Feedforward Control in Linear Multivariable Systems

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by

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ABSTRACT

A constructive solution is given to the general feedforward problem in a geometric framework. Necessary and sufficient conditions are given and the minimal amount of differentiation needed for the solution is identified. The case of stable feedforward controllers is also resolved. The way in which feedback and feedforward control should be combined is discussed. A generic analysis of the feedforward problem reveals that the class of systems for which a feedforward controller exists is large.

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1. INTRODUCTION.

A basic control objective is to make the controlled system insensitive to exogenous disturbances. For this purpose, there are two main control principles available, feedback and feedforward control . The feedback principle has been exploited in a vast number of papers. As an example, there is by now a more or less complete theory for steady state regulation of disturbances by means of feedback control [5, 7, 14, 15]. Although of high significance for process control, see e.g. [3, 4, 11], the feedforward principle has not been subject to the same theoretical development. The best control is often obtained by a combination of feedback and feedforward, see e.g. [11]. The possibility of using feedforward for steady state regulation is discussed in [5]. In [6, 14] (disturbance localization) the output is made independent of the exogenous signals by means of linear state feedback, i.e. not by a feedforward compensator in the classical sense.

The feedforward principle is the intelligent use of available plant models to compensate for exogenous signals before the process is upset. The purpose of this paper is to develop a theory for feedforward control of multivariable linear systems, or more precisely to find a feedforward compensator so that the transfer matrix between the exogenous signals and the controlled variables is zero. The basic mathematical tool is the geometric state space theory developed by Wonham and Morse, [10], see also Wonham [14]. One obstacle is that feedforward control may need noncausal compensators, i.e. compensators which require pure differentiators for its implementation. It is shown that compensators of this type can be included in the standard geometric machinery. In practise, a differentiator is approximated by a causal transfer function as is done e.g. in the classical PID regulator, where the deficiency depends on the quality of the measured signal and the amount of differentiation. The approximation can be made arbitrary close over e.g. any finite frequency interval. Since our primary purpose is to investigate what can be achieved by feedforward control, there is therefore no reason to a priori exclude noncausal controllers.

To minimize the trade off, it is desirable to keep the amount of differentiation in the compensator as low as possible. It is also desirable to have a stable feedforward compensator. Both these problems are solved. Specifically, we give necessary and sufficient conditions for the existence of a feedforward compensator and identify the minimal amount of differentiation needed for the solution. Thereby, we also give necessary and sufficient conditions for the existence of a causal feedforward compensator. To have an efficient controller, it is also necessary to combine the feedforward controller by a suitable feedback controller. A controller structure which allows the inclusion of feedback is suggested, where the feedforward part is fixed and the feedback part is free to choose according to standard techniques.

To demonstrate that feedforward compensation is a useful tool in synthesis, we perform a generic analysis of the problem. This analysis reveals that the class of system for which feedforward compensation is applicable is large, provided only that the number of control inputs is larger than or equal to the number of controlled outputs.

The paper is organized as follows. In section 2 we define the concept of "noncausal" control in terms of an underlying state representation. The feedforward problem is solved in section 3. In section 4 the feedforward compensator is combined with a suitable feedback compensator. Generic solvability is treated in section 5.

Notations

Script letters $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$, denote linear vector spaces of finite dimensions and capital roman letters A, B, C, linear maps. If $A:\mathcal{X} \to \mathcal{Y}$ is a linear map and $\mathcal{V} \subset \mathcal{X}$ and $\mathcal{W} \subset \mathcal{X}$ subspaces, the subspaces $A\mathcal{V} = \{Av | v \in \mathcal{V}\}$ and $A^{-1}\mathcal{W} = \{x | x \in \mathcal{X}, Ax \in \mathcal{W}\}$ denote the image of $\mathcal{V}$ and the inverse image of $\mathcal{W}$ respectively. The range space of A is written Im(A) and sometimes $\mathcal{A}$. The null space of A is denoted ker(A).

Let $A:\mathcal{X} \to \mathcal{X}$ and $B:\mathcal{U} \to \mathcal{X}$ be a pair of linear maps. The controllable subspace for the pair (A,B) is written $\langle A|\mathcal{B}\rangle$ where $\langle A|\mathcal{B}\rangle = \mathcal{B} + A\mathcal{B} + \ldots + A^{n-1}\mathcal{B}$ with $n = \dim(\mathcal{X})$. A subspace $\mathcal{V}$ is said to be (A,B)-invariant if $(A+BF)\,\mathcal{V} \subset \mathcal{V}$ for some linear map $F:\mathcal{X} \to \mathcal{U}$. The subspace $\mathcal{R}$ is a controllability subspace if $\mathcal{R} = \langle A + BF|\mathcal{B} \cap \mathcal{R}\rangle$ for some linear map F. The family of maps F such that $(A+BF)\,\mathcal{V} \subset \mathcal{V}$ is written $\underline{F}(\mathcal{V})$. The family of all (A,B)-invariant subspaces contained in a given subspace $\mathcal{D}$ is denoted $\mathcal{I}(A,B,\mathcal{D}) = \{\mathcal{V} | \mathcal{V} \subset \mathcal{D}, A\mathcal{V} \subset \mathcal{V} + \mathcal{B}\}$. Analogously, the family of all controllability subspaces contained in $\mathcal{D}$ is written $\mathcal{C}(A,B,\mathcal{D}) = \{\mathcal{R} | \mathcal{R} \subset \mathcal{D}, \mathcal{R} = \langle A + BF|\mathcal{B} \cap \mathcal{R}\rangle \text{ for some } F\}$. It can be shown that these families have unique supremal elements $\mathcal{V}^*$ and $\mathcal{R}^*$ which can easily be computed from the given data [14,2].

Let $A:\mathcal{X} \to \mathcal{X}$ be a linear map and $\alpha(s)$ the minimal polynomial of A. Factorize $\alpha(s)$ as $\alpha(s) = \alpha^+(s)\alpha^-(s)$ where $\alpha^+(s)$ and $\alpha^-(s)$ have all their roots in the closed right half plane and the open left half plane respectively. The stable subspace $X^-(A)$ and the unstable subspace $X^+(A)$ are defined by

$$X^{\pm}(A) = \ker\left(\alpha^{\pm}(A)\right)$$

It will be assumed that the reader is familiar with the basic results and ideas of geometric state space theory as expressed e.g. in [10,14].

Model.

The plant is assumed to be described by a state equation

$$\begin{aligned} \dot{x} &= Ax + Bu + Ew \\ y &= Cx \end{aligned} \qquad (1)$$

where $x \in \mathcal{X}$ is the state, $y \in \mathcal{Y}$ the controlled output, $u \in \mathcal{U}$ the control input and $w \in \mathcal{W}$ the disturbance input. The spaces $\mathcal{X}$, $\mathcal{U}$, $\mathcal{Y}$ and $\mathcal{W}$ are all vector spaces of dimensions n, m, q and r respectively. Here, A, B, C and E are linear time invariant maps between the appropriate vector spaces. We also assume that the maps B and C are monic and epic respectively.

