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ABSTRACT

Adaptive controllers of a certain structure are considered in this report. The parameters in a difference equation model of the process are estimated on-line using the least squares method. The current parameter estimates are used to calculate the parameters of the feedback law that governs the process. The resulting adaptive controller is called a self-tuning algorithm. It is shown that the convergence properties of such algorithms can be investigated by analysing an associated ordinary differential equation. The analysis is applied to specific examples of self-tuning algorithms.

TABLE OF CONTENTS

1. SELF-TUNING REGULATORS

Adaptive control of dynamic systems has been extensively discussed during the last ten years. An important special case is when the process parameters are known to be time invariant, but the values are unknown. For this case the control algorithms should be such that they converge to the optimal control algorithms that could be derived if the system characteristics were known. Such an algorithm is called a self-tuning regulator.

Most suggested adaptive and self-tuning regulators are based on the assumption that the real time estimation of the parameters of the process can be separated from the determination of the control signal. See fig. 1.1. In many cases the control signal is determined without taking into consideration that the estimates of the parameters are uncertain. A more sophisticated type of controllers are those which

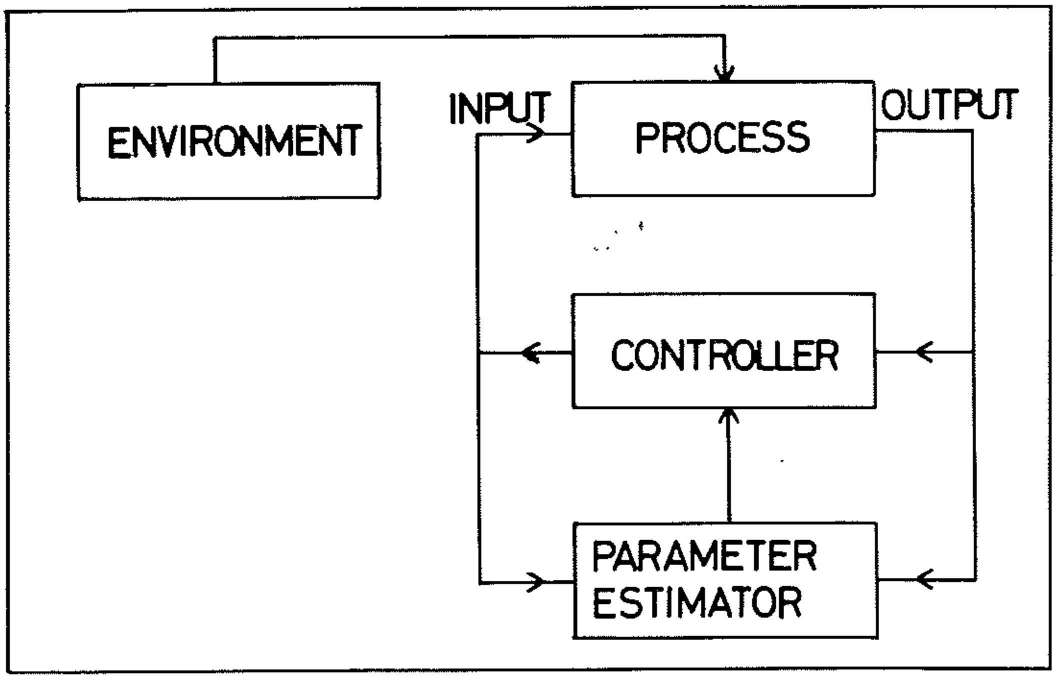


Fig. 1.1 - Schematic block diagram for adaptive regulators.

consider the parameter uncertainties when the control signal is determined, but do not make anything to obtain better estimates. The dual controllers, see Feldbaum (1960, 1961), represent a further degree of sophistication where the input signal is chosen to increase the accuracy of the parameters in the same time as the controller tries to make as good control as possible. The two activities of the dual controller are mutually contradictory and there must be a compromise between the identification and control activities of the controller. The dual controllers have attractive features, but it is very difficult to get practical solutions to the dual control problem.

In this report the behaviour of self-tuning regulators with a structure as in fig. 1.1 is discussed in the case when the identifier is based on the least squares (LS) method. However, the techniques put forward are applicable to more general identification schemes.

Special attention will be paid to the case when the controller is a minimum variance control law based on current estimates. This self-tuning regulator is discussed in Åström - Wittenmark (1973).

The central question for the analysis of such a regulator is of course: Will the regulator converge to the desired one? Techniques and basic theorems to answer this question are presented in this report. Specific examples of self-tuning regulators are analysed using these tools.

In Chapter 2 the LS method is defined, and known results are repeated for easy reference. A class of self-tuning regulators is strictly defined. The main problem in the analysis of the regulators is that the feedback is time varying. This makes the input and output sequences non-stationary and the usual consistency results for the LS method are inapplicable.

A theorem on consistency that is valid in the present case is proved in Chapter 3.

In general the result of the identification depends on the feedback law. This introduces essential non-linearities into the identification process. To handle this problem an ordinary differential equation (ODE), which is connected with the regulator, is derived in Chapter 4. Stability of this ODE is shown to be equivalent to convergence of the regulator. Also, the paths of the ODE define "expected behaviour" of the regulator.

In Chapters 3 and 4 the controller is not specified. In Chapters 5, 6, and 7 most of the analysis is concerned with the self-tuning regulators discussed in Wittenmark (1973). In Chapter 5 the behaviour near or outside the stability boundary of the closed loop system is discussed. The regulators are shown to stabilize the closed loop system even if the model noise does not agree with the true noise characteristics.

In Chapter 6 the ODE defined in Chapter 4 is investigated for some self-tuning regulators. It is shown that the regulators do not converge for general noise structures. Actually, it was indicated by extensive simulations, Åström - Wittenmark (1973), that the regulators converge in general. Only after using the analysis of Chapter 6 could examples be constructed for which the regulators do not converge.

In Chapter 7 the ODE is solved numerically for a number of cases of interest.

In Appendix A the proof of Lemma 4.1 is given. The results of the report hold for several different model structures. In Appendix B it is shown how other model structures can be handled, and the modifications of the results shown are also given there.

2. PRELIMINARIES

The class of self-tuning regulators to be treated is formally defined in this chapter. In Section 2.1 some different models for least squares identification are discussed. Off-line and on-line algorithms for least squares identification are given in Section 2.2. In Section 2.3 algorithms of stochastic approximation type are introduced. The class of self-tuning regulators is defined in Section 2.4, and in Section 2.5 the special algorithms "STURE0" and "STURE1" (self-tuning regulator) are introduced.

2.1 Models.

Assume that the system can be described by the difference equation

$$y(t+1) + a_1 y(t) + \dots + a_n y(t+1-n) = \\ = b_0 u(t-k) + \dots + b_m u(t-k-m) + v(t+1) \qquad (2.1)$$

where $k \geq 0$, $\{v(t)\}$ is a sequence of random variables and where $\{y(t)\}$ is the output and $\{u(t)\}$ the input of the system. The usual model for least squares identification has the same structure as (2.1) and in general all the constants $a_1,\dots,$ a_n, $b_0,\dots,b_m$ are estimated. In connection with self-tuning regulators, cf. Åström - Wittenmark (1973), it is in some cases meaningful not to estimate b_0.

It is possible to rewrite (2.1) as

$$y(t+k+1) + \alpha_1 y(t) + \dots + \alpha_n y(t-n+1) = \beta_0 u(t) + \dots + \\ + \beta_{m'} u(t-m') + \varepsilon(t+k+1) \qquad (2.2)$$

where $\{\varepsilon(t)\}$ is a process formed as a moving average from $v(t),\dots,v(t-k)$. The variable m' equals $m+k$, see Åström - Wittenmark (1973). This modification proves to be of great value when control laws are synthesized as shown in Section 2.5.

Also in this case it is suitable to consider β_o as an a priori known constant. Then, by introducing $\beta_i = \beta_i'/\beta_o$, (2.2) can be written as

$$y(t+k+1) + \alpha_1 y(t) + \ldots + \alpha_n y(t-n+1) = \beta_o[u(t) + \beta_1 u(t-1) +$$

$$+ \ldots + \beta_{m'} u(t-m')] + \varepsilon(t+k+1) \qquad (2.3)$$

The model of the system then is

$$y(t+\hat{k}+1) + \hat{\alpha}_1 y(t) + \ldots + \hat{\alpha}_{\hat{n}} y(t-\hat{n}+1) = \hat{\beta}_o[u(t) + \ldots +$$

$$+ \hat{\beta}_{\hat{m}} u(t-\hat{m})] + \hat{\varepsilon}(t+\hat{k}+1) \qquad (2.4)$$

Here $\hat{\alpha}_i$ is the estimate of α_i and $\hat{\beta}_i$ the estimate of β_i. β_o is regarded as a priori known and is not estimated. It is, however, not realistic to assume that the value of β_o is exactly known. Therefore, some of the analysis in this report will deal with the case when β_o is assumed to have the value $\hat{\beta}_o$, which may be different from β_o. The orders $\hat{n}$, $\hat{m}$ and the time delay $\hat{k}$ may not be the same as the true ones.

In this report (2.4) is used as the basic model, since this structure is used in the self-tuning algorithms STURE0 and STURE1, defined in Section 2.5. However, the other model structures can be treated formally in exactly the same way. This is shown in Appendix B.

