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Stochastic Control of Critical Processes

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To my parents

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“Stufen

Wie jede Blüte welkt und jede Jugend
Dem Alter weicht, blüht jede Lebensstufe,
Blüht jede Weisheit auch und jede Tugend
Zu ihrer Zeit und darf nicht ewig dauern.
Es muß das Herz bei jedem Lebensrufe
Bereit zum Abschied sein und Neubeginne,
Um sich in Tapferkeit und ohne Trauern
In andre, neue Bindungen zu geben.
Und jedem Anfang wohnt ein Zauber inne,
Der uns beschützt und der uns hilft, zu leben.

Wir sollen heiter Raum um Raum durchschreiten,
An keinem wie an einer Heimat hängen,
Der Weltgeist will nicht fesseln uns und engen,
Er will uns Stuf' um Stufe heben, weiten.
Kaum sind wir heimisch einem Lebenskreise
Und traulich eingewohnt, so droht Erschlaffen,
Nur wer bereit zu Aufbruch ist und Reise,
Mag lähmender Gewöhnung sich entraffen.

Es wird vielleicht auch noch die Todesstunde
Uns neuen Räumen jung entgegensenden,
Des Lebens Ruf an uns wird niemals enden ...
Wohlan denn, Herz, nimm Abschied und gesunde!”

*Hermann Hesse**

* Das Glasperlenspiel, Suhrkamp Taschenbuch Verlag, Frankfurt am Main, 1972

Preface

My interest in control of critical processes and extreme values began in the autumn of 1990 as I was following a course on extremes in random processes given by Professor Georg Lindgren at the Department of Mathematical Statistics in Lund. Initial results were in the area of linear feedback control of Gaussian random processes. As was pointed out to me by Professor Torsten Söderström in Uppsala, the generic controller for critical processes should be nonlinear. This is indeed the case. Some work has been done in this area. However, it has been difficult to obtain general results for nonlinear controllers that are easy to implement in terms of numerical routines. Most of the work presented in this thesis will be on linear controllers.

The thesis is somewhat interdisciplinary and in the borderland of automatic control and mathematical statistics. It is primarily written for a reader with knowledge of automatic control at a graduate level, but I hope that the references will help any other reader with some mathematical background to read it.

Acknowledgements

This work has been carried out at the Department of Automatic Control, Lund Institute of Technology, Sweden. I would like to thank all my colleagues at the department. It is a great pleasure to work in the creative, friendly and highly stimulating atmosphere which they are all contributing to.

I am very happy to express my gratitude to my supervisors Per Hagander and Karl Johan Åström. Per Hagander has been my main supervisor for the last couple of years. He has shown a lot of tolerance and patience. I much admire his skillful supervision. Per has made my sloppy ideas and writing much clearer. The following words could have been said by him: "It has long been an axiom of mine that the little things are infinitely the most important."* Especially I would like to mention

* Sir Arthur Conan Doyle, The Adventures of Sherlock Holmes

his contributions to parts III and IV of the thesis. Karl Johan Åström was my main supervisor during the first years. Thanks to his brilliant lectures on adaptive control I started working at the department. When I got the idea of controlling extreme values, he was extremely encouraging. His great knowledge of people and broad overview on different research areas has been very helpful. I am especially grateful to him for letting me work also on other projects than my thesis-project. This has given me a broad background in automatic control.

I am very grateful to Lennart Andersson who did his M.Sc. project under my supervision. The results presented in the latter part of Chapter 3 are mainly due to him. I would also like to express my gratitude to Bo Bernhardsson for suggested improvements and stimulating discussions. He has been a valuable source of inspiration. I am very grateful to Mark Davis at Imperial College, London, for taking good care of me when I visited him. He suggested many interesting references related to stochastic control of extreme values. I am also indebted to Georg Lindgren for stimulating discussions, and for his interesting lectures on extremes in random processes. I would like to thank Lars Nielsen for being a great supervisor of my M.Sc. project, and for making me aware of the research on extreme values at the Department of Mathematical Statistics. I would also like to thank Leif Andersson and Anders Blomdell for maintaining excellent computer facilities at the department. Many thanks goes to Eva Dagnegård who has helped me with TEX-nicalities and the final layout of the thesis, to Britt-Marie Mårtensson for her nice figures drawn with great skill, and to Eva Schildt and Agneta Tuszynski for always being very helpful.

For financial support I would like to thank mainly Lund University, but also SAAB Scania AB, Landshövding Nils Hörjels Forskningsfond, The Swedish Institute, Kungliga Fysiografiska Sällskapet, Knut och Alice Wallenbers stiftelse, Hans Emil Hanssons forskningsfond, and Jacob Letterstedts stipendiefond. The latter sources have made it possible for me to visit international institutes and conferences. This has been highly stimulating.

Finally, I would like to thank my parents for their encouragement and support.

Lund, March 1995

Anders Hansson

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Introductory Part

Background and Motivation

MANY processes in industry are critical. They are often critical in the sense that they have a limiting level. This can be either physical or artificial. Examples of the former are such levels that cannot be exceeded without catastrophic consequences, e.g. explosion. One example on the latter is alarm levels, which if they are exceeded will initiate emergency shutdown or a change in operational conditions. Another example is quality levels, which if they are exceeded will cause unsatisfied customers. Common to the critical processes are that they enter their critical region abruptly as a signal exceeds a limiting level.

The distance between the limiting or critical level and the reference value is normally not small, since otherwise the number of exceedances of the level by the controlled signal would be intolerably high. However, there may be other control-objectives that make it undesirable or impossible to choose the distance large. An example of problems of this kind can be found in Borisson and Syding (1976), where the power of an ore crusher should be kept as high as possible but not exceed a certain level, in order that the overload protection does not cause shutdown. Another example is moisture control of a paper machine, where it is desired to keep the moisture content as high as possible without causing wet streaks, Åström (1970) pp. 188–209. Yet another example is power control of wind power plants, where the supervisory system initiates emergency shutdown if the generated power exceeds 140% of rated power, Mattsson (1984). Other examples can be found in sensor-based robotics and force control, Hansson and Nielsen (1991), and control of non-linear plants, where the stability may be state dependent, Shinskey (1967).

The main contributions of this thesis is the formulation of critical control problems in terms of the Minimum Upcrossing (MU) controller. The upcrossing criterion goes back to Rice (1939). It will be seen that this stochastic optimization problem can be solved in terms of a one-parametric optimization over Linear Quadratic Gaussian (LQG) control problems. Another contribution of the thesis is simple necessary and sufficient conditions for existence of solutions to LQG problems. These are used to derive necessary and sufficient conditions for existence of the MU controller.

The purpose of the Introductory Part is to give a background, motivation and introduction to the other parts. In Chapter 1 previous work by other authors in control of critical processes will be discussed. Both deterministic and stochastic setups will be reviewed. In Chapter 2 different nonlinear optimal stochastic control problem formulations will be given. For some examples explicit solutions will be obtained. In general, however, the resort seems to be numerical computations. Then in Chapter 3 the specific problems considered in parts I–IV of the thesis will be described. They will all be partial information linear time invariant optimal stochastic control problems. The risk criterion is the probability that the largest value of the controlled process during a fixed time interval exceeds the critical level. The Mean Time Between Failures (MTBF) criterion is the expected value of the time between two consecutive upcrossings of the critical level. It will be seen that the upcrossing criterion can be used to approximate the two former criteria, and it seems possible to obtain explicit solutions only for the latter criterion. In Chapter 4 some examples will be investigated. Finally, in Chapter 5 an outline of the remaining parts of the thesis will be given.

CHAPTER 1

PREVIOUS WORK

CRITICAL control systems, the expression was coined in Zakian (1989), has attained much interest during the last couple of years. In the sense of control of processes with constraints on the states and the control signals this has, however, been an area of research for more than 30 years. Some of the earliest contributions are Manne (1960), who used linear programming to compute state feedback laws, and Andreev (1961), who considered stochastic feed-forward control problems. Most work has been done in a deterministic setup, see Chang and Seborg (1983) for early references on constrained control problems, and Gutman (1986) for linear programming references. The aim of this chapter is to give a brief overview of the status of current and previous research in the area of critical control systems. In Section 1 general deterministic optimal control problem setups are discussed. In Section 2 some stochastic problem formulations are reviewed. Finally, in Section 3 some concluding remarks about analysis of crossing problems in stochastic processes are given. This is an old research area that goes back to Rice (1939), and it is in this field that the present work has its main roots. It is not claimed that the references mentioned below constitute a complete list of work done in control of critical processes.

1.1 Deterministic Problem Setup

In a deterministic framework critical optimal control problems can be formulated in terms of the criterion

$$\max_{d} ||z||_{\infty}$$

where z is the controlled signal and d is a disturbance acting on z. Problems of this type have been studied extensively. Depending on what assumptions are made on d several different formulations are obtained.

The H_2-Controller

Assuming a linear process and bounded energy on the disturbance, i.e. $||d||_2 \leq 1$, gives the well-known H_2-controller, Vidyasagar (1986). This is perhaps not the most well-known derivation of the H_2 performance index. Usually it is derived in a stochastic context, where it is called the LQG-controller. This controller was launched already in the fifties, and it is regarded as one of the milestones in modern control theory. It has been widely applied in practice. Although it has been thoroughly studied in theory for more than 30 years, new interesting results still emerge, Chen *et al.* (1993), Trentelman and Stoorvogel (1993), Trentelman and Stoorvogel (1994), where existence and construction of the H_2 controller are investigated. This will be discussed in more detail in Part III.

The L_1-Controller

By assuming that the disturbance has bounded supremum norm, i.e. $||d||_\infty \leq 1$, the L_1 performance index in continuous time, Vidyasagar (1986) and the l_1 performance index in discrete time, Dahleh and Pearson (1987), are obtained. A book will soon be published on the topic, Dahleh and Diaz-Bobillo (1995).

The Sup Regulator

In Liu and Zakian (1990) the disturbance has bounded increments, i.e. $||\Delta d||_\infty \leq 1$. The controller that minimizes the performance index under this assumption on the disturbance is called the sup regulator. It is treated in more detail in Whidborne and Liu (1993), where also some applications concerning critical control systems are described. One of the examples, the control of an electro-magnetic suspension, has also been described in Whidborne (1993).

1.2 *Probabilistic Problem Setup*

Common to the deterministic criteria is the design for worst case disturbances, which may seem somewhat too conservative. The classical way to overcome this is to introduce a stochastic formulation. This could be done by considering e.g. the criterion

$$\mathrm{E}\left\{f(||z||_\infty)\right\} \tag{1.1}$$

where E denotes expectation with respect to a probability measure induced by some probabilistic characterization of a disturbance acting on the controlled signal z.

Andreev and Åström

By taking the function f to be the indicator function for the set $\{x \geq z_0\}$, the following performance index is obtained:

$$\mathrm{P}\{||z||_\infty \geq z_0\} \tag{1.2}$$

where P is a probability measure, and where z_0 is a critical level that the controlled signal z should not exceed. This criterion is described already in Andreev (1961), and extensively discussed in Andreev (1969). There feed-forward problems are considered, for which approximate solutions are obtained. In Åström (1961) exact solutions are obtained for a first order full information feedback example. This will be treated in more detail in Chapter 2. In Chapter 3 approximations of the criterion in (1.2) will be discussed.

Heinricher and Stockbridge

In Heinricher and Stockbridge (1991) full information feedback solutions to problems similar to (1.1) are obtained by introducing the so called running max $\xi(t) = \max\{z(s) : 0 \leq s \leq t\}$. Via dynamic programming a Bellman-equation is obtained for a stopping problem. This equation sometimes has analytic solutions. This will be discussed in more detail in Chapter 2.

1.3 Concluding Remarks

In this chapter a brief review of previous work in optimal control of critical processes has been given. However, some more work has to be mentioned at this point, and it concerns the approximation of performance indices such as (1.2). In the context of stochastic processes this is known as extreme value analysis. The limiting distributions of the maxima for independent and identically distributed random variables were discussed already in Tippet (1925), Fréchet (1927), Fisher and Tippet (1928). The results were generalized to dependent variables by Watson (1954), Berman (1964), Loynes (1965), Leadbetter (1974). A good book in the topic is Leadbetter *et al.* (1982), where also continuous time is covered. There distributions of extrema are approximated with upcrossing intensities. These were discussed already in Rice (1939), Rice (1944), Rice (1945). Rice's celebrated formula for the mean number of upcrossings of a level z_0 per unit time by a stationary Gaussian process, with zero mean value and covariance function $r(\tau)$, is given by

$$\mu = \frac{1}{2\pi}\sqrt{\frac{-r''(0)}{r(0)}}\exp\left[-\frac{z_0^2}{2r(0)}\right]$$

The origin of Rice's Formula is well described in Rainal (1988). The minimization of this formula is the topic of Part I. In discrete time the corresponding formula is less explicit

$$\mu = \mathrm{P}\{z(k) \le z_0 \cap z(k+1) > z_0\} \tag{1.3}$$

Cramér and Leadbetter (1967). This makes the analysis somewhat harder in discrete time than in continuous time. The minimization of this formula is treated in parts II and IV.

CHAPTER 2

NONLINEAR CONTROL

In this chapter nonlinear stochastic control of critical processes will be investigated. Nonlinear stochastic control in general has been a research area for many years. Mostly full-information problems are treated, which is the case also in this presentation. Both continuous time formulations, Section 1, and discrete time formulations, Section 2, are given. They both utilize the so called running max, Heinricher and Stockbridge (1991), to formulate critical control systems in a stochastic context. Dynamic programming will be used to derive a Bellman-equation. For some examples explicit solutions of this equation are obtained. In general, however, the resort seams to be numerical computations. Finally, in Section 3 some concluding remarks are given.

2.1 Continuous Time Problems

In this section continuous time stochastic control of critical processes will be discussed. To get a feeling for what problem-formulations are relevant some examples will be investigated. The first example will be a simple continuous-time linear first order process controlled with a proportional controller.

EXAMPLE 2.1—Proportional Control
Let the process to be controlled be given by the stochastic differential equation

$$dx(t) = [ax(t) + bu(t)]dt + \sigma dw(t), \quad x(0) = x$$

where x is the state of the process and where w is a standard Wiener-process. Assume that the controller has full information, i.e. that the control signal $u(t)$ is a function of $x(t)$. Now, consider a proportional controller $u(t) = -kx(t)$. The closed loop system is then governed by

$$dx(t) = (a - bk)x(t)dt + \sigma dw(t), \quad x(0) = x$$

If k is chosen such that $a - bk < 0$, then the closed loop will be stable. Then, for any initial value x, it is easy to show that the solution to this equation in stationarity is a Gaussian process with zero mean and covariance $P = \sigma^2/[2(bk-a)]$, Åström (1970). By letting bk go to infinity it follows that $x(t)$ can be made equal to zero in mean square. Notice that there is no problem with respect to stability in doing this. However, the variance of the control signal is $k^2\sigma^2/[2(bk-a)]$ and it converges to infinity as bk goes to infinity. □

In the next example it will be shown that, in order to prevent the closed loop system to enter a critical region, infinite gain only has to be applied at the boundary of the critical region.

EXAMPLE 2.2—Singular Control

Consider the same process as before, but let the controller be $u(t)dt = -a/bx(t)dt + dg[x(t)]$, where $g = g^+ - g^-$, and where g^+ and g^- are defined as in Karatzas (1983)

$$g^+(t) = \max[0, \max_{0\le s\le t}[-x(s) + g^+(s) - x_0]]$$

$$g^-(t) = \max[0, \max_{0\le s\le t}[x(s) + g^-(s) - x_0]]$$

The closed loop system will now be governed by

$$dx(t) = dg(x(t)) + \sigma dw(t), \quad x(0) = x$$

This equation and similar ones were studied already in Åström (1961), where it was found by solving the Fokker-Planck equation, assuming the initial value x to be in $I = (-x_0, x_0)$, that the density function $p(t,\xi)$ of $x(t)$ has compact support on I and that it is given by

$$p(t,\xi) = \frac{1}{2x_0}\sqrt{\frac{T}{\pi t}}\sum_{n=-\infty}^{\infty}\exp\left[-\frac{T}{t}\left(\frac{\xi}{2x_0}+n\right)^2\right]$$

where $T = 2x_0^2/\sigma^2$. This density converges, as t approaches infinity, to a uniform distribution on I. Thus the probability of $x(t)$ being in the critical region $R \setminus I$ is zero for each t. In Åström (1961) no equations or explicit expressions for g were given. It was only assumed that there existed a g such that the density of $x(t)$ would have compact support on I. It was later shown that such a g indeed existed, and that it was uniquely given by the above equations. The total variation of g is given by $g^+ + g^-$, and it is bounded for all t. Had g been differentiable, which is not the case, then $u(t) = -a/bx(t) + \dot{g}(t)$ and the total variation of g would have been

$$\int_0^t |\dot{g}(s)|ds$$

Hence, formally, the control signal above is such that the integrated absolute value of it is bounded. This type of control problems are known as singular stochastic control problems, since the control signal is not absolutely continuous with respect to Lebesgue measure, see Karatzas (1983) for a good survey. In fact g behaves lake a Wiener-process when it is not constant. Thus since a Wiener process is a.s. nowhere differentiable, formally, it holds that $\dot{g}$ is either 0, $+\infty$ or $-\infty$. The deterministic counterpart to this type of control is known as impulse control. □

From a practical point of view it seems strange that such good performance can be obtained. This is due to the fact that infinite control signals may be applied to the process without causing instability. In order to get more interesting problems different approaches can be taken. Considering discrete time problems often removes the pathological behavior encountered in the examples above. This is due to the fact that high gain usually will cause instability. When considering continuous time problems one attractive way of ruling out infinite control signals is to limit the control signal to a certain set. Another way is to consider optimal control problems with sufficiently large weighting on the control signal, e.g. quadratic weighting. Sometimes, as in Heinricher and Stockbridge (1991), non-trivial problems can be obtained by considering non-controllable processes. Another way of obtaining well-formulated problems is to consider the case of partial information.

In this section full information continuous time optimal stochastic control of critical processes will be treated. The processes considered will be stochastic differential equations. For an introduction to these see e.g. Øksendal (1989). A more complete treatment is given in Karatzas and Shreve (1991). In order to address critical processes the notion of the running max will be introduced as in Heinricher and Stockbridge (1991). Different relevant control objectives will be discussed by considering fairly general optimization problems. Sufficient conditions for these problems in terms of Hamilton-Jacobi-Bellman (HJB) equations will be obtained. It will be seen how it is possible to solve the HJB-equation explicitly for an example.

Model

Let the open loop system be modeled by the following stochastic differential equation:

$$dx(t) = f[x(t), u(t)]dt + \sigma[x(t), u(t)]dw(t), \quad x(0) = x \tag{2.1}$$

where w is a standard n-dimensional Wiener process, and where f and σ are n-dimensional vector functions and $n \times n$-matrix functions of the n-

dimensional state $x(t)$ and the m-dimensional control $u(t)$. The assumptions to be imposed on f and σ for (2.1) to have a well-defined solution can be found in e.g. Fleming and Soner (1993).

In order to be able to address critical processes, introduce the running max of $g(x(t))$, which is defined as

$$\xi(t) = \max\{g[x(s)] : 0 \leq s \leq t\} \vee \xi, \quad \xi(0) = \xi \geq g[x]$$

where g is a real-valued differentiable function of the n-dimensional state $x(t)$, and where $\vee$ denotes max of the left and right hand side. By defining the set $A = \{t \in R : g(x(t)) = \xi(t) \cap dg[x(t)] > 0\}$ it is possible to express $d\xi$ as

$$d\xi(t) = I_A(t)\,\frac{dg[x(t)]}{dx}\,\{f[x(t), u(t)]dt + \sigma[x(t), u(t)]dw(t)\}$$

where $I_A(t)$ denotes the indicator function of the set A. In the sequel (2.1) will be augmented with this equation. The augmented state $\left(x(t)^T \ \xi(t)\right)^T$ is a strong Markov process, but it is not a diffusion due to the fact that $I_A(t)$ is not adapted to the σ-algebra $\mathcal{F}(t) = \sigma\{w(s) : 0 \leq s \leq t\}$. Notice however, that $\xi(t)$ is adapted to $\mathcal{F}(t)$; it is also increasing. These facts will be used later on.

Control Objectives

One control objective relevant for critical processes is obtained by considering the following criterion function

$$J_f[x, \xi, u(\cdot)] = \mathrm{E}\left\{\int_0^T h[x(t), \xi(t), u(t)]dt + \Psi[T, x(T), \xi(T)]\right\} \tag{2.2}$$

where h and Ψ are real-valued functions of the state, the running max and the control. Another possible control objective is the so called discounted cost criterion

$$J_d[x, \xi, u(\cdot)] = \mathrm{E}\left\{\int_0^\infty e^{-\beta t} h[x(t), \xi(t), u(t)]dt\right\} \tag{2.3}$$

where $\beta > 0$. The set of controls over which the minimization of the criterion functions is to be performed will be the set of admissible controls as defined in Fleming and Soner (1993). The former criterion function will result in time-dependent control laws, whereas the latter will result in control laws independent of time. The conditions that have to be imposed on h and Ψ for the control problem to be well defined are given in Fleming and Soner (1993).

The Hamilton-Jacobi-Bellman Equation

Now sufficient conditions for optimality in terms of HJB-equations will be given. The results are variants of the result in Heinricher and Stockbridge (1991), and they follow the path of standard verification theory as presented in Fleming and Soner (1993).

First consider the problem of minimizing J_f. Introduce the following partial differential equation, called the Hamilton-Jacobi-Bellman (HJB) equation

$$V_t + \min_u \left\{ V_x^T f + \frac{1}{2} \mathrm{tr} V_{xx} \sigma \sigma^T + h \right\} = 0 \tag{2.4}$$

for $V = V[t, x, \xi]$ on $\mathcal{D}_f = \{(t, x, \xi) \in R^{n+2} : g[x] \leq \xi,\ 0 \leq t \leq T\}$ with terminal condition $V[T, x, \xi] = \Psi[T, x, \xi]$ and boundary condition $V_\xi[t, x, \xi] = 0$ for $g[x] = \xi$ and $0 < t < T$. Assume that this equation has a solution V on $\mathcal{D}_f$ that fulfills all the assumptions for a classical solution as defined in Fleming and Soner (1993).

Since $\xi(t)$ is adapted to $\mathcal{F}(t)$ and increasing, it follows by the Itô-formula, Karatzas and Shreve (1991) Theorem 3.6, that

$$V[T, x(T), \xi(T)] = V[0, x, \xi] + \int_0^T \left[V_t + V_x^T f + \frac{1}{2} \mathrm{tr} V_{xx} \sigma \sigma^T + I_A V_\xi g_x^T f \right] dt + \int_0^T \left[V_x^T + I_A V_\xi g_x^T \right] \sigma dw$$

Noting that $V_\xi[t, x, \xi] = 0$ for $g[x] = \xi$ and $0 < t < T$, and that $V_x^T[t, x(t), \xi(t)]\sigma[x(t), u(t)]$ is adapted to $\mathcal{F}(t)$, it follows by taking expectations that

$$\mathrm{E}\{V[T, x(T), \xi(T)]\} = V[0, x, \xi] + \mathrm{E}\left\{ \int_0^T \left[V_t + V_x^T f + \frac{1}{2} \mathrm{tr} V_{xx} \sigma \sigma^T \right] dt \right\}$$

Now, by adding and subtracting h in the integral, using (2.4), and noting that $V[T, x(T), \xi(T)] = \Psi[T, x(T), \xi(T)]$ it holds by (2.2) that

$$V[0, x, \xi] \leq J[x, \xi, u(\cdot)]$$

with equality for the $u(\cdot)$ that solves (2.4). This shows that the optimal control can be obtained by solving the HJB-equation under the condition that this solution fulfills the assumptions that justifies the calculations above, i.e. has a classical solution in the sense of Fleming and Soner (1993). Notice that the differentiability assumption on g is not necessary, since $I_A(t) V_\xi[t, x(t), \xi(t)] = 0$.

Similar techniques as above can be used to show that the existence of a classical solution to

$$-\beta V + \min_u \left\{ V_x^T f + \frac{1}{2}\mathrm{tr} V_{xx}\sigma\sigma^T + h \right\} = 0 \tag{2.5}$$

for $V = V[x, \xi]$ on $\mathcal{D}_d = \{(x, \xi) \in R^{n+1} : g[x] \leq \xi\}$ with boundary condition $V_\xi[x, \xi] = 0$ for $g[x] = \xi$ and terminal condition

$$\lim_{t\to\infty} e^{-\beta t}\mathrm{E}\{V[x(t), \xi(t)]\} = 0$$

is a sufficient condition for minimizing J_d. In Heinricher and Stockbridge (1991) a stopping problem is considered where the sufficient condition is the same as the one for the discounted const criterion above with $\beta = 0$ and with the additional assumption of $V[\xi, \xi] = 0$.

Example

It turns out that the time-independent HJB-equations are much easier to solve than the time-dependent. Thus a discounted cost criterion problem will be considered.

EXAMPLE 2.3—Discounted LQ Control in the Running Max
Let the process be linear, i.e. let

$$dx(t) = [ax(t) + bu(t)]dt + \sigma dw(t), \quad x(0) = x$$

and let the criterion be given by

$$\mathrm{E}\left\{ \int_0^\infty e^{-\beta t}\frac{1}{2}\left[\xi^2(t) + \rho u^2(t)\right] dt \right\}.$$

which is a type of discounted Linear Quadratic (LQ) control problem, but not in the state x as is usual, but in the running max ξ. Assume that $g(x) = x$. Easy calculations show that the optimal control is given by $u = -b/\rho V_x$, and that the HJB-equation for this control becomes

$$-\beta V + axV_x - \frac{b^2}{2\rho}V_x^2 + \frac{1}{2}\sigma^2 V_{xx} + \frac{1}{2}\xi^2 = 0$$

Inspired by the solution to the standard discounted LQ problem the solution $V = K_1x^2 + K_2x\xi + K_3\xi^2 + K_4$ for K_i, $i = 1, \ldots 4$ being some constants, will be investigated. Some calculations show that this is indeed a solution, if $a = 0$, and it is given by

$$K_1 = -\frac{\rho\beta}{2b^2}; \qquad K_2 = -2K_3$$

$$K_3 = \frac{-\rho\beta - \sqrt{\rho^2\beta^2 + 4\rho b^2}}{4b^2}; \qquad K_4 = -\frac{\rho\sigma^2}{2b^2}$$

The resulting control signal is given by

$$u(t) = \frac{\beta}{b}x(t) - \frac{\rho\beta + \sqrt{\rho^2\beta^2 + 4\rho b^2}}{2\rho b}\xi(t)$$

and it is always negative. Strangely enough bu is increasing as a function of the state. It is interesting to note that the standard discounted LQ-controller is given by, Fleming and Soner (1993)

$$u(t) = -\frac{\rho\beta + \sqrt{\rho^2\beta^2 + 4\rho b^2}}{2\rho b}x(t)$$

Hence the discounted LQ-controller for the running max just replaces the state in the standard problem with the running max and adds a new feedback from the state. If $\beta = 0$, then the state feedback is not present and the controller is given by

$$u(t) = -\frac{\mathrm{sign}(b)}{\sqrt{\rho}}\xi(t)$$

The assumptions that justifies the use of the HJB-equation to derive the optimal controller for the discounted cost problem has to be shown to hold for the solution obtained. The only assumption that is difficult to verify for the discounted cost problem is

$$\lim_{t\to\infty} e^{-\beta t}\mathrm{E}\left\{V[x(t), \xi(t)]\right\} = 0$$

where $\begin{pmatrix} x(t) & \xi(t) \end{pmatrix}^T$ is the state obtained by applying the candidate optimal control law above. It might be possible to show this. □

Summary

Optimal stochastic control problems for the running max have been treated. Sufficient conditions in terms of HJB-equations have been given. For an LQ type of problem for the running max, assuming an integrator process, i.e. $a = 0$, the HJB-equation has been solved explicitly. For more complicated processes the solution is not known. This seems to be an inherent problem in control of the running max, see Heinricher and Stockbridge (1991), where explicit solutions also only are obtained for integrator processes.

The condition $V_\xi[x,\xi]$ for $g[x] = \xi$ was imposed above in order to use Itô-calculus for obtaining the sufficient condition in terms of the HJB-equation. At a first glance it seems to be interesting to instead try to obtain the backward evolution operator for the Markov process $\begin{pmatrix} x(t)^T & \xi(t) \end{pmatrix}^T$ and the corresponding Dynkin formula to derive a sufficient condition for the problem. It, however, turns out that the condition $V_\xi[x,\xi]$ for $g[x] = \xi$ is a necessary condition for the backward evolution operator to exist. Thus nothing is gained by this alternative approach.

In this section only full information control has been treated. The partial information case is much more complicated. For an introductory treatment of partial information optimal stochastic control in continuous time see e.g. Wonham (1968). Most of the material in this section is taken from Hansson (1993c).

2.2 *Discrete Time Problems*

In this section full information discrete time optimal stochastic control of critical processes will be treated. The processes considered will be stochastic difference equations. For a simple introduction to optimal stochastic control in discrete time see Åström (1977). A more rigorous treatment is given in Bertsekas (1978). As in the previous section the running max will be introduced. In discrete time the running max will actually obey a difference equation. This will simplify things as compared to the continuous time case. Some control objectives relevant to critical processes will be discussed by considering different optimization problems involving the running max. For the case of linear relative degree one processes it will actually be possible to solve the Bellman-equation related to a special optimal control problem explicitly. The solution will be the full information Minimum Variance (MV) controller. In general, however, the resort seems to be numerical computations. This will be demonstrated for a relative degree two example. It will be seen that the optimal controller is nonlinear. This seems to be the generic case.

Model

Let the open loop system be modeled by the following stochastic difference equation:

$$x(k+1) = f[x(k), u(k), v(k)], \quad x(0) = x \tag{2.6}$$

where $u(\cdot)$ is the control signal, and where $v(\cdot)$ is a sequence of independent Gaussian random variables with zero mean and unit covariance, i.e. $\mathrm{E}v(k)v^T(k) = I$. Define $\xi(k)$ to be the discrete time running running max of $g[x(k)]$ by

$$\xi(k) = \max\{g[x(i)] : 0 \le i \le k\} \vee \xi, \quad \xi(0) = \xi \ge g[x]$$

Notice that

$$\xi(k+1) = \max\{\xi(k), g[x(k+1)]\} = \max\{\xi(k), g[f[x(k), u(k), v(k)]]\} \tag{2.7}$$

Due to this difference equation for the running max it is possible to describe the behavior of the augmented system with the new state $\bar{x}(k) = \begin{pmatrix} x^T(k) & \xi(k) \end{pmatrix}^T$ by the stochastic difference equation

$$\begin{aligned} x(k+1) &= f[x(k), u(k), v(k)] \\ \xi(k+1) &= \max\{\xi(k), g[f[x(k), u(k), v(k)]]\} \end{aligned} \tag{2.8}$$

with initial value $\bar{x} = \begin{pmatrix} x^T & \xi \end{pmatrix}^T$. This description will be used in the sequel. It should be noted that the difference equation above defines a discrete time Markov process.

Control Objectives

The control objectives that will be considered can all be expressed in the general form of

$$J[x, \xi, u(\cdot)] = \mathrm{E}\left\{\sum_{k=0}^{N} h[k, x(k), \xi(k), u(k)]\right\}$$

where h is a real-valued function of time, the augmented state, and the control. The admissible controls u, which J will be minimized over, will be functions of the augmented state.

The Bellman Equation

In this subsection the dynamic programming equation, or the Bellman-equation, which gives a solution procedure for the minimization of the cost function J above, will be derived. It should be noted that it is difficult to make this derivation rigorous due to the fact that the minima computed in the sequel may not be measurable. These questions will not be addressed here. For a discussion about the measurability problem see Bertsekas (1978).

Let $\mathcal{Y}(k)$ be the sequence of information available to the controller at time k, i.e. $\mathcal{Y}(k) = \{x(i), \xi(i) : 0 \le i \le k\} = \{x(i) : 0 \le i \le k\}$. Further introduce

$$V[k, x(k), \xi(k)] = \min_{\{u(i):k \le i \le N\}} \mathrm{E}\left\{ \sum_{i=k}^{N} h[i, x(i), \xi(i), u(i)] \Big| \mathcal{Y}(k) \right\}$$

It is now obvious that

$$V[0, x, \xi] = \min_{\{u(k):0 \le k \le N\}} J[x, \xi, u(\cdot)]$$

Further by the principle of optimality and the fact that $\{\bar{x}(k)\}$ is a Markov process it holds that

$$\begin{aligned} V[k, x(k), \xi(k)] &= \min_{u(k)} \mathrm{E}\Big\{ h[k, x(k), \xi(k), u(k)] \\ &\quad + V[k+1, x(k+1), \xi(k+1)] \Big| x(k), \xi(k) \Big\} \\ &= \min_{u(k)} \Big\{ h[k, x(k), \xi(k), u(k)] \\ &\quad + \mathrm{E}\Big\{ V[k+1, x(k+1), \xi(k+1)] \Big| x(k), \xi(k) \Big\}\Big\} \end{aligned}$$

for $0 \le k \le N-1$ with final value

$$V[N, x(N), \xi(N)] = \min_{u(N)} h[N, x(N), \xi(N), u(N)]$$

This equation is called the Bellman-equation and gives a recursion for the optimal value of the cost function J. The first step in computing the recursion is to evaluate the expectation

$$E[x(k), \xi(k), u(k)] = \mathrm{E}\Big\{ V[k+1, x(k+1), \xi(k+1)] \Big| x(k), \xi(k) \Big\}$$

and express it in terms of $x(k)$, $\xi(k)$ and $u(k)$. The second step is to perform the minimization with respect to $u(k)$. The expectation can be computed as

$$E[x(k),\xi(k),u(k)] = \int p\left[x(k+1),\xi(k+1)\middle| x(k),\xi(k)\right] \cdot V[k+1,x(k+1),\xi(k+1)]dx(k+1)d\xi(k+1)$$

where p is the density function for the extended state at time $k+1$ conditioned on the extended state at time k. The integration is to be performed over the value space of the augmented state. This will be illustrated more in detail in the special case that follows.

Relative Degree One Problems

Consider the case when the loss function is given by

$$J = \mathrm{P}\{\xi(N) > \xi_0\} \tag{2.9}$$

for the critical level ξ_0, and when the model dynamics is linear, i.e.

$$f[x(k),u(k),v(k)] = Ax(k) + B_u u(k) + B_v v(k) \tag{2.10}$$

Further let $g[x] = |Cx|$. This is easily seen to be a special case of the problem formulation above by taking $h[k,x(k),\xi(k),u(k)] = 0$ for $0 \le k \le N-1$ and $h[N,x(k),\xi(k),u(k)] = I_{\{\xi(N)>\xi_0\}}$ For this case the Bellman-equation becomes

$$V[k,x(k),\xi(k)] = \min_{u(k)} \int p\left[x(k+1),\xi(k+1)\middle| x(k),\xi(k)\right] \cdot V[k+1,x(k+1),\xi(k+1)]dx(k+1)d\xi(k+1) \tag{2.11}$$

for $0 \le k \le N-1$ with final value

$$V[N,x(N),\xi(N)] = I_{\{\xi(N)>\xi_0\}}$$

Assume that there exists a solution such that $V[k,x(k),\xi(k)]$ is not a function of $x(k)$. Note that this assumption holds for $k = N$. Then by integrating out the state variable $x(k+1)$ the Bellman-equation reads

$$V[k,\xi(k)] = \min_{u(k)} \int p\left[\xi(k+1)\middle| x(k),\xi(k)\right] \cdot V[k+1,\xi(k+1)]d\xi(k+1)$$

Some calculations show that the conditioned density in the equation above is

$$p\left[\xi(k+1)\middle|x(k),\xi(k)\right] = \frac{1}{\sigma}\varphi\left[\frac{\xi(k+1)-m(k)}{\sigma}\right] + \frac{1}{\sigma}\varphi\left[\frac{\xi(k+1)+m(k)}{\sigma}\right]$$

if $\xi(k) \le \xi(k+1)$ and that it is zero if $\xi(k) > \xi(k+1)$, where

$$\begin{aligned} m(k) &= C[Ax(k) + B_u u(k)] \\ \sigma^2 &= CB_v B_v^T C^T \end{aligned}$$

and where φ is the standardized normal density function. Some further calculations show that the optimal choice of $u(k)$ is given by $m(k) = 0$ if there exist a solution to this equation. It is easily seen that the resulting $V[k, \xi(k)]$ is indeed independent of $x(k)$. Thus by induction the optimal control law is given by the equation above for all $0 \le k \le N$. The existence of a solution to $m(k) = 0$ is e.g. in the case of a single-input system implied by $CB_u \neq 0$, and for this case the solution is given by

$$u(k) = -\frac{CA}{CB_u}x(k)$$

If, however, $CB_u = 0$, then $u(k)$ can be taken arbitrarily, and the assumption made above about $V[k, \xi(k)]$ being independent of $x(k)$ may not hold. It should be noted that the resulting control law above is the same as the MV control law for the full information case when minimizing the variance of Cx.

A Special Relative Degree Two Problem

Consider the same problem setup as in the previous subsection, and let the system in (2.10) be described by

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}; \quad B_u = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$B_v = \begin{pmatrix} c \\ 1 \end{pmatrix}\sigma; \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

The process is a double integrator influenced by colored noise, i.e. with $z(k) = x_1(k) = Cx(k)$ and $w(k) = \sigma v(k)$ it holds that

$$(q-1)^2 z(k) = u(k) + (cq + 1 - c)w(k)$$

where q is the forward shift operator. Further (2.11) can be reformulated using the system equations in (2.8) as

$$V[k, x(k), \xi(k)] = \min_{u(k)} \int p(w) V[k+1, x(k+1), \xi(k+1)]\, dw \qquad (2.12)$$

where $p(w)$ is given by

$$p(w) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{w^2}{2\sigma^2}\right)$$

which is the density function for $w(k)$. Since the optimization defined above cannot be solved analytically, numerical methods have been used based on the integral in equation (2.12), see Andersson (1993). Notice that the $\xi(k)$-dependence in V and the optimal control $u(k)$ is a function only of $I_{\{\xi(k)>\xi_0\}}$ see Andersson (1993) Appendix A. This can be proved by induction. Since the values of V and the optimal control $u(k)$ are trivial for $\xi(k) \geq \xi_0$, these values will not be discussed in the sequel.

The optimal control law is time varying but depends only on $N-k$ and not on N due to the recursive property of the Bellman equation. Also note that the output up to time N cannot be affected by neither $u(N)$ nor $u(N-1)$, since the system is a double integrator. The control law at time $N-2$ as a function of the state x_2 for different values of x_1 is given in Figure 2.1. The control law as a function of the state x_1 for different values of x_2 is given in Figure 2.2. The standard deviation of the noise σ and the parameter c have been taken to 0.5 and 0.7, respectively. The figures show that the control law is almost linear when $x_1 + x_2 \ll \xi_0 = 1$. It is in fact very close to the minimum variance controller for the variance of z, see Åström (1970). However, near the line $x_1 + x_2 = \xi_0 = 1$ the optimal control law becomes nonlinear.

