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To my parents

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“Stufen

Wie jede Blüte welkt und jede Jugend
Dem Alter weicht, blüht jede Lebensstufe,
Blüht jede Weisheit auch und jede Tugend
Zu ihrer Zeit und darf nicht ewig dauern.
Es muß das Herz bei jedem Lebensrufe
Bereit zum Abschied sein und Neubeginne,
Um sich in Tapferkeit und ohne Trauern
In andre, neue Bindungen zu geben.
Und jedem Anfang wohnt ein Zauber inne,
Der uns beschützt und der uns hilft, zu leben.

Wir sollen heiter Raum um Raum durchschreiten,
An keinem wie an einer Heimat hängen,
Der Weltgeist will nicht fesseln uns und engen,
Er will uns Stuf' um Stufe heben, weiten.
Kaum sind wir heimisch einem Lebenskreise
Und traulich eingewohnt, so droht Erschlaffen,
Nur wer bereit zu Aufbruch ist und Reise,
Mag lähmender Gewöhnung sich entraffen.

Es wird vielleicht auch noch die Todesstunde
Uns neuen Räumen jung entgegensenden,
Des Lebens Ruf an uns wird niemals enden ...
Wohlan denn, Herz, nimm Abschied und gesunde!”

*Hermann Hesse**

---

* Das Glasperlenspiel, Suhrkamp Taschenbuch Verlag, Frankfurt am Main, 1972

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Abstract

A new optimal stochastic control problem is posed. The criterion is to minimize the probability for a signal's largest value to be above a level given a certain reference value. It is shown that this control problem is closely related both to the problem of minimizing the variance of the signal—minimum variance control—and to the problem of minimizing the so called upcrossing probability. It is made plausible that the upcrossing probability is a better approximating criterion to minimize than the minimum variance criterion. The problem of minimizing the upcrossing probability can be thought of as finding optimal weighting-matrices in an LQG-problem. The new controller is compared with the minimum variance controller for a first order process. It is seen that the new controller causes a lower upcrossing intensity and a smaller probability for the largest value of the controlled signal to be above the critical level. The improvement in the example is up to about 10%. This makes it possible to choose the reference value closer to the critical level when using the minimum risk controller, than when using the minimum variance controller. Further it seen that the control signal is more well-behaved. The only drawback of the new controller is the larger computational burden.

Key words
Stochastic control, Optimal control, Minimum variance control, Level Crossings, Extremes

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Preface

My interest in control of extreme values began in the autumn of 1990 as I was following a course on extremes in random processes given by Professor Georg Lindgren at the Department of Mathematical Statistics in Lund. The course was suggested to me by Professor Lars Nielsen to give me a basis for our common interest in investigating prediction methods for level crossings. After some time it was evident to me that theoretical work about predictions in closed loop was not what a control engineer primarily should work with. Why predict catastrophes, when it is possible to control to avoid them. In fact, good control that avoids level crossings, makes prediction of level crossings less interesting. The closed loop system modes are then fast, and multi-step predictions will rapidly converge to the mean of the process. However, due to practical considerations, such as uncertainties in process models or actuator constraints, it is not always possible to control in a satisfactory way, and in these cases supervision may be useful. This has been described for the continuous time case in [Hansson and Nielsen, 1991]. In this thesis, only control of level crossings and extreme values for the discrete time case will be considered. The continuous time case has been described in [Hansson, 1991a], [Hansson, 1991b] and [Hansson, 1992].

As is obvious from what is said above, the thesis is somewhat interdisciplinary and in the borderland of automatic control and mathematical statistics. It is primarily written for a reader with knowledge of automatic control at a graduate level, but I hope that the references will help any other reader with some mathematical background to read it.

Acknowledgements

This work has been carried out at the Department of Automatic Control, Lund Institute of Technology, Sweden. I would like to thank all my colleagues at the department. It is a great pleasure to work in the creative, friendly and highly stimulating atmosphere which they are all contributing to.

I am very happy to thank Professor Karl Johan Åström, and Dr. Per Hagander for their encouraging support and guidance, and for their valuable comments and suggestions, which have improved my work considerably. Professor Åström's great enthusiasm for my work moderated by his sincere interest in controllers well adapted to industrial control problems has been most encouraging, and served as a very good example on a serious attitude towards scientific research in an applied field. Dr. Hagander not only has the invaluable ability to find most, I hope, errors in a manuscript, but also a persistent will to understand and discuss problems, which is very helpful when trying to write for the uninitiated reader. I am also very grateful to Professor Björn Wittenmark for his comments and criticism on the first version of my manuscript. I wish to express my gratitude to M.Sc. Bo Bernhardsson for suggested improvements and stimulating discussions. He has been a valuable source of inspiration. I am also indebted to Professor Georg Lindgren for stimulating discussions, and for his interesting lectures on extremes in random processes. I would like to thank Professor Lars Nielsen for suggesting the topic of investigating prediction of catastrophes, for having had time to discuss related topics, and for making me aware of the research on extreme values at the Department of Mathematical Statistics. I am also grateful to L.Sc. Ola Dahl and M.Sc. Klas Nilsson for interesting discussions about applications in robotics.

I would like to thank L.Sc. Kjell Gustafsson for providing good LQG-routines in Matlab, and M.Sc. Leif Andersson for providing excellent TeXmacros, which have made the type-setting much smoother than it otherwise would have been.

Finally, I would like to thank my parents for their encouragement and support.

Lund, November 1991
Anders Hansson

Table of Contents

1. Introduction 1

2. The Control Problem 4
- 2.1 Problem Formulation 4
- 2.2 Bounds for the Criterion Probability 5
- 2.3 Approximation of the Problem Formulation 7
- 2.4 Summary 8

3. Regulator Design 9
- 3.1 Solution 9
- 3.2 LQG-equations 16
- 3.3 The General Case 18
- 3.4 Summary 20

4. Example 21
- 4.1 Process 21
- 4.2 LQG-Controllers 22
- 4.3 MR and MV Controllers 22
- 4.4 Summary 30

5. Conclusions 31

References 33

A. More General Process Model 35

1

Introduction

There are many control problems where the goal is not only to keep the controlled signal near a certain reference value, but in addition to keep it below a critical level—critical in the sense that if the signal crosses the level, a failure in the system under control is caused. The distance between the critical level and the reference value is normally not small, since otherwise the failure rate will be intolerably high. However, there may be other control-objectives that make it undesirable or impossible to choose the distance large. An example of problems of this kind can be found in [Borisson and Syding, 1976], where the power of an ore crusher should be kept as high as possible but not exceed a certain level. Another example is moisture control of a paper machine, where it is desired to keep the moisture content as high as possible without causing wet streaks. Yet another example is power control of wind power plants, where the supervisory system initiates emergency shutdown, if the generated power exceeds 140% of rated power, [Mattsson, 1984]. Other examples can be found in sensor-based robotics and force control, [Hansson and Nielsen, 1991], and control of non-linear plants, where the stability may be state-space dependent, [Shinskey, 1967].

This type of problems has previously been solved approximately by minimum variance control, [Åström and Wittenmark, 1990, p. 203], [Åström, 1970, pp. 159–209], and [Borisson and Syding, 1976]—the intuitively best controller. The gain of the minimum variance controller depends critically on the sampling period. Too small a sampling period leads to large variations in the control signal, [Åström and Wittenmark, 1990, pp. 316–317]. This problem has been solved by introducing weighting on the control signal—LQG-design. However, there has been no good criteria on how to choose the weighting. The proposed controller can be interpreted as a choice of optimal weightings in an LQG-problem, chosen in such a way that they minimizes the mean number of upcrossings of the critical level per unit time. The problem of level crossings in the context of stochastic processes was already studied by Rice, [Rice, 1936].

In [Hansson, 1991a], [Hansson, 1991b] and [Hansson, 1992] the problem was solved in the continuous time case; here the discrete time case is treated, which previously to some extent has been described in [Hansson, 1991c]. Only the case of a linear process controlled with a linear controller will be treated. It is still an open question whether the best controller for a linear processes is linear or not. This question is difficult to answer in the stochastic framework used here, since a Gaussian stochastic process controlled by a non-linear controller will most likely not have a closed loop that is Gaussian.

In Chapter 2 the problem of keeping a signal's largest values below a level given a certain reference value is related to the minimum variance controller and to the controller that minimizes the so called upcrossing probability. It is also made plausible that the upcrossing probability criterion better captures the control-objectives in the problems described above than does the minimum variance criterion.

In Chapter 3 the controller that minimizes the upcrossing probability—the minimum risk controller— is determined. It is obtained by solving a one-parametric optimization problem over a set of LQG-problem solutions. Thus the complexity is not significantly larger than for an ordinary LQG-problem. It can be interpreted as choosing optimal weighting-matrices in an LQG-problem, provided that the solutions to the LQG-problems are unique.

In Chapter 4 the minimum risk controller found in Chapter 3 is compared with the minimum variance controller for a first order process. It is seen that the new controller causes a lower upcrossing probability and smaller probability for the largest value of being above the dangerous level. Further it is seen that the control signal is more well-behaved.

Both theory and experiments show that the minimum risk controller and the minimum variance controller are approximately the same for large values of the distance between the reference value and the critical level. However, in an example it is seen that the minimum risk controller can have up to about 10% better performance for moderate values of the level, which is the interesting case for the examples described above.

Finally in Chapter 5 the results of the previous chapters are summarized.

2

The Control Problem

The control problems described in Chapter 1 will be mathematically formalized in a stochastic framework. In Section 2.1 the control criterion is defined; it is defined such that the controller should minimize the probability for the largest value of the controlled signal to be above a level given a certain reference value. Two bounds for the criterion probability are investigated in Section 2.2. One of them is minimized by minimum variance control, the other one—tighter than the first one—is approximately minimized by minimizing the so called upcrossing probability. In Section 2.3 the approximate control criterion to minimize the upcrossing probability is further motivated and stated. The results are summarized in Section 2.4.

2.1 Problem Formulation

Let the controlled signal, z, be a stationary Gaussian sequence with mean

$$m_z = \mathrm{E}\{z(k)\}$$

and with covariance

$$r_z(\tau) = \mathrm{E}\{(z(k+\tau) - m_z)(z(k) - m_z)\}$$

Denote the variance of z by σ_z^2, i.e. let $\sigma_z^2 = r_z(0)$. Consider a time-invariant controller H, linear in both the measurement signal y and in the constant reference value r. The problems mentioned in Chapter 1 are captured in the following criterion:

$$\min_H \mathrm{P}\left\{\max_{0\le k\le N} z(k) > z_0\right\} \tag{2.1}$$

subject to $m_z = r$ and to a stable closed loop system, where $\mathrm{P}\{\cdot\}$ denotes probability measure, and where $z_0 > m_z$ is the critical level. The reason for constraining the minimization to $m_z = r$ is that it may be profitable to not having $m_z - z_0$ too large; e.g. in the paper machine example it was desired to keep the moisture content as high as possible without causing wet streaks. The time horizon N and the distance $m_z - z_0$ have to be chosen in such a way that the probability in (2.1) is small, otherwise the failure rate will be too high. The larger N is, the larger $m_z - z_0$ must be. Without loss of generality it may be assumed that $m_z = r = 0$, which can be obtained with a change of coordinates. To simplify the notations this will be assumed in the sequel.

2.2 Bounds for the Criterion Probability

To simplify the problem upper bounds for the probability in (2.1) will be given. It will be shown that these bounds are tight, if N and z_0/σ_z are large and the probability in (2.1) is small.

The problem of level crossings is a classical problem in stochastic processes. Initial results were given in [Rice, 1936]. Good references to crossing problems are [Cramér and Leadbetter, 1967] and [Leadbetter *et al.*, 1982]. The latter book also treats the problem of extreme values in stochastic processes. The results developed in this section will make extensive use of the results in these books.

THEOREM 2.1
If $z(k)$ is a stationary random sequence, then

$$\mathrm{P}\left\{\max_{0\le k\le N} z(k) > z_0\right\} \le P_1(z_0) \le P_2(z_0) \tag{2.2}$$

where

$$P_1(z_0) = \mathrm{P}\{z(0) > z_0\} + N\mu(z_0)$$
$$P_2(z_0) = (N+1)\mathrm{P}\{z(0) > z_0\}$$

and where

$$\mu(z_0) = \mathrm{P}\{z(0) \le z_0 \cap z(1) > z_0\} \tag{2.3}$$

Proof: The proof is easy:

$$\begin{aligned}\mathrm{P}\left\{\max_{0\le k\le N} z(k) > z_0\right\} &= \mathrm{P}\left\{z(0) > z_0 \bigcup_{k=0}^{N-1} (z(k) \le z_0 \cap z(k+1) > z_0)\right\} \\ &\le \mathrm{P}\{z(0) > z_0\} + \sum_{k=0}^{N-1} \mathrm{P}\{z(k) \le z_0 \cap z(k+1) > z_0\} \\ &= \mathrm{P}\{z(0) > z_0\} + N\mathrm{P}\{z(0) \le z_0 \cap z(1) > z_0\} \\ &\le (N+1)\mathrm{P}\{z(0) > z_0\}\end{aligned}$$

□

Remark 1. For large values of N and large values of z_0/σ_z the first term in P_1 is negligible.

Remark 2. Note that P_2 is minimized by minimum variance control.

The quaintity μ in (2.3) will be called the upcrossing probability, and it is equal to the mean number of upcrossings in the interval $[0, 1)$, see e.g. [Cramér and Leadbetter, 1967, p. 281]. The bound P_1 is well known in the context of continuous time extreme value analysis, see e.g. [Leadbetter *et al.*, 1982, Lemma 8.2.1], since there the bound P_2 is infinite, and thus not usable for investigating the behavior of extreme values as the time horizon and the critical level approaches infinity. However, the bound P_2 is good enough for investigating this behavior in the discrete time domain, but for the purposes in this work—focused on finite time horizons and levels—it is interesting also to consider a tighter bound such as P_2.

It will now be shown that the bounds in Theorem 2.1 are tight.

THEOREM 2.2
If $z(k)$ is a stationary Gaussian sequence and $z_0^{(N)}$ is chosen such that

$$\lim_{N\to\infty} P_2\left(z_0^{(N)}\right) = L$$

then

$$\lim_{N\to\infty}\left|\frac{M\left(z_0^{(N)}\right)-P_1\left(z_0^{(N)}\right)}{M\left(z_0^{(N)}\right)}\right| \leq \lim_{N\to\infty}\left|\frac{M\left(z_0^{(N)}\right)-P_2\left(z_0^{(N)}\right)}{M\left(z_0^{(N)}\right)}\right| \leq \frac{L}{2}$$

where $M(x) = \mathrm{P}\left\{\max_{0\leq k\leq N} z(k) > x\right\}$.

Proof: The first inequality follows by Theorem 2.1. Further by Theorem 2.1 and since by [Leadbetter *et al.*, 1982, Theorem 4.3.3]

$$\lim_{N\to\infty} P_2\left(z_0^{(N)}\right) = L$$

if and only if

$$\lim_{N\to\infty} 1 - M\left(z_0^{(N)}\right) = e^{-L}$$

it follows that

$$\lim_{N\to\infty}\left|\frac{M\left(z_0^{(N)}\right)-P_2\left(z_0^{(N)}\right)}{M\left(z_0^{(N)}\right)}\right| = \left|\frac{1-e^{-L}-L}{1-e^{-L}}\right| \leq \frac{L}{2}$$

which concludes the proof. □

Related problems of convergence have been investigated for other approximations of extremal-probabilities, see e.g. [Leadbetter *et al.*, 1982, Chapter 4.6], but these approximations are not upper bounds as the ones discussed here.

2.3 Approximation of the Problem Formulation

Now by theorems 2.1 and 2.2 it is obvious that the probability in (2.1) can be approximately minimized for large values of N and z_0/σ_z and for small values of the probability in (2.1) by minimizing either the variance or the upcrossing probability μ. However, for moderate values of N and z_0/σ_z, the results of Theorem 2.1 still holds, and it is tempting to believe that the upcrossing probability is a better criterion to minimize in this case, which is the interesting one for the problems described in Chapter

1. Therefore the following approximation of the criterion (2.1) will be considered from now on:

$$\min_{H} \mu(z_0) \tag{2.4}$$

subject to a stable closed loop system. There may be some problems with this approximation, since there are two ways of making μ small—either by keeping z well below z_0 or by keeping it well above z_0. It is clear that the probability in (2.1) will not be small, if μ is made small by keeping z well above z_0. To exclude this possibility, the minimization of μ will also be restricted to $\sigma_z \leq z_0$. The validity of the approximation of the problem formulation will be investigated further in Chapter 4.

2.4 Summary

The control problems described in Chapter 1 have been mathematically formalized in a stochastic framework. The control criterion has been defined such that the controller should minimize the probability for the largest value of the controlled signal to be above a level given a certain reference value. Two bounds for the criterion probability have been investigated. One of them is minimized by minimum variance control, the other one—tighter than the first one—is approximately minimized by minimizing the so called upcrossing probability. It has been made plausible that minimizing the upcrossing probability is a better approximation to the original problem than minimizing the variance.

3

Regulator Design

The problem of minimizing the upcrossing probability will now be solved. In the Section 3.1 the problem is reformulated as a one-parameter minimization over solutions to LQG-problems. Thus the complexity is not significantly larger than for an ordinary LQG-problem. The solution can be interpreted as a choice of optimal weighting-matrices in an LQG-problem. The equations for solving the LQG-problems are given in Section 3.2. In Section 3.3 the results of the previous sections are generalized to more general process models. Finally in Section 3.4 the results are summarized.

3.1 Solution

Let the stationary Gaussian sequence z be defined by

$$\begin{cases} x(k+1) = Ax(k) + B_1 u(k) + B_2 v(k) \\ \quad y(k) = C_1 x(k) + De(k) \\ \quad z(k) = C_2 x(k) \end{cases} \tag{3.1}$$

where v and e are zero mean, Gaussian, white noise sequences with $\mathrm{E}vv^T = R_1$, $\mathrm{E}ee^T = R_2$ and $\mathrm{E}ve^T = R_{12} = 0$. The signal y is the measured signal, and u is the control signal. The signal z is the signal that is desirable to control. The reason for not having $C_1 = C_2$ can be motivated by the examples in Chapter 1, where e.g. in the ore crusher example, the measured power y is not the desired signal to control, but instead some filtered version z of it, due to the filtering behavior of the thermal overload protection. More general process models than (3.1) may be considered. The treatment of a more general process model will be given later in Section 3.3 and in Appendix A. Introduce

$$\begin{cases} \alpha(k) = z(k+1) + z(k) \\ \beta(k) = z(k+1) - z(k) \end{cases} \tag{3.2}$$

which are independent variables due to the stationarity of z. Let $\mathcal{D}$ be the set of linear time-invariant stabilizing controllers of (3.1), and let $\mathcal{D}_z$ be the set of linear time-invariant stabilizing controllers of (3.1) for which

$$\sigma_z \leq z_0 \tag{3.3}$$

holds, where σ_z is the variance of z. Note that the sets $\mathcal{D}$ and $\mathcal{D}_z$ may be empty. It will be seen that the minimization of μ in (2.3) over $\mathcal{D}_z$ can be done by first minimizing

$$J = \mathrm{E}\{(1-\rho)\alpha^2 + \rho\beta^2\} \tag{3.4}$$

for $\rho \in [0,1]$ over $\mathcal{D}$, and then minimizing μ over the solutions obtained in the first minimization, i.e. over $\mathcal{V}_J \cap \mathcal{V}_z$, where

$$\begin{aligned} \mathcal{V}_J &= \Big\{ (\sigma_\alpha(H), \sigma_\beta(H)) \in R^2 \Big| H \in \mathcal{D}_J \Big\} \\ \mathcal{V}_z &= \Big\{ (\sigma_\alpha, \sigma_\beta) \in R^2 \Big| \sigma_z \leq z_0,\ \sigma_\alpha \geq 0,\ \sigma_\beta \geq 0 \Big\} \\ \mathcal{D}_J &= \Big\{ H \in \mathcal{D} \Big| H = \operatorname{argmin} J(H,\rho),\ \rho \in [0,1] \Big\} \end{aligned}$$

and where σ_α^2 and σ_β^2 are the variances of α and β.

In the following lemma J is rewritten to fit the standard LQG-problem formulation.

LEMMA 3.1
The loss function J in (3.4) can be written

$$J = \bar{J} + \mathrm{E}\{v^T B_2^T C_2^T C_2 B_2 v\}$$

where

$$\bar{J} = \mathrm{E}\{x^T Q_1 x + 2x^T Q_{12} u + u^T Q_2 u\}, \tag{3.5}$$

and where

$$\begin{aligned} Q_1 &= C_2^T C_2 + A^T C_2^T C_2 A + (1-2\rho)(C_2^T C_2 A + A^T C_2^T C_2) \\ Q_{12} &= (A^T + (1-2\rho)I) C_2^T C_2 B_1 \\ Q_2 &= B_1^T C_2^T C_2 B_1 \end{aligned} \tag{3.6}$$

Proof: The result follows immediately from the definitions of z in (3.1), and α and β in (3.2), and by noting that v is uncorrelated with x and u, since u is a functional of $y(k), y(k-1), \ldots$, and since $R_{12} = 0$. □

Remark. For $\rho = 0.5$ it follows that the controller that minimizes $J = \mathrm{E}\{z(k+1)^2 + z(k)^2\}$. This case thus corresponds to minimum variance control of z.

Next it will be shown that all jointly minimal variances of α and β can be obtained by minimizing J in (3.4) for $\rho \in [0,1]$. A precise definition of jointly minimal will first be given.