2.2 Least squares identification.

Introduce

$$\theta = (\hat{\alpha}_1, \ldots \hat{\alpha}_{\hat{n}}, \quad \hat{\beta}_1, \ldots \hat{\beta}_{\hat{m}})^T$$

and

$$x(t) = (-y(t), \ldots, -y(t-\hat{n}+1), \hat{\beta}_o u(t-1), \ldots, \hat{\beta}_o u(t-\hat{m}))^T$$

Then (2.4) can be written

$$y(t+\hat{k}+1) = \theta^T x(t) + \hat{\beta}_o u(t) + \hat{\varepsilon}(t+\hat{k}+1) \tag{2.5}$$

The LS criterion for this model is (initial value effects are neglected):

$$V_t(\theta) = \frac{1}{t} \sum_{s=1}^{t} [y(s) - \hat{\beta}_o u(s-\hat{k}-1) - \theta^T x(s-\hat{k}-1)]^2 \tag{2.6}$$

This function is minimized by

$$\theta(t) = P(t)h(t) \tag{2.7}$$

where

$$P^{-1}(t) = \frac{1}{t} \sum_{s=1}^{t} x(s-\hat{k}-1)\, x(s-\hat{k}-1)^T \tag{2.8}$$

and

$$h(t) = \frac{1}{t} \sum_{s=1}^{t} [y(s) - \hat{\beta}_o u(s-\hat{k}-1)]\, x(s-\hat{k}-1) \tag{2.9}$$

If $\{v(t)\}$ in (2.1) is a sequence of independent random variables, then the LS noise condition is said to be satisfied. Then $\varepsilon(t)$ and $\varepsilon(s)$ are independent for $|t-s| > k$. If the LS noise condition is satisfied, $\hat{n} \geq n$, $\hat{m} \geq m$, $\hat{k} = k$ and $\hat{\beta}_o = \beta_o$, then it can be shown that $\theta(t)$ tends to the true value $(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m)^T$ w.p.1 (with probability one) as t tends to infinity, see Åström (1968). This question is further discussed in Chapter 3.

The solution (2.7) can be written recursively as

$$\theta(t+1) = \theta(t) + \frac{1}{t+1} P(t)\, x(t-\hat{k})[y(t+1) - \hat{\beta}_o u(t-\hat{k}) - \theta(t)^T x(t-\hat{k})] \cdot$$

$$\frac{t+1}{t+x(t-k)^T P(t) x(t-\hat{k})} \qquad (2.10)$$

$$P^{-1}(t+1) = P^{-1}(t) + \frac{1}{t+1} [x(t-\hat{k})\, x(t-\hat{k})^T - P^{-1}(t)]$$

In asymptotic analysis the factor $\frac{t+1}{t+x(t-k)^T P(t)x(t-k)}$ will be replaced by 1. Recursive formulas can also be given for P(t) directly, Åström (1968).

2.3 Identification using stochastic approximation.

A suitable identification criterion for model (2.4) is

$$J(\theta) = E[y(t+1) - \hat{\beta}_o u(t-\hat{k}) - \theta^T x(t-\hat{k})]^2 \qquad (2.11)$$

where the expectation is taken with respect to $\varepsilon(t+1)$, $\varepsilon(t)$, ... Naturally (2.11) cannot be computed when only a finite number of data y(t), u(t) are known. One approach is to replace (2.11) with the estimated mean value (2.6). This has been discussed in Section 2.2. Another approach is to apply the Robbins-Monro scheme, see e.g. Tsypkin (1973) to the derivative of (2.11). This gives

$$\theta(t+1) = \theta(t) + \gamma(t+1)\, x(t-\hat{k})[y(t+1) - \hat{\beta}_o u(t-\hat{k}) - x(t-\hat{k})^T \theta(t)] \qquad (2.12)$$

The sequence of scalars $\{\gamma(t)\}$ must satisfy certain conditions, Tsypkin (1973), which are further discussed in Chapter 4. Common choices of the sequence are

$$\gamma(t) = \frac{1}{t} \qquad (2.13a)$$

$$\gamma(t) = \left[\sum_{s=1}^{t-\hat{k}} |x(s)|^2\right]^{-1} \qquad (2.13b)$$

$$\gamma(t) = \frac{1}{t} (x(t-\hat{k})^T x(t-\hat{k}))^{-1} \qquad (2.13c)$$

Algorithm (2.12) is clearly quite similar to (2.10). The latter one requires more computation and more memory storage than (2.12). In return it converges more rapidly.

2.4 Self-tuning regulators.

Suppose that the input to the process, u(t), is determined as a feedback from old inputs and outputs. Suppose also that the coefficients of the feedback law are calculated from the current LS-estimates of the process parameters:

$$u(t) = f(\theta(t), x(t)) \tag{2.14}$$

This type of adaptive controllers, which are based on a straightforward separation between identification and control, is discussed by e.g. Kalman (1958), Peterka (1970), Åström - Wittenmark (1971), (1973) and Peterka - Åström (1973).

The equations (2.10), (2.12) and (2.14) thus define a class of self-tuning regulators. They have the form:

$$\theta(t+1) = \theta(t) + \gamma(t+1)S(t)x(t-\hat{k})[y(t+1) - \hat{\beta}_o u(t-\hat{k}) - \theta(t)^T x(t-\hat{k})] \tag{2.15a}$$

$$u(t) = f(\theta(t), x(t)) \tag{2.15b}$$

Linear regulators

$$u(t) = F(\theta(t))\, x(t) \tag{2.15c}$$

form an important subclass of the algorithm. Two choices of S(t) will be considered. Let P(t) be defined by

$$P^{-1}(t+1) = P^{-1}(t) + \gamma(t+1)[x(t-\hat{k})x(t-\hat{k})^T - P^{-1}(t)] \tag{2.15d}$$

Then S(t) is taken either as

$$S(t) = \frac{P(t)}{1+\gamma(t+1)\ [x(t-\hat{k})^T P(t) x(t-\hat{k})-1]} \qquad (2.15e)$$

or

$$S(t) = \frac{1}{\operatorname{tr} P^{-1}(t+1)} \qquad (2.15f)$$

The sequence $\{\gamma(t)\}$ is a sequence of deterministic scalars. If $\gamma(t) = 1/t$, the choice (2.15e) gives the recursive least squares algorithm, and (2.15f) gives the stochastic approximation algorithm (2.12) with (2.13b).

For future reference, the regulators with S(t) as in (2.15e) will be called self-tuning regulators of LS type. Correspondingly, with S(t) as in (2.15f) they will be called self-tuning regulators of SA type. Equation (2.15) will be used as a basic reference.

Various conditions on the noise sequence $\{\varepsilon(t)\}(\{v(t)\})$ will be considered. In Chapter 3 and in Section 6.2 the LS noise condition is assumed to be satisfied. Then $\varepsilon(t+\hat{k}+1)$ and x(t) are independent if $\hat{k} \geq k$. In the rest of the analysis the conditions are much less restrictive. They will be defined for each case, cf. (4.12) and (5.3).

2.5 Minimum variance control and self-tuning regulators.

Consider the system (2.3). Suppose that the LS noise condition is satisfied. If the input is chosen as

$$u(t) = -\frac{1}{\beta_0}\ [-\alpha_1 y(t) - \ldots - \alpha_n y(t-n+1) + \beta_1 u(t-1) + \ldots +$$

$$+ \beta_{m'} u(t-m')] \qquad (2.16)$$

then the output is

$$y(t) = \varepsilon(t)$$

Obviously, no other control law can yield lower variance of the output. The feedback law (2.16) is therefore called the minimum variance controller. It is discussed at length in Åström (1970).

If the parameters of model (2.3) are unknown, the minimum variance control law (2.16) cannot be computed. It is suggested in Åström - Wittenmark (1973) that the coefficients of (2.16) should be chosen as the LS-estimates of the system parameters. This means that $f(\theta(t), x(t))$ in (2.15) is chosen as

$$f(\theta(t), x(t)) = -\frac{1}{\hat{\beta}_o} \theta(t)^T x(t) \tag{2.17}$$

The self-tuning regulator of SA type with this feedback law is called STURE0 (self-tuning regulator). The corresponding algorithm of LS-type is called STURE1. These regulators are discussed in e.g. Peterka (1970) and Wittenmark (1973). They have also been applied to industrial processes, see e.g. Cegrell - Hedqvist (1973) and Borisson - Wittenmark (1973).

Example 2.1. The behaviour of STURE1 will be illustrated on the system

$$y(t+1) + a\, y(t) = b\, u(t-1) + v(t+1) \tag{2.18}$$

where $a = -0.9$, $b = 1$ and $v(t) = e(t) - 0.4\, e(t-1)$, $\{e(t)\}$ being a sequence of independent, random variables with distribution $N(0,1)$. The LS noise condition is thus not satisfied for (2.18). The minimum variance controller for the system (2.18) is given by

$$u(t) = -0.45\, y(t) - 0.5\, u(t-1) \tag{2.19}$$

This controller gives in each step an expected loss

$$E\, y(t)^2 = 1.25$$

Without any control the loss is 2.32 per step. When using the self-tuning regulator the parameters $\hat{\alpha}$ and $\hat{\beta}$ are estimated from the model

$$y(t+2) + \hat{\alpha} y(t) = \hat{\beta}_o [u(t) + \hat{\beta} u(t-1)] + \hat{\varepsilon}(t+2)$$

using a stochastic approximation method, STURE0, or the least squares method, STURE1. Hence, in this case $\hat{m} = m'$, $\hat{n} = n$. The control law is then simply

$$u(t) = \frac{1}{\hat{\beta}_o} [\hat{\alpha} y(t) - \hat{\beta} u(t-1)]$$

The system has been simulated with STURE1 using $\hat{\beta}_o = 1$. The sequence $\{\gamma(t)\}$ was chosen as $1/t$ and the initial covariance matrix $P(0)$ was 0.1 times a unit matrix. The parameter estimates from one simulation are given in figure 2.1. Very quickly the estimates are quite close to the optimal ones. The quality of the control can be determined from the accumulated loss $\sum_{s=1}^{t} y(s)^2$. The accumulated loss at $t = 2000$ when using STURE1 is 2447, while when using the optimal regulator (2.19) the loss is 2430. The accumulated loss is shown in figure 2.2.

The good behaviour is somewhat unexpected. The LS-identification gives biased estimates in case the variables $\{v(t)\}$ are dependent, Åström - Eykhoff (1971). Also, for dependent noise the minimum variance control law is not given by (2.16). These two effects, however, compensate each other. It is shown in Åström - Wittenmark (1973) that if the regulator converges it must converge to the minimum variance control regulator. One of the main questions treated in this report is whether the algorithm actually converges.