The optimization has also been done for other parameter values and values of k. These solutions show that the deviation from the MV solution is greater for $k = N-3$ than it is for $k = N-2$. For $k > N-3$ the control law has converged to a control law almost identical to that for $k = N-3$. The solutions show that the nonlinearity begins at lower values of the states for higher noise variances than it does for lower noise variances. The same is recognized when the parameter c is increased, but then the slope of the nonlinearity decreases as well.

The loss function at time $k = N-2$ is shown in Figure 2.3. At $x_1 = 1$ there is a discontinuous behavior corresponding to Equation 2.7, which describes the way in which the maximum is updated. There is also a smooth edge in the loss function at the line $x_1 + x_2 = 1$. These characteristics are typical also for other parameter values.

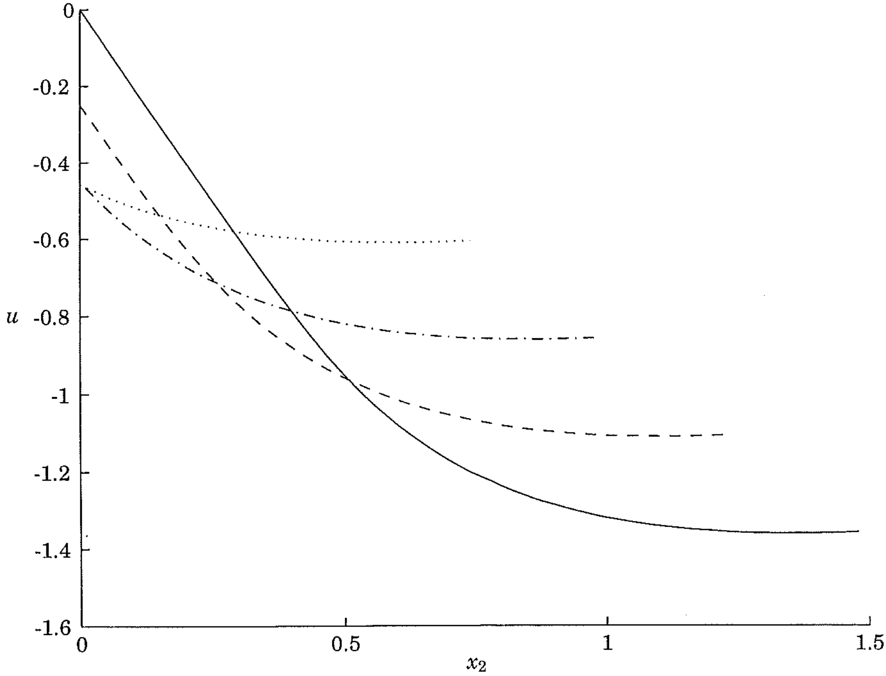

Figure 2.1 The control signal u as a function of x_2 for the x_1 values 0—solid, 0.25—dashed, 0.5—dash dotted, and 0.75—dotted line.

In general it is impossible to solve the problem analytically. For the case with $N = 2$, however, much insight can be gained by analyzing the equations. It holds that the loss function (2.9) becomes

$$J = \mathrm{P}\{\xi(2) \geq \xi_0\}$$

This expression can be reformulated using $z(k) = x_1(k)$ and (2.7). It holds that

$$J = \mathrm{P}\{|x_1(2)| > \xi_0 \cup |x_1(1)| > \xi_0 \cup |x_1(0)| > \xi_0\} \tag{2.13}$$

To simplify the analysis introduce the following notation

$$\begin{aligned} A &= \{x_1(2) : \ |x_1(2)| > \xi_0\} \\ B &= \{x_1(1) : \ |x_1(1)| > \xi_0\} \\ C &= \{x_1(0) : \ |x_1(0)| > \xi_0\} \end{aligned} \tag{2.14}$$

Using this notation the loss function (2.13) can be written as

$$J = \mathrm{P}\{A \cup B \cup C\}$$

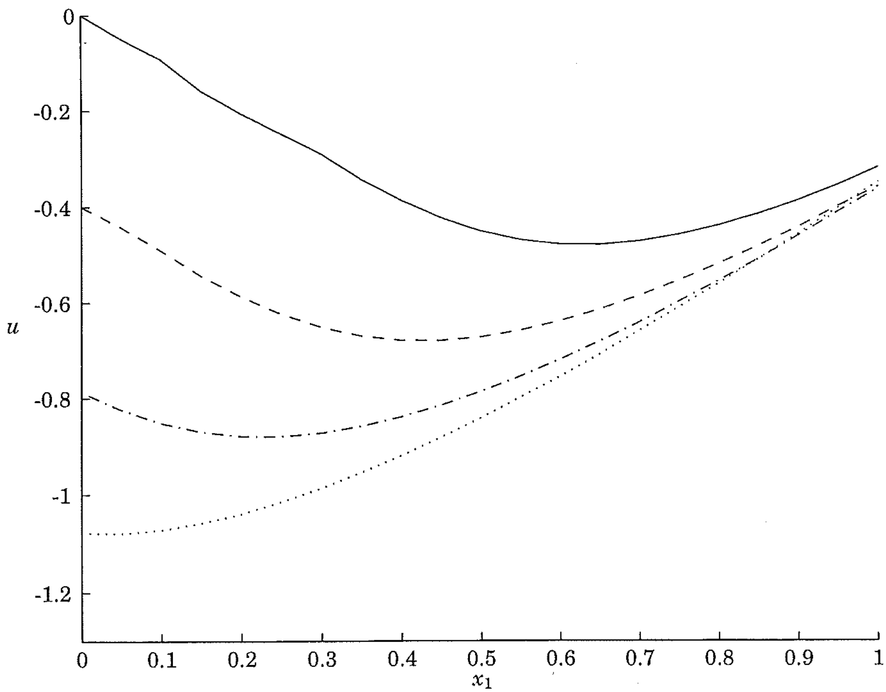

Figure 2.2 The control signal u as a function of x_1 for the x_2 values 0—solid, 0.2—dashed, 0.4—dash dotted, and 0.6—dotted line.

By mutual exclusion it holds that

$$J = \mathrm{P}\left\{A\bar{B}\,\bar{C}\right\} + \mathrm{P}\left\{B\bar{C}\right\} + \mathrm{P}\{C\}$$

As explained above the control signal at time $k = 1$ and $k = 2$ cannot affect the loss function. Therefore the loss function only has to be minimized with respect to the control $u(0)$. By similar arguments it holds that the control $u(0)$ cannot affect the outputs $x_1(0)$ and $x_1(1)$. Therefore the minimization problem can be written as

$$\begin{aligned}
\min_{u(0)} \mathrm{P}\left\{A\bar{B}\,\bar{C}\right\} &= \min_{u(0)} \int_{R^2} f(x)\mathrm{P}\left\{A\bar{B}\,\bar{C}|x\right\}\,dx \\
&= \int_{R^2} f(x) \min_{u(0)} \mathrm{P}\left\{A\bar{B}\,\bar{C}|x\right\}\,dx \\
&= \min_{u(0)} \mathrm{P}\left\{A\bar{B}\,\bar{C}|x\right\}
\end{aligned}$$

where $f(x)$ is the density function for the initial value x. The first equality follows from the formula of total probability. The second equality holds

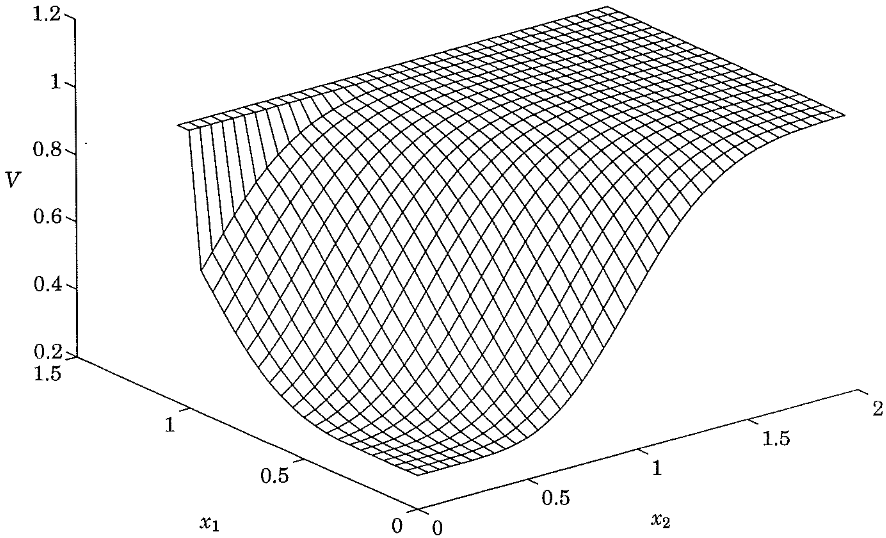

Figure 2.3 The loss function V in one quadrant when $k = N - 2$

since x cannot be affected by $u(0)$. The third equality follows from the fact that $f(x) = \delta(x)$, since the initial value x is known. Further it holds that

$$\begin{aligned}\min_{u(0)} \mathrm{P}\left\{A\bar{B}\bar{C}|x\right\} &= \min_{u(0)} \mathrm{P}\left\{A\bar{B}|x\right\} \\ &= \min_{u(0)} \mathrm{P}\left\{\bar{B}\right\}\mathrm{P}\left\{A|\bar{B},x\right\} \\ &= \mathrm{P}\left\{\bar{B}\right\}\min_{u(0)} \mathrm{P}\left\{A|\bar{B},x\right\}\end{aligned}$$

where the first equality holds since $\bar{C}$ is known when x is known. The second equality follows from the conditional probability formula. The last equality holds since the output $x_1(1)$ cannot be affected by $u(0)$. The problem can therefore be reformulated as

$$\min_{u(0)} \mathrm{P}\left\{A|\bar{B},x\right\}$$

It has now been shown that the minimization of (2.13) with respect to the control signals $\{u(0), u(1), u(2)\}$ is equivalent to the minimization of

$$\mathrm{P}\left\{|x_1(2)| > \xi_0 \;\middle|\; |x_1(1)| \le \xi_0, x_1(0), x_2(0)\right\}$$

with respect to $u(0)$. It should be noted that this probability is closely related to the upcrossing probability in (1.3).

The reformulation can be interpreted in the following way. The control $u(0)$ is based on the state $\begin{pmatrix} x_1(0) & x_2(0) \end{pmatrix}^T$, and since the system is a double integrator it can only affect the state $x_1(2)$ and not $x_1(1)$. This means that there is nothing to be done about a level crossing appearing before $k = 2$. If a level crossing appears, the loss is equal to one independent of $x_1(2)$. This means that it can be assumed that the level has not been crossed before $k = 2$ when the control $u(0)$ is chosen.

From (2.6) it is seen that the given information about $x_1(1)$ is equivalent to information about the noise, i.e.

$$|x_1(1)| < \xi_0 \Leftrightarrow |x_1(0) + x_2(0) + cw(0)| < \xi_0$$

Assume that the noise level is low, i.e. $c\sigma \ll \xi_0$. Then it is seen that only little information about the noise $w(0)$ is obtained, when $x_1(0) + x_2(0)$ is near zero. Near the line $|x_1(0) + x_2(0)| = \xi_0$ a lot of information about the noise is obtained. The system output at time $k = 2$ is given by

$$x_1(2) = x_1(0) + 2x_2(0) + u(0) + (1 + c)w(0) + cw(1)$$

which shows that information about $w(0)$ can be used to chose $u(0)$. The MV strategy as well as the optimal control strategy for small values of $x_1(0) + x_2(0)$ is to make $x_1(2)$ small by taking

$$u(0) = -x_1(0) - 2x_2(0)$$

implying

$$x_1(2) = (1 + c)w(0) + cw(1) \tag{2.15}$$

When $x_1(0) + x_2(0)$ is not small the optimal controller will make use of the information about $w(0)$ to further decrease its influence in (2.15). This explains why the optimal control strategy is nonlinear. In the discussion above $N = 2$, which is a special case easy to analyze. When N is larger the analysis is more complicated.

Simulations have been done to compare the optimal control strategy with the MV controller. The parameters c and ξ_0 has been taken to $c = 0.7$ and $\xi_0 = 1$, respectively. In order to compare the optimal controller with the MV controller the influence of the initial state and the noise level has been studied. As a measure empirical loss values have been used. For simplicity only the case with $N = 2$ was studied. This means that only one control, $u(0)$, can affect the maximum $\xi(2)$ and therefore also the loss. The simulations have been done for different initial conditions

x_1	x_2	$\hat{J}_{optimal}$	$\hat{J}_{minvar}$	$J_{optimal}$
0	0	0.2780	0.2780	0.2767
0.1	0.1	0.2818	0.2818	0.2766
0.2	0.2	0.2760	0.2761	0.2778
0.3	0.3	0.2992	0.3051	0.2989
0.35	0.35	0.3231	0.3439	0.3320
0.4	0.4	0.3833	0.4242	0.3861
0.45	0.45	0.4587	0.5254	0.4605
0.5	0.5	0.5511	0.6391	0.5493
0.3	0.5	0.3871	0.4274	0.3861
0.5	0.3	0.3871	0.4238	0.3861

Table 2.1 Table showing the true loss function $J_{optimal}$ in (2.9) and estimates of it for the optimal controller—$\hat{J}_{optimal}$, and the MV controller—$\hat{J}_{minvar}$ for different values of the initial state.

and different values of the noise level σ. The loss values $\hat{J}_{optimal}$ obtained when $\sigma = 0.5$ are presented in Table 2.1. Also presented are the simulated loss values $\hat{J}_{minvar}$ for the MV controller and the computed optimal loss values $J_{optimal}$ for the optimal controller. Every estimated loss value is based on 20000 simulations. The conclusions that can be drawn from the table is that the optimal controller performs better than the MV controller near the line $x_1 + x_2 = 1$. Far from the line the two controllers behave almost identically. This result agrees well with the result of the previous analysis. The same simulations were also done for the noise level $\sigma = 0.2$, and in that case the loss was almost the same for the two controllers.

The estimated loss, based on 1000 simulations, as a function of N is shown in Figure 2.4 for the noise level $\sigma = 0.5$. The solid line corresponds to the optimal controller whereas the dashed line corresponds to the MV controller. The figure shows that the optimal controller has a lower loss than the MV controller when the minimization horizon is short. The loss has also been estimated for $\sigma = 0.2$. In this case the two curves are almost identical. This shows that the optimal controller has an advantage only when the noise level is high as compared to the bounds within which the output is to be kept.

Figure 2.5 shows an example of how realizations of the state x_1 and the control signal u may look like when $\sigma = 0.5$. The solid lines correspond to the optimal controller whereas the dashed lines correspond to the MV controller. The noise sequence is the same for both controllers. The reference signal is zero, and it is seen that the noise level is high as compared to the critical levels given by $\xi_0 = 1$. Therefore it may be

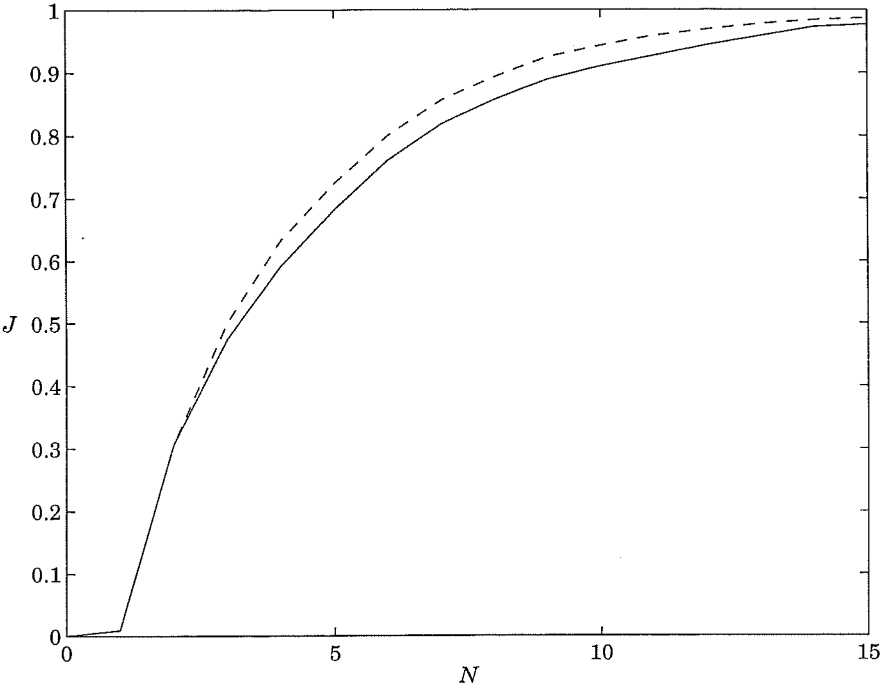

Figure 2.4 The loss as a function of N, MV controller—dashed, and optimal controller—solid.

expected to be advantageous to use the optimal controller. In this particular simulation this is seen to be the case. The MV controller passes the critical level already at $k = 6$. The violation at $k = 29$ also points out that the high noise level makes it impossible also for the optimal controller to avoid the bounds over a longer period of time. It is also seen that the control signal of the optimal controller is smaller than the one obtained from the MV controller.

Summary

In this section discrete time stochastic optimal control of the running max has been discussed. Only the full information case has been treated. In a special case an explicit solution of the Bellman equation was obtained. The resulting controller for this linear relative degree one case was the same as the MV controller. However, in general the resort seems to be numerical computations, and the optimal controller is generically nonlinear. This has been demonstrated in a relative degree two example. Analysis showed that the criterion function was closely related to the upcrossing probability. This will be elaborated more in the next chapter, where the

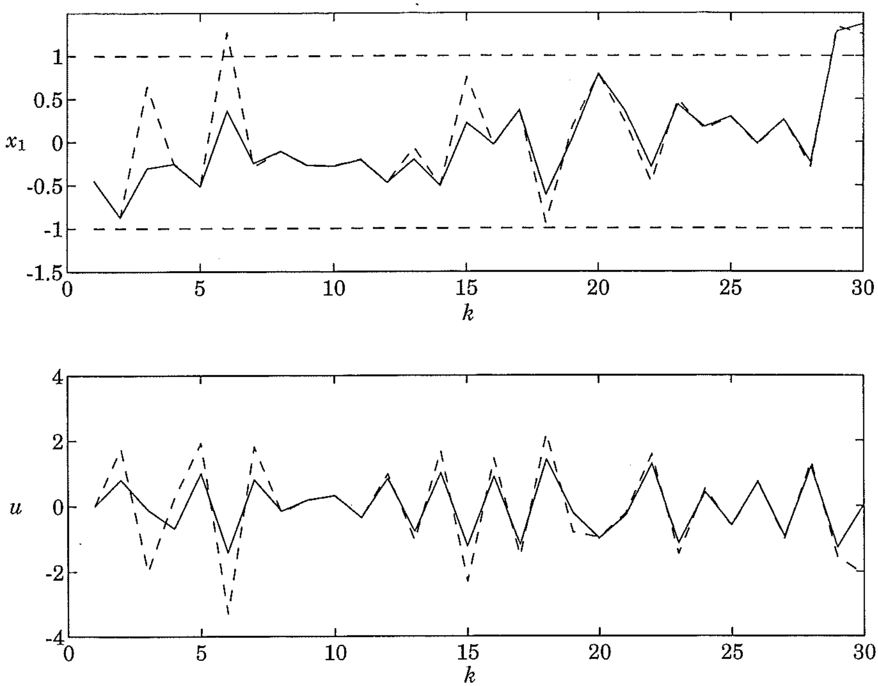

Figure 2.5 Simulation example with optimal controller—solid, and MV controller—dashed.

approximation of the criterion in (2.9) by means of the upcrossing criterion is described. The material in this section was taken from Hansson (1993c), Andersson (1993), Andersson and Hansson (1994).

2.3 Concluding Remarks

In this chapter nonlinear continuous and discrete time formulations of control of critical processes in a stochastic context have been given. Optimal continuous time stochastic control problems for the running max have been treated. Sufficient conditions in terms of HJB-equations have been given. For an LQ type of problem in the running max, assuming an integrator process, the HJB-equation has been solved explicitly. Also discrete time optimal stochastic control of the running max has been discussed. Only the full information case has been treated. In a special case an explicit solution to the Bellman equation was obtained. The resulting controller for this linear relative degree one case was the same as the MV controller. However, in general the resort seems to be numerical computations, and the optimal controller is generically nonlinear.

CHAPTER 3

PROBLEM FORMULATIONS

THE generic controller for critical processes was in the previous chapter shown to be nonlinear. It was, however, difficult to find general easily implementable solution procedures. In this chapter the criteria and controllers considered in parts I–IV will be defined. The goal is to formulate problems which are possible to solve, at least approximately, with standard numerical routines. To this end only linear controllers will be considered. One of the criteria, the risk criterion, can under the constraint of linear controllers be shown to be the same as the one in (2.9). Only discrete time will be treated. The continuous time upcrossing criterion is treated in an analogous way in Part I. The continuous time risk and MTBF criteria have been described and approximated in Hansson (1991c).

In Section 1 the model of the process to be controlled will be defined. It is a linear time-invariant Gaussian stochastic difference equation. The admissible controllers will be linear time invariant causal feedbacks from the measurement signal. This implies that the case of partial information also will be treated. Then the different control objectives will be discussed. In Section 2 the upcrossing criterion is defined. The equations for deriving the controller minimizing the objective function are given. It will be seen that the solution can be obtained as a one-parametric optimization over a set of LQG problem solutions. In sections 3 and 4 the risk and MTBF criteria are defined. Approximations of these criteria by means of the upcrossing criterion are investigated. Finally, in Section 5 some concluding remarks are given.

3.1 Model

Let the process to be controlled be described by

$$\begin{pmatrix} x(k+1) \\ z(k) \\ y(k) \end{pmatrix} = \begin{pmatrix} A & B_w & B_u \\ C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & D_{yu} \end{pmatrix} \begin{pmatrix} x(k) \\ w(k) \\ u(k) \end{pmatrix} \tag{3.1}$$

where $w(k) \in R^l$ is a sequence of independent zero mean Gaussian random variables with covariance I, $u(k) \in R^m$ is the control signal, $x(k) \in R^n$ is the state, $y(k) \in R^p$ is the measurement signal, and $z(k) \in R$ is the signal to be controlled. It will be assumed that $D_{yu} = 0$. Denote by σ_z^2 and m_z the variance and mean of $z(k)$ in stationarity. Let the control signal be given by

$$\begin{aligned} \xi(k+1) &= A_H\xi(k) + B_H y(k) \\ u(k) &= -C_H\xi(k) - D_H y(k) + D_r r(k) \end{aligned} \tag{3.2}$$

where $r(k) \in R$ is the reference value, where ξ is the finite dimensional state of the controller, and where A_H, B_H, C_H, D_H, and D_r are matrices of consistent dimensions. Since only constant reference values will be considered, it is no loss in generality to assume that $r(k) = 0$ by a change of coordinates. This implies that $m_z = 0$ provided a stabilizing controller is used. Introduce the notation $H = \{A_H, B_H, C_H, D_H\}$ for the controller defined by (3.2). Further denote by $\mathcal{D}$ the set of all H as defined in (3.2), i.e. the set of all linear, causal, and time-invariant controllers, and by $\mathcal{D}_s$ the subset of $\mathcal{D}$ which stabilizes (3.1), i.e. the set of controllers which are such that the eigenvalues of the closed loop system have absolute values strictly less than one. Further denote by $\mathcal{D}_z$ the set of controllers for which the closed loop standard deviation of z satisfies $\sigma_z \leq z_0$.

3.2 *The Upcrossing Criterion*

Introduce the following performance index evaluated in stationarity:

$$\mu(H; z_0) = \mathrm{P}\{z(k) \leq z_0 \cap z(k+1) > z_0\}, \quad H \in \mathcal{D}_s \tag{3.3}$$

where z_0 is the critical level that should not be upcrossed by z. The quantity μ will in the sequel be called the upcrossing probability, and it is equal to the mean number of upcrossings during a sample interval, see e.g. Cramér and Leadbetter (1967) p. 281. Another good references to crossing problems is Leadbetter *et al.* (1982). The solution to

$$\min_{H \in \mathcal{D}_z} \mu(H) \tag{3.4}$$

will in the sequel be called the MU controller. This most probably exists under very weak conditions.

In Part IV it will be assumed that (A, B_u) is stabilizable, that (C_y, A) is detectable, that

$$\mathrm{rank}_{|z|=1} \begin{pmatrix} zI - A & -B_u \\ C_z & D_{zu} \end{pmatrix} = n + 1$$

and that

$$\text{rank}_{|z|=1}\begin{pmatrix} zI - A & -B_w \\ C_y & D_{yw} \end{pmatrix} = n + p$$

Under these conditions it holds that there exist a solution to (3.4) if and only if there exists a MV controller for z with closed loop standard deviation of z smaller than or equal to z_0. Notice that the first rank-condition implies that the control signal must be scalar, i.e. $m = 1$, since $C_z \in R^{1\times n}$.

The optimal controller can be obtained by performing a one-dimensional optimization over a set of LQG problem solutions parameterized by a scalar. To this end introduce the independent variables

$$\begin{cases} \alpha(k) = z(k) + z(k-1) \\ \beta(k) = z(k) - z(k-1) \end{cases} \tag{3.5}$$

It then holds that

$$\mu = \int_0^{\infty} \phi(y) \int_{x_l(y)}^{x_u(y)} \phi(x)dxdy$$

where $\phi(x) = (2\pi)^{-1/2}\exp(-x^2/2)$, $x_l(y) = (2z_0 - \sigma_\beta y)/\sigma_\alpha$, and $x_u(y) = (2z_0 + \sigma_\beta y)/\sigma_\alpha$, and where σ_α^2 and σ_β^2 are the variances of α and β in stationarity. Hence the upcrossing probability can be computed from knowledge of the closed loop variances of α and β and the value of z_0. Not all variances of α and β corresponding to controllers in $\mathcal{D}_z$ have to be considered, but only the ones obtained by solving a set of LQG-problems parameterized by a scalar $\rho \in [0, 1]$. To this end introduce

$$\begin{aligned}
\bar{A} &= \begin{pmatrix} A & 0 \\ bC_z & 0 \end{pmatrix} & \bar{B}_u &= \begin{pmatrix} B_u \\ bD_{zu} \end{pmatrix} \\
\bar{C}_z &= \begin{pmatrix} aC_z & 1 \end{pmatrix} & \bar{C}_y &= \begin{pmatrix} C_y & 0 \end{pmatrix} \\
\bar{D}_{zu} &= aD_{zu} & \bar{B}_w &= \begin{pmatrix} B_w \\ bD_{zw} \end{pmatrix}. \\
\bar{D}_{yw} &= D_{yw} & \bar{D}_{zw} &= aD_{zw}
\end{aligned}$$

where $a(\rho) = \sqrt{1-\rho} + \sqrt{\rho}$ and $b(\rho) = \sqrt{1-\rho} - \sqrt{\rho}$. Further let S, L, L_v, and L_w be the solutions of

$$\begin{aligned}
S &= (\bar{A} - \bar{B}_u L)^T S (\bar{A} - \bar{B}_u L) \\
&\quad + (\bar{C}_z - \bar{D}_{zu} L)^T (\bar{C}_z - \bar{D}_{zu} L) \\
G_S &= \bar{B}_u^T S \bar{B}_u + \bar{D}_{zu}^T \bar{D}_{zu} \\
G_S \begin{pmatrix} L & L_v & L_w \end{pmatrix} &= \begin{pmatrix} \bar{B}_u^T S \bar{A} & \bar{B}_u^T S & \bar{D}_{zu}^T \end{pmatrix}
\end{aligned} \tag{3.6}$$

and let P, K, K_x, K_v, and K_w be the solutions of

$$\begin{aligned}
P &= (\bar{A} - K\bar{C}_y)P(\bar{A} - K\bar{C}_y)^T + (\bar{B}_w - K\bar{D}_{yw})(\bar{B}_w - K\bar{D}_{yw})^T \\
H_P &= \bar{D}_{yw}\bar{D}_{yw}^T + \bar{C}_y P \bar{C}_y^T
\end{aligned}$$

$$H_P \begin{pmatrix} K \\ K_x \\ K_v \\ K_w \end{pmatrix}^T = \begin{pmatrix} \bar{B}_w\bar{D}_{yw}^T + \bar{A}P\bar{C}_y^T \\ P\bar{C}_y^T \\ \bar{B}_w\bar{D}_{yw}^T \\ \bar{D}_{zw}\bar{D}_{yw}^T \end{pmatrix}^T \tag{3.7}$$

where S and P should be the maximal symmetric real-valued positive semi-definite solutions. Further let

$$\begin{aligned}
A_{co} &= \bar{A} - \bar{B}_u L - K\bar{C}_y + \bar{B}_u D_c \bar{C}_y \\
B_c &= K - \bar{B}_u D_c \\
C_c &= L - D_c\bar{C}_y \\
D_c &= LK_x + L_v K_v
\end{aligned}$$

and let $H(\rho) = \{A_H, B_H, C_H, D_H\}$ be a minimal realization of the controller $\{A_{co}, B_c, C_c, D_c\}$, which can be shown to be the controller that makes the LQG-criterion

$$J = \mathrm{E}\{(1-\rho)\alpha^2(k) + \rho\beta^2(k)\} \tag{3.8}$$

attain its infimal value. Then it is sufficient to minimize μ over the closed loop variances of α and β obtained when $H(\rho), \rho \in [0, 1]$ are used. These variances are given by

$$\begin{aligned}
\sigma_\alpha^2(\rho) &= C_t(A_t + I)X(A_t + I)^T C_t^T + (C_tB_t + G_t)(C_tB_t + G_t)^T + G_tG_t^T \\
\sigma_\beta^2(\rho) &= C_t(A_t - I)X(A_t - I)^T C_t^T + (C_tB_t - G_t)(C_tB_t - G_t)^T + G_tG_t^T
\end{aligned}$$

where X is the minimal solution of the Lyapunov equation

$$X = A_tXA_t + B_tB_t^T$$

and where

$$\begin{aligned}
A_t &= \begin{pmatrix} \bar{A} - \bar{B}_u L & B_c\bar{C}_y \\ 0 & \bar{A} - K\bar{C}_y \end{pmatrix} \\
B_t &= \begin{pmatrix} B_c\bar{D}_{yw} \\ \bar{B}_w - K\bar{D}_{yw} \end{pmatrix} \\
C_t &= \left(\left\{ \begin{pmatrix} C_z & 0 \end{pmatrix} - D_{zu}L \right\} \quad \left\{ \begin{pmatrix} C_z & 0 \end{pmatrix} - D_{zu}D_c\bar{C}_y \right\} \right) \\
G_t &= D_{zw} - D_{zu}D_c\bar{D}_{yw}
\end{aligned}$$

In Part II other equations for computing the LQG controllers and the closed loop variances of α and β are given. Those equations are only valid for the case when $D_{zu} = 0$. They are sometimes easier to work with, since they are of the same dimension as the original process model. However, they are no good when analyzing the MU controller, as in Part IV, and they cannot be extended to cover the case $D_{zu} \neq 0$.

The algorithm for minimizing the upcrossing probability can be summarized as: (1) solve the associated LQG-problems, and (2) minimize the upcrossing probability over the variances obtained in the first step. It must be stressed that if $\sigma_z > z_0$, then no solution exists. In order to obtain a solution, the distance between the reference value and the critical level z_0 must be sufficiently large.

It has been seen that the computation of the variances is not more complicated than solving a linear system of equations. Further the upcrossing probability can easily be obtained with some numerical integration routine. The complexity of this latter problem does not depend on the size of the process model. Thus the computations performed for each value of ρ is not significantly larger than for an ordinary LQG-problem. Moreover by adopting some numerical routine for minimizing the upcrossing probability, it may not be necessary to solve that many LQG-problems. A good choice of starting value for ρ is 0.5, which corresponds to MV control. In this sense the computational burden for obtaining the MU controller is not significantly larger than for the LQG controller that corresponds to MV control. Further it should be noted that it can be shown that the minimizing value of ρ is always greater or equal to 0.5. This follows from the fact that it is possible to show that $0 < \partial\mu/\partial\sigma_\alpha^2 < \partial\mu/\partial\sigma_\beta^2$, and that the slope of the curve $(\sigma_\alpha^2(\rho), \sigma_\beta^2(\rho))$, defined by the values of the variances of α and β that minimizes (3.8), is given by $1 - 1/\rho$, which is smaller or equal to -1 for for $\rho \leq 0.5$. An illustration of this is shown in Figure 3.1, where level-curves of μ are shown together with a curve $(\sigma_\alpha^2(\rho), \sigma_\beta^2(\rho))$ for an example. Also is drawn the tangent of the latter curve for $\rho = 0.5$. A stringent proof will not be done in this presentation. It is nowhere utilized that the optimal value of ρ is greater or equal to 0.5.

3.3 The Risk Criterion

Consider the following control problem

$$\min_{H \in \mathcal{D}_z} \mathrm{P}\left\{ \max_{0 \leq k \leq N} z(k) > z_0 \right\} \tag{3.9}$$

where $z_0 > 0$ is the critical level. The probability should be evaluated in stationarity. The time horizon N and the critical level z_0 have to be

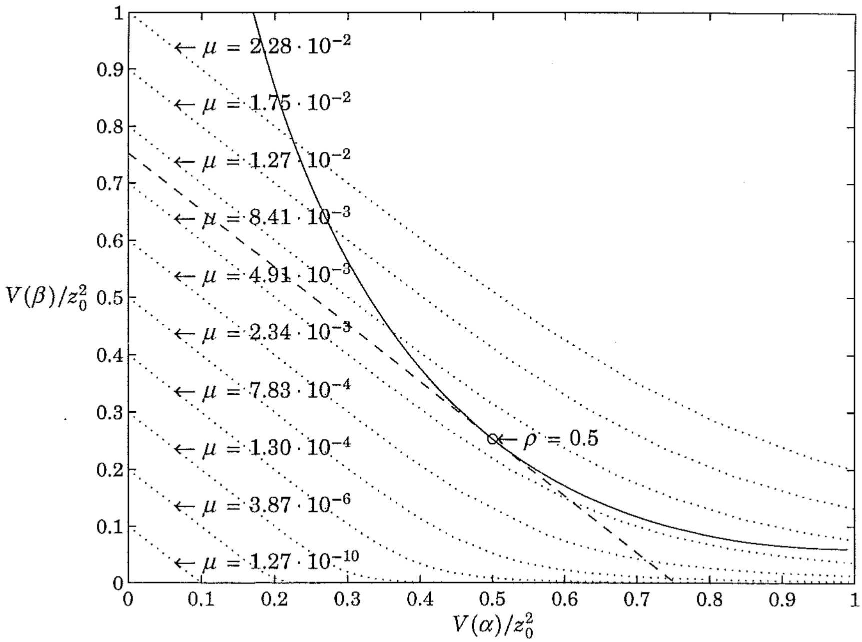

Figure 3.1 Plot showing level-curves of μ—dotted lines, a curve $(\sigma_\alpha^2(\rho), \sigma_\beta^2(\rho))$ for an example—solid line, and the tangent of $(\sigma_\alpha^2(\rho), \sigma_\beta^2(\rho))$ for $\rho = 0.5$—dashed line.

chosen in such a way that the probability in (3.9) is small, otherwise the failure rate will be too high. The larger N is, the larger z_0 must be.

To simplify the problem upper bounds for the probability in (3.9) will be given. It will be shown that these bounds are tight, if N and z_0/σ_z are large and the probability in (3.9) is small. The tighter bound is obtained by considering level crossings.

If $z(k)$ is a stationary random sequence, then

$$\mathrm{P}\left\{\max_{0\le k\le N} z(k) > z_0\right\} = \mathrm{P}\left\{z(0) > z_0 \bigcup_{k=0}^{N-1} (z(k) \le z_0 \cap z(k+1) > z_0)\right\}$$
$$\le \mathrm{P}\{z(0) > z_0\} + N\mathrm{P}\{z(0) \le z_0 \cap z(1) > z_0\} \le (N+1)\mathrm{P}\{z(0) > z_0\}$$

Hence it holds that

$$\mathrm{P}\left\{\max_{0\le k\le N} z(k) > z_0\right\} \le P_1(H; z_0) \le P_2(H; z_0) \tag{3.10}$$

where

$$P_1(H; z_0) = \mathrm{P}\{z(0) > z_0\} + N\mu(H; z_0)$$
$$P_2(H; z_0) = (N+1)\mathrm{P}\{z(0) > z_0\}$$

Notice that for large values of N and large values of z_0/σ_z the first term in P_1 is negligible. Further notice that P_2 is minimized by MV control. The bound P_1 is well known in the context of continuous time extreme value analysis, see e.g. Leadbetter *et al.* (1982) Lemma 8.2.1. There the bound P_2 is infinite, and thus not usable for investigating the behavior of extreme values as the time horizon and the critical level approaches infinity. However, the bound P_2 is good enough for investigating this behavior in the discrete time domain, but for the purposes in this work—focused on finite time horizons and levels—it is interesting also to consider a tighter bound such as P_1.

It will now be shown that the bounds described above are tight. Suppose that $z(k)$ is a stationary Gaussian sequence with covariance function satisfying

$$\lim_{\tau\to\infty} r_z(\tau)\ln\tau = 0$$

which can be shown to hold if the closed loop system is stable. Further take $z_0^{(N)}$ such that

$$\lim_{N\to\infty} P_2\left(z_0^{(N)}\right) = L$$

It then holds that

$$\lim_{N\to\infty}\left|\frac{M\left(z_0^{(N)}\right) - P_1\left(z_0^{(N)}\right)}{M\left(z_0^{(N)}\right)}\right| \le \lim_{N\to\infty}\left|\frac{M\left(z_0^{(N)}\right) - P_2\left(z_0^{(N)}\right)}{M\left(z_0^{(N)}\right)}\right| \le \frac{L}{2}$$

where $M(x) = \mathrm{P}\{\max_{0\le k\le N} z(k) > x\}$. The first inequality follows from (3.10). Since by Leadbetter *et al.* (1982) Theorem 4.3.3

$$\lim_{N\to\infty} P_2\left(z_0^{(N)}\right) = L$$

if and only if

$$\lim_{N\to\infty} 1 - M\left(z_0^{(N)}\right) = e^{-L}$$

it follows that

$$\lim_{N\to\infty}\left|\frac{M\left(z_0^{(N)}\right) - P_2\left(z_0^{(N)}\right)}{M\left(z_0^{(N)}\right)}\right| = \left|\frac{1-e^{-L}-L}{1-e^{-L}}\right| \le \frac{L}{2}$$

which proves the second inequality. Now the bounds are tight if L is small. This is obtained by taking N and z_0/σ_z large and in such a way that $M\left(z_0^{(N)}\right)$ is small.

Related problems of convergence have been investigated for other approximations of extremal-probabilities, see e.g. Leadbetter *et al.* (1982) Chapter 4.6, but these approximations are not upper bounds as the ones discussed here.

