure for the model cannot be validated from data, (cf. Section 8). The only possible way to achieve identifiability is to change the experimental conditions, e.g. the structure of the regulator.

The main part of the results are derived for direct identification with prediction error methods. The main concern has been to develop conditions for Strong System Identifiability. The conditions are summarized as follows (cf. Section 5).

- o If the system is in open loop or in closed loop with a deliberately injected input or in closed loop with noise acting in the feedback loop, the system is SSI(I,X), provided that the perturbation or noise is persistently exciting.

- o If the feedback law is linear, time invariant, noise free and no extra perturbation is injected, the system is not SSI(I,X). For this case necessary and sufficient conditions on the system, the regulator and the model for SI(M,I,X) and PI(M,I,X) are given in Section 9 for single-input single-output systems. The conditions for SI(M,I,X) and PI(M,I,X) for the most favourable choice of model structure are also derived, eq. (9.17). These conditions reveal that the question whether it is possible to identify a single-input single-output system depends only on the structures of the system and of the regulator. A large time delay in the system or a regulator of high order is most often sufficient for identifiability.

- o If the feedback law is non linear or time varying but noise free and if no extra perturbation is injected, the system is SSI(I,X) except in degenerated cases. A suitable and practically feasible way to guarantee SSI(I,X) is suggested. It is shown to be sufficient to switch between a number of different feedback laws. This number depends in a simple way on the number of inputs and outputs (7.4).

The relations between indirect and direct identification have been analysed (Section 6). The results can be summarized as follows.

- o Direct identification with prediction error methods is not restricted in its use. However, for certain experiment conditions identifiability will not be achieved.

- Direct identification with noncausal methods cannot be used for closed loop systems (cf. Section 2).

- Indirect identification can be used for closed loop systems only in certain cases. One such case is if the feedback is noise free and time invariant. It can also be used if the feedback is time varying in a special way, e.g. if the regulator is switched between different known feedback laws.

- Additional knowledge, viz. the regulator, is always necessary for indirect identification. If the feedback is noise free, time invariant and linear, this knowledge can, however, be easily obtained.

- Direct identification has the same identifiability properties as indirect identification in all cases when indirect identification is applicable.

In order to exemplify the possibilities to identify closed loop systems and to illustrate what can happen in different situations some numerical simulations are shown (Section 10).

The best way to perform identification experiments is often in open loop with an injected perturbation signal. If this is not possible, there are in principle three different ways to achieve identifiability of the system operating in closed loop. Firstly to use an extra perturbation signal which, however, may not be allowed e.g. because of the increased output fluctuations. Secondly to use a very complex regulator, which may be difficult to implement. Thirdly to switch between different simple, linear feedback laws during the experiment, which seems to be an attractive alternative in many situations. This may often be possible without violating the demands on the security and the production quality. Several different regulators may control the process with approximately the same result. In most applications in process industries the implementation of this strategy would not be difficult either. However, for many systems e.g. in economy and biology, it is clear that the regulator is not disposable. It can often be expected that the regulator is time varying.

Feedback is generally introduced in order to decrease output fluctuations. The information content of the signal is thus reduced. It is then obvious that in practice longer data records usually are needed to obtain accurate models for systems operating in closed loop during the experiments than for open loop systems. The accuracy properties are under current study. Such a study would give valuable information on the applicability in practice of identification of systems operating in closed loop.

The analysis in this report has shown that it is always possible to use direct identification. Identifiability properties cannot be improved by using indirect identification. Therefore it seems to be most attractive to analyse the data via direct identification with a prediction error method since such algorithms often are available for identification of open loop systems.

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REFERENCES

Akaike, H. (1967)
Some problems in the application of the cross-spectral method. In Spectral Analysis of Time Series (ed. B. Harris), John Wiley, New York, pp 81-107

Akaike, H. (1968)
On the use of a linear model for the identification of feedback systems. Ann. of the Inst. of Stat. Math., Vol. 20, No.3, pp 425-439

Åström, K.J. and Bohlin T. (1965)
Numerical identification of linear dynamic systems from normal operating records. Paper at the IFAC Symp. on Theory of self-adaptive control systems, Teddington, England. Also in Theory of Self-Adaptive Control Systems (ed. P. Hammond), Plenum Press, New York (1966)

Åström, K.J. (1970)
Introduction to Stochastic Control Theory. Academic Press, New York and London

Åström, K.J. and Eykhoff, P. (1971)
System identification - a survey. Automatica, Vol. 7, No. 2, pp 123-162

Baeyens, R. and Jacquet, B. (1973)
Survey of applications of identification in power generation and distribution. Preprints of the 3rd IFAC Symp. on Identification and System Parameter Estimation, The Hague

Balakrishnan, A.V. (1973)
A theory of adaptive control of linear dynamic systems. Report UCLA-ENG 7372, School of Engineering and Applied Science, University of California

Bekey, G.A. (1973)
Survey of applications of identification in biological systems. Preprints of the 3rd IFAC Symp. on Identification and System Parameter Estimation, The Hague.