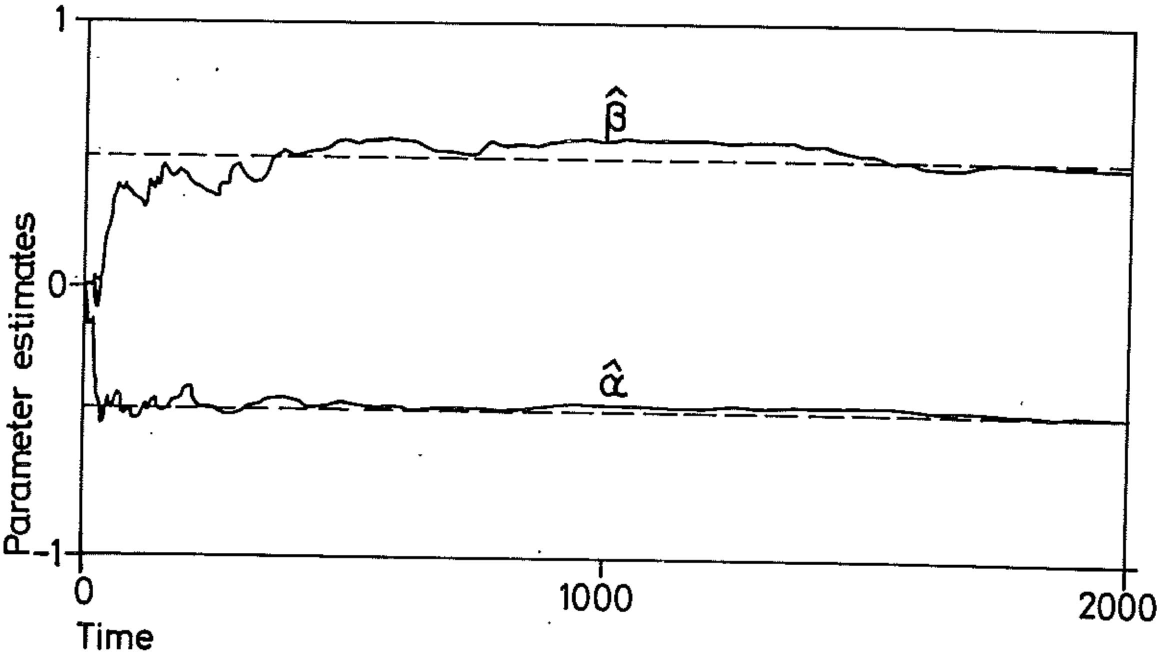


Fig. 2.1 Parameter estimates when the system (2.18) is controlled with STURE1. The dashed lines show the values corresponding to the optimal regulator (2.19).

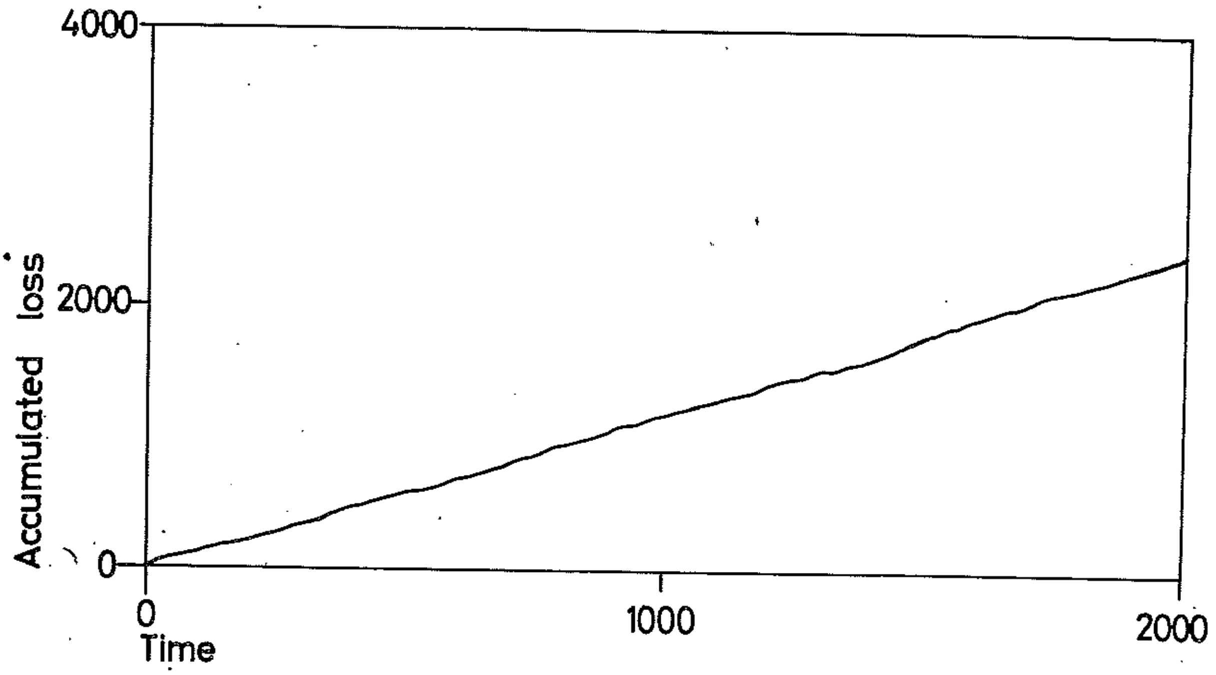


Fig. 2.2 The accumulated loss when the system (2.18) is controlled by STURE1.

3. LEAST SQUARES IDENTIFICATION OF CLOSED LOOP SYSTEMS

In Section 2.2 least squares identification of parameters in different equations was described. The convergence properties of the method are well known in case the input is persistently exciting and independent of the noise, Åström - Eykhoff (1971). However, for adaptive regulators, the input is determined as output feedback and will consequently be correlated with the noise. Moreover, the coefficients in the feedback law are time varying and depend in a complex way on previous input and output. The convergence under such conditions is treated in Section 3.1.

In Section 3.2 the results are applied to the self-tuning regulator STURE1. This analysis concerns basically the convergence of the regulator parameters.

It should be emphasized that the results of this chapter are valid only if the assumptions made about the model structure are true. This means that the LS noise condition is assumed to be satisfied. Furthermore, β_o and the time delay k must be known, and the model orders must not be underestimated (i.e. $\hat{n} \geq n$, $\hat{m} \geq m'$). The convergence properties when these assumptions no longer are true are discussed in the following chapters.

3.1 Consistency of least squares estimates.

Consider a system that is described by (2.3):

$$y(t+k+1) + \alpha_1 y(t) + \ldots + \alpha_n y(t-n+1) = \beta_o[u(t) + \beta_1 u(t-1) +$$

$$+ \ldots + \beta_{m'} u(t-m')] + \varepsilon(t+k+1) \qquad (3.1)$$

Suppose that k and β_o are known constants, and that upper bounds $\hat{n}$ and $\hat{m}$ respectively for n and m' are known. Then the model of (3.1) is

$$y(t+k+1) + \hat{\alpha}_1 y(t) + \ldots + \hat{\alpha}_{\hat{n}} y(t-\hat{n}+1) = \hat{\beta}_o[u(t) + \ldots +$$

$$+ \hat{\beta}_{\hat{m}} u(t-\hat{m})] + \hat{\varepsilon}(t+k+1) \quad (3.2)$$

Introduce

$\theta_o = (\alpha_1, \ldots \alpha_n, 0,\ldots,0, \beta_1 \ldots \beta_{m'}, 0,\ldots,0)$; $\hat{n} - n$ and $\hat{m} - m'$ zeros respectively.

Then (3.1) and (3.2) can be written

$$y(t+k+1) = \theta_o^T x(t) + \beta_o u(t) + \varepsilon(t+k+1) \quad (3.4)$$

$$y(t+k+1) = \theta^T x(t) + \beta_o u(t) + \hat{\varepsilon}(t+k+1) \quad (3.5)$$

To show consistency (with probability one, w.p.1) of θ the following assumptions are usually made (see e.g. Åström - Eykhoff (1971)):

- $\{\varepsilon(t)\}$ is a stationary sequence of random variables with zero mean values and bounded fourth moments, such that $\varepsilon(s)$ and $\varepsilon(t)$ are independent for $|t - s| > k$. (3.6a)
- The polynomial (3.6b)
  $z^{n+k} + \alpha_1 z^{n-1} + \ldots + \alpha_n$
  is stable, i.e. all roots have magnitude less than one.
- The input sequence $\{u(t)\}$ is independent of $\{\varepsilon(t)\}$. (3.6c)
- The input sequence is persistently exciting of order $\hat{m}$. (3.6d)

Condition (3.6c) excludes systems with feedback. To show consistency also for closed loop systems the following assumptions are introduced:

- $\{\varepsilon(t)\}$ is a sequence (not necessarily stationary) of random variables with uniformly bounded second moments and zero mean values, such that $\varepsilon(t)$ and $\varepsilon(s)$ are independent for $|s - t| > k$. (3.7a)

- $$\limsup_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} y^2(t) < \infty \qquad \text{w.p.1} \qquad (3.7b)$$
  $$\limsup_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} u^2(t) < \infty \qquad \text{w.p.1}$$

- $u(t)$ is independent of $\varepsilon(s)$ $s > t + k$ (3.7c)

- $$\{\theta \mid \liminf_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} [\theta^T x(t)]^2 = 0\} = \{0\} \text{ w.p.1} \qquad (3.7d)$$

Clearly, (3.7abc) corresponds to (3.6abc) and are weaker conditions. Condition (3.7d) deserves some discussion. Let

$$\underline{u}(t) = [u(t-1) \quad u(t-2) \ \dots \ u(t-\hat{m})]^T$$

$$\underline{y}(t) = [y(t) \quad y(t-1) \ \dots \ y(t-\hat{n}+1)]^T$$

Thus

$$x(t) = \begin{bmatrix} -\underline{y}(t) \\ \hat{\beta}_o \underline{u}(t) \end{bmatrix}$$

Then condition (3.6d) can be written

$\lim_{N\to\infty} \frac{1}{N} \sum_{1}^{N} [\theta_u^T \underline{u}(t)]^2$ exists w.p.1 and is strictly positive for any non-zero vector θ_u of dimension $\hat{m}$.