Now by the inequalities proven above it is obvious that the probability in (3.9) can be approximately minimized for large values of N and z_0/σ_z and small values of the upcrossing probability in (3.9) by minimizing either the variance or the upcrossing probability μ. However, for moderate values of N and z_0/σ_z it is tempting to believe that the upcrossing probability is a better criterion to minimize. In Chapter 4 this approximation will be justified in an example, where it is seen that the MU controller performs up to 10% better than the MV controller with respect to the risk criterion.

3.4 *The Mean Time Between Failures Criterion*

Consider the following problem formulation:

$$\max_{H \in \mathcal{D}_z} \mathrm{E}\{T\} \tag{3.11}$$

where T is the time between two consecutive upcrossings of z_0 by z. The expectation should be evaluated in stationarity.

To simplify the problem, approximations for the expectation in (3.11) are given in Part II. These rely on asymptotic results relating the mean number of exceedances to the mean number of upcrossings. They imply that the expectation in (3.11) for large values of z_0/σ_z can be approximately expressed as

$$\mathrm{E}\{T\} \approx \frac{1}{P\{z(0) > z_0\}} \approx \frac{1}{\mu(z_0)} .$$

which is maximized by minimizing either the variance of z or the upcrossing probability μ. However, for $z_0/\sigma_z < \infty$ it holds that

$$\frac{1}{P\{z(0) > z_0\}} < \frac{1}{\mu(z_0)}$$

and it is tempting to believe that the upcrossing probability is a better criterion to minimize for moderate values of z_0/σ_z. In Part II this approximation will be justified in an example.

3.5 *Concluding Remarks*

In this chapter the control problems that will be considered in the remaining parts of the thesis have been mathematically formalized in a stochastic framework. It has been discussed how both the risk criterion and the MTBF criterion may be approximated by both the variance criterion and with the upcrossing criterion. It has been made plausible that the upcrossing criterion is a better approximating criterion than the variance criterion. This will for the MTBF criterion be justified in an example in Part II, and for the risk criterion in Chapter 4. The material in this chapter is based on Part IV, Hansson (1991d), Hansson (1994).

CHAPTER 4

EXAMPLES

IN this chapter two examples will be investigated. The first one is an example that shows that the MU controller is advantageous as compared with the MV controller when evaluating them with respect to the risk criterion. It is the same process model, the same MU controller, and the same MV controller as in the example of Part II. However, there the comparison is done with respect to the MTBF criterion. The presentation is based on Hansson (1991d). The second example is concerned with control of an active automotive suspension. The objective in this example is to show how it is possible to cast an application example as a stochastic critical control problem. The example is very special in that the infimum of the upcrossing criterion is not attainable by any stabilizing controller. However, the machinery used for computing the MU controller, i.e. LQG controller design, makes it possible to consider also other criteria than the upcrossing criterion. By adding a suitable weighting on the control signal to the LQG criterion corresponding to the MU controller, a stabilizing controller is obtained.

4.1 Illustrative Example

To evaluate the performance of the MU controller obtained by minimizing the upcrossing probability a first order process will be investigated. In this section the equations in Part II for computing the LQG-controllers will be used, since it is easier to get an analytic expression for the optimal controller with these equations than with the ones presented in Chapter 3. In the first subsection the process is defined. The set of LQG-solutions is calculated analytically in the second subsection. In the third subsection the MU controller is computed and compared with the MV controller. It is seen that the MU controller causes a lower upcrossing probability and smaller probability for the largest value of the controlled signal of being above the critical level. Further it is seen that it has a control signal that is more well-behaved. In the fourth subsection the results of the previous sections are summarized.

Process

Let the process be given by

$$\begin{cases} x(k+1) = x(k) + 0.04u(k) + 0.2v(k) \\ y(k) = x(k) + 5e(k) \\ z(k) = x(k) \end{cases} \tag{4.1}$$

where v and e are zero mean Gaussian white noise sequences with $\mathrm{E}v^2 = R_1 = 1$, $\mathrm{E}e^2 = R_2 = 1$ and $\mathrm{E}ve = R_{12} = 0$. The signal y is the measurement signal, and u is the control signal. This process can be obtained approximately by fast sampling of a continuous time integrator process.

LQG-Controllers

The weighting-matrices in Part II are

$$\begin{aligned} Q_1 &= 4(1-\rho) \\ Q_{12} &= 0.08(1-\rho) \\ Q_2 &= 0.0016 \end{aligned}$$

and the solutions to the Riccati-equations in Part II are

$$\begin{aligned} S &= 2\sqrt{\rho(1-\rho)} \\ P &= \frac{0.04+\sqrt{4.0016}}{2} \end{aligned}$$

Some more tedious calculations will give the LQG controller $H(q)$ to be

$$H(q) = -\frac{s_0 q}{r_0 q + r_1}$$

where

$$\begin{aligned} s_0 &= \left[2\sqrt{\rho(1-\rho)} + 2(1-\rho)\right]\left[0.04+\sqrt{4.0016}\right] \\ r_0 &= 0.04\left[2\sqrt{\rho(1-\rho)} + 1\right]\left[50.04+\sqrt{4.0016}\right] \\ r_1 &= 2(1-2\rho) \end{aligned}$$

It is interesting to note that for $\rho = 0.5$—MV control—the controller is a proportional controller.

MU and MV Controllers

The MU controller will now be compared with the MV controller.

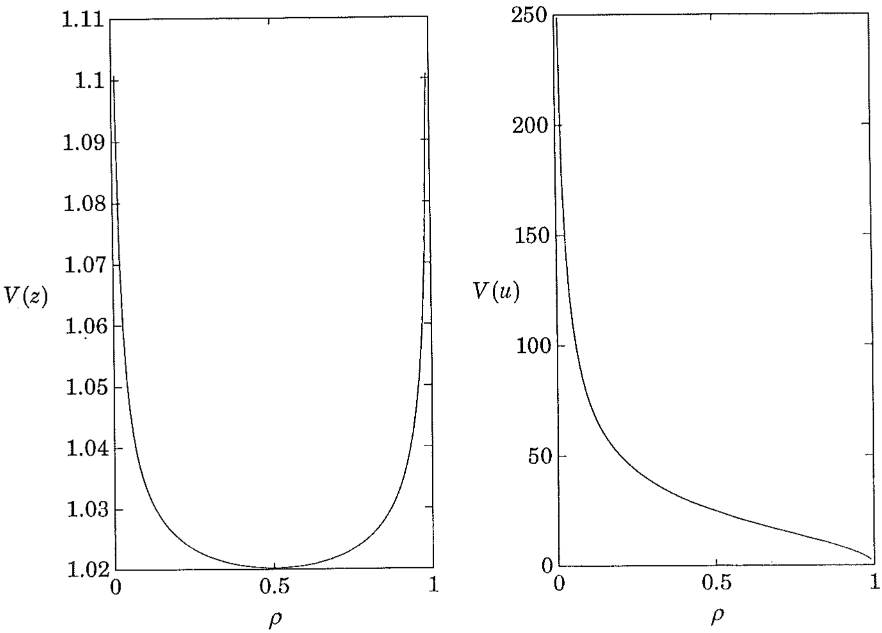

Figure 4.1 The variances of z—left, and u—right, as functions of ρ.

VARIANCE AND UPCROSSING PROBABILITY. The variances of z and u have been calculated numerically for values of ρ with a step of 0.01 in the range of 0.01 to 0.99. It is seen in Figure 4.1 that the variance of z does not depend so much on ρ as does the variance of u.

The probability μ has been calculated for the values $z_0 = 2, 3, 4$ and 5 of the critical level. The result is seen in Figure 4.2. The minimum value of the probability μ is obtained for ρ greater than 0.5. The variance of the control signal is smaller the larger ρ is, and the controller obtained for $\rho = 0.5$ is the MV controller. Thus the MU controller not only minimize the upcrossing probability, but it also has a control signal that is more well-behaved than that of the MV controller.

SIMULATIONS. The controllers have also been compared by simulations. The same noise sequences were used for both controllers in all cases. Figure 4.3 shows plots of z and u as functions of time for the MV controller and the MU controller for $z_0 = 3$. It is seen that that the MU controller manages to keep the signal z below the critical level, while the MV controller does not. Further it is seen that the variance of u is smaller for the MU controller than for the MV controller. Note that z is not white noise for the MV controller although y is, since y is correlated with e.

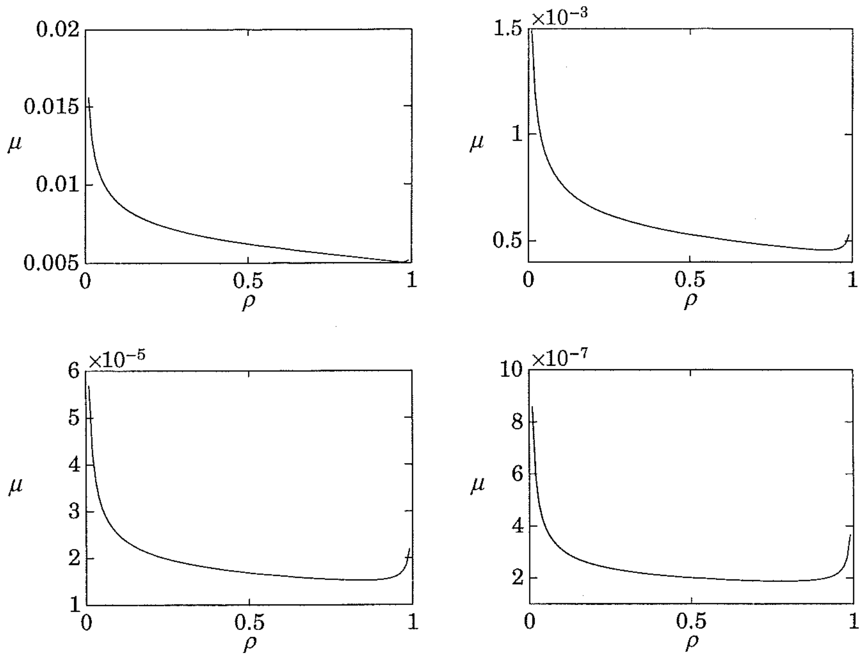

Figure 4.2 The probability μ as function of ρ for $z_0 = 2$—top left, $z_0 = 3$—top right, $z_0 = 4$—bottom left, and $z_0 = 5$—bottom right. Notice the different scales.

ROBUSTNESS. To investigate the robustness against unmodeled non-linearities the process-dynamics was changed to

$$x(k+1) = 0.33x^2(k) + x(k) + 0.04u(k) + 0.2v(k) \tag{4.2}$$

Thus the process for which the controllers are designed can be thought of as a linearization of the non-linear process round $x(k) = 0$. If $v(k)$ is zero, and if the minimum variance control strategy is applied, then the nonlinear process is stable for initial values of x that are smaller than approximately 3. Therefore it is interesting to compare the MU controller designed for $z_0 = 3$ with the MV controller. Plots of y, z, and u for the two different control strategies with the same noise sequences are shown in figures 4.4 and 4.5. It is seen that the signals start to diverge to infinity earlier for the MV controller than for the MU controller.

TRANSFER FUNCTIONS. The MU controller for $z_0 = 3$ ($\rho = 0.92$) is given by:

$$H(q) = -\frac{0.4901q}{q - 0.4804}$$

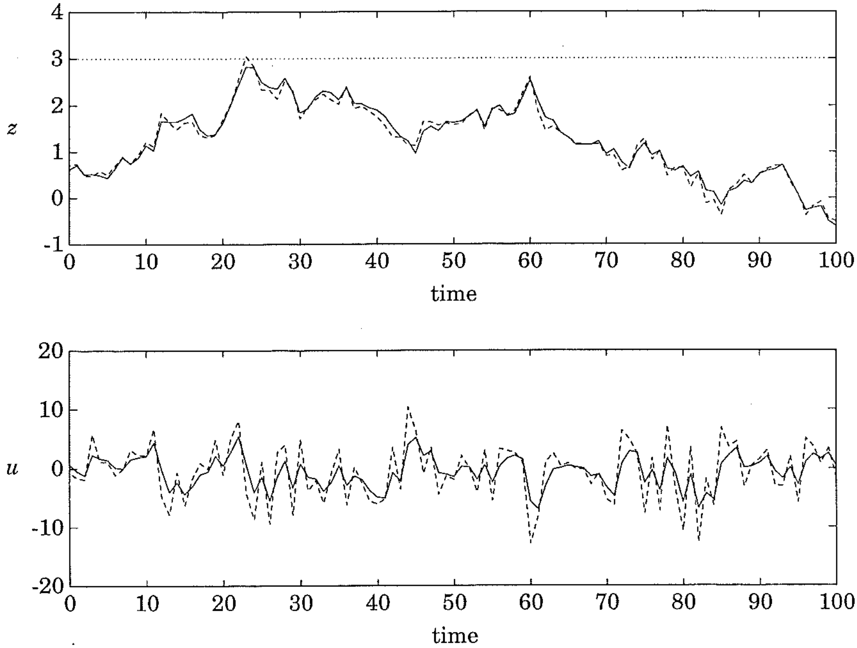

Figure 4.3 The signals $z(t)$—top, and $u(t)$—bottom, as function of time for the optimal controller—solid line, and the minimum variance controller—dashed line, when controlling the linear process (4.1).

and the MV controller is given by:

$$H(q) = -0.9802$$

It is interesting to note that the difference between the MV controller and the MU controller is that the MU controller has a 3 times lower gain for high frequences ($q = -1$). The MU controller is a first order system while the MV controller is only a proportional controller. This explains why the variance of the control signal is much smaller for the MU controller. Some calculations give that

$$(q - 0.9608)z = 0.2v - 0.196e$$

for the MV controller and

$$[(q-1)(q-0.4804) + 0.0196]z = 0.2(q-0.4804)v - 0.098e$$

for the MU controller. It is seen that the main difference in the closed loop behavior between the MV controller and the MU controller is the lower high frequency gain ($q = -1$) from e to z for the MU controller.

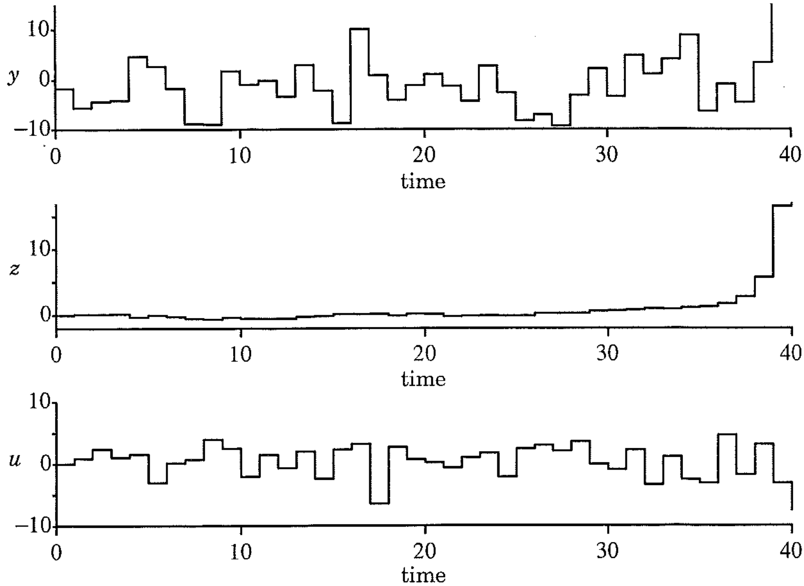

Figure 4.4 The signals $y(t)$, $z(t)$ and $u(t)$ as functions of time for the MU controller, when controlling the non-linear process (4.2).

APPROXIMATION-VALIDITY. The validity of the assumptions made in the approximation of the problem formulation in Section 3.3 will now be investigated further; one positive indication has already been seen in Figure 4.3. In Figure 4.6 it is seen how the optimal value of ρ, and how the relative decrease of upcrossing probability between the MV controller and the MU controller decreases as z_0 increases. This indicates that the MU controller and the MV controller are approximately the same for large values of z_0.

To investigate the behavior of the controllers for moderate values of z_0, Monte Carlo simulations have been performed to estimate the probability $\mathrm{P}\{\max_{0\le k\le N} z(k) > z_0\}$ for the MU controller—$\hat{P}_{\mathrm{opt}}$, and for the MV controller—$\hat{P}_{\mathrm{mv}}$. The estimated values all have 90 % confidence intervals that are smaller than plus minus 2.2 % ($z_0 = 2$), 9.5 % ($z_0 = 3$), and 17 % ($z_0 = 4$) of the estimated values. These intervals have been computed as in Waerden (1969) p. 33. In Figure 4.7 these estimates of the probabilities are compared with the bounds P_1 and P_2 of Section 3.3, where for short reference

$$P_1(z_0) = \mathrm{P}\{z(0) > z_0\} + N\mu(z_0)$$
$$P_2(z_0) = (N+1)\mathrm{P}\{z(0) > z_0\}$$

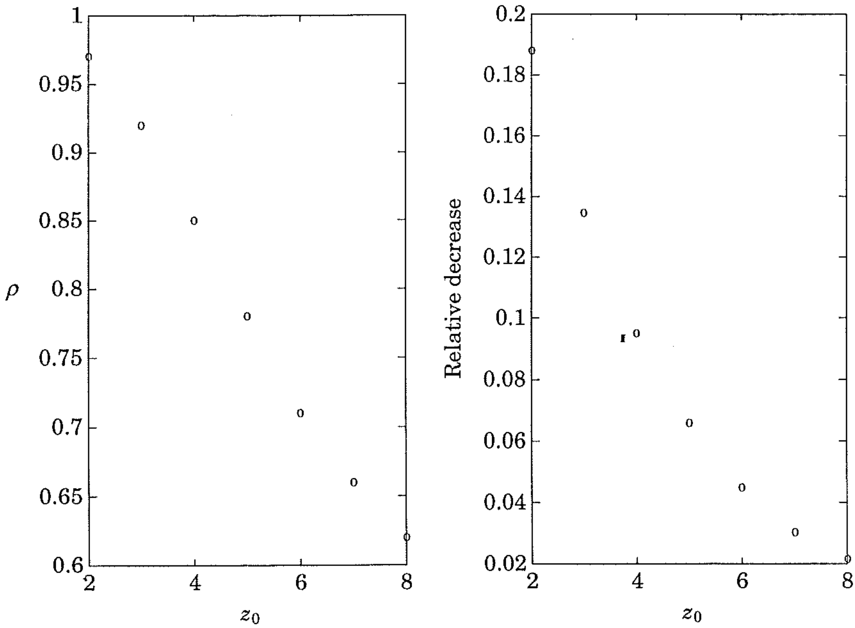

Figure 4.6 The optimal values of ρ as function of z_0—left, and $(\mu_{\text{mv}}-\mu_{\text{opt}})/\mu_{\text{mv}}$ as function of z_0—right, where μ_{mv} is the upcrossing probability for the MV controller and μ_{opt} is the upcrossing probability for the MU controller.

MV controller is only a proportional controller. The latter has a higher high-frequency gain. The variance of z is slightly larger but the variance of u is much smaller for the MU controller as compared with the MV controller. Further it has been seen in simulations that the probability for the largest value of z of being above the critical level is smaller for the MU controller. It has also been seen that the MU controller is more robust against unmodeled non-linearities than the MV controller. When comparing the differences between the MU controller and the MV controller for varying distances to the critical level, it has been seen that these are larger for moderate values of the distance and smaller for larger values of the distance.

4.2 Active Automotive Suspension Control

In this section MU active control of suspensions for cars will be investigated. Active suspension control has been described in e.g. Hrovat (1982),

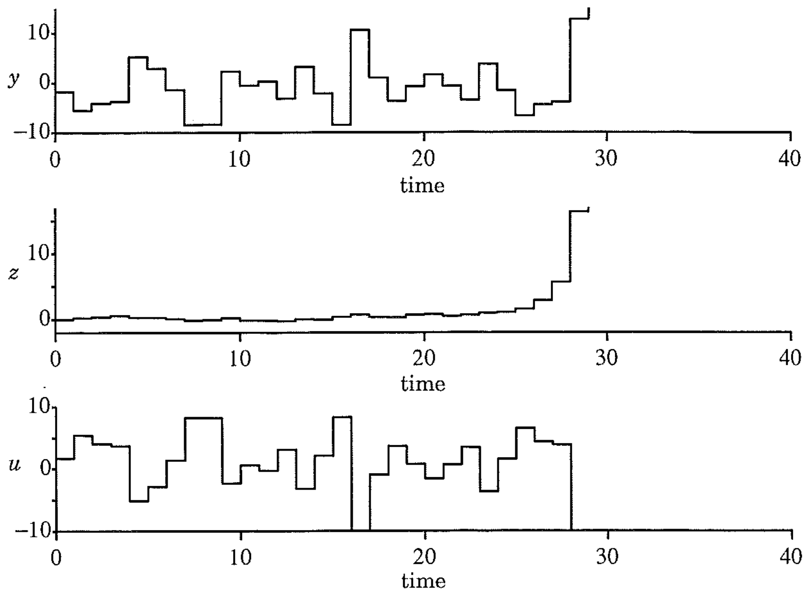

Figure 4.5 The signals $y(t)$, $z(t)$ and $u(t)$ as functions of time for the the MV controller, when controlling the non-linear process (4.2).

The bound P_1, which is approximately minimized by the MU controller, has been computed for the MU controller. The bound P_2, which is minimized by MV control, has been computed for the MV controller. The values of N and z_0 has been chosen such that the bound P_2 is about 0.1. The values are $(z_0, N) = (2, 10), (3,100)$ and $(4,1000)$. The result is shown in Figure 4.7. It is seen in the left plot that the bound P_1 is much lower than the bound P_2, and that the estimate $\hat{P}_{\rm opt}$ is lower than estimate $\hat{P}_{\rm mv}$. The latter is seen more clearly in the right plot, where the relative decrease of the probability of being above the critical level between the MV controller and the MU controller— $(\hat{P}_{\rm mv} - \hat{P}_{\rm opt})/\hat{P}_{\rm mv}$—is plotted versus z_0. Thus the MU controller performs about 5% to 10% better than the minimum variance controller for moderate values of the critical level in this example.

Summary

The theory presented in the previous chapter has been evaluated using a first order process. In spite of the simplicity of the process many interesting features of the MU controller have been demonstrated. It has been shown that the MU controller is a first order system whereas the

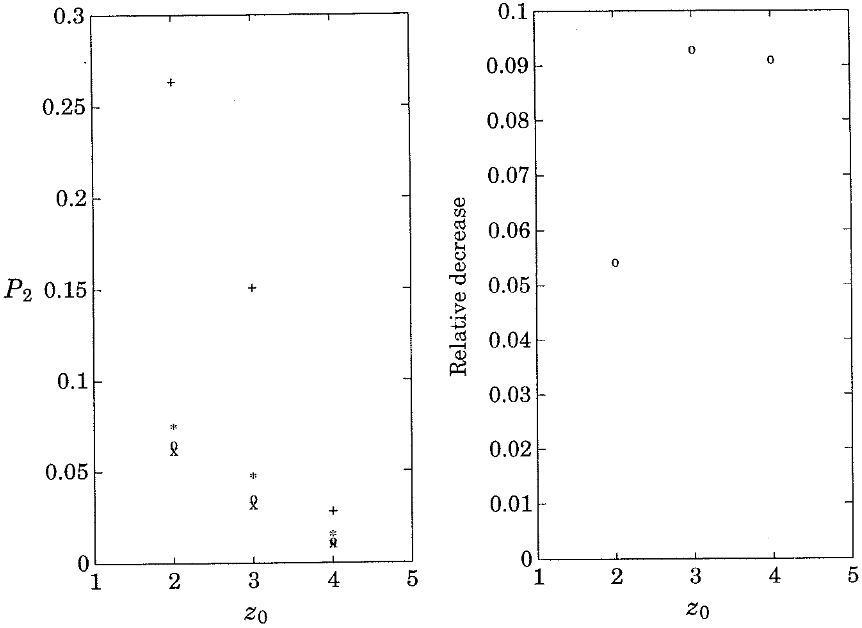

Figure 4.7 The left plot shows the bound P_2 for the MV controller—'+', the bound P_1 for the MU controller—'*', $\hat{P}_{mv}$—'o', and $\hat{P}_{opt}$ —'x', as functions of z_0. The values of N has been 10 for $z_0 = 2$, 100 for $z_0 = 3$ and 1000 for $z_0 = 4$. The right plot shows $(\hat{P}_{mv} - \hat{P}_{opt})/\hat{P}_{mv}$ as function of z_0.

Yamashita *et al.* (1990), Ando *et al.* (1993), Cai and Konik (1993), Titli *et al.* (1993), Obinata *et al.* (1993), Roukieh and Titli (1993). Here control of the normal force of the road acting on the wheel will be investigated. This is of special interest for sports cars. It will be seen that this type of control problem is difficult, since it is not possible to change the normal force of the road in stationarity by applying a constant control force between the wheel and the car body. This difficulty will also show up as a theoretical problem in terms of a zero on the unit circle in the transfer function from control signal to normal force.

In the first subsection the process model together with the control objectives will be given. The process model is a linear discrete time stochastic difference equation with four states describing the dynamics of the road and a quarter model car, i.e. a car with one wheel. The control objective is to prevent the normal force between the road and the wheel from upcrossing zero, i.e. the wheel should not lose its contact with the road. In the second subsection the MU controller is computed for different vari-

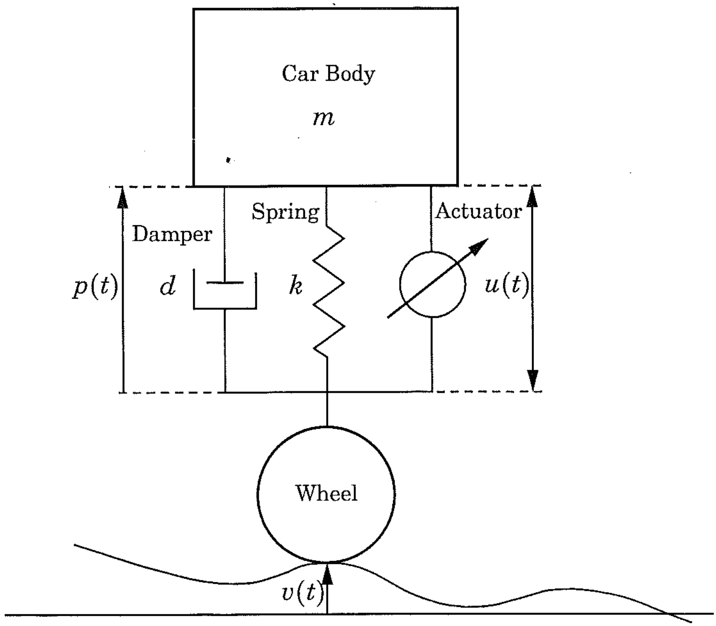

Figure 4.8 Process model of a car with one wheel and active damping.

ability in the road. It will be seen that there exists no optimal controller, since the infimal value of the upcrossing probability is attained for a controller that does not stabilize the closed loop system. This is due to the zero on the unit circle. Simulations are performed to evaluate the design. One of the infimal controllers is compared with a suboptimal controller that stabilizes the closed loop system. Finally, in the last subsection some concluding remarks are given.

Process Model and Control Objective

A car with one wheel can be modeled as indicated in Figure 4.8. The differential equation describing the motion of the distance $p(t)$ between the wheel and the car body due to the influence of the road profile $v(t)$ and the damping force $u(t)$ used for control is:

$$m\left[\frac{d^2p(t)}{dt^2}+\frac{d^2v(t)}{dt^2}\right] = u(t) - k\left[p(t)-p_0\right] - mg - d\frac{dp(t)}{dt}$$

where m is the mass of the car body, k is the spring coefficient, p_0 is the unsprung length of the spring, $g = 9.81\text{m/s}^2$, and where d is the damping ratio of the passive damper. The normal force $N(t)$ from the road acting on the wheel is given by

$$N(t) = u(t) - k\left[p(t) - p_0\right] - d\frac{dp(t)}{dt}$$

The control objective is to prevent this signal from downcrossing 0. Notice that the motion of the car body and the normal force are not described by the equations given above in case the car leaves the road. This will, however, not be addressed in the presentation given. Had this been done, then the model would have been nonlinear, and it would not have been possible to apply the available theory about MU controllers which assumes linear processes.

Introduce the following states for the differential equation:

$$x_1(t) = p(t) - p^0; \qquad x_2(t) = \frac{dx_1(t)}{dt}$$

where $p^0 = p_0 - mg/k$ is the stationary value of $p(t)$ for $u(t) = 0$ and $d^2v(t)/dt^2 = 0$. With $x_c(t) = \begin{pmatrix} x_1(t) & x_2(t) \end{pmatrix}^T$ the equation for the motion of the car body can be written

$$\frac{dx_c(t)}{dt} = \begin{pmatrix} 0 & 1 \\ -k/m & -d/m \end{pmatrix} x_c(t) + \begin{pmatrix} 0 \\ 1/m \end{pmatrix} u(t) + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \frac{d^2v(t)}{dt^2}$$

Notice that the normal force can be expressed in terms of the states and the control signal as $N(t) = mg - kx_1(t) - dx_2(t) + u(t)$. Hence the downcrossing of zero by $N(t)$ is equivalent to the upcrossing of mg by $z(t) = \begin{pmatrix} k & d \end{pmatrix} x_c(t) - u(t)$. Furthermore it is now seen that the transfer function from u to z is given by

$$\frac{ms^2}{ms^2 + ds + k} \tag{4.3}$$

which has two zeros at the origin. Let the measurement signal $y(t)$ that will be used for control be the state $x_1(t)$, which can be expressed as $y(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} x_c(t)$. It now only remains to model the road profile $v(t)$. If the vehicle is moving with constant speed along the road it can be modeled as a locally stationary Gaussian process with spectral density

$$S(\omega) = c/\omega^a, \quad 2 \le a \le 3, \quad \omega_{\min} \le |\omega| \le \omega_{\max}$$

and zero otherwise, see e.g. Bormann (1978), Lindgren (1981). In this presentation $\omega_{\min} = 0$ and $a = 2$. Furthermore to be able to model the

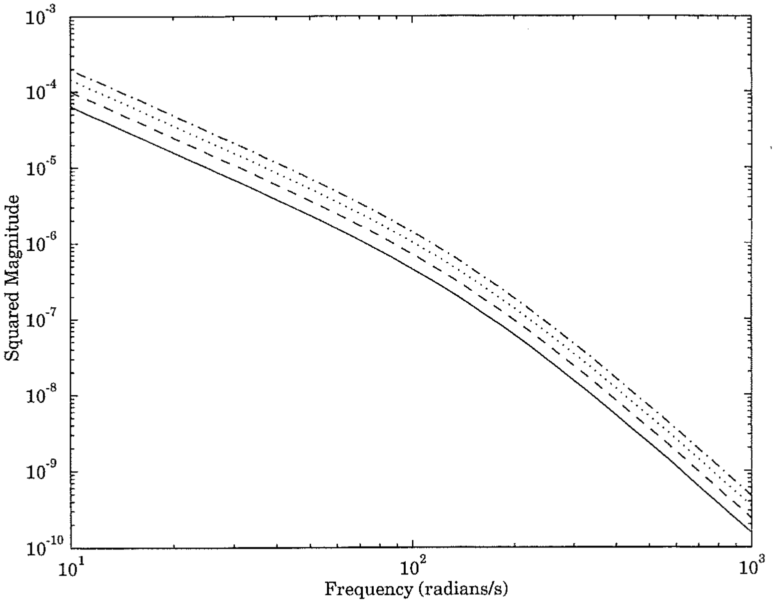

Figure 4.9 The spectral density of $v(t)$ for the parameter values $\omega_{\max} = 157\mathrm{s}^{-1}$ and $c = 0.08$—solid line, 0.10—dashed line, 0.12—dotted line, and 0.14—dash-dotted line.

road profile as a linear filtration of a standard Wiener process, the spectral density will not be taken to be zero outside the cutoff frequency $\omega_{\max}$, but instead it will just asymptotically decay two times faster than before the cutoff frequency. This can be accomplished by the following equation

$$\begin{aligned} dx_v(t) &= \begin{pmatrix} 0 & 1 \\ 0 & -\omega_{\max} \end{pmatrix} x_v(t)dt + \begin{pmatrix} 0 \\ c\omega_{\max} \end{pmatrix} de(t) \\ v(t) &= \begin{pmatrix} 1 & 0 \end{pmatrix} x_v(t) \end{aligned}$$

where $x_v(t) = \begin{pmatrix} x_3(t) & x_4(t) \end{pmatrix}^T$, and where $e(t)$ is a standard Wiener process. The spectral densities for different values of c are plotted in Figure 4.9. Let $x(t) = \begin{pmatrix} x_c^T(t) & x_v^T(t) \end{pmatrix}^T$. Then it holds that

$$\begin{aligned} dx(t) &= Ax(t)dt + B_u du(t) + B_w de(t) \\ z(t) &= C_z x(t) + D_{zu} u(t) \\ v(t) &= C_v x(t) \\ y(t) &= C_y x(t) \end{aligned} \tag{4.4}$$

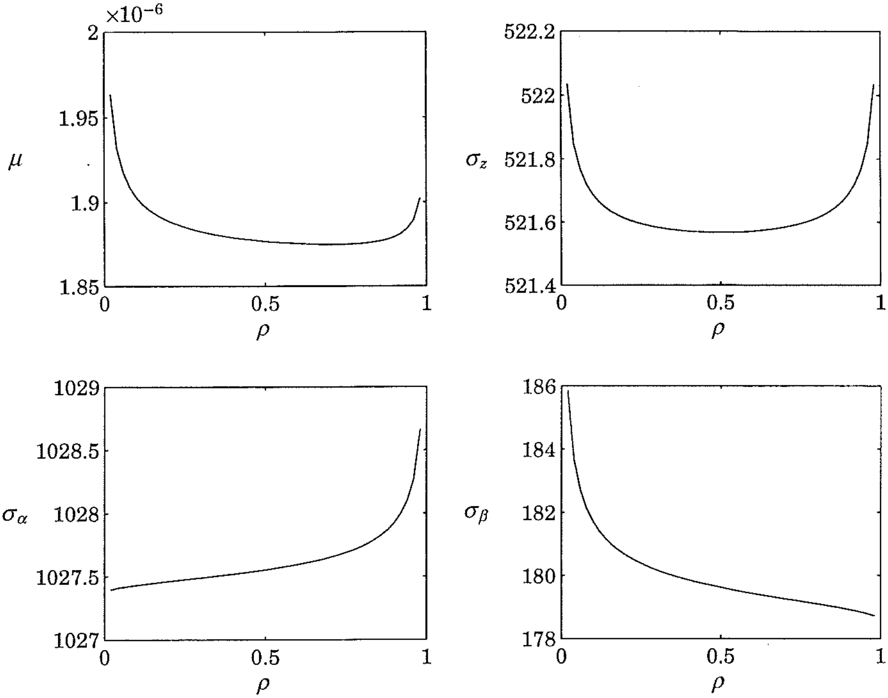

Figure 4.10 The upcrossing probability and the standard deviations of z, α, and β for the LQG controllers as functions of ρ for $c = 0.08$.

where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -k/m & -d/m & 0 & \omega_{\max} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\omega_{\max} \end{pmatrix} \quad B_u = \begin{pmatrix} 0 \\ 1/m \\ 0 \\ 0 \end{pmatrix}$$

$$B_w = \begin{pmatrix} 0 \\ -c\omega_{\max} \\ 0 \\ c\omega_{\max} \end{pmatrix} \quad C_z = \begin{pmatrix} k & d & 0 & 0 \end{pmatrix}$$

$$C_v = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} \quad D_{zu} = -1$$

$$Cy = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}$$

Now some reasonable parameter values are needed. From Ando *et al.* (1993) the values $m = 240$kg, $k = 16000$N/m, and $d = 980$Ns/m have

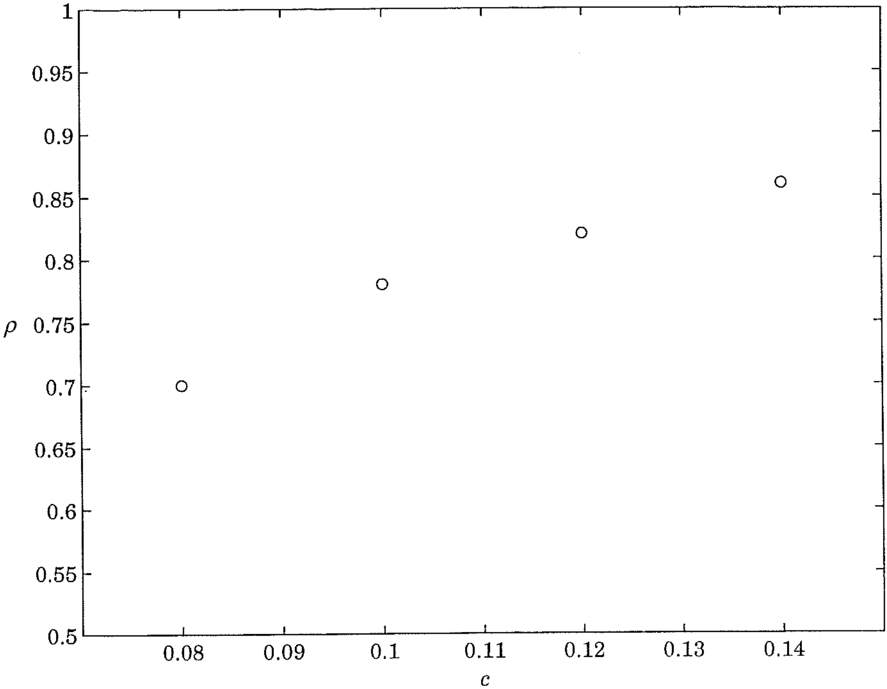

Figure 4.11 The optimal values of ρ for different values of c.

been taken. Furthermore assume that the car is moving with the constant speed of 90km/h = 25m/s, and that the shortest period-length of variations in the road is 1m. Then the corresponding period measured in seconds is $T = 1/25\text{s} = 0.04\text{s}$. Hence it would be reasonable to choose $\omega_{\max} = 2\pi/T = 157\text{s}^{-1}$. Notice that $mg = 2354\text{N}$, which is the level that the signal z should not upcross.