DEFINITION 3.1—Pareto Optimality
Let $\mathcal{X}$ denote an arbitrary nonempty set. Let $f_i : \mathcal{X} \to R^+, \quad 1 \le i \le s$ be s nonnegative functionals defined on $\mathcal{X}$. A point x^0 is said to be Pareto optimal with respect to the vector-valued criterion $f = (f_1, f_2, \cdots, f_s)$ if there does not exist $x \in \mathcal{X}$ such that $f_i(x) \le f_i(x^0)$ for all $i, \quad 1 \le i \le s$, and $f_k(x) < f_k(x^0)$ for some $k, \quad 1 \le k \le s$. □

The concept of Pareto optimality is illuminated in Figure 3.1. The set of achievable variances of α and β is the set of points in the plane that are above and to the right of or on the solid curve. The controller corresponding to the variances at B is not Pareto optimal, since there exist e.g. controllers corresponding to strictly lower variance of β without having larger variance of α—the controllers with variances on the line connecting A with B. Moreover it is seen that the controller corresponding to the variances at A is Pareto optimal, since by picking any other

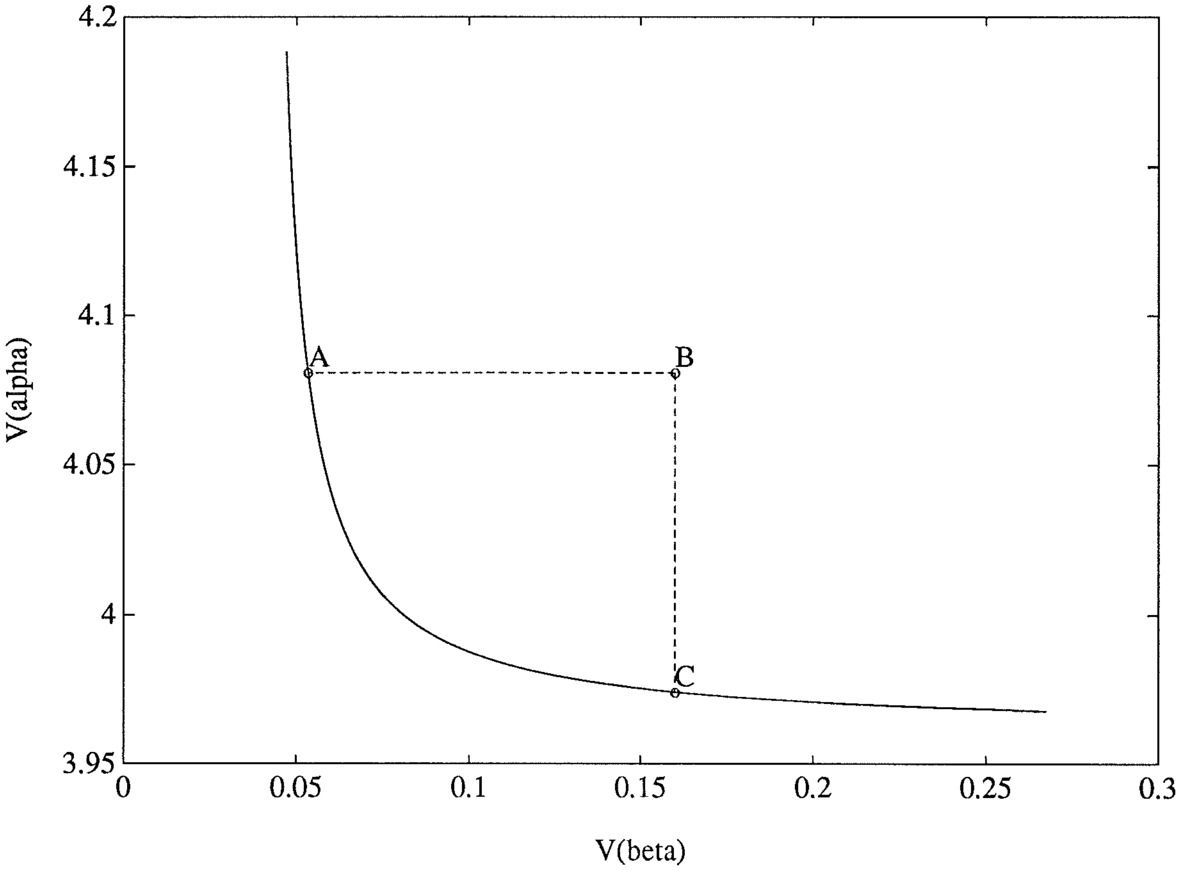

Figure 3.1 Illustration of Pareto optimality.

point, to the right of, above the curve, or on it, will either increase the variance of α or the variance of β. This reasoning holds for all points on the curve, and thus they are all Pareto optimal. Equivalent definitions of Pareto optimality can be found in [Leitmann, 1981, p. 292].

LEMMA 3.2
Suppose that (A, B_1) is stabilizable, and that (C_1, A) is detectable. Then the set $\mathcal{D}_P$ of Pareto optimal controllers with respect to $(\sigma_\alpha^2, \sigma_\beta^2)$ is a subset of $\mathcal{D}_J$.

Proof: Using the Youla parametrization, [Boyd and Barratt, 1991, Chapter 7.4], it follows that all stabilizing controllers of (3.1) can be parameterized by a stable transfer-function matrix Q. Thus to minimize J over $\mathcal{D}$ is equivalent to minimize J over Q, where Q belongs to the linear space of stable transfer-function matrices. Further it follows from [Boyd and Barratt, 1991, Chapter 7.4] that the transfer-function matrices from v and e to z are affine in Q. Since the variances of α and β are convex in the transfer-function matrices, it follows that the variances are convex in Q. The result now follows by [Khargonekar and Rotea, 1991, Theorem 1]. □

Remark 1. All controllers obtained by minimizing J for $\rho \in (0,1)$ are Pareto optimal by [Leitmann, 1981, Lemma 17.1]. If the controllers obtained for $\rho = 0$ and $\rho = 1$ are unique, then they are also Pareto optimal by [Leitmann, 1981, Lemma 17.2].

Remark 2. Remark 1 and Definition 3.1 imply that $\mathcal{V}_J$ can be parameterized by a scalar. This is not necessarily the case for $\mathcal{D}_J$.

Remark 3. Remark 1 implies that if the controllers obtained by minimizing J for $\rho \in [0,1]$ are unique, then a parameterization of $\mathcal{D}_P = \mathcal{D}_J$ by ρ is obtained, [Khargonekar and Rotea, 1991, p. 16].

The next lemma gives an expression for the upcrossing probability μ in (2.3) in terms of a double integral.

LEMMA 3.3
It holds that

$$\mu = \mathrm{P}\left\{z(0) \leq z_0 \cap z(1) > z_0\right\} = \int_0^\infty \phi(y) \int_{x_l}^{x_u} \phi(x) dx dy$$

where $\phi(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$, $x_l = (2z_0 - \sigma_\beta y)/\sigma_\alpha$, and $x_u = (2z_0 + \sigma_\beta y)/\sigma_\alpha$.

Proof: Since α and β are independent it holds that

$$\begin{aligned} \mu =& \mathrm{P}\left\{|\alpha - 2z_0| < \beta\right\} \\ =& \int\int_{|x-2z_0|<y} \frac{1}{\sigma_\alpha}\phi\left(\frac{x}{\sigma_\alpha}\right)\frac{1}{\sigma_\beta}\phi\left(\frac{y}{\sigma_\beta}\right) dx dy \end{aligned}$$

from which the result follows by a change of variables. □

In the following lemma it will be shown that the upcrossing probability μ in (2.3) has strictly positive partial derivatives with respect to σ_α and σ_β.

LEMMA 3.4
Let

$$\mathcal{V}(r) = \left\{ (\sigma_\alpha, \sigma_\beta) \in R^2 \middle| \sigma_z \le r,\ \sigma_\alpha > 0,\ \sigma_\beta > 0 \right\}$$

where $r > 0$. Then the upcrossing probability μ in (2.3) has strictly positive partial derivatives with respect to both σ_α and σ_β on $\mathcal{V}(r)$, if and only if $r \le z_0$.

Proof: It holds that

$$\frac{\partial \mu}{\partial \sigma_\beta} = \int_0^\infty \phi(y) \left(\frac{y}{\sigma_\alpha} \phi(x_u) + \frac{y}{\sigma_\alpha} \phi(x_l) \right) dy > 0$$

Further let $x_l = (2z_0 - \sigma_\beta y)/\sigma_\alpha$, and $x_u = (2z_0 + \sigma_\beta y)/\sigma_\alpha$. Using Lemma 3.3 gives

$$\frac{\partial \mu}{\partial \sigma_\alpha} = \int_0^\infty \phi(y) \left(\frac{x_l}{\sigma_\alpha} \phi(x_l) - \frac{x_u}{\sigma_\alpha} \phi(x_u) \right) dy$$

By completing the squares in the exponents and by a change of coordinates it is possible to express the integral in terms of $\Phi(x) = \int_{-\infty}^{x} \phi(t) dt$, and $\sigma_z^2 = (\sigma_\alpha^2 + \sigma_\beta^2)/4$

$$\frac{\partial \mu}{\partial \sigma_\alpha} = \frac{\sigma_\alpha}{8\pi\sigma_z^2} \exp\left(-\frac{\gamma^2}{2}\right) \left[\sqrt{2\pi}\gamma \left(2\Phi(\eta) - 1\right) - 2\frac{\eta}{\gamma} \exp\left(-\frac{\eta^2}{2}\right) \right]$$

where $\eta = \gamma\sqrt{\xi}$, $\xi = (\sigma_\beta/\sigma_\alpha)^2$, and $\gamma = z_0/\sigma_z > 0$. It is seen that $\frac{\partial \mu}{\partial \sigma_\alpha} > 0$ if and only if

$$2\Phi(\eta) - 1 > \sqrt{\frac{2}{\pi}} \frac{\eta}{\gamma^2} \exp\left(-\frac{\eta^2}{2}\right)$$

So if $\frac{\partial \mu}{\partial \sigma_\alpha} > 0$ on $\mathcal{V}(r)$, then the inequality above holds for all values of $\eta > 0$, since $\gamma > 0$, and since it must hold for all values of $\xi > 0$. A Taylor-expansion round $\eta = 0$ gives

$$\sqrt{\frac{2}{\pi}}\eta > \sqrt{\frac{2}{\pi}} \frac{\eta}{\gamma^2} + \mathcal{O}(\eta^2)$$

So for the inequality to hold for small values of η, it must be that $\gamma \ge 1$, which is equivalent to $r \le z_0$.

Now suppose that $r \leq z_0$, which implies $\gamma \geq 1$. Then

$$\begin{aligned}(2\Phi(\eta)-1)^2 &\geq 1-\exp\left(-\frac{2\eta^2}{\pi}\right)-\frac{2(\pi-3)}{3\pi^2}\eta^4\exp\left(-\frac{\eta^2}{2}\right)\\ &\geq 1-\exp\left(-\frac{2\xi}{\pi}\right)-\frac{2(\pi-3)}{3\pi^2}\xi^2\exp\left(-\frac{\xi}{2}\right)\end{aligned}$$

where the first inequality follows from [Abromowitzm and Stegun, 1968, Formula 26.2.25] and the second one from $\gamma \geq 1$. Further

$$\left(\sqrt{\frac{2}{\pi}}\frac{\eta}{\gamma^2}\exp\left(-\frac{\eta^2}{2}\right)\right)^2 \leq \frac{2}{\pi}\xi\exp(-\xi)$$

To show $\frac{\partial\mu}{\partial\sigma_\alpha} > 0$, it is now sufficient to show $L > R$ for $\xi > 0$, where

$$\begin{aligned}L &= \exp\left(\frac{\xi}{2}\right)\\ R &= \frac{2}{\pi}\xi\exp\left(-\frac{\xi}{2}\right)+\exp\left(\left(\frac{1}{2}-\frac{2}{\pi}\right)\xi\right)+\frac{2(\pi-3)}{3\pi^2}\xi^2\end{aligned}$$

Some calculations give

$$\begin{aligned}L &\geq 1+\frac{1}{2}\xi+\frac{1}{8}\xi^2\\ R &\leq 1+\frac{1}{2}\xi+\left(\frac{1}{8}-\frac{1}{3\pi}\right)\xi^2\end{aligned}$$

From this it follows that $L > R$ for $\xi > 0$, so $\frac{\partial\mu}{\partial\sigma_\alpha} > 0$. □

Remark. The largest region $\mathcal{V}(r)$ in which both $\frac{\partial\mu}{\partial\sigma_\alpha} > 0$ and $\frac{\partial\mu}{\partial\sigma_\beta} > 0$ is $\mathcal{V}(z_0)$. So if the constraint $\sigma_z \leq z_0$ is not considered, then it may well be that μ is minimized by $\sigma_\alpha = \infty$.

It will now be shown how the minimization of μ in (2.3) can be rephrased to a minimization over a set of LQG-problem-solutions.

THEOREM 3.1
Suppose that (A, B_1) is stabilizable, and that (C_1, A) is detectable. Then

$$\left\{ H \in \mathcal{D}_z \middle| H = \text{argmin}\ \mu\left(\sigma_\alpha(H), \sigma_\beta(H)\right) \right\} \subseteq \mathcal{D}_P \cap \mathcal{D}_z$$

Proof: Assume that the minimum of μ on $\mathcal{D}_z$ is attained for some $H \notin \mathcal{D}_P \cap \mathcal{D}_z$. For all $H \notin \mathcal{D}_P \cap \mathcal{D}_z$ there exist by Definition 3.1 $\bar{H} \in \mathcal{D}_z$ such that $\sigma_i(\bar{H}) < \sigma_i(H)$ for at lest one of $i = \alpha, \beta$. Since μ is differentiable and by Lemma 3.4 has strictly positive partial derivatives with respect to σ_α and σ_β on $\mathcal{V}(z_0)$, it follows that $\mu(\sigma_\alpha(\bar{H}), \sigma_\beta(\bar{H})) < \mu(\sigma_\alpha(H), \sigma_\beta(H))$. This is a contradiction, and thus the minimum of μ is attained on $\mathcal{D}_P \cap \mathcal{D}_z$, if it exists on $\mathcal{D}_z$. □

Remark 1. By Lemma 3.2 $\mathcal{D}_P \subseteq \mathcal{D}_J$, which implies

$$\left\{ (\sigma_\alpha(H), \sigma_\beta(H)) \in \mathcal{V}_z \middle| H = \text{argmin}\ \mu(\sigma_\alpha(H), \sigma_\beta(H)) \right\} \subseteq \mathcal{V}_J \cap \mathcal{V}_z$$

Thus the minimization of μ can be done over $\mathcal{V}_J \cap \mathcal{V}_z$. This is a one-parametric optimization problem by Remark 2 of Lemma 3.2.

Remark 2. If for each $\rho \in [0, 1]$ the minimizing H of J is unique, then by Lemma 3.1 and Remark 3 of Lemma 3.2 the minimization of μ can be thought of as finding optimal weights in an LQG-problem.

3.2 LQG-equations

For short reference the equations for deriving the solution that minimizes $\bar{J}$ in (3.5) in Lemma 3.1 when the controller H is allowed to have a direct-term are given below. More stringent proofs of the results can be found in Appendix A, which also covers a more general process model. The transfer function from measurement to control is

$$H(q) = -L_x(qI - A + B_1 L_x + K C_1)^{-1} K_y - L_y \tag{3.7}$$

where L_x, L_y and K are given by

$$
\begin{aligned}
L_x &= L - L_y C_1 \\
L_y &= L K_f \\
L &= (Q_2 + B_1^T S B_1)^{-1}(B_1^T S A + Q_{12}^T) \\
K_y &= K - B_1 L_y \\
K &= A K_f \\
K_f &= P C_1^T (D R_2 D^T + C_1 P C_1^T)^{-1}
\end{aligned}
$$

where S and P are the solutions to the Riccati-equations, [Åström and Wittenmark, 1990, Chapter 11.4], and [Gustafsson and Hagander, 1991],

$$
\begin{aligned}
A^T S A - S - (A^T S B_1 + Q_{12})(Q_2 + B_1^T S B_1)^{-1}(Q_{12}^T + B_1^T S A) + Q_1 = 0 \\
A P A^T - P - (A P C_1^T + B_2 R_{12} D^T)(D R_2 D^T + C_1 P C_1^T)^{-1} \\
(C_1 P A^T + D R_{12}^T B_2^T) + B_2 R_1 B_2^T = 0
\end{aligned}
\tag{3.8}
$$

and where Q_1, Q_2 and Q_{12} are given by (3.6) in Lemma 3.1. To calculate σ_z, σ_u, σ_α and σ_β the following Lyapunov-equation for the closed loop system should be solved, [Åström, 1970, p. 49],

$$
A_c X A_c^T + B_c R B_c^T = X \tag{3.9}
$$

where

$$
\begin{aligned}
A_c &= \begin{pmatrix} A - B_1 L & B_1 L_x \\ 0 & A - K C_1 \end{pmatrix} \\
B_c &= \begin{pmatrix} B_2 & -B_1 L_y D \\ B_2 & -K D \end{pmatrix} \\
R &= \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}
\end{aligned}
$$

Then σ_α, σ_β, σ_z and σ_u are given by

$$
\begin{aligned}
\sigma_\alpha^2 &= (\,C_2 \quad 0\,)\,((A_c + I) X (A_c + I)^T + B_c R B_c^T)\,(\,C_2 \quad 0\,)^T \\
\sigma_\beta^2 &= (\,C_2 \quad 0\,)\,((A_c - I) X (A_c - I)^T + B_c R B_c^T)\,(\,C_2 \quad 0\,)^T \\
\sigma_z^2 &= (\,C_2 \quad 0\,)\,X\,(\,C_2 \quad 0\,)^T \\
\sigma_u^2 &= (\,-L \quad L_x\,)\,X\,(\,-L \quad L_x\,)^T + L_y D R_2 D^T L_y^T
\end{aligned}
\tag{3.10}
$$

Since A_c is triangular, Equation (3.9) can be split up into three equations, where one of the solutions is P in (3.8), which reduces the complexity of the problem.

3.3 The General Case

The results of the previous section are now generalized to the more general process model:

$$\begin{cases} x(k+1) = Ax(k) + B_1u(k) + B_2v(k) \\ y(k) = C_1x(k) + D_1e(k) \\ z(k) = C_2x(k) + D_2w(k) \end{cases} \tag{3.11}$$

where v, e and w are zero mean Gaussian white noise sequences with the positive semidefinite covariance matrix

$$\mathrm{E}\left\{ \begin{pmatrix} v \\ e \\ w \end{pmatrix} \begin{pmatrix} v^T & e^T & w^T \end{pmatrix} \right\} = \begin{pmatrix} R_1 & R_{12} & R_{13} \\ R_{12}^T & R_2 & R_{23} \\ R_{13}^T & R_{23}^T & R_3 \end{pmatrix}$$

The signal y is the measurement signal, and u is the control signal. The proofs of the results in this section can be found in Appendix A. Let

$$
\begin{aligned}
Q_{23} &= B_1^T C_2^T C_2 B_2 \\
Q_{24} &= (1-2\rho) B_1^T C_2^T D_2 \\
\bar{Q} &= (Q_2 + B_1^T S B_1) \\
L &= \bar{Q}^{-1}(B_1^T S A + Q_{12}^T) \\
L_v &= \bar{Q}^{-1}(B_1^T S B_2 + Q_{23}) \\
L_w &= \bar{Q}^{-1} Q_{24} \\
R_y &= C_1 P C_1^T + D_1 R_2 D_1^T \\
K_f &= P C_1^T R_y^{-1} \\
K_v &= R_{12} D_1^T R_y^{-1} \\
K_w &= R_{23} D_1^T R_y^{-1} \\
K &= A K_f + B_2 K_v \\
K_y &= K - B_1 L_y \\
L_y &= L K_f + L_v K_v + L_w K_w \\
L_x &= L - L_y C_1
\end{aligned}
$$

where S and P are the solution to the Riccati-equation in (3.8), and where Q_1, Q_2, and Q_{12} are given by (3.6) in Lemma 3.1. The transfer function $H(q)$ for the optimal controller with direct-term that minimizes J in (3.4) is then given by

$$
H(q) = -L_x(qI - A + B_1 L_x + K C_1)^{-1} K_y - L_y \tag{3.12}
$$

Further let

$$
\begin{aligned}
A_c &= \begin{pmatrix} A - B_1 L & B_1 L_x \\ 0 & A - K C_1 \end{pmatrix} \\
B_c &= \begin{pmatrix} B_2 & -B_1 L_y D_1 \\ B_2 & -K D_1 \end{pmatrix} \\
R &= \begin{pmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{pmatrix}
\end{aligned}
$$

The variances of α, β, z and u are then given by

$$
\begin{aligned}
\sigma_\alpha^2 &= (\,C_2 \quad 0\,)\,((A_c+I)X(A_c+I)^T + B_cRB_c^T)\,(\,C_2 \quad 0\,)^T + 2D_2R_3D_2^T \\
\sigma_\beta^2 &= (\,C_2 \quad 0\,)\,((A_c-I)X(A_c-I)^T + B_cRB_c^T)\,(\,C_2 \quad 0\,)^T + 2D_2R_3D_2^T \\
\sigma_z^2 &= (\,C_2 \quad 0\,)\,X\,(\,C_2 \quad 0\,)^T + D_2R_3D_2^T \\
\sigma_u^2 &= (\,-L \quad L_x\,)\,X\,(\,-L \quad L_x\,)^T + L_yD_1R_2D_1^TL_y^T
\end{aligned}
\tag{3.13}
$$

where X is the solution to the Lyapunov equation in (3.9). Notice that (3.9) is triangular also in the general case.

3.4 Summary

It has been shown that the minimization of the upcrossing probability can be expressed as minimization over a set of LQG-problem solutions parameterized by a scalar, regardless of the uniqueness of the solutions to the LQG-problems. If the solutions to the LQG-problems are unique, then the problem of minimizing the upcrossing probability can be thought of as finding optimal weightings in an LQG-problem. Note that the Lyapunov equation (3.9) is linear, and thus does not add any significant complexity compared to an ordinary LQG-problem.

The algorithm for minimizing the upcrossing probability can be summarized as: 1) solve the associated LQG-problems, and 2) minimize the upcrossing probability over the variances obtained in the first step. It has been seen that the computation of the variances is not more complicated than solving a linear system of equations. Further the upcrossing probability can easily be obtained with some numerical integration routine. The complexity of this latter problem does not depend on the size of the process model. Thus the computations performed for each value of ρ is not significantly larger than for an ordinary LQG-problem. Moreover by adopting some numerical routine for minimizing the upcrossing probability, it may not be necessary to solve that many LQG-problems. A good choice of starting value for ρ is 0.5, which corresponds to the minimum variance controller. In this sense the computational burden for obtaining the minimum risk controller is not significantly larger than for the LQG controller that corresponds to minimum variance control.

4

Example

To evaluate the performance of the minimum risk controller obtained by minimizing the upcrossing probability a first order process will be investigated. In the Section 4.1 the process is defined. The set of LQG-solutions is calculated analytically in Section 4.2. In the Section 4.3 the minimum risk controller is computed and compared with the minimum variance controller. It is seen that the new controller causes a lower upcrossing probability and smaller probability for the largest value of the signal of being above the critical level. Further it is seen that it has a control signal that is more well-behaved. In the Section 4.4 the results of the previous sections are summarized.

4.1 Process

Let the process be given by

$$\begin{cases} x(k+1) = x(k) + 0.04u(k) + 0.2v(k) \\ y(k) = x(k) + 5e(k) \\ z(k) = x(k) \end{cases}$$

where v and e ar zero mean Gaussian white noise sequences with $\mathrm{E}v^2 = R_1 = 1$, $\mathrm{E}e^2 = R_2 = 1$ and $\mathrm{E}ve = R_{12} = 0$. The signal y is the measurement signal, and u is the control signal.

4.2 LQG-Controllers

The weighting-matrices in (3.6) are

$$\begin{aligned} Q_1 &= 4(1-\rho) \\ Q_{12} &= 0.08(1-\rho) \\ Q_2 &= 0.16 \end{aligned}$$

and the solutions to the Riccati-equations in (3.8) are

$$\begin{aligned} S &= 2\sqrt{\rho(1-\rho)} \\ P &= \frac{0.04+\sqrt{4.0016}}{2} \end{aligned}$$

Some more tedious calculations will give the controller $H(q)$ in (3.7) to be

$$H(q) = -\frac{s_0 q}{r_0 q + r_1}$$

where

$$\begin{aligned} s_0 &= 2\sqrt{\rho(1-\rho)} + 2(1-\rho))(0.04+\sqrt{4.0016}) \\ r_0 &= 0.04(2\sqrt{\rho(1-\rho)}+1)(50.04+\sqrt{4.0016}) \\ r_1 &= 2(1-2\rho) \end{aligned}$$

It is interesting to note that for $\rho = 0.5$—minimum variance control by the remark of Lemma 3.1—the controller is a proportional controller.

4.3 MR and MV Controllers

The minimum risk (MR) controller will now be compared with the minimum variance (MV) controller.