The relationship between (3.6d) and (3.7d) is now quite clear. We have:

$$\theta^T x(t) = \theta_y^T \underline{y}(t) + \hat{\beta}_o \theta_u^T \underline{u}(t)$$

If the system is open loop, i.e. if (3.5c) is satisfied, it is easy to show that

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{1}^{N} [\theta^T x(t)]^2 = 0$$ implies w.p.1 that $\theta_y = 0$. Thus,

in this case (3.7d) means

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{1}^{N} [\theta_u{}^T \underline{u}(t)]^2 = 0 \Rightarrow \theta_u = 0$$

which essentially means that {u(t)} must be persistently exciting. It is, however, somewhat weaker, since it does not require that the limit exists. In case the system operates under output feedback, relationships between u and y exist. Condition (3.7d) states that these relations must not be of a certain kind. For example, if $\hat{n} = 2$, $\hat{m} = 1$ and the feedback is $u(t) = -y(t)$, then (3.7d) is not satisfied. The vector θ can be chosen as $(0\ 1\ 1)^T$ and all terms are identically zero.

The condition is formulated for limit inferior, since it is not known that the limit of

$$\frac{1}{N} \sum_{1}^{N} [\theta^T x(t)]^2$$

exists. This would require stationarity of the closed loop system, and might not be true in a number of applications.

Of the conditions (3.7) only (3.7d) is restrictive in practice. It can be interpreted as an identifiability condition for systems operating in closed loop. Such problems are discussed in a more general context in Gustavsson-Ljung-Söderström (1974). Conditions that are sufficient for (3.7d) to be satisfied are given there.

A theorem on consistency for LS estimates can now be formulated. The main tool to overcome the difficulties with non-stationary processes $\{y(t)\}$ and $\{u(t)\}$ is the convergence theorem for martingales, see Doob (1953).

Theorem 3.1 Consider the system (3.1). The system parameters θ_o are estimated using an ordinary least squares criterion (cf. Section 2.2). Suppose that (3.7) is satisfied. Then the estimates $\theta(t)$ converge with probability one to their true values as the number of data tends to infinity.

Proof: Let k = 0 for convenience. The LS criterion to be minimized with respect to θ at step N is

$$V_N(\theta) = \frac{1}{N}\sum_1^N [y(t+1) - \beta_o u(t) - \theta^T x(t)]^2 =$$

$$= \frac{1}{N}\sum_1^N [(\theta_o-\theta)^T x(t) + \varepsilon(t+1)]^2 =$$

$$= \frac{1}{N}\sum_1^N \varepsilon(t+1)^2 + 2\frac{1}{N}\sum_1^N \varepsilon(t+1)\ (\theta_o-\theta)^T x(t) + \frac{1}{N}\sum_1^N [(\theta_o-\theta)^T x(t)]^2 \hat{=}$$

$$\hat{=} V_N^{(1)} + 2\, V_N^{(2)}(\theta) + V_N^{(3)}(\theta)$$

Let the minimizing θ be denoted by $\theta(N)$. Let F_t be the σ-algebra generated by $\{\varepsilon(0),\varepsilon(1),\ldots,\varepsilon(t)\}$. It is no loss of generality to assume that $E\varepsilon(t)^2 = 1$. Then

$$E\{[\varepsilon(t+1)y(t)]^2 | F_t\} = y(t)^2$$

Let

$$s(t) = \sum_{r=1}^t y(r)^2 + 1 \qquad s(0) = 1$$

and consider

$$z(t+1) = \sum_{r=1}^t \varepsilon(r+1)y(r)/s(r), \quad z(1) = 0$$

The sequence $[z(t), F_t]$ is a martingale, since

$$E[z(t+1)|F_t] = z(t) + E[y(t)\varepsilon(t+1)/s(t)|F_t] =$$

$$= z(t) + \frac{y(t)}{s(t)} E(\varepsilon(t+1)|F_t) = z(t)$$

Consider

$$E\, z(N)^2 = \sum_{r=2}^{N} E(z(r)^2 - z(r-1)^2) = E \sum_{r=2}^{N} E[z(r)^2 - z(r-1)^2|F_{r-1}] =$$

$$= E \sum_{r=2}^{N} E[(z(r) - z(r-1))^2|F_{r-1}] = E \sum_{1}^{N-1} y(r)^2 / s(r)^2$$

$$\leq E \sum_{1}^{N-1} [s(r) - s(r-1)]/s(r)s(r-1) = E \sum_{1}^{N-1} \left[\frac{1}{s(r-1)} - \frac{1}{s(r)}\right] \leq 1$$

Hence z(N) converges with probability one due to the martingale convergence theorem. Kronecker's lemma (see e.g. Chung (1968)) now gives that

$$\frac{1}{s(N)} \sum_{1}^{N} \varepsilon(t+1)\, y(t) \to 0 \quad \text{w.p.1 as} \quad N\to\infty.$$

Since

$$\frac{N}{s(N)} > \delta \quad \text{for } N > N_\delta \quad \text{from (3.7b)}$$

this means that

$$\frac{1}{N} \sum_{1}^{N} \varepsilon(t+1)\, y(t) \to 0 \quad \text{w.p.1} \quad \text{as } N\to\infty.$$

This is the first element in the column vector

$$\frac{1}{N} \sum_{1}^{N} \varepsilon(t+1)\, x(t)$$

Repeating the argument, it can be shown that also the other elements of the vector tend to zero w.p.1. Hence, the term

$$V_N^{(2)}(\theta) = \frac{1}{N} [\theta_o - \theta]^T \sum_1^N \varepsilon(t+1)\ x(t)$$

tends to zero uniformly in θ, w.p.1.

As the second step in the proof it will now be shown that

$\forall\varepsilon$, $\exists\ N_o$ (that depends on the realization) and δ_1 such that
if $N > N_o(\omega)$, then $|\theta - \theta_o| > \varepsilon \Rightarrow V_N^{(3)}(\theta) > \delta_1$ (3.8)

If (3.8) is true, then

$V_N(\theta) > V_N(\theta_o) + \delta_1/2$ (where $V_N(\theta_o) = V_N^{(1)}$) for $|\theta - \theta_o| > \varepsilon$ and $N > N(\omega)$. Since $\theta(N)$ minimizes $V_N(\theta)$, this implies that $\theta(N) \to \theta_o$ w.p.1 as $N\to\infty$, i.e. the assertion of the theorem.

Suppose that (3.8) is not true. Then there exists a sequence $\{\tilde{\theta}_N\}$ such that

$$C > |\tilde{\theta}_N| > \delta$$

and

$$V_{N_i}(\theta_o + \tilde{\theta}_{N_i}) = \frac{1}{N_i} \sum_1^{N_i} [\tilde{\theta}_{N_i}^T\ x(t)]^2 \to 0 \text{ as } i\to\infty \text{ for a subsequence}$$

$\{\tilde{\theta}_{N_i}\}$.

Let $\bar{\theta}$ be a cluster point to this subsequence. Then with

$$R_N = \frac{1}{N} \sum_1^N x(t)x(t)^T$$

$$V_{N_i}^{(3)}(\theta_o + \tilde{\theta}_{N_i}) = \bar{\theta}^T R_{N_i} \bar{\theta} - 2(\bar{\theta} - \tilde{\theta}_{N_i})^T R_{N_i} \bar{\theta} + (\bar{\theta} - \tilde{\theta}_{N_i})^T R_{N_i} (\bar{\theta} - \tilde{\theta}_{N_i})$$

But R_N is bounded according to (3.7b), $|\theta - \tilde{\theta}_{N_i}|$ tends to zero along a subsequence and $\bar{\theta}^T R_{N_i} \bar{\theta} > \delta_3$ according to (3.7d).

Hence $V_{N_i}^{(3)}(\theta_o+\theta_{N_i})$ cannot tend to zero and (3.8) follows.

This concludes the proof of the theorem. □

Remark: In case k > 0, the sum $V_N^{(2)}$ has to be split up into k sums:

$$V_N^{(2)}(\theta) = \frac{1}{N} \sum_{r=0}^{k-1} \sum_{t=1}^{N} \varepsilon(kt+r+1)[\theta_o-\theta]^T x(kt+r)$$

in order to apply the martingale theorem.

In case (3.7d) does not hold, the estimates may converge to several different limits depending on the realization. The set of possible convergence points is characterized in the following corollary, which is obtained by a slight modification of the proof of the theorem.

Corollary: Suppose (3.7abc) holds. Define the set

$$D_I = \{\theta \mid \lim_{N\to\infty} \inf \frac{1}{N} \sum_{1}^{N} [(\theta-\theta_o)^T x(t)]^2 = 0\}$$

(which depends on the realization ω).

Then $\theta(N) \to D_I$ w.p.1.[†]

Furthermore,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{1}^{N} [\theta(N)^T x(t)]^2 = 0 \quad \text{w.p.1}$$

□

Remark. The corollary can be seen as a special case of Theorem 5.1 in Ljung (1974b).

[†] By this it is meant that $\inf_{\theta \in D_I(\omega)} |\theta(N,\omega) - \theta| \to 0$ as $N \to \infty$ a.e. ω.

3.2 Self-tuning regulators.

In this section Theorem 3.1 is applied to the self-tuning regulator STURE1, described in Section 2.5. According to the corollary of this theorem,

$$\theta(N) \rightarrow D_I \quad \text{w.p.1 as } N\rightarrow\infty$$

It will be shown that all elements in D_I actually give the desired minimum variance control law. That is, if $\tilde{\theta}\in D_I$, then the feedback law (let $\beta_o = 1$)

$$u(t) = -\tilde{\theta}^T x(t)$$

gives the output

$$y(t) = \varepsilon(t)$$

Since

$$y(t+1) = \theta_o^T x(t-k) + u(t-k) + \varepsilon(t+1)$$

this implies that

$$(\theta_o-\tilde{\theta})^T x(t) = 0 \quad \text{all } t.$$

Consider a feedback law

$$F(q^{-1})\, u(t) = G(q^{-1})\, y(t)$$

where

$$F(z) = 1 + f_1 z + \dots + f_{\hat{m}} z^{\hat{m}}$$

$$G(z) = g_1 + g_2 z + \dots + g_{\hat{n}} z^{\hat{n}}$$

and q^{-1} is the backward shift operator.