In order to make discrete-time synthesis a sampled version of the above model is necessary. A reasonable choice of sample interval is obtained by considering the distance of the eigenvalues of the the system matrix to the origin. These distances are given by $\sqrt{k/m} = 8.1650\text{s}^{-1}$, 0s^{-1}, and $\omega_{\max} = 157\text{s}^{-1}$. In order to be able to see the influence of $\omega_{\max}$, a reasonable choice is to take the sample interval to $h = 2\pi/\omega_{\max}/50 = 0.0008\text{s}$, see e.g. Åström and Wittenmark (1990) Chapter 3 for upper limits. A sampled version with sample interval h of the equation above can be obtained by computing a matrix exponential as described in van Loan (1978), also see Åström and Wittenmark (1990) for the relation to

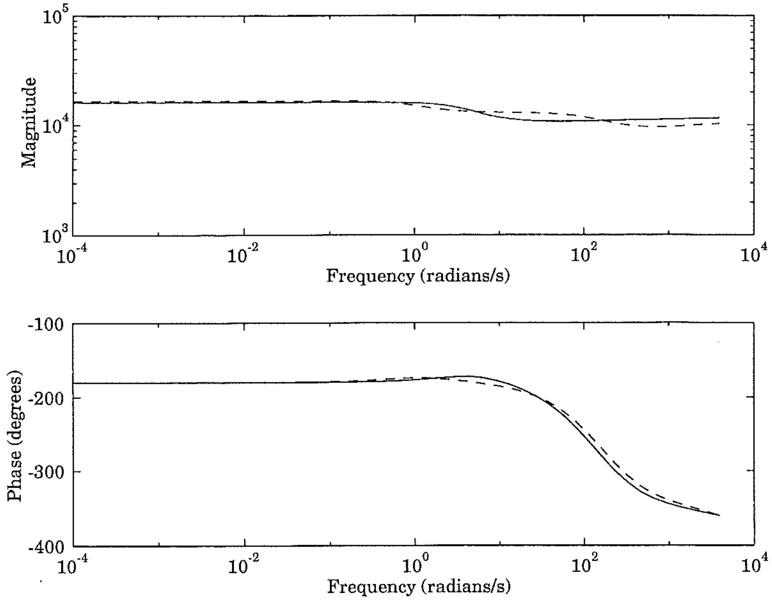

Figure 4.12 Bode-diagram for the transfer function of the MU controller—solid line, and the suboptimal LQG controller—dashed line, for $c = 0.08$.

sampling. To this end introduce the matrix

$$C = \begin{pmatrix} -A^T & B_w B_w^T & 0 \\ 0 & A & B_u \\ 0 & 0 & 0 \end{pmatrix},$$

and let

$$e^{Ch} = \begin{pmatrix} F_1 & G_1 & H_1 \\ 0 & F_2 & G_2 \\ 0 & 0 & F_3 \end{pmatrix}.$$

Furthermore let $\Phi = F_2$, $\Gamma_u = G_2$, let Γ_w be a matrix that solves $\Gamma_w \Gamma_w^T = F_2^T G_1$, and let $x(k)$ be governed by

$$x(k+1) = \Phi x(k) + \Gamma_u u(k) + \Gamma_w w(k)$$

where $w(k)$ is a sequence of independent Gaussian random vectors of appropriate dimension with zero mean and covariance I. Then it holds that $x(k)$ has the same mean and covariance as $x(t)$ defined by (4.4) at

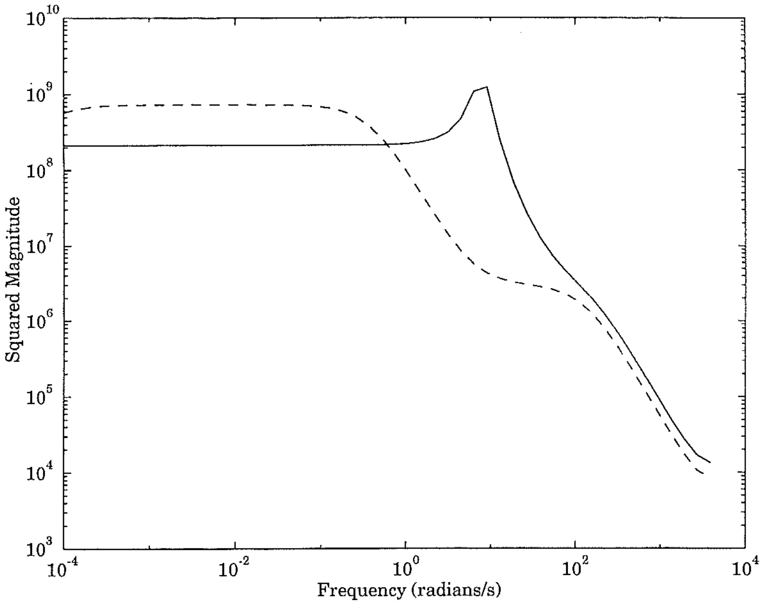

Figure 4.13 The open loop—solid line, and closed loop (MU controller)—dashed line, spectral densities of z for $c = 0.08$.

the sample instances. The overall discrete time model becomes

$$\begin{aligned}
x(k+1) &= \Phi x(k) + \Gamma_u u(k) + \Gamma_w w(k) \\
z(k) &= C_z x(k) + D_{zu} u(k) \\
v(k) &= C_v x(k) \\
y(k) &= C_y x(k)
\end{aligned}$$

where for sample interval $h = 0.0008$s and parameter value $c = 0.08$ it holds that

$$\Phi = \begin{pmatrix} 1 & 0.0008 & 0 & 0.0000 \\ -0.0532 & 0.9967 & 0 & 0.1179 \\ 0 & 0 & 1 & 0.0008 \\ 0 & 0 & 0 & 0.8819 \end{pmatrix}$$

$$\Gamma_u = 10^{-5} \begin{pmatrix} 0.0001 \\ 0.3328 \\ 0 \\ 0 \end{pmatrix}; \qquad \Gamma_w = \begin{pmatrix} 0.0093 & -0.0058 & 0.0001 \\ -0.3547 & 0.0118 & 0.0000 \\ 0 & 0 & 0 \\ 0.3132 & 0.0135 & 0.0000 \end{pmatrix}$$

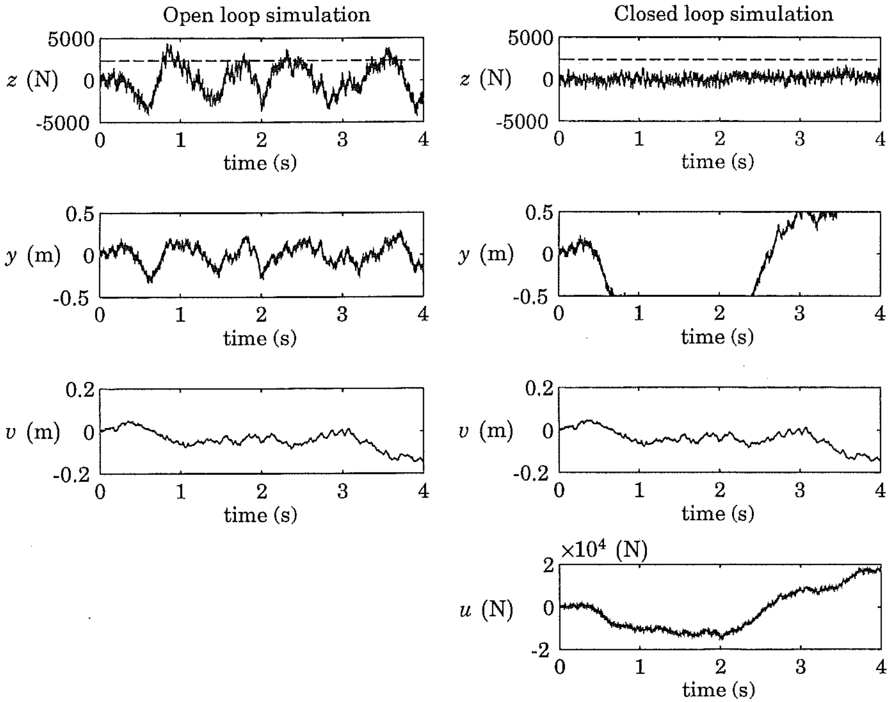

Figure 4.14 Simulation of open and closed loop system for the MU controller for $c = 0.08$.

$$C_z = \begin{pmatrix} 16000 & 980 & 0 & 0 \end{pmatrix}; \qquad D_{zu} = -1$$
$$C_v = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}; \qquad C_y = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}$$

The zero entries that are written 0 are not only approximately equal to zero, but exactly equal to zero due to the structure of the model. The same goes for the one entries. Especially notice that the state x_3 is not stabilizable. However, it is not influencing any other state, and it is only observable in the signal v. Hence this state can be removed when designing controllers for objectives related only to z. A reduced order model is given by

$$\begin{aligned} \xi(k+1) &= \Phi\xi(k) + \Gamma_u u(k) + \Gamma_w w(k) \\ z(k) &= C_z\xi(k) + D_{zu}u(k) \\ y(k) &= C_y\xi(k) \end{aligned}$$

where $\xi^T(k) = \begin{pmatrix} x_1(k) & x_2(k) & x_4(k) \end{pmatrix}^T$, and where

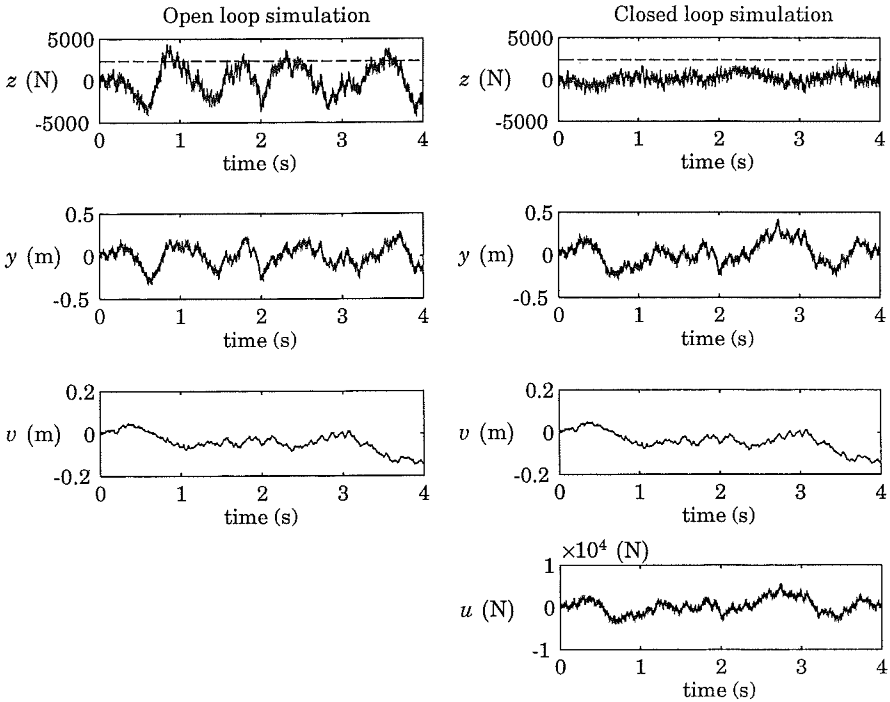

Figure 4.15 Simulation of open and closed loop system for the suboptimal controller for $c = 0.08$.

$$\Phi = \begin{pmatrix} 1.0000 & 0.0008 & 0.0000 \\ -0.0532 & 0.9967 & 0.1179 \\ 0 & 0 & 0.8819 \end{pmatrix}$$

$$\Gamma_u = 10^{-5} \begin{pmatrix} 0.0001 \\ 0.3328 \\ 0 \end{pmatrix}; \qquad \Gamma_w = \begin{pmatrix} 0.0093 & -0.0058 & 0.0001 \\ -0.3547 & 0.0118 & 0.0000 \\ 0.3132 & 0.0135 & 0.0000 \end{pmatrix}$$

$$C_y = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}; \qquad C_z = \begin{pmatrix} 16000 & 980 & 0 \end{pmatrix}; \qquad D_{zu} = -1$$

Notice that it is trivial to compute $v(k)$ from the recursion $v(k+1) = v(k) + h\xi_3(k)$, $h = 0.0008$. Thus there is no need to use any other model than the reduced order one with three states. This is the model that will be used from now on.

The control objective to be considered is the MU criterion. Notice that the assumptions on stabilizability and detectability are fulfilled, but that one of the rank conditions on the unit circle is not. The zeros of

$$\begin{pmatrix} zI - \Phi & -\Gamma_u \\ C_z & D_{zu} \end{pmatrix}$$

are located at 1, 0.99998, and 0.8819, which implies that the rank condition is violated for $z = 1$. The zero at 1 comes from the zeros at the origin of (4.3), see Åström *et al.* (1984). This implies that the theoretical results of Part IV may not be valid. In Part III there is no direct problem with zeros on the unit circle, and hence it will still be possible to compute the LQG controllers, which may only be infimal and not optimal. Also notice that the necessary condition of Part II still is valid. i.e. if there exists a MU controller then it can be obtained by performing one-dimensional optimization over the set of LQG optimal variances of α and β. From a practical point of view this is not an inherently difficult control problem. The zero at $z = 1$ just means that it is not possible to change the normal force of the road acting on the wheel in stationary by applying a constant force between the wheel and the car body. The theoretical formulation is, however, ill-posed, because the zero at $z = 1$ may turn up as an eigenvalue of the closed loop system, and then there will be no optimal controller, only an infimal controller. The result is still useful because it states the limit of achievable performance, i.e. no stabilizing controller can do better. It is most likely possible to obtain a stabilizing controller that is arbitrarily close the infimal one with respect to the upcrossing criterion, but that is not yet formally proven.

Computations and Simulations

Now the MU controller will be computed for different values of the road parameter c. This parameter is closely related to the variability of the road—remember Figure 4.9. The values of the upcrossing probability and the standard deviation of z for the LQG controllers are in Figure 4.10 plotted versus ρ for $c = 0.08$. It is seen that the infimal value of the MU probability is attained for a value of $\rho \geq 0.5$. However, the difference in upcrossing probability between the MV controller and the MU controller is not significant—only about 0.1%. To obtain a larger difference larger values of c have to be considered, but then the value of the upcrossing probability will be too high from a practical point of view—remember that an approximate value of the mean time between upcrossings is given by h/μ, which implies the value 420s for $c = 0.08$. In Figure 4.11 the optimal values of ρ are plotted versus c. It is seen that the infimal value of the upcrossing probability is attained for higher values of ρ the larger c is. In Figure 4.12 the transfer function of the MU controller for $c = 0.08$ is plotted. The controller is unstable and has negative low frequency gain.

Figure 4.13 gives the open loop and closed loop spectral densities of the normal force. It is seen that the controller has lower gain for higher frequences than for lower frequences. Furthermore it is seen that this controller moves some of the higher frequency content in the spectral density to lower frequences. Especially the open loop peak at about $10s^{-1}$ is removed , i.e. the eigenvalues corresponding to the spring have been damped by the MU controller. In Figure 4.14 the results of open loop and closed loop simulations for the MU controller with $c = 0.08$ are presented. Notice that the infimal controller is not stabilizing. It is intuitive that there is no reason for the control signal to remain bounded when only considering the normal force as control criterion. This follows from the fact that when the control signal has changed in order to compensate for a normal force that was different from the reference value, then the mean of the normal force will in stationarity not be affected by this change in control signal. Hence there is no reason to move the control signal back to zero after a compensation has been done. In order to obtain a stabilizing controller an LQG controller with extra weighting on the control signal in the performance index has been computed, i.e. a term $0.01u^2(k)$ has been added to the LQG criterion corresponding to the MU controller. Its transfer function is seen in Figure 4.12. It is close to the MU controller. In Figure 4.15 simulation results are presented, which are not too far from the ones obtained in Figure 4.14 with respect to normal force. Notice that the closed loop system is stable for this suboptimal controller. This was not the case for the infimal MU controller in Figure 4.14. Further notice that the measurement signal $y(k)$ is also more well-behaved.

Summary

The theory presented in the previous chapter was evaluated on an application example concerned with active automotive suspension control. The control objective was to prevent the car from losing its contact with the road. It was seen that this problem is possible to cast as a MU control problem. There is no solution to the optimization problem in terms of an optimal controller. However, it is possible to obtain the limit of achievable performance in terms of an infimal controller. Also a suboptimal controller was computed which stabilizes the closed loop system. To summarize, in this example it has been seen that the MU controller may have the same draw-backs as the MV controller, i.e. large variations in the control signal. However, it is possible by modifying the associated LQG problem with extra weighting on the control signal to obtain practically usable controllers, and it is of course also possible to compute the upcrossing probability for this suboptimal controller in order to evaluate its performance.

4.3 Concluding Remarks

The theory presented in the previous chapter has been evaluated using two examples. In spite of the simplicity of the examples many interesting features of the MU controller have been demonstrated. It has been seen that the infimal value of the upcrossing probability is always obtained for a value of ρ that is greater or equal to 0.5. Furthermore it seems to be the case that the difference between MU control and MV control with respect to the upcrossing criterion is larger the closer the distance to the critical level is. Also it has been seen that the MU controller usually has a smaller variance of the control signal than has the MV controller. However, some times the MV and MU controllers share the same bad properties.

CHAPTER 5

OUTLINE

THE rest of the thesis consists of four parts. They are published in international journals or have been submitted to such a journal for possible publication. The scope of this outline is to give a brief description of what is treated in each part and how the different parts are related. The parts have been written independently of one another, and the notation is not fully consistent.

Part I—Control of Level Crossings in Stationary Gaussian Random Processes

In this part the continuous time upcrossing criterion in minimized. The solution is given as a one-parametric optimization over a set of LQG-problem solutions. It can sometimes be thought of as finding optimal weightings in an LQG-problem.

This part has been published in Hansson (1993a). It is a revised version of Hansson (1991a). Related work is Hansson (1991c).

Part II—Control of Mean Time Between Failures

In this part the discrete time MTBF control problem is treated. It is shown that this control problem is closely related both to the problem of minimizing the variance of the signal—MV control—and to the problem of minimizing the so called upcrossing probability—MU control. It is made plausible that the upcrossing probability is a better approximating criterion to minimize than the variance criterion. The problem of minimizing the upcrossing probability can be thought of as finding optimal weighting-matrices in an LQG-problem.

This part has been published in Hansson (1994). It is a revised version of Hansson (1991d). Related work is Hansson (1991b), Hansson (1992a).

Part III—Existence of Discrete-Time LQG-Controllers

In this part existence results for the discrete time LQG-controller are investigated. Assuming left and right invertibility gives a unique Riccati equation based controller, potentially with closed loop eigenvalues on the unit circle. It is shown that this controller is optimal if and only if it stabilizes the system after removal of all its unobservable and uncontrollable modes. This condition is a considerable simplification of the more general geometric condition recently derived by Trentelman and Stoorvogel, Trentelman and Stoorvogel (1993).

This part will appear in Systems & Control Letters during 1995. It is coauthored with Dr. Per Hagander, and it is a revised version of Hagander and Hansson (1994b). Related work is Hagander and Hansson (1994a).

Part IV—Existence of Minimum Upcrossing Controllers

In this part the existence of the discrete time MU controller is investigated. As in parts I and II the optimal controller can be obtained from a one-parametric optimization over a set of LQG control problem solutions. However, in this part the LQG formulation is slightly different, and finding the MU controller can be interpreted as finding an optimal costing transfer function. The existence of the MU controller is investigated in a constructive way. To this end rank-conditions on the unit circle are imposed, and the results of Part III are used extensively. It is shown that the existence of the MU controller is equivalent to the existence of a MV controller with sufficiently small closed loop variance.

This part has been submitted for possible publication. It is coauthored with Dr. Per Hagander, and it is a revised version of Hansson and Hagander (1994). Related work is Hansson (1992b), Hansson (1993b).

CHAPTER 6

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Part I

Control of Level-Crossings in Stationary Gaussian Random Processes

A new optimal stochastic control problem that minimizes the intensity for a signal to upcross a level is solved by rewriting it as a one-parametric optimization problem over a set of LQG-problem solutions. The solution can sometimes be thought of as finding optimal weightings in an LQG-problem.

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1. Introduction

There are a lot of control problems where the goal is not only to keep the controlled signal near a certain reference value, but also to prevent it from upcrossing a level. The distance between the level and the reference value is normally not small, since otherwise the upcrossing intensity will be intolerably high. However, there may be other control-objectives that make it undesirable or impossible to choose the distance large. Examples of problems of this kind can be found for example in sensor-based robotics and force control, Hansson and Nielsen (1991).

The controller designed below is obtained by solving a one-parametric optimization problem over a set of LQG-problem solutions, and it can sometimes be interpreted as choosing optimal weighting matrices in an LQG-problem. In Hansson (1991b) and Hansson (1991c) the problem is solved for the discrete time case; here the continuous time case is treated, which previously has been described in Hansson (1991a).

In Section 2 the control problem is formulated. It is an optimal stochastic control problem. In Section 3, the problem presented in Section 2 is solved. In Section 4, the optimal controller found in Section 3 is computed for a second order process. Finally, in Section 5, the results are summarized.

2. Control Problem

Let z be a stationary Gaussian process defined by

$$\begin{cases} dx = Axdt + B_1du + B_2dv \\ dy = C_1xdt + Dde \\ z = C_2x \end{cases} \tag{1}$$

where v and e are zero-mean Wiener-processes with $\mathrm{E}dvdv^T = R_1dt$, $\mathrm{E}dede^T = R_2dt$ and $\mathrm{E}dvde^T = R_{12} = 0$. The results below are easyly generalized to $R_{12} \neq 0$. The signal y is the measurement, of which the control signal u is constrained to be a linear time-invariant feedback. It is assumed that the mean $m_z = \mathrm{E}\{z\}$ of z is equal to a predescribed constant reference value. Further, it is assumed that the covariance $r_z(\tau) = \mathrm{E}\{(z(t+\tau)-m_z(t+\tau))(z(t)-m_z(t))\}$ has a finite second derivative at the origin. This implies that C_2B_2 must be equal to zero.

LEMMA 1—Rice's Formula
If $r_z(\tau)$ has a finite second derivative for $\tau = 0$, then the mean number of upcrossings of the level z_0 per unit time is given by

$$\mu = \frac{1}{2\pi}\frac{\sigma_{\dot{z}}}{\sigma_z}\exp\left(-\frac{(z_0 - m_z)^2}{2\sigma_z^2}\right) \tag{2}$$

where $\sigma_z^2 = r_z(0)$ and $\sigma_{\dot{z}}^2 = -r_z''(0)$.

Proof: See Theorem 7.3.2 in Leadbetter *et al.* (1982) or Chapter 10.4 in Cramér and Leadbetter (1967). □

Remark. The quaintity μ in (2) will be called the upcrossing intensity.

Let $\mathcal{D}$ be the set of linear time-invariant stabilizing feedbacks of (1), and let $\mathcal{D}_z$ be the set of linear time-invariant stabilizing feedbacks of (1) for which $\sigma_z \leq z_0 - m_z$ holds.

The control problems mentioned in Section 1 are captured in the following problem formulation:

$$\min_{H \in \mathcal{D}_z} \mu \tag{3}$$

where μ is given by (2). The restriction on σ_z will exclude the trivial solution $\sigma_z = \infty$ for minimizing μ.

3. *Regulator Design*

In the first subsection, the problem of minimizing the upcrossing intensity is rephrased to a one-parametric minimization problem over a set of solutions to LQG-problems. The equations for solving the LQG-problems are given in the second subsection. In the last subsection, the results are summarized.

Solution

It will be seen that the minimization of μ in (2) over $\mathcal{D}_z$ can be done by first minimizing

$$J = \mathrm{E}\{z^2 + \rho^2\dot{z}^2\} \tag{4}$$

for $\rho \in [0,\infty]$ and $m_z = 0$ over $\mathcal{D}$, and then minimizing μ over the solutions obtained in the first minimization, i.e. over $\mathcal{V}_J \cap \mathcal{V}_z$, where

$$\mathcal{V}_J = \left\{(\sigma_z(H), \sigma_{\dot{z}}(H)) \in R^2 \middle| H \in \mathcal{D}_J\right\}$$
$$\mathcal{V}_z = \left\{(\sigma_z, \sigma_{\dot{z}}) \in R^2 \middle| 0 \leq \sigma_z \leq z_0 - m_z,\ 0 \leq \sigma_{\dot{z}}\right\}$$
$$\mathcal{D}_J = \left\{H \in \mathcal{D} \middle| H = \operatorname{argmin} J(H,\rho),\ \rho \in [0,\infty]\right\}$$

Note that it is only assumed that $m_z = 0$ when J is minimized, not when μ is minimized.

In the following lemma J is rewritten to fit into the usual LQG-problem formulation.

LEMMA 2
The loss-function J in (4) can be written

$$J = \mathrm{E}\{x^T Q_1 x + 2x^T Q_{12}\dot{u} + \dot{u}^T Q_2 \dot{u}\}$$

where

$$\begin{aligned} Q_1 &= C_2^T C_2 + \rho^2 A^T C_2^T C_2 A \\ Q_{12} &= \rho^2 A^T C_2^T C_2 B_1 \\ Q_2 &= \rho^2 B_1^T C_2^T C_2 B_1 \end{aligned}$$

Proof: The result follows immediately by using the definition of z in (1). □

The next lemma shows how all jointly minimal variances σ_z^2 and $\sigma_{\dot{z}}^2$ can be obtained by minimizing J. But first a precise definition of jointly minimal will be given.

DEFINITION 1—Pareto optimality
Let $\mathcal{X}$ denote an arbitrary nonempty set. Let $f_i : \mathcal{X} \to R^+, \quad i \in \underline{s}$ be s nonnegative functionals defined on $\mathcal{X}$. Then a point x^0 is said to be Pareto optimal with respect to the vector-valued criterion $f = (f_1, f_2, \cdots, f_s)$ if there does not exist $x \in \mathcal{X}$ such that $f_i(x) \le f_i(x^0)$ for all $i \in \underline{s}$, and $f_k(x) < f_k(x^0)$ for some $k \in \underline{s}$. □

LEMMA 3
Suppose that (A, B_1) is stabilizable, and that (C_1, A) is detectable. Then the set $\mathcal{D}_P$ of Pareto optimal controllers with respect to $(\sigma_z^2, \sigma_{\dot{z}}^2)$, where $\sigma_i^2 : \mathcal{D} \to R^+, \quad i = z, \dot{z}$, is a subset of $\mathcal{D}_J$.

Proof: Let

$$\begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} = \begin{pmatrix} (\,C_2(sI-A)^{-1}B_2 \;\; 0\,) & C_2(sI-A)^{-1}B_1 \\ (\,C_1(sI-A)^{-1}B_2 \;\; D\,) & C_1(sI-A)^{-1}B_1 \end{pmatrix}$$

and let $P_{22} = N_r D_r^{-1} = D_l^{-1} N_l$ be right- and left-coprime factorizations of P_{22} with

$$\begin{pmatrix} V_r & -U_r \\ -N_l & D_l \end{pmatrix} \begin{pmatrix} D_r & U_l \\ N_r & V_l \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

Then, by Theorem 1, p. 38 in Francis (1986), all stabilizing controllers $U = HY$ of (4), where U and Y are Laplace transforms of $\dot{u}$ and $\dot{y}$, can be written $H = H_1 H_2^{-1}$, where

$$\begin{pmatrix} H_1 \\ H_2 \end{pmatrix} = \begin{pmatrix} U_l & -D_r \\ V_l & -N_r \end{pmatrix} \begin{pmatrix} I \\ Q \end{pmatrix}$$

with Q being a stable transfer-function matrix. Thus, the minimization of J over $\mathcal{D}$ can be rephrased to a minimization over Q, where Q belongs to the linear space of stable transfer-function matrices. By Theorem 1 p. 43 in Francis (1986)

$$Z = (P_{11} + P_{12} D_r U_r P_{21} - P_{12} D_r Q D_l P_{21}) \begin{pmatrix} V \\ E \end{pmatrix}$$

where Z, V and E are Laplace transforms of z, $\dot{v}$ and $\dot{e}$. It is seen that the transfer-function matrices from V and E to Z are affine in Q, and since the variances of z and $\dot{z}$ are convex in the transfer-function matrices, it follows that the variances are convex in Q. The result now follows by Theorem 1 in Khargonekar and Rotea (1991). □

Remark 1. All controllers obtained by minimizing J for $\rho \in (0, \infty)$ are Pareto optimal by Lemma 17.1 in Leitmann (1981). If the solutions obtained for $\rho = 0$ and $\rho = \infty$ are unique, then they are also Pareto optimal by Lemma 17.2 in Leitmann (1981).

Remark 2. Remark 1 and Definition 1 implies that $\mathcal{V}_J$ can be parameterized by a scalar. This is not necessarily the case for $\mathcal{D}_J$.

Remark 3. Remark 1 implies that if the controllers obtained by minimizing J for $\rho \in [0, \infty]$ are unique, then a parameterization of $\mathcal{D}_P = \mathcal{D}_J$ by ρ is obtained, Khargonekar and Rotea (1991), p. 16.

It will now be shown how the minimization of μ in (2) can be rephrased to a minimization over a set of LQG-problem solutions.

THEOREM 1

Suppose that (A, B_1) is stabilizable, and that (C_1, A) is detectable. Then

$$\left\{ H \in \mathcal{D}_z \middle| H = \operatorname{argmin} \mu\left(\sigma_z(H), \sigma_{\dot{z}}(H)\right) \right\} \subseteq \mathcal{D}_P \cap \mathcal{D}_z$$

and

$$\left\{ (\sigma_z(H), \sigma_{\dot{z}}(H)) \in \mathcal{V}_z \middle| H = \operatorname{argmin} \mu(\sigma_z(H), \sigma_{\dot{z}}(H)) \right\} \subseteq \mathcal{V}_J \cap \mathcal{V}_z$$

Proof: Assume that the minimum of μ on $\mathcal{D}_z$ is attained for some $H \notin \mathcal{D}_P \cap \mathcal{D}_z$. For all $H \notin \mathcal{D}_P \cap \mathcal{D}_z$ there exist by Definition 1 $\bar{H} \in \mathcal{D}_z$ such that $\sigma_i(\bar{H}) < \sigma_i(H)$ for at lest one of $i = z, \dot{z}$. Since μ is differentiable and has strictly positive partial derivatives with respect to σ_z and $\sigma_{\dot{z}}$ for $\sigma_z < z_0 - m_z$, it follows that $\mu(\sigma_z(\bar{H}), \sigma_{\dot{z}}(\bar{H})) < \mu(\sigma_z(H), \sigma_{\dot{z}}(H))$. This is a contradiction, and thus the minimum of μ is attained on $\mathcal{D}_P \cap \mathcal{D}_z$, if it exists on $\mathcal{D}_z$. Further, by Lemma 3, $\mathcal{D}_P \subseteq \mathcal{D}_J$, which concludes the proof. □

Remark 1. Note that the minimization of μ can be done over $\mathcal{V}_J \cap \mathcal{V}_z$. This is a one-parametric optimization problem by Remark 2 of Lemma 3.

Remark 2. If for each $\rho \in [0,1]$ the minimizing H of J is unique, then by Lemma 2 and Remark 3 of Lemma 3 the minimization of μ can be thought of as finding optimal weights in an LQG-problem.

LQG-equations

For short reference the equations for deriving the LQG-solution when Q_2 and R_2 are invertible are given below. The transfer function from measurement to control is

$$H(s) = -L(sI - A + B_1 L + KC_1)^{-1} K \tag{5}$$

where L and K are given by

$$\begin{aligned} L &= Q_2^{-1}(Q_{12}^T + B_1^T S) \\ K &= P C_1^T R_2^{-1} \end{aligned} \tag{6}$$

and where S and P are the solutions to the Riccati-equations, Andersson and Moore (1990) p. 56–58, and p. 168,

$$\begin{aligned} (A - B_1 Q_2^{-1} Q_{12}^T)^T S + S(A - B_1 Q_2^{-1} Q_{12}^T) & \\ - S B_1 Q_2^{-1} B_1^T S + Q_1 - Q_{12} Q_2^{-1} Q_{12}^T &= 0 \\ AP + PA^T + B_2 R_1 B_2^T - P C_1^T (D R_2 D^T)^{-1} C_1 P &= 0 \end{aligned} \tag{7}$$

To calculate σ_z and $\sigma_{\dot{z}}$ the following Lyapunov-equation for the closed loop system should be solved, Åström (1970) p. 66 and pp. 290–291,

$$A_c X + X A_c^T + R_c = 0, \tag{8}$$

where

$$A_c = \begin{pmatrix} A - B_1 L & B_1 L \\ 0 & A - KC_1 \end{pmatrix}$$

$$R_c = \begin{pmatrix} B_2 R_1 B_2^T & B_2 R_1 B_2^T \\ B_2 R_1 B_2^T & B_2 R_1 B_2^T + K D R_2 D^T K^T \end{pmatrix}$$

Then σ_z and $\sigma_{\dot{z}}$ are given by

$$\begin{aligned}\sigma_z^2 &= (\,C_2 \quad 0\,)\,X\,(\,C_2 \quad 0\,)^T \\ \sigma_{\dot{z}}^2 &= C_2\,(\,A - B_1L \quad B_1L\,)\,X\,(\,A - B_1L \quad B_1L\,)^T\,C_2^T\end{aligned}$$

Due to the block triangularity of A_c it is possible to split up (8) into three equations, where one of the solutions is P in (7), which reduces the complexity of the problem.

Summary

It has been shown how the minimization of the upcrossing intensity can be rephrased to a minimization over a set of LQG-problem solutions parameterized by a scalar, regardless of the uniqueness of the solutions to the LQG-problems. However, if the solutions to the LQG-problems are unique, then the problem of minimizing the upcrossing intensity can be thought of as finding optimal weightings in an LQG-problem. Note that the Lyapunov equation (8) is linear, and thus does not add any significant complexity compared to an ordinary LQG-problem.

4. *Evaluation*

To evaluate the performance of the optimal controller obtained by minimizing (2) a second order process will be investigated. The set of LQG-solutions is calculated analytically, and then $\mu(\rho)$ is calculated numerically and plotted.

Let the process be given by

$$\begin{cases} dx = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x dt + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} du + \begin{pmatrix} 0 \\ 1 \end{pmatrix} dv \\ dy = (\,1 \quad 0\,)\,x dt + de \\ \;\; z = (\,1 \quad 0\,)\,x \end{cases}$$

$R_1 = \sigma_1^2 > 0$, and $R_2 = \sigma_2^2 > 0$. The solutions to the Riccati-equations in (7) are

$$S = \begin{pmatrix} \rho & 0 \\ 0 & 0 \end{pmatrix}$$

$$P = \begin{pmatrix} \sigma_2\sqrt{2\sigma_1\sigma_2} & \sigma_1\sigma_2 \\ \sigma_1\sigma_2 & \sigma_1\sqrt{2\sigma_1\sigma_2} \end{pmatrix}$$

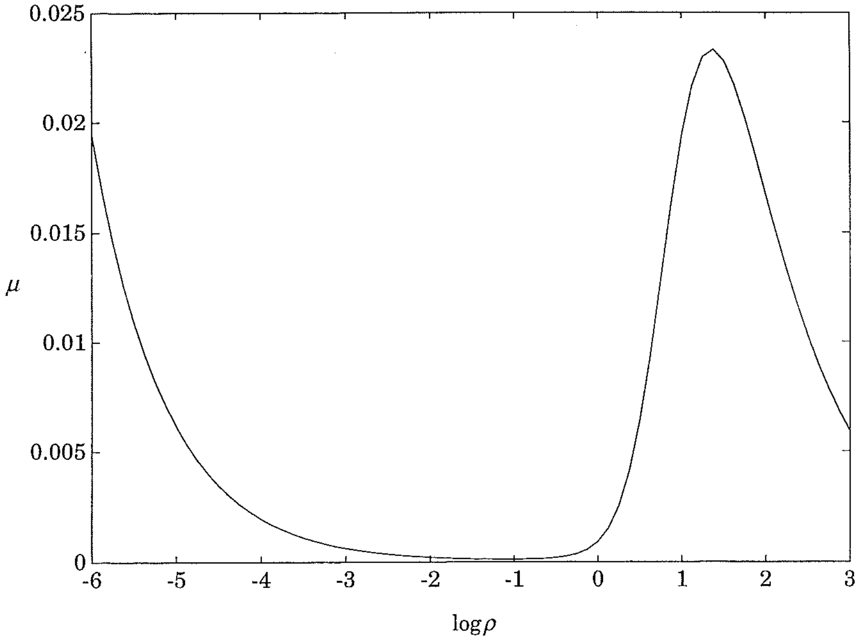

Figure 1. The crossing intensity μ as a function of $\log(\rho)$.

By using (6) it is found that

$$L = \begin{pmatrix} \frac{1}{\rho b_1} & \frac{1}{b_1} \end{pmatrix}$$

$$K = \begin{pmatrix} \sqrt{2\frac{\sigma_1}{\sigma_2}} \\ \frac{\sigma_1}{\sigma_2} \end{pmatrix}$$

Some more tedious calculations will give the controller $H(s)$ in (5) to be

$$\begin{aligned} H(s) = & -[(\sqrt{2\sigma_1\sigma_2} + \rho\sigma_1)s + \sigma_1]/ \\ & [(b_1\rho\sigma_2 s^2 + (b_1\sigma_2 + b_1\rho\sqrt{2\sigma_1\sigma_2} + b_2\rho\sigma_2)s \\ & + b_2(\sigma_2 + \rho\sqrt{2\sigma_1\sigma_2})] \end{aligned}$$

It is interesting to note that if $b_1 \neq 0$, then the controller is proper for all values of ρ. For $\rho > 0$ the controller is strictly proper. When $b_1 = 0$ and $b_2 \neq 0$, the controller is proper only for $\rho > 0$. It is also seen how an integrator can be forced into the controller by having a Wiener process as load-disturbance, i.e. $b_1 \neq 0$ and $b_2 = 0$.

The intensity μ has been calculated numerically for values of ρ in the range of 10^{-6} to 10^{3}, $m_z = 0$, $z_0 = 5$ and $b_1 = b_2 = \sigma_1 = \sigma_2 = 1$. The result is shown in Figure 1. The intensity has a minimum for $\rho = 0.1$, which is $\mu = 1.1334 \cdot 10^{-4}$.

5. Conclusions

A new optimal stochastic control problem that minimizes the intensity for a signal to upcross a level has been solved.

The new controller is obtained as the solution to a one-parametric optimization problem over a set of LQG-problem solutions, and thus the complexity is not significantly larger than for an ordinary LQG-problem. Further it can sometimes be thought of as finding optimal weightings in an LQG-problem.

The optimal controller has been computed for a second order process. It has been seen that it is fairly easy to compute.

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Part II

Control of Mean Time Between Failures

A new optimal stochastic control problem is posed. The criterion is to maximize the mean-time-between-failures criterion given a certain reference value. It is shown that this control problem is closely related both to the problem of minimizing the variance of the signal—minimum variance control—and to the problem of minimizing the so-called upcrossing probability—minimum upcrossing control. It is made plausible that the upcrossing probability is a better approximating criterion to minimize than the variance criterion. The problem of minimizing the upcrossing probability can be thought of as finding optimal weighting-matrices in an LQG-problem. The new controller is compared with the minimum variance controller for a first-order process. It is seen that the new controller causes a lower upcrossing probability and a larger mean time between failures. The improvement in the example is up to about 25%. This makes it possible to choose the reference value closer to the critical level without causing smaller mean time between failures. Further, it seen that the control signal is more well behaved.

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1. *Introduction*

There are many control problems where the goal is not only to keep the controlled signal near a specified reference value, but in addition to prevent it from upcrossing a critical level. The word critical is used in the sense of a severe failure, which may imply that the process has to be repaired and restarted. The distance between the critical level and the reference value is normally not too small. Otherwise the failure rate will be intolerably high. However, there may be other control-objectives that make it undesirable or impossible to choose the distance large. An example of problems of this kind can be found in the work of Borisson and Syding (1976), where the power of an ore crusher should be kept as high as possible but not exceed a certain level, in order that the overload protection does not cause shutdown. Another example is moisture control of a paper machine, where it is desired to keep the moisture content as high as possible without causing wet streaks. Yet another example is control of wind power plants, where the supervisory system initiates emergency shutdown if the generated power exceeds 140% of rated power, Mattsson (1984). Further examples can be found in sensor-based robotics and force control, Hansson and Nielsen (1991), and control of nonlinear plants in which the stability may be state dependent, Shinskey (1967).