2. PRELIMINARIES.

In ordinary linear systems the internal consequences of applying control can be described in terms of the controllable subspace. This is a key concept used in the geometric state space theory, [10, 14], to describe input-output properties using internal concepts only. In this paper, we will permit controls containing pure differentiators. This leads to the analysis of differential equations of the form

$$\begin{aligned} sx &= Ax + B(s)u \\ y &= Cx \end{aligned} \qquad s = d/dt \tag{2}$$

where

$$B(s) = B_\sigma + sB_{\sigma-1} + \dots + s^\sigma B_0$$

and $A: X \to X$, $C: X \to Y$ and $B_i: U \to X$ are linear maps. The differential equation (2) is <u>not a system</u> in the usual sense, which means that the notions of controllability and observability, [8], cannot be directly used.

To be able to use the standard framework of linear systems, let us introduce a system Σ which is regarded as a state description of (2). Let $\bar{u} = u^\sigma$ be the input and $\bar{x} = \left(x; u^{(0)}, u^{(1)}, \dots, u^{(\sigma-1)}\right)$ be the state. Then from (2)

$$\begin{aligned} \dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}\bar{u} \\ y &= \bar{C}\bar{x} \end{aligned} \tag{3a}$$

where

$$\bar{A} = \begin{bmatrix} A & B_\sigma & B_{\sigma-1} & \cdots & B_1 \\ 0 & 0 & I & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & I \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \qquad \bar{B} = \begin{bmatrix} B_0 \\ 0 \\ \vdots \\ I \end{bmatrix} \tag{3b}$$

$$\bar{C} = [C \quad 0 \quad 0 \quad \cdots \quad 0]$$

which is a system of the usual form. Using Σ as a representative of (2), some immediate conclusions can be made. The transfer function of (2) is zero if and only if

$$\langle\bar{A}|\bar{B}\rangle \subset \text{Ker } \bar{C} \tag{4}$$

Let $P: X \oplus U^\sigma \to X$ be the projection such that $P(x+z) = x$ for all $x \in X$ and all $z \in U^\sigma$. It is readily seen that $\bar{C} = CP$. An equivalent statement of condition (4) is thus

$$P\langle\bar{A}|\bar{B}\rangle \subset \text{Ker } C$$

The subspace $P\langle\bar{A}|\bar{B}\rangle$ thus plays the same role as the ordinary controllable subspace for the purpose of describing noninteraction between inputs and outputs in (2). Therefore, denote

$$\{A|B(s)\} \underline{\Delta} P\langle\bar{A}|\bar{B}\rangle \tag{5}$$

We may <u>formally</u> regard $\{A|B(s)\}$ as the controllable subspace for (2) with the understanding that it is defined through the state description Σ.

A closer examination of the controllability matrix for the pair $(\bar{A},\bar{B})$ shows that

$$\{A|B(s)\} = S_0 + S_1 + \ldots + S_{\sigma-1} + \langle A|S_\sigma\rangle \qquad (6a)$$

where $S_i = \mathrm{Im}(S_i)$ and S_i is recursively given by

$$S_0 = B_0$$
$$S_i = AS_{i-1} + B_i \qquad (6b)$$

Let us summarize this discussion into

<u>Lemma 1.</u> <u>The transfer function of (2) is zero if and only if</u>

$$\{A|B(s)\} \subset \mathrm{Ker}\, C$$ □

<u>Remark.</u> The system (2) is controllable in the sense of [12, 13] if and only if $\mathrm{rank}[s-A;\ B(s)] = n$ for all $s \in \mathbb{C}$. Note that controllability in this sense <u>is not implied</u> by $\{A|B(s)\} = X$. Consider e.g.

$$A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \qquad B(s) = \begin{pmatrix} s-a_1 & 0 \\ 0 & 1 \end{pmatrix}$$

In this case $\{A|B(s)\} = R^2$ but $\mathrm{rank}[a_1-A;\ B(a_1)] = 1$.

3. FEEDFORWARD CONTROL

In this section a solution to the general feedforward problem is presented. The purpose is to find, in a suitable state representation, a control of the form

$$u = K(s)w \tag{7}$$

where K(s) may be a nonproper rational matrix such that the transfer function from the disturbance to the output vanishes identically, i.e. from (1)

$$G_*(s) = C(s-A)^{-1}(BK(s) + E) = 0 \tag{8}$$

It would also be possible to consider more general types of control laws as $u = N(s)x + K(s)w$. This is, however, an unnecessary complification since whenever such a control solves the problem, it can also be solved by a control of the form (7). The approach will instead be to investigate to what extent (7) can be supplemented by a feedback control once it is fixed. This is the topic of section 4.

The solution to the problem (8) will be given in two steps. In the first step (8) is solved, permitting K(s) to be only a polynomial matrix. In the second step we obtain the general solution for a nonproper transfer matrix K(s).

Nondynamic Feedforward Control.

The admissible control is of the form (7) where in this case K(s) is a matrix of polynomials in s, i.e.

$$K(s) = K_\sigma + K_{\sigma-1}s + \dots + K_0 s^\sigma \tag{9}$$

where $K_i : \mathcal{W} \to \mathcal{U}$ are linear maps. The matrix K(s) shall be chosen so that the transfer function w to y in (2) vanishes, i.e. so that (8) holds. Since K(s) is a polynomial matrix, it is by section 2 well defined what is meant by (8) in terms of internal concepts. Using Lemma 1, we can formulate the nondynamic feedforward problem as: Find a polynomial matrix K(s) such that

$$\{A \mid BK(s) + E\} \subset \text{Ker } C \tag{10}$$

Introduce the following nondecreasing sequence of linear subspaces

$$\mathcal{B}_0 = \mathcal{B} \qquad \mathcal{B}_{i+1} = \mathcal{B} + A(\mathcal{B}_i \cap \text{Ker } C) \tag{11}$$

where $\mathcal{B} = \text{Im } B$. The solution to the nondynamic feedforward problem is given in

Theorem 1. There exists a solution K(s) with $\deg(K(s)) \leq \sigma$ to the nondynamic feedforward problem if and only if

$$\mathcal{V}_0^* + \mathcal{B}_\sigma \supset \mathcal{E}$$

where B_σ is given by (11) and $\mathcal{V}_0^* = \sup \mathcal{I}(A,0,\text{Ker } C)$ and $\mathcal{E} = \text{Im } E$.

To prove this theorem we need

Lemma 2. Let $B: \mathcal{U} \to \mathcal{X}$ and $E: \mathcal{W} \to \mathcal{X}$ be linear maps and $\mathcal{V} \subset \mathcal{X}$ a linear subspace. There is a linear map N such that $\mathrm{Im}(BN+E) \subset \mathcal{V}$ if and only if $\mathcal{V} + \mathcal{B} \supset \mathcal{E}$.

Proof. (if) Let w_i, $i \in \underline{q}$, be a basis for $\mathcal{W}$. Then there exist $v_i \in \mathcal{V}$ and $u_i \in \mathcal{U}$, $i \in \underline{q}$, so that $v_i + Bu_i = Ew_i$. Let $N: \mathcal{W} \to \mathcal{U}$ be a map such that $Nw_i = -u_i$. Since w_i, $i \in \underline{q}$, is a basis, N is well defined. Obviously, $(BN+E)w_i = v_i$, so $\mathrm{Im}(BN+E) \subset \mathcal{V}$. (only if) Let w_i, $i \in \underline{q}$ be a basis for $\mathcal{W}$. Then there exist $v_i \in \mathcal{V}$, $i \in \underline{q}$, so that $(BN+E)w_i = v_i$, i.e. $Ew_i = -BNw_i + v_i \in \mathcal{B} + \mathcal{V}$.