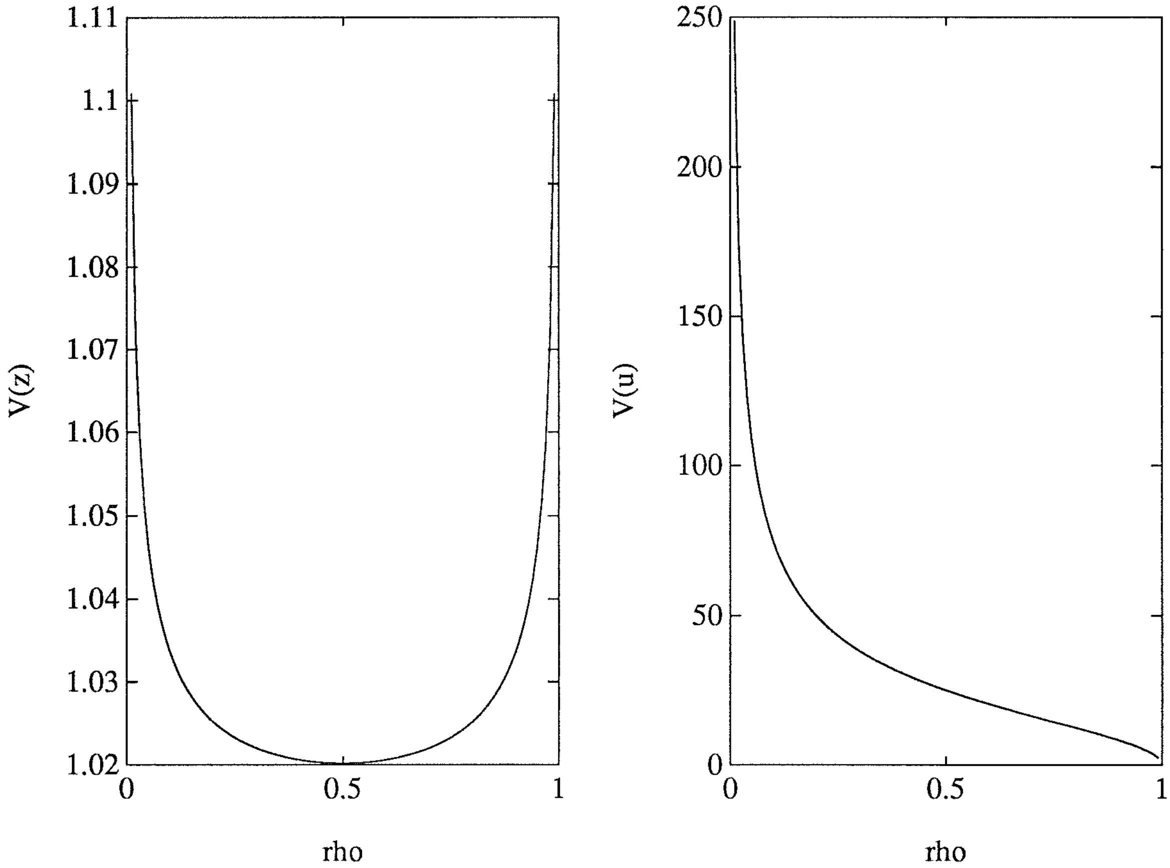

Figure 4.1 The variances of z—left, and u—right, as functions of ρ.

Variance and Upcrossing Probability

The variances of z and u have been calculated numerically for values of ρ with a step of 0.01 in the range of 0.01 to 0.99. It is seen in Figure 4.1 that the variance of z does not depend so much on ρ as does the variance of u.

The probability μ has been calculated for $m_z = 0$ and the values $z_0 = 2$, 3 ,4 and 5 of the critical level. The result is seen in Figure 4.2. The minimum value of the probability μ is obtained for ρ greater than 0.5. The variance of the control signal is smaller the larger ρ is, and the controller obtained for $\rho = 0.5$ is the MV controller by the remark of Lemma 3.1. Thus the MR controller not only minimize the upcrossing probability, but that it also has a control signal that is more well-behaved than that of the MV controller.

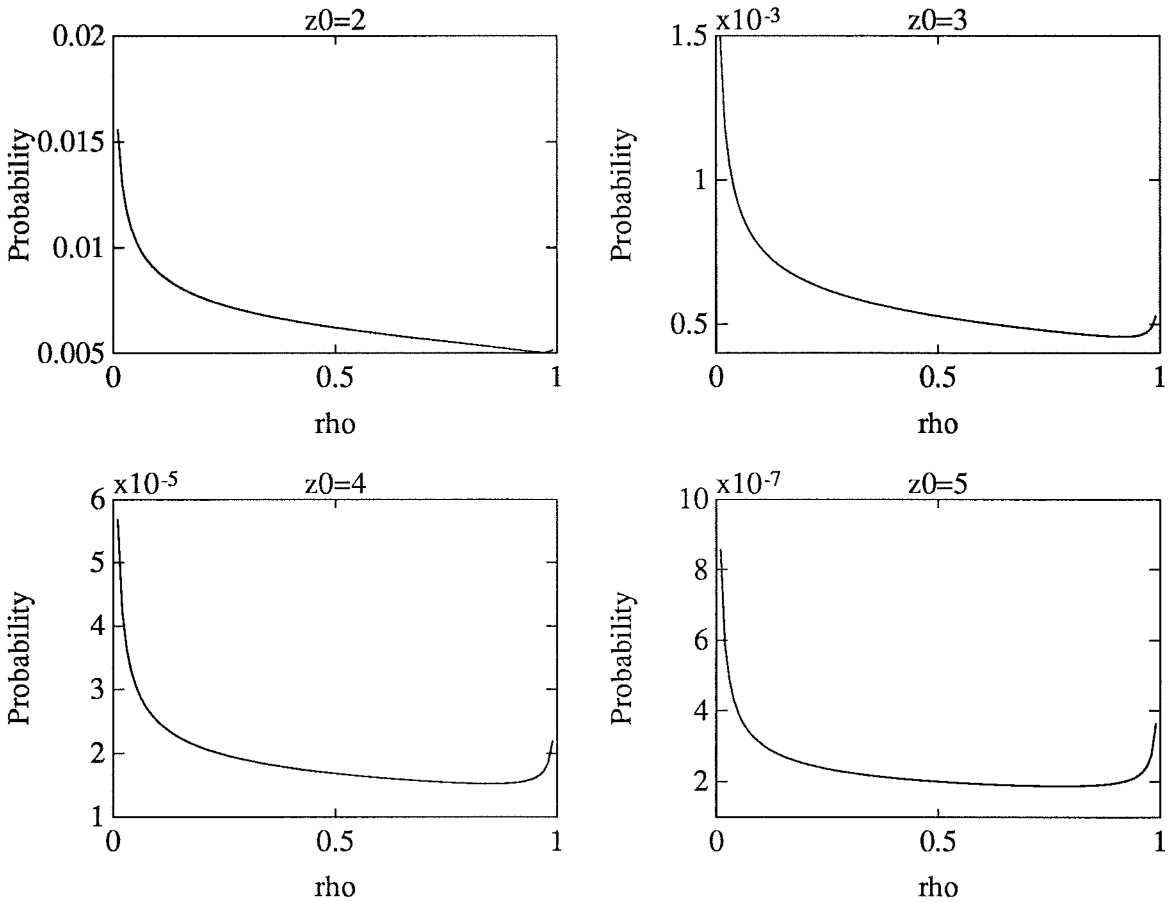

Figure 4.2 The probability μ as function of ρ for $z_0 = 2$—top left, $z_0 = 3$—top right, $z_0 = 4$—bottom left, and $z_0 = 5$—bottom left.

Simulations

The controllers have also been compared by simulations. The same noise sequences were used for both the MR controller and the MV controller in all cases. Figure 4.3 shows plots of z and u as functions of time for the MV controller and the MR controller for $z_0 = 3$. It is seen that that the MR controller manages to keep the signal z below the critical level, while the MV controller does not. Further it is seen that the variance of u is smaller for the MR controller than for the MV controller. Note that z is not white noise for the MV controller although y is, since y is correlated with e.

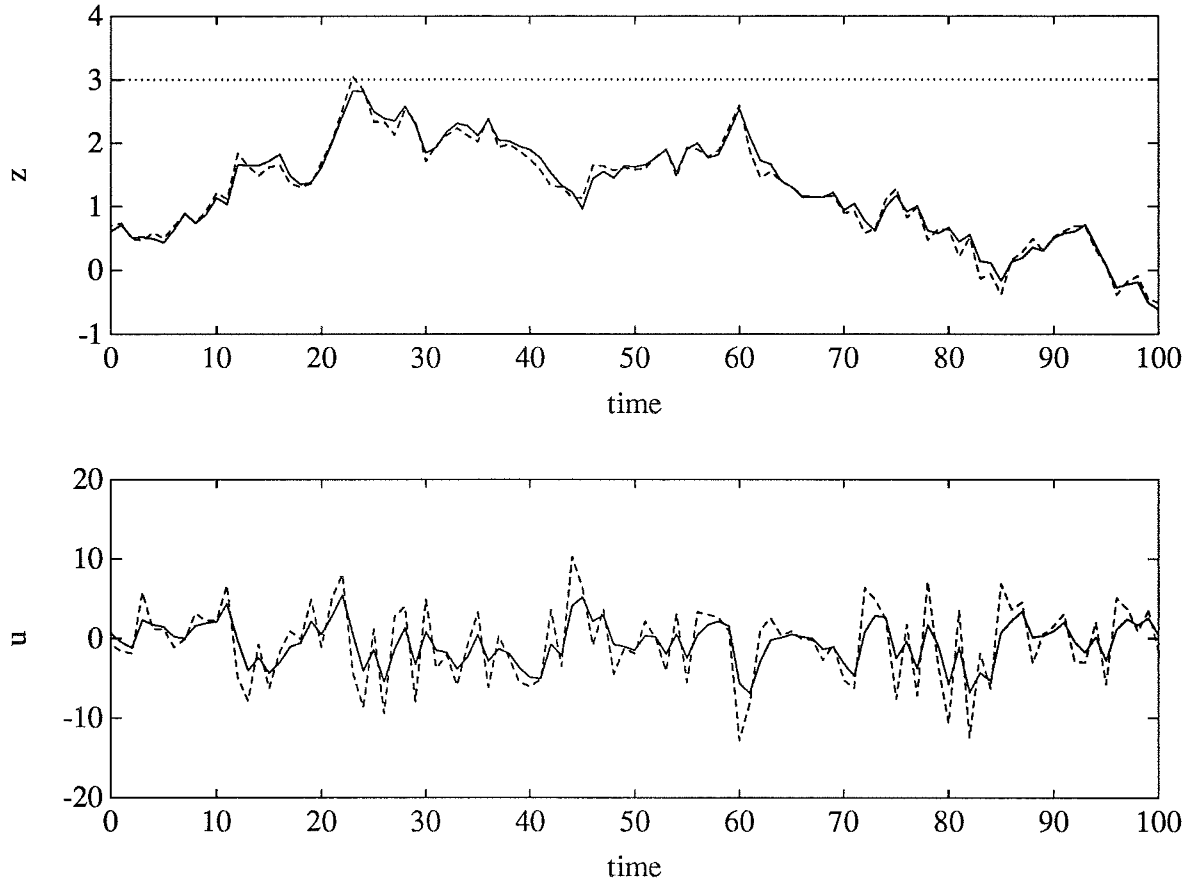

Figure 4.3 The signals $z(t)$—top, and $u(t)$—bottom, as function of time for the optimal controller—solid line, and the minimum variance controller—dashed line.

Robustness

To investigate the robustness against unmodeled non-linearities the process-dynamics was changed to

$$x(k+1) = 0.33x^2(k) + x(k) + 0.04u(k) + 0.2v(k)$$

Thus the process for which the controllers are designed can be thought of as a linearization of the non-linear process round $x(k) = 0$. If $v(k)$ is zero, and if the minimum variance control strategy is applied, then the nonlinear process is stable for initial values of x that are smaller than approximately 3. Therefore it is interesting to compare the MR controller designed for $z_0 = 3$ with the MV controller. Plots of y, z, and u for the two different control strategies with the same noise sequences are shown in figure 4.4 and 4.5. It is seen that the MV controller has

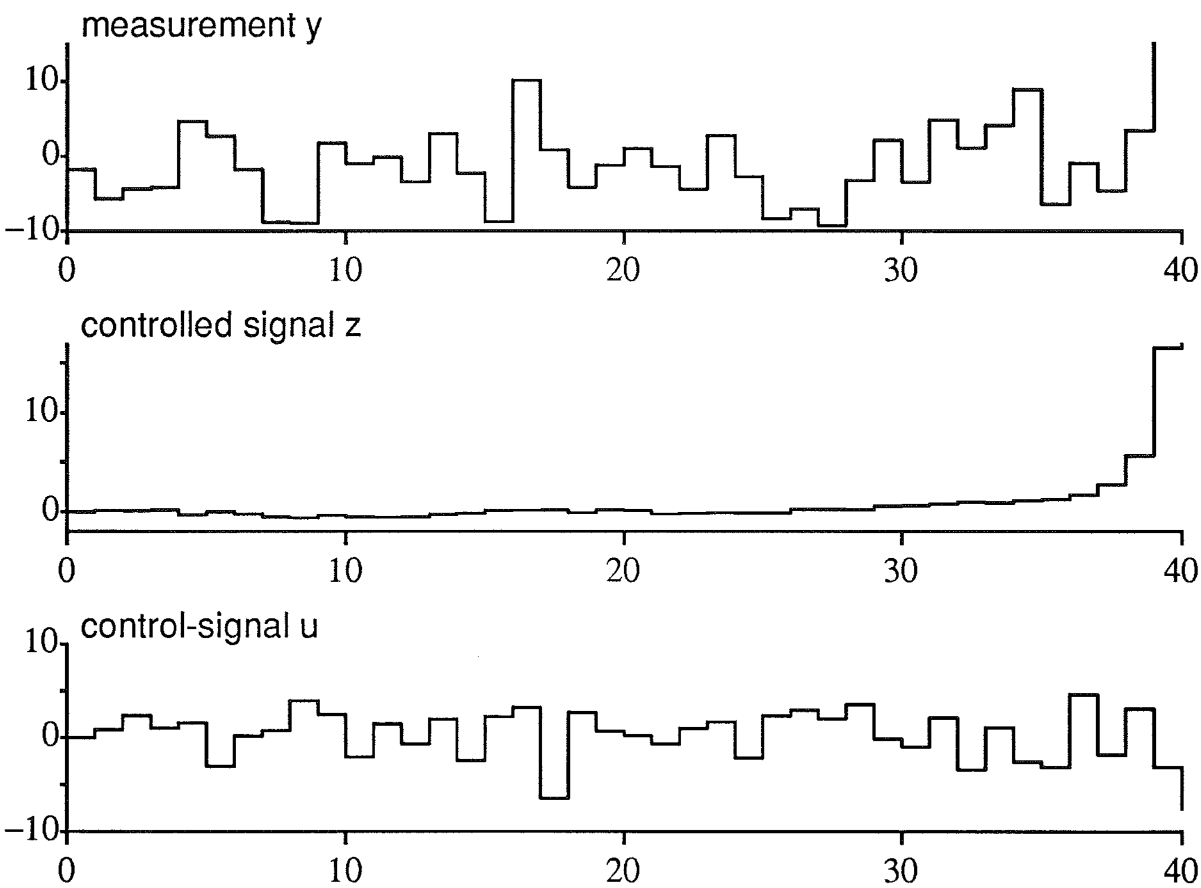

Figure 4.4 The signals $y(t)$, $z(t)$ and $u(t)$ as functions of time for the minimum risk controller, when controlling a non-linear process.

more difficulties to stabilize the process than the MR controller has.

Transfer Functions

The MR controller for $z_0 = 3$ $(\rho = 0.92)$ is given by:

$$H(q) = -\frac{0.4901q}{q - 0.4804}$$

and the MV controller is given by:

$$H(q) = -0.9802$$

It is interesting to note that the difference between the MV controller and the MR controller is that the MR controller has a 3 times lower gain for high frequences $(q = -1)$ due to the MR controller being a first order system while the MV controller being only a proportional controller. This

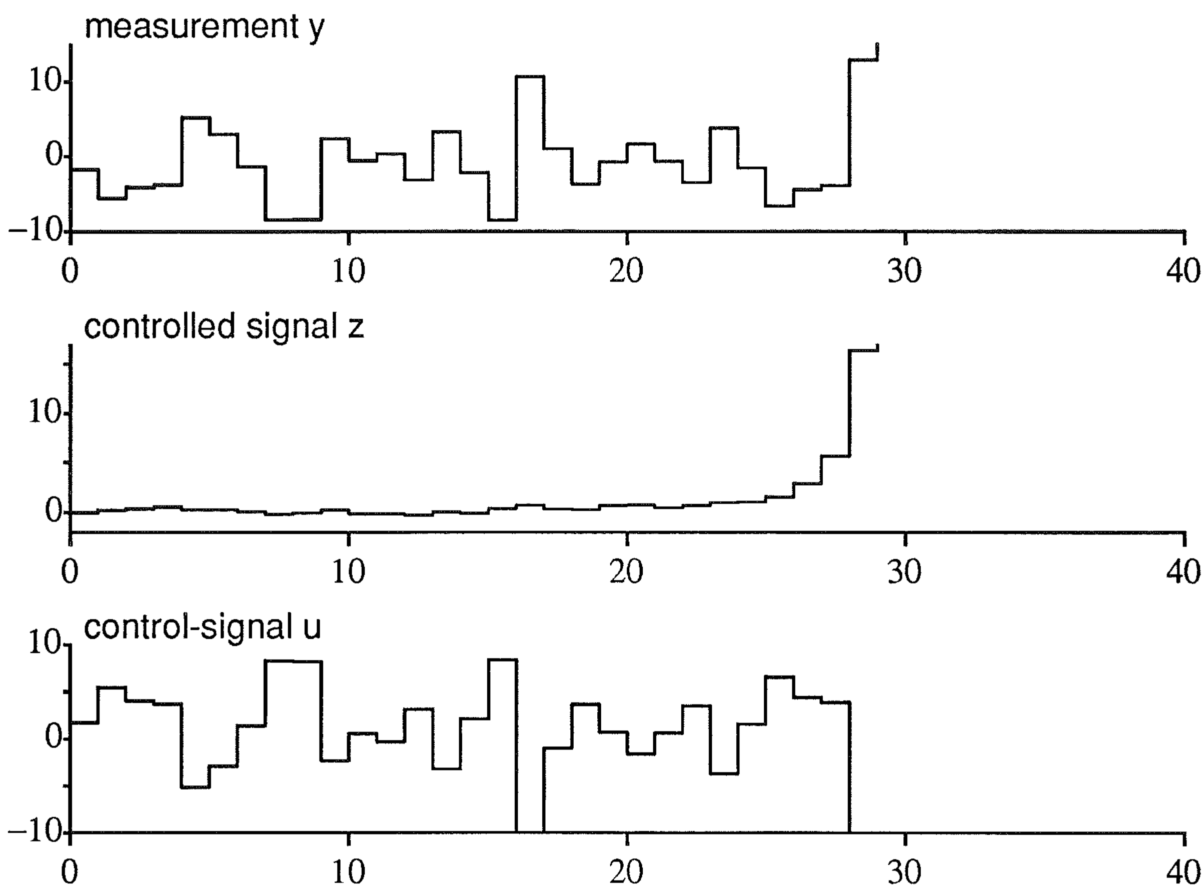

Figure 4.5 The signals $y(t)$, $z(t)$ and $u(t)$ as functions of time for the the minimum variance controller, when controlling a non-linear process.

explains why the variance of the control signal is much smaller for the MR controller. Some calculations give that

$$(q - 0.9608)z = 0.2v - 0.196e$$

for the MV controller and

$$[(q-1)(q-0.4804) + 0.0196]z = 0.2(q-0.4804)v - 0.098e$$

for the MR controller. It is seen that the main difference in the closed loop behavior between the MV controller and the MR controller is the lower high frequency gain ($q = -1$) from e to z for the MR controller.

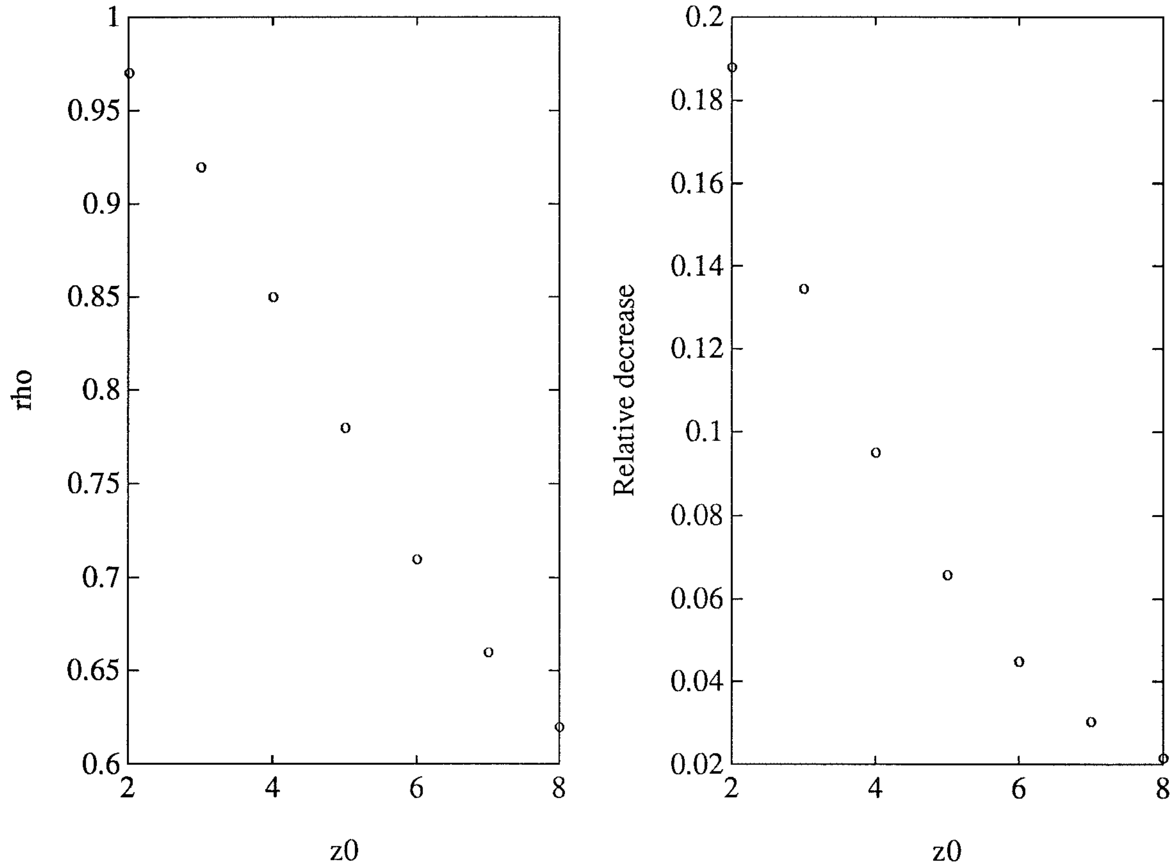

Figure 4.6 The optimal values of ρ as function of z_0—left, and $(\mu_{\text{mv}} - \mu_{\text{opt}})/\mu_{\text{mv}}$ as function of z_0—right, where μ_{mv} is the upcrossing probability for the minimum variance controller and μ_{opt} is the upcrossing probability for the minimum risk controller.