An appealing criterion to maximize for the problems described above is the 'mean-time-between-failures' (MTBF) criterion, i.e. the mean time between upcrossings of the critical level. This type of problems can intuitively be solved by minimum variance (MV) control, Åström (1970) pp. 159–209, Åström and Wittenmark (1990) p. 203, and Borisson and Syding (1976). The gain of the MV controller depends critically on the sampling period. Too small a sampling period leads to large variations in the control signal, Åström and Wittenmark (1990) pp. 316–317. This problem has been solved by introducing weighting on the control signal—LQG-design. There are, however, no good criteria for choosing the weighting.

The proposed controller can be interpreted as a choice of optimal weightings in an LQG-problem, chosen in such a way that they minimize the mean number of upcrossings of the critical level per unit time—the upcrossing probability. The idea of considering upcrossing probabilities originates from continuous time extreme value analysis, where initial results were given in Rice (1936), Rice (1939), and Rice (1944). Rice's celebrated formula for the mean number of upcrossings of a level z_0 per unit time by a stationary gaussian process z, with zero mean value and covariance function r, is given by

$$\mu = \frac{1}{2\pi}\left(-\frac{r''(0)}{r(0)}\right)^{1/2} \exp\left(-\frac{z_0^2}{2r(0)}\right)$$

It has been shown that the number of upcrossings for large values of z_0 is approximately a Poisson process with intensity μ, Leadbetter *et al.* (1982) Section 9.1. Thus, the time between failures is exponentially distributed with mean $1/\mu$, which is maximized by minimizing μ. In discrete time, the corresponding formula is

$$\mu = \mathrm{P}\{z(0) \leq z_0 \cap z(1) > z_0\}$$

where $\mathrm{P}\{\cdot\}$ denotes probability measure, Cramér and Leadbetter (1967). It will be seen that it is possible to obtain similar Poisson results in discrete time.

The present author has previously, Hansson (1991a, b), Hansson (1993a), solved the problem of minimizing the mean number of upcrossings per unit time in the continuous time case; he also solved it in the discrete time case, Hansson (1991c). He also, to some extent, described it in other work, Hansson (1992) and Hansson (1993b). Here, the relation to maximizing the MTBF criterion will be discussed. Only the case of a linear process controlled by a linear controller will be treated since, then, if the disturbances acting on the process are gaussian the closed-loop system will also be gaussian. It is very likely that a nonlinear controller will do better. However, the analysis would then be much harder since the signals are not gaussian.

In §2, the problem of maximizing the MTBF criterion given a certain reference value is related to the MV controller and to the controller that minimizes the upcrossing probability—the minimum upcrossing (MU) controller. It is also made plausible that the upcrossing probability criterion captures the control-objectives better than the minimum variance criterion.

In §3, the MU controller is determined. It is obtained by solving a one-parametric optimization problem over a set of LQG-problem solutions. The complexity is thus only one order of magnitude larger than for an ordinary LQG-problem. It can be interpreted as choosing optimal weighting-matrices in an LQG-problem, provided that the solutions to the LQG-problems are unique.

In §4 the MU controller found in §3 is compared with the MV controller for a first order process. It is seen that the new controller causes a lower upcrossing probability and larger MTBF. Further, it is seen that the control signal is more well-behaved. Both theory and simulations show that the MU controller and the MV controller are approximately the same for large values of the distance between the reference value and the critical level. However, in an example it is seen that the MU controller can have up to about 25% better performance for moderate values of the distance. This is the interesting case for the examples described above.

Finally, in §5 the results of the previous sections are summarized.

2. *The Control Problem*

The control problems described in §1 is mathematically formalized in a stochastic framework. The control criterion is defined such that the controller should maximize the MTBF criterion given a certain reference value. Two approximations for the criterion are derived. One of them is maximized by MV control, and the other is maximized by MU control. The favorability of the approximate control criterion to minimize the upcrossing probability is made plausible.

Problem Formulation

Let the controlled signal z be a stationary gaussian sequence with constant mean

$$m_z = \mathrm{E}\{z(k)\}$$

and with covariance function

$$r_z(\tau) = \mathrm{E}\{(z(k+\tau) - m_z)(z(k) - m_z)\}$$

Denote the variance of z by σ_z^2, i.e. let $\sigma_z^2 = r_z(0)$. Consider a time-invariant controller H, linear in both the measurement signal y and in the constant reference value r. The problems mentioned in §1 are captured in the following problem formulation:

$$\max_H \mathrm{E}\{T\} \tag{1}$$

subject to $m_z = r$ and to a stable closed-loop system, where $\mathrm{E}\{\cdot\}$ denotes expectation, and where T is the time between two consecutive upcrossings of z_0 by z. The reason for constraining the minimization to $m_z = r$ is that it may be profitable not to have $m_z - z_0$ too large; e.g. in the paper machine example it was desired to keep the moisture content as high as possible without causing wet streaks. Without loss of generality, it may be assumed that $m_z = r = 0$, which can be obtained with a change of coordinates. To simplify the notation, this will be assumed in what follows.

Poisson-Convergence

To simplify the problem, approximations for the expectation in (1) will be given, but first some asymptotic results relating the mean number of exceedances to the mean number of upcrossings will be derived.

LEMMA 1

Let z be a stationary gaussian random sequence with covariance function r_z satisfying $|\rho| < 1$, where $\rho = r_z(1)/r_z(0)$. Then

$$\lim_{z_0 \to \infty} \mathrm{P}\{z_{k+1} \le z_0 | z_k > z_0\} = 1$$

Proof: The case when $\rho = 0$ is trivial. Suppose $0 < |\rho| < 1$ and let $\xi_k = z_{k+1} - \rho z_k$. Then

$$\begin{aligned}
\mathrm{P}\{z_{k+1} \le z_0 | z_k > z_0\} &= \mathrm{P}\{\xi_k \le z_0 - \rho z_k | z_k > z_0\} \\
&\ge \mathrm{P}\left\{\xi_k \le z_0 - \rho z_k \bigcap z_0 < z_k \le \frac{z_0 - \varepsilon}{|\rho|} \middle| z_k > z_0\right\} \\
&\ge \mathrm{P}\left\{\xi_k \le z_0 - \rho \frac{z_0 - \varepsilon}{|\rho|} \bigcap z_0 < z_k \le \frac{z_0 - \varepsilon}{|\rho|} \middle| z_k > z_0\right\} \\
&\ge \mathrm{P}\left\{\xi_k \le \varepsilon \bigcap z_0 < z_k \le \frac{z_0 - \varepsilon}{|\rho|} \middle| z_k > z_0\right\}
\end{aligned}$$

where $0 < \varepsilon < (1 - |\rho|)z_0$. Now, choose $\varepsilon = (1 - |\rho|)z_0 - \varepsilon'$, where $0 < \varepsilon' < (1 - |\rho|)z_0$. Notice that ξ_k is independent of z_k. Thus it follows that

$$\begin{aligned}
\mathrm{P}\{z_{k+1} \le z_0 | z_k > z_0\} \ge &\mathrm{P}\{\xi_k \le (1 - |\rho|)z_0 - \varepsilon'\} \\
&\times \mathrm{P}\{z_0 < z_k \le z_0 + \varepsilon'/|\rho| | z_k > z_0\}
\end{aligned}$$

It is obvious that it only remains to be shown that the last factor converges to 1 as z_0 approaches infinity. The last factor can be written as $D(z_0)/N(z_0)$, where

$$\begin{aligned}
D(z_0) &= \Phi\left(\frac{z_0 + \varepsilon'/|\rho|}{\sigma_\xi}\right) - \Phi\left(\frac{z_0}{\sigma_\xi}\right) \\
N(z_0) &= 1 - \Phi\left(\frac{z_0}{\sigma_\xi}\right) \\
\sigma_\xi^2 &= (1 - \rho^2) r_z(0)
\end{aligned}$$

and where

$$\Phi(x) = \int_{-\infty}^{x} \phi(t)dt \quad \text{and} \quad \phi(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$$

By l'Hospital's Theorem, it follows that

$$\lim_{z_0\to\infty}\frac{D(z_0)}{N(z_0)} = \lim_{x\downarrow 0}\frac{D(1/x)}{N(1/x)} = \lim_{x\downarrow 0}\frac{\frac{dD(1/x)}{dx}}{\frac{dN(1/x)}{dx}} = \lim_{x\downarrow 0}\frac{\phi\left(\frac{1/x+\varepsilon'/|\rho|}{\sigma_\xi}\right)-\phi\left(\frac{1/x}{\sigma_\xi}\right)}{-\phi\left(\frac{1/x}{\sigma_\xi}\right)}$$

$$= \lim_{z_0\to\infty}\left[1-\exp\left(-\frac{1}{2\sigma_\xi^2}\left(\frac{2\varepsilon'}{|\rho|}z_0+\frac{\varepsilon'^2}{|\rho|^2}\right)\right)\right] = 1$$

This concludes the proof. □

Remark 1. Thus exceedances of an infinitely high level will almost surely last for only one time-instant. □

LEMMA 2

Let z be a stationary gaussian random sequence with covariance function r_z satisfying $|\rho| < 1$, where $\rho = r_z(1)/r_z(0)$. Then

$$\lim_{n\to\infty} nP\left(z_0^{(n)}\right) = L$$

if and only if

$$\lim_{n\to\infty} n\mu\left(z_0^{(n)}\right) = L$$

where

$$\left.\begin{aligned} P(x) &= \mathrm{P}\{z(1) > x\} \\ \mu(x) &= \mathrm{P}\{z(0) \le x \cap z(1) > x\} \end{aligned}\right\} \qquad (2)$$

Proof: Since $\lim_{n\to\infty} z_0^{(n)} = \infty$, it follows by Lemma 1 that

$$\lim_{n\to\infty}\left[nP\left(z_0^{(n)}\right) - n\mu\left(z_0^{(n)}\right)\right]$$
$$= \lim_{n\to\infty}\left[nP\left(z_0^{(n)}\right)\left(1-\mathrm{P}\left\{z(0)\le z_0^{(n)}\middle| z(1) > z_0^{(n)}\right\}\right)\right] = 0$$

if $\lim_{n\to\infty} nP\left(z_0^{(n)}\right) = L$. Similar arguments can be used for the necessity part of the proof. □

Remark 2. Notice that $nP\left(z_0^{(n)}\right)$ and $n\mu\left(z_0^{(n)}\right)$ are the mean number of exceedances and upcrossings, respectively, of z by z_0 in the time interval $[1, n)$. □

The quaintity μ in (2) will be called the upcrossing probability, and it is equal to the mean number of upcrossings in the time interval $[0, 1)$, see e.g. Cramér and Leadbetter (1967) p. 281.

THEOREM 1
Suppose that z is a stationary gaussian sequence with covariance function r_z satisfying

$$\lim_{\tau\to\infty} r_z(\tau)\ln\tau = 0$$

and suppose that $z_0^{(n)}(L)$ is chosen such that

$$\lim_{n\to\infty} n\mu\left(z_0^{(n)}(L)\right) = L, \quad \forall L > 0$$

Let the time normalized process $\zeta_n(t), t = k/n, k = 1, 2, \ldots; n = 1, 2, \ldots$ be defined by $\zeta_n(k/n) = z(k)$, and let $N_n(t)$ be the number of upcrossings of $z_0^{(n)}$ by ζ_n in $(0, t]$. Then, for any fixed $L > 0$, N_n converges in distribution to a Poisson process with intensity L on $(0, \infty)$ as n approaches infinity.

Proof: Notice that the condition of Lemma 2, i.e. that $|r_z(1)| < |r_z(0)|$, follows from $\lim_{\tau\to\infty} r_z(\tau)\ln\tau = 0$, since if $|r_z(1)| = |r_z(0)|$, it holds that $|r_z(\tau)| = |r_z(0)|$, for all τ, which is in contradiction with $\lim_{\tau\to\infty} r_z(\tau)\ln\tau = 0$. The proof for the case when N_n is the number of exceedances can be found in the work of Leadbetter *et al.* (1982) Theorem 5.2.1, by noting that the conditions in that theorem are fulfilled by Lemma 2 and Leadbetter *et al.* (1982) Lemma 4.4.1. Further, by Lemma 2 and by examining the proof in Leadbetter *et al.* (1982) Theorem 5.2.1 it follows that the Poisson convergence result also holds for N_n being the number of upcrossings □

Remark 3. The time T_n between two consecutive upcrossings of $z_0^{(n)}$ by ζ_n converges in distribution to an exponential distribution with mean value $1/L$ as n approaches infinity. □

Approximation of the Problem Formulation

Now by Theorem 1 it is obvious that the expectation in (1) for large values of z_0/σ_z can be approximately expressed as

$$\mathrm{E}\{T\} \approx \frac{1}{P(z_0)} \approx \frac{1}{\mu(z_0)}$$

which is maximized by minimizing either the variance of z or the upcrossing probability μ. However, for $z_0/\sigma_z < \infty$ it holds that

$$\frac{1}{P(z_0)} < \frac{1}{\mu(z_0)}$$

and it is tempting to believe that the upcrossing probability is a better criterion to minimize for moderate values of z_0/σ_z, which is the interesting case for the problems described in § 1. Therefore, the following approximation of the criterion (1) will be considered from now on:

$$\min_H \mu(z_0) \tag{3}$$

subject to a stable closed-loop system. There may be some problems with this approximation, since there are two ways of making μ small—either by keeping z well below z_0 or by keeping it well above z_0. To exclude the latter possibility, the minimization of μ will also be restricted to $\sigma_z \leq z_0$. The validity of the approximation of the problem formulation will be investigated further in § 4.

It is interesting to note that the approximate criterion could also have been obtained by approximating another interesting criterion:

$$\mathrm{P}\left\{\max_{0\leq k\leq N} z(k) > z_0\right\}$$

The present author has previously, Hansson (1991c), described how this criterion can be approximately minimized by minimizing the upcrossing probability.

Summary

The control problems described in § 1 have been mathematically formalized in a stochastic framework. The control criterion has been defined such that the controller should maximize the MTBF criterion given a certain reference value. Two approximations for the criterion have been investigated. One of them is maximized by MV control, and the other one is maximized by MU control. It has been made plausible that minimizing the upcrossing probability is a better approximation to the original problem than MV control.

3. *Regulator Design*

The problem of minimizing the upcrossing probability will now be solved. The problem is reformulated as a one-parameter minimization over solutions to LQG-problems. Thus the complexity is not significantly larger than for an ordinary LQG-problem. The solution can be interpreted as a choice of optimal weighting-matrices in an LQG-problem. The equations for solving the LQG-problems are then given.

Solution

Let the stationary gaussian sequence z be defined by

$$\left.\begin{aligned} x(k+1) &= Ax(k) + B_1u(k) + B_2v(k) \\ y(k) &= C_1x(k) + De(k) \\ z(k) &= C_2x(k) \end{aligned}\right\} \tag{4}$$

where v and e are zero mean, gaussian, white noise sequences with $\mathrm{E}vv^T = R_1$, $\mathrm{E}ee^T = R_2$ and $\mathrm{E}ve^T = R_{12} = 0$.† The signal y is the measurement signal and u is the control signal. The signal z is the signal that is desirable to control. The reason for not having $C_1 = C_2$ can be motivated by the examples in § 1, where, for example, in the ore crusher example, the measured power y is not the desired signal to control, but instead some filtered version z of it, due to the filtering behavior of the thermal overload protection. More general process models than (4) may be considered, see Hansson (1991c). Introduce

$$\left.\begin{aligned} \alpha(k) &= z(k+1) + z(k) \\ \beta(k) &= z(k+1) - z(k) \end{aligned}\right\} \tag{5}$$

which are independent variables due to the stationarity of z. Let $\mathcal{D}$ be the set of linear time-invariant stabilizing controllers of (4), and let $\mathcal{D}_z$ be the set of linear time-invariant stabilizing controllers of (4) for which

$$\sigma_z \le z_0 \tag{6}$$

holds, where σ_z^2 is the variance of z. Note that the sets $\mathcal{D}$ and $\mathcal{D}_z$ may be empty, if the process is not stabilizable or if z_0 is too small. It will be seen that the minimization of μ in (2) over $\mathcal{D}_z$ can be done by first minimizing

$$J = \mathrm{E}\left\{(1-\rho)\alpha^2 + \rho\beta^2\right\} \tag{7}$$

for $\rho \in [0,1]$ over $\mathcal{D}$, and then minimizing μ over the solutions obtained

† The condition $\lim_{\tau\to\infty} r_z(\tau)\ln\tau = 0$ of Theorem 1 is easily shown to hold if the closed loop system is stable. The covariance function of z is then given by $r_z(\tau) = C_2A_c^\tau R_x C_2^T$ where A_c is the closed loop system matrix and R_x is the covariance matrix of x. Let $\|\cdot\|$ be any self-consistent matrix norm. Then it holds that $|r_z(\tau)| \le M\|A_c^\tau\|$ for some finite M. Further there exist r satisfying $\rho < r < 1$, where ρ is the spectral radius of A_c. It can be shown that $\|A_c^\tau\| \le Nr^\tau$, where N is finite. Since $\lim_{\tau\to\infty} r^\tau \ln\tau = 0$, the result follows.

in the first minimization, i.e. over $\mathcal{V}_J \cap \mathcal{V}_z$, where

$$\begin{aligned}\mathcal{V}_J &= \left\{ (\sigma_\alpha(H), \sigma_\beta(H)) \in R^2 \middle| H \in \mathcal{D}_J \right\}\\ \mathcal{V}_z &= \left\{ (\sigma_\alpha, \sigma_\beta) \in R^2 \middle| \sigma_z \leq z_0,\ \sigma_\alpha \geq 0,\ \sigma_\beta \geq 0 \right\}\\ \mathcal{D}_J &= \left\{ H \in \mathcal{D} \middle| H = \operatorname{argmin} J(H, \rho),\ \rho \in [0,1] \right\}\end{aligned}$$

and where σ_α^2 and σ_β^2 are the variances of α and β.

In the following lemma J is rewritten to fit the standard LQG-problem formulation.

LEMMA 3
The loss function J in (7) can be written

$$J = \bar{J} + \mathrm{E}\left\{ v^T B_2^T C_2^T C_2 B_2 v \right\}$$

where

$$\bar{J} = \mathrm{E}\left\{ x^T Q_1 x + 2x^T Q_{12} u + u^T Q_2 u \right\}, \tag{8}$$

and where

$$\left.\begin{aligned} Q_1 &= C_2^T C_2 + A^T C_2^T C_2 A + (1-2\rho)\left(C_2^T C_2 A + A^T C_2^T C_2\right)\\ Q_{12} &= \left(A^T + (1-2\rho)I\right) C_2^T C_2 B_1\\ Q_2 &= B_1^T C_2^T C_2 B_1 \end{aligned}\right\} \tag{9}$$

Proof: The result follows immediately from the definitions of z in (4), and α and β in (5), and by noting that v is uncorrelated with x and u, since u is a functional of $y(k), y(k-1), \ldots$, and since $R_{12} = 0$. □

Remark 4. For $\rho = 0.5$ it follows that $J = \mathrm{E}\left\{ z(k+1)^2 + z(k)^2 \right\}$. This case thus corresponds to minimum variance control of z. □

Next it will be shown that all jointly minimal variances of α and β can be obtained by minimizing J in (7) for $\rho \in [0,1]$, but first a precise definition of joint minimality due to Pareto (1896) will be given.

DEFINITION 1—Pareto Optimality
Let $\mathcal{X}$ denote an arbitrary nonempty set. Let $f_i : \mathcal{X} \to R^+$, $1 \leq i \leq s$ be s nonnegative functionals defined on $\mathcal{X}$. A point x^0 is said to be Pareto optimal with respect to the vector-valued criterion $f = (f_1, f_2, \cdots, f_s)$ if there does not exist $x \in \mathcal{X}$ such that $f_i(x) \leq f_i(x^0)$ for all i, $1 \leq i \leq s$, and $f_k(x) < f_k(x^0)$ for some k, $1 \leq k \leq s$. □

The concept of Pareto optimality is illuminated in Figure 1. The set of achievable variances of α and β is the set of points in the plane

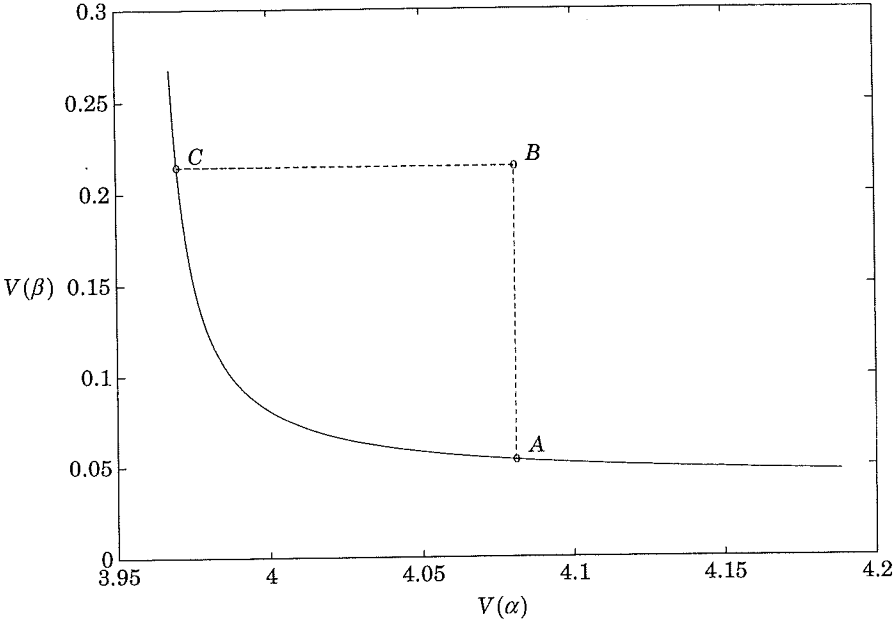

Figure 1. Illustration of Pareto optimality.

that are above and to the right of or on the solid curve. The controller corresponding to the variances at B is not Pareto optimal, since there exist, for example, controllers corresponding to strictly lower variance of β without having larger variance of α—the controllers with variances on the line connecting A with B. Moreover, it is seen that the controller corresponding to the variances at A is Pareto optimal, since by picking any other point to the right of, above or on the curve will either increase the variance of α or the variance of β. This reasoning holds for all points on the curve, and thus they are all Pareto optimal. Equivalent definitions of Pareto optimality can be found in the work of Leitmann (1981) p. 292.

LEMMA 4
Suppose that (A, B_1) is stabilizable, and that (C_1, A) is detectable. Then the set $\mathcal{D}_P$ of Pareto optimal controllers with respect to $(\sigma_\alpha^2, \sigma_\beta^2)$ is a subset of $\mathcal{D}_J$.

Proof: Using the Youla parametrization, Boyd and Barratt (1991) Chapter 7.4, it follows that all stabilizing controllers of (4) can be parameterized by a stable transfer-function matrix Q. Thus, to minimize J over $\mathcal{D}$ is equivalent to minimize J over Q, where Q belongs to the linear space of stable transfer-function matrices. Further it follows from the work of Boyd and Barratt (1991) Chapter 7.4 that the transfer-function matrices from v and e to z are affine in Q. Since the variances of α and β are convex in the transfer-function matrices, it follows that the variances are convex in Q. The result now follows by Khargonekar and Rotea (1991) Theorem 1. □

Remark 5. All controllers obtained by minimizing J for $\rho \in (0,1)$ are Pareto optimal by Leitmann (1981) Lemma 17.1. If the controllers obtained for $\rho = 0$ and $\rho = 1$ are unique, then they are also Pareto optimal by Leitmann (1981) Lemma 17.2. □

Remark 6. Remark 1 and Definition 1 imply that $\mathcal{V}_J$ can be parameterized by a scalar. This is not necessarily the case for $\mathcal{D}_J$. □

Remark 7. Remark 1 implies that if the controllers obtained by minimizing J for $\rho \in [0,1]$ are unique, then a parameterization of $\mathcal{D}_P = \mathcal{D}_J$ by ρ is obtained, Khargonekar and Rotea (1991) p. 16. □

The next lemma gives an expression for the upcrossing probability μ in (2) in terms of a double integral.

LEMMA 5
It holds that

$$\mu = \mathrm{P}\{z(0) \le z_0 \cap z(1) > z_0\} = \int_0^\infty \phi(y) \int_{x_l}^{x_u} \phi(x)dxdy$$

where

$$\phi(x) = (2\pi)^{-1/2}\exp(-x^2/2), \qquad x_l = (2z_0 - \sigma_\beta y)/\sigma_\alpha$$

and

$$x_u = (2z_0 + \sigma_\beta y)/\sigma_\alpha$$

Proof: Since α and β are independent it holds that

$$\begin{aligned}\mu =& \mathrm{P}\{|\alpha - 2z_0| < \beta\} \\ =& \int\int_{|x-2z_0|<y} \frac{1}{\sigma_\alpha}\phi\left(\frac{x}{\sigma_\alpha}\right)\frac{1}{\sigma_\beta}\phi\left(\frac{y}{\sigma_\beta}\right) dxdy\end{aligned}$$

from which the result follows by a change of variables. □

In the following lemma it will be shown that the upcrossing probability μ in (2) has strictly positive partial derivatives with respect to σ_α and σ_β.

LEMMA 6
Let

$$\mathcal{V}(r) = \left\{(\sigma_\alpha, \sigma_\beta) \in R^2 \middle| \sigma_z \le r,\ \sigma_\alpha > 0,\ \sigma_\beta > 0\right\}$$

where $r > 0$. Then the upcrossing probability μ in (2) has strictly positive partial derivatives with respect to both σ_α and σ_β on $\mathcal{V}(r)$, if and only if $r \le z_0$.

Proof: It holds that

$$\frac{\partial \mu}{\partial \sigma_\beta} = \int_0^\infty \phi(y) \left(\frac{y}{\sigma_\alpha}\phi(x_u) + \frac{y}{\sigma_\alpha}\phi(x_l)\right) dy > 0$$

Further let $x_l = (2z_0 - \sigma_\beta y)/\sigma_\alpha$, and $x_u = (2z_0 + \sigma_\beta y)/\sigma_\alpha$. Using Lemma 5 gives

$$\frac{\partial \mu}{\partial \sigma_\alpha} = \int_0^\infty \phi(y) \left(\frac{x_l}{\sigma_\alpha}\phi(x_l) - \frac{x_u}{\sigma_\alpha}\phi(x_u)\right) dy$$

By completing the squares in the exponents and by a change of coordinates it is possible to express the integral in terms of $\Phi(x) = \int_{-\infty}^x \phi(t)dt$, and $\sigma_z^2 = (\sigma_\alpha^2 + \sigma_\beta^2)/4$

$$\frac{\partial \mu}{\partial \sigma_\alpha} = \frac{\sigma_\alpha}{8\pi\sigma_z^2} \exp\left(-\frac{\gamma^2}{2}\right) \left[\sqrt{2\pi}\gamma\,(2\Phi(\eta) - 1) - 2\frac{\eta}{\gamma} \exp\left(-\frac{\eta^2}{2}\right)\right]$$

where $\eta = \gamma\sqrt{\xi}$, $\xi = (\sigma_\beta/\sigma_\alpha)^2$, and $\gamma = z_0/\sigma_z > 0$. It is seen that $\partial\mu/\partial\sigma_\alpha > 0$ if and only if

$$2\Phi(\eta) - 1 > \sqrt{\frac{2}{\pi}}\frac{\eta}{\gamma^2} \exp\left(-\frac{\eta^2}{2}\right)$$

So if $\partial\mu/\partial\sigma_\alpha > 0$ on $\mathcal{V}(r)$, then the inequality above holds for all values of $\eta > 0$, since $\gamma > 0$, and since it must hold for all values of $\xi > 0$. A Taylor expansion round $\eta = 0$ gives

$$\sqrt{\frac{2}{\pi}}\eta > \sqrt{\frac{2}{\pi}}\frac{\eta}{\gamma^2} + O(\eta^2)$$

So, for the inequality to hold for small values of η, it must be that $\gamma \ge 1$, which is equivalent to $r \le z_0$.

Now suppose that $r \le z_0$, which implies $\gamma \ge 1$. Then

$$\begin{aligned}(2\Phi(\eta) - 1)^2 &\ge 1 - \exp\left(-\frac{2\eta^2}{\pi}\right) - \frac{2(\pi - 3)}{3\pi^2}\eta^4 \exp\left(-\frac{\eta^2}{2}\right) \\ &\ge 1 - \exp\left(-\frac{2\xi}{\pi}\right) - \frac{2(\pi - 3)}{3\pi^2}\xi^2 \exp\left(-\frac{\xi}{2}\right)\end{aligned}$$

where the first inequality follows from Abromowitz and Stegun (1968) Formula 26.2.25 and the second one from $\gamma \geq 1$. Further,

$$\left(\sqrt{\frac{2}{\pi}}\frac{\eta}{\gamma^2}\exp\left(-\frac{\eta^2}{2}\right)\right)^2 \leq \frac{2}{\pi}\xi\exp(-\xi)$$

To show $\partial\mu/\partial\sigma_\alpha > 0$, it is now sufficient to show $L > R$ for $\xi > 0$, where

$$\begin{aligned} L &= \exp\left(\frac{\xi}{2}\right) \\ R &= \frac{2}{\pi}\xi\exp\left(-\frac{\xi}{2}\right) + \exp\left(\left(\frac{1}{2}-\frac{2}{\pi}\right)\xi\right) + \frac{2(\pi-3)}{3\pi^2}\xi^2 \end{aligned}$$

Some calculations give

$$\begin{aligned} L &\geq 1 + \frac{1}{2}\xi + \frac{1}{8}\xi^2 \\ R &\leq 1 + \frac{1}{2}\xi + \left(\frac{1}{8} - \frac{1}{3\pi}\right)\xi^2 \end{aligned}$$

From this it follows that $L > R$ for $\xi > 0$, so $\partial\mu/\partial\sigma_\alpha > 0$. □

Remark 8. The largest region $\mathcal{V}(r)$ in which both $\partial\mu/\partial\sigma_\alpha > 0$ and $\partial\mu/\partial\sigma_\beta > 0$ is $\mathcal{V}(z_0)$. So, if the constraint $\sigma_z \leq z_0$ is not considered, then it may well be that μ is minimized by $\sigma_\alpha = \infty$. □

It will now be shown how the minimization of μ in (2) can be rephrased to a minimization over a set of LQG-problem-solutions. Figure 2 illuminates the proof of the following theorem.

THEOREM 2
Suppose that (A, B_1) is stabilizable, and that (C_1, A) is detectable. Then

$$\left\{H \in \mathcal{D}_z \middle| H = \operatorname{argmin}\mu\left(\sigma_\alpha(H), \sigma_\beta(H)\right)\right\} \subseteq \mathcal{D}_P \cap \mathcal{D}_z$$

and

$$\left\{(\sigma_\alpha(H), \sigma_\beta(H)) \in \mathcal{V}_z \middle| H = \operatorname{argmin}\mu(\sigma_\alpha(H), \sigma_\beta(H))\right\} \subseteq \mathcal{V}_J \cap \mathcal{V}_z$$

Proof: Assume that the minimum of μ on $\mathcal{D}_z$ is attained for some $H \notin \mathcal{D}_P \cap \mathcal{D}_z$. For all $H \notin \mathcal{D}_P \cap \mathcal{D}_z$ there exist by Definition 1 $\bar{H} \in \mathcal{D}_z$ such that $\sigma_i(\bar{H}) < \sigma_i(H)$ for at lest one of $i = \alpha, \beta$. Since μ is differentiable and by Lemma 6 has strictly positive partial derivatives with respect to σ_α and

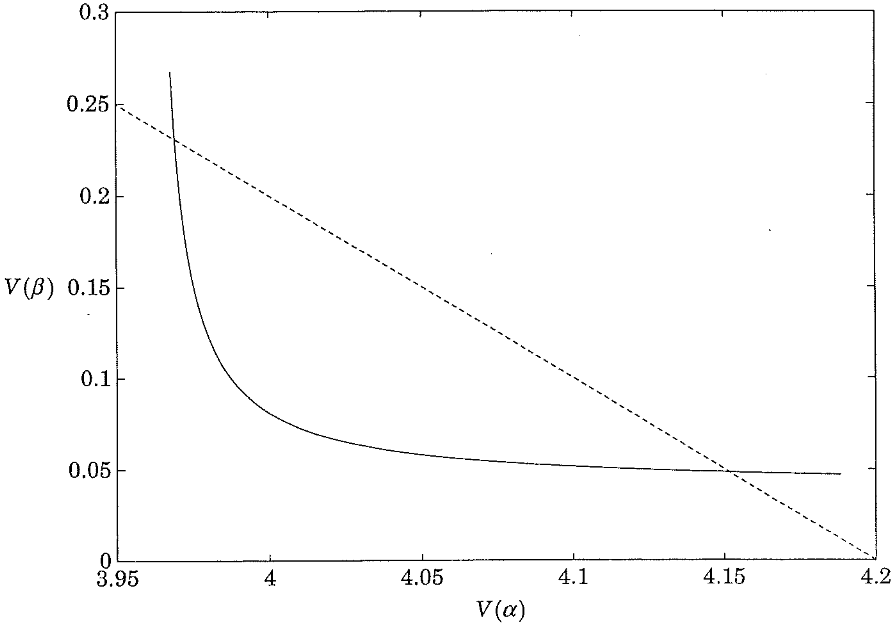

Figure 2. The solid line is $\mathcal{V}_J$, and the dashed line is $\sigma_z = z_0$ for $z_0 = 1.05$.

σ_β on $\mathcal{V}(z_0)$, it follows that $\mu(\sigma_\alpha(\bar{H}), \sigma_\beta(\bar{H})) < \mu(\sigma_\alpha(H), \sigma_\beta(H))$. This is a contradiction, and thus the minimum of μ is attained on $\mathcal{D}_P \cap \mathcal{D}_z$, if it exists on $\mathcal{D}_z$. Further $\mathcal{D}_P \subseteq \mathcal{D}_J$ by Lemma 4, which concludes the proof. □

Remark 9. Note that the minimization of μ can be done over $\mathcal{V}_J \cap \mathcal{V}_z$. This is a one-parametric optimization problem by Remark 6 on Lemma 4. □

Remark 10. If for each $\rho \in [0, 1]$ the minimizing $\bar{H}$ of J is unique, then by Lemma 3 and Remark 7 on Lemma 4 the minimization of μ can be thought of as finding optimal weights in an LQG-problem. This is apparent from the following explanation of the optimization procedure: in the first step, the weightings in the LQG-problem, as well as the solutions to them together with the resulting closed-loop variances, are all parameterized by ρ; in the second step the optimal controller, together with its corresponding optimal weighting, is found by minimizing the upcrossing probability over the closed-loop variances obtained in the first step. □

LQG-equations

For short reference, the equations for deriving the solution that minimizes $\bar{J}$ in (8) in Lemma 3 when the controller H is allowed to have a direct-term are given below. More stringent proofs of the results can be found in previous work of the present author: Hansson (1991c), which also covers a more general process model. The transfer function from measurement to control is

$$H(q) = -L_x(qI - A + B_1L_x + KC_1)^{-1}K_y - L_y \tag{10}$$

where L_x, L_y and K are given by

$$\begin{aligned} L_x &= L - L_yC_1 \\ L_y &= LK_f \\ L &= \left(Q_2 + B_1^TSB_1\right)^{-1}\left(B_1^TSA + Q_{12}^T\right) \\ K_y &= K - B_1L_y \\ K &= AK_f \\ K_f &= PC_1^T\left(DR_2D^T + C_1PC_1^T\right)^{-1} \end{aligned}$$

where S and P are the solutions to the Riccati-equations, Åström and Wittenmark (1990) Chapter 11.4, and Gustafsson and Hagander (1991),

$$\left.\begin{aligned} A^TSA - S - (A^TSB_1 + Q_{12})(Q_2 + B_1^TSB_1)^{-1}(Q_{12}^T + B_1^TSA) + Q_1 &= 0 \\ APA^T - P - APC_1^T(DR_2D^T + C_1PC_1^T)^{-1}C_1PA^T + B_2R_1B_2^T &= 0 \end{aligned}\right\} \tag{11}$$

and where Q_1, Q_2 and Q_{12} are given by (9) in Lemma 3. To calculate σ_z, σ_u, σ_α and σ_β the following Lyapunov-equation for the closed-loop system should be solved, Åström (1970) p. 49,

$$A_cXA_c^T + B_cRB_c^T = X \tag{12}$$

where

$$A_c = \begin{pmatrix} A - B_1L & B_1L_x \\ 0 & A - KC_1 \end{pmatrix}$$
$$B_c = \begin{pmatrix} B_2 & -B_1L_yD \\ B_2 & -KD \end{pmatrix}$$
$$R = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}$$

Then σ_α, σ_β, σ_z and σ_u are given by

$$\left.\begin{aligned}
\sigma_\alpha^2 &= \begin{pmatrix} C_2 & 0 \end{pmatrix} \left((A_c+I)X(A_c+I)^T + B_cRB_c^T\right) \begin{pmatrix} C_2 & 0 \end{pmatrix}^T \\
\sigma_\beta^2 &= \begin{pmatrix} C_2 & 0 \end{pmatrix} \left((A_c-I)X(A_c-I)^T + B_cRB_c^T\right) \begin{pmatrix} C_2 & 0 \end{pmatrix}^T \\
\sigma_z^2 &= \begin{pmatrix} C_2 & 0 \end{pmatrix} X \,(C_2 \quad 0)^T \\
\sigma_u^2 &= \begin{pmatrix} -L & L_x \end{pmatrix} X \begin{pmatrix} -L & L_x \end{pmatrix}^T + L_yDR_2D^TL_y^T
\end{aligned}\right\} \qquad (13)$$

Since A_c is block-triangular, (12) can be split up into three equations, one of which has P in (11) as its solution. This reduces the complexity of the problem.

Summary

It has been shown that the minimization of the upcrossing probability can be expressed as a minimization over a set of LQG-problem solutions parameterized by a scalar, regardless of the uniqueness of the solutions to the LQG-problems. If the solutions to the LQG-problems are unique, then the problem of minimizing the upcrossing probability can be thought of as finding optimal weightings in an LQG-problem. Note that the Lyapunov equation (12) is linear, and thus does not add any significant complexity compared to an ordinary LQG-problem.

The algorithm for minimizing the upcrossing probability can be summarized as: (1) solve the associated LQG-problems; and (2) minimize the upcrossing probability over the variances obtained in the first step. It must be stressed that if $\sigma_z > z_0$, then no solution exist. In order to obtain a solution, the distance between the reference value and the critical level z_0 must be sufficiently large.