□

Proof of Theorem 1. (If) Consider the sequence (11) and let $\mathcal{R}_i = \mathcal{B}_i \cap \mathrm{Ker}(C)$. Then

$$\begin{aligned} \mathcal{R}_0 &= \mathcal{B} \cap \mathrm{Ker}\, C \\ \mathcal{R}_i &= (\mathcal{B} + A\mathcal{R}_{i-1}) \cap \mathrm{Ker}\, C \end{aligned} \qquad (12)$$

Let $R_i: \mathcal{X}_i \to \mathcal{X}$ be linear maps such that $\mathrm{Im}(R_i) = \mathcal{R}_i$. From (12) there are maps $Q_i: \mathcal{X}_i \to \mathcal{U}$ and $P_{i-1}: \mathcal{X}_i \to \mathcal{X}_{i-1}$, such that

$$\begin{aligned} R_0 &= BQ_0 \\ R_i &= AR_{i-1}P_{i-1} + BQ_i; \qquad i \in [1, 2, \ldots, \sigma-1] \end{aligned} \qquad (13)$$

Using (11), the condition of the theorem can be rewritten as

$$\mathcal{V}_0^* + \mathcal{B} + A(\mathcal{B}_{\sigma-1} \cap \mathrm{Ker}\, C) \supset \mathcal{E}$$

By Lemma 2, there are linear maps $Q_\sigma : \mathcal{W} \to \mathcal{U}$ and $P_{\sigma-1} : \mathcal{W} \to \mathcal{X}_{\sigma-1}$ such that

$$\mathrm{Im}\left(BQ_\sigma + AR_{\sigma-1}P_{\sigma-1} + E\right) \subset \mathcal{V}_0^*$$

Set

$$\begin{aligned} S_\sigma &= BQ_\sigma + AR_{\sigma-1}P_{\sigma-1} + E \\ S_i &= R_i P_i P_{i+1} \dots P_{\sigma-1} \qquad\qquad i \in [0, 1, \dots, \sigma-1] \end{aligned} \tag{14a}$$

and

$$\begin{aligned} K_\sigma &= Q_\sigma \\ K_i &= Q_i P_i P_{i+1} \dots P_{\sigma-1} \qquad\qquad i \in [0, 1, 2, \dots, \sigma-1] \end{aligned} \tag{14b}$$

It then follows from (13) and (14) that

$$\begin{aligned} S_0 &= BK_0 \\ S_i &= AS_{i-1} + BK_i \\ S_\sigma &= AS_{\sigma-1} + BK_\sigma + E \end{aligned} \tag{15}$$

Take $K(s) = K_\sigma + K_{\sigma-1}s + \dots + K_0 s^\sigma$. Since $\mathcal{S}_\sigma \subset \mathcal{V}_0^*$ and $\mathcal{S}_i \subset \mathcal{R}_i = \mathcal{B}_i \cap \mathrm{Ker}\, C$, $i \in [0, 1, 2, \dots, \sigma-1]$, it follows that

$$\begin{aligned} \{A|BK(s) + E\} &= \mathcal{S}_0 + \mathcal{S}_1 + \dots + \mathcal{S}_{\sigma-1} + \langle A|\mathcal{S}_\sigma\rangle \subset \\ &\subset \mathrm{Ker}\, C + \langle A|\mathcal{V}_0\rangle = \mathrm{Ker}\, C \end{aligned}$$

and K(s) solves the problem.

(Only if) Assume u = K(s)w solves the problem. Then

$$\{A|BK(s) + E\} = S_0 + S_1 + \dots + S_{\sigma-1} + \langle A|\mathrm{Im}(S_\sigma + E)\rangle \subset \mathrm{Ker}\, C \tag{16}$$

where

$$\begin{aligned} S_0 &= BK_0 \\ S_i &= AS_{i-1} + BK_i \qquad i \in [1, 2, \dots, \sigma] \end{aligned} \tag{17}$$

Set $V = \langle A|\mathrm{Im}(S_\sigma + E)\rangle$. Then $V \in I(A,0,\mathrm{Ker}C)$, so $V \subset V_0^*$. Moreover, $\mathrm{Im}(S_\sigma + E) \subset$ $\subset V$, which by Lemma 2 implies $S_\sigma + V \supset E$. Hence

$$S_\sigma + V_0^* \supset E$$

Moreover, by (16) and (17)

$$S_0 \subset B \cap \mathrm{Ker}\, C \subset B_0$$

and by induction

$$\begin{aligned} S_i &\subset AS_{i-1} + B = A(S_{i-1} \cap \mathrm{Ker}\, C) + B \\ &\subset A(B_{i-1} \cap \mathrm{Ker}\, C) + B = B_i \end{aligned}$$

Hence, $B_\sigma \supset S_\sigma$ and thus $V_0^* + B_\sigma \supset E$. □

Corollary 1. There exists a solution K(s) (with unfixed degree) to the non-dynamic feedforward problem if and only if $B_{n-m} + V_0^* \supset E$ where $n = \dim(X)$ and $m = \dim(U)$.

Proof. Just note that $\mathcal{B}_0 \subset \mathcal{B}_1 \subset \mathcal{B}_2 \subset \ldots$ is nondecreasing sequence of linear subspaces contained in $\mathcal{X}$ and containing $\mathcal{B}$ and must thus converge in at most dim($\mathcal{X}$) - dim($\mathcal{B}$) = n - m steps. The results then follow from Theorem 1. □

Dynamic Feedforward Control.

In the sequel it will be assumed that the system (1) is observable. If this is not the case, the results below can with no loss of generality be applied to the observable subsystem, i.e. the original system modulus the unobservable subspace

$$\mathcal{N}_C = \bigcap_{i=1}^{n} \mathrm{Ker}(CA^{i-1})$$

The admissible control is of the form (7) where in this case K(s) is an arbitrary matrix of rational functions in s. Specificly, K(s) is permitted to be nonproper. The matrix K(s) shall be chosen so that the transfer function from the disturbance w to the output y is zero, i.e. so that (8) holds.

Following [12], any system of the form (7) has a differential representation of the following form

$$\begin{aligned} sx_a &= A_a x_a + N(s)w \qquad\qquad s = d/dt \\ u &= F_a x_a + M(s)w \end{aligned} \tag{18}$$

where $x_a \in \mathcal{X}_a$ and N(s) and M(s) are polynomial matrices in s. This observation enables us to reformulate the problem as an ordinary dynamic extension problem along the lines of [10] using the results of section 2.