Bellman, R. and Åström, K.J. (1970)
On structural identifiability. Math. Biosc., Vol. 7, pp 329-339

Bohlin, T. (1971a)
Using statistical identification methods. Technical paper TP 18.206, IBM Nordic Laboratory, Lidingö, Sweden

Bohlin, T. (1971b)
On the problem of ambiguities in maximum likelihood identification. Automatica, Vol. 7, No. 2, pp 199-210

Box, G.E.P. and MacGregor, J.F. (1972)
The analysis of closed-loop dynamic-stochastic systems. Department of Statistics, University of Wisconsin, Madison, Wisconsin, Technical report No. 309 (the fourth chapter in a series Topics in Control)

Caines, P.E. (1970)
The parameter estimation of state variable models of multivariable linear systems. Ph. D. dissertation, Imperial College Sci. Technol., London

Caines, P.E. (1971)
The parameter estimation of state variable models of multivariable linear systems. Proceedings of the U.K.A.C. Conference on Multivariable Systems, Manchester, England

Caines, P.E. and Wall, K.D. (1972)
Theoretical foundations for methods of estimation and control in economic policy optimization. Discussion paper No. 3, Programme of Research into Econometric Methods, Department of Economics, Queen Mary College, University of London, London

Chan, C-W. (1973)
University of Manchester, Institute of Science and Technology, Manchester, England. Personal communication

Eaton, J. (1967)
Identification for control purposes. Paper IEEE Winter Meeting, New York

Eykhoff, P. (1973)
System Parameter and State Estimation. In manuscript

Fisher, E.E. (1965)
Identification of linear systems. Paper JACC, pp 473-475

Glover, K. and Willems, J.C. (1973)
On the identifiability of linear dynamic systems. Preprints of the 3rd IFAC Symp. on Identification and System Parameter Estimation, The Hague

Glover, K. (1973)
Structural aspects of system identification. Report ESL-R-516, Electronic Systems Laboratory, Decision and Control Sciences Group, Department of Electrical Engineering, MIT, Cambridge, Mass.

Goodwin, G.C., Payne, R.L. and Murdoch, J.C. (1973)
Optimal test signal design for linear single input-single output closed loop identification. Conference on Computer Aided Control System Design, IEE Conference Publication No. 98, Cambridge, England

Gustavsson, I. (1973)
Survey of applications of identification in chemical and physical processes. Preprints of the 3rd IFAC Symp. on Identification and System Parameter Estimation, The Hague

Gustavsson, I, Selander, S. and Wieslander, J. (1973)
IDPAC - User's guide. Report 7331, Division of Automatic Control, Lund Institute of Technology, Lund, Sweden

Itoh, T., Saito, H., Takumi, Y. and Shimizu, T. (1971)
Process analysis techniques for cement kilns. IEEE IGA, Vol. 7, No. 4, pp 470-479

Leonhard, W. (1972)
Some results with identification procedures using linear discrete models, Part II. Lecture notes from the NATO Summer School on Theory and Practice of Systems Modelling and Identification, Toulouse, France

Lindberger, N.A. (1973a)
Stochastic identification of computer-regulated linear plants in noisy environments. Int. J. Control, Vol. 17, No. 1, pp 65-80

Lindberger, N.A. (1973b)
Stochastic identification of digital feedback loops using a table approach. Preprints of the 3rd IFAC Symp. on Identification and System Parameter Estimation, The Hague

Ljung, L. (1974)
Consistency for prediction error identification methods. Forthcoming report, Division of Automatic Control, Lund Institute of Technology, Lund, Sweden

Lobbia, R.N. and Saridis, G.N. (1972)
On-line identification of multivariable stochastic feedback systems. Paper JACC, pp 802-810

Mayne, D.Q. (1972)
A canonical form for identification of multivariable linear systems. IEEE AC, Vol. 17, No. 5, pp 728-729

Otomo, T., Nakagawa, T. and Akaike, H. (1969)
Implementation of computer control of a cement rotary kiln through data analysis. Proceedings of the 4th IFAC Congress, Warsaw, paper 66.6