It has been seen that the computation of the variances is not more complicated than solving a linear system of equations. Further, the upcrossing probability can easily be obtained with some numerical integration routine. The complexity of this latter problem does not depend on the size of the process model. Thus the computations performed for each value of ρ is not significantly larger than for an ordinary LQG-problem. Moreover by adopting some numerical routine for minimizing the upcrossing probability, it may not be necessary to solve that many LQG-problems. A good choice of starting value for ρ is 0.5, which corresponds to the MV controller. In this sense, the computational burden for obtaining the MU controller is not significantly larger than for the LQG controller that corresponds to MV control.

4. Example

To evaluate the performance of the MU controller obtained by minimizing the upcrossing probability, a first order process will be investigated. The process is defined, the set of LQG-solutions is calculated analytically, and the MU controller is computed and compared with the MV controller. It is seen that the new controller causes a lower upcrossing probability and a larger MTBF. It is also seen that it has a control signal that is more well-behaved.

Process

Let the process be given by

$$\begin{cases} x(k+1) = x(k) + 0.04u(k) + 0.2v(k) \\ y(k) = x(k) + 5e(k) \\ z(k) = x(k) \end{cases}$$

where v and e ar zero mean gaussian white noise sequences with $\mathrm{E}v^2 = R_1 = 1$, $\mathrm{E}e^2 = R_2 = 1$ and $\mathrm{E}ve = R_{12} = 0$. The signal y is the measurement signal, and u is the control signal. This process can be obtained approximately by sampling a continuous time integrator process with sampling interval 0.04.

LQG-Controllers

The weighting-matrices in (9) are

$$\begin{aligned} Q_1 &= 4(1-\rho) \\ Q_{12} &= 0.08(1-\rho) \\ Q_2 &= 0.0016 \end{aligned}$$

and the solutions to the Riccati-equations in (11) are

$$\begin{aligned} S &= 2[\rho(1-\rho)]^{1/2} \\ P &= \frac{0.04 + 4.0016^{1/2}}{2} \end{aligned}$$

Some more tedious calculations will give the controller $H(q)$ in (10) to be

$$H(q) = -\frac{s_0 q}{r_0 q + r_1}$$

where

$$
\begin{aligned}
s_0 &= \left(2[\rho(1-\rho)]^{1/2} + 2(1-\rho)\right)\left(0.04 + 4.0016^{1/2}\right)\\
r_0 &= 0.04\left(2[\rho(1-\rho)]^{1/2} + 1\right)\left(50.04 + 4.0016^{1/2}\right)\\
r_1 &= 2(1-2\rho)
\end{aligned}
$$

It is interesting to note that for $\rho = 0.5$—minimum variance control by Remark 4 to Lemma 3—the controller is a proportional controller.

MU and MV Controllers

The MU controller will now be compared with the MV controller.

VARIANCE AND UPCROSSING PROBABILITY. The variances of z and u have been calculated numerically for values of ρ with a step of 0.01 in the range 0.01 to 0.99. It is seen in Figure 3 that the variance of z does not depend so much on ρ as does the variance of u.

The probability μ has been calculated according to Lemma 5 for m_z = 0 and for values of the critical level given by z_0 = 2, 3 ,4 and 5. The result is seen in Figure 4. The minimum values of the probability μ are obtained for values of ρ greater than 0.5. The variance of the control

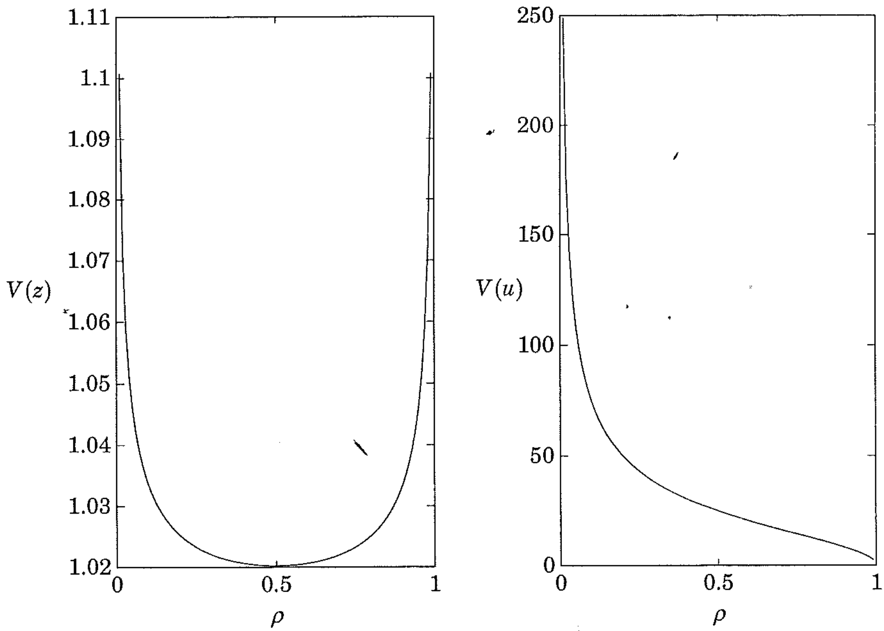

Figure 3. The variances of z—left, and u—right, as functions of ρ.

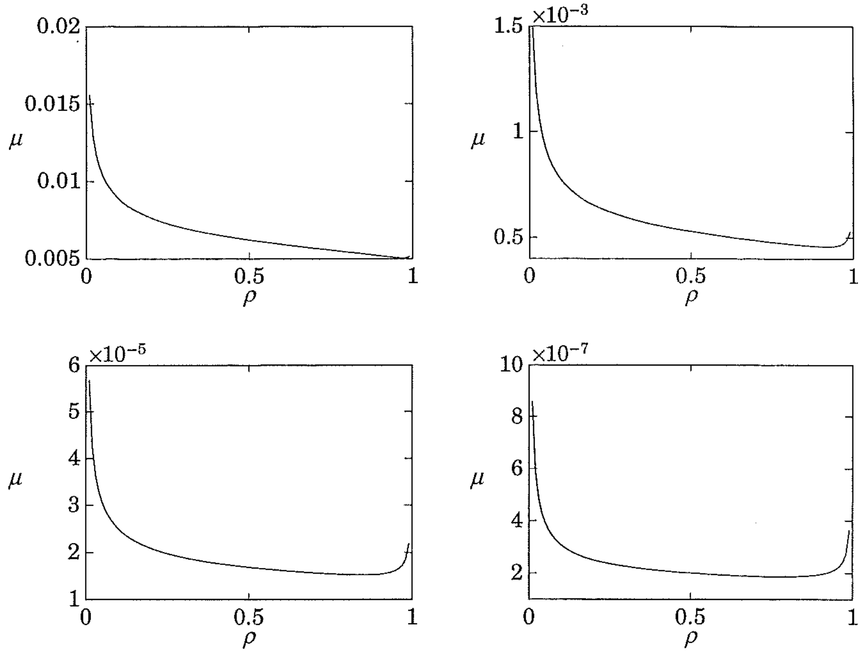

Figure 4. The probability μ as function of ρ for $z_0 = 2$—top left, $z_0 = 3$—top right, $z_0 = 4$—bottom left, and $z_0 = 5$—bottom right.

signal is smaller the larger ρ is, and the controller obtained for $\rho = 0.5$ is the MV controller by Remark 4 to Lemma 3. Thus the MU controller not only minimizes the upcrossing probability, but it also has a control signal that is more well-behaved than that of the MV controller.

SIMULATIONS. The controllers have also been compared in simulations. The same noise sequences were used for both controllers in all cases. Figure 5 shows plots of z and u as functions of time for the MV controller and the MU controller when $z_0 = 3$. It is seen that that the MU controller manages to keep the signal z below the critical level, while the MV controller does not. Further, it is seen that the variance of u is smaller for the MU controller than for the MV controller. Note that z is not white noise for the MV controller although y is. This is due to the fact that y is correlated with e.

TRANSFER FUNCTIONS. The MU controller for $z_0 = 3$ ($\rho = 0.92$) is given by

$$H(q) = -\frac{0.4901q}{q - 0.4804}$$

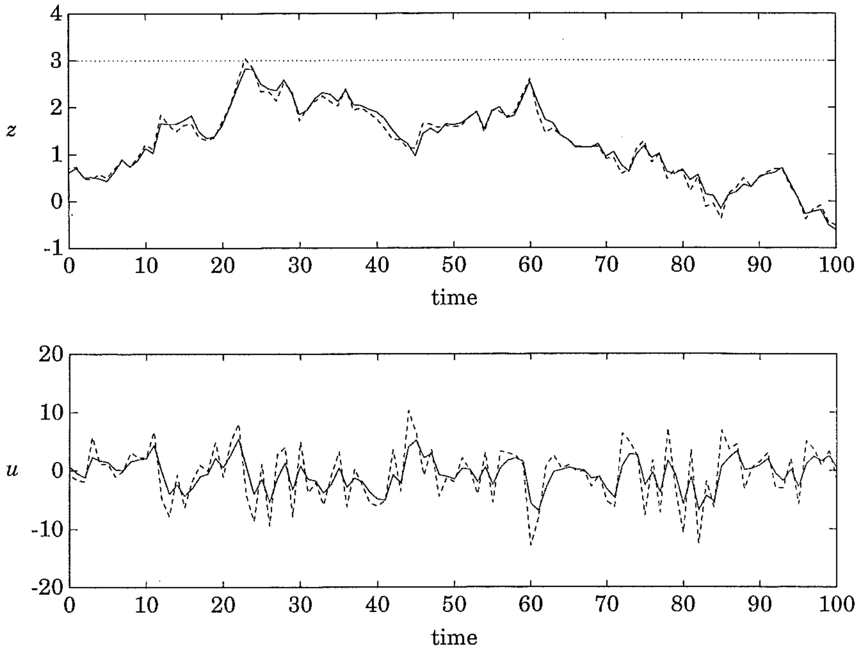

Figure 5. The signals $z(t)$—top, and $u(t)$—bottom, as function of time for: (1) the minimum upcrossing controller—solid line; and (2) the minimum variance controller—broken line.

and the MV controller is given by:

$$H(q) = -0.9802$$

It is interesting to note that the main difference between the MV controller and the MU controller is that the MU controller has a three times lower gain for high frequences ($q = -1$) owing to the MU controller being a first-order system while the MV controller being only a proportional controller. This explains why the variance of the control signal is much smaller for the MU controller. Some calculations show that the closed-loop system is governed by

$$(q - 0.9608)z = 0.2v - 0.196e$$

for the MV controller and

$$[(q-1)(q-0.4804) + 0.0196]z = 0.2(q-0.4804)v - 0.098e$$

for the MU controller. It is seen that the main difference between the MV controller and the MU controller is the lower high frequency gain ($q = -1$) from e to z for the MU controller.

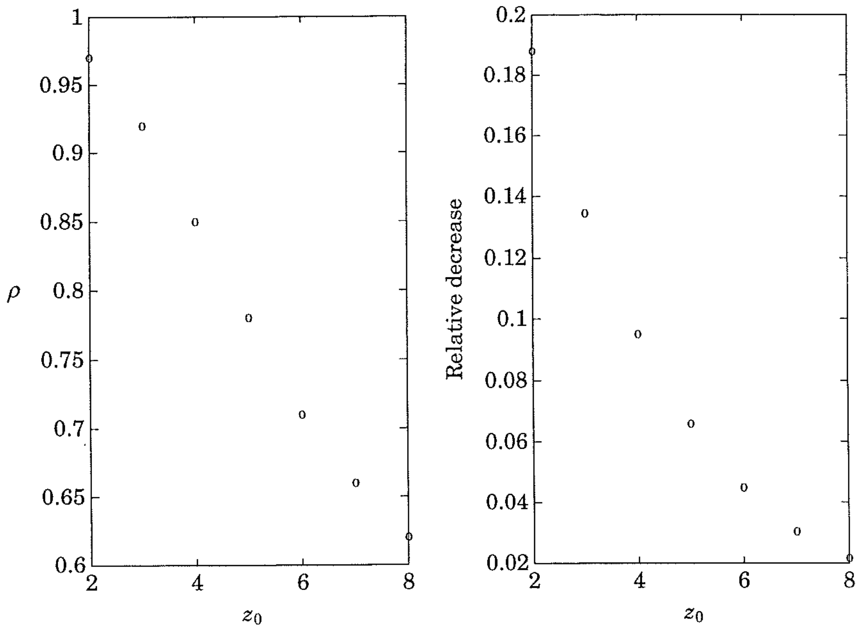

Figure 6. The optimal values of ρ as function of z_0—left, and $(\mu_{\rm mv} - \mu_{\rm mu})/\mu_{\rm mv}$ as function of z_0—right, where $\mu_{\rm mv}$ is the upcrossing probability for the MV controller and $\mu_{\rm mu}$ is the upcrossing probability for the MU controller.

APPROXIMATION-VALIDITY. The validity of the assumptions made in the approximation of the problem formulation in § 2 will now be investigated further; one positive indication has already been seen in Figure 5. In Figure 6 it is seen how the optimal value of ρ, and how the relative decrease of upcrossing probability between the MV controller and the MU controller decreases as z_0 increases. This indicates that the MU controller and the MV controller are approximately the same for large values of z_0.

To investigate the behavior of the controllers for moderate values of z_0, Monte Carlo-simulations have been performed to estimate the distribution function for the time between failures. The result is seen in Figure 7. The x-axis is time-normalized as in Theorem 1 in such a way that $n\mu(z_0^{(n)}(L)) = L = 0.1$ for the MU controller. Notice that the distribution function of the MU controller is always below that of the MV controller. It is also seen that the deviation from the limiting distribution function is smaller the larger is z_0, and that the convergence is better the larger are the values of k/n. Notice further that the larger is z_0, the smaller is the

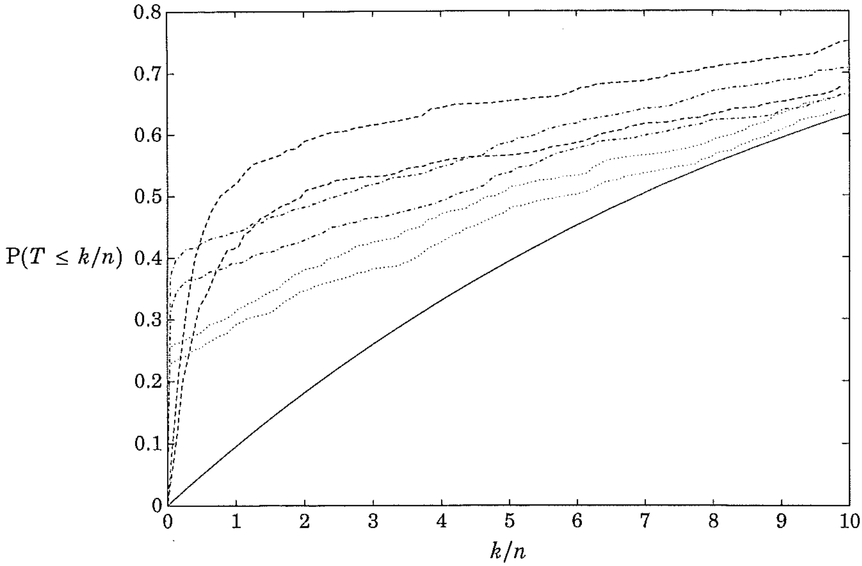

Figure 7. The distribution function for the time between failures. The x-axis is time-normalized as in Theorem 1 in such a way that $L = 0.1$ for the MU controller. The solid line is the theoretical limiting distribution function. The dashed lines correspond to $z_0 = 2$, the dash-dotted to $z_0 = 3$ and the dotted to $z_0 = 4$. The lines with the higher values within each group corresponds to MV control, and the lines with the lower values correspond to MU control.

difference between the distribution functions for the different controllers. The mean values of the distributions, i.e. the MTBF are given in Table 1. Here time is not normalized. It is seen that the relative increase of MTBF between the MV controller and the MU controller decreases as z_0 increases, and that the MU controller in this example performs up to 25% better than the MV controller for moderate values of the critical level.

z_0	$\mathrm{MTBF}_{\mathrm{MV}}$	$\mathrm{MTBF}_{\mathrm{MU}}$	$\frac{\mathrm{MTBF}_{\mathrm{MU}}-\mathrm{MTBF}_{\mathrm{MV}}}{\mathrm{MTBF}_{\mathrm{MV}}}$
2	151	190	0.256
3	1845	2216	0.201
4	61050	68664	0.125

Table 1. Estimates of the MTBF for different values of z_0. The right-most column shows the relative increase in MTBF between the MV controller and the MU controller.

Summary

The theory developed in the previous chapters has been evaluated using a first-order process. In spite of the simplicity of the process, many interesting features of the new controller have been demonstrated.

It has been shown that the MU controller is a first-order system whereas the MV controller is only a zero-order system—a proportional controller. The former has a lower high-frequency gain. The variance of z is slightly larger but the variance of u is much smaller for the MU controller as compared with the MV controller. The simulations have given further insight into the consequences of the approximations made to derive the new controller. When comparing the MU controller and the MV controller with respect to the MTBF criterion for varying distances to the critical level, it has been seen that the difference in MTBF is larger for moderate values of the distance and smaller for larger values of the distance. For the examples in the introduction the distance is typically moderate, and thus it has been justified that the MU controller may well be a good alternative to the MV controller for this class of problems.

5. *Conclusions*

A new optimal stochastic control problem has been posed. The solution maximizes the MTBF criterion. There are many examples of control problems for which this approach is appealing, i.e. problems for which there exist a level such that a failure in the controlled system occurs when the controlled signal upcrosses the level. One important class of such problems is processes equipped with supervision, where upcrossings of alarm levels may initiate emergency shutdown causing loss in production.

It has been seen that the control problem posed is closely related both to the problem of minimizing the variance of the signal—MV control—and to the problem of minimizing the upcrossing probability—MU control. The latter relation is novel, whereas the former relation has been known for a long time, but the motivations given here are believed to be new. It has been made plausible that the upcrossing probability is a better criterion to minimize than the variance criterion.

The problem of minimizing the upcrossing probability over the set of stabilizing linear time-invariant controllers has been rephrased to a minimization over LQG-problem solutions parameterized by a scalar, and thus the complexity is only one order of magnitude larger than for an ordinary LQG-problem. If the solutions to the LQG-problems are unique, then the problem of minimizing the upcrossing probability can be thought of as finding optimal weighting-matrices in an LQG-problem. The key to the new method is the reformulation using the independent variables α and β making it possible to quantify by Lemma 5 the upcrossing probability in terms of the variances of α and β.

The new controller has been compared with the MV controller for a first-order process. It has been seen that the new controller causes a lower upcrossing intensity and a larger MTBF. Further, it has been seen that the control signal is more well behaved.

Both theory and simulations have shown that the MU controller and the MV controller are approximately the same for large values of the dangerous level. However, in the example it has been seen that the MU controller can have up to about 25% better performance for moderate values of the critical level. This is the interesting case for the examples in § 1. Thus MU control makes it possible to choose the reference value closer to the critical level without causing smaller MTBF. This will in many cases increase profit.

To summarize the advantages of the new controller, the following features should be mentioned—larger MTBF, a control-signal that is more well behaved, and an interpretation as weighting-optimal LQG. The only drawback is the slightly larger computational burden in computing it.

This concludes the work of proving the raison d'être of the MU controller and demonstrating its advantages as compared to the MV controller for control of processes with critical levels.

Acknowledgements

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Part III

Existence of Discrete-Time LQG-Controllers

EXISTENCE results for the LQG-controller are investigated. An infimal Riccati equation based controller may potentially give closed loop eigenvalues on the unit circle. Assuming left and right invertibility it is shown that there exists an optimal controller if and only if the Riccati equation based controller stabilizes the closed loop system after removal of all its unobservable and uncontrollable modes. Furthermore this reduced controller is the optimal controller, and its transfer function is unique. This existence condition is a considerable simplification of the more general geometric condition recently derived by Trentelman and Stoorvogel.

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1. *Introduction*

The LQG- or H_2-controller has a long history. The topic is treated in many text-books, among others Kucera (1991). The singular cases were, however, not fully described. When people started to work on the optimal H_∞-controller the subject got renewed interest. A good understanding of singular LQG controllers also facilitates the analysis of the so called minimum upcrossing controller, as in Hansson and Hagander (1994).

Two different types of singularities are encountered in H_2-problems. The first type is related to non-uniqueness due to redundant control or measurement signals. This type of singularity will not be discussed here. For LQ-problems it has been discussed in e.g. Hagander and Hansson (1994). The second type of singularity is related to closed loop poles on the stability boundary. In most such cases there exists no optimal controller corresponding to the infimal cost. It will, however, be shown that if and only if all the unstable modes of the closed loop system are in the controller and such that they are canceled by zeros of the controller, then this reduced order controller is indeed an optimal controller. This was actually discussed already in Kucera (1980), and it is illustrated by an example:

EXAMPLE 1—Minimum Variance Control
Consider the process model

$$A(q)y(k) = B(q)u(k) + C(q)e(q)$$

where $y(k)$ is the measurement signal, $u(k)$ the control signal, $e(k)$ is a sequence of independent zero mean Gaussian distributed random variables, and $A(q)$, $B(q)$, and $C(q)$ are polynomials in the forward shift operator q. Assume that $C(q)$ has all its zeros inside or on the unit circle, that $\deg A(q) = \deg C(q) = n$, and that $\deg B(q) = n - d$. It is well-known, see Åström and Wittenmark (1990), that the controller that minimizes

$$\mathrm{E}\left\{y^2(k)\right\}$$

in stationarity is given by $u(k) = -S(q)/R(q)y(k)$, where $S(q)$ and $R(q)$ satisfy the following Diophantine equation

$$A(q)R(q) + B(q)S(q) = P(q)C(q)$$

with $R(0) = S(0) = 0$, and $P(q) = q^d \prod_{i=1}^{s}(q - z_i) \prod_{i=s+1}^{n-d}(q - 1/z_i)$, where z_i are the stable and unstable zeros of $B(q)$ respectively. If in addition $B(q)$ and $C(q)$ have no zeros on the unit circle, then the controller will also be stabilizing. The converse is, however, not always true, as the following example shows. Consider

$$A(q) = q^4; \quad B(q) = (q-1)(q-2)^2; \quad C(q) = q(q-1)(q^2 + 8/21q + 4/21)$$

There are two closed loop poles at $q = 1$ due to the presence of a factor $(q-1)$ in both $B(q)$ and $C(q)$. The Diophantine equation

$$q^4R(q) + (q-1)(q-2)^2S(q) = q(q-1/2)^2(q-1)q(q-1)(q^2+8/21q+4/21)$$

has the solution $R(q) = q(q-1)^2(q+51/84)$ and $S(q) = -1/84q^2(q-1)^2$. Here the two closed loop poles at $q = 1$ appear in the controller, i.e. $(q-1)^2$ are factors of $R(q)$, and they are canceled by the same factors in $S(q)$. The reduced order controller

$$u(k) = -\frac{q}{84q+51}y(k) \tag{1}$$

is thus optimal. □

The cancellation of the factor in $B(q)$ can be interpreted as loss of controllability in the controller, and the cancellation of the factor in $C(q)$ can be interpreted as loss of observability in the controller.

It is straight forward to see that the presence of all unstable closed loop poles in the H_2-controller together with cancellation is a sufficient condition for it to be stabilizing. That it is also a necessary condition, is more tricky. The derivation will rely on a general version of the separation principle. In Trentelman and Stoorvogel (1993) a geometric approach utilizing Stoorvogel and van der Woude (1991) is taken to give necessary and sufficient conditions for a stabilizing H_2-controller. It will be seen that the conditions given there are closely related to the more explicit one given in this paper.

In Chen *et al.* (1993) an algorithm is given for constructing all stabilizing H_2-controllers. There the modes of the controller are not canceled, but instead moved to an arbitrary position inside the unit circle. Then the use of Q-parametrization gives a necessary and sufficient condition for uniqueness of the optimal controller. The specialization in this paper implies the uniqueness.

2. Control Problem and Solution

Consider the following state space description

$$\begin{pmatrix} x(k+1) \\ z(k) \\ y(k) \end{pmatrix} = \begin{pmatrix} A & B_w & B_u \\ C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & D_{yu} \end{pmatrix} \begin{pmatrix} x(k) \\ w(k) \\ u(k) \end{pmatrix} \tag{2}$$

where $w(k) \in R^l$ is a sequence of independent zero mean Gaussian random variables with covariance I, $u(k) \in R^m$ is the control signal, $x(k) \in R^n$ is the state, $y(k) \in R^p$ is the measurement signal, and $z(k) \in R^q$ is the signal to be controlled. It will be assumed that $D_{yu} = 0$. Let the control signal be given by

$$\begin{aligned} \xi(k+1) &= A_H\xi(k) + B_H y(k) \\ -u(k) &= C_H\xi(k) + D_H y(k) \end{aligned} \tag{3}$$

where A_H, B_H, C_H, and D_H are real-valued matrices of appropriate dimensions. The state ξ may have any finite dimension. Introduce the notation $H = \{A_H, B_H, C_H, D_H\}$ for a controller such as (3). Denote by $\mathcal{D}$ the set of all H as defined above, and by $\mathcal{D}_s$ the subset of $\mathcal{D}$ which stabilizes (2), i.e. the set of controllers which are such that the eigenvalues of the closed loop system have absolute values strictly less than one. Introduce the following performance index:

$$J(H) = \lim_{k\to\infty} \mathrm{E}\left\{z^T(k)z(k)\right\}, \quad H \in \mathcal{D}_s \tag{4}$$

Since $H \in \mathcal{D}_s$, it is no loss in generality to assume that $x(0) = 0$ when evaluating J. Consider the following optimal control problem

$$\min_{H\in\mathcal{D}_s} J(H) \tag{5}$$

which is known as the H_2-problem. This is a convex problem, and hence the infimum of $J(H)$ always exists. However, the set $\mathcal{D}_s$ is open, and thus the infimum will not always be a minimum, i.e. the smallest value of the performance index J may be attained by a controller which does not internally stabilize the closed loop system.

Introduce the following standing assumptions:

(A1) : (A, B_u) stabilizable

(A2) : (C_y, A) detectable

If (A1) or (A2) does not hold, then $\mathcal{D}_s$ will be empty, i.e. there is no optimal controller either. Also introduce the matrices

$$P_c(z) = \begin{pmatrix} zI - A & -B_u \\ C_z & D_{zu} \end{pmatrix}; \quad P_o(z) = \begin{pmatrix} zI - A & -B_w \\ C_y & D_{yw} \end{pmatrix}$$

and the non-standing assumptions

(A3) : $\max_z \operatorname{rank} P_c(z) = n + m$

(A4) : $\max_z \operatorname{rank} P_o(z) = n + p$

Conditions (A3) and (A4) are actually equivalent to the uniqueness of the optimal controller and usually referred to as left and right invertibility of $\{A, B_u, C_z, D_{zu}\}$ and $\{A, B_w, C_y, D_{yw}\}$ respectively, e.g. Silverman (1976).

It will be shown that the optimal controller, whenever it exists, can be obtained by solving the Riccati equations

$$\begin{aligned} S &= (A - B_u L)^T S (A - B_u L) + (C_z - D_{zu} L)^T (C_z - D_{zu} L) \\ G &= B_u^T S B_u + D_{zu}^T D_{zu} \\ G \begin{pmatrix} L & L_v & L_w \end{pmatrix} &= \begin{pmatrix} B_u^T S A + D_{zu}^T C_z & B_u^T S & D_{zu}^T \end{pmatrix} \end{aligned} \tag{6}$$

and

$$\begin{aligned} P &= (A - KC_y) P (A - KC_y)^T + (B_w - KD_{yw})(B_w - KD_{yw})^T \\ (C_y P C_y^T + D_{yw} D_{yw}^T) \begin{pmatrix} K \\ K_x \\ K_v \\ K_w \end{pmatrix}^T &= \begin{pmatrix} A P C_y^T + B_w D_{yw}^T \\ P C_y^T \\ B_w D_{yw}^T \\ D_{zw} D_{yw}^T \end{pmatrix}^T \end{aligned} \tag{7}$$

These are the unique real symmetric matrices $S \geq 0$ and $P \geq 0$ such that there exist L and K with all eigenvalues of $A - B_u L$ and $A - KC_y$ inside or on the unit circle. They are also the maximal solutions. The existence of these solutions are guaranteed by conditions (A1)–(A4). Let

$$\begin{aligned} A_{co} &= A - B_u L - KC_y + B_u D_c C_y; & B_c &= K - B_u D_c \\ C_c &= L - D_c C_y; & D_c &= LK_x + L_v K_v + L_w K_w \end{aligned} \tag{8}$$

and define the controller

$$H_{nom} = \{A_{nom}, B_{nom}, C_{nom}, D_{nom}\} \tag{9}$$

where $\{A_{nom}, B_{nom}, C_{nom}, D_{nom}\}$ is a minimal realization of $\{A_{co}, B_c, C_c, D_c\}$.

THEOREM 1
Under assumptions (A1)–(A4) there exists a solution to (5) if and only if H_{nom} as defined in (9) is in $\mathcal{D}_s$, i.e. is stabilizing. Further, this solution has a unique transfer function. □

The proof of this theorem will be carried out in Section 3, where the structure of the cancellations is also further investigated. The existence of an H_2-controller is easily investigated. Just solve the Riccati equations (6) and (7), compute the controller as in (8), and then obtain a minimal realization as in (9). Then there exists an optimal controller if and only if the minimal realization is stabilizing, and furthermore the optimal controller is given by this minimal realization. The existence conditions in Trentelman and Stoorvogel (1993) available for the general case, i.e. also when conditions (A3) and (A4) do not hold, are much more involved. As discussed in the next section the existence conditions in Trentelman and Stoorvogel (1993) imply that the closed loop unstable modes from $A_o = A - KC_y$ are unobservable in C_c and that the ones from $A_c = A - B_u L$ are uncontrollable from B_c. This also suggests an explicit reduced order controller.

3. *Derivation of the Results*

In this section Theorem 1 will be proved. Furthermore the conditions derived in this paper will be related to the ones presented in Trentelman and Stoorvogel (1993).

The Separation Principle

The approach taken in this paper is the classical stochastic approach for solving LQG-problems utilizing separation. To this end introduce the following observer

$$\begin{aligned} \hat{x}(k+1) &= A\hat{x}(k) + B_u u(k) + K\tilde{y}(k), \quad \hat{x}(0) = 0 \\ \tilde{y}(k) &= y(k) - C_y \hat{x}(k) \end{aligned} \tag{10}$$

where K is a solution of (7) such that $\text{eig}(A - KC_y) \leq 1$. Define $\tilde{x}(k) = x(k) - \hat{x}(k)$. It then holds, since $x(0) = 0$, that

$$\begin{aligned} \tilde{x}(k+1) &= A_o\tilde{x}(k) + B_N w(k), \quad \tilde{x}(0) = 0 \\ \tilde{y}(k) &= C_y\tilde{x}(k) + D_{yw}w(k) \end{aligned} \tag{11}$$

where $A_o = A - KC_y$, and $B_N = B_w - KD_{yw}$. Since there is no guarantee for A_o being stable, some care has to be taken in order to get orthogonality between $\hat{x}(k)$ and $\tilde{x}(k)$.

LEMMA 1
Assume that (A2) and (A4) hold. Then it holds that $\hat{x}(k)$ and $\tilde{x}(k)$ as defined in (10) and (11) are orthogonal in stationarity, i.e. $\mathrm{E}\left\{\hat{x}(k)\tilde{x}^T(k)\right\} \rightarrow 0, \quad k \rightarrow \infty$. Further the stationary covariance of $\tilde{x}(k)$ is given by the solution P of (7).

Proof: It is well-known that there exists a time-varying Kalman filter, i.e. $K = K(k)$, which computes an estimate $\hat{x}(k)$ of $x(k)$ such that $\hat{x}(k)$ and $\tilde{x}(k)$ are orthogonal, provided that the initial value $\hat{x}(0)$ is correctly chosen. Further if A_o is stable then there also exists a stationary Kalman-filter with the properties given above. However, if $\tilde{x}(0) = 0$, then it can be shown that there still exists a stationary Kalman-filter, even if A_o is not stable. This follows from the fact that the unstable modes of A_o are not controllable from B_N, and hence these modes will be identically zero, provided that $\tilde{x}(0) = 0$, which make them uncorrelated with the corresponding $\hat{x}(k)$-modes. That the unstable modes of A_o are not controllable from B_N is the dual of Lemma 7 in the appendix. □

Remark. It can be shown that $\hat{x}(k) = \mathrm{E}\{x(k)|\mathcal{Y}(k-1)\}$, where $\mathcal{Y}(k-1) = \begin{pmatrix} y(k-1) & y(k-2) & \cdots \end{pmatrix}$.

In order to allow for not only strictly proper controllers but also for proper controllers, an estimate of $x(k)$ based on $\mathcal{Y}(k)$ is needed as well as estimates of $e_x(k) = B_w w(k)$ and $e_z(k) = D_{zw} w(k)$.

LEMMA 2
Assume that (A2) and (A4) hold. Then with K_x, K_v, and K_w from (7) it holds that

$$\mathrm{E}\left\{\left[\begin{pmatrix} x(k) \\ e_x(k) \\ e_z(k) \end{pmatrix} \middle| \mathcal{Y}(k)\right\} - \left\{\begin{pmatrix} K_x \\ K_v \\ K_w \end{pmatrix} \tilde{y}(k) + \begin{pmatrix} \hat{x}(k) \\ 0 \\ 0 \end{pmatrix}\right]\right\} \rightarrow 0, \quad k \rightarrow \infty$$

Proof: The result follows by Lemma 1 and Åström (1970) Theorem 3.2. and Theorem 3.3. □

THEOREM 2—Separation Principle
Assume that (A1)–(A4) hold. Then for any $H \in \mathcal{D}_s$ it holds that

$$J(H) = \lim_{k\rightarrow\infty} \mathrm{E}\left\{[u(k) + L\hat{x}(k) + D_c\tilde{y}(k)]^T G\,[u(k) + L\hat{x}(k) + D_c\tilde{y}(k)]\right\} + J^*$$

where J^* is independent of H.

Proof: The first step of the proof is a tedious completion of squares utilizing (6) and the fact that $\lim_{k\rightarrow\infty} \mathrm{E}\left\{x^T(k+1)Sx(k+1) - x^T(k)Sx(k)\right\} = 0$

for any stabilizing controller, yielding

$$J(H) = \lim_{k\to\infty} \mathrm{E}\Big\{[u(k) + Lx(k) + L_v e_x(k) + L_w e_z(k)]^T \cdot G[u(k) + Lx(k) + L_v e_x(k) + L_w e_z(k)]\Big\} + \bar{J}$$

where $\bar{J}$ is independent of H. Then by (7), (8), Lemma 2, and the orthogonality between the estimates and the estimation errors the result follows. □

Sufficient and Necessary Conditions for Existence

Let $A_c = A - B_u L$, and temporarily introduce the following assumptions

(A5) : $\mathrm{rank}_{|z|=1} P_c(z) = n + m$

(A6) : $\mathrm{rank}_{|z|=1} P_o(z) = n + p$

which are equivalent to no zeros on the unit circle. Notice that these conditions were not fulfilled in Example 1. Then the following sufficient condition holds:

LEMMA 3

Under assumptions (A1)–(A6) the controller H_{nom}, as defined in (9), is a solution to the optimization problem (5). Further its transfer function is unique.

Proof: Notice that by Lemma 8 in the appendix there exists only one solution (S, P) to the Riccati equations such that A_c and A_o are stable. Further by Theorem 2, and since $G > 0$ by Lemma 6, it holds that $u(k) = -L\hat{x}(k) - D_c\tilde{y}(k)$ is the unique control signal that minimizes the performance index J. This is the same control signal as the one defined by H_{nom}. Further this controller is stabilizing, since the closed loop system is governed by

$$\begin{pmatrix} x(k+1) \\ \tilde{x}(k+1) \end{pmatrix} = \begin{pmatrix} A_c & B_u C_c \\ 0 & A_o \end{pmatrix} \begin{pmatrix} x(k) \\ \tilde{x}(k) \end{pmatrix} + \begin{pmatrix} B_N + B_c D_{yw} \\ B_N \end{pmatrix} w(k) \quad (12)$$

where A_c and A_o are stable under conditions (A5) and (A6) by Lemma 6 and its dual version. □

Now allow zeros on the unit circle, i.e assume only (A1)–(A4), and drop (A5) and (A6).

PROOF OF THEOREM 1

The proof follows ideas from Trentelman and Stoorvogel (1994). Consider the perturbation of (2) obtained by the following replacements

$$B_w \leftrightarrow \begin{pmatrix} B_w & \varepsilon I \end{pmatrix}; \qquad D_{yw} \leftrightarrow \begin{pmatrix} D_{yw} & 0 \end{pmatrix}$$
$$C_z \leftrightarrow \begin{pmatrix} C_z \\ \varepsilon I \end{pmatrix}; \qquad D_{zu} \leftrightarrow \begin{pmatrix} D_{zu} \\ 0 \end{pmatrix}$$

First it will be shown that the solutions S_ε and P_ε of the Riccati-equations associated with the perturbed problem converge to limits S and P satisfying the algebraic Riccati equations for the original problem as $\varepsilon \to 0$. To this end introduce the performance index V_ε associated with the LQ-problem related to S_ε, i.e. let

$$\begin{aligned} x(k+1) &= Ax(k) + B_u u(k) + B_v v(k) \\ z_\varepsilon(k) &= \begin{pmatrix} C_z \\ \varepsilon I \end{pmatrix} x(k) + \begin{pmatrix} D_{zu} \\ 0 \end{pmatrix} u(k) \\ V_\varepsilon(u) &= \lim_{k\to\infty} \mathrm{E}\left\{z_\varepsilon^T(k) z_\varepsilon(k)\right\} \end{aligned}$$

where $v(k) \in R$ is a sequence of independent Gaussian random variables with unit covariance and zero mean. Notice that the solution S_ε of the Riccati-equation does not depend on B_v. It holds that $V_{\varepsilon_1}(u) \le V_{\varepsilon_2}(u)$ for $0 < \varepsilon_1 \le \varepsilon_2$, since $z_\varepsilon^T(k)z_\varepsilon(k) = z^T(k)z(k) + \varepsilon^2 x^T(k)x(k)$. Further it holds that

$$0 \le B_v^T S_{\varepsilon_1} B_v = V_{\varepsilon_1}(u^*_{\varepsilon_1}) \le V_{\varepsilon_1}(u^*_{\varepsilon_2}) \le V_{\varepsilon_2}(u^*_{\varepsilon_2}) = B_v^T S_{\varepsilon_2} B_v$$

where $u^*_{\varepsilon_i}$ is the control signal minimizing V_{ε_i}. This control signal is well-defined for all $\varepsilon_i > 0$ by (A5). By considering suitable B_v:s it follows that $S_\varepsilon \downarrow S \ge 0$ as $\varepsilon \downarrow 0$ for some S. Further it holds that $G_\varepsilon > 0$ for all $\varepsilon \ge 0$ by (A3). Thus the Riccati-equation is well-defined for all $\varepsilon \ge 0$, and hence the limit S will satisfy the Riccati-equation for $\varepsilon = 0$. Dual arguing shows the same result for P_ε.