Approximation-validity

The validity of the assumptions made in the approximation of the problem formulation in Section 2.3 will now be investigated further; one positive indication has already been seen in Figure 4.3. In Figure 4.6 it is seen how the optimal value of ρ, and how the relative decrease of upcrossing probability between the MV controller and the MR controller decreases as z_0 increases. This indicates that the MR controller and the MV controller are approximately the same for large values of z_0.

To investigate the behavior of the controllers for moderate values of z_0, Monte Carlo simulations have been performed to estimate the probability $\text{P}\left\{\max_{0 \le k \le N} z(k) > z_0\right\}$ in (2.1) for the MR controller—$\hat{P}_{\text{opt}}$, and for the MV controller—$\hat{P}_{\text{mv}}$. In Figure 4.7 these estimates of the proba-

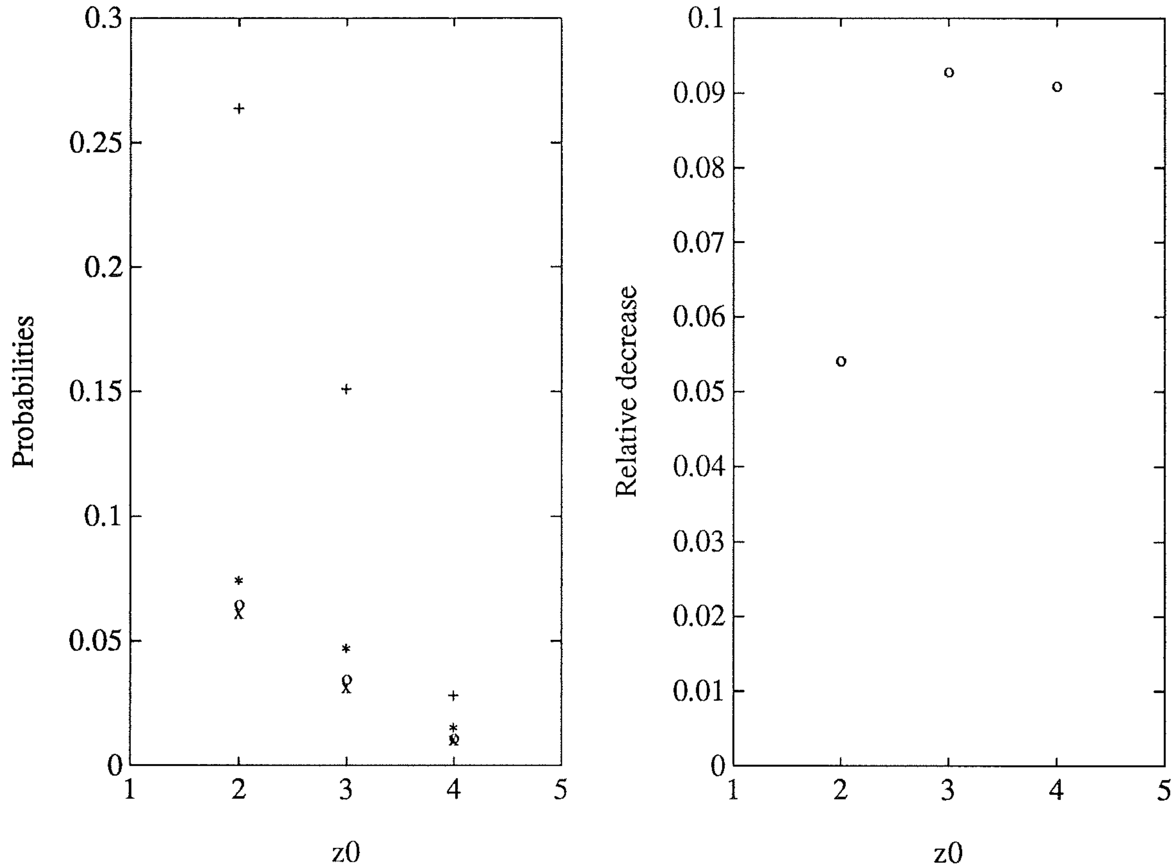

Figure 4.7 The left plot shows the bound P_2 for the minimum variance controller—'+', the bound P_1 for the minimum risk controller—'*', $\hat{P}_{\mathrm{mv}}$—'o', and $\hat{P}_{\mathrm{opt}}$ —'x', as functions of z_0. The values of N has been 10 for $z_0 = 2$, 100 for $z_0 = 3$ and 1000 for $z_0 = 4$. The right plot shows $(\hat{P}_{\mathrm{mv}} - \hat{P}_{\mathrm{opt}})/\hat{P}_{\mathrm{mv}}$ as function of z_0.

bilities are compared with the bounds P_1 and P_2 of Theorem 2.1, where for short reference

$$P_1(z_0) = \mathrm{P}\{z(0) > z_0\} + N\mu(z_0)$$
$$P_2(z_0) = (N+1)\mathrm{P}\{z(0) > z_0\}$$

The bound P_1, which by Remark 1 of Theorem 2.1 is approximately minimized by the MR controller, has been computed for the MR controller. The bound P_2, which by Remark 2 of Theorem 2.1 is minimized by MV control, has been computed for the MV controller. The values of N and z_0 has been chosen such that the bound P_2 is about 0.1. The values are $(z_0, N) = (2, 10)$, (3,100) and (4,1000). The result is shown in Figure 4.7. It is seen in the left plot that the bound P_1 is much lower than the bound

P_2, and that the estimate $\hat{P}_{\mathrm{opt}}$ is lower than estimate $\hat{P}_{\mathrm{mv}}$. The latter is seen more clearly in the right plot, where the relative decrease of the probability of being above the critical level between the MV controller and the MR controller— $(\hat{P}_{\mathrm{mv}} - \hat{P}_{\mathrm{opt}})/\hat{P}_{\mathrm{mv}}$—is plotted versus z_0. Thus the MR controller is about 5% to 10% better than the minimum variance controller for moderate values of the critical level in this example.

4.4 Summary

The theory developed in the previous chapters has been evaluated using a first order process. In spite of the simplicity of the process many interesting features of the new controller have been demonstrated.

It has been shown that the MR controller is a first order system whereas the MV controller is only a proportional controller. The latter has a higher high-frequency gain. The variance of z is slightly larger but the variance of u is a lot smaller for the MR controller as compared with the MV controller. Further it has been seen in simulations that the probability for the largest value of z of being above the critical level is smaller for the MR controller. It has also been seen that the new controller is more robust against unmodeled non-linearities than the MV controller. The simulations have also given insight into the consequences of the approximations made to derive the new controller. When comparing the differences between the MR controller and the MV controller for varying distances to the critical level, it has been seen that these are larger for moderate values of the distance and smaller for larger values of the distance. For the examples in Chapter 1 the distance is typically moderate, and thus it has been justified that the MR controller may well be superior to the MV controller for this class of interesting problems.

5

Conclusions

A new optimal stochastic control problem has been posed. The solution minimizes the probability for a signal's largest value to be above a level given a certain reference value. There are many examples of control problems for which this approach is appealing, i.e. problems for which there exist a level such that a failure in the controlled system occurs when the controlled signal is above the level.

It has been seen that this control problem is closely related both to the problem of minimizing the variance of the signal—minimum variance control—and to the problem of minimizing the upcrossing probability. The latter relation is novel, whereas the former relation has been known for a long time, but the motivation given here is believed to be new. It has been made plausible that the upcrossing probability is a better criterion to minimize than the minimum variance criterion.

The problem of minimizing the upcrossing probability over the set of stabilizing linear time-invariant controllers has been rephrased to a minimization over LQG-problem solutions parameterized by a scalar, and thus the complexity is not significantly larger than for an ordinary LQG-problem. If the solutions to the LQG-problems are unique, then the problem of minimizing the upcrossing probability can be thought of as finding optimal weighting-matrices in an LQG-problem. The key to the new method is the reformulation using the independent variables α and β

making it possible to quantify by Lemma 3.3 the upcrossing probability in terms of the variances of α and β.

The new controller has been compared with the minimum variance controller for a first order process. It has been seen that the new controller causes a lower upcrossing intensity and a smaller probability for the largest value of the controlled signal to be above the dangerous level. Further it has been seen that the control signal is more well-behaved.

Both theory and simulations have shown that the minimum risk controller and the minimum variance controller are approximately the same for large values of the dangerous level. However, in the example it has been seen that the minimum risk controller can have up to about 10% better performance for moderate values of the critical level, which is the interesting case for the examples in Chapter 1. This makes it possible to choose the reference value closer to the critical level when using the minimum risk controller, than when using the minimum variance controller. This will in many cases increase the profit.

Thus the new controller has many advantages as compared with the minimum variance controller—a smaller probability of being above the dangerous level, a control-signal that is more well-behaved, and an interpretation as weighting-optimal LQG. The only drawback is the slightly larger computational burden.

This concludes the work of proving the raison d'être of the minimum risk controller and demonstrating its advantages as compared to the minimum variance controller for a large class of control problems.

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A

More General Process Model

Let the stationary Gaussian sequence z be defined by a process model more general than (3.1):

$$\begin{cases} x(k+1) = Ax(k) + B_1 u(k) + B_2 v(k) \\ \quad y(k) = C_1 x(k) + D_1 e(k) \\ \quad z(k) = C_2 x(k) + D_2 w(k) \end{cases} \tag{A.1}$$

where v, e and w are zero mean Gaussian white noise sequences with the positive semidefinite covariance matrix

$$\mathrm{E}\left\{ \begin{pmatrix} v \\ e \\ w \end{pmatrix} \begin{pmatrix} v^T & e^T & w^T \end{pmatrix} \right\} = \begin{pmatrix} R_1 & R_{12} & R_{13} \\ R_{12}^T & R_2 & R_{23} \\ R_{13}^T & R_{23}^T & R_3 \end{pmatrix}$$

The signal y is the measurement signal, and u is the control signal. The following lemma is a generalization of Lemma 3.1.

LEMMA A.1
The loss function J in (3.4) can be written

$$J = \bar{J} + \mathrm{E}\{v^T Q_3 v + 2v^T Q_{34} w + w^T Q_4 w\}$$

where

$$\bar{J} = \mathrm{E}\{x^T Q_1 x + 2x^T Q_{12} u + u^T Q_2 u + 2u^T Q_{23} v + 2u^T Q_{24} w\}, \quad \text{(A.2)}$$

and where Q_1, Q_{12} and Q_2 are as in (3.6) in Lemma 3.1 and where

$$\begin{aligned} Q_{23} &= B_1^T C_2^T C_2 B_2 \\ Q_{24} &= (1-2\rho) B_1^T C_2^T D_2 \\ Q_3 &= B_2^T C_2^T C_2 B_2 \\ Q_{34} &= (1-2\rho) B_2^T C_2^T D_2 \\ Q_4 &= 2D_2^T D_2 \end{aligned} \quad \text{(A.3)}$$

Proof: The result follows immediately by using the definitions of z in (A.1), and of α and β in (3.2) and by noting that $v(k)$ and $w(k)$ are uncorrelated with $x(k)$, and that $u(k)$ is uncorrelated with $w(k+1)$. □
Remark. For $\rho = 0.5$ it follows that $J = \mathrm{E}\{z(k+1)^2 + z(k)^2\}$. This case thus corresponds to minimum variance control of z.

The problem of minimizing $\bar{J}$ in (A.2) is not a standard LQG-problem, therefor the following lemma is needed.

LEMMA A.2
If $\bar{Q}$ is invertible, then the loss-function $\bar{J}$ in (A.2) can be written

$$\bar{J} = \tilde{J} + \mathrm{E}\{v^T(S - L_v^T \bar{Q} L_v)v - 2v^T L_v^T \bar{Q} L_w w - w^T L_w^T \bar{Q} L_w w\}$$

where

$$\tilde{J} = \mathrm{E}\{(u + Lx + L_v v + L_w w)^T \bar{Q}(u + Lx + L_v v + L_w w)\}$$

where S is the solution to the Riccati-equation in (3.8), and where

$$\begin{aligned} \bar{Q} &= (Q_2 + B_1^T S B_1) \\ L &= \bar{Q}^{-1}(B_1^T S A + Q_{12}^T) \\ L_v &= \bar{Q}^{-1}(B_1^T S B_2 + Q_{23}) \\ L_w &= \bar{Q}^{-1} Q_{24} \end{aligned}$$

Proof: The result follows by a generalization of [Åström and Wittenmark, 1990, Theorem 11.2] to $Q_{12} \neq 0$, see e.g. [Gustafsson and Hagander, 1991, p. 3], by completing the squares in $\bar{J}$, and by noting that v and w are uncorrelated with x. □

To obtain the optimal controller that minimizes J, estimates of $x(k)$, $v(k)$, and $w(k)$ based on observations of y up to time k are needed. The ones for x and v can be found in [Åström and Wittenmark, 1990, Eq. (11.49), p. 352]. The following lemma gives the estimate of w.

LEMMA A.3
If R_y is invertible, then the estimate

$$\hat{w}(k|k) = \mathrm{E}\{w(k)|\mathcal{Y}_k\}$$

of $w(k)$, where $\mathcal{Y}_k$ is the σ-algebra generated by all past observations of y up to time k, is given by

$$\hat{w}(k|k) = R_{23}^T D_1^T R_y^{-1} \tilde{y}(k)$$

where

$$\begin{aligned} R_y &= C_1 P C_1^T + D_1 R_2 D_1^T \\ \tilde{y}(k) &= y(k) - C_1 \hat{x}(k|k-1) \\ \hat{x}(k|k-1) &= \mathrm{E}\{x(k)|\mathcal{Y}_{k-1}\} \end{aligned}$$

and where P is the solution to the Riccati-equation in (3.8).

Proof: Since $\mathcal{Y}_k$ is the same σ-algebra as the one generated by all past observations of y up to time $k-1$ and by $\tilde{y}(k)$, since $\tilde{y}(k)$ is independent of $\mathcal{Y}_{k-1}$ by [Åström, 1970, Theorem 3.2, p. 219], and since $\mathrm{E}\{w(k)\tilde{y}(k)^T\} = \mathrm{E}\{w(k)(D_1 e(k))^T\}$, it follows by [Åström, 1970, Th. 3.2 and Th. 3.3, pp. 219–220] that

$$\begin{aligned} \hat{w}(k|k) &= \mathrm{E}\{w(k)|\mathcal{Y}_{k-1}, \tilde{y}(k)\} \\ &= \mathrm{E}\{w(k)|\mathcal{Y}_{k-1}\} + \mathrm{E}\{w(k)|\tilde{y}(k)\}\} \\ &= \mathrm{E}\{w(k)|\tilde{y}(k)\} \\ &= \mathrm{E}\{w(k)\tilde{y}(k)^T\} \left[\mathrm{E}\{\tilde{y}(k)\tilde{y}(k)^T\}\right]^{-1} (\tilde{y}(k) - \mathrm{E}\{\tilde{y}(k)\}) \\ &= R_{23}^T D_1^T R_y^{-1} \tilde{y}(k) \end{aligned}$$

□

LEMMA A.4

If $\bar{Q}$ and R_y are invertible, then the optimal controller that minimizes J in (3.4) is given by

$$u(k) = -L\hat{x}(k|k) - L_v\hat{v}(k|k) - L_w\hat{w}(k|k)$$

where $\hat{x}(k|k)$ and $\hat{v}(k|k)$ are given by [Åström and Wittenmark, 1990, Eq. (11.49)] and $\hat{w}(k|k)$ is given by Lemma A.3.

Proof: The result follows by lemmas A.1 and A.2 and by the separation principle, see e.g. [Åström, 1970, p. 282]. □

THEOREM A.1

If $\bar{Q}$ and R_y are invertible, then the transfer function $H(q)$ for the optimal controller that minimizes J in (3.4) is given by

$$H(q) = -L_x(qI - A + B_1L_x + KC_1)^{-1}K_y - L_y \tag{A.4}$$

where

$$\begin{aligned}
L_x &= L - L_yC_1\\
L_y &= LK_f + L_vK_v + L_wK_w\\
K_y &= K - B_1L_y\\
K &= AK_f + B_2K_v\\
K_f &= PC_1^TR_y^{-1}\\
K_v &= R_{12}D_1^TR_y^{-1}\\
K_w &= R_{23}D_1^TR_y^{-1}
\end{aligned}$$

and where P is the solution to the Riccati-equation in (3.8).

Proof: The proof is straight forward calculations making use of lemmas A.3 and A.4 and the equations in [Åström and Wittenmark, 1990, Theorem 11.6]. □

LEMMA A.5

The closed loop system behavior for the optimal controller is governed by

$$\bar{x}(k+1) = A_c\bar{x}(k) + B_c\bar{v}(k)$$

where $\bar{x}(k) = (\, x^T(k) \quad \tilde{x}^T(k)\,)^T$, $\bar{v}(k) = (\, v^T(k) \quad e^T(k)\,)^T$, and where

$$A_c = \begin{pmatrix} A - B_1 L & B_1 L_x \\ 0 & A - K C_1 \end{pmatrix}$$
$$B_c = \begin{pmatrix} B_2 & -B_1 L_y D_1 \\ B_2 & -K D_1 \end{pmatrix}$$

Proof: The proof is straight forward calculations making use of lemmas A.3 and A.4 and the equations in [Åström and Wittenmark, 1990, Theorem 11.6]. □

THEOREM A.2
The variances of α, β, z and u are given by

$$\begin{aligned}
\sigma_\alpha^2 &= (\, C_2 \quad 0\,)\,((A_c + I)X(A_c + I)^T + B_c R B_c^T)\,(\, C_2 \quad 0\,)^T + 2 D_2 R_3 D_2^T \\
\sigma_\beta^2 &= (\, C_2 \quad 0\,)\,((A_c - I)X(A_c - I)^T + B_c R B_c^T)\,(\, C_2 \quad 0\,)^T + 2 D_2 R_3 D_2^T \\
\sigma_z^2 &= (\, C_2 \quad 0\,)\, X\, (\, C_2 \quad 0\,)^T + D_2 R_3 D_2^T \\
\sigma_u^2 &= (\, -L \quad L_x\,)\, X\, (\, -L \quad L_x\,)^T + L_y D_1 R_2 D_1^T L_y^T
\end{aligned} \tag{A.5}$$

where X is the solution to the Lyapunov equation in (3.9), and where R is given by

$$R = \begin{pmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{pmatrix}$$

Proof: The result follows from Lemma A.5 and [Åström, 1970, p. 49]. □

Remark. Due to the triangularity of A_c it is possible to split up (3.9) into three equations, where one of the solutions is P in (3.8), which reduces the complexity of the problem.

Minimum Risk Control

Anders Hansson

Second Edition

Department of Automatic Control, Lund Institute of Technology

Minimum Risk Control

Anders Hansson

ERRATA

Page	Line	Real	Should be
vii	+8 in Abstract	intensity	probability
vii	+11 in Abstract	risk	upcrossing
11	+4	the Section	Section
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Minimum Risk Control

Anders Hansson

Second Edition

Department of Automatic Control
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To my parents

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"Stufen

Wie jede Blüte welkt und jede Jugend
Dem Alter weicht, blüht jede Lebensstufe,
Blüht jede Weisheit auch und jede Tugend
Zu ihrer Zeit und darf nicht ewig dauern.
Es muß das Herz bei jedem Lebensrufe
Bereit zum Abschied sein und Neubeginne,
Um sich in Tapferkeit und ohne Trauern
In andre, neue Bindungen zu geben.
Und jedem Anfang wohnt ein Zauber inne,
Der uns beschützt und der uns hilft, zu leben.

Wir sollen heiter Raum um Raum durchschreiten,
An keinem wie an einer Heimat hängen,
Der Weltgeist will nicht fesseln uns und engen,
Er will uns Stuf' um Stufe heben, weiten.
Kaum sind wir heimisch einem Lebenskreise
Und traulich eingewohnt, so droht Erschlaffen,
Nur wer bereit zu Aufbruch ist und Reise,
Mag lähmender Gewöhnung sich entraffen.

Es wird vielleicht auch noch die Todesstunde
Uns neuen Räumen jung entgegensenden,
Des Lebens Ruf an uns wird niemals enden ...
Wohlan denn, Herz, nimm Abschied und gesunde!"

*Hermann Hesse**

---

* Das Glasperlenspiel, Suhrkamp Taschenbuch Verlag, Frankfurt am Main, 1972

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Abstract

A new optimal stochastic control problem is posed. The criterion is to minimize the probability for a signal's largest value to be above a level given a certain reference value. It is shown that this control problem is closely related both to the problem of minimizing the variance of the signal—minimum variance control—and to the problem of minimizing the so called upcrossing probability. It is made plausible that the upcrossing probability is a better approximating criterion to minimize than the minimum variance criterion. The problem of minimizing the upcrossing probability can be thought of as finding optimal weighting-matrices in an LQG-problem. The new controller is compared with the minimum variance controller for a first order process. It is seen that the new controller causes a lower upcrossing intensity and a smaller probability for the largest value of the controlled signal to be above the critical level. The improvement in the example is up to about 10%. This makes it possible to choose the reference value closer to the critical level when using the minimum risk controller, than when using the minimum variance controller. Further it seen that the control signal is more well-behaved. The only drawback of the new controller is the larger computational burden.

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Preface

My interest in control of extreme values began in the autumn of 1990 as I was following a course on extremes in random processes given by Professor Georg Lindgren at the Department of Mathematical Statistics in Lund. The course was suggested to me by Professor Lars Nielsen to give me a basis for our common interest in investigating prediction methods for level crossings. After some time it was evident to me that theoretical work about predictions in closed loop was not what a control engineer primarily should work with. Why predict catastrophes, when it is possible to control to avoid them. In fact, good control that avoids level crossings, makes prediction of level crossings less interesting. The closed loop system modes are then fast, and multi-step predictions will rapidly converge to the mean of the process. However, due to practical considerations, such as uncertainties in process models or actuator constraints, it is not always possible to control in a satisfactory way, and in these cases supervision may be useful. This has been described for the continuous time case in [Hansson and Nielsen, 1991]. In this thesis, only control of level crossings and extreme values for the discrete time case will be considered. The continuous time case has been described in [Hansson, 1991a], [Hansson, 1991b] and [Hansson, 1992a].

As is obvious from what is said above, the thesis is somewhat interdisciplinary and in the borderland of automatic control and mathematical statistics. It is primarily written for a reader with knowledge of automatic control at a graduate level, but I hope that the references will help any other reader with some mathematical background to read it.

Acknowledgements

This work has been carried out at the Department of Automatic Control, Lund Institute of Technology, Sweden. I would like to thank all my colleagues at the department. It is a great pleasure to work in the creative, friendly and highly stimulating atmosphere which they are all contributing to.

I am very happy to thank Professor Karl Johan Åström, and Dr. Per Hagander for their encouraging support and guidance, and for their valuable comments and suggestions, which have improved my work considerably. Professor Åström's great enthusiasm for my work moderated by his sincere interest in controllers well adapted to industrial control problems has been most encouraging, and served as a very good example on a serious attitude towards scientific research in an applied field. Dr. Hagander not only has the invaluable ability to find most, I hope, errors in a manuscript, but also a persistent will to understand and discuss problems, which is very helpful when trying to write for the uninitiated reader. I am also very grateful to Professor Björn Wittenmark for his comments and criticism on the first version of my manuscript. I wish to express my gratitude to M.Sc. Bo Bernhardsson for suggested improvements and stimulating discussions. He has been a valuable source of inspiration. I am also indebted to Professor Georg Lindgren for stimulating discussions, and for his interesting lectures on extremes in random processes. I would like to thank Professor Lars Nielsen for suggesting the topic of investigating prediction of catastrophes, for having had time to discuss related topics, and for making me aware of the research on extreme values at the Department of Mathematical Statistics. I am also grateful to L.Sc. Ola Dahl and M.Sc. Klas Nilsson for interesting discussions about applications in robotics.