If the polynomials F(z) and G(z) have common factors, this feedback law will generate x(t)-vectors that lie in a certain subspace of $R^{\hat{m}+\hat{n}}$. Let the subspace corresponding to the minimum variance control law be denoted by H. The dimension of this subspace is $\hat{m} - m' + \hat{n} - n$.

We will by a somewhat heuristic argument show that

$$\tilde{\theta} \in D_I \Rightarrow (\theta_0 - \tilde{\theta}) \perp H$$

This implies that all elements in D_I give the minimum variance control law, since

$$u(t) = \tilde{\theta}^T x(t) = \theta_0^T x(t) + (\tilde{\theta} - \theta_0)^T x(t) = \theta_0^T x(t)$$

and

$$y(t) = \varepsilon(t)$$

for $x(t) \in H$.

Suppose first that $\theta(N)$ gives the minimum variance control law for some N. The produced x(t) then belong to H and the obtained estimates $\theta(k)$, $k=N,\ldots$ must then satisfy

$$(\theta(k) - \theta_0) \perp H$$

since, according to the corollary of Theorem 3.1

$$\frac{1}{k} \sum_{t=1}^{k} [\theta(k)^T x(t)]^2 \to 0 \text{ as } k \to \infty$$

These estimates consequently also give the minimum variance control law, which shows that minimum variance control will continue if it once has started.

Suppose now that the estimates $\theta(N_k)$ tend to a point θ^* that does not give minimum variance control. Then the produced $x(t)$ are not orthogonal to $\theta_0 - \theta^*$, and the obtained new estimates will move away from θ^*. When doing so, the produced $x(t)$ will eventually span either the whole space or H. In both cases an estimate that gives minimum variance control results, and according to the discussion above, it will continue thereafter.

Hence, if $\hat{n} \geq n$, $\hat{m} \geq m'$, D_I consists of parameters which give the desired minimum variance control law, so even if the estimates do not converge to their true values, they will still give the desired controller. It also follows that if $\hat{n} = n$, $\hat{m} = m'$, the estimates will actually converge to the true parameter θ_0.

4. TOOLS FOR CONVERGENCE ANALYSIS

4.1 Background.

In Chapter 3 the self-tuning regulator STURE1 was analysed in case the assumptions about orders and noise characteristics were true. It was remarked in example 2.1 that this self-tuning regulator has desired behaviour also in some cases when the assumptions are not satisfied. The analysis of these cases cannot be formulated as consistency questions for the identification. This is clear, since there no longer are any "true" parameter values and no consistent estimates.

When the LS noise condition is not satisfied, the estimates will in general be biased. The bias depends on the feedback law. The effect is clearly seen from the following example:

Example 4.1 Consider the system

$$y(t+1) + ay(t) = u(t) + e(t+1) + ce(t) \tag{4.1}$$

where $\{e(t)\}$ is white noise with variance λ and $|c| < 1$. The following model is assumed:

$$y(t+1) + \hat{\alpha}y(t) = u(t) + \varepsilon(t+1) \tag{4.2}$$

It is straightforward to show that if u(t) is chosen as white noise with variance μ, independent of e, then the LS estimate of $\hat{\alpha}$ tends to

$$\hat{\alpha} = a - \frac{c\lambda(1-a^2)}{\lambda(1+c^2-2ac)+\mu} \tag{4.3}$$

On the other hand, if u(t) is determined as output feedback,

$$u(t) = g\, y(t)$$

then the LS estimate of $\hat{\alpha}$ tends to

$$\hat{\alpha} = a - \frac{c\,(1 - (a-g)^2)}{1 + c^2 - 2(a-g)\,c} \tag{4.4}$$

For the self-tuning regulator, STURE1, the feedback coefficient g at time t is chosen as the current estimate of α. It is thus time varying, which makes it difficult to analyse the behaviour of the algorithm. In Åström - Wittenmark (1973) an attempt is made to heuristically analyse it when applied to the system (4.1). The feedback coefficient is assumed to be fixed $= \hat{\alpha}_k$ over a long time period. During this period the estimated of α converges to

$$\hat{\alpha}_{k+1} = a - \frac{c(1 - (a-\hat{\alpha}_k)^2)}{1 + c^2 - 2(a-\hat{\alpha}_k)\,c} \tag{4.5}$$

which is taken as the next feedback coefficient, etc. It is then argued that if (4.5) converges to the desired regulator ($\hat{\alpha} = a-c$), this should be taken as an indication of convergence of the self-tuning regulator.

In this heuristic analysis the important feature, that the feedback coefficient actually changes in every step, is neglected. To include it, consider the change of the estimate over just one step, instead of over a very long time period. The estimate of α at time t, $\hat{\alpha}_t$ is given by (2.15).

$$\hat{\alpha}_{t+1} = \hat{\alpha}_t + \frac{y(t)}{\sum_1^t y^2(k)} \{y(t+1) + \hat{\alpha}_t y(t) - u(t)\}$$

Since u(t) is chosen as $\hat{\alpha}_t y(t)$ we have

$$\hat{\alpha}_{t+1} - \hat{\alpha}_t = \frac{y(t)y(t+1)}{\sum_1^t y^2(k)} = \frac{1}{\frac{1}{t}\sum_1^t y^2(k)} \cdot \frac{1}{t}\, y(t)y(t+1) \tag{4.6}$$

The first factor is disregarded in this intuitive discussion. Consider $Ey(t)y(t+1)$. This value of the covariance function depends on the feedback coefficient, since the closed loop behaviour is affected by the feedback. The expectation exists only if the closed loop system is stable. Let the feedback be

$$u(t) = \alpha y(t)$$

and denote

$$E\, y(t)y(t+1) = f(\alpha)$$

In (4.6) the difference $\hat{\alpha}_{t+1} - \hat{\alpha}_t$ tends to zero as t tends to infinity. Hence an increasing number of sample points, say T, are required to change α a given small distance. Let $\Delta\tau = \sum_{t}^{t+T} 1/k$. The change is caused by a large number of random variables $y(t)y(t+1)$, which all have approximately the mean value $f(\hat{\alpha}_t)$. It is reasonable to assume that due to some "law of large numbers," the change is proportional to $f(\hat{\alpha}_t)$:

$$\hat{\alpha}_{t+T} = \hat{\alpha}_t + \Delta\tau\, f(\hat{\alpha}_t)$$

This scheme can be seen as an approximation to the ordinary differential equation (ODE):

$$\frac{d}{d\tau}\hat{\alpha} = f(\hat{\alpha}) \qquad (4.7)$$

□

From the example it seems plausible that the trajectories of (4.7) in some sense describe the sequence of estimates. In fact, in Section 4.2 it is shown that stability of (4.7) implies convergence of the algorithm. In Section 4.3 it is shown that the trajectories of (4.7) actually can be interpreted as "expected paths" for the sequence of estimates. The results are shown for the general linear self-tuning regulator (2.15) with (2.15c). The regulator is treated as a general recursive algorithm:

$$\theta(t+1) = \theta(t) + \gamma(t)\ Q(t,\ \theta(t),\ \ldots,\ \theta(0),\ e(t+1)) \qquad (4.8)$$

Similar convergence results for such algorithms are shown in Ljung (1974).

4.2 Convergence.

Consider the class of self-tuning algorithms (2.15) with linear feedback (2.15c). Some additional assumptions about the noise, the gain sequence $\{\gamma(t)\}$ and the closed loop behaviour are first introduced.

Introduce

$$f(\theta) = E\ x(t)\ [y(t+\hat{k}+1) - \theta^T x(t) - \hat{\beta}_0 u(t)] \qquad (4.9)$$

$$G_1(\theta) = E\ x(t)x(t)^T$$

$$G_2(\theta) = E\ x(t)^T x(t) \qquad (4.10)$$

where the expectation shall be taken, assuming that the system is regulated by the time invariant feedback law

$$u(t) = F(\theta)\ x(t)$$

It shall be assumed that the input and output sequences have reached stationarity, i.e. effects of initial values are neglected. This statement deserves some discussion. In the algorithm (2.15) strict stationarity for the input and output sequences is never achieved, mainly due to the time varying feedback. The expected change in the variables actually depends on all previous feedback laws. It would be quite impossible to calculate the expectation values in (4.9), (4.10) taking such dependences into account. It is therefore a significant result if these effects can be neglected. The functions f and G_i are simple functions of certain covariances.

Stationarity can be achieved, and hence the functions f and G_i defined, only if the closed loop system obtained with $u(t) = F(\theta)\, x(t)$ is stable. Therefore a condition that assures that the closed loop system is not unstable all the time must be introduced:

> The feedback regulator is such that there w.p.1 exists a subsequence N_k, (which may depend on the realization) such that $\theta(N_k)$ belongs to a closed subset of the area that gives stable closed loop systems, and such that $|x(N_k)|$ is bounded. If the area which gives stable closed loop systems is unbounded, it is assumed that the estimates are prevented from tending to infinity by some suitable projection algorithm, cf. Ljung (1974) Chapter 5. (4.11)

The convergence of the algorithm (2.15) also depends on the sequence $\{\gamma(t)\}$ and on the noise $\{\varepsilon(t)\}$. Assume that $\varepsilon(t)$ is obtained as filtered white noise:

$$\varepsilon(t) = \frac{C(q^{-1})}{D(q^{-1})}\, e(t) \tag{4.12}$$

where $\{e(t)\}$ is white noise, C(z) and D(z) are polynomials and q^{-1} is the backward shift operator. The polynomial D(z) is assumed to have all zeroes outside the unit circle.