Now consider the LQG-problem. Denote the value of the performance index for the perturbed system and any $H \in \mathcal{D}_s$ by $J_\varepsilon(H)$. Notice that the perturbation does not influence the transfer function from u to y, so $\mathcal{D}_s$ is also the set of controllers that stabilize the perturbed system. This system satisfies (A5) and (A6) and has an optimal controller $H_\varepsilon \in \mathcal{D}_s$ for all $\varepsilon > 0$ by Lemma 3. Denote the corresponding minimal value of the performance index by J^*_ε. Since the perturbation is linear it holds that

$J(H) \le J_\varepsilon(H)$ for all $H \in \mathcal{D}_s$. Further $J(H) \ge J^*$ for all $H \in \mathcal{D}_s$ by Theorem 2. Especially this holds for $H = H_\varepsilon$, which sums up to

$$J^* \le J(H_\varepsilon) \le J_\varepsilon^*$$

Now S_ε and P_ε converge to S and P, which are the by Lemma 8 unique solutions of (6) and (7) such that A_c and A_o have all their eigenvalues inside or on the unit circle. Hence $J_\varepsilon^* \to J^*$, $\varepsilon \to 0$, which by Theorem 2 implies that the infimal value of J is attained by

$$u(k) = -L\hat{x}(k) - D_c\tilde{y}(k)$$

This control signal is unique, since $G > 0$. It can also be expressed as

$$\begin{aligned} \hat{x}(k+1) &= A_{co}\hat{x}(k) + B_c y(k) \\ -u(k) &= C_c\hat{x}(k) + D_c y(k) \end{aligned}$$

□

Relation to Trentelman and Stoorvogel

Let $\mathcal{V}_g$ be the invariant subspace associated with the stable eigenvalues of A_c, and let $\mathcal{S}_g$ be the invariant subspace associated with the unstable eigenvalues of A_o.

THEOREM 3—Trentelman and Stoorvogel
Assume that (A1)–(A4) hold. Then there exists a solution to (5) if and only if

(C1): $\text{Im}B_c \subset \mathcal{V}_g$

(C2): $\mathcal{S}_g \subset \text{Ker}\,C_c$

(C3): $(A - B_u D_c C_y)\mathcal{S}_g \subset \mathcal{V}_g$

(C4): $\mathcal{S}_g \subset \mathcal{V}_g$

Proof: This is an immediate specialization of the conditions given in Trentelman and Stoorvogel (1993). □

It will now be shown how these conditions are closely related to the results of Theorem 1. To this end let U_c and U_o be transformations that bring A_c and A_o, respectively, to block diagonal form

$$A_c U_c = U_c \begin{pmatrix} J_{cs} & 0 \\ 0 & J_{cu} \end{pmatrix}; \quad A_o U_o = U_o \begin{pmatrix} J_{os} & 0 \\ 0 & J_{ou} \end{pmatrix} \tag{13}$$

where J_{cs} and J_{os} contain the stable parts. Further let

$$U_c = \begin{pmatrix} U_{cs} & U_{cu} \end{pmatrix}; \quad U_o = \begin{pmatrix} U_{os} & U_{ou} \end{pmatrix}$$

be partitionings corresponding to the blocking in (13), and denote the inverses of these transformations by

$$V_c^T = \begin{pmatrix} V_{cs}^T \\ V_{cu}^T \end{pmatrix}; \quad V_o^T = \begin{pmatrix} V_{os}^T \\ V_{ou}^T \end{pmatrix}$$

LEMMA 4

The conditions (C1)–(C4) of Theorem 3 are equivalent to :

(I): All unstable modes of A_c are uncontrollable from B_c, i.e. $V_{cu}^T B_c = 0$.

(II): All unstable modes of A_o are unobservable from C_c, i.e. $C_c U_{ou} = 0$.

(III): $\mathrm{Im}\, U_{ou} \subset \mathrm{Im}\, U_{cs}$, i.e. $V_{cu}^T U_{ou} = 0$.

Proof: It is trivial that (C2) is equivalent to (II), and that (C4) is equivalent to (III). That (C1) is equivalent to (I) follows from the fact that (C1) is equivalent to $\mathrm{Im} B_c \subset \mathrm{Im} U_{cs}$, which is equivalent to $\mathrm{Im} V_{cu} \subset \mathrm{Ker} B_c^T$, which is equivalent to (I). Further it holds that $(A - B_u D_c C_y) U_{ou} = (A_c + B_u C_c) U_{ou} = A_c U_{ou}$ by (II) or equivalently by (C2). By (III) or equivalently by (C4) there exist α such that $U_{ou} = U_{cs}\alpha$. Hence $(A - B_u D_c C_y) U_{ou} = A_c U_{cs}\alpha = U_{cs} J_{cs}\alpha$, where the second equality follows by the definition of U_{cs}. Hence conditions (C2) and (C4) imply condition (C3). □

LEMMA 5

Conditions (I)–(III) of Lemma 4 are equivalent to the existence of a state transformation T such that

$$\begin{aligned} A_{co} T &= T \begin{pmatrix} J_{co} & 0 & * \\ * & J_{ou} & * \\ 0 & 0 & J_{cu} \end{pmatrix} \\ T^{-1} B_c &= \begin{pmatrix} B_1 \\ B_2 \\ 0 \end{pmatrix} \\ C_c T &= \begin{pmatrix} C_1 & 0 & C_3 \end{pmatrix} \end{aligned} \tag{14}$$

where J_{cu} and J_{ou} are given by (13).

Proof: Assume that there exists a state transformation T as defined above. Then multiply the first equation of (14) by $\begin{pmatrix} 0 & I & 0 \end{pmatrix}^T$ from the right. This implies that $U_{ou} = T \begin{pmatrix} 0 & I & 0 \end{pmatrix}^T$. Multiplying the first equation of (14) by $\begin{pmatrix} 0 & 0 & I \end{pmatrix} T^{-1}$ from the left implies that $V_{cu}^T =$

$\begin{pmatrix} 0 & 0 & I \end{pmatrix} T^{-1}$. Hence $C_c U_{ou} = 0$, $V_{cu}^T B_c = 0$ and $V_{cu}^T U_{ou} = 0$, which are equivalent to conditions (I)–(III). Now assume the converse, i.e. that the conditions of Lemma 4 hold. Condition (III) implies that there exists α such that $U_{ou} = U_{cs}\alpha$. Notice that the columns of α are linearly independent, and that

$$V_c^T U_o = \begin{pmatrix} V_{cs}^T U_{os} & \alpha \\ V_{cu}^T U_{os} & 0 \end{pmatrix}$$

Consider

$$\begin{aligned} V_c^T A_{co} U_o &= V_c^T U_o \left\{ \begin{pmatrix} J_{os} & 0 \\ 0 & J_{ou} \end{pmatrix} - V_o^T B_u \begin{pmatrix} C_c U_{os} & 0 \end{pmatrix} \right\} \\ &= \begin{pmatrix} * & \alpha J_{ou} \\ * & 0 \end{pmatrix} = \left\{ \begin{pmatrix} J_{cs} & 0 \\ 0 & J_{cu} \end{pmatrix} - \begin{pmatrix} V_{cs}^T B_c \\ 0 \end{pmatrix} C_y U_c \right\} V_c^T U_o \\ &= \begin{pmatrix} * & (J_{cs} - V_{cs}^T B_c C_y U_{cs})\alpha \\ * & 0 \end{pmatrix} \end{aligned}$$

This implies that

$$(J_{cs} - V_{cs}^T B_c C_y U_{cs})\alpha = \alpha J_{ou}$$

Let β be such that $\begin{pmatrix} \beta & \alpha \end{pmatrix}$ is a basis, and such that

$$(J_{cs} - V_{cs}^T B_c C_y U_{cs}) \begin{pmatrix} \beta & \alpha \end{pmatrix} = \begin{pmatrix} \beta & \alpha \end{pmatrix} \begin{pmatrix} J_{co} & 0 \\ * & J_{ou} \end{pmatrix}$$

Now, consider

$$\begin{aligned} U_c^{-1} A_{co} U_c \begin{pmatrix} \beta & \alpha & 0 \\ 0 & 0 & I \end{pmatrix} &= \begin{pmatrix} J_{cs} - V_{cs}^T B_c C_y U_{cs} & * \\ 0 & J_{cu} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} \beta & \alpha \end{pmatrix} & 0 \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} \begin{pmatrix} \beta & \alpha \end{pmatrix} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \begin{pmatrix} J_{co} & 0 \\ * & J_{ou} \end{pmatrix} & * \\ 0 & J_{cu} \end{pmatrix} \end{aligned}$$

Hence with

$$T = U_c \begin{pmatrix} \begin{pmatrix} \beta & \alpha \end{pmatrix} & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} U_{cs}\beta & U_{ou} & U_{cu} \end{pmatrix}$$

it holds that

$$A_{co} T = T \begin{pmatrix} J_{co} & 0 & * \\ * & J_{cu} & * \\ 0 & 0 & J_{ou} \end{pmatrix}$$

and by condition (II) it holds that

$$C_c T = \begin{pmatrix} C_1 & 0 & C_3 \end{pmatrix}$$

for some C_1 and C_3. Further

$$T^{-1} = \begin{pmatrix} \begin{pmatrix} \beta & \alpha \end{pmatrix}^{-1} & 0 \\ 0 & I \end{pmatrix} V_c^T$$

and hence it follows by condition (I) that

$$T^{-1} B_c = \begin{pmatrix} B_1 \\ B_2 \\ 0 \end{pmatrix}$$

for some B_1 and B_2. This concluses the proof. □

Remark 1. In words this means that there exists an optimal controller if and only if all unstable closed loop modes are in the controller (8) and such that the ones from A_o are unobservable in C_c and the ones from A_c are uncontrollable from B_c. This can also be formalized in the following way. Introduce the reduced order controller

$$\begin{aligned} q\xi &= J_{co}\xi + B_1 y \\ -u &= C_1 \xi + D_c y \end{aligned}$$

and the state transformation

$$z = \begin{pmatrix} V_{cs}^T & 0 \\ V_{os}^T & -V_{os}^T U_{cs} \beta \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix}$$

Then it holds that

$$qz = \begin{pmatrix} J_{cs} & * \\ 0 & J_{os} \end{pmatrix} z + \begin{pmatrix} * \\ * \end{pmatrix} w$$

Remark 2. The interpretation of the conditions given by Trentelman and Stoorvogel implies that not all uncontrollable and unobservable modes of the controller have to be removed as is the case in (9), i.e. the optimal controller can be implemented as

$$H_1 = \{J_{co}, B_1, C_1, D_c\}$$

Remark 3. It should be stressed that what is different in the approach in this paper as compared to the approach in Chen *et al.* (1993) is that the unstable modes of the closed loop system are not only moved but that they are actually removed from the closed loop system by implementing a reduced order controller.

4. Conclusions

In this paper the existence of H_2-controllers has been investigated. Special attention has been given to the case of "zeros on the unit circle". Intuition about cancellation in the controller from the polynomial SISO minimum variance case has been shown to carry over to the multivariable case.

For the case of uniqueness of the controller it has been shown that there exists an H_2-controller if and only if all unstable modes of the closed loop system, when applying the controller obtained by solving the Riccati equations, are modes also of the controller and such that they are unobservable or uncontrollable. This condition is a very intuitive interpretation of the conditions (C1)–(C4) in Trentelman and Stoorvogel (1993), and it is easy to check. Further it shows that the optimal controller given in Chen *et al.* (1993) is nonminimal. There unobservable or uncontrollable modes of the controller are not removed, just moved inside the unit circle.

When the controller is not unique, i.e. when assumptions (A3) or (A4) are not fulfilled, the approach taken in this paper has not yet been fruitful.

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6. *Appendix—Some Results on Riccati Equations*

Some results on solutions of Riccati equations are collected in this appendix. Consider the Riccati equation (6) which could be rewritten as

$$\begin{pmatrix} I & 0 \\ L & I \end{pmatrix}^T \begin{pmatrix} S & 0 \\ 0 & G \end{pmatrix} \begin{pmatrix} I & 0 \\ L & I \end{pmatrix} = \begin{pmatrix} A & B_u \\ C_z & D_{zu} \end{pmatrix}^T \begin{pmatrix} S & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B_u \\ C_z & D_{zu} \end{pmatrix} \tag{15}$$

With the notation $A_c = A - B_u L$ and $C_N = D_{zu} L - C_z$ the 1,1-block of (15) can be written as

$$S = A_c^T S A_c + C_N^T C_N$$

LEMMA 6
Assume that (A1) and (A3) hold. Then there always exists a solution (S, L) to (15) such that S is real, symmetric and positive semidefinite, and such that the eigenvalues of A_c are inside or on the unit circle. It also holds that $G > 0$. The eigenvalues of A_c are strictly inside the unit circle if and only if assumption (A5) holds.

Proof: See Hagander and Hansson (1994). □

For any real symmetric solution $S \geq 0$ to (15) and any corresponding A_c introduce the state transformation $T = \begin{pmatrix} T_- & T_0 & T_+ \end{pmatrix}$ such that

$$A_c T = T \operatorname{diag}(J_-, J_0, J_+)$$

where J_-, J_0, and J_+ are blocks with eigenvalues outside the unit circle, on the unit circle, and inside the unit circle, respectively.

LEMMA 7
For any real symmetric solution $S \geq 0$ to (15) it holds that

$$C_N \begin{pmatrix} T_- & T_0 \end{pmatrix} = 0; \quad S \begin{pmatrix} T_- & T_0 \end{pmatrix} = 0$$

Proof: Left out due to space limitations. □

LEMMA 8
Let (S, L) be a solution to (15) such that S is real, symmetric and positive semidefinite and such that the eigenvalues of A_c are inside or on the unit circle. Then S is unique. Under assumption (A3) the corresponding L is also unique.
Proof: Left out due to space limitations. □

7. *Addendum—Proofs*

In this section the proofs that were omitted will be given.

PROOF OF LEMMA 6
Define the following Kleinman-like recursion, e.g. Kucera (1991),

$$\begin{cases} S_i = (A - B_u L_i)^T S_i (A - B_u L_i) + (C_z - D_{zu} L_i)^T (C_z - D_{zu} L_i) \\ G_i = D_{zu}^T D_{zu} + B_u^T S_i B_u \\ G_i L_{i+1} = D_{zu}^T C_z + B_u^T S_i A \end{cases} \tag{16}$$

for $i = 0, 1, \ldots$ with initial value L_0 such that $A - B_u L_0$ is stable. It will first be shown that the sequence of L_i is well defined, and then the question about convergence will be investigated. Assume that $A - B_u L_i$ is stable. Then there exists a unique $S_i \geq 0$ that solves the first equation in (16), since it is a Lyapunov-equation, and there exists an L_{i+1} that solves the third equation in (16), since

$$\begin{pmatrix} A & B_u \\ C_z & D_{zu} \end{pmatrix}^T \begin{pmatrix} S_i & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B_u \\ C_z & D_{zu} \end{pmatrix} \geq 0$$

If it can be concluded that $A - B_u L_{i+1}$ is stable, it thus follows by induction that $A - B_u L_i$ is stable for all $i \geq 0$. Assume that $A - B_u L_{i+1}$ is not stable. Then there exist λ and x such that $|\lambda| \geq 1$ and

$$(A - B_u L_{i+1})x = x\lambda \tag{17}$$

Now rewrite (16) using $\Delta_i = (L_i - L_{i+1})^T G_i (L_i - L_{i+1})$ to obtain

$$S_i = (A - B_u L_{i+1})^T S_i (A - B_u L_{i+1}) + (C_z - D_{zu} L_{i+1})^T (C_z - D_{zu} L_{i+1}) + \Delta_i \tag{18}$$

Combining (17) and (18) gives

$$\begin{aligned}(1 - |\lambda|^2) x^* S_i x &= x^* (C_z - D_{zu} L_{i+1})^T (C_z - D_{zu} L_{i+1}) x \\ &+ x^* (L_i - L_{i+1})^T G_i (L_i - L_{i+1}) x\end{aligned}$$

Since $|\lambda| \geq 1$ and $S_i \geq 0$ it follows that $x^*(L_i - L_{i+1})^T G_i (L_i - L_{i+1})x = 0$. If it can be shown that $G_i > 0$ it follows that $L_i x = L_{i+1} x$, and hence that λ is also an eigenvalue of $A - B_u L_i$, which is a contradiction. That this actually holds will now be shown. Rewrite (16) and (18) as

$$\begin{aligned}&\begin{pmatrix} I & 0 \\ L_{i+1} & I \end{pmatrix}^T \begin{pmatrix} S_i - \Delta_i & 0 \\ 0 & G_i \end{pmatrix} \begin{pmatrix} I & 0 \\ L_{i+1} & I \end{pmatrix} \\ &= \begin{pmatrix} A & B_u \\ C_z & D_{zu} \end{pmatrix}^T \begin{pmatrix} S_i & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B_u \\ C_z & D_{zu} \end{pmatrix}\end{aligned} \tag{19}$$

Let $\Psi(z) = (zI - A)^{-1} B_u$, and let $H(z) = \begin{pmatrix} C_z & D_{zu} \end{pmatrix} \begin{pmatrix} \Psi(z) \\ I \end{pmatrix}$. Notice that $\begin{pmatrix} A & B_u \end{pmatrix} \begin{pmatrix} \Psi(z) \\ I \end{pmatrix} = -z\Psi(z)$. Thus by multiplying (19) by $\begin{pmatrix} \Psi(z) \\ I \end{pmatrix}$ from the right and its adjoint from the left the following equality is obtained

$$H^*(z) H(z) + \Psi^*(z) \Delta_i \Psi(z) = [I + L_{i+1} \Psi(z)]^* G_i [I + L_{i+1} \Psi(z)] \tag{20}$$

Now the rank condition (A3) implies that there exists z such that rank $H(z) = m$, which by (20) and $\Delta_i \geq 0$ implies that $G_i > 0$. Thus it is proven that the sequence of L_i is well defined and that $A - B_u L_i$ is stable for all $i \geq 0$.

It will now be shown that the sequence S_i converges to some limit S. Further manipulations show that the following Lyapunov-equation holds

$$S_i - S_{i+1} = (A - B_u L_{i+1})^T (S_i - S_{i+1})(A - B_u L_{i+1}) + \Delta_i \tag{21}$$

Since $A - B_u L_{i+1}$ is stable and since $\Delta_i \geq 0$ it follows that $S_i - S_{i+1} \geq 0$. Thus it holds that $0 \leq S_{i+1} \leq S_i$, which implies that the sequence of S_i converges to some limit $S \geq 0$ as i goes to infinity. The second equation in (16) implies that $G_i \to G = D_{zu}^T D_{zu} + B_u^T S B_u$. Since

$$\begin{pmatrix} A & B_u \\ C_z & D_{zu} \end{pmatrix}^T \begin{pmatrix} S & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B_u \\ C_z & D_{zu} \end{pmatrix} \geq 0$$

there exists L such that $GL = D_{zu}^T C_z + B_u^T S A$. From (21) it also follows that $(A - B_u L_{i+1})^T (S_i - S_{i+1})(A - B_u L_{i+1}) \to 0$ and $\Delta_i \to 0$, since both matrices are positive semidefinite. Thus S solves the algebraic Riccati equation (15), and similarly to (20) it holds that

$$H^*(z)H(z) = [I + L\Psi(z)]^* G [I + L\Psi(z)] \tag{22}$$

Now the rank condition (A3) implies that $G > 0$, and hence L is a unique solution. The sequence L_i therefore converges to L. Since the eigenvalues of $A - B_u L_i$ are inside the unit circle, it follows that in the limit the eigenvalues of $A - B_u L$ are inside or on the unit circle. Now these closed loop poles are the zeros of $I + L\Psi(z) = I + L(zI - A)^{-1} B_u$, since

$$\begin{pmatrix} zI - A & -B_u \\ L & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -L & I \end{pmatrix} = \begin{pmatrix} zI - A + B_u L & -B_u \\ 0 & I \end{pmatrix}$$

From (22) it now follows that any closed loop pole on the unit circle would also be a zero of $H(z)$. Conversely it also follows that any zero of $H(z)$ on the unit circle would show up as a closed loop pole (Here is discussed zeros before cancellations by possible poles. Notice that the poles on the unit circle are the same on both sides of the equality in the spectral factorization). Therefore L is stablizing if and only if $H(z)$ has no zeros on the unit circle, or equivalently that

$$\text{rank}_{|z|=1} P(z) = n + m$$

This concludes the proof of Lemma 6. □

PROOF OF LEMMA 7

For any real symmetric $S \geq 0$ solving (15) it holds by

$$S = (A_c^T)^k S A_c^k + \sum_{i=0}^{k-1} (A_c^T)^i C_N^T C_N A_c^i, \quad k \geq 1$$

that $C_N \begin{pmatrix} T_- & T_0 \end{pmatrix} = 0$ and $ST_- = 0$. To prove that $ST_0 = 0$ notice that the equation for L in (15) implies that $B_u^T SA_c = D_{zu}^T C_N$. Since $C_N T_0 = 0$, it follows that $B_u^T ST_0 = 0$. Further

$$T_0^T S = J_0^T T_0^T SA_c + T_0^T C_N^T C_N = J_0^T T_0^T SA_c$$

Now by the stabilizability of (A, B_u) there exist L_0 such that $A - B_u L_0$ is stable. Summing up gives

$$T_0^T S = J_0^T \left[T_0^T S(A - B_u L_0) + T_0^T SB_u(L_0 - L)\right] = J_0^T T_0^T S(A - B_u L_0)$$

which is equivalent to $X(A - B_u L_0) = JX$, where $X = T_0^T S$ and $J = J_0^{-1}$. Since the eigenvalues of J are on the unit circle, and since $A - B_u L_0$ is stable, it follows that $X = 0$. □

PROOF OF LEMMA 8

Assume the contrary, i.e. let S_1 and S_2 be two solutions of the Riccati equation. Then with $A_1 = A - B_u L_1$ and $A_2 = A - B_u L_2$ it holds that

$$A_2^T (S_1 - S_2) A_1 = S_1 - S_2$$

Let $T_1 = \begin{pmatrix} T_{1_0} & T_{1_+} \end{pmatrix}$ and $T_2 = \begin{pmatrix} T_{2_0} & T_{2_+} \end{pmatrix}$ be transformations such that

$$A_1 T_1 = T_1 \operatorname{diag}(J_{1_0}, J_{1_+})$$
$$A_2 T_2 = T_2 \operatorname{diag}(J_{2_0}, J_{2_+})$$

where J_{1_+}, and J_{2_+} are Jordan blocks with eigenvalues inside the unit circle. It then holds that

$$T_{2_+}^T (S_1 - S_2) T_{1_+} = (J_{2_+}^T)^k T_{2_+}^T (S_1 - S_2) T_{1_+} J_{1_+}^k \to 0, \quad k \to \infty$$

Now, since $ST_{1_0} = 0$ and $ST_{2_0} = 0$ by Lemma 7, and since T_1 and T_2 are invertible, it follows that S is unique. Further, by Lemma 6 and the definition of L it follows that L is unique under condition (A3). □

Remark. The solution S considered is actually the maximal solution.

Part IV

Existence of Minimum Upcrossing Controllers

An optimal stochastic control problem that minimizes the probability that a signal upcrosses a level is solved by rewriting it as a one-parametric optimization problem over a set of LQG control problem solutions. Finding the optimal controller can be interpreted as finding an optimal costing transfer function. The existence of the optimal controller is here investigated in a constructive way, and it is shown that it is equivalent to the existence of a minimum variance controller with sufficiently small closed loop variance.

Coauthored with Per Hagander

1. *Introduction*

In many control problems the primary goal is to keep the controlled signal near a certain reference value. Sometimes it is also of interest to consider a secondary goal of preventing the controlled signal from upcrossing a level, where the upcrossing would cause some undesirable event such as e.g. emergency shutdown or instability. The distance between the level and the reference value is normally not small, since otherwise the upcrossing intensity will be intolerably high. However, there may be other control-objectives that make it undesirable or impossible to choose the distance large. An example of problems of this kind can be found in bor+syd76, where the power of an ore crusher should be kept as high as possible but not exceed a certain level, in order that the overload protection does not cause shutdown. Another example is moisture control of a paper machine, where it is desired to keep the moisture content as high as possible without causing wet streaks, ast70 pp. 188–209. Yet another example is power control of wind power plants, where the supervisory system initiates emergency shutdown, if the generated power exceeds 140% of rated power, mat84. Other examples can be found in sensor-based robotics and force control, han+nie91, and control of non-linear plants, where the stability may be state dependent, shi67.

The proposed controller—the minimum upcrossing (MU) controller—is obtained by minimizing the mean number of upcrossings of the critical level during a sample interval.

In han93a the problem was solved in the continuous time case in terms of necessary conditions; here the discrete time case is treated in terms of both sufficient and necessary conditions. Necessary conditions have previously been described in han91c, han92, han93b, han94. In han91c, han94 the MU controller is used to approximate the so called risk criterion and the mean time between failures criterion. In han93b sufficient and necessary conditions for the MU controller is given, when the controlled process is a scalar *ARMAX*-process. It is also shown that the optimal controller in this case can be found by solving a set of minimum variance control problems parameterized by a scalar. Further the optimal controller can be interpreted as finding an optimal costing transfer function for the system output. For the more general multivariable process models treated in han94 the solution can be found by solving a set of LQG-problems parameterized by a scalar, and the solution can be interpreted as finding optimal weightings in an LQG-problem. The existence of the optimal controller is more difficult to investigate for the general case. The aim of this paper is to extend the existence results from the scalar *ARMAX*-case to the case with several measurement signals. This paper is a revised version of han+hag94. It also covers the case with a direct term from control signal to controlled signal. This question has not previously been addressed.

Only the case of a linear process controlled with a linear controller will be treated, since then, if the disturbances acting on the system are Gaussian, the closed loop process will also be Gaussian. It is very likely that a nonlinear controller will do better. However, then the non-Gaussianity of the closed loop process makes the analysis much harder. Preliminary results are given in and+han94

The paper is organized as follows. In Section 2 the control problem is formulated. It is an optimal stochastic control problem. In Section 3 the problem presented in Section 2 is solved, i.e. a solution procedure together with necessary and sufficient conditions for existence of a solution are given. It will be seen that finding the optimal controller can be interpreted as finding an optimal costing transfer function for an LQG-problem, and that the existence of the optimal controller is equivalent to the existence of a minimum variance controller with sufficiently small closed loop variance. In Section 4 an example is investigated. Finally, in Section 5 the results are summarized.

2. *Control Problem*

Let the stationary Gaussian sequence z be defined by

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) + Ee(k) \\ z(k) = C_1x(k) + Du(k) + Ge(k) \\ y(k) = C_2x(k) + Fe(k) \end{cases} \tag{1}$$

where $x(k) \in R^n$ is the state, $u(k) \in R$ is the scalar control signal, $z(k) \in R$ is the scalar signal to be controlled, $y(k) \in R^p$ is the measurement signal, $e(k) \in R^l$ is a sequence of independent zero mean Gaussian random variables with covariance I, and where A, B, C_1, C_2, D, E, F, and G are matrices of consistent dimensions. It will be assumed that (A, B) is stabilizable, that (C_2, A) is detectable, that

$$\operatorname{rank}_{|z|=1} \begin{pmatrix} zI - A & -B \\ C_1 & D \end{pmatrix} = n + 1 \tag{2}$$

and that

$$\operatorname{rank}_{|z|=1} \begin{pmatrix} zI - A & -E \\ C_2 & F \end{pmatrix} = n + p \tag{3}$$

Let the control signal be given by

$$\begin{aligned} \xi(k+1) &= A_H \xi(k) + B_H y(k) \\ u(k) &= -C_H \xi(k) - D_H y(k) + D_r r(k) \end{aligned} \tag{4}$$

where $r(k) \in R$ is the reference value, where ξ is the finite dimensional state of the controller, and where A_H, B_H, C_H, D_H, and D_r are matrices of consistent dimensions. Since only constant reference values will be considered, it is no loss in generality to assume that $r(k) = 0$ by a change of coordinates. This implies that the mean of z in stationarity is zero if a stabilizing controller is used. Denote by σ_z^2 the variance of $z(k)$ in stationarity. Introduce the notation $H = \{A_H, B_H, C_H, D_H\}$ for the controller defined by (4). Further let

$$\begin{aligned} \mathcal{D} &= \{H \text{ as defined in } (4)\} \\ \mathcal{D}_s &= \{H \in \mathcal{D} \mid H \text{stabilizes } (1)\} \\ \mathcal{D}_z &= \{H \in \mathcal{D}_s \mid \sigma_z \leq z_0\} \end{aligned}$$

where $z_0 \in R^+$ is the critical level that should not be upcrossed. Introduce the following performance index evaluated in stationarity:

$$\mu(H; z_0) = \mathrm{P}\{z(k) \leq z_0 \cap z(k+1) > z_0\}, \quad H \in \mathcal{D}_s \tag{5}$$

where P denotes the probability measure induced by the random process e. The quantity μ will in the sequel be called the upcrossing probability, and it is equal to the mean number of upcrossings during a sample interval, see e.g. cra+lea67 p. 281. The solution to

$$\min_{H \in \mathcal{D}_z} \mu(H) \tag{6}$$

will in the sequel be called the MU controller. It should be stressed that μ is to be evaluated in stationarity. Since the controllers in $\mathcal{D}_z$ are stabilizing, there is hence no loss in generality to assume that $x(0) = 0$. The restriction on σ_z will exclude the degenerated solution $\sigma_z = \infty$ for minimizing μ. Also assume that there exist no controller in $\mathcal{D}_z$ such that $\sigma_z = 0$. Had this been the case, then this controller would trivially minimize μ, and the minimal value would be zero.

The following lemma gives an expression for the upcrossing probability μ in (5) in terms of a double integral.

LEMMA 1
It holds that

$$\mu = \mathrm{P}\{z(0) \le z_0 \cap z(1) > z_0\} = \int_0^\infty \phi(y) \int_{x_l(y)}^{x_u(y)} \phi(x)dxdy$$

where $\phi(x) = (2\pi)^{-1/2}\exp(-x^2/2)$, $x_l(y) = (2z_0 - \sigma_\beta y)/\sigma_\alpha$, and $x_u(y) = (2z_0+\sigma_\beta y)/\sigma_\alpha$, and where σ_α^2 and σ_β^2 are the variances of the independent variables

$$\begin{cases} \alpha(k) = z(k) + z(k-1) \\ \beta(k) = z(k) - z(k-1) \end{cases} \tag{7}$$

Proof: First notice that $\sigma_z > 0$ implies that $\sigma_\alpha^2 > 0$ and $\sigma_\beta^2 > 0$. Assume the contrary. Start with $\sigma_\alpha^2 = 0$. Then

$$0 = \sigma_\alpha^2 = \mathrm{E}\left\{(z(k) + z(k-1))^2\right\} = 2\sigma_z^2 + 2\mathrm{E}\{z(k)z(k-1)\}$$

where E denotes expectation with respect to the probability measure induced by the random process e. This implies that the correlation coefficient between $z(k)$ and $z(k-1)$ is equal to -1. This contradicts that z is a stable process. The proof goes along the lines of han94 Proof of Theorem 1 and footnote on page 1491. If $\sigma_\beta^2 = 0$, then the correlation coefficient between $z(k)$ and $z(k-1)$ is equal to 1, which also contradicts that z is a stable process. Hence the expressions in the lemma involving the inverses of σ_α and σ_β are well defined. Furthermore, since α and β are independent it holds that

$$\mu = \mathrm{P}\{|\alpha - 2z_0| < \beta\} = \int\int_{|x-2z_0|<y} \frac{1}{\sigma_\alpha}\phi\left(\frac{x}{\sigma_\alpha}\right)\frac{1}{\sigma_\beta}\phi\left(\frac{y}{\sigma_\beta}\right) dxdy$$

from which the result follows by a change of variables. □

Thus μ is easily calculated with some numerical routine. Further μ only depends on the variances of α and β and the critical level z_0. This dependence will be further investigated in the following lemma.

LEMMA 2
Let

$$\mathcal{V}(r) = \left\{(\sigma_\alpha^2, \sigma_\beta^2) \in R^2 \middle| \sigma_z \le r,\ \sigma_\alpha > 0,\ \sigma_\beta > 0\right\}$$

where $r > 0$. Then the upcrossing probability μ in (5) has strictly positive partial derivatives with respect to both σ_α and σ_β on $\mathcal{V}(r)$, if and only if $r \le z_0$.

Proof: It holds that

$$\frac{\partial \mu}{\partial \sigma_\beta} = \int_0^\infty \phi(y) \left(\frac{y}{\sigma_\alpha}\phi(x_u) + \frac{y}{\sigma_\alpha}\phi(x_l) \right) dy > 0$$

Further let $x_l = (2z_0 - \sigma_\beta y)/\sigma_\alpha$, and $x_u = (2z_0 + \sigma_\beta y)/\sigma_\alpha$. Using Lemma 1 gives

$$\frac{\partial \mu}{\partial \sigma_\alpha} = \int_0^\infty \phi(y) \left(\frac{x_l}{\sigma_\alpha}\phi(x_l) - \frac{x_u}{\sigma_\alpha}\phi(x_u) \right) dy$$

By completing the squares in the exponents and by a change of coordinates it is possible to express the integral in terms of $\Phi(x) = \int_{-\infty}^{x} \phi(t)dt$, and $\sigma_z^2 = (\sigma_\alpha^2 + \sigma_\beta^2)/4$:

$$\frac{\partial \mu}{\partial \sigma_\alpha} = \frac{\sigma_\alpha}{2\sigma_z^2}\gamma\phi(\gamma) \left[\Phi(\eta) - \frac{1}{2} - \frac{\eta}{\gamma^2}\phi(\eta) \right]$$

where $\gamma = z_0/\sigma_z$ and $\eta = \gamma\sigma_\beta/\sigma_\alpha$. It is seen that $\partial\mu/\partial\sigma_\alpha > 0$ if and only if

$$F(\eta) = \Phi(\eta) - \frac{1}{2} - \frac{\eta}{\gamma^2}\phi(\eta) > 0, \quad \text{for } \eta > 0.$$

Since $F(0) = 0$ a necessary condition is that

$$F'(0) = \phi(0)(1 - 1/\gamma^2) \geq 0, \qquad \text{i.e. } \gamma \geq 1.$$

On the other hand if $\gamma \geq 1$ then

$$F(\eta) = \int_0^\eta \phi(t)\, dt - \frac{\eta}{\gamma^2}\phi(\eta) > \eta\phi(\eta) - \eta\phi(\eta)/\gamma^2 \geq 0$$

follows directly, since $\phi(t)$ is decreasing for $t \geq 0$. □

This lemma indicates that the optimal controller can be found among all the controllers which make the variances of α and β jointly minimal. This will be further elaborated in the following section.

3. *Regulator Design*

In this section the problem of minimizing the upcrossing probability is rephrased to a minimization over a set parameterized by a scalar. It is also shown how this set can be obtained by solving LQG control problems. Furthermore the existence of the optimal controller is investigated. It is shown that the existence is equivalent to the existence of a minimum variance controller with sufficiently small closed loop variance.

Introduce

$$J(H,\rho) = \mathrm{E}\{(1-\rho)\alpha^2(k) + \rho\beta^2(k)\}, \quad H \in \mathcal{D}_s, \quad \rho \in [0,1] \tag{8}$$

and consider the optimization problem

$$\min_{H \in \mathcal{D}_s} J(H,\rho), \quad \rho \in [0,1] \tag{9}$$

Notice that J is to be evaluated in stationarity. It will be seen that the minimization of μ in (5) over $\mathcal{D}_z$ can be done by first solving the optimization problem (9) for all $\rho \in [0,1]$ and then minimizing μ over the solutions obtained in the first minimization, i.e. over $\mathcal{V}_J \cap \mathcal{V}_z$, where

$$\begin{aligned}
\mathcal{D}_J &= \left\{ H \in \mathcal{D}_s \middle| H = \operatorname{argmin} J(H,\rho),\ \rho \in [0,1] \right\} \\
\mathcal{V}_J &= \left\{ \left(\sigma_\alpha^2(H), \sigma_\beta^2(H)\right) \in R^2 \middle| H \in \mathcal{D}_J \right\} \\
\mathcal{V}_z &= \left\{ \left(\sigma_\alpha^2, \sigma_\beta^2\right) \in R^2 \middle| \sigma_z \le z_0,\ \sigma_\alpha > 0,\ \sigma_\beta > 0 \right\}
\end{aligned}$$

and where σ_α^2 and σ_β^2 are the variances of α and β. More precisely the following theorem holds:

THEOREM 1

Let

$$\begin{aligned}
\mathcal{D}_\mu &= \left\{ H \in \mathcal{D}_z \middle| H = \operatorname{argmin} \mu\left(\sigma_\alpha(H), \sigma_\beta(H)\right) \right\} \\
\mathcal{V}_\mu &= \left\{ (\sigma_\alpha(H), \sigma_\beta(H)) \in \mathcal{V}_z \middle| H \in \mathcal{D}_\mu \right\}
\end{aligned}$$

Then it holds that

$$\mathcal{D}_\mu \subseteq \mathcal{D}_J \cap \mathcal{D}_z \text{ and } \mathcal{V}_\mu \subseteq \mathcal{V}_J \cap \mathcal{V}_z$$

where the left hand sides are non-empty if and only if $\mathcal{D}_z$ is non-empty.

Remark 1. Note that the minimization of μ can be done over $\mathcal{V}_J \cap \mathcal{V}_z$. This will be shown to be a one-parametric optimization problem over $\rho \in [0,1]$.

Remark 2. It will be seen that the elements of $\mathcal{V}_J$ can be obtained by solving LQG-problems.

Remark 3. Notice that an alternative formulation of the existence result is that there exist an MU-controller if and only if there exist a minimum variance controller, obtained for $\rho = 0.5$, with closed loop standard deviation satisfying $\sigma_z \le z_0$.

The remaining part of the paper is concerned with proving the above theorem and with showing how to compute the controllers that minimize J. The difficult part of the proof, lemmas 5–7, is to establish the continuity of $\sigma_\alpha^2(H_\rho)$ and $\sigma_\beta^2(H_\rho)$ for $\rho = 0, 1$ when these variances are bounded. This is then used to show that the set $\mathcal{V}_J \cap \mathcal{V}_z$ is connected, closed, and bounded, from which the proof can be derived fairly easy. This way of proof gives as a byproduct some useful properties of $\mathcal{V}_J$.

First, however, the minimization of J will be shown to be equivalent to solving LQG problems. Sufficient and necessary conditions for when these LQG problems have solutions will be given. It will also be seen that the solutions, whenever they exist, are unique with respect to the transfer function relating the measurement signal to the control signal. This will be done in Lemma 3 and Theorem 2.