Extend the original system (1) by a set of integrators

$$\dot{x}_a = B_a u_a$$

where $x_a \in \mathcal{X}_a$ and $u_a \in \mathcal{U}_a$. It may be assumed that $\mathcal{X}_a \approx \mathcal{U}_a$. Let $n_a = \dim(\mathcal{X}_a)$. The extended system has state space $\mathcal{X}_e = \mathcal{X} \oplus \mathcal{X}_a$, input spaces $\mathcal{U}_e = \mathcal{U} \oplus \mathcal{U}_a$ and $\mathcal{W}_e = \mathcal{W}$ and output space $\mathcal{Y}_e = \mathcal{Y}$, and is described by the following system matrices

$$A_e = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \qquad B_e = \begin{pmatrix} B & 0 \\ 0 & B_a \end{pmatrix} \qquad E_e = \begin{pmatrix} E \\ 0 \end{pmatrix}$$
$$C_e = [C \quad 0] \tag{19}$$

In the extended system, the control (18) is of the form

$$u_e = F_e x_e + N_e(s)w$$

where $N_e(s)$ is a polynomial matrix in s and Ker $F_e \supset \mathcal{X}$. It is also easily checked that any control of this form can be represented in the form (18).

Using Lemma 1, we are then able to formulate the <u>dynamic feedforward problem</u>: Find a dimension n_a, a map $F_e:\mathcal{X}_e \to \mathcal{U}_e$ and a polynomial matrix $N_e(s)$ such that

(i) $\quad \mathrm{Ker}\, F_e \supset X$ (20a)

(ii) $\quad \{A_e + B_eF_e | B_eN_e(s) + E_e\} \subset \mathrm{Ker}\, C_e$ (20b)

The solution to this problem is given in a form in which the attainable degree of $N_e(s)$, i.e. the number of pure differentiators in the controller, is minimized.

Theorem 2. <u>There exists a solution to the dynamic feedforward problem such that</u> $\deg(N_e(s)) \leq \sigma$ <u>if and only if</u> $V^* + B_\sigma \supset E$ <u>where</u> B_σ <u>is given by</u> (11), $V^* = \sup I(A,B,\mathrm{Ker}\, C)$ <u>and</u> $E = \mathrm{Im}(E)$. <u>Moreover, the order</u> n_a <u>of dynamic extension required is bounded by</u> $n_a \leq \dim(V^*)$.

Lemma 3. <u>Assume that</u> S_i, $i \in [0, 1, 2, \ldots, \sigma]$, <u>and</u> N_i, $i \in [0, 1, 2, \ldots, \sigma]$, <u>are linear maps satisfying</u> $S_i = AS_{i-1} + BN_i$, $S_0 = BN_0$, <u>for</u> $i \in [0, 1, \ldots, \sigma]$. <u>Let</u> $S(s) = S_{\sigma-1} + S_{\sigma-2}s + \ldots + S_0s^{\sigma-1}$ <u>and</u> $N(s) = N_\sigma + N_{\sigma-1}s + \ldots + N_0s^\sigma$. <u>Then</u> $S(s)$ <u>and</u> $N(s)$ <u>satisfy the identity</u> $(A-s)\,S(s) + BN(s) = S_\sigma$.

Proof. Just compute the powers of s in $(A-s)S(s) + BN(s)$. □

Proof of Theorem 2. (if) Let $F \in \underline{F}(V^*)$. Then $V^* \in I(A+BF,0,\mathrm{Ker}\, C)$ and Theorem 1 can be used to find an $N(s)$ such that $\{A + BF | BN(s) + E\} \subset \mathrm{Ker}\, C$. The control is

$$u = Fx + N(s)w \qquad s = d/dt \tag{21}$$

and the controlled system

$$\begin{aligned} sx &= (A+BF)x + \big(BN(s) + E\big)w \\ y &= Cx \end{aligned} \tag{22}$$

In the construction of N(s) in the proof of Theorem 1, consider the maps N_i (replacing K_i) and S_i, $i \in [0, 1, 2, \ldots, \sigma]$, defined by (14a) and (14b) (where it is assumed that A is replaced by A + BF). They satisfy (15) with A replaced by A + BF. If S(s) is defined as

$$S(s) = S_{\sigma-1} + S_{\sigma-2}s + \ldots + S_0 s^{\sigma-1} \tag{23}$$

it follows from Lemma 3 that

$$(A+BF-s)S(s) + \big(BN(s) + E\big) = S_\sigma \tag{24}$$

Moreover, since $S_i \subset \text{Ker } C$ for all i

$$CS(s) = 0 \tag{25}$$

Take in (22)

$$z = x - S(s)w \tag{26}$$

Then from (24)

$$\begin{aligned} (S-A-BF)z &= (s-A-BF)x - (s-A-BF)S(s)w \\ &= \big(BN(s) + E\big)w + S_\sigma w - \big(BN(s) + E\big)w \\ &= S_\sigma w \end{aligned}$$

and from (25)

$$y = Cx = Cz$$

Hence, the controlled system is described by a state equation

$$\dot{z} = (A+BF)z + S_\sigma w$$
$$y = Cz$$

and

$$u = Fz + \left(N(s) - FS(s)\right)w$$

Consider the state transformation

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = Tz \qquad (27)$$

which describes the system in a basis compatible with $X = \hat{X} \oplus V^*$ for some extension space $\hat{X}$. Since $S_\sigma \subset V^*$ we have

$$\begin{aligned} \dot{q}_1 &= \bar{A}_{11}q_1 \\ \dot{q}_2 &= \bar{A}_{21}q_1 + \bar{A}_{22}q_2 + \bar{S}_\sigma w \\ y &= \bar{C}_1 q_1 \\ u &= \bar{F}_2 q_1 + \bar{F}_2 q_2 + \bar{N}_2(s)w \end{aligned} \qquad (28)$$

i.e. the control is generated by

$$\dot{q}_2 = \bar{A}_{22}q_2 + \bar{S}_\sigma w$$
$$u = \bar{F}_2 q_2 + \bar{N}_2(s)w \tag{29}$$

Take a <u>copy</u> as the dynamic compensator, i.e.

$$\dot{x}_a = \bar{A}_{22}x_a + \bar{S}_\sigma w$$
$$u = \bar{F}_2 x_a + \bar{N}_2(s)w \tag{30}$$

since this control is <u>identical</u> (for zero initial states) to the control (29) which produces zero output for all $w(\cdot)$, it follows that (30) is a solution to the problem. Moreover, in this case $n_a = \dim(\mathcal{V}^*)$.

(only if) We need

<u>Lemma 4</u>. <u>Assume that</u> $(\mathcal{V}_1+\mathcal{V}_2) \cap \mathcal{X}_a = 0$. <u>Then</u> $(\mathcal{V}_1+\mathcal{X}_a) \cap (\mathcal{V}_2+\mathcal{X}_a) = \mathcal{V}_1 \cap \mathcal{V}_2 + \mathcal{X}_a$.