Further assume that

$$E|\varepsilon(t)|^{4p} \leq C_1 \qquad p \text{ integer} \tag{4.13}$$

The sequence $\{\gamma(t)\}$ is taken as

$$\gamma(t) = c_\gamma t^{-s} \qquad 1/p < s \leq 1 \qquad c_\gamma > 0, \tag{4.14}$$

Theorem 4.1 Consider the algorithm (2.15) with linear feedback (2.15c). Suppose that the feedback law $F(\theta)$ is Lipschitz continuous and such that the stability condition (4.11) is satisfied. Let (4.12) - (4.14) hold for the noise and for the gain sequences. Let $f(\theta)$, $G_1(\theta)$ and $G_2(\theta)$ be defined by (4.9), (4.10). Consider the ODE

$$\frac{d}{d\tau}\,\theta(\tau) = S_i(\tau)f(\theta(\tau)) \tag{4.15a}$$

$$\frac{d}{d\tau}\,S_i(\tau) = S_i(\tau) - S_i(\tau)G_i(\theta(\tau))S_i(\tau) \tag{4.15b}$$

where for the LS algorithm (2.15e) $i = 1$, and for the SA algorithm (2.15f) $i = 2$. Assume that it has a stationary point (θ^*,S^*) that is globally asymptotically stable.[†] Then the solution of (2.15), $\theta(t)$, tends to θ^* w.p.1 as t tends to infinity.

Remark: Notice that for the SA algorithm, S_2 is a positive scalar. Then, instead of (4.15), it is sufficient to require that the ODE

$$\frac{d}{d\tau}\,\theta(\tau) = f(\theta(\tau)) \tag{4.16}$$

is asymptotically stable. Notice also that (4.15b) can be written

$$\frac{d}{d\tau}\,S_i^{-1}(\tau) = G_i(\theta(\tau)) - S_i^{-1}(\tau)$$

Proof: The theorem is proved quite analogously to theorems 3.1 and 4.1 in Ljung (1974).

[†] In the region where S_i is strictly positive (definite).

The proof is technically involved. The basic idea is however, simple. The idea is that the sequence $\{\theta(t), S_i(t)\}$ behaves like solutions to the ODE (4.15). In the proof the intuitive arguments on page 26 are formalized. The following lemma characterizes the local behaviour of the estimate sequence and a connection with the ODE (4.15) is established.

Lemma 4.1 Suppose $\theta(n)$ and $\bar{\theta}$ belong to the area where f and G_i are defined. Let $m(n,\Delta\tau)$ satisfy

$$\sum_{n}^{m(n,\Delta\tau)} \gamma(k) \to \Delta\tau \quad \text{as } n\to\infty$$

Suppose that $|x(n)| < C$ (C may depend on the realization). Then for sufficiently small $\Delta\tau$ and $(\theta(n),S_i(n))$ sufficiently close to $(\bar{\theta},\bar{S}_i)$

$$\theta(m(n,\Delta\tau)) = \theta(n) + \Delta\tau\bar{S}_i f(\bar{\theta}) + R_1(n,\Delta\tau,\bar{\theta},\bar{S}_i) + R_2(n,\Delta\tau,\bar{\theta},\bar{S}_i) \tag{4.17}$$

$$S_i(m(n,\Delta\tau)) = S_i(n) + \Delta\tau[-\bar{S}_i G_i(\bar{\theta})\bar{S}_i + \bar{S}_i] + R_1'(n,\Delta\tau,\bar{\theta},\bar{S}_i) +$$

$$+ R_2'(n,\Delta\tau,\bar{\theta},\bar{S}_i)$$

where

$$|R_1^{(')}(n,\Delta\tau,\bar{\theta},\bar{S}_i)| \leq \Delta\tau\cdot K\{|\theta(n) - \bar{\theta}| + |S_i(n) - \bar{S}_i|\} + A(\Delta\tau)^2$$

and

$$R_2^{(')}(n,\Delta\tau,\bar{\theta},\bar{S}_i) \to 0 \text{ w.p.1 as } n\to\infty. \qquad \square$$

The lemma is proved in appendix A. It implies that for large n, $\theta(n)$ will follow the ODE (4.15) locally. Consider from now on the SA algorithm and assume that (4.16) is asymptotically stable. The LS case is treated analogously.

From Krasovskij (1963) the existence of a Lyapunov function $V(\theta)$ for the ODE (4.16) is inferred. The function V is infinitely differentiable, positive definite and has a negative definite time derivative along solutions to (4.16). It is readily shown that (4.17) implies that, for sufficiently small $\Delta\tau$ and large n, and $\theta(n)$, $S_2(n)$ sufficiently close to $\bar{\theta}$, $\bar{S}_2$ we have

$$V[\theta(m(n,\Delta\tau))] < V(\bar{\theta}) - \Delta\tau\bar{S}_2\delta/2 \tag{4.18}$$

where

$$-\delta = \frac{d}{d\tau} V(\theta(\tau))\Big|_{\theta=\bar{\theta}} = V'(\bar{\theta})\, f(\bar{\theta})$$

Consider from now on a fixed realization ω. In order to use eq. (4.17) a sequence $\{\theta(n_k,(\omega)\}$ tending to $\bar{\theta}(\omega)$ as n_k tends to infinity will be considered. The existence of such a sequence follows from (4.11). The argument ω is suppressed in the sequel. Now, applying (4.17) and (4.18) to the sequence $\{\theta(n_k)\}$ gives, cf. Ljung (1974)

$$V(\theta(m(n_k;\Delta\tau))) < V(\bar{\theta}) - S_2(n_k)\Delta\tau\delta/64$$

It is quite clear that $S_2(n_k)$ is, possibly after extraction of a new subsequence n_k, bounded from below by a positive constant. Hence

$$V[\theta(m(n_k,\Delta\tau))] < V(\bar{\theta}) - \Delta\tau\cdot\delta'$$

In lemmas A.2 and A.3 in Ljung (1974) it is formally shown that the inequality above, which holds for any clusterpoint $\bar{\theta} \neq \theta^*$, and sufficiently large n_k implies that $\theta(n) \to \theta^*$ as $n\to\infty$ for the chosen realization. This holds for almost every realization (cf. Ljung (1974)) and the theorem follows. □

Let us apply the theorem to the system in Example 4.1, governed by the self-tuning regulator STURE1. Clearly, $F(\theta) = -\theta$ is Lipschitz continuous. It is quite straightforward to show that (4.11) is satisfied, see Section 5.1. Let the noise $\{\varepsilon(t)\}$ be normally distributed and take $\gamma(t) = t^{-1}$. Then (4.12), (4.14) are satisfied. Let the feedback be $u(t) = \alpha y(t)$. The function

$$f(\alpha) = E\, y(t)y(t+1)$$

is then easily calculated and the ODE (4.16) is

$$\dot{\alpha} = -\frac{(c-a+\alpha)\ (1-c(a-\alpha))}{1-(a-\alpha)^2} \qquad \text{defined for } |\alpha-a| < 1 \qquad (4.19)$$

With $z = \alpha - a + c$

$$\dot{z} = -z\,\frac{1-c(c-z)}{1-(c-z)^2} \qquad \text{defined for } |z-c| < 1$$

Clearly, the solution $z^* = 0$ is globally asymptotically stable. It now follows from the theorem that

$$\hat{\alpha}(t) \rightarrow a-c \quad \text{as } t\rightarrow\infty$$

which gives the minimum variance control law $u(t) = (a-c)\, y(t)$. In Chapter 6 more general systems are analysed.

The theorem is formulated for linear feedback laws. In a number of applications the input is limited. If, in such a case, the open loop system is stable, then (4.11) is trivially satisfied and the ODE's are defined everywhere. This kind of nonlinear regulator requires some minor modifications of the proof of Theorem 4.1. Limitation of the input signals naturally affects the function $f(\theta)$. Therefore limitation may affect both convergence and limit point.

It is relevant to ask what connection Theorem 4.1 has to stochastic Lyapunov functions. These have been discussed e.g. by Kushner (1967). A stochastic Lyapunov function is a positive supermartingale. It is assumed that the process for which convergence shall be shown (i.e. $\{\theta(t)\}$ in the present case) is a Markov process. However, for the present application $\{\theta(t)\}$ is not a Markov process. Also, a Lyapunov function for (4.15) is generally not a stochastic Lyapunov function for the sequence of estimates. This means that it is more difficult to find a stochastic Lyapunov function for the sequence of estimates than to show stability for (4.15). This is illustrated in the following simple example.

Example 4.2 Consider the algorithm

$$c_{n+1} = c_n + \gamma_n(e_n - c_n); \quad \gamma_n = 1/n$$

where $\{e_n\}$ is a sequence of independent, random variables with zero mean values and unit variances. This example is much simpler than the algorithms considered in Theorem 4.1. It is readily shown that the corresponding ODE is

$$\dot{c} = -c$$

the stability of which easily is shown, e.g. by means of the Lyapunov function $V(c) = 1/2\ c^2$. However, $V(c)$ is not a stochastic Lyapunov function for c_n since

$$E\{V(c_{n+1}) - V(c_n)\ |c_n, \ldots, c_o\} = -1/n\ c_n^2 + \frac{1}{n^2}(c_n^2+1)/2$$

The RHS is greater than zero for $c_n = 0$.

