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Abstract: Linear filtering with unknown initial values is the dual of the linear quadratic control problem with fixed endpoint.

The Kalman filter would require infinite initial gain and finite gain would give infinite error covariance. It is here shown both for continuous and discrete time that an optimal estimate with finite error covariance can be obtained after a nonzero time interval giving starting values for the Kalman filter. The requirements on observability are also specified. The structure of the optimal solution is similar to the smoothing estimate.

Introduction: At the start up of a Kalman filter it is assumed that the statistics of the initial state are known. It is often unrealistic to rely upon that information. If no second order moment of the probability distribution of the initial state is assumed to exist, this paper gives the necessary modification for an optimal solution. It is usually suggested that the inverse of the covariance matrix should be used to start up the covariance equation, e.g. [5] , but this does not solve the problem with the filter. A finite gain will not give the correct start up, and the error covariance will never exist!

Statement of the problem: Regard the system

$$\begin{cases} dx = Axdt + dv & x(0) = x_o \\ dy = Cxdt + de & y(0) = 0 \end{cases}$$

where $Ex_o = 0$, and where v, e wiener processes with incremental covariance R_1dt, R_2dt, independent of each other and of x_o. Assume also that the probability distribution of the initial state is degenerated, so that there exists no finite second order moment in some linear subspace of the state space, while it is finite in the rest of the state space. Let the projections of x_o into the two subspaces be given by the projection matrix N_1

$$x_o = x_o^1 + x_o^2 = N_1x_o + (I - N_1)x_o$$

where x_o^1 has the finite covariance R_o^1.

Determine the best linear estimate $\hat{x}(t_1)$ of $x(t_1)$ based on y(t) during $[0, t_1]$.

First estimate linear combinations of $x(t_1)$ say $a^Tx(t_1)$ and let best mean minimizing the mean square error

$$J = E(a^Tx(t_1) - a^T\hat{x}(t_1))^2 \tag{1}$$

Admissible estimators are

$$a^T\hat{x}(t_1) = - \int_o^{t_1} u^T(t)dy(t)$$

The notation $a^T\hat{x}(t_1)$ will be justified later. There is really a $\hat{x}(t_1)$ such that $a^T\hat{x}(t_1)$ is the best estimate of $a^Tx(t_1)$ for all a.

<u>Solution by duality</u>: The problem will be solved using the duality with linear quadratic control problems [1,4].

Rewrite (1) using the vector $z(t)$ defined by

$$\begin{cases} -\dot{z} = A^T z + C^T u \\ z(t_1) = a \end{cases} \tag{2}$$

then

$$a^T x(t_1) - a^T \hat{x}(t_1) = z^T(t_o)x(t_o) + \int_o^{t_1} [z^T(t)dv(t) + u^T(t)de(t)]$$

In order to get a finite J, i.e. to have a finite covariance of $z^T(0)x(0)$, $z(0)$ has to be orthogonal to all x_o^2, requiring

$$Nz(0) = (I-N_1)^T z(0) = 0 \tag{3}$$

so that

$$J = z^T(0)R_o^1 z(0) + \int_o^{t_1} (z^T(t)R_1 z(t) + u^T(t)R_2 u(t))dt \tag{4}$$

J should now be minimized with respect to u under the constraints (2) and (3) giving, e.g. [2] or [3],

$$u(t) = -R_2^{-1}C\Pi(t)z(t) - R_2^{-1}C\psi(t,0)N^T(NM(0,t_1)N^T)^{-1}N\psi^T(t_1,0)a \tag{5}$$

where

$$\begin{cases} \dot{\Pi} = A\Pi + \Pi A^T + R_1 - \Pi C^T R_2^{-1} C\Pi \\ \Pi(0) = R_o^1 \end{cases} \tag{6}$$

and where the fundamental matrix ψ satisfies

$$\begin{cases} \frac{d}{dt}\,\psi(t,s) = (A-\Pi(t)C^TR_2^{-1}C)\psi(t,s) \\ \psi(s,s) = I \end{cases}$$

and

$$M(t,t_1) = \int_t^{t_1} \psi^T(s,t)C^TR_2^{-1}C\psi(s,t)ds \tag{7}$$

the observability Gramian.