Priestley, M.B. (1969)
Estimation of transfer functions in closed loop stochastic systems. Automatica, Vol. 5, No. 5, pp 623-632

Rissanen, J. (1972)
Basis of invariants for linear dynamic systems. Report IBM Research Laboratory, San Jose, California

Rissanen, J. (1973)
Estimation of parameters in multi-variate random processes. Report IBM Research Laboratory, San Jose, California

Saridis, G.N. and Lobbia, R.N. (1972)
Parameter identification and control of linear discrete-time system. IEEE AC, Vol. 17, No. 1, pp 52-60

Schultze, H. (1971)
Anwendung von Schätzverfahren für die Kenngrössen von Regelstrecken aufgrund von Messungen im geschlossenen Regelkreis. Regelungstechnik und Prozess-Datenverarbeitung, 19. Jahrgang, Heft 3, pp 113-119

Söderström, T. (1973)
On the uniqueness of maximum likelihood identification for different structures. Report 7307, Division of Automatic Control, Lund Institute of Technology, Lund, Sweden

Tse, E. and Anton, J. (1972)
On the identifiability of parameters. IEEE AC, Vol. 17, No. 5, pp 637-645

Turtle, D.P. and Phillipson, P.H. (1971)
Simultaneous identification and control. Automatica, Vol. 7, No. 4, pp 445-453

APPENDIX

In this appendix the first part of Theorem 9.1 is proved. The purpose is to show that (9.10) and (9.15) are equivalent. The calculations will be somewhat technical.

Introduce the new variables

$$n_a^* = \hat{n}_a - n_a$$
$$n_b^* = \hat{n}_b + \hat{k} - n_b - k$$
$$n_c^* = \hat{n}_c - n_c$$
$$n_q^* = n_q - n_p$$
$$x = (n_f - n_b) - (k + n_g - n_a)$$

Now (9.10) is rewritten as follows.

$$\max[n_f+n_a+n_c+\max(n_a^*, n_c^*-r), n_g+k+n_b+n_c+\max(n_b^*, n_c^*-r)]-$$
$$- n_p + \min[n_a^*, n_b^*, n_c^*] \geq n_a+n_a^*+n_b^*-\hat{k}+n_b+k+n_c+n_c^*$$

$$n_p+k-\hat{k} \leq \max[n_f-n_b+\max(n_a^*, n_c^*-r), k+n_g-n_a+\max(n_b^*, n_c^*-r)]+$$
$$+ \min[n_a^*, n_b^*, n_c^*] - n_a^* - n_b^* - n_c^*$$

$$n_p+k-\hat{k} \leq (k+n_g-n_a) + \max[x+\max(n_a^*, n_c^*-r), \max(n_b^*, n_c^*-r)]+$$
$$+ \min[n_a^*, n_b^*, n_c^*] - n_a^* - n_b^* - n_c^* \qquad (A.1)$$

The expression for n_q^* is written as.

$$n_q{}^* = \min[n_c{}^*+n_p,\ \max(n_a+n_a{}^*+n_f,\ k+n_b+n_b{}^*+n_g) - n_h] - n_p$$

$$= \min[n_c{}^*,\ \max(n_f-n_b+n_a{}^*,\ k+n_g-n_a+n_b{}^*) -$$

$$-\max(n_f-n_b,\ k+n_g-n_a) + r]$$

$$= \min[n_c{}^*,\ \max(x+n_a{}^*,\ n_b{}^*) - \max(x,0)+r] \qquad (A.2)$$

The sufficient condition (9.15) is rewritten as

$$\max[n_f-\max(n_b{}^*+n_b+k-\hat{k}+n_p,\ n_b+k-\hat{k}+n_q{}^*+n_p),$$

$$\tilde{k}+n_g-\max(n_a{}^*+n_a+n_p,\ n_a+n_q{}^*+n_p)] \geq 0$$

$$\max[n_f-n_b-k+\hat{k}-n_p-\max(n_b{}^*,\ n_q{}^*),\ \hat{k}+n_g-n_a-n_p-\max(n_a{}^*,\ n_q{}^*)] \geq 0$$

$$n_p+k-\hat{k} \leq \max[n_f-n_b-\max(n_b{}^*,\ n_q{}^*),\ k+n_g-n_a-\max(n_a{}^*,\ n_q{}^*)]$$

$$n_p+k-k \leq (k+n_g-n_a) + \max[x-\max(n_b{}^*,\ n_q{}^*),\ -\max(n_a{}^*,\ n_q{}^*)] \qquad (A.3)$$