4.3 Behaviour of the algorithm.

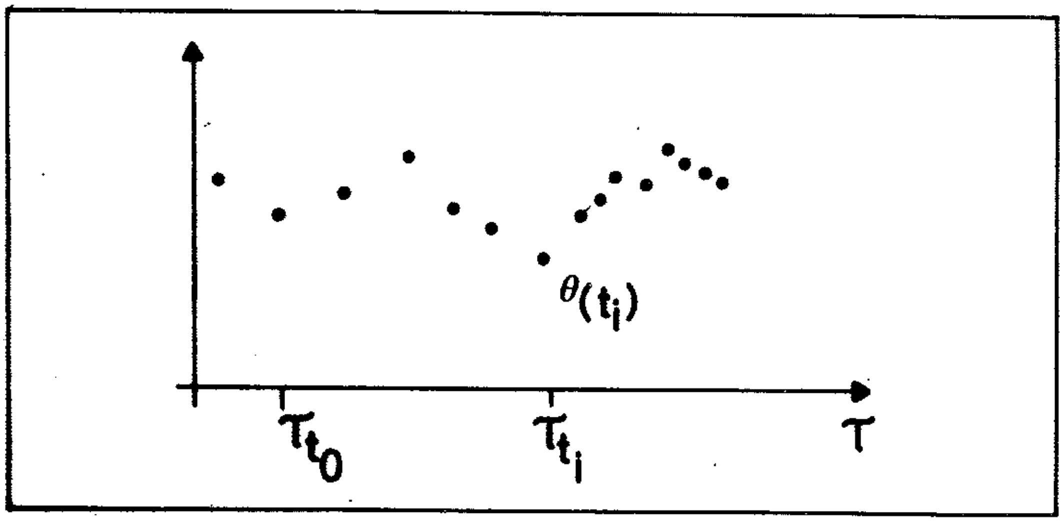
The ODE (4.15) is important not only for the question of convergence. It can, in fact, be shown that the trajectories of (4.15) also govern the behaviour of the sequence of estimates

$\{\theta(t)\}$ obtained from (2.15).

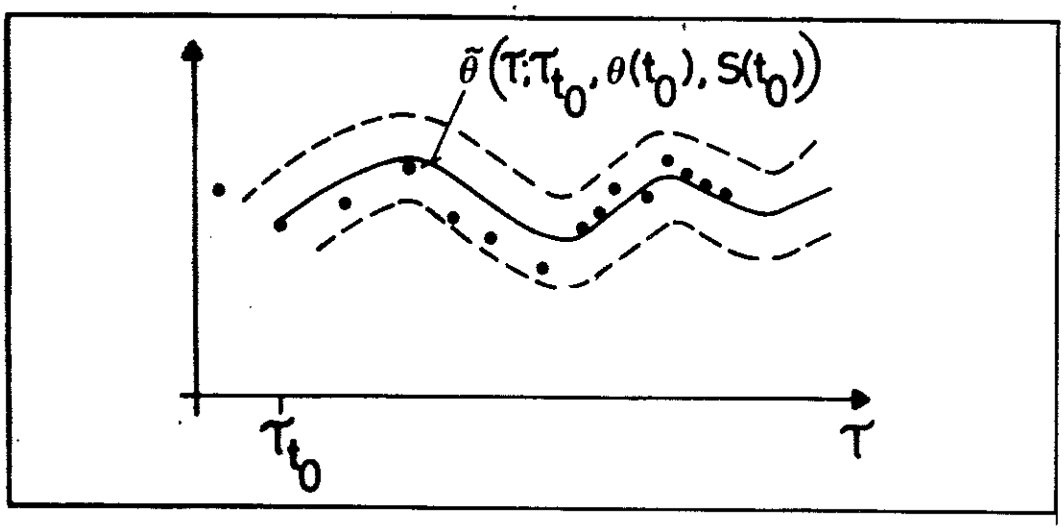
The result is formulated as follows. Let $\{\theta(t)\}$ be generated by (2.15). Introduce a fictitious time τ by

$$\tau_n = \sum_1^n \gamma(t)$$

Suppose that the estimates $\theta(t)$ are plotted against τ_t:



Let $\tilde{\theta}(\tau;\ \tau_{t_o},\ \theta(t_o),\ S(t_o))$ be the solution of (4.15) with initial value $\theta(t_o)$, $S(t_o)$ at time τ_{t_o}. Plot also this solution in the same diagram:



Let I be a set of integers. The probability that all points $\theta(t)$; $t\in I$ simultaneously are within a certain distance ε from the trajectory is estimated in the following theorem:

Theorem 4.2 Consider algorithm (2.15) with the same conditions on $\{e(k)\}$ and $\{\gamma(k)\}$ as in Theorem 4.1. Denote $\tau_t = \sum_1^t \gamma(s)$. Assume that the right hand side of (4.15) is continously differentiable. Denote the solution of (4.15) with initial condition $\theta(t_o)$, $S(t_o)$ at $\tau = \tau_{t_o}$ by $\tilde{\theta}(\tau;\tau_{t_o}, \theta(t_o), S(t_o))$. Consider the ODE (4.15) linearized around this solution. Suppose that there exists a quadratic Lyapunov function for this linear, time varying ODE. (See e.g. Brockett (1970)). Let I be a set of integers, such that $\inf |\tau_i - \tau_j| = D > 0$ where $i \neq j$ and $i,j\in I$. Then there exists a K, δ_o and ε_o, such that for $\varepsilon < \varepsilon_o$, $|\gamma(t_o)x(t_o)| < \delta_o$.

$$P\{\sup_{\substack{t\in I\\ t>t_o}} |\theta(t) - \tilde{\theta}(\tau_t;\tau_{t_o}, \theta(t_o), S(t_o))| > \varepsilon\} \leq$$

$$\leq \frac{K^r}{\varepsilon^{4r}} \sum_{j=t_o}^{N} \gamma(j)^r \qquad r \leq p \tag{4.20}$$

where $N = \sup_{t\in I} t$, which may be ∞. □

The proof is based on Lemma 4.1 and follows from this lemma in exactly the same way as Theorem 6.1 in Ljung (1974) is proved.

Since the sum

$$\sum_1^{\infty} \gamma(j)^p$$

is convergent, the RHS of (4.20) can, for fixed ε, be chosen arbitrarily small by taking t_o sufficiently large. Thus, the theorem states that the trajectories of the ODE (4.15) arbitrarily well describe the behaviour of the algorithm (2.15) for sufficiently large time points.

It should be remarked that, although the proof of Theorem 4.2 provides an estimate of K, it is not practically feasible to use the theorem to obtain numerical bounds for the probability. The estimates are too crude. The main value of the theorem is that a basic relationship between the trajectories and the algorithm is established.

To summarize, Theorems 4.1 and 4.2 state that analysis of the time invariant, deterministic ODE (4.15) gives valuable insight into the behaviour of the time varying, nonlinear stochastic difference equation (2.15). In Chapter 6 the ODE's that correspond to the self-tuning algorithms, STURE0 and STURE1, are derived and analysed.

5. STABILIZATION PROPERTIES

The self-tuning regulators STURE1 and STURE0, were defined in Section 2.5. They were originally, Wieslander - Wittenmark (1971), designed for control of system (2.1) when the LS noise condition is satisfied. The analysis in Chapter 3 shows that the regulators have desired behaviour in this case. If the noise has a more general structure, the parameter estimates will be biased. The bias depends on the control law and this in turn depends on the current estimates. This makes it quite difficult to follow the estimation process. The performance outside the stability region of the closed loop system is considered in this chapter.

To apply Theorem 4.1 it is required that the stability condition (4.11) is satisfied. In Section 5.1 it is shown that both self-tuning regulators have this desired property for quite general noise sequences. The time delay k must be known, and the orders of the system must not be underestimated.

In Section 5.2 a stronger result is shown for STURE1. It is shown that

$$\frac{1}{N}\sum_{k=1}^{N} y(k)^2$$

is uniformly bounded in N w.p.1. This result ensures a stable behaviour of the closed loop system. This holds also if the open loop system is unstable. Thus STURE1 stabilizes any system, provided the time delay is known.

5.1 A general stability property.

The stability properties of the self-tuning regulators, STURE1 and STURE0, are investigated in this section. It is shown that these regulators satisfy the condition (4.11). To make the discussion easier to follow, some of the arguments are kept on a somewhat heuristic level. It should, however, meet no difficulties to convert the discussion into a formal proof.

Consider the system (2.3) and the model (3.4). Assume that $\hat{k} = k$, and that $\hat{m} \geq m'$, $\hat{n} \geq n$. For simplicity in the formal treatment, the time delay k is supposed to be zero in this section.

Form the following vector from the parameter values α_i, β_i and the estimates $\hat{\alpha}_i(t)$, $\hat{\beta}_i(t)$

$$\tilde{\theta}(t) = [\alpha_1 - \frac{\beta_o}{\hat{\beta}_o}\hat{\alpha}_1(t), \ldots, \alpha_n - \frac{\beta_o}{\hat{\beta}_o}\hat{\alpha}_n(t), \ldots, -\frac{\beta_o}{\hat{\beta}_o}\hat{\alpha}_{\hat{n}}(t),$$

$$, \frac{\beta_1}{\hat{\beta}_o} - \frac{\beta_o}{\hat{\beta}_o}\hat{\beta}_1(t), \ldots, \frac{\beta_{m'}}{\hat{\beta}_o} - \frac{\beta_o}{\hat{\beta}_o}\beta_{m'}(t), \ldots, -\frac{\beta_o}{\hat{\beta}_o}\hat{\beta}_{\hat{m}}(t)]^T$$

Then, with $u(t) = -\frac{1}{\hat{\beta}_o}\theta(t)^T x(t)$, (2.15a) and (2.3) can be rewritten as

$$\tilde{\theta}(t+1) = \tilde{\theta}(t) - \gamma(t+1)\, S(t)\, x(t)[x(t)^T \frac{\beta_o}{\hat{\beta}_o}\tilde{\theta}(t) - \varepsilon(t+1)] \quad (5.1)$$

$$y(t+1) = \tilde{\theta}(t)^T x(t) + \varepsilon(t+1) \quad (5.2)$$

The sequence $\varepsilon(t)$ is supposed to satisfy

$$\frac{1}{N}\sum_{t=1}^{N} \varepsilon(t)^2 < K \qquad N > N_o \quad (5.3)$$

where K may depend on the realizaticn. For STURE1

$$S(t) = \frac{P(t)}{1 + \gamma(t+1)[x(t)^T P(t) x(t) - 1]}$$

where P(t) is given by (2.15d). For STUREO

$$S(t) = \frac{1}{\sum_{s=1}^{t} \xi_s^t x(s)^T x(s)}$$

where

$$\xi_s^t = \gamma(s) \prod_{j=s+1}^{t} (1-\gamma(j)); \quad \xi_t^t = \gamma(t)$$