By solving (2),(5) expressions for $z(t)$ and $u(t)$ are obtained

$$z(t) = \psi^T(t_1,t)a-\int_t^{t_1} \psi^T(s,t)C^TR_2^{-1}C\psi(s,0)dsN^T(NMN^T)^{-1}N\psi^T(t_1,0)a =$$

$$= \psi^T(t_1,t)a-M(t,t_1)\psi(t,0)N^T(NMN^T)^{-1}N\psi^T(t_1,0)a$$

and

$$u(t) = [-R_2^{-1}C\Pi(t)\psi^T(t_1,t)+R_2^{-1}C\Pi(t)M(t,t_1)\psi(t,0)N^T(NMN^T)N\psi^T(t_1,0)-$$

$$- R_2^{-1}C\psi(t,0)N^T(NMN^T)^{-1}N\psi^T(t_1,0)]a$$

Thus

$$x(t_1)= \int_0^{t_1} \{\psi(t_1,t)\Pi(t)-\psi(t_1,0)N^T(NM(0,t_1)N^T)^{-1}N\psi^T(t,0)M(t,t_1)\Pi(t) +$$

$$+ \psi(t_1,0)N^T(NM(0,t_1)N^T)^{-1}N\psi^T(t,0)\}C^TR_2^{-1}dy(t) \tag{8}$$

independent of a.

Note also that the minimal J is

$$J = a^T\Pi(t_1)a+a^T\psi(t_1,0)N^T(NM(0,t_1)N^T)^{-1}N\psi^T(t_1,0)a$$

The covariance of the estimation error, $\tilde{x}(t_1)$, is thus

$$P(t_1) = \Pi(t_1)+\psi(t_1,0)N^T(NM(0,t_1)N^T)^{-1}N\psi^T(t_1,0) = \tag{9}$$

$$= \Pi(t_1) + \Sigma(t_1)$$

Further differentiate (8) to obtain a recursive formula for $\hat{x}(t_1)$ and use

$$\frac{d}{dt_1}(NM(0,t_1)N^T)^{-1} = -(NMN^T)^{-1}\{N\psi(t_1,0)C^TR_2^{-1}C\psi^T(t_1,0)N^T\}(NMN)^{-1}$$

so that

$$d\hat{x}(t_1) = A\hat{x}(t_1)dt_1+\{\Pi(t_1)+\psi(t_1,0)N^T(NMN^T)^{-1}N\psi(t_1,0)\}C^TR_2^{-1}[dy(t_1)-C\hat{x}(t_1)dt_1] =$$

$$= A\hat{x}(t_1)dt_1+P(t_1)C^TR_2^{-1}\{dy(t_1)-C\hat{x}(t_1)dt_1\} \tag{10}$$

This is the usual Kalman Bucy filter.

However (10) could not be applied from t=0 since $\Sigma(t)$ and thus $P(t)$ are infinite for t=0.

It is possible to rewrite $\hat{x}(t_1)$ using $\hat{x}_\Pi(t_1)$:

$$\begin{cases} d\hat{x}_\Pi(t) = A\hat{x}(t)dt+\Pi(t)C^TR_2^{-1}(dy(t)-C\hat{x}_\Pi(t)dt) \\ \hat{x}_\Pi(0) = 0 \end{cases} \tag{11}$$

or

$$\hat{x}_\Pi(t_1) = \int_0^{t_1} \psi\,(t_1,t)\Pi(t)C^TR_2^{-1}dy(t)$$

then

$$\hat{x}(t_1)=\hat{x}_\Pi(t_1)+\psi(t_1,0)N^T(NMN^T)^{-1}N[\int_0^{t_1}\psi^T(t,0)C^TR_2^{-1}dy(t) -$$

$$-\int_0^{t_1}\psi^T(t,0)(\int_t^{t_1}\psi^T(s,t)C^TR_2^{-1}C\psi(s,t)\Pi(t)C^TR_2^{-1}dy(t))ds$$

$$= \hat{x}_\Pi(t_1)+\psi(t_1,0)N^T(NMN^T)^{-1}N[\int_0^{t_1}\psi^T(t,0)C^TR_2^{-1}dy(t) -$$

$$-\int_0^{t_1}\psi^T(s,0)C^TR_2^{-1}C[\int_0^{s}\psi(s,t)\Pi(t)C^TR_2^{-1}dy(t)]ds =$$