Introduce the difference of the right hand sides of (A.1) and (A.3) as

$$F(x,\ n_a{}^*,\ n_b{}^*) = \max[x-\max(n_b{}^*,\ n_q{}^*),\ -\max(n_a{}^*,\ n_q{}^*)] -$$

$$-\max[x+\max(n_a{}^*,\ n_c{}^*-r),\ \max(n_b{}^*,\ n_c{}^*-r)] -$$

$$-\min[n_a{}^*,\ n_b{}^*,\ n_c{}^*] + n_a{}^*+n_b{}^*+n_c{}^* \qquad (A.4)$$

It will be shown that F is identical to zero.

First it is noticed that $F(x,\ n_b{}^*,\ n_a{}^*) = F(-x,\ n_a{}^*,\ n_b{}^*)$
Thus it is no restriction to assume that $n_a{}^* \leq n_b{}^*$.

The general situation will be divided into a number of cases. The second and third argument in $F(x, n_a^*, n_b^*)$ are dropped for convenience.

Case 1. Assume $r > 0$. Then $x = 0$, and

$$n_q^* = \min[n_c^*, n_b^*+r] \text{ and}$$

$$F(x) = -\max(n_a^*, n_q^*) - \max(n_b^*, n_c^*-r) - \min(n_a^*, n_c^*) +$$

$$+ n_a^*+n_b^*+n_c^*$$

Case 1.1 Assume further that $n_c^* \leq n_b^*+r$. Then $n_q^* = n_c^*$ and

$$F(x) = -\max(n_a^*, n_c^*) - n_b^* - \min(n_a^*, n_c^*) + n_a^*+n_b^*+n_c^* = 0$$

Case 1.2. Assume instead $n_c^* \geq n_b^*+r$. Then $n_q^* = n_b^* + r$ and

$$F(x) = -\max(n_a^*, n_b^*+r) - n_c^*+r-n_a^*+n_a^*+n_b^*+n_c^*$$

$$= -(n_b^*+r) + r + n_b^* = C$$

Case 2. Assume $r = 0$. Then

$$n_q^* = \min[n_c^*, \max(x+n_a^*, n_b^*) - \max(x,0)]$$

$$F(x) = \max[x-\max(n_b^*, n_q^*), -\max(n_a^*, n_q^*)]$$

$$- \max[x+\max(n_a^*, n_c^*), \max(n_b^*, n_c^*)]$$

$$- \min[n_a^*, n_b^*, n_c^*] + n_a^*+n_b^*+n_c^*$$

This case will also be considered in two subcases

Case 2.1. Assume further $n_c^* \leq n_a^*$. Then

$$n_q^* = n_c^*$$

$$F(x) = \max[x-n_b^*, -n_a^*] - \max[x+n_a^*, n_b^*] - n_c^* + n_a^* + n_b^* + n_c^*$$

$$= \max[x+n_a^*, n_b^*] - \max[x+n_a^*, n_b^*] = 0$$

Case 2.2. Assume instead $n_c^* \geq n_a^*$. For this subcase the analysis is further divided by considering x in two different intervals. For this case

$$n_a^* \leq n_q^* \leq \min(n_b^*, n_c^*)$$

$$F(x) = \max[x-n_b^*, -n_q^*] - \max[x+n_c^*, \max(n_b^*, n_c^*)] -$$

$$-n_a^* + n_a^* + n_b^* + n_c^*$$

Case 2.2.1. Assume especially $x \leq \max(0, n_b^*-n_c^*)$. Then

$$n_q^* = \min(n_b^*, n_c^*)$$

$$F(x) = -n_b^* + \max[x, n_b^* - \min(n_b^*, n_c^*)] -$$

$$-n_c^* - \max[x, \max(0, n_b^*-n_c^*)] + n_b^* + n_c^*$$

$$= \max[x, \max(n_b^*-n_c^*, 0)] - \max[x, \max(0, n_b^*-n_c^*)]$$

$$= 0$$

Case 2.2.2. Assume especially $x \geq \max(0, n_b^*-n_c^*)$. Then

$$x+n_q^* = \min[x+n_c^*, \max(x+n_a^*, n_b^*)]$$

$$\geq \min[x+n_c^*, n_b^*] = n_b^*$$

This gives

$$F(x) = x-n_b^* - \max[x+n_c^*, n_b^*] + n_b^* + n_c^*$$

$$= x-n_c^* - x + n_c^* = 0$$

□