Decompose $\tilde{\theta}(t)$ into one component parallell to x(t), denoted by $\tilde{\theta}_{\parallel}(t)$ and one orthogonal to x(t), $\theta_{\perp}(t)$:

$$\tilde{\theta}(t) = \tilde{\theta}_{\parallel}(t) + \tilde{\theta}_{\perp}(t) \tag{5.4}$$

The symmetrical matrix $x(t)\, x(t)^T$ has all eigenvalues but one equal to zero. The non-zero eigenvalue is $x(t)^T\, x(t)$, and the corresponding eigenvector is x(t). Hence

$$x(t)\, x(t)^T\, \tilde{\theta}(t) = x(t)^T\, x(t)\, \tilde{\theta}_{\parallel}(t)$$

Introduce also

$$\lambda(t) = \frac{\gamma(t+1)\; x(t)^T\; x(t)}{\sum_{1}^{t} \xi_s^t x(s)^T\, x(s)} \tag{5.5}$$

Clearly

$$0 \leqslant \lambda(t) \leqslant 1 \tag{5.6}$$

Eq. (5.1) can now be written

$$\tilde{\theta}(t+1) = \tilde{\theta}(t) - \lambda(t) \frac{\beta_o}{\hat{\beta}_o} \tilde{\theta}_{\parallel}(t) + \gamma(t+1)S(t)x(t)\varepsilon(t+1) \tag{5.7}$$

When $|x(t)|$ is sufficiently large, say $\geq K'$, the last term in (5.7) can be neglected, and the following relation is obtained:

$$|\tilde{\theta}(t+1)|^2 \approx |\tilde{\theta}(t)|^2 - 2\lambda(t)\frac{\beta_o}{\hat{\beta}_o}|\tilde{\theta}_{\parallel}(t)|^2 + \lambda(t)^2\frac{\beta_o}{\hat{\beta}_o}^2|\tilde{\theta}_{\parallel}(t)|^2 \tag{5.8}$$

If $0 < \frac{\beta_o}{\hat{\beta}_o} < 2$, it follows from (5.8) and (5.5) that

$$|\tilde{\theta}(t+1)|^2 \leq |\tilde{\theta}(t)|^2 - c\lambda(t)|\tilde{\theta}_{\parallel}(t)|^2 \qquad c > 0 \tag{5.9}$$

From (5.2) follows that for large $|x(t)|$

$$|y(t+1)|^2 \approx |\tilde{\theta}_{\parallel}(t)|^2\,|x(t)|^2 \leq |\tilde{\theta}_{\parallel}(t)|^2 \cdot \max_{t\leq s\leq t-\hat{n}}(y(s)^2, u(s)^2)\cdot 4\hat{n}^2 \tag{5.10}$$

Assume that the system is minimum phase, i.e. that the polynomial

$$B(z) = \beta_o + \beta_1 z + \beta_2 z^2 + \dots + \beta_{m'} z^{m'}$$

has all its zeroes outside the unit circle. Then $|u(t)|$ is not significantly much larger than $|y(t)|$.

Assume now that (4.11) does not hold, i.e. that the feedback

$$u(t) = -\frac{1}{\hat{\beta}_o}\,\theta(t)^T x(t)$$

gives an unstable, closed loop system for all $t > N$. We will lead this assumption into contradiction. Distinguish between the following two cases:

Case a.

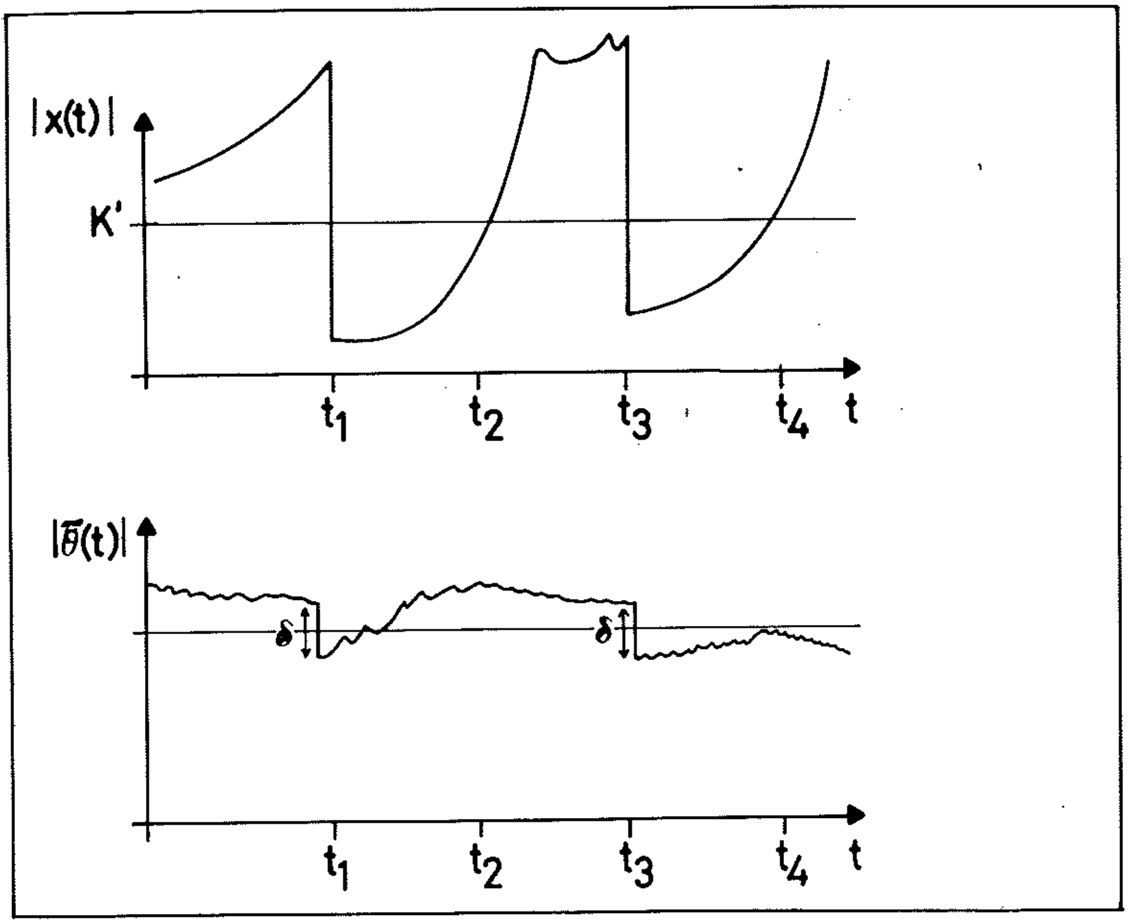
$$\limsup_{t\to\infty} |\tilde{\theta}(t+1) - \tilde{\theta}(t)| = 0$$

In this case the feedback law and hence the parameters of the closed loop system change arbitrarily slowly as t increases. Consequently $|x(t)|$ increases exponentially for sufficiently large t. This implies that (5.9) holds, and also that $\lambda(t)$ is bounded from below by a strictly positive number. From (5.9) now follows that $|\tilde{\theta}_{\parallel}(t)| \to 0$, which from (5.10) implies that $|y(t)|$ and $|x(t)|$ start to decrease when $|\tilde{\theta}_{\parallel}(t)|$ is sufficiently small. This contradicts the exponential increase of $|x(t)|$.

Case b.

$$\limsup_{t\to\infty} |\tilde{\theta}(t+1) - \tilde{\theta}(t)| = \delta > 0$$

According to (5.7) such "jumps" in the estimates are possible only if $|x(t)|$ assumes arbitrarily large values. Therefore (5.9) is valid as soon as the estimate jumps. A jump may cause $|x(t)|$ to decrease drastically. In fact, it is the only way for $|x(t)|$ to decrease if the closed loop system is unstable. Consider the following figure



The only possibility for $|\tilde{\theta}(t)|$ to increase is when (5.9) is not valid, i.e. when $|x(t)| < K'$. Now, each such period must be preceded by a jump; $|\tilde{\theta}(t+1)| \leq |\tilde{\theta}(t)| - \delta$. Also, the length of a period when $|x(t)| < K'$ is essentially bounded by a fixed length. The unstable modes are excited by the noise terms and $|x(t)|$ quickly starts to increase. During such a period $|\tilde{\theta}(t+1) - \tilde{\theta}(t)|$ is arbitrarily small, and the possible increase in $|\tilde{\theta}(t)|$ becomes eventually less than $\delta/2$. Hence, it follows that $|\tilde{\theta}(t)|$ decreases with the net amount of at least $\delta/2$ infinitely many times which, of course, is impossible.

Consequently the assumption that $\theta(t)$ belongs to the area which gives unstable, closed loop systems for all $t > N$ is contradicted and (4.11) follows.

The self-tuning regulator STURE1 is treated analogously.

To summarize, the regulators STURE0 and STURE1 satisfy condition (4.11) in case the time delay, k, is known (only the case k = 0 has been treated in this section), and in case the system orders are not underestimated. The estimate $\hat{\beta}_o$ must be so good that

$$0 < \frac{\beta_o}{\hat{\beta}_o} < 2 \qquad (5.11)$$

The process has been assumed to be minimum phase. The noise $\varepsilon(t)$ may be quite general as long as

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{1}^{N} \varepsilon(t)^2 < \infty \qquad \text{w.p.1}$$

It has so far only been shown that (5.11) is a sufficient condition for the stability condition to hold. The following simple example shows that (5.11) is in fact also necessary.

Example 5.1 Consider the system

$$y(t+1) + \alpha y(t) = u(t) + e(t+1)$$

and the model

$$y(t+1) + \hat{\alpha} y(t) = \hat{\beta} u(t) + \varepsilon(t+1)$$

where

$$\hat{\beta} = \frac{1}{2+\delta}$$

Then, with $\gamma(t) = 1/t$