I would like to thank L.Sc. Kjell Gustafsson for providing good LQG-routines in Matlab, and M.Sc. Leif Andersson for providing excellent TEXmacros, which have made the type-setting much smoother than it otherwise would have been.

Finally, I would like to thank my parents for their encouragement and support.

Lund, November 1991
Anders Hansson

Preface to second Edition

The major changes in the second edition are in chapters 1 and 2. In Chapter 1 the relation of my work to the extensively studied deterministic case is described, and in Chapter 2 the relation of the mean time between failures criterion to the upcrossing probability criterion is discussed.

I would like to thank my opponent Professor Torsten Söderström for his constructive opposition, which has greatly inspired the revision of this second edition.

Lund, February 1992
Anders Hansson

Table of Contents

1. Introduction 1

2. The Control Problem 5
2.1 Problem Formulation 5
2.2 Bounds for the Criterion Probability 6
2.3 Approximation of the Problem Formulation 9
2.4 Summary 10

3. Regulator Design 11
3.1 Solution 11
3.2 LQG-equations 19
3.3 The General Case 20
3.4 Summary 22

4. Example 24
4.1 Process 24
4.2 LQG-Controllers 25
4.3 MU and MV Controllers 26
4.4 Summary 33

5. Conclusions 34

References 36

A. More General Process Model 39

1

Introduction

There are many control problems where the goal is not only to keep the controlled signal near a specified reference value, but in addition to keep it below a critical level. The word critical is used in the sense of a severe failure, which may imply that the process has to be repaired and restarted. The distance between the critical level and the reference value is normally not too small. Otherwise the failure rate will be intolerably high. However, there may be other control-objectives that make it undesirable or impossible to choose the distance large. An example of problems of this kind can be found in [Borisson and Syding, 1976], where the power of an ore crusher should be kept as high as possible but not exceed a certain level, in order that the overload protection does not cause shutdown. Another example is moisture control of a paper machine, where it is desired to keep the moisture content as high as possible without causing wet streaks. Yet another example is control of wind power plants, where the supervisory system initiates emergency shutdown, if the generated power exceeds 140% of rated power, [Mattsson, 1984]. Other examples can be found in sensor-based robotics and force control, [Hansson and Nielsen, 1991], and control of non-linear plants, where the stability may be state dependent, [Shinskey, 1967].

In a deterministic framework this type of problems could be solved by minimizing

$$\max_{d} ||z||_{\infty}$$

where z is the controlled signal and d is a disturbance acting on z. Problems of this type have been studied extensively. Assuming bounded energy on the disturbance gives the well-known H_2-controller, [Vidyasagar, 1986]. Other types of disturbances have also been considered. In [Vidyasagar, 1986] and [Dahleh and Pearson, 1987] the disturbance has bounded supremum norm, and in [Liu and Zakian, 1990] it has bounded increments.

Common to the deterministic criteria is the design for worst case disturbances, which may seem somewhat too conservative. The classical way to overcome this is to consider a stochastic formulation, where the natural criterion is to minimize

$$\mathrm{P}\left\{\max_{0\leq k\leq N} z(k) > z_0\right\}$$

where z_0 is the distance to the critical level. This type of problems has previously been solved approximately by minimum variance control, [Åström, 1970, pp. 159–209], [Åström and Wittenmark, 1990, p. 203], and [Borisson and Syding, 1976]—the intuitively best controller. The gain of the minimum variance controller depends critically on the sampling period. Too small a sampling period leads to large variations in the control signal, [Åström and Wittenmark, 1990, pp. 316–317]. This problem has been solved by introducing weighting on the control signal—LQG-design. However, there has been no good criteria for choosing the weighting.

It should be noted that the solution to the deterministic problem with bounded disturbance energy, i.e. the H_2-controller, and the solution to the minimum variance problem are the same. Thus the deterministic problem, with disturbances having bounded energy, will also suffer from the drawbacks of the minimum variance controller. Another way to express this, is that the minimum variance controller is just as conservative as the controller designed for worst case disturbances.

The proposed controller is less conservative than the minimum variance controller. It can be interpreted as a choice of optimal weightings in an LQG-problem, chosen in such a way that they minimize the mean number of upcrossings of the critical level per unit time—the upcrossing probability. This controller will also be an approximate solution to the stochastic problem above, however obtained by considering a tighter bound for the criterion probability than the one which corresponds to the minimum variance controller

The limiting distributions of the maxima for independent and identically distributed random variables were discussed already in [Tippet, 1925], [Fréchet, 1927], and [Fisher and Tippet, 1928]. The results were generalized to dependent variables by [Watson, 1954], [Berman, 1964], [Loynes, 1965], and [Leadbetter, 1974]. A good book in the topic is [Leadbetter *et al.*, 1982]. However, the idea of approximating distributions of extrema with upcrossing probabilities originates from continuous time extreme value analysis. In that context, the problem of upcrossings was first studied in [Rice, 1936]—Rice's private notes. His results were only briefly mentioned in his own publications—a sentence in [Rice, 1939] and a footnote in [Rice, 1944 1945, Section 3.8]. Rice's celebrated formula for the mean number of upcrossings of a level z_0 per unit time by a stationary Gaussian process, with zero mean value and covariance function $r(\tau)$, is given by

$$\mu = \frac{1}{2\pi}\sqrt{\frac{-r''(0)}{r(0)}}\exp\left(-\frac{z_0^2}{2r(0)}\right)$$

The origin of Rice's Formula is well described in [Rainal, 1988]. In discrete time the corresponding formula is less explicit

$$\mu = \mathrm{P}\left\{z(0) \leq z_0 \cap z(1) > z_0\right\}$$

[Cramér and Leadbetter, 1967]. This will make the analysis somewhat harder than in continuous time.

In [Hansson, 1991a], [Hansson, 1991b] and [Hansson, 1992a] the problem was solved in the continuous time case; here the discrete time case is treated, which previously to some extent has been described in [Hansson, 1992b]. Only the case of a linear process controlled with a linear controller will be treated, since then, if the disturbances acting on the process are Gaussian, the closed loop system will also be Gaussian. It is very likely that a nonlinear controller will do better. However, the analysis will then be much harder, since the signals are not Gaussian.

In Chapter 2 the problem of keeping a signal's largest values below a level given a certain reference value is related to the minimum variance controller and to the controller that minimizes the so called upcrossing probability. It is also made plausible that the upcrossing probability criterion captures the control-objectives better than the minimum variance criterion.

In Chapter 3 the controller that minimizes the upcrossing probability—the minimum upcrossing controller— is determined. It is obtained by solving a one-parametric optimization problem over a set of LQG-problem solutions. The complexity is thus only one order of magnitude larger than for an ordinary LQG-problem. It can be interpreted as choosing optimal weighting-matrices in an LQG-problem, provided that the solutions to the LQG-problems are unique.

In Chapter 4 the minimum upcrossing controller found in Chapter 3 is compared with the minimum variance controller for a first order process. It is seen that the new controller causes a lower upcrossing probability and smaller probability for the largest value of being above the dangerous level. Further it is seen that the control signal is more well-behaved. Both theory and simulations show that the minimum upcrossing controller and the minimum variance controller are approximately the same for large values of the distance between the reference value and the critical level. However, in an example it is seen that the minimum upcrossing controller can have up to about 10% better performance for moderate values of the level. This is the interesting case for the examples described above.

Finally, in Chapter 5 the results of the previous chapters are summarized.

2

The Control Problem

The control problems described in Chapter 1 will be mathematically formalized in a stochastic framework. In Section 2.1 the control criterion is defined; it is defined such that the controller should minimize the probability for the largest value of the controlled signal to be above a level given a certain reference value. Two bounds for the criterion probability are investigated in Section 2.2. One of them is minimized by minimum variance control, the other one—tighter than the first one—is approximately minimized by minimizing the so called upcrossing probability. In Section 2.3 the approximate control criterion to minimize the upcrossing probability is further motivated and stated. The results are summarized in Section 2.4.

2.1 Problem Formulation

Let the controlled signal, z, be a stationary Gaussian sequence with mean

$$m_z = \mathrm{E}\{z(k)\}$$

and with covariance function

$$r_z(\tau) = \mathrm{E}\{(z(k+\tau) - m_z)(z(k) - m_z)\}$$

Denote the variance of z by σ_z^2, i.e. let $\sigma_z^2 = r_z(0)$. Consider a time-invariant controller H, linear in both the measurement signal y and in the constant reference value r. The problems mentioned in Chapter 1 are captured in the following problem:

$$\min_{H} \mathrm{P}\left\{\max_{0\leq k\leq N} z(k) > z_0\right\} \tag{2.1}$$

subject to $m_z = r$ and to a stable closed loop system, where $\mathrm{P}\{\cdot\}$ denotes probability measure, and where $z_0 > m_z$ is the critical level. The reason for constraining the minimization to $m_z = r$ is that it may be profitable to not having $m_z - z_0$ too large; e.g. in the paper machine example it was desired to keep the moisture content as high as possible without causing wet streaks. The time horizon N and the distance $m_z - z_0$ have to be chosen in such a way that the probability in (2.1) is small, otherwise the failure rate will be too high. The larger N is, the larger $m_z - z_0$ must be. Without loss of generality it may be assumed that $m_z = r = 0$, which can be obtained with a change of coordinates. To simplify the notations this will be assumed in the sequel.

2.2 Bounds for the Criterion Probability

To simplify the problem upper bounds for the probability in (2.1) will be given. It will be shown that these bounds are tight, if N and z_0/σ_z are large and the probability in (2.1) is small. The tighter bound is obtained by considering level crossings.

The problem of level crossings is a classical problem in stochastic processes. Initial results were given in [Rice, 1936]. Good references to crossing problems are [Cramér and Leadbetter, 1967] and [Leadbetter *et al.*, 1982]. The latter book also treats the problem of extreme values in stochastic processes. The results developed in this section will make extensive use of the results in these books.

THEOREM 2.1
If $z(k)$ is a stationary random sequence, then

$$\mathrm{P}\left\{\max_{0\leq k\leq N} z(k) > z_0\right\} \leq P_1(z_0) \leq P_2(z_0) \tag{2.2}$$

where

$$P_1(z_0) = \mathrm{P}\{z(0) > z_0\} + N\mu(z_0)$$
$$P_2(z_0) = (N+1)\mathrm{P}\{z(0) > z_0\}$$

and where

$$\mu(z_0) = \mathrm{P}\{z(0) \leq z_0 \cap z(1) > z_0\} \tag{2.3}$$

Proof: The proof is easy:

$$\mathrm{P}\left\{\max_{0\leq k\leq N} z(k) > z_0\right\} = \mathrm{P}\left\{z(0) > z_0 \bigcup_{k=0}^{N-1} (z(k) \leq z_0 \cap z(k+1) > z_0)\right\}$$
$$\leq \mathrm{P}\{z(0) > z_0\} + \sum_{k=0}^{N-1} \mathrm{P}\{z(k) \leq z_0 \cap z(k+1) > z_0\}$$
$$= \mathrm{P}\{z(0) > z_0\} + N\mathrm{P}\{z(0) \leq z_0 \cap z(1) > z_0\}$$
$$\leq (N+1)\mathrm{P}\{z(0) > z_0\}$$

□

Remark 1. For large values of N and large values of z_0/σ_z the first term in P_1 is negligible.

Remark 2. Note that P_2 is minimized by minimum variance control.

The quaintity μ in (2.3) will be called the upcrossing probability, and it is equal to the mean number of upcrossings in the interval $[0, 1)$, see e.g. [Cramér and Leadbetter, 1967, p. 281]. The bound P_1 is well known in the context of continuous time extreme value analysis, see e.g. [Leadbetter *et al.*, 1982, Lemma 8.2.1], since there the bound P_2 is infinite, and thus not usable for investigating the behavior of extreme values as the time horizon and the critical level approaches infinity. However, the bound P_2 is good enough for investigating this behavior in the discrete time domain, but for the purposes in this work—focused on finite time horizons and levels—it is interesting also to consider a tighter bound such as P_1.

It will now be shown that the bounds in Theorem 2.1 are tight.

THEOREM 2.2
If $z(k)$ is a stationary Gaussian sequence with covariance function satisfying

$$\lim_{\tau\to\infty} r_z(\tau)\ln\tau = 0$$

and if $z_0^{(N)}$ is chosen such that

$$\lim_{N\to\infty} P_2\left(z_0^{(N)}\right) = L$$

then

$$\lim_{N\to\infty}\left|\frac{M\left(z_0^{(N)}\right)-P_1\left(z_0^{(N)}\right)}{M\left(z_0^{(N)}\right)}\right| \le \lim_{N\to\infty}\left|\frac{M\left(z_0^{(N)}\right)-P_2\left(z_0^{(N)}\right)}{M\left(z_0^{(N)}\right)}\right| \le \frac{L}{2}$$

where $M(x) = \mathrm{P}\left\{\max_{0\le k\le N} z(k) > x\right\}$.

Proof: The first inequality follows by Theorem 2.1. Further by Theorem 2.1 and since by [Leadbetter *et al.*, 1982, Theorem 4.3.3]

$$\lim_{N\to\infty} P_2\left(z_0^{(N)}\right) = L$$

if and only if

$$\lim_{N\to\infty} 1 - M\left(z_0^{(N)}\right) = e^{-L}$$

it follows that

$$\lim_{N\to\infty}\left|\frac{M\left(z_0^{(N)}\right)-P_2\left(z_0^{(N)}\right)}{M\left(z_0^{(N)}\right)}\right| = \left|\frac{1-e^{-L}-L}{1-e^{-L}}\right| \le \frac{L}{2}$$

which concludes the proof. □

Remark. Let n_N be the number of upcrossings of $z_0^{(N)}$ by the time normalized process $\zeta_N(t)$, $t = k/N$, $k = 1, 2, \ldots$ defined by $\zeta_N(k/N) = z(k)$. Then it can be shown, under the conditions above, that n_N converges in distribution to a Poisson process with intensity L on $(0, 1]$, [Leadbetter *et al.*, 1982, Theorem 5.2.1]. From this it follows that the number of upcrossings of a sufficiently large fixed z_0 by z will be approximately a Poisson process with intensity $L/N \approx \mathrm{P}\{z(0) > z_0\} \approx \mu(z_0)$.

Related problems of convergence have been investigated for other approximations of extremal-probabilities, see e.g. [Leadbetter *et al.*, 1982, Chapter 4.6], but these approximations are not upper bounds as the ones discussed here.

2.3 Approximation of the Problem Formulation

Now by theorems 2.1 and 2.2 it is obvious that the probability in (2.1) can be approximately minimized for large values of N and z_0/σ_z and for small values of the probability in (2.1) by minimizing either the variance or the upcrossing probability μ. However, for moderate values of N and z_0/σ_z, the results of Theorem 2.1 still hold, and it is tempting to believe that the upcrossing probability is a better criterion to minimize in this case, which is the interesting one for the problems described in Chapter 1. Therefore the following approximation of the criterion (2.1) will be considered from now on:

$$\min_H \mu(z_0) \tag{2.4}$$

subject to a stable closed loop system. There may be some problems with this approximation, since there are two ways of making μ small—either by keeping z well below z_0 or by keeping it well above z_0. It is clear that the probability in (2.1) will not be small, if μ is made small by keeping z well above z_0. To exclude this possibility, the minimization of μ will also be restricted to $\sigma_z \leq z_0$. The validity of the approximation of the problem formulation will be investigated further in Chapter 4.

It is interesting to note that the approximate criterion could also have been obtained by approximating another interesting criterion. Let T be the time interval between two consecutive upcrossings of z_0 by z. Then the mean time between failures (MTBF) defined as $\mathrm{E}\{T\}$ would be an interesting quantity to maximize. By the remark of Theorem 2.2 it holds

$$\mathrm{E}\{T\} \approx \frac{1}{\mathrm{P}\{z(0) > z_0\}} \approx \frac{1}{\mu(z_0)}$$

Thus MTBF is approximately maximized by minimizing either the variance of z or the upcrossing probability μ. It is however difficult to get a feeling for which approximation is the best to minimize with the MTBF-criterion.

2.4 Summary

The control problems described in Chapter 1 have been mathematically formalized in a stochastic framework. The control criterion has been defined such that the controller should minimize the probability for the largest value of the controlled signal to be above a level given a certain reference value. Two bounds for the criterion probability have been investigated. One of them is minimized by minimum variance control, the other one—tighter than the first one—is approximately minimized by minimizing the so called upcrossing probability. It has been made plausible that minimizing the upcrossing probability is a better approximation to the original problem than minimizing the variance. One drawback with the criterion in (2.1) is the time horizon N, which seams strange when minimizing over time-invariant controllers. This can be overcome by considering the mean time between failures criterion. It is however more difficult to get a feeling for which approximation is the best to minimize with this criterion.

3

Regulator Design

The problem of minimizing the upcrossing probability will now be solved. In the Section 3.1 the problem is reformulated as a one-parameter minimization over solutions to LQG-problems. Thus the complexity is not significantly larger than for an ordinary LQG-problem. The solution can be interpreted as a choice of optimal weighting-matrices in an LQG-problem. The equations for solving the LQG-problems are given in Section 3.2. In Section 3.3 the results of the previous sections are generalized to more general process models. Finally in Section 3.4 the results are summarized.

3.1 Solution

Let the stationary Gaussian sequence z be defined by

$$\begin{cases} x(k+1) = Ax(k) + B_1 u(k) + B_2 v(k) \\ \quad\; y(k) = C_1 x(k) + De(k) \\ \quad\; z(k) = C_2 x(k) \end{cases} \tag{3.1}$$

where v and e are zero mean, Gaussian, white noise sequences with $\mathrm{E}vv^T = R_1$, $\mathrm{E}ee^T = R_2$ and $\mathrm{E}ve^T = R_{12} = 0$. The signal y is the measured signal, and u is the control signal. The signal z is the signal that is desirable to control. The reason for not having $C_1 = C_2$ can be motivated by the examples in Chapter 1, where e.g. in the ore crusher example, the measured power y is not the desired signal to control, but instead some filtered version z of it, due to the filtering behavior of the thermal overload protection. More general process models than (3.1) may be considered. The treatment of a more general process model will be given later in Section 3.3 and in Appendix A. Introduce

$$\begin{cases} \alpha(k) = z(k+1) + z(k) \\ \beta(k) = z(k+1) - z(k) \end{cases} \tag{3.2}$$

which are independent variables due to the stationarity of z. Let $\mathcal{D}$ be the set of linear time-invariant stabilizing controllers of (3.1), and let $\mathcal{D}_z$ be the set of linear time-invariant stabilizing controllers of (3.1) for which

$$\sigma_z \leq z_0 \tag{3.3}$$

holds, where σ_z^2 is the variance of z. Note that the sets $\mathcal{D}$ and $\mathcal{D}_z$ may be empty. It will be seen that the minimization of μ in (2.3) over $\mathcal{D}_z$ can be done by first minimizing

$$J = \mathrm{E}\left\{(1-\rho)\alpha^2 + \rho\beta^2\right\} \tag{3.4}$$

for $\rho \in [0,1]$ over $\mathcal{D}$, and then minimizing μ over the solutions obtained in the first minimization, i.e. over $\mathcal{V}_J \cap \mathcal{V}_z$, where

$$\begin{aligned} \mathcal{V}_J &= \left\{(\sigma_\alpha(H), \sigma_\beta(H)) \in R^2 \middle| H \in \mathcal{D}_J\right\} \\ \mathcal{V}_z &= \left\{(\sigma_\alpha, \sigma_\beta) \in R^2 \middle| \sigma_z \leq z_0,\ \sigma_\alpha \geq 0,\ \sigma_\beta \geq 0\right\} \\ \mathcal{D}_J &= \left\{H \in \mathcal{D} \middle| H = \operatorname{argmin} J(H,\rho),\ \rho \in [0,1]\right\} \end{aligned}$$

and where σ_α^2 and σ_β^2 are the variances of α and β.

In the following lemma J is rewritten to fit the standard LQG-problem formulation.

LEMMA 3.1
The loss function J in (3.4) can be written

$$J = \bar{J} + \mathrm{E}\left\{v^T B_2^T C_2^T C_2 B_2 v\right\}$$

where

$$\bar{J} = \mathrm{E}\left\{x^T Q_1 x + 2x^T Q_{12} u + u^T Q_2 u\right\}, \tag{3.5}$$

and where

$$\begin{aligned} Q_1 &= C_2^T C_2 + A^T C_2^T C_2 A + (1-2\rho)\left(C_2^T C_2 A + A^T C_2^T C_2\right) \\ Q_{12} &= \left(A^T + (1-2\rho)I\right) C_2^T C_2 B_1 \\ Q_2 &= B_1^T C_2^T C_2 B_1 \end{aligned} \tag{3.6}$$

Proof: The result follows immediately from the definitions of z in (3.1), and α and β in (3.2), and by noting that v is uncorrelated with x and u, since u is a functional of $y(k), y(k-1), \ldots$, and since $R_{12} = 0$. □

Remark. For $\rho = 0.5$ it follows that $J = \mathrm{E}\left\{z(k+1)^2 + z(k)^2\right\}$. This case thus corresponds to minimum variance control of z.

Next it will be shown that all jointly minimal variances of α and β can be obtained by minimizing J in (3.4) for $\rho \in [0,1]$. A precise definition of jointly minimal due to Pareto, [Pareto, 1896], will first be given.