$$= \hat{x}_\Pi(t_1)+\psi(t_1,0)N^T(NMN^T)^{-1}\int_0^{t_1}\psi^T(t,0)C^TR_2^{-1}[dy(t)-C\hat{x}_\Pi(t)dt]$$

$$= \hat{x}_\Pi(t_1)+\psi(t_1,0)N^T(N\,M\,N^T)^{-1}\lambda(0) \tag{12}$$

where

$$\begin{cases} -d\lambda(t) = (A-\Pi C^TR_2^{-1})^T\lambda(t)dt + C^TR_2^{-1}[dy(t) - C\hat{x}_\Pi(t)dt] \\ \lambda(t_1) = 0 \end{cases} \tag{13}$$

Note the ressemblance with the smoothing estimate formulas e.g. [3]. Thus if x_o^2 is observable at $t=\tau$, i.e. $(NM(0,\tau)N^T)^{-1}$ exist, $\hat{x}(\tau)$ and $P(\tau)$ could be obtained using (11), (12), (13) and (9). What approximately happens as $\tau\to 0$ is demonstrated in Example 1.

<u>Discrete time results</u>: The discrete time case

$$\begin{cases} x(t+1) = \phi(t+1,t)x(t) + v(t) \\ y(t) = \theta x(t) + e(t) \end{cases}$$

with analog v, e and x(0), can be treated in the same way. The best one step predictor $\hat{x}(t_1|t_1-1)$ is

$$\hat{x}(t_1|t_1-1) = \hat{x}_\Pi(t_1|t_1-1)+\psi(t_1,t_o)N^T(NM(t_o,t_1)N^T)^{-1}N\lambda(t_o-1)$$

where

$$\begin{cases} \hat{x}_\Pi(t+1|t) = \phi(t+1,t)\hat{x}_\Pi(t|t-1)+K(t)[y(t)-\theta\hat{x}_\Pi(t|t-1)] \\ \hat{x}_\Pi(t_o|t_o-1) = 0 \end{cases}$$

$$K(t) = \phi(t+1,t)\Pi(t)\theta^T[\theta\Pi\theta^T+R_2]^{-1}$$

$$\psi(t+1,t) = \phi(t+1,t) - K(t)\theta(t)$$

$$\begin{cases} \Pi(t+1) = \phi(t+1,t)\Pi(t)\phi^T(t+1,t)+R_1-K(t)\theta\Pi(t)\phi^T(t+1,t) \\ \Pi(t_o) = R_o^1 \end{cases}$$

$$M(t_o,t_1) = \sum_{t=t_o}^{t_1-1} \psi^T(t,t_o)\theta^T(R_2+\theta\Pi(t)\theta^T)^{-1}\theta\psi(t,t_o)$$

$$\begin{cases} \lambda(t-1) = \psi^T(t+1,t)\lambda(t)+\theta^T[R_2+\theta\Pi\theta^T]^{-1}[y(t)-\theta\hat{x}_\Pi(t|t-1)] \\ \lambda(t_1-1) = 0 \end{cases}$$

The covariance of the error $\tilde{x}(t_1|t_1-1)$ is

$$P(t_1)= \Pi(t_1)+\Sigma(t_1) = \Pi(t_1)+\psi(t_1,t_o)N^T(NM(t_o,t_1)N^T)^{-1}N\psi^T(t_1,t_o)$$

which is finite provided that the part of the initial state that does not have any finite second moment, is observable for the interval $[t_o,t_1]$. This assures that the inverse exists. The proof is done in the same way using duality but is rather lengthy.