<u>Proof</u>. Obviously $\mathcal{S} = (\mathcal{V}_1+\mathcal{X}_a) \cap (\mathcal{V}_2+\mathcal{X}_a) \supset \mathcal{V}_1 \cap \mathcal{V}_2 + \mathcal{V}_1 \cap \mathcal{X}_a + \mathcal{V}_2 \cap \mathcal{X}_a + \mathcal{X}_a \cap \cap \mathcal{X}_a = \mathcal{V}_1 \cap \mathcal{V}_2 + \mathcal{X}_a$. Moreover, let $x \in \mathcal{S}$. Then $x = x_1 + x_{a1} = x_2 + x_{a2}$, where $x_1 \in \mathcal{V}_1$, $x_2 \in \mathcal{V}_2$, $x_{ai} \in \mathcal{X}_a$. Then $x_1 - x_2 = x_{a2} - x_{a1} \in \mathcal{X}_a$, which implies that $x_1 = x_2$ since $(\mathcal{V}_1+\mathcal{V}_2) \cap \mathcal{X}_a = 0$. Hence, $x = x_1 + x_{a1} \in \mathcal{V}_1 \cap \mathcal{V}_2 + \mathcal{X}_a$ so $\mathcal{S} \subset \subset \mathcal{V}_1 \cap \mathcal{V}_2 + \mathcal{X}_a$. Thus $\mathcal{S} = \mathcal{V}_1 \cap \mathcal{V}_2 + \mathcal{X}_a$. □

Let F_e, $N_e(s)$ be a solution where

$$N_e(s) = N_{e,\sigma} + N_{e,\sigma-1}s + \dots + N_{e,0}s^\sigma$$

It follows from (20) and the definition of $\{A_e + B_eF_e|B_eN_e(s) + E_e\}$ that

$$\langle A_e + B_eF_e|\mathrm{Im}(S_{e,\sigma}+E_e)\rangle \subseteq \mathrm{Ker}\, C_e \tag{31}$$

$$\mathcal{S}_{e,i} \subset \mathrm{Ker}\, C_e \qquad i \in [0, 1, \ldots, \sigma-1]$$

where

$$S_{e,0} = B_eN_{e,0}$$
$$S_{e,i+1} = (A_e + B_eF_e)S_{e,i} + B_eN_{e,i+1} \tag{32}$$

Then

$$\mathcal{S}_{e,0} \subset \mathcal{B}_e = \mathcal{B} \oplus \mathcal{X}_a = \mathcal{B}_0 \oplus \mathcal{X}_a$$

where $\mathcal{B}_i$ is defined by (12). By induction

$$\mathcal{S}_{e,i+1} \subset A_e\mathcal{S}_{e,i} + \mathcal{B}_e \tag{33a}$$
$$= A_e(\mathcal{S}_{e,i} \cap \mathrm{Ker}\, C_e) + \mathcal{B}_e \tag{33b}$$
$$\subset A_e\big((\mathcal{B}_i \oplus \mathcal{X}_a) \cap (\mathrm{Ker}\, C \oplus \mathcal{X}_a)\big) + \mathcal{B}_e \tag{33c}$$
$$= A_e\big((\mathcal{B}_i \cap \mathrm{Ker}\, C) + \mathcal{X}_a\big) + \mathcal{B} + \mathcal{X}_a \tag{33d}$$
$$= \big(A(\mathcal{B}_i \cap \mathrm{Ker}\, C) + \mathcal{B}\big) + \mathcal{X}_a \tag{33e}$$
$$= \mathcal{B}_{i+1} \oplus \mathcal{X}_a$$

where (33a) follows from (32), (33b) from (31), (33c) by induction, (33d) from Lemma 4 and (33e) from the fact that $A_e\mathcal{X}_a = 0$ and $A_e|\mathcal{X} = A$. Hence

$$\mathcal{S}_{e,\sigma} \subset \mathcal{B}_\sigma \oplus \mathcal{X}_a \subset \mathcal{B}_\sigma + \mathcal{B}_e$$

which implies that there are maps R and Q such that

$$S_{e,\sigma} = B_\sigma R + B_e Q$$

From (31)

$$\mathcal{R}_e = \langle A_e + B_e F_e | \mathrm{Im}(B_e Q + B_\sigma R + E_e) \rangle \subset \mathrm{Ker}\, C_e \tag{34}$$

Now, let $P: \mathcal{X} \oplus \mathcal{X}_a \to \mathcal{X}$ be the projection along $\mathcal{X}_a$. It is easily verified that

$$PB_e = BS; \qquad CP = C_e; \qquad PE_e = E; \qquad A_e P = PA \tag{35}$$

for some linear map S. Note that $\mathcal{R}_e$ is (A_e, B_e)-invariant. Using (35) it follows that $\mathcal{R} \triangleq P\mathcal{R}_e$ is (A,B)-invariant. Moreover, $C\mathcal{R} = CP\mathcal{R}_e = C_e\mathcal{R}_e = 0$, and therefore $\mathcal{R} \in \mathfrak{I}(A,B,\mathrm{Ker}\, C)$. Moreover, by (34)

$$\mathcal{R}_e \supset \mathrm{Im}(B_e Q + B_\sigma R + E_e)$$

and taking images of both sides under P

$$\mathcal{R} \supset \mathrm{Im}(BSQ + B_\sigma R + E)$$

Using Lemma 2,

$$\mathcal{R} + \mathcal{B} + \mathcal{B}_\sigma \supset \mathcal{E}$$

Since $\mathcal{B}_\sigma \supset \mathcal{B}$ and $\mathcal{R} \subset \mathcal{V}^*$, the result follows. □

For completeness we also include

Corollary 2. There exists a solution F_e, $N_e(s)$ to the dynamic feedforward problem (with no fixed degree on $N_e(s)$) if and only if $\mathcal{B}_{n-m} + \mathcal{V}^* \supset \mathcal{E}$ where $\mathcal{V}^*$, $\mathcal{B}_i$ and $\mathcal{E}$ are the same as in the theorem and $n = \dim(X)$ and $m = \dim(\mathcal{U})$.

Proof. Apply the same arguments as in the proof of Corollary 1. □

Corollary 3. There exist a causal solution F_e, N_e (i.e. independent of s) to the dynamic feedforward problem if and only if $\mathcal{B} + \mathcal{V}^* \supset \mathcal{E}$ where $\mathcal{V}^*$ is the same as in the theorem.

Proof. Just put $\sigma = 0$ in Theorem 2. □

Stable Feedforward Controllers.

A realistic feedforward control must be achieved by a stable dynamical system. The natural stability condition in this case is to require that K(s) in (8) with representation (18) be stable, i.e. that the eigenvalues of A_a must all be contained in the open left halfplane of the complex plane. An equivalent way of expressing this property is to require that the unstable subspace of the extended system be contained in the original state space, i.e.

$$X_e^+(A_e+B_eF_e) \subset X \tag{36}$$

A feedforward controller having this property is said to be stable. The problem of overall stability is solved by the controller structure introduced in the next section.

Let

$$\mathcal{V}^* = \sup \mathcal{I}(A,B,\mathrm{Ker}\ C); \qquad \mathcal{R}^* = \sup \mathcal{C}(A,B,\mathrm{Ker}\ C)$$

Moreover, let $P:\mathcal{X} \to \mathcal{X}/\mathcal{R}^*$ be the canonical projection and let $\bar{A}_F$ be the map induced by $A + BF$ in $\mathcal{X}/\mathcal{R}^*$ where $F \in \underline{F}(\mathcal{V}^*)$, i.e. the unique map satisfying

$$P(A+BF) = \bar{A}_F P \tag{37}$$

Introduce the following subspaces

$$\bar{\mathcal{V}}^* = P\mathcal{V}^* \tag{38a}$$

$$\bar{\mathcal{B}}_\sigma = P\mathcal{B}_\sigma \tag{38b}$$

$$\bar{\mathcal{E}} = P\mathcal{E} \tag{38c}$$

$$\begin{aligned}\bar{\mathcal{V}}^- &= \bar{\mathcal{V}}^* \cap \bar{\mathcal{X}}^-(\bar{A}_F) \\ &= P\big(\mathcal{V}^* \cap \mathcal{X}^-(A+BF)\big)\end{aligned} \tag{38d}$$

The subspace $\bar{\mathcal{V}}^-$ and the map $\bar{A}_F|\bar{\mathcal{V}}^*$ are independent of the choice of $F \in \underline{F}(\mathcal{V}^*)$ cf [14].