DEFINITION 3.1—Pareto Optimality
Let $\mathcal{X}$ denote an arbitrary nonempty set. Let $f_i : \mathcal{X} \to R^+, \quad 1 \le i \le s$ be s nonnegative functionals defined on $\mathcal{X}$. A point x^0 is said to be Pareto optimal with respect to the vector-valued criterion $f = (f_1, f_2, \cdots, f_s)$ if there does not exist $x \in \mathcal{X}$ such that $f_i(x) \le f_i(x^0)$ for all $i, \quad 1 \le i \le s$, and $f_k(x) < f_k(x^0)$ for some $k, \quad 1 \le k \le s$. □

The concept of Pareto optimality is illuminated in Figure 3.1. The set of achievable variances of α and β is the set of points in the plane that are above and to the right of or on the solid curve. The controller corresponding to the variances at B is not Pareto optimal, since there exist e.g. controllers corresponding to strictly lower variance of β without having larger variance of α—the controllers with variances on the line connecting A with B. Moreover it is seen that the controller corresponding to the variances at A is Pareto optimal, since by picking any other

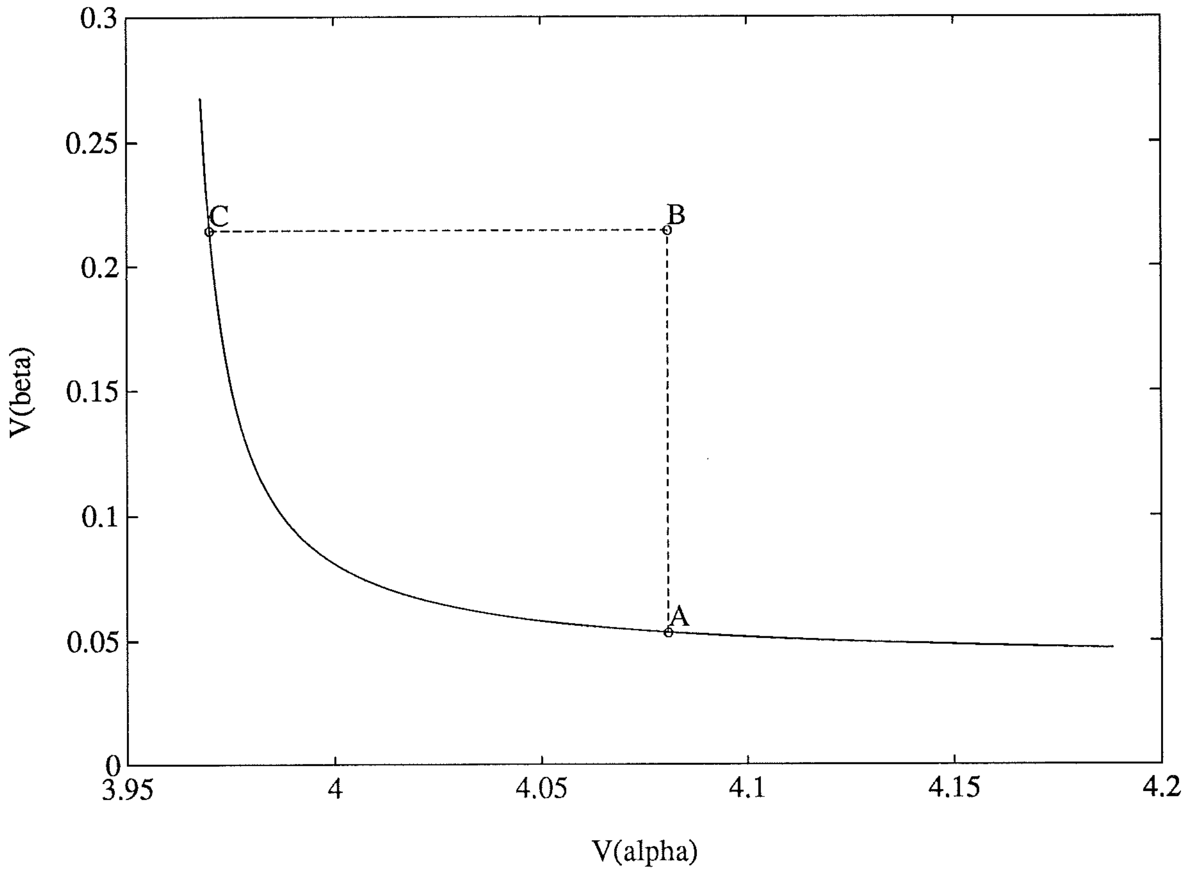

Figure 3.1 Illustration of Pareto optimality.

point, to the right of, above the curve, or on it, will either increase the variance of α or the variance of β. This reasoning holds for all points on the curve, and thus they are all Pareto optimal. Equivalent definitions of Pareto optimality can be found in [Leitmann, 1981, p. 292].

LEMMA 3.2
Suppose that (A, B_1) is stabilizable, and that (C_1, A) is detectable. Then the set $\mathcal{D}_P$ of Pareto optimal controllers with respect to $(\sigma_\alpha^2, \sigma_\beta^2)$ is a subset of $\mathcal{D}_J$.

Proof: Using the Youla parametrization, [Boyd and Barratt, 1991, Chapter 7.4], it follows that all stabilizing controllers of (3.1) can be parameterized by a stable transfer-function matrix Q. Thus to minimize J over $\mathcal{D}$ is equivalent to minimize J over Q, where Q belongs to the linear space of stable transfer-function matrices. Further it follows from [Boyd and Barratt, 1991, Chapter 7.4] that the transfer-function matrices from v and e to z are affine in Q. Since the variances of α and β are convex in the transfer-function matrices, it follows that the variances are convex in Q. The result now follows by [Khargonekar and Rotea, 1991, Theorem 1]. □

Remark 1. All controllers obtained by minimizing J for $\rho \in (0,1)$ are Pareto optimal by [Leitmann, 1981, Lemma 17.1]. If the controllers obtained for $\rho = 0$ and $\rho = 1$ are unique, then they are also Pareto optimal by [Leitmann, 1981, Lemma 17.2].

Remark 2. Remark 1 and Definition 3.1 imply that $\mathcal{V}_J$ can be parameterized by a scalar. This is not necessarily the case for $\mathcal{D}_J$.

Remark 3. Remark 1 implies that if the controllers obtained by minimizing J for $\rho \in [0,1]$ are unique, then a parameterization of $\mathcal{D}_P = \mathcal{D}_J$ by ρ is obtained, [Khargonekar and Rotea, 1991, p. 16].

The next lemma gives an expression for the upcrossing probability μ in (2.3) in terms of a double integral.

LEMMA 3.3
It holds that

$$\mu = \mathrm{P}\left\{z(0) \leq z_0 \cap z(1) > z_0\right\} = \int_0^\infty \phi(y) \int_{x_l}^{x_u} \phi(x) dx dy$$

where $\phi(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$, $x_l = (2z_0 - \sigma_\beta y)/\sigma_\alpha$, and $x_u = (2z_0 + \sigma_\beta y)/\sigma_\alpha$.

Proof: Since α and β are independent it holds that

$$\begin{aligned}\mu =& \mathrm{P}\left\{|\alpha - 2z_0| < \beta\right\} \\ =& \int\int_{|x-2z_0|<y} \frac{1}{\sigma_\alpha}\phi\left(\frac{x}{\sigma_\alpha}\right)\frac{1}{\sigma_\beta}\phi\left(\frac{y}{\sigma_\beta}\right) dx dy\end{aligned}$$

from which the result follows by a change of variables. □

In the following lemma it will be shown that the upcrossing probability μ in (2.3) has strictly positive partial derivatives with respect to σ_α and σ_β.

LEMMA 3.4
Let

$$\mathcal{V}(r) = \left\{ (\sigma_\alpha, \sigma_\beta) \in R^2 \middle| \sigma_z \leq r,\ \sigma_\alpha > 0,\ \sigma_\beta > 0 \right\}$$

where $r > 0$. Then the upcrossing probability μ in (2.3) has strictly positive partial derivatives with respect to both σ_α and σ_β on $\mathcal{V}(r)$, if and only if $r \leq z_0$.

Proof: It holds that

$$\frac{\partial \mu}{\partial \sigma_\beta} = \int_0^\infty \phi(y) \left(\frac{y}{\sigma_\alpha} \phi(x_u) + \frac{y}{\sigma_\alpha} \phi(x_l) \right) dy > 0$$

Further let $x_l = (2z_0 - \sigma_\beta y)/\sigma_\alpha$, and $x_u = (2z_0 + \sigma_\beta y)/\sigma_\alpha$. Using Lemma 3.3 gives

$$\frac{\partial \mu}{\partial \sigma_\alpha} = \int_0^\infty \phi(y) \left(\frac{x_l}{\sigma_\alpha} \phi(x_l) - \frac{x_u}{\sigma_\alpha} \phi(x_u) \right) dy$$

By completing the squares in the exponents and by a change of coordinates it is possible to express the integral in terms of $\Phi(x) = \int_{-\infty}^{x} \phi(t)dt$, and $\sigma_z^2 = (\sigma_\alpha^2 + \sigma_\beta^2)/4$

$$\frac{\partial \mu}{\partial \sigma_\alpha} = \frac{\sigma_\alpha}{8\pi\sigma_z^2} \exp\left(-\frac{\gamma^2}{2}\right) \left[\sqrt{2\pi}\gamma\,(2\Phi(\eta) - 1) - 2\frac{\eta}{\gamma} \exp\left(-\frac{\eta^2}{2}\right) \right]$$

where $\eta = \gamma\sqrt{\xi}$, $\xi = (\sigma_\beta/\sigma_\alpha)^2$, and $\gamma = z_0/\sigma_z > 0$. It is seen that $\frac{\partial \mu}{\partial \sigma_\alpha} > 0$ if and only if

$$2\Phi(\eta) - 1 > \sqrt{\frac{2}{\pi}} \frac{\eta}{\gamma^2} \exp\left(-\frac{\eta^2}{2}\right)$$

So if $\frac{\partial \mu}{\partial \sigma_\alpha} > 0$ on $\mathcal{V}(r)$, then the inequality above holds for all values of $\eta > 0$, since $\gamma > 0$, and since it must hold for all values of $\xi > 0$. A Taylor-expansion round $\eta = 0$ gives

$$\sqrt{\frac{2}{\pi}}\eta > \sqrt{\frac{2}{\pi}} \frac{\eta}{\gamma^2} + \mathcal{O}(\eta^2)$$

So for the inequality to hold for small values of η, it must be that $\gamma \geq 1$, which is equivalent to $r \leq z_0$.

Now suppose that $r \leq z_0$, which implies $\gamma \geq 1$. Then

$$\begin{aligned}(2\Phi(\eta)-1)^2 &\geq 1-\exp\left(-\frac{2\eta^2}{\pi}\right)-\frac{2(\pi-3)}{3\pi^2}\eta^4\exp\left(-\frac{\eta^2}{2}\right)\\ &\geq 1-\exp\left(-\frac{2\xi}{\pi}\right)-\frac{2(\pi-3)}{3\pi^2}\xi^2\exp\left(-\frac{\xi}{2}\right)\end{aligned}$$

where the first inequality follows from [Abromowitz and Stegun, 1968, Formula 26.2.25] and the second one from $\gamma \geq 1$. Further

$$\left(\sqrt{\frac{2}{\pi}}\frac{\eta}{\gamma^2}\exp\left(-\frac{\eta^2}{2}\right)\right)^2 \leq \frac{2}{\pi}\xi\exp(-\xi)$$

To show $\frac{\partial\mu}{\partial\sigma_\alpha} > 0$, it is now sufficient to show $L > R$ for $\xi > 0$, where

$$\begin{aligned}L &= \exp\left(\frac{\xi}{2}\right)\\ R &= \frac{2}{\pi}\xi\exp\left(-\frac{\xi}{2}\right)+\exp\left(\left(\frac{1}{2}-\frac{2}{\pi}\right)\xi\right)+\frac{2(\pi-3)}{3\pi^2}\xi^2\end{aligned}$$

Some calculations give

$$\begin{aligned}L &\geq 1+\frac{1}{2}\xi+\frac{1}{8}\xi^2\\ R &\leq 1+\frac{1}{2}\xi+\left(\frac{1}{8}-\frac{1}{3\pi}\right)\xi^2\end{aligned}$$

From this it follows that $L > R$ for $\xi > 0$, so $\frac{\partial\mu}{\partial\sigma_\alpha} > 0$. □

Remark. The largest region $\mathcal{V}(r)$ in which both $\frac{\partial\mu}{\partial\sigma_\alpha} > 0$ and $\frac{\partial\mu}{\partial\sigma_\beta} > 0$ is $\mathcal{V}(z_0)$. So if the constraint $\sigma_z \leq z_0$ is not considered, then it may well be that μ is minimized by $\sigma_\alpha = \infty$.

It will now be shown how the minimization of μ in (2.3) can be rephrased to a minimization over a set of LQG-problem-solutions. Figure 3.2 illuminates the proof of the following theorem.

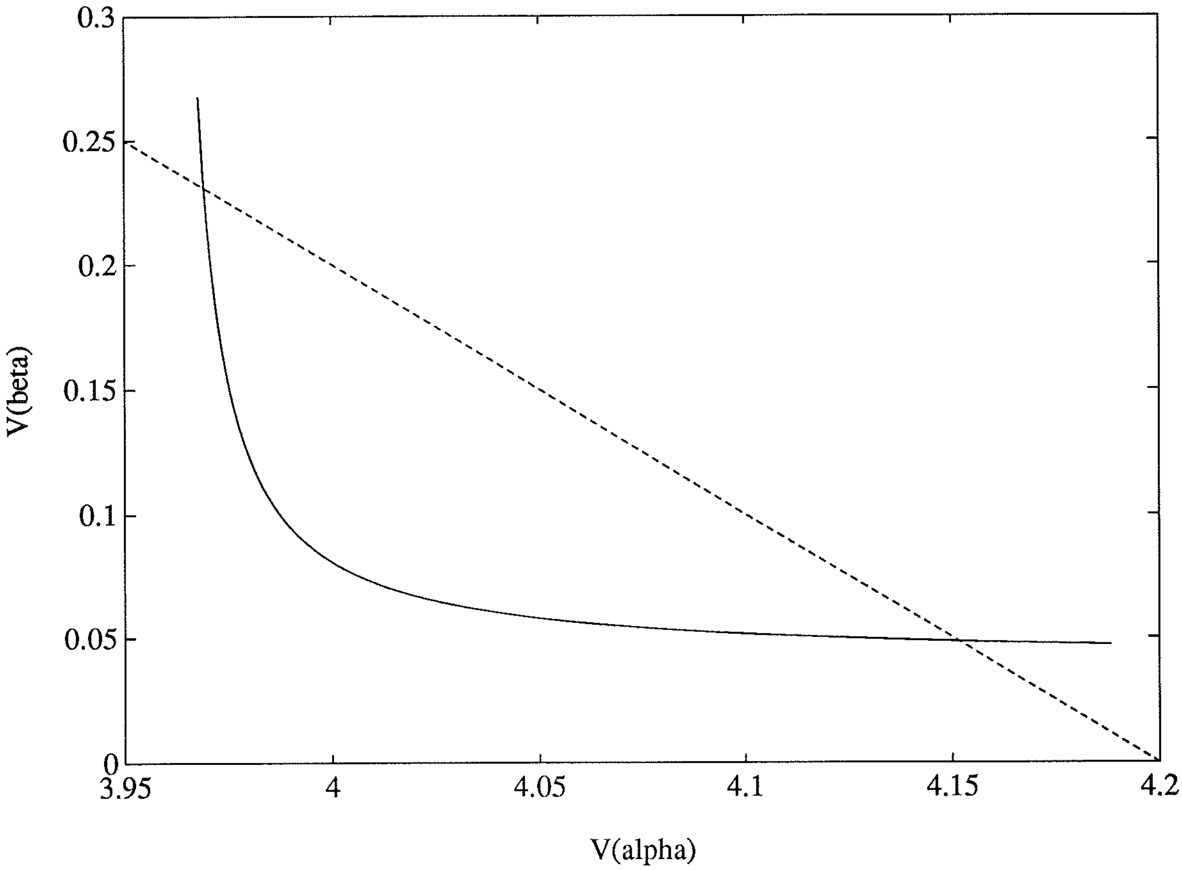

Figure 3.2 The solid line is $\mathcal{V}_J$, and the dashed line is $\sigma_z = z_0$ for $z_0 = 1.05$.

THEOREM 3.1
Suppose that (A, B_1) is stabilizable, and that (C_1, A) is detectable. Then

$$\left\{H \in \mathcal{D}_z \middle| H = \text{argmin } \mu\left(\sigma_\alpha(H), \sigma_\beta(H)\right)\right\} \subseteq \mathcal{D}_P \cap \mathcal{D}_z$$

and

$$\left\{(\sigma_\alpha(H), \sigma_\beta(H)) \in \mathcal{V}_z \middle| H = \text{argmin } \mu(\sigma_\alpha(H), \sigma_\beta(H))\right\} \subseteq \mathcal{V}_J \cap \mathcal{V}_z$$

Proof: Assume that the minimum of μ on $\mathcal{D}_z$ is attained for some $H \notin \mathcal{D}_P \cap \mathcal{D}_z$. For all $H \notin \mathcal{D}_P \cap \mathcal{D}_z$ there exist by Definition 3.1 $\bar{H} \in \mathcal{D}_z$ such that $\sigma_i(\bar{H}) < \sigma_i(H)$ for at lest one of $i = \alpha, \beta$. Since μ is differentiable and by Lemma 3.4 has strictly positive partial derivatives with respect to σ_α and σ_β on $\mathcal{V}(z_0)$, it follows that $\mu(\sigma_\alpha(\bar{H}), \sigma_\beta(\bar{H})) < \mu(\sigma_\alpha(H), \sigma_\beta(H))$.

This is a contradiction, and thus the minimum of μ is attained on $\mathcal{D}_P \cap \mathcal{D}_z$, if it exists on $\mathcal{D}_z$. Further $\mathcal{D}_P \subseteq \mathcal{D}_J$ by Lemma 3.2, which concludes the proof. □

Remark 1. Note that the minimization of μ can be done over $\mathcal{V}_J \cap \mathcal{V}_z$. This is a one-parametric optimization problem by Remark 2 of Lemma 3.2.

Remark 2. If for each $\rho \in [0, 1]$ the minimizing H of J is unique, then by Lemma 3.1 and Remark 3 of Lemma 3.2 the minimization of μ can be thought of as finding optimal weights in an LQG-problem.

3.2 LQG-equations

For short reference the equations for deriving the solution that minimizes $\bar{J}$ in (3.5) in Lemma 3.1 when the controller H is allowed to have a direct-term are given below. More stringent proofs of the results can be found in Appendix A, which also covers a more general process model. The transfer function from measurement to control is

$$H(q) = -L_x(qI - A + B_1 L_x + KC_1)^{-1} K_y - L_y \tag{3.7}$$

where L_x, L_y and K are given by

$$\begin{aligned}
L_x &= L - L_y C_1 \\
L_y &= L K_f \\
L &= \left(Q_2 + B_1^T S B_1\right)^{-1} \left(B_1^T S A + Q_{12}^T\right) \\
K_y &= K - B_1 L_y \\
K &= A K_f \\
K_f &= P C_1^T \left(D R_2 D^T + C_1 P C_1^T\right)^{-1}
\end{aligned}$$

where S and P are the solutions to the Riccati-equations, [Åström and Wittenmark, 1990, Chapter 11.4], and [Gustafsson and Hagander, 1991],

$$\begin{aligned}
A^T S A - S - (A^T S B_1 + Q_{12})(Q_2 + B_1^T S B_1)^{-1}(Q_{12}^T + B_1^T S A) + Q_1 = 0 \\
A P A^T - P - A P C_1^T (D R_2 D^T + C_1 P C_1^T)^{-1} C_1 P A^T + B_2 R_1 B_2^T = 0
\end{aligned} \tag{3.8}$$

and where Q_1, Q_2 and Q_{12} are given by (3.6) in Lemma 3.1. To calculate σ_z, σ_u, σ_α and σ_β the following Lyapunov-equation for the closed loop system should be solved, [Åström, 1970, p. 49],

$$A_c X A_c^T + B_c R B_c^T = X \tag{3.9}$$

where

$$A_c = \begin{pmatrix} A - B_1 L & B_1 L_x \\ 0 & A - KC_1 \end{pmatrix}$$
$$B_c = \begin{pmatrix} B_2 & -B_1 L_y D \\ B_2 & -KD \end{pmatrix}$$
$$R = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}$$

Then σ_α, σ_β, σ_z and σ_u are given by

$$\begin{aligned}
\sigma_\alpha^2 &= (\,C_2 \quad 0\,)\left((A_c + I)X(A_c+I)^T + B_c R B_c^T\right)(\,C_2 \quad 0\,)^T \\
\sigma_\beta^2 &= (\,C_2 \quad 0\,)\left((A_c - I)X(A_c-I)^T + B_c R B_c^T\right)(\,C_2 \quad 0\,)^T \\
\sigma_z^2 &= (\,C_2 \quad 0\,)\,X\,(\,C_2 \quad 0\,)^T \\
\sigma_u^2 &= (\,-L \quad L_x\,)\,X\,(\,-L \quad L_x\,)^T + L_y D R_2 D^T L_y^T
\end{aligned} \tag{3.10}$$

Since A_c is block-triangular, Equation (3.9) can be split up into three equations, one of which has P in (3.8) as its solution. This reduces the complexity of the problem.

3.3 The General Case

The results of the previous section are now generalized to the more general process model:

$$\begin{cases} x(k+1) = Ax(k) + B_1 u(k) + B_2 v(k) \\ y(k) = C_1 x(k) + D_1 e(k) \\ z(k) = C_2 x(k) + D_2 w(k) \end{cases} \tag{3.11}$$

where v, e and w are zero mean Gaussian white noise sequences with the positive semidefinite covariance matrix

$$\mathrm{E}\left\{\begin{pmatrix} v \\ e \\ w \end{pmatrix}\begin{pmatrix} v^T & e^T & w^T \end{pmatrix}\right\} = \begin{pmatrix} R_1 & R_{12} & R_{13} \\ R_{12}^T & R_2 & R_{23} \\ R_{13}^T & R_{23}^T & R_3 \end{pmatrix}$$

The signal y is the measurement signal, and u is the control signal. The proofs of the results in this section can be found in Appendix A. Let

$$\begin{aligned}
Q_{23} &= B_1^T C_2^T C_2 B_2 \\
Q_{24} &= (1-2\rho) B_1^T C_2^T D_2 \\
\bar{Q} &= \left(Q_2 + B_1^T S B_1\right) \\
L &= \bar{Q}^{-1}\left(B_1^T S A + Q_{12}^T\right) \\
L_v &= \bar{Q}^{-1}\left(B_1^T S B_2 + Q_{23}\right) \\
L_w &= \bar{Q}^{-1} Q_{24} \\
R_y &= C_1 P C_1^T + D_1 R_2 D_1^T \\
K_f &= P C_1^T R_y^{-1} \\
K_v &= R_{12} D_1^T R_y^{-1} \\
K_w &= R_{23} D_1^T R_y^{-1} \\
K &= A K_f + B_2 K_v \\
K_y &= K - B_1 L_y \\
L_y &= L K_f + L_v K_v + L_w K_w \\
L_x &= L - L_y C_1
\end{aligned}$$

where S is the solution to the Riccati-equation in (3.8), P is the solution to the Riccati-equation in (A.4), and where Q_1, Q_2, and Q_{12} are given by (3.6) in Lemma 3.1. The transfer function $H(q)$ for the optimal controller with direct-term that minimizes J in (3.4) is then given by

$$H(q) = -L_x(qI - A + B_1 L_x + K C_1)^{-1} K_y - L_y \tag{3.12}$$

Further let

$$A_c = \begin{pmatrix} A - B_1 L & B_1 L_x \\ 0 & A - KC_1 \end{pmatrix}$$
$$B_c = \begin{pmatrix} B_2 & -B_1 L_y D_1 \\ B_2 & -KD_1 \end{pmatrix}$$
$$R = \begin{pmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{pmatrix}$$

The variances of α, β, z and u are then given by

$$\begin{aligned} \sigma_\alpha^2 &= (\,C_2 \quad 0\,)\left((A_c + I)X(A_c + I)^T + B_c R B_c^T\right)(\,C_2 \quad 0\,)^T + 2D_2 R_3 D_2^T \\ \sigma_\beta^2 &= (\,C_2 \quad 0\,)\left((A_c - I)X(A_c - I)^T + B_c R B_c^T\right)(\,C_2 \quad 0\,)^T + 2D_2 R_3 D_2^T \\ \sigma_z^2 &= (\,C_2 \quad 0\,)X(\,C_2 \quad 0\,)^T + D_2 R_3 D_2^T \\ \sigma_u^2 &= (\,-L \quad L_x\,)X(\,-L \quad L_x\,)^T + L_y D_1 R_2 D_1^T L_y^T \end{aligned} \tag{3.13}$$

where X is the solution to the Lyapunov equation in (3.9). Notice that (3.9) is block-triangular also in the general case.