Example 1 (Continuous time)

$$\begin{cases} \dot{x} = -ax + v \\ y = x + e \end{cases} \qquad Ex_o = 0,\ Ex_o^2 = 1;\ R_2=1,\ R_1=r_1$$

$$\begin{cases} \dot{\Pi} = -2a\Pi + r_1 - \Pi^2 \\ \Pi(0) = 0 \end{cases}$$

$$\Pi(t) = [\Lambda_{21}(t,0)+\Lambda_{22}(t,0)\Pi(0)]/[\Lambda_{11}(t,0) + \Lambda_{12}(t,0)\Pi(0)]$$

$$\frac{d}{dt}\Lambda(t,0) = \begin{bmatrix} a & 1 \\ r_1 & -a \end{bmatrix}\Lambda(t,0),\ \Lambda(0,0) = I,\ \lambda = \sqrt{r_1+a^2}$$

$$\Lambda(t,0) = \begin{bmatrix} \cosh\lambda t + \frac{a}{\lambda}\,\mathrm{ainh}\,\lambda t & \frac{1}{\lambda}\sinh\lambda t \\ \frac{r_1}{\lambda}\sinh\lambda t & \cosh\lambda t - \frac{a}{\lambda}\sinh\lambda t \end{bmatrix}$$

$$\Pi(t) = r_1\tanh\lambda t/(\lambda+a\tanh\lambda t)$$

$$\Sigma(t) = \psi^2(t,0)/\int_0^t \psi^2(s,0)ds$$

$$\psi(t,0) = \phi(t,0) - \phi(t,0)\Pi(t)(\Pi(t)+1)^{-1}$$

For $t = \varepsilon \to 0$: $\psi(\varepsilon,0) \approx 1,\ M(0,\varepsilon) \approx \varepsilon,\ \Sigma(\varepsilon) \approx 1/\varepsilon$

$$\lambda(0) \approx \int_0^\varepsilon y(s)ds,\ \hat{x}_\Pi(\varepsilon) \approx 0$$

and thus

$$\begin{cases} \hat{x}(\varepsilon) = \frac{1}{\varepsilon}\int_0^\varepsilon y(s)ds \approx y(0) \\ P(\varepsilon) = 1/\varepsilon \end{cases}$$

This gives the starting values for

$$\dot{\hat{x}}(t) = -a\hat{x}(t) + P(t)(y(t)-\hat{x}(t))$$

when started from $t = \varepsilon$.

Example 2 (Discrete time)

$$x(t+1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(t) + v(t) \qquad N = I \quad R_1 = I \ R_2 = 1$$

$$y(t) = [1 \quad 0] \ x(t) + e(t)$$

$\Pi(0) = 0$, $K(0) = 0$, $\psi(1,0) = \phi$, $\Pi(1) = I$

$\hat{x}(1|0) = 0$

$$M(0,1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \lambda_1(-1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} y(0)$$

$$\hat{x}(1|0) = \begin{bmatrix} ? \\ 1 \end{bmatrix} y(0), \ P(1) = \begin{bmatrix} \infty & 0 \\ 0 & 2 \end{bmatrix}$$

$$K(1) = \frac{1}{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \psi(2,1) = \phi \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1/2 & 0 \end{bmatrix}$$

$$\hat{x}_{\Pi}(2|1) = \frac{1}{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix} y(1) \qquad \Pi(2) = \begin{bmatrix} 2 & 0 \\ 0 & 3/2 \end{bmatrix}$$

$$M(0,2) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{2}\phi^T \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \phi = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} > 0$$

$$\lambda_2(0) = \frac{1}{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} y(1)$$

$$\lambda_2(-1) = \phi^T \lambda_2(0) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} y(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} y(0) + \frac{1}{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix} y(1)$$

$$\hat{x}(2|1) = \frac{1}{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix} y(1) + \psi(2,1)\psi(1,0)M^{-1}(0,2)\lambda_2(-1) =$$

$$= \frac{1}{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix} y(1) + \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}^{-1} \lambda_2(-1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} y(0) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} y(1)$$

$$P(2) = \begin{bmatrix} 2 & 0 \\ 0 & 3/2 \end{bmatrix} + \phi \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \phi \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}^{-1} \phi^T \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \phi^T =$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 3/2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

Remark: The expression can be obtained also by starting with $R_o = cI$ and $c \to \infty$ but for large systems the expressions of $P(t_1)$ and $x(t_1)$ in c would be extremely complicated. Formel manipulations on computer would be necessary. Numerical solution for large parameters c would give large round off errors.

Conclusions: The optimal filters for systems with unknown initial values have been deduced both for continuous and discrete time. The error covariance becomes finite as soon as the unknown initial modes become observable. Using a formula similar to smoothing the estimate and its error covariance could then be obtained, and a usual Kalman filter started. The error covariance still does not contain the actual measurements.

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