With these notations, it is possible to state

Theorem 3. There exist a stable solution F_e, $N_e(s)$ with $\deg(N_e(s)) \leq \sigma$ to the dynamic noncausal feedforward problem if and only if $\bar{\mathcal{V}}_\sigma^- + \bar{\mathcal{B}}_\sigma \supset \bar{\mathcal{E}}$ where $\bar{\mathcal{V}}^-$, $\bar{\mathcal{B}}_\sigma$ and $\bar{\mathcal{E}}$ are defined by (38).

Lemma 3. Let $\mathcal{V}^* = \sup \mathcal{I}(A,B,\mathcal{D})$ and $\mathcal{R}^* = \sup \mathcal{C}(A,B,\mathcal{D})$. Then $\mathcal{X}^-(A+BF_1) \cap \mathcal{V}^* + \mathcal{R}^* = \mathcal{X}^-(A+BF_2) \cap \mathcal{V}^* + \mathcal{R}^*$ for all $F_1, F_2 \in \underline{F}(\mathcal{V}^*)$.

Proof. Follows from [14], Thm. 5.7. □

Proof. Let $F \in F(\mathcal{V}^*)$ be such that $(A+BF)|\mathcal{R}^*$ is stable. Such an F exists, see [14]. Consider the subspace $\mathcal{V} = \mathcal{X}^-(A+BF) \cap \mathcal{V}^*$. Then $\mathcal{V}$ is (A+BF)-invariant and $\mathcal{V} \supset \mathcal{R}^*$. Let $P:\mathcal{X} \to \mathcal{X}/\mathcal{R}^*$ be the canonical projection. From (38d), $P\mathcal{V} = \bar{\mathcal{V}}^-$. It then follows from the condition in the theorem that

$$P^{-1}(\bar{\mathcal{V}}^- + \bar{\mathcal{B}}_\sigma) \supset P^{-1}(\bar{\mathcal{E}})$$

$$P^{-1}P(\mathcal{V} + \mathcal{B}_\sigma) \supset P^{-1}P(\mathcal{E})$$

i.e. since Ker $P = \mathcal{R}^*$,

$$\mathcal{V} + \mathcal{B}_\sigma + \mathcal{R}^* \supset \mathcal{E} + \mathcal{R}^*$$

which implies $\mathcal{V} + \mathcal{B}_\sigma \supset \mathcal{E}$ since $\mathcal{R}^* \subset \mathcal{V}$. We may then proceed as in the proof of Theorem 2, replacing $\mathcal{V}^*$ by $\mathcal{V}$. The controller then has the form (30), where in this case $\bar{A}_{22} = (A+BF)|\mathcal{V}$, which is stable by construction.

(Only if) We may use the same procedure as in the necessity part of the proof of Theorem 2 to show that (34) holds.

Let R_e be as in (34) and introduce $W \triangleq X_e^+(A_e+B_eF_e) \cap R_e$. Also let $P: X \oplus X_a \rightarrow X$ be the projection such that $P(x+x_a) = x$ for all $x \in X$, $x_a \in X_a$. This projection satisfies (35). **Using** (20a) , there follows $A_e W \subset W$. Taking image of both sides under P and using (35) yields $APW \subset PW$. Also $0 = C_e W = CPW$ and therefore $PW = 0$ since (A,C) is an observable pair. Hence, $W = 0$ since $W \subset X$ and P is the identity on X. There follows

$$R_e \subset X_e^-(A_e+B_eF_e)$$

i.e.

$$R_e \subset X_e^-(A_e+B_eF_e) \cap R_e$$

Now, $R_e \in I(A_e,B_e,\text{Ker } C_e)$ and take $\hat{F}_e \in \underline{F}(V_e^*)$ such that $\hat{F}_e|R_e = F_e|R_e$ where $V_e^* = \sup I(A_e,B_e,\text{Ker } C_e)$. Then

$$R_e \subset X_e^-(A_e+B_e\hat{F}_e) \cap R_e \subset X_e^-(A_e+B_e\hat{F}_e) \cap V_e^*$$

Take $\tilde{F}_e = TFP$ where $F \in \underline{F}(V^*)$ and T is such that $B_eT = B$. It is easily verified that $\tilde{F}_e \in \underline{F}(V_e^*)$. Using Lemma 3

$$R_e \subset X_e^-(A_e+B_e\hat{F}_e) \cap V_e^* + R_e^*$$

$$= X_e^-(A_e+BFP) \cap V_e^* + R_e^* \qquad (39a)$$

where $R_e^* = \sup C(A_e, B_e, \mathrm{Ker}\, C_e)$. Now, $P(A_e+BFP) = (A+BF)P$ using (35). Taking image of both sides of (39) under P and using the fact that $PV_e^* = V^*$ and $PR_e^* = R^*$, cf. [14]. Lemma 10.6, we have

$$PR_e \subset X^-(A+BF) \cap V^* + R^*$$

Since $R_e \supset \mathrm{Im}(B_eQ+B_\sigma R+E_e)$, we have using (35)

$$\mathrm{Im}(BSQ+B_\sigma R+E) \subset X^-(A+BF) \cap V^* + R^*$$

Take image of both sides under P_1 where $P_1: X \to X/R^*$ is the canonical projection. Then

$$\mathrm{Im}(\bar{B}SQ+\bar{B}_\sigma R+\bar{E}) \subset \bar{V}^-$$

and using Lemma 2

$$\bar{B}_\sigma + \bar{B} + \bar{V}^- \supset \bar{E}$$

The result follows since $B_\sigma \supset B$. □

For completeness we also include

<u>Corollary 4</u>. <u>There exists a stable solution</u> F_e, $N_e(s)$ <u>(with unfixed degree) to the dynamic feedforward problem if and only if</u> $\bar{V}^- + \bar{B}_{n-m} \supset \bar{E}$ <u>where</u> $n = \dim(X)$ <u>and</u> $m = \dim(U)$.

Corollary 5. There exists a stable and causal solution F_e, N_e (i.e. independent of s) to the dynamic feedforward problem if and only if $\bar{V}^- + \bar{B} \supset E$.

Remark. The construction of the controller follows essentially the procedure outlined in the sufficiency part of Theorem 2 (with some slight modifications to take care of the internal stability requirement). In fact, the controller is constructed by means of a sequence of operations of linear algebra with real numbers. It is well known and fairly straightforward to construct computational algorithms for such operations, cf. e.g. [2].

Remark. A feedforward control with a minimal amount of differentiations is straightforwardly obtained by successively computing B_i from (11) and taking the first i for which the condition of Theorem 3 (Theorem 2) holds. This integer is then the minimal number.

Remark. The subspace B_{n-m} used in Corollaries 1, 2 and 4 is in fact the dual of V^*, i.e. the minimal controllable subspace obtainable by output injection transformation (A,B,C) → (A+KC,B,C). This follows e.g. by taking formally the orthogonal complements of (11) which shows that $B_{n-m}^{\perp} = \sup I(A^T, \mathrm{Ker}\ B^T, C^T)$. The output injection transformation was introduced in [9].

It is possible, using the results above, to give conditions which ensure solvability for all linear maps $E: W \to X$ irrespective of the dimension of W. This is, in fact, a strong result since it shows that the class of systems for which a feedforward control exists is in a sense very large.

For this purpose, a system (A,B,C) is said to be right invertible if its transfer function $G(s) = C(s-A)^{-1}B$ is right invertible over the field of rational functions. The transmission zeros of (A,B,C) [1, 9] are the zeros of the characteristic polynomial of

$$\bar{A}_F|\bar{V}^* \tag{39b}$$

where $\bar{A}_F$ and $\bar{V}^*$ are defined by (37). Moreover, a system (A,B,C) is said to be minimum phase if all its transmission zeros are in the open left half-plane of $\mathbb{C}$.

With these notations, we can state

Corollary 6. Consider system (1). If the system (A,B,C) is right invertible and minimum phase, then to every linear map $E: W \to X$ there exists an internally stable feedforward controller.

To prove this corollary we need

Lemma 6. The system (A,B,C) is right invertible if and only if $R_d^* = 0$, where $R_d^* = \sup C(A^T, \mathrm{Ker}\, B^T, C^T)$ (Proof see [10].) □

Lemma 7. Let $V^* = \sup I(A,B,\mathrm{Ker}\, C)$, $V_d^* = \sup I(A^T, C^T, \mathrm{Ker}\, B^T)$ and $R_d^* = \sup C \cdot (A^T, C^T, \mathrm{Ker}\, B^T)$. Then $R_d^* = V_d^* \cap (V^*)^{\perp}$. (Proof see [1, 9].)

Proof of Corollary 6. Taking orthogonal complements of (12) we have

$$\mathcal{B}_0^{\perp} = \text{Ker } B^T \; ; \qquad \mathcal{B}_{i+1}^{\perp} = \text{Ker } B^T \cap (A^T)^{-1}\left(\mathcal{B}_i^{\perp} + \text{Im}(C^T)\right) \tag{40}$$

which shows that $\mathcal{B}_{n-m}^{\perp} = \mathcal{V}_d^*$ where $\mathcal{V}_d^*$ is defined as in Lemma 7. From Lemma 6, $\mathcal{R}_d^* = 0$, and from Lemma 7 by taking orthonogal complements

$$\mathcal{X} = \mathcal{V}^* + (\mathcal{V}_d^*)^{\perp} = \mathcal{V}^* + \mathcal{B}_{n-m} \tag{41}$$

Since the system is minimum phase, it follows that $P\mathcal{V}^* = \bar{\mathcal{V}}^* = \bar{\mathcal{V}}^-$ where P and the subspaces are defined as in (36). Hence from (41),

$$P(\mathcal{V}^* + \mathcal{B}_{n-m}) = \bar{\mathcal{V}}^- + \bar{\mathcal{B}}_{n-m} = P\mathcal{X} \supset P\mathcal{E}$$

where $\mathcal{E}$ = Im E. Moreover, this holds for all linear maps E. The result then follows from Corollary 4. □

Corollary 7. Consider the system (1). If the system (A,B,C) is minimum phase and the linear maps B and C are such that rank(CB) = q where q = = dim($\mathcal{Y}$) then to every linear map $E:\mathcal{W} \to \mathcal{X}$ there exists a stable causal feedforward controller.

Proof. From rank(CB) = q it follows that Ker C + $\mathcal{B}$ = $\mathcal{X}$, i.e. $\mathcal{V}^*$ = Ker C and $\mathcal{V}^* + \mathcal{B} = \mathcal{X}$. Since the system is minimum phase we also have $P\mathcal{V}^* = \bar{\mathcal{V}}^-$ where $P:\mathcal{X} \to \mathcal{X}/\mathcal{R}^*$ is the canonical projection. Thus $\bar{\mathcal{V}}^- + \bar{\mathcal{B}} = P\mathcal{X} \supset P\mathcal{E}$ for all E. The result follows from Theorem 3. □

4. FEEDBACK STRUCTURES

A pure feedforward solution is sensitive with respect to inaccuracies in the model and noise effects which are not included in the model. This means that a feedforward compensator should be supplemented by some feedback action. This can be done using the observation that the feedforward controller actually contains a copy of a suitable subsystem of the original system, cf. the sufficiency part of the proof of Theorem 2 (3).

More precisely, write the control as

$$u = u_r + u_f \tag{42}$$

where u_f is a feedback control to be chosen and u_r is a feedforward compensator constructed as in Theorem 2, eq. (30). Using (1) and (30), the controlled system becomes

$$\begin{aligned}\dot{x} &= Ax + Bu_r + Bu_f + Ew\\ \dot{x}_a &= \bar{A}_{22}x_a + \tilde{S}_o w\\ u_r &= \bar{F}_2 x_a + \bar{N}_2(s)w\end{aligned} \tag{43}$$

Denote the nominal response of (1) with feedforward control u_r and $u_f = 0$ by x_r, i.e.

$$\dot{x}_r = Ax_r + Bu_r + Ew \tag{44}$$

Since the feedforward compensator is chosen as a copy of the controlled plant, cf. (28) and (30), there is a simple relationship between x_r, x_a

and w:

$$x_r = T^{-1}\left(\begin{pmatrix} 0 \\ \vdots \\ x_a \end{pmatrix} + S(s)w\right) \tag{45}$$

If we perform the state transformation $x \rightarrow \Delta x = x - x_r$, where x_r is given by (45), the controlled system (43) becomes

$$\Delta\dot{x} = A\Delta x + Bu_f$$
$$\dot{x}_a = A_{22}x_a + \bar{S}_\sigma w$$
$$y = C\Delta x$$

From this we see that there is no response in y for disturbance inputs w (zero initial state) if the feedback is chosen as

$$u_f = R(\Delta x)$$

where R is any dynamic compensator (including nonlinear) such that $R(0) = 0$.

Let us summarize this into

Theorem 4. Assume that the feedforward compensator is constructed as in the proof of Theorem 2 (3), i.e.

$$\dot{x}_a = \bar{A}_{22}x_a + \bar{S}_\sigma w$$
$$u_r = \bar{F}_2 x_a + \bar{N}_2(s)w$$

Then the control

$$u = u_r + R(x-x_r)$$

$$x_r = T^{-1}\left(\begin{pmatrix}0\\x_a\end{pmatrix} + S(s)w\right)$$

where T and S(s) are given by (27) and (23) and R(·) is any dynamic operator (including nonlinear) such that R(0) = 0, is also a solution to the feedforward problem. □

The feedback part in (43) does not become active until the nominal value x_r deviates from the actual value x on the state due to some inadequateness in the system description. Moreover, the feedback part can be synthesized separately, using any standard technique.

From the "separation property" it also follows that the stability of the overall system (i.e. including both original system and compensator) is determined by the spectra of (assuming R is a linear state feedback)

(i) $\bar{A}_{22}$ i.e. the dynamics of the compensator

(ii) A + BR

Hence

Corollary 8. *There exists a feedforward controller which achieves overall stability if and only if*

(i) *There exists a stable feedforward controller*

(ii) (A,B) *is a stabilizable pair.*

5. GENERIC SOLVABILITY

To be applicable, the class of systems for which a feedforward controller exists should in some sense be large. The Corollaries 6 and 7 showed that there is a broad class of systems for which feedforward control can be applied. In this section we will perform a more detailed analysis on this topic using some elementary topology, along the lines of [14].

Consider the system (1) and regard $p = (A,B,C,E)$ as a data point in R^N where $N = n^2 + nm + nq + nr$. Following Wonham [14], the feedforward problem is generically solvable if it is solvable at all data points in the complement of a proper algebraic variety V in R^N.

We will distinguish between four cases corresponding to different requirements on the control $u = K(s)w$ in (7):

(a) K(s) is proper, i.e. the causal feedforward problem (CFP)

(b) K(s) is nonproper, i.e. the noncausal feedforward problem (NFP)

(c) K(s) is proper and stable, i.e. the causal feedforward problem with stability (CFPS)

(d) K(s) is nonproper and stable, i.e. the noncausal feedforward problem with stability (NFPS)

All these cases are treated in Theorem 2 and 3 with corollaries. Note that stability of the compensator is the concept of interest since overall stability can always be achieved using "separation property" of section 3, provided (A,B) is a stabilizable pair, cf. Corollary 8.

Theorem 5. (Generic solvability) Let m and q be the numbers of inputs and outputs respectively to the system (A,B,C). Then

(a) CFP is generically solvable if and only if $m \geq q$.

(b) NFP is generically solvable if and only if $m \geq q$.

(c) CFP$\mathcal{S}$ is generically solvable if and only if at least one of the following conditions holds

(i) $m > q$

(ii) $m = q$ and the system (A,B,C) is minimum phase

(d) NFP$\mathcal{S}$ is generically solvable if and only if at least one of (i) and (ii) holds.

Proof.

(a) If $m \geq q$, then $\text{Ker } C + \mathcal{B} = \mathcal{X}$ except for data points p such that $\text{rank}(CB) < q$, i.e. except for data points belonging to a proper algebraic variety V defined by

$$\det(CBB^TC^T) = 0 \tag{44}$$

Thus for all $p \in V^C$, $\text{Ker } C + \mathcal{B} = \mathcal{X} \supset A \text{ Ker } C$, i.e. $\mathcal{V}^* = \text{Ker } C$. Hence, $\mathcal{V}^* + \mathcal{B} = \mathcal{X} \supset \mathcal{E}$ for all $p \in V^C$. Conversely, if CFP is generically solvable it is solvable for all $p = (A,B,C,E) \in V^C$, where V is a proper algebraic variety. It then follows that $\mathcal{V}^* + \mathcal{B} = \mathcal{X} = \text{Ker } C + \mathcal{B}$ since otherwise we can find a $\tilde{p} = (A,B,C,\tilde{E})$ with E arbitrarily near E such that $\mathcal{X} \neq \mathcal{V}^* + \mathcal{B} \not\supset \tilde{\mathcal{E}}$ which means that any neighbourhood of p contains a point belonging to V which is a contradiction. Hence $\text{Ker } C + \mathcal{B} = \mathcal{X}$ which implies that $\dim(\text{Ker } C) + \dim(\mathcal{B}) = n - q + m \geq n$, i.e. $m \geq q$.

(b) From (a) we immediately conclude that $m \geq q$ implies generic solvability of NFP since solvability of NFP is implied by solvability of CFP. Conversely, if NFP is generically solvable, it is solvable for all $p = (A,B,C,E) \in V^C$ where V is a proper algebraic variety. Then $\mathcal{V}^* + \mathcal{B}_{n-m} = \mathcal{X}$ since otherwise we can find data points $\tilde{p} = (A,B,C,\tilde{E})$ with $\tilde{E}$ arbitrarily near E such that $\mathcal{X} \neq \mathcal{V}^* + \mathcal{B}_{n-m} \not\supset \tilde{\mathcal{E}}$. Hence since $(\mathcal{V}_d^*)^\perp = \mathcal{B}_{n-m}$, cf. the proof of Cor. 6, we have $\mathcal{V}^* + (\mathcal{V}_d^*)^\perp = \mathcal{X}$ for all $p \in V^C$. Thus from Lemma 7, $\mathcal{R}_d^* = \mathcal{V}_d^* \cap (\mathcal{V}^*)^\perp = \left(\mathcal{V}^* + (\mathcal{V}_d^*)^\perp\right)^\perp = \mathcal{X}^\perp = 0$, which implies by Lemma 6 that the system is right invertible. Since $G(s) = C(s-A)^{-1}B$ is a $q \times m$ matrix, it follows that $m \geq q$ for all $p \in V^C$.

(c) Assume first that (i) holds. If $m > q$ then $\mathcal{R}^* = \text{Ker } C$ except for points $p = (A,B,C,E) \in V_1$, some proper algebraic variety, cf. Wonham [3] Lemma 11.1. Moreover, $\mathcal{V}^* + \mathcal{B} = \mathcal{X}$ except for $p \in V_2$, where V_2 is a proper algebraic variety defined by (44). Hence $\mathcal{R}^* + \mathcal{B} = \mathcal{X}$ for all $p \in (V_1 \cup V_2)^C$. For all these points we have $P\mathcal{R}^* + P\mathcal{B} = \bar{\mathcal{B}} = \bar{\mathcal{X}} \supset P\mathcal{E}$, i.e. that CFPIS is solvable.

Assume now that (ii) holds. Then $\text{Ker } C + \mathcal{B} = \mathcal{X}$ for data points $p \in V_1^C$ where V_1 is defined by (44), and thus, since $m = q$, $\text{Ker } C \cap \mathcal{B} = 0$ which implies that $\text{rank}(CB) = q$.

It follows from Corollary 7 that CFPIS_o^σ is solvable for all $p \in V_1^C$. Conversely, if CFPIS_o^σ is generic with respect to some proper algebraic variety V, then using the same arguments in (a), $\mathcal{V}^* + \mathcal{B} = \text{Ker } C + \mathcal{B} = \mathcal{X}$ for all $p \in V^C$. Hence, $n - q + m \geq n$, i.e. $m \geq q$. Assume that $m = q$. Then $\mathcal{R}^* = 0$ by the same arguments as above. Assume that (A,B,C) is not minimum phase. Then since $P = I$, $\bar{\mathcal{V}}^- = \mathcal{X}^-(A+BF) \cap \mathcal{V}^* \neq \mathcal{V}^*$, $\bar{\mathcal{B}} = \mathcal{B}$, $\bar{\mathcal{E}} = \mathcal{E}$ and thus $\mathcal{X} \neq \bar{\mathcal{V}}^- + \bar{\mathcal{B}}$. This implies that we can find a point $p = (A,B,C,\tilde{E})$ with $\tilde{E}$ arbitrarily near E such that $\bar{\mathcal{V}}^{-} + \bar{\mathcal{B}} \not\supset \tilde{E}$, which is a contradiction. Hence, the system is minimum phase.

(d) It follows from (c) that (i) and (ii) imply generic solvability. Conversely, it follows from (b) that $m \geq q$. If $m = q$ we may use the same approach as in (c) to show that the system must be minimum phase. □

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