3.4 Summary

It has been shown that the minimization of the upcrossing probability can be expressed as minimization over a set of LQG-problem solutions parameterized by a scalar, regardless of the uniqueness of the solutions to the LQG-problems. If the solutions to the LQG-problems are unique, then the problem of minimizing the upcrossing probability can be thought of as finding optimal weightings in an LQG-problem. Note that the Lyapunov equation (3.9) is linear, and thus does not add any significant complexity compared to an ordinary LQG-problem.

The algorithm for minimizing the upcrossing probability can be summarized as: 1) solve the associated LQG-problems, and 2) minimize the upcrossing probability over the variances obtained in the first step. It must be stressed that if $\sigma_z > z_0$, then no solution exist. In order to obtain a solution, the distance between the reference value and the critical level z_0 must be increased.

It has been seen that the computation of the variances is not more complicated than solving a linear system of equations. Further the upcrossing probability can easily be obtained with some numerical integration routine. The complexity of this latter problem does not depend on the size of the process model. Thus the computations performed for each value of ρ is not significantly larger than for an ordinary LQG-problem. Moreover by adopting some numerical routine for minimizing the upcrossing probability, it may not be necessary to solve that many LQG-problems. A good choice of starting value for ρ is 0.5, which corresponds to the minimum variance controller. In this sense the computational burden for obtaining the minimum risk controller is not significantly larger than for the LQG controller that corresponds to minimum variance control.

4

Example

To evaluate the performance of the minimum upcrossing controller obtained by minimizing the upcrossing probability a first order process will be investigated. In the Section 4.1 the process is defined. The set of LQG-solutions is calculated analytically in Section 4.2. In the Section 4.3 the minimum upcrossing controller is computed and compared with the minimum variance controller. It is seen that the new controller causes a lower upcrossing probability and smaller probability for the largest value of the signal of being above the critical level. Further it is seen that it has a control signal that is more well-behaved. In the Section 4.4 the results of the previous sections are summarized.

4.1 Process

Let the process be given by

$$\begin{cases} x(k+1) = x(k) + 0.04u(k) + 0.2v(k) \\ y(k) = x(k) + 5e(k) \\ z(k) = x(k) \end{cases}$$

where v and e ar zero mean Gaussian white noise sequences with $\mathrm{E}v^2 = R_1 = 1$, $\mathrm{E}e^2 = R_2 = 1$ and $\mathrm{E}ve = R_{12} = 0$. The signal y is the measurement signal, and u is the control signal. This process can be obtained approximately by fast sampling of a continuous time integrator process.

4.2 LQG-Controllers

The weighting-matrices in (3.6) are

$$
\begin{aligned}
Q_1 &= 4(1-\rho) \\
Q_{12} &= 0.08(1-\rho) \\
Q_2 &= 0.0016
\end{aligned}
$$

and the solutions to the Riccati-equations in (3.8) are

$$
\begin{aligned}
S &= 2\sqrt{\rho(1-\rho)} \\
P &= \frac{0.04 + \sqrt{4.0016}}{2}
\end{aligned}
$$

Some more tedious calculations will give the controller $H(q)$ in (3.7) to be

$$
H(q) = -\frac{s_0 q}{r_0 q + r_1}
$$

where

$$
\begin{aligned}
s_0 &= \left(2\sqrt{\rho(1-\rho)} + 2(1-\rho)\right)\left(0.04 + \sqrt{4.0016}\right) \\
r_0 &= 0.04\left(2\sqrt{\rho(1-\rho)} + 1\right)\left(50.04 + \sqrt{4.0016}\right) \\
r_1 &= 2(1-2\rho)
\end{aligned}
$$

It is interesting to note that for $\rho = 0.5$—minimum variance control by the remark of Lemma 3.1—the controller is a proportional controller.

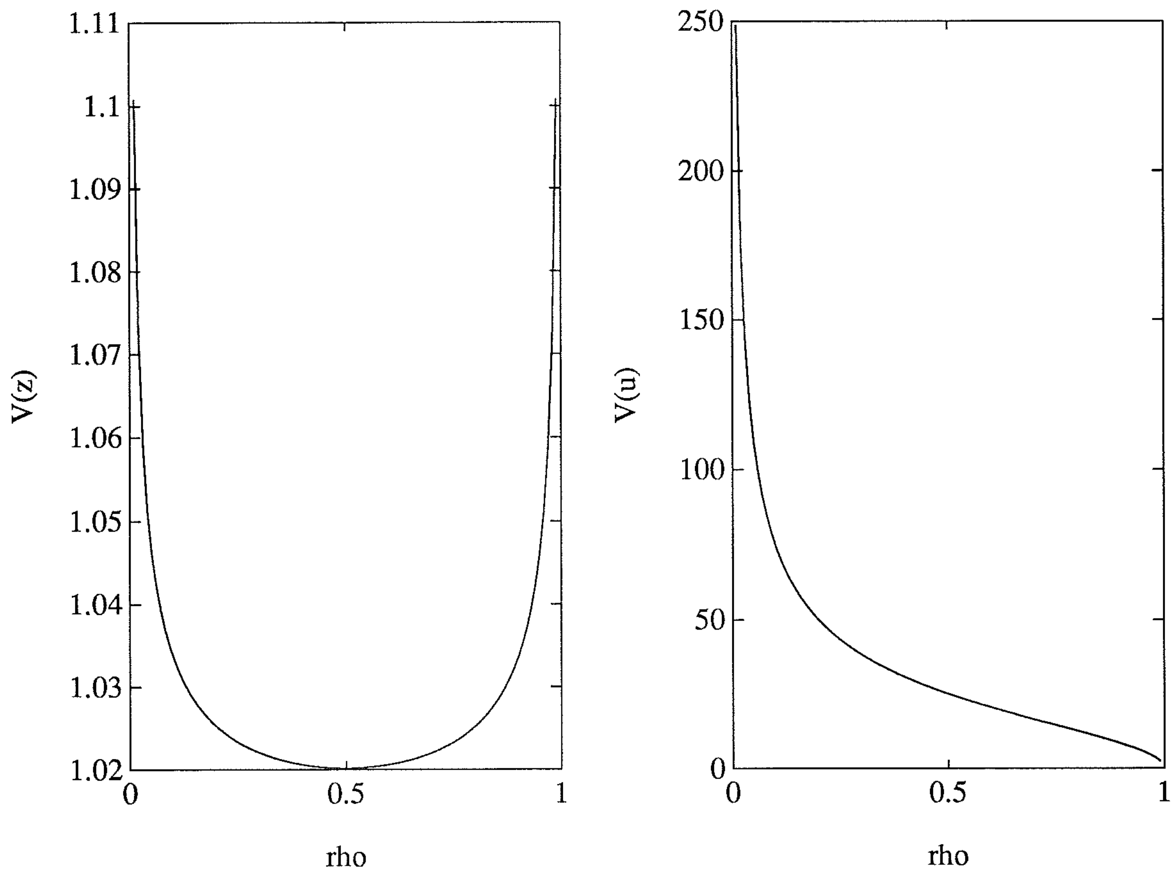

Figure 4.1 The variances of z—left, and u—right, as functions of ρ.

4.3 MU and MV Controllers

The minimum upcrossing (MU) controller will now be compared with the minimum variance (MV) controller.

Variance and Upcrossing Probability

The variances of z and u have been calculated numerically for values of ρ with a step of 0.01 in the range of 0.01 to 0.99. It is seen in Figure 4.1 that the variance of z does not depend so much on ρ as does the variance of u.

The probability μ has been calculated according to Lemma 3.3 for $m_z = 0$ and the values $z_0 = 2$, 3 ,4 and 5 of the critical level. The result is seen in Figure 4.2. The minimum value of the probability μ is obtained for ρ greater than 0.5. The variance of the control signal is smaller the larger ρ is, and the controller obtained for $\rho = 0.5$ is the MV controller

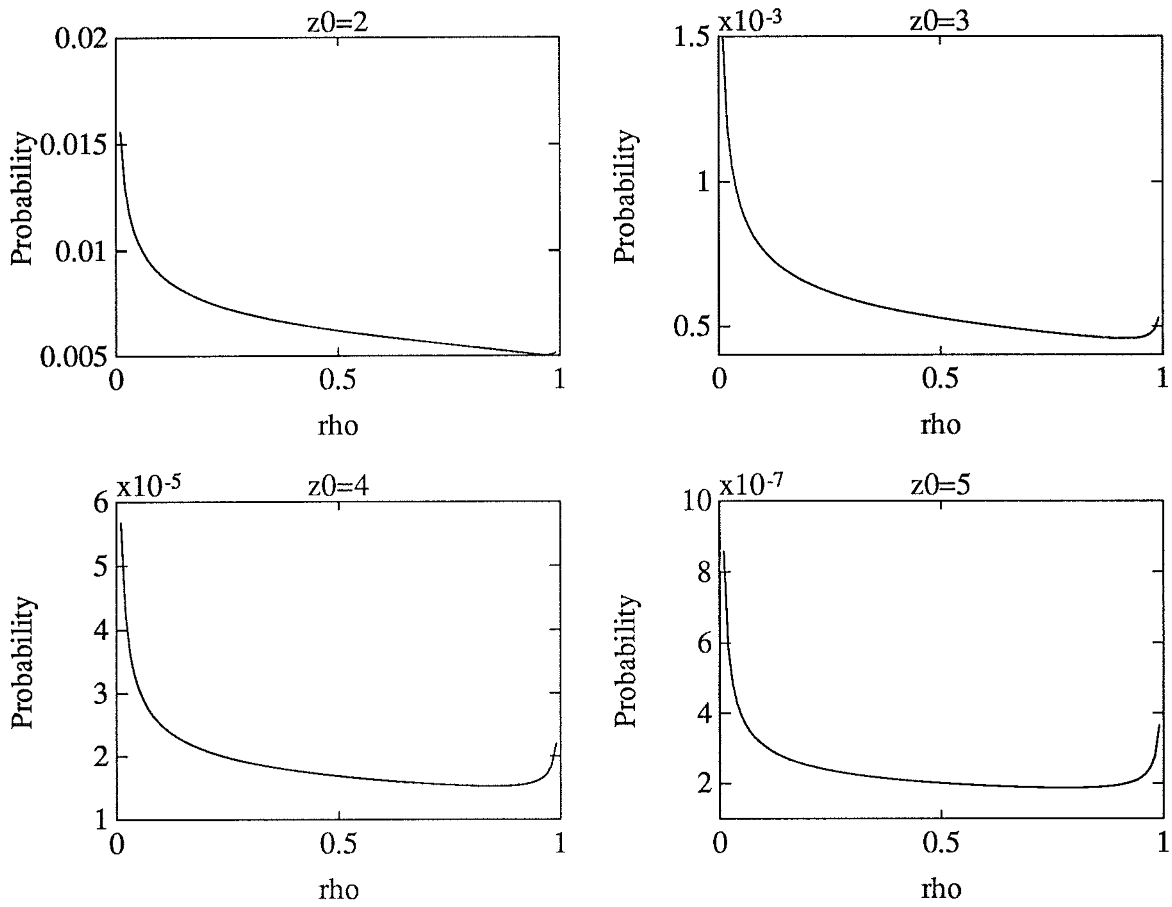

Figure 4.2 The probability μ as function of ρ for $z_0 = 2$—top left, $z_0 = 3$—top right, $z_0 = 4$—bottom left, and $z_0 = 5$—bottom right.

by the remark of Lemma 3.1. Thus the MU controller not only minimize the upcrossing probability, but it also has a control signal that is more well-behaved than that of the MV controller.

Simulations

The controllers have also been compared by simulations. The same noise sequences were used for both the MU controller and the MV controller in all cases. Figure 4.3 shows plots of z and u as functions of time for the MV controller and the MU controller for $z_0 = 3$. It is seen that that the MU controller manages to keep the signal z below the critical level, while the MV controller does not. Further it is seen that the variance of u is smaller for the MU controller than for the MV controller. Note that z is not white noise for the MV controller although y is, since y is correlated with e.

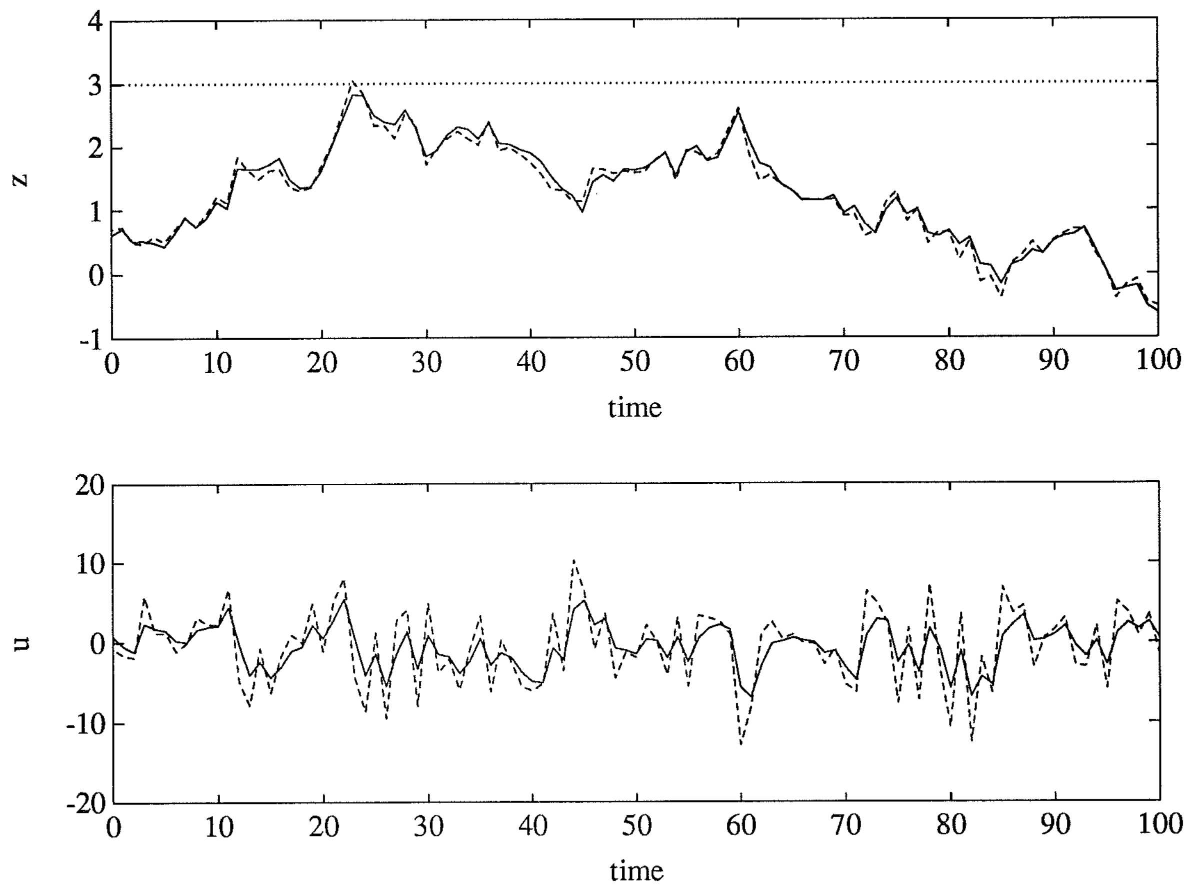

Figure 4.3 The signals $z(t)$—top, and $u(t)$—bottom, as function of time for the optimal controller—solid line, and the minimum variance controller—dashed line.

Robustness

To investigate the robustness against unmodeled non-linearities the process-dynamics was changed to

$$x(k+1) = 0.33x^2(k) + x(k) + 0.04u(k) + 0.2v(k)$$

Thus the process for which the controllers are designed can be thought of as a linearization of the non-linear process round $x(k) = 0$. If $v(k)$ is zero, and if the minimum variance control strategy is applied, then the nonlinear process is stable for initial values of x that are smaller than approximately 3. Therefore it is interesting to compare the MU controller designed for $z_0 = 3$ with the MV controller. Plots of y, z, and u for the two different control strategies with the same noise sequences are shown in figure 4.4 and 4.5. It is seen that the MV controller has

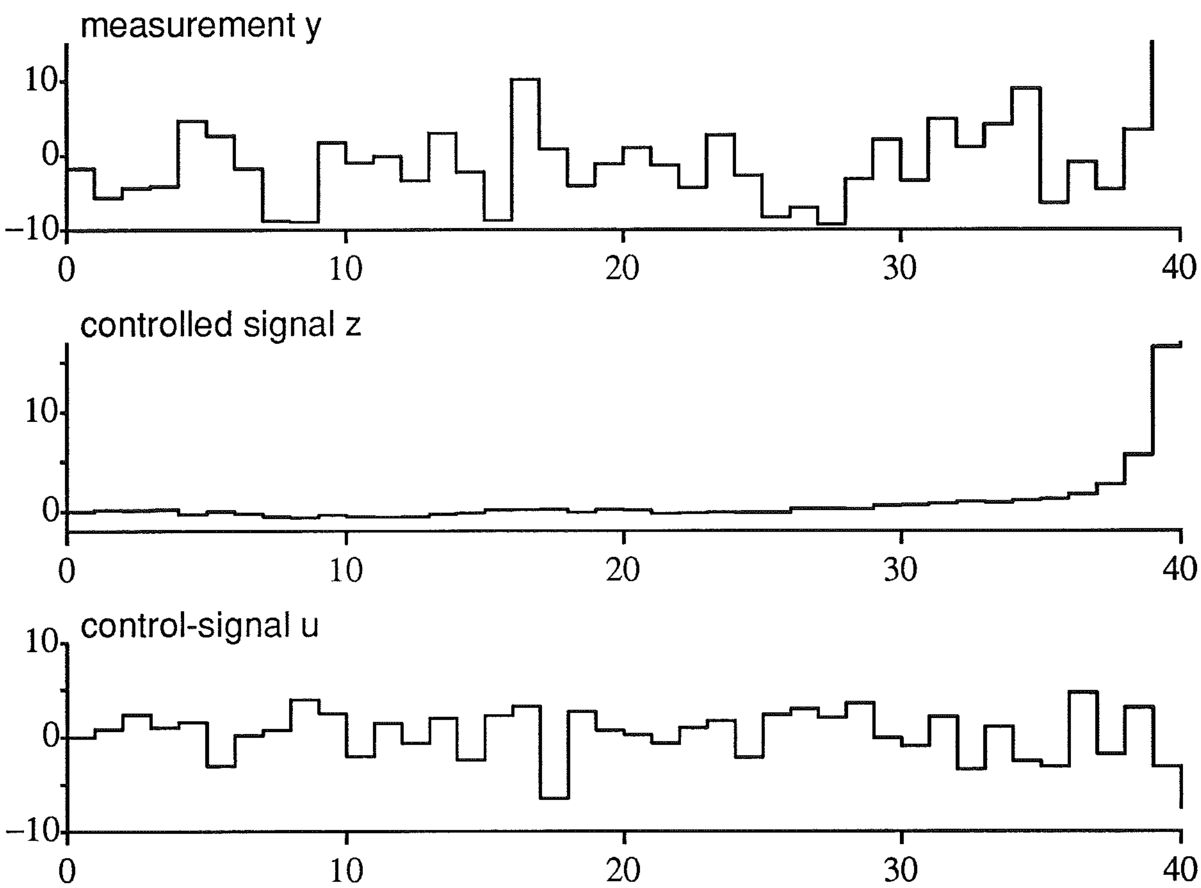

Figure 4.4 The signals $y(t)$, $z(t)$ and $u(t)$ as functions of time for the minimum risk controller, when controlling a non-linear process.

more difficulties to stabilize the process than the MU controller has.

Transfer Functions

The MU controller for $z_0 = 3$ ($\rho = 0.92$) is given by:

$$H(q) = -\frac{0.4901q}{q - 0.4804}$$

and the MV controller is given by:

$$H(q) = -0.9802$$

It is interesting to note that the difference between the MV controller and the MU controller is that the MU controller has a 3 times lower gain for high frequences ($q = -1$) due to the MU controller being a first order system while the MV controller being only a proportional controller. This

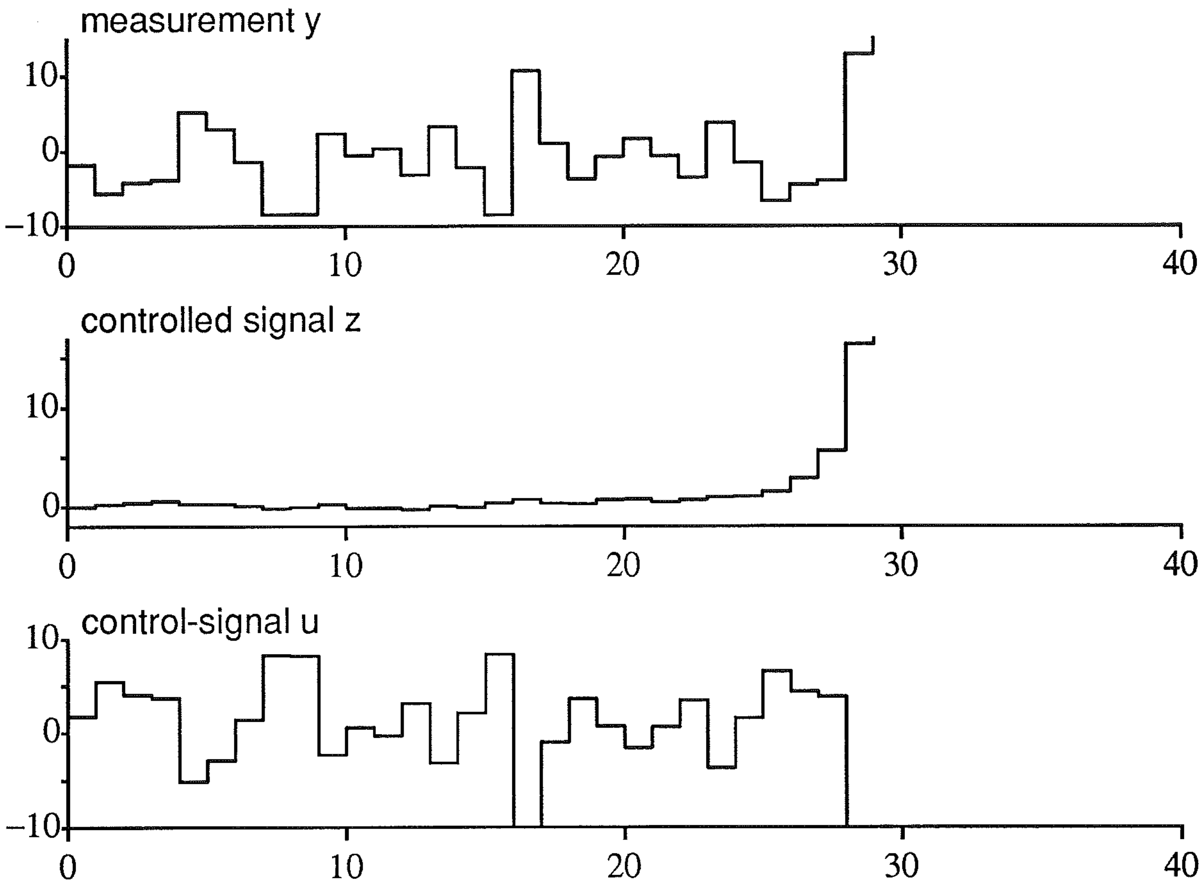

Figure 4.5 The signals $y(t)$, $z(t)$ and $u(t)$ as functions of time for the the minimum variance controller, when controlling a non-linear process.

explains why the variance of the control signal is much smaller for the MU controller. Some calculations give that

$$(q - 0.9608)z = 0.2v - 0.196e$$

for the MV controller and

$$[(q-1)(q-0.4804) + 0.0196]z = 0.2(q-0.4804)v - 0.098e$$

for the MU controller. It is seen that the main difference in the closed loop behavior between the MV controller and the MU controller is the lower high frequency gain ($q = -1$) from e to z for the MU controller.