LEMMA 3
Let $a(\rho) = \sqrt{1-\rho} + \sqrt{\rho}$ and $b(\rho) = \sqrt{1-\rho} - \sqrt{\rho}$, and introduce the filtered output

$$\bar{z}(k) = \frac{a(\rho)q + b(\rho)}{q} z(k) = \frac{X_\rho(q)}{q} z(k)$$

Then it holds that the loss function J in (8) can be written

$$J = \mathrm{E}\left\{\bar{z}^2(k)\right\}$$

Proof: It holds that

$$\begin{aligned} J &= \mathrm{E}\left\{\left(\sqrt{1-\rho}\,\alpha(k)\right)^2 + \left(\sqrt{\rho}\,\beta(k)\right)^2\right\} \\ &= \mathrm{E}\left\{\left(\sqrt{1-\rho}\,\alpha(k) + \sqrt{\rho}\,\beta(k)\right)^2\right\} = \mathrm{E}\left\{\bar{z}^2(k)\right\} \end{aligned}$$

where the second equality follows from noting that $\mathrm{E}\{\alpha\beta\} = 0$ in stationarity. □

Remark 1. The case $\rho = 0.5$ with $J = \mathrm{E}\{2z^2(k)\}$ corresponds to minimum variance control.

Remark 2. Notice that the problem of minimizing the upcrossing probability can be interpreted as finding an optimal costing transfer function for the signal $z(k)$ to be controlled. Costing transfer functions were proposed e.g. in cla+gaw79 to generalize minimum variance control. Typical

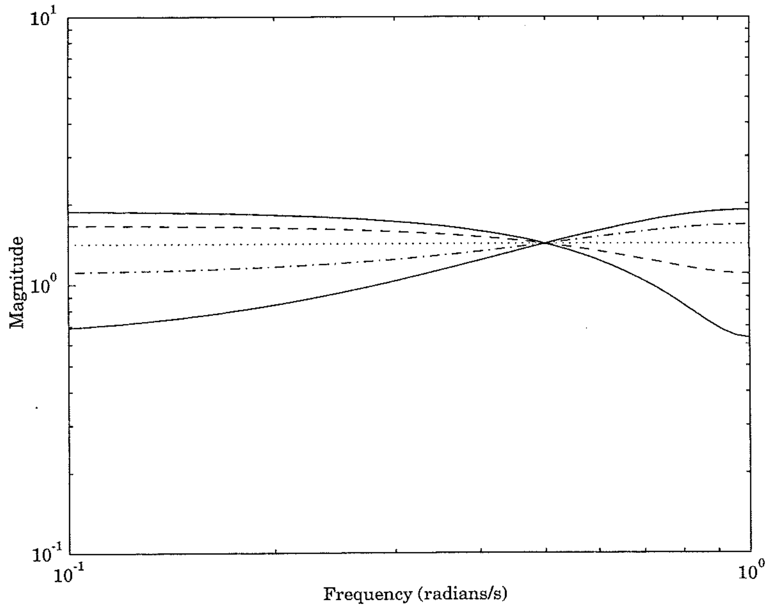

Figure 1. Amplitude diagram for the costing transfer function $X_\rho(q)/q$ for $\rho = 0.1, 0.3, 0.5, 0.7$, and 0.9. The transfer function is low-pass for $\rho \leq 0.5$ and it is high-pass for $\rho \geq 0.5$.

$X_\rho(q)/q$ are seen in Figure 1. Notice that for $\rho \geq 0.5$ the amplification is higher the higher the frequency is.

Remark 3. A state space realization for $\bar{z}$ is e.g. given by the augmented system

$$\begin{cases} \bar{x}(k+1) = \bar{A}\bar{x}(k) + \bar{B}u(k) + \bar{E}e(k) \\ \bar{z}(k) = \bar{C}_1\bar{x}(k) + \bar{D}u(k) + \bar{G}e(k) \\ y(k) = \bar{C}_2\bar{x}(k) + \bar{F}e(k) \end{cases} \tag{10}$$

where $\bar{x}^T(k) = \begin{pmatrix} x^T(k) & \xi^T(k) \end{pmatrix}$ and where

$$\begin{aligned} \bar{A} &= \begin{pmatrix} A & 0 \\ bC_1 & 0 \end{pmatrix}; & \bar{B} &= \begin{pmatrix} B \\ bD \end{pmatrix} \\ \bar{C}_1 &= \begin{pmatrix} aC_1 & 1 \end{pmatrix}; & \bar{C}_2 &= \begin{pmatrix} C_2 & 0 \end{pmatrix} \\ \bar{D} &= aD; & \bar{E} &= \begin{pmatrix} E \\ bG \end{pmatrix} \\ \bar{F} &= F; & \bar{G} &= aG \end{aligned}$$

The equations for solving LQG problems are summarized below. Let S, L, L_v, and L_w be solutions of

$$\begin{aligned} S &= (\bar{A}-\bar{B}L)^T S(\bar{A}-\bar{B}L) + (\bar{C}_1-\bar{D}L)^T(\bar{C}_1-\bar{D}L) \\ G_S &= \bar{B}^T S\bar{B} + \bar{D}^T\bar{D} \\ G_S \begin{pmatrix} L & L_v & L_w \end{pmatrix} &= \begin{pmatrix} \bar{B}^T S\bar{A} + \bar{D}^T\bar{C}_1 & \bar{B}^T S & \bar{D}^T \end{pmatrix} \end{aligned} \tag{11}$$

and let P, K, K_x, K_v, and K_w be solutions of

$$\begin{aligned} P &= (\bar{A}-K\bar{C}_2)P(\bar{A}-K\bar{C}_2)^T + (\bar{E}-K\bar{F})(\bar{E}-K\bar{F})^T \\ H_P &= \bar{F}\bar{F}^T + \bar{C}_2 P\bar{C}_2^T \\ H_P \begin{pmatrix} K \\ K_x \\ K_v \\ K_w \end{pmatrix}^T &= \begin{pmatrix} \bar{E}\bar{F}^T + \bar{A}P\bar{C}_2^T \\ P\bar{C}_2^T \\ \bar{E}\bar{F}^T \\ \bar{G}\bar{F}^T \end{pmatrix}^T \end{aligned} \tag{12}$$

where S and P are the maximal real symmetric positive semidefinite solutions of the Riccati equations. Introduce the control signal

$$u(k) = -L\hat{x}(k) - D_c\tilde{y}(k) \tag{13}$$

with $\hat{x}(k)$ defined by

$$\hat{x}(k+1) = (\bar{A}-K\bar{C}_2)\hat{x}(k) + \bar{B}u(k) + Ky(k)$$

and where $\tilde{y}(k) = y(k) - \bar{C}_2\hat{x}(k)$. Equivalently it can be expressed as

$$u(k) = -\left[C_c(qI - A_{co})^{-1}B_c + D_c\right] y(k) \tag{14}$$

where

$$\begin{aligned} A_{co} &= \bar{A} - \bar{B}L - K\bar{C}_2 + \bar{B}D_c\bar{C}_2; \qquad & B_c &= K - \bar{B}D_c \\ C_c &= L - D_c\bar{C}_2; & D_c &= LK_x + L_vK_v + L_wK_w \end{aligned}$$

THEOREM 2

If $\rho \in (0,1)$ then the controller defined by (14) is a solution to (9). If $\rho = 0, 1$, then there exists a solution to (9) if and only if rank $\begin{pmatrix} zI - A_{co} & B_c \end{pmatrix} < n+1$ for $z = -1, 1$, respectively. In case of existence the optimal controller is given by e.g. a minimal realization of $\{A_{co}, B_c, C_c, D_c\}$. Furthermore the transfer-function of the optimal controller is unique for all $\rho \in [0,1]$.

Proof: By Lemma 3 the problem in (9) is an LQG problem. This will have a unique solution in terms of the transfer function from y to u if $(\bar{A}, \bar{B})$ is stabilizable, $(\bar{C}_2, \bar{A})$ is detectable,

$$\operatorname{rank}_{|z|=1} \begin{pmatrix} zI - \bar{A} & -\bar{B} \\ \bar{C}_1 & \bar{D} \end{pmatrix} = n + 2 \tag{15}$$

and

$$\operatorname{rank}_{|z|=1} \begin{pmatrix} zI - \bar{A} & -\bar{E} \\ \bar{C}_2 & \bar{F} \end{pmatrix} = n + 1 + p$$

see e.g. hag+han95. The matrices for the PBH rank test for stabilizability and detectability are given by

$$\begin{pmatrix} zI - \bar{A} & \bar{B} \end{pmatrix} = \begin{pmatrix} zI - A & 0 & B \\ -bC_1 & z & bD \end{pmatrix}$$

$$\begin{pmatrix} zI - \bar{A} \\ \bar{C}_2 \end{pmatrix} = \begin{pmatrix} zI - A & 0 \\ -bC_1 & z \\ C_2 & 0 \end{pmatrix}$$

which have rank $n + 1$ for $|z| \geq 1$, since

$$\begin{pmatrix} zI - A & B \end{pmatrix} \text{ and } \begin{pmatrix} zI - A \\ C_2 \end{pmatrix}$$

have rank n for $|z| \geq 1$ by the stabilizability and detectability of (A, B) and (C_2, A), respectively. This proves that $(\bar{A}, \bar{B})$ is stabilizable and that $(\bar{C}_2, \bar{A})$ is detectable. Furthermore it holds that

$$\begin{pmatrix} zI - \bar{A} & -\bar{B} \\ \bar{C}_1 & \bar{D} \end{pmatrix} \sim \begin{pmatrix} zI - A & -B & 0 \\ C_1 & D & 1/a \\ 0 & 0 & X_\rho(z) \end{pmatrix}$$

and that

$$\begin{pmatrix} zI - \bar{A} & -\bar{E} \\ \bar{C}_2 & \bar{F} \end{pmatrix} \sim \begin{pmatrix} zI - A & -E & 0 \\ C_2 & F & 0 \\ -bC_1 & -bG & z \end{pmatrix}$$

where $\sim$ denotes equivalence with respect to elementary row and column operations. Thus, if $\rho \in (0, 1)$, the rank conditions are fulfilled by assumptions (2) and (3). This implies that the conditions for existence of a unique solution are fulfilled. When $\rho = 0, 1$, $X_\rho(z)$ has a zero at $z = -1, 1$. Then there exist by e.g. hag+han95 an optimal controller if and only if rank $\begin{pmatrix} zI - A_{co} & B_c \end{pmatrix} < n+1$ for $z = -1, 1$, respectively. In case of existence, the optimal controller is given by e.g. a minimal realization of $\{A_{co}, B_c, C_c, D_c\}$. The uniqueness of the transfer function from measurement signal to control signal also follows from the same reference. □

Remark 1. Notice that there always exist maximal solutions to the Riccati equations, see e.g. hag+han95. Further it holds that the controller defined above always makes the performance index attain its infimal value. The only problem is that it may not be stabilizing for $\rho = 0, 1$. This is due to the fact that $-1, 1$ will be an eigenvalue of A_c.

Remark 2. The condition for existence of a solution for $\rho = 0, 1$ is equivalent to that the unstable eigenvalue of A_c at $-1, 1$ is uncontrollable from B_c, since the same eigenvalue is uncontrollable in the controller by $A_{co} = A_c - B_c\bar{C}_2$.

Remark 3. Notice that the LQG controller for $\rho = 0, 1$ is a minimal realization of $\{A_{co}, B_c, C_c, D_c\}$. Hence the uncontrollable modes do not have to be removed when computing variances as long as $x(0) = \hat{x}(0) = \tilde{x}(0) = 0$. Remember that $x(0) = 0$ was assumed in Section 2. Further $\hat{x}(0)$ is the initial value of the controller which should be chosen to be zero. This then implies that $\tilde{x}(0) = 0$.

Remark 4. The closed loop is governed by

$$\begin{pmatrix} \hat{x}(k+1) \\ \tilde{x}(k+1) \end{pmatrix} = A_t \begin{pmatrix} \hat{x}(k) \\ \tilde{x}(k) \end{pmatrix} + B_t e(k) \tag{16}$$

where $\tilde{x}(k) = \bar{x}(k) - \hat{x}(k)$,

$$A_t = \begin{pmatrix} A_c & B_c\bar{C}_2 \\ 0 & A_o \end{pmatrix}; \qquad B_t = \begin{pmatrix} B_c\bar{F} \\ \bar{E} - K\bar{F} \end{pmatrix}$$
$$A_c = \bar{A} - \bar{B}L; \qquad A_o = \bar{A} - K\bar{C}_2$$

The closed loop behavior of α and β is governed by

$$\begin{aligned} \alpha(k+1) &= C_t(A_t + I)\begin{pmatrix} \hat{x}(k) \\ \tilde{x}(k) \end{pmatrix} + (C_tB_t + G_t)e(k) + G_te(k+1) \\ \beta(k+1) &= C_t(A_t - I)\begin{pmatrix} \hat{x}(k) \\ \tilde{x}(k) \end{pmatrix} + (C_tB_t - G_t)e(k) + G_te(k+1) \end{aligned} \tag{17}$$

where

$$C_t = \begin{pmatrix} \left\{ \begin{pmatrix} C_1 & 0 \end{pmatrix} - DL \right\} & \left\{ \begin{pmatrix} C_1 & 0 \end{pmatrix} - DD_c\bar{C}_2 \right\} \end{pmatrix}$$
$$G_t = G - DD_c\bar{F}$$

Now, the observability in α and β of the unstable mode of A_c for $\rho = 0, 1$ will be investigated. To this end the equations given above are not suitable. Instead the following equations for α and β will be used:

$$\begin{aligned} \alpha(k) &= (C_\alpha - DL)\hat{x}(k) + (C_\alpha - DD_c\bar{C}_2)\tilde{x}(k) + (G - DD_c\bar{F})e(k) \\ \beta(k) &= (C_\beta - DL)\hat{x}(k) + (C_\beta - DD_c\bar{C}_2)\tilde{x}(k) + (G - DD_c\bar{F})e(k) \end{aligned} \tag{18}$$

where $C_\alpha = \begin{pmatrix} C_1 & 1/b \end{pmatrix}$ and $C_\beta = \begin{pmatrix} C_1 & -1/b \end{pmatrix}$, $\rho \neq 0.5$.

LEMMA 4

It holds that the unstable eigenvalue of A_c at -1 for $\rho = 0$ is observable in β but not in α. Further it holds that the unstable eigenvalue of A_c at 1 for $\rho = 1$ is observable in α but not in β.

Proof: Only the observability in α will be investigated, since the observability in β can be investigated analogously. From (16) and (18) the matrix for the PBH-rank test for observability in α is given by

$$\begin{pmatrix} zI - A_c & B_c\bar{C}_2 \\ 0 & zI - A_o \\ C_\alpha - DL & C_\alpha - DD_c\bar{C}_2 \end{pmatrix} \sim \begin{pmatrix} zI - A_c & B_c\bar{C}_2 \\ C_\alpha - DL & C_\alpha - DD_c\bar{C}_2 \\ 0 & zI - A_o \end{pmatrix}$$

Due to the fact that the unstable eigenvalue is in A_c and never in A_o it is sufficient to consider the rank of of

$$\begin{pmatrix} zI - A_c \\ C_\alpha - DL \end{pmatrix} \tag{19}$$

i.e. the observability in $C_\alpha - DL$ of the unstable eigenvalue in A_c. Further it holds that

$$\begin{pmatrix} zI - A_c & -\bar{B} \\ C_\alpha - DL & D \end{pmatrix} = \begin{pmatrix} zI - \bar{A} & -\bar{B} \\ C_\alpha & D \end{pmatrix} \begin{pmatrix} I & 0 \\ -L & I \end{pmatrix}$$

and that

$$\begin{pmatrix} zI - \bar{A} & -\bar{B} \\ C_\alpha & D \end{pmatrix} = \begin{pmatrix} zI - A & 0 & -B \\ -bC_1 & z & -bD \\ C_1 & 1/b & D \end{pmatrix} \sim \begin{pmatrix} zI - A & -B & 0 \\ C_1 & D & 1/b \\ 0 & 0 & z+1 \end{pmatrix}$$

By (2) it now follows that the matrix in (19) has full rank for $z = 1$, i.e. the unstable eigenvalue of A_c at 1 for $\rho = 1$ is observable in $C_\alpha - DL$.

Further it holds that

$$\begin{pmatrix} zI - A_c \\ C_\alpha - DL \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} zI - A & 0 \end{pmatrix} + BL \\ -b\left\{\begin{pmatrix} C_1 & -z/b \end{pmatrix} - DL\right\} \\ \begin{pmatrix} C_1 & 1/b \end{pmatrix} - DL \end{pmatrix}$$

For $\rho = 0$ it holds that $b = 1$ and that -1 is an eigenvalue of A_c, and hence the matrix above loses rank for $z = -1$, which implies that there is unobservability in $C_\alpha - DL$. □

Lemma 5

Let H_1 and H_2 be two controllers that solve (9) for ρ_1 and ρ_2, respectively, where $0 \le \rho_1 < \rho_2 \le 1$. Let the corresponding variances of α and β be $\sigma_\alpha^2(H_1)$, $\sigma_\alpha^2(H_2)$, $\sigma_\beta^2(H_1)$, and $\sigma_\beta^2(H_2)$. It then holds that

$$\sigma_\alpha^2(H_2) \ge \sigma_\alpha^2(H_1) \text{ and } \sigma_\beta^2(H_1) \le \sigma_\beta^2(H_2)$$

It also follows that

$$\begin{aligned} \sigma_\alpha^2(H_2) - \sigma_\beta^2(H_2) &\ge \sigma_\alpha^2(H_1) - \sigma_\beta^2(H_1) \\ J(H_1,\rho_1) &\le J(H_2,\rho_2) + (\rho_2 - \rho_1)\left[\sigma_\alpha^2(H_2) - \sigma_\beta^2(H_2)\right] \\ J(H_2,\rho_2) &\le J(H_1,\rho_1) + (\rho_1 - \rho_2)\left[\sigma_\alpha^2(H_1) - \sigma_\beta^2(H_1)\right] \end{aligned}$$

Proof: Introduce $a_i = \sigma_\alpha^2(H_i)$, and $b_i = \sigma_\beta^2(H_i)$, $i = 1,2$. Then it holds that

$$\begin{aligned} J(H_1,\rho_1) &= (1-\rho_1)a_1 + \rho_1 b_1 \le (1-\rho_1)a_2 + \rho_1 b_2 = J(H_2,\rho_1) \\ J(H_2,\rho_2) &= (1-\rho_2)a_2 + \rho_2 b_2 \le (1-\rho_2)a_1 + \rho_2 b_1 = J(H_1,\rho_2) \end{aligned}$$

i.e.

$$\begin{aligned} (1-\rho_1)(a_2 - a_1) + \rho_1(b_2 - b_1) &\ge 0 \\ (1-\rho_2)(a_1 - a_2) + \rho_2(b_1 - b_2) &\ge 0 \end{aligned}$$

so that $(\rho_2 - \rho_1)[(a_2 - a_1) + (b_1 - b_2)] \ge 0$. Since $\rho_2 - \rho_1 > 0$ it follows that

$$\begin{aligned} a_2 - a_1 &\ge \rho_1[(a_2 - a_1) + (b_1 - b_2)] \ge 0 \\ b_1 - b_2 &\ge (1-\rho_1)[(a_2 - a_1) + (b_1 - b_2)] \ge 0 \end{aligned}$$

which proves the first two inequalities. The remaining ones are immediate. □

LEMMA 6
It holds that the solutions S of (11) and P of (12) are continuous functions of ρ on $[0,1]$.

Proof: Application of Lemma 8 in the appendix to the Riccati equations in (11) and (12) implies that P is a continuous function of ρ on the closed interval $[0,1]$, but since A_c has an eigenvalue on the unit circle for $\rho = 0,1$ this lemma only implies that S is a continuous function of ρ on the open interval $(0,1)$. In the remaining part of the proof the continuity of S will be shown to hold also at $\rho = 0,1$. To this end introduce the following auxiliary LQ-problem, which has the same Riccati equation for S associated with it as the original LQG problem:

$$\begin{aligned} \bar{x}(k+1) &= \bar{A}\bar{x}(k) + \bar{B}u(k) + \bar{B}_v v(k) \\ z(k) &= C_1 x(k) + Du(k) + \frac{1}{b}B_2 v(k) \\ z_\rho(k) &= \bar{C}_1\bar{x}(k) + \bar{D}u(k) + \frac{a}{b}B_2 v(k) \end{aligned}$$

for $\rho \neq 0.5$ with performance index $V_\rho(u) = \mathrm{E}z_\rho^2$, where v is a sequence of scalar zero mean Gaussian random variables with covariance one, and where

$$\bar{B}_v = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$$

can be chosen freely. It also holds with new $\alpha = (1+q^{-1})z$ and $\beta = (1-q^{-1})z$ that

$$V_\rho(u) = (1-\rho)\mathrm{E}\alpha^2 + \rho\mathrm{E}\beta^2$$

The performance index is to be minimized over the set of all linear time-invariant causal stabilizing feedbacks of the system. For any u that stabilizes the closed loop system it holds that

$$V_\rho(u) = \mathrm{E}\left\{(u(k) + L_\rho\bar{x}(k))^T G_S (u(k) + L_\rho\bar{x}(k))\right\} + \bar{B}_v^T S_\rho \bar{B}_v$$

Hence the infimal value is $V_\rho^* = \bar{B}_v^T S_\rho \bar{B}_v$.

Consider the continuity for $\rho = 0$. By applying Lemma 5 to the auxiliary LQ-problem it is possible to distinguish two cases: (I): There exist $\varepsilon > 0$ such that $\sigma_\alpha^2(L_\varepsilon) - \sigma_\beta^2(L_\varepsilon) \leq 0$, which implies that $\sigma_\alpha^2(L_\rho) - \sigma_\beta^2(L_\rho) \leq 0$ for all $\rho \in (0,\varepsilon]$; (II): $\sigma_\alpha^2(L_\rho) - \sigma_\beta^2(L_\rho) > 0$ for all $\rho \in (0, 0.5)$. In Case I it follows by Lemma 5 that

$$0 \leq \bar{B}_v^T S_{\rho_1} \bar{B}_v \leq \bar{B}_v^T S_{\rho_2} \bar{B}_v, \quad 0 < \rho_1 < \rho_2 \leq \varepsilon$$

and in Case II it follows by Lemma 5 and the fact that $(\bar{A}, \bar{B})$ is stabilizable that

$$B_v^T S_{\rho_2} B_v \leq B_v^T S_{\rho_1} B_v \leq K < \infty, \quad 0 < \rho_1 < \rho_2 < 0.5$$

where K is some constant.

Now, first consider Case I. Then it holds that $\bar{B}_v^T S \bar{B}_v$ is bounded from below at $\rho = 0$ and increasing as a function of ρ on $(0, \varepsilon]$. Hence $\lim_{\rho \downarrow 0} \bar{B}_v^T S_\rho \bar{B}_v$ exist. By considering suitable $\bar{B}_v$:s it follows that the limit $\lim_{\rho \downarrow 0} S_\rho$ exist. This limit satisfies the Riccati equation in the limit, since $G_S > 0$ uniformly on $[0, 1]$. Hence S is continuous at $\rho = 0$, since the Riccati equation has a unique solution such that A_c has all its eigenvalues inside or on the unit circle. Similar arguing can be used to show the continuity at $\rho = 0$ for Case II. Finally, the continuity at $\rho = 1$ is shown analogously. □

LEMMA 7
The set $\mathcal{V}_J \cap \mathcal{V}_z$ is connected, bounded, and closed.

Proof: It is sufficient to show that the variances of α and β for the optimal LQG-controllers are continuous functions of ρ on $(0, 1)$ and that they are continuous at 0 and 1 in the case the corresponding variances are both finite. Furthermore it has to be shown that there exist optimal LQG controllers on the set where the continuity has been shown to hold.

From Lemma 6 it is known that S and P are continuous functions of ρ on $[0, 1]$. Since G_S and H_P are uniformly positive definite on $[0, 1]$ it holds that L, L_v and L_w of (11) and K, K_x, K_v and K_w of (12) also are continuous functions of ρ on $[0, 1]$. This implies that $\hat{P} = \mathrm{E}\left\{\hat{x}(k)\hat{x}^T(k)\right\}$ is a continuous function of ρ on $(0, 1)$. This follows from the fact that A_c, B_c, and H_P are continuous functions of ρ on $[0, 1]$, and that $\hat{P}$ by (16) is the limit as k approaches infinity of

$$\hat{P}(k+1) = A_c \hat{P}(k) A_c^T + B_c H_P B_c^T, \quad \hat{P}(0) = 0 \tag{20}$$

which is a uniformly convergent sequence on $[\delta, 1-\delta]$ for all $0 < \delta \leq 0.5$. Notice that $\hat{P}$ may not necessarily be continuous at $\rho = 0, 1$ since A_c is not stable for $\rho = 0, 1$.

It now follows that the variances of α and β are continuous functions of ρ on $(0, 1)$, since these variances are affine in P and $\hat{P}$ by (17). It only remains to show the continuity at $\rho = 0, 1$. Remember Lemma 5. This implies that the limits $\lim_{\rho \to 0,1} \sigma_i^2(\rho)$, $i = \alpha, \beta$ exist, possibly unbounded, and it only remains to show that the values in the limit are equal to the limiting values. The remaining part of the proof will only focus on proving that $\sigma_\alpha^2(0) = \lim_{\rho \to 0} \sigma_\alpha^2(\rho)$ and $\sigma_\beta^2(0) = \lim_{\rho \to 0} \sigma_\beta^2(\rho)$. The other two equalities are proven analogously.

Let $U_c = \begin{pmatrix} U_{cs} & U_{cu} \end{pmatrix}$ be a transformation defined for $\rho \in [0, \varepsilon]$ for some $\varepsilon > 0$ such that $A_c U_c = U_c \bar{A}_c$, where

$$\bar{A}_c = \begin{pmatrix} J_{cs} & 0 \\ 0 & J_{cu} \end{pmatrix}$$

with $J_{cu} = -1$ for $\rho = 0$. Such a transformation exists since A_c has only one eigenvalue on the unit circle for $\rho = 0$. Further the transformation U_c is continuous by e.g. gol+loan83 Theorem 7.2–4, since A_c is continuous. Let

$$V_c^T = \begin{pmatrix} V_{cs}^T \\ V_{cu}^T \end{pmatrix}$$

be the inverse of U_c which is also continuous. Define

$$\bar{B}_c = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} V_{cs}^T B_c \\ V_{cu}^T B_c \end{pmatrix}$$

With $\bar{P} = V_c^T \hat{P} V_c$ it then holds that

$$\bar{P} = \bar{A}_c \bar{P} \bar{A}_c^T + \bar{B}_c H_P \bar{B}_c^T$$

implying

$$\begin{aligned} \bar{P}_1 &= J_{cs} \bar{P} J_{cs}^T + B_1 H_P B_1^T \\ \bar{P}_{12} &= -J_{cs} \bar{P}_{12} J_{cu} + B_1 H_P B_2^T \\ \bar{P}_2 &= J_{cu}^2 \bar{P}_2 + B_2 H_P B_2^T \end{aligned} \tag{21}$$

for the elements of $\bar{P}$. The first and second equations have continuous solutions $\bar{P}_1$ and $\bar{P}_{12}$ on $[0, \varepsilon]$, since J_{cs} is stable. The third equation is more tricky, and will be dealt with later on. However, notice that $\bar{P}_2 = 0$ for $\rho = 0$ if $B_2 = 0$ by (20).

Let $\Psi = C_\beta - DL$, and define

$$\bar{\Psi} = \begin{pmatrix} \Psi_1 & \Psi_2 \end{pmatrix} = \begin{pmatrix} \Psi U_{cs} & \Psi U_{cu} \end{pmatrix}$$

Notice that there exist $\varepsilon' \in (0, \varepsilon]$ such that $\Psi_2^2 > 0$ and continuous for $\rho \in [0, \varepsilon']$. This follows from the fact that U_c is continuous and that $\Psi_2 \neq 0$ for $\rho = 0$ by Lemma 4. Notice that (18) can be written

$$\begin{aligned} \sigma_\beta^2 &= \bar{\Psi} \bar{P} \bar{\Psi}^T + (C_\beta - DD_c \bar{C}_2) P (C_\beta - DD_c \bar{C}_2)^T \\ &\quad + (G - DD_c \bar{F})(G - DD_c \bar{F})^T \end{aligned}$$

or equivalently

$$\sigma_\beta^2(\rho) = \Psi_2^2(\rho)\bar{P}_2(\rho) + k(\rho) \tag{22}$$

for some continuous $k(\rho)$.

Now, start with the case for which the unstable eigenvalue of A_c at -1 for $\rho = 0$ is controllable from B_c, i.e. when $B_2(0) \neq 0$. Then it holds that there exist $\varepsilon'' \in (0, \varepsilon']$ such that $B_2(\rho) \neq 0$ for $\rho \in [0, \varepsilon'']$. Hence $\lim_{\rho\to 0} \bar{P}_2(\rho) = \infty$ by (21), which implies $\lim_{\rho\to 0} \sigma_\beta^2(\rho) = \infty$ by (22). Notice that the continuity of $\sigma_\alpha^2(\rho)$ and $\sigma_\beta^2(\rho)$ for $\rho = 0$ does not have to be investigated for this case, since $\sigma_z^2 = \left(\sigma_\alpha^2 + \sigma_\beta^2\right)^2 /4 \leq z_0^2$ for elements of $\mathcal{V}_J \cap \mathcal{V}_z$. Finally, notice that there exist optimal controllers for $\rho = (0, 1)$ by Theorem 2.

Now consider the case when the unstable eigenvalue of A_c at -1 for $\rho = 0$ is uncontrollable from B_c, i.e. when $B_2(0) = 0$. Remember that $\bar{P}_2(0) = 0$. Suppose that $\bar{P}_2$ is not continuous for $\rho = 0$. Then (22) shows that $\sigma_\beta^2(\rho) > \sigma_\beta^2(0)$ for some $\rho > 0$, which contradicts Lemma 5. This together with the fact that $\bar{P}_1$, $\bar{P}_{12}$, and U_c are continuous functions of ρ implies that $\hat{P}$ is continuous for $\rho = 0$, and hence that σ_β^2 is continuous for $\rho = 0$. The continuity of $\hat{P}$ now implies that σ_α^2 is also continuous for $\rho = 0$ by (18). Finally, notice that there exist optimal controllers for $\rho = [0, 1)$ by Theorem 2. □

PROOF OF THEOREM 1

Consider the case when $\mathcal{D}_z = \emptyset$. Then it is trivial that $\mathcal{D}_\mu = \emptyset$ by the definition of the optimization problem, and hence the theorem is trivial. Now, consider the case when $\mathcal{D}_z \neq \emptyset$. Then $\mathcal{V}_J \cap \mathcal{V}_z$ is non-empty, since the controller that minimizes J for $\rho = 0.5$ is the minimum variance controller for the variance of z. Let $\left(\sigma_\alpha^2(H^*), \sigma_\beta^2(H^*)\right)$ be an element of $\mathcal{V}_J \cap \mathcal{V}_z$ that minimizes μ on this set, which exists by Lemma 7. Denote the corresponding minimal value by μ^*. It is now sufficient to show that for any H in $\mathcal{D}_z$ for which $(a, b) = \left(\sigma_\alpha^2(H), \sigma_\beta^2(H)\right)$ is not in $\mathcal{V}_J \cap \mathcal{V}_z$ it holds that $\mu(a, b) > \mu^*$. This is implied by the existence of H_ρ in $\mathcal{D}_z$ for which $(a_\rho, b_\rho) = \left(\sigma_\alpha^2(\bar{H}), \sigma_\beta^2(\bar{H})\right)$ is in $\mathcal{V}_J \cap \mathcal{V}_z$ and such that $a_\rho \leq a$ and $b_\rho \leq b$ with at least one strict inequality, since for such an H_ρ it holds that $\mu(a, b) > \mu(a_\rho, b_\rho) \geq \mu^*$, where the first inequality follows from Lemma 2 and the second by the definition of μ^*. Hence it only remains to show the existence of an H_ρ with the above properties.

Notice that $\mathcal{V}_J \cap \mathcal{V}_z = \{(a_\rho, b_\rho) \mid \rho \in [\rho_{\min}, \rho_{\max}]\}$, where $\rho_{\min} = 0$ and $\rho_{\max} = 1$ if $\mathcal{V}_J \cap \mathcal{V}_z = \mathcal{V}_J$, and where $\rho_{\min} > 0$ or $\rho_{\max} < 1$ if

$\mathcal{V}_J \cap \mathcal{V}_z \neq \mathcal{V}_J$. Also notice that

$$\begin{aligned} a_{\rho_{\max}} + b_{\rho_{\max}} &= 4z_0^2, \quad \rho_{\max} < 1 \\ a_{\rho_{\min}} + b_{\rho_{\min}} &= 4z_0^2, \quad \rho_{\min} > 0 \end{aligned} \tag{23}$$

Furthermore the existence of an H_ρ with the desired properties is equivalent to the existence of a $\rho \in [\rho_{\min}, \rho_{\max}]$ such that $a - a_\rho \geq 0$ and $b - b_\rho \geq 0$ with at least one strict inequality. Now assume that such a ρ does not exist. This implies by the fact that $\mathcal{V}_J \cap \mathcal{V}_z$ is connected that $a - a_\rho \leq 0$ or $b - b_\rho \leq 0$ for all $\rho \in [\rho_{\min}, \rho_{\max}]$. Furthermore it holds for any $\rho \in [\rho_{\min}, \rho_{\max}]$ by the optimality that

$$(1-\rho)a_\rho + \rho b_\rho < (1-\rho)a + \rho b$$

which can be rewritten as

$$(1-\rho)(a - a_\rho) + \rho(b - b_\rho) > 0 \tag{24}$$

Start with the case when $a - a_\rho \leq 0$. If $\rho_{\min} = 0$, then there is an immediate contradiction to (24). If $\rho_{\min} > 0$, then by (23) it holds that $a + b \leq a_{\rho_{\min}} + b_{\rho_{\min}}$, since $a + b \leq 4z_0^2$ for any H in $\mathcal{D}_z$. This implies that

$$(1-\rho_{\min})(a - a_{\rho_{\min}}) + \rho_{\min}(b - b_{\rho_{\min}}) \leq (1 - \rho_{\min} - \rho_{\min})(a - a_{\rho_{\min}}) \leq 0$$

where the second inequality is implied by $\rho_{\min} \leq 0.5$, which follows from the fact that the minimum variance controller for z has closed loop variances $(a_{0.5}, b_{0.5})$ which is an element of $\mathcal{V}_J \cap \mathcal{V}_z$. The case when $b - b_\rho \leq 0$ is proven analogously by using $\rho_{\max}$. □

Remark 1. Notice that the minimization of μ can be thought of as finding an optimal value $\rho \in [\rho_{\min}, \rho_{\max}]$ for the LQG-problem.

Remark 2. Notice that an alternative formulation of the existence result is that there exist an MU-controller if and only if there exist a minimum variance controller, obtained for $\rho = 0.5$, with closed loop standard deviation satisfying $\sigma_z \leq z_0$.

4. *Example*

In this section an example will be given to illuminate the sometimes degenerated behavior of the MU controller. Let the process to be controlled be defined by

$$A = \begin{pmatrix} -a & b-a \\ 0 & 0 \end{pmatrix}; \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$C_1 = \begin{pmatrix} 1 & 1 \end{pmatrix}; \quad C_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$D = 0; \quad E = \begin{pmatrix} c-a & 0 \\ 0 & 0 \end{pmatrix}$$

$$F = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad G = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

Assume that $a = 7/8$, $b = 2$, and $c = 1/2$. Then it can be shown that the LQG controllers solving (9) are independent of ρ and given by

$$u(k) = -\frac{1}{8}\begin{pmatrix} 1 & 0 \end{pmatrix} y(k)$$

Hence the MU controller will be equal to this controller provided $\sigma_z \leq z_0$. Further it will be the same as the minimum variance controller. It should be stressed that this behavior is not the generic one. It is also an example of the difficult case that there may exist controllers with bounded σ_α^2 and σ_β^2 for $\rho = 0, 1$.

5. *Conclusions*

The existence of the MU controller has been investigated. This controller minimizes the probability for the controlled signal to upcross a level given a certain reference value. There are many examples of control problems for which this approach is appealing, i.e. problems for which there exist a level such that a failure in the controlled system occurs when the controlled signal upcrosses the level. One important class of such problems is processes equipped with supervision, where upcrossings of alarm levels may initiate emergency shutdown causing loss in production.

The problem of minimizing the upcrossing probability over the set of stabilizing linear time-invariant controllers has been rephrased to a minimization over LQG problem solutions parameterized by a scalar, and thus the complexity is only one order of magnitude larger than for an ordinary LQG problem. It has also been seen that the optimal controller can be interpreted as finding an optimal costing transfer function for the system output. The key to the method is the reformulation using the independent variables α and β making it possible to quantify by Lemma 1 the upcrossing probability in terms of the variances of α and β.

The set of closed loop variances of α and β obtained by solving the set of LQG problems has been characterized. This made it possible to give a necessary and sufficient condition for the existence of the minimum upcrossing controller, which is equivalent to the existence of a minimum variance controller with sufficiently small closed loop variance. This point, first solved for the scalar *ARMAX*-case in han93b was here generalized to the more general state space case with several measurement signals.

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7. *Appendix*

LEMMA 8
Assume that $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}$, and $D \in R^{p\times m}$ are differentiable functions of a real-valued parameter ρ on an interval $I \subset R$, and that (A, B) is stabilizable on I and that

$$\operatorname{rank}_{|z|=1} \begin{pmatrix} zI - A & -B \\ C & D \end{pmatrix} = n + m$$

holds on I. Consider the Riccati-equation

$$\begin{aligned} S &= (A - BL)^T S (A - BL) + (C - DL)^T (C - DL) \\ GL &= B^T S A + D^T C \\ G &= B^T S B + D^T D \end{aligned}$$

Then the real, symmetric, and positive semidefinite solution S of the Riccati-equation such that $A_c = A - BL$ is stable on I will also be a differentiable function of ρ on I.

Proof: The Riccati equation has a well defined solution $S(\rho)$ on I. Formal implicit derivation with respect to ρ gives

$$\begin{aligned} \dot S + \dot L^T G L + L^T \dot G L + L^T G \dot L &= \dot A^T S A + A^T \dot S A + A^T S \dot A + \dot C^T C + C^T \dot C \\ \dot G L + G \dot L &= \dot B^T S A + B^T \dot S A + B^T S \dot A + \dot D^T C + D^T \dot C \\ \dot G &= \dot B^T S B + B^T \dot S B + B^T S \dot B + \dot D^T D + D^T \dot D \end{aligned} \tag{25}$$

Substitution of the second and third equation into the first equation gives

$$\begin{aligned}\dot{S} &= A_c^T \dot{S} A_c \\ &+ (\dot{A} - \dot{B}L)^T S (A - BL) + (A - BL)^T S (\dot{A} - \dot{B}L) \\ &+ (\dot{C} - \dot{D}L)^T (C - DL) + (C - D)^T (\dot{C} - \dot{D}L)\end{aligned}$$

Now, A_c is stable by assumption and the equation for $\dot{S}$ is a Lyapunov-equation. Hence it follows that $\dot{S}$ is unique. By the third equation of (25) it follows that $\dot{G}$ is unique. Further, since $G > 0$, it follows by the second equation that $\dot{L}$ is unique as well. Hence S is differentiable on I by the implicit function theorem. □