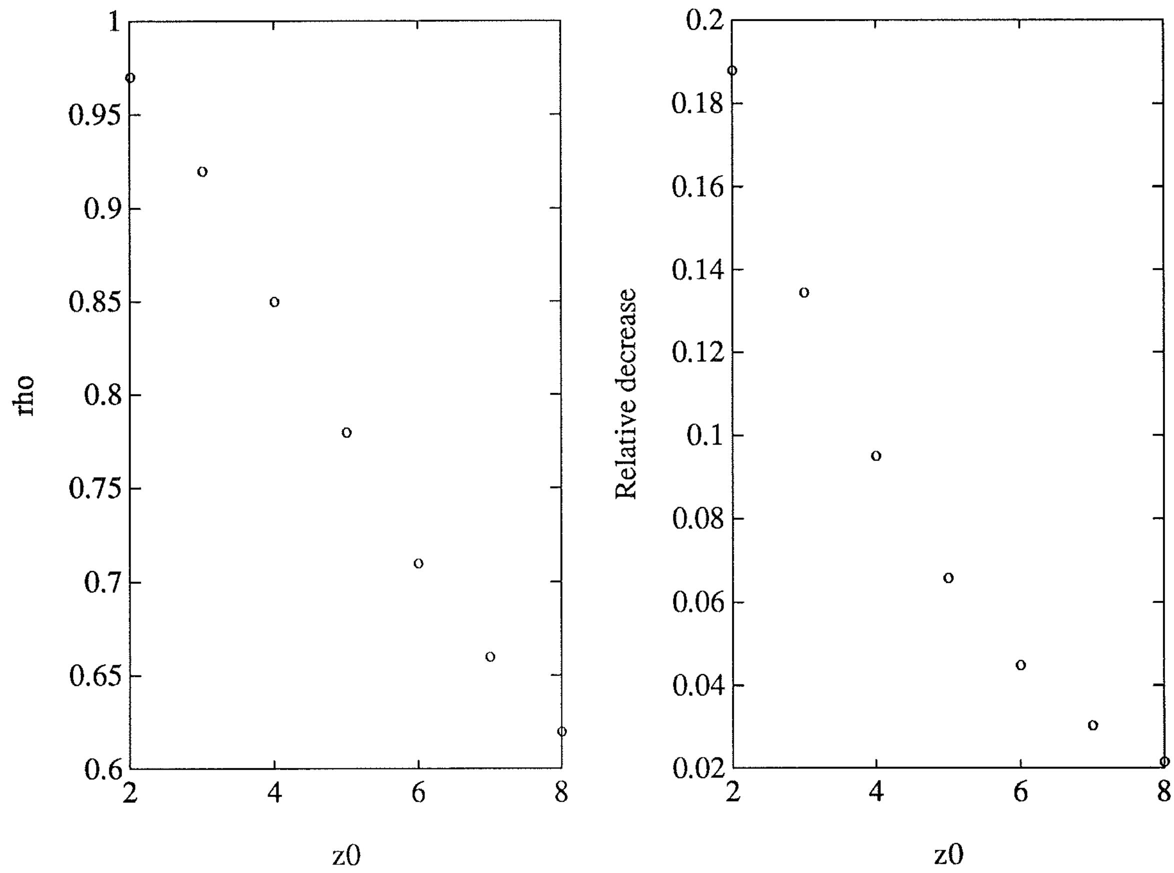

Figure 4.6 The optimal values of ρ as function of z_0—left, and $(\mu_{\text{mv}} - \mu_{\text{opt}})/\mu_{\text{mv}}$ as function of z_0—right, where μ_{mv} is the upcrossing probability for the minimum variance controller and μ_{opt} is the upcrossing probability for the minimum risk controller.

Approximation-validity

The validity of the assumptions made in the approximation of the problem formulation in Section 2.3 will now be investigated further; one positive indication has already been seen in Figure 4.3. In Figure 4.6 it is seen how the optimal value of ρ, and how the relative decrease of upcrossing probability between the MV controller and the MU controller decreases as z_0 increases. This indicates that the MU controller and the MV controller are approximately the same for large values of z_0.

To investigate the behavior of the controllers for moderate values of z_0, Monte Carlo simulations have been performed to estimate the probability $\mathrm{P}\left\{\max_{0\leq k\leq N} z(k) > z_0\right\}$ in (2.1) for the MU controller—$\hat{P}_{\text{opt}}$, and for the MV controller—$\hat{P}_{\text{mv}}$. The estimated values all have 90 % confi-

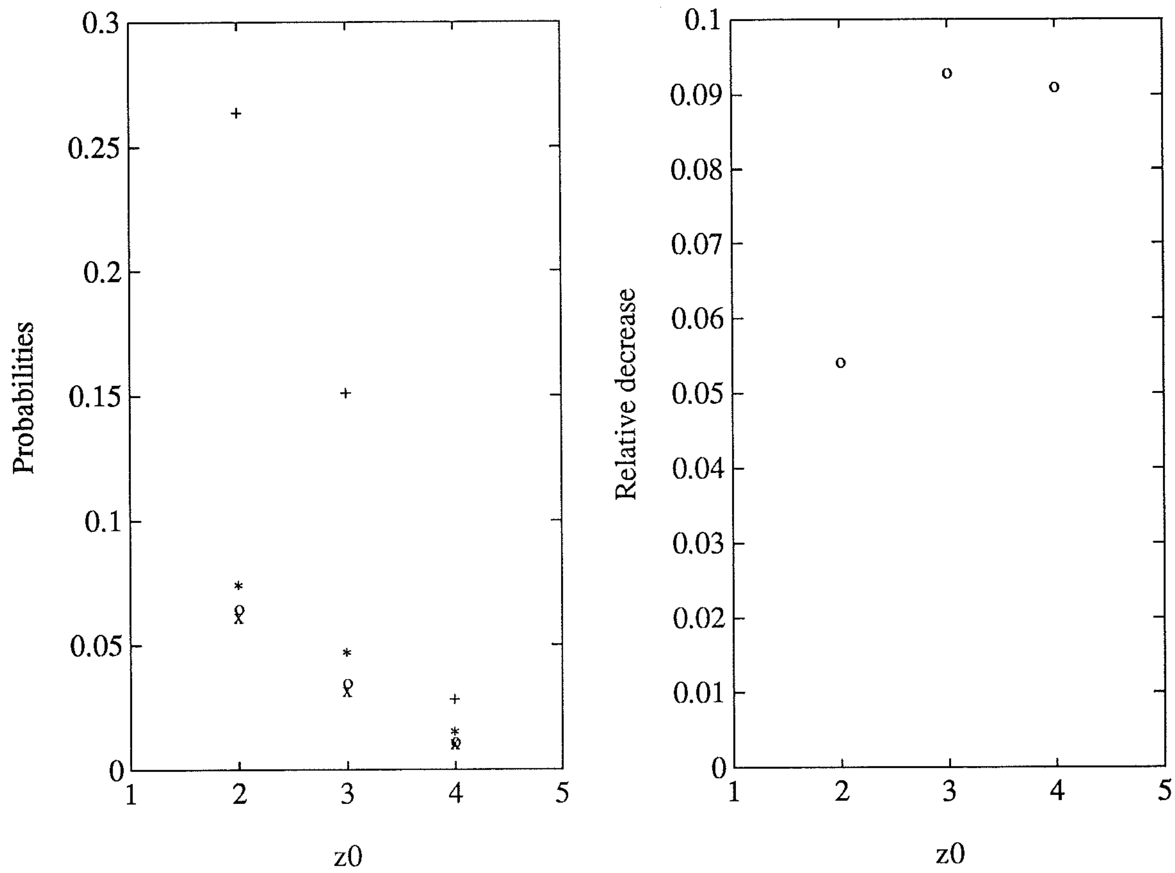

Figure 4.7 The left plot shows the bound P_2 for the minimum variance controller—'+', the bound P_1 for the minimum risk controller—'*', $\hat{P}_{\mathbf{mv}}$—'o', and $\hat{P}_{\mathbf{opt}}$ —'x', as functions of z_0. The values of N has been 10 for $z_0 = 2$, 100 for $z_0 = 3$ and 1000 for $z_0 = 4$. The right plot shows $(\hat{P}_{\mathbf{mv}} - \hat{P}_{\mathbf{opt}})/\hat{P}_{\mathbf{mv}}$ as function of z_0.

dence intervals that are smaller than plus minus 2.2 % ($z_0 = 2$), 9.5 % ($z_0 = 3$), and 17 % ($z_0 = 4$) of the estimated values. These intervals have been computed as in [Waerden, 1969, p. 33]. In Figure 4.7 these estimates of the probabilities are compared with the bounds P_1 and P_2 of Theorem 2.1, where for short reference

$$P_1(z_0) = \mathrm{P}\left\{z(0) > z_0\right\} + N\mu(z_0)$$
$$P_2(z_0) = (N+1)\mathrm{P}\left\{z(0) > z_0\right\}$$

The bound P_1, which by Remark 1 of Theorem 2.1 is approximately minimized by the MU controller, has been computed for the MU controller. The bound P_2, which by Remark 2 of Theorem 2.1 is minimized by MV control, has been computed for the MV controller. The values of N and z_0 has been chosen such that the bound P_2 is about 0.1. The values are $(z_0, N) = (2, 10)$, (3,100) and (4,1000). The result is shown in Figure 4.7. It is seen in the left plot that the bound P_1 is much lower than the bound P_2, and that the estimate $\hat{P}_{\text{opt}}$ is lower than estimate $\hat{P}_{\text{mv}}$. The latter is seen more clearly in the right plot, where the relative decrease of the probability of being above the critical level between the MV controller and the MU controller— $(\hat{P}_{\text{mv}} - \hat{P}_{\text{opt}})/\hat{P}_{\text{mv}}$—is plotted versus z_0. Thus the MU controller is about 5% to 10% better than the minimum variance controller for moderate values of the critical level in this example.

4.4 Summary

The theory developed in the previous chapters has been evaluated using a first order process. In spite of the simplicity of the process many interesting features of the new controller have been demonstrated.

It has been shown that the MU controller is a first order system whereas the MV controller is only a proportional controller. The latter has a higher high-frequency gain. The variance of z is slightly larger but the variance of u is a lot smaller for the MU controller as compared with the MV controller. Further it has been seen in simulations that the probability for the largest value of z of being above the critical level is smaller for the MU controller. It has also been seen that the new controller is more robust against unmodeled non-linearities than the MV controller. The simulations have also given insight into the consequences of the approximations made to derive the new controller. When comparing the differences between the MU controller and the MV controller for varying distances to the critical level, it has been seen that these are larger for moderate values of the distance and smaller for larger values of the distance. For the examples in Chapter 1 the distance is typically moderate, and thus it has been justified that the MU controller may well be superior to the MV controller for this class of interesting problems.

5

Conclusions

A new optimal stochastic control problem has been posed. The solution minimizes the probability for a signal's largest value to be above a level given a certain reference value. There are many examples of control problems for which this approach is appealing, i.e. problems for which there exist a level such that a failure in the controlled system occurs when the controlled signal is above the level. One important class of such problems is processes equipped with supervision, where upcrossings of alarm levels may initiate emergency shutdown causing loss in production.

It has been seen that this control problem is closely related both to the problem of minimizing the variance of the signal—minimum variance control—and to the problem of minimizing the upcrossing probability. The latter relation is novel, whereas the former relation has been known for a long time, but the motivations given here are believed to be new. It has been made plausible that the upcrossing probability is a better criterion to minimize than the minimum variance criterion.

The problem of minimizing the upcrossing probability over the set of stabilizing linear time-invariant controllers has been rephrased to a minimization over LQG-problem solutions parameterized by a scalar, and thus the complexity is only one order of magnitude larger than for an ordinary LQG-problem. If the solutions to the LQG-problems are unique, then the problem of minimizing the upcrossing probability can be thought of as finding optimal weighting-matrices in an LQG-problem. The key to the new method is the reformulation using the independent variables α and β making it possible to quantify by Lemma 3.3 the upcrossing probability in terms of the variances of α and β.

The new controller has been compared with the minimum variance controller for a first order process. It has been seen that the new controller causes a lower upcrossing intensity and a smaller probability for the largest value of the controlled signal to be above the dangerous level. Further it has been seen that the control signal is more well-behaved.

Both theory and simulations have shown that the minimum risk controller and the minimum variance controller are approximately the same for large values of the dangerous level. However, in the example it has been seen that the minimum risk controller can have up to about 10% better performance for moderate values of the critical level, which is the interesting case for the examples in Chapter 1. This makes it possible to choose the reference value closer to the critical level when using the minimum risk controller, than when using the minimum variance controller. This will in many cases increase the profit.

Thus the new controller has many advantages as compared with the minimum variance controller—a smaller probability of being above the dangerous level, a control-signal that is more well-behaved, and an interpretation as weighting-optimal LQG. The only drawback is the slightly larger computational burden.

This concludes the work of proving the raison d'être of the minimum risk controller and demonstrating its advantages as compared to the minimum variance controller for a large class of control problems.

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A

More General Process Model

Let the stationary Gaussian sequence z be defined by a process model more general than (3.1):

$$\begin{cases} x(k+1) = Ax(k) + B_1u(k) + B_2v(k) \\ y(k) = C_1x(k) + D_1e(k) \\ z(k) = C_2x(k) + D_2w(k) \end{cases} \tag{A.1}$$

where v, e and w are zero mean Gaussian white noise sequences with the positive semidefinite covariance matrix

$$\mathrm{E}\left\{ \begin{pmatrix} v \\ e \\ w \end{pmatrix} \begin{pmatrix} v^T & e^T & w^T \end{pmatrix} \right\} = \begin{pmatrix} R_1 & R_{12} & R_{13} \\ R_{12}^T & R_2 & R_{23} \\ R_{13}^T & R_{23}^T & R_3 \end{pmatrix}$$

The signal y is the measurement signal, and u is the control signal. The following lemma is a generalization of Lemma 3.1.

Lemma A.1
The loss function J in (3.4) can be written

$$J = \bar{J} + \mathrm{E}\left\{v^T Q_3 v + 2v^T Q_{34} w + w^T Q_4 w\right\}$$

where

$$\bar{J} = \mathrm{E}\left\{x^T Q_1 x + 2x^T Q_{12} u + u^T Q_2 u + 2u^T Q_{23} v + 2u^T Q_{24} w\right\}, \quad \text{(A.2)}$$

and where Q_1, Q_{12} and Q_2 are as in (3.6) in Lemma 3.1 and where

$$\begin{aligned} Q_{23} &= B_1^T C_2^T C_2 B_2 \\ Q_{24} &= (1-2\rho) B_1^T C_2^T D_2 \\ Q_3 &= B_2^T C_2^T C_2 B_2 \\ Q_{34} &= (1-2\rho) B_2^T C_2^T D_2 \\ Q_4 &= 2 D_2^T D_2 \end{aligned} \quad \text{(A.3)}$$

Proof: The result follows immediately by using the definitions of z in (A.1), and of α and β in (3.2) and by noting that $v(k)$ and $w(k)$ are uncorrelated with $x(k)$, and that $u(k)$ is uncorrelated with $w(k+1)$. □
Remark. For $\rho = 0.5$ it follows that $J = \mathrm{E}\{z(k+1)^2 + z(k)^2\}$. This case thus corresponds to minimum variance control of z.

The problem of minimizing $\bar{J}$ in (A.2) is not a standard LQG-problem, therefor the following lemma is needed.

Lemma A.2
If $\bar{Q}$ is invertible, then the loss-function $\bar{J}$ in (A.2) can be written

$$\bar{J} = \tilde{J} + \mathrm{E}\left\{v^T(S - L_v^T \bar{Q} L_v)v - 2v^T L_v^T \bar{Q} L_w w - w^T L_w^T \bar{Q} L_w w\right\}$$

where

$$\tilde{J} = \mathrm{E}\left\{(u + Lx + L_v v + L_w w)^T \bar{Q}(u + Lx + L_v v + L_w w)\right\}$$

where S is the solution to the Riccati-equation in (3.8), and where

$$\begin{aligned} \bar{Q} &= \left(Q_2 + B_1^T S B_1\right) \\ L &= \bar{Q}^{-1}\left(B_1^T S A + Q_{12}^T\right) \\ L_v &= \bar{Q}^{-1}\left(B_1^T S B_2 + Q_{23}\right) \\ L_w &= \bar{Q}^{-1} Q_{24} \end{aligned}$$

Proof: The result follows by a generalization of [Åström and Wittenmark, 1990, Theorem 11.2] to $Q_{12} \neq 0$, see e.g. [Gustafsson and Hagander, 1991, p. 3], by completing the squares in $\bar{J}$, and by noting that v and w are uncorrelated with x. □

To obtain the optimal controller that minimizes J, estimates of $x(k)$, $v(k)$, and $w(k)$ based on observations of y up to time k are needed. The ones for x and v can be found in [Åström and Wittenmark, 1990, Eq. (11.49), p. 352]. The following lemma gives the estimate of w.

LEMMA A.3
If R_y is invertible, then the estimate

$$\hat{w}(k|k) = \mathrm{E}\{w(k)|\mathcal{Y}_k\}$$

of $w(k)$, where $\mathcal{Y}_k$ is the σ-algebra generated by all past observations of y up to time k, is given by

$$\hat{w}(k|k) = R_{23}^T D_1^T R_y^{-1} \tilde{y}(k)$$

where

$$\begin{aligned} R_y &= C_1 P C_1^T + D_1 R_2 D_1^T \\ \tilde{y}(k) &= y(k) - C_1 \hat{x}(k|k-1) \\ \hat{x}(k|k-1) &= \mathrm{E}\{x(k)|\mathcal{Y}_{k-1}\} \end{aligned}$$

and where P is the solution to the Riccati-equation

$$\begin{aligned} APA^T - P - \left(APC_1^T + B_2 R_{12} D^T\right)\left(DR_2 D^T + C_1 P C_1^T\right)^{-1} \\ \left(C_1 P A^T + D R_{12}^T B_2^T\right) + B_2 R_1 B_2^T = 0 \end{aligned} \tag{A.4}$$

Proof: Since $\mathcal{Y}_k$ is the same σ-algebra as the one generated by all past observations of y up to time $k-1$ and by $\tilde{y}(k)$, since $\tilde{y}(k)$ is independent of $\mathcal{Y}_{k-1}$ by [Åström, 1970, Theorem 3.2, p. 219], and since $\mathrm{E}\{w(k)\tilde{y}(k)^T\} = \mathrm{E}\{w(k)(D_1 e(k))^T\}$, it follows by [Åström, 1970, Th. 3.2 and Th. 3.3, pp. 219–220] that

$$\begin{aligned} \hat{w}(k|k) &= \mathrm{E}\{w(k)|\mathcal{Y}_{k-1}, \tilde{y}(k)\} \\ &= \mathrm{E}\{w(k)|\mathcal{Y}_{k-1}\} + \mathrm{E}\{w(k)|\tilde{y}(k)\}\} \\ &= \mathrm{E}\{w(k)|\tilde{y}(k)\} \\ &= \mathrm{E}\left\{w(k)\tilde{y}(k)^T\right\}\left[\mathrm{E}\left\{\tilde{y}(k)\tilde{y}(k)^T\right\}\right]^{-1}\left(\tilde{y}(k) - \mathrm{E}\{\tilde{y}(k)\}\right) \\ &= R_{23}^T D_1^T R_y^{-1} \tilde{y}(k) \end{aligned}$$

□

LEMMA A.4
If $\bar{Q}$ and R_y are invertible, then the optimal controller that minimizes J in (3.4) is given by

$$u(k) = -L\hat{x}(k|k) - L_v\hat{v}(k|k) - L_w\hat{w}(k|k)$$

where $\hat{x}(k|k)$ and $\hat{v}(k|k)$ are given by [Åström and Wittenmark, 1990, Eq. (11.49)] and $\hat{w}(k|k)$ is given by Lemma A.3.

Proof: The result follows by lemmas A.1 and A.2 and by the separation principle, see e.g. [Åström, 1970, p. 282]. □

THEOREM A.1
If $\bar{Q}$ and R_y are invertible, then the transfer function $H(q)$ for the optimal controller that minimizes J in (3.4) is given by

$$H(q) = -L_x(qI - A + B_1L_x + KC_1)^{-1}K_y - L_y \tag{A.5}$$

where

$$\begin{aligned}
L_x &= L - L_yC_1 \\
L_y &= LK_f + L_vK_v + L_wK_w \\
K_y &= K - B_1L_y \\
K &= AK_f + B_2K_v \\
K_f &= PC_1^TR_y^{-1} \\
K_v &= R_{12}D_1^TR_y^{-1} \\
K_w &= R_{23}D_1^TR_y^{-1}
\end{aligned}$$

and where P is the solution to the Riccati-equation in (A.4).

Proof: The proof is straight forward calculations making use of lemmas A.3 and A.4 and the equations in [Åström and Wittenmark, 1990, Theorem 11.6]. □

LEMMA A.5
The closed loop system behavior for the optimal controller is governed by

$$\bar{x}(k+1) = A_c\bar{x}(k) + B_c\bar{v}(k)$$

where $\bar{x}(k) = (\, x^T(k) \quad \tilde{x}^T(k)\,)^T$, $\bar{v}(k) = (\, v^T(k) \quad e^T(k)\,)^T$, and where

$$A_c = \begin{pmatrix} A - B_1 L & B_1 L_x \\ 0 & A - KC_1 \end{pmatrix}$$
$$B_c = \begin{pmatrix} B_2 & -B_1 L_y D_1 \\ B_2 & -KD_1 \end{pmatrix}$$

Proof: The proof is straight forward calculations making use of lemmas A.3 and A.4 and the equations in [Åström and Wittenmark, 1990, Theorem 11.6]. □

THEOREM A.2
The variances of α, β, z and u are given by

$$\begin{aligned}
\sigma_\alpha^2 &= (\, C_2 \quad 0\,)\left((A_c + I)X(A_c + I)^T + B_c R B_c^T\right)(\, C_2 \quad 0\,)^T + 2D_2R_3D_2^T \\
\sigma_\beta^2 &= (\, C_2 \quad 0\,)\left((A_c - I)X(A_c - I)^T + B_c R B_c^T\right)(\, C_2 \quad 0\,)^T + 2D_2R_3D_2^T \\
\sigma_z^2 &= (\, C_2 \quad 0\,)\, X\, (\, C_2 \quad 0\,)^T + D_2R_3D_2^T \\
\sigma_u^2 &= (\, -L \quad L_x\,)\, X\, (\, -L \quad L_x\,)^T + L_y D_1 R_2 D_1^T L_y^T
\end{aligned} \tag{A.6}$$

where X is the solution to the Lyapunov equation in (3.9), and where R is given by

$$R = \begin{pmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{pmatrix}$$

Proof: The result follows from Lemma A.5 and [Åström, 1970, p. 49]. □

Remark. Due to the block-triangularity of A_c it is possible to split up (3.9) into three equations, one of which has P as its solution. This reduces the complexity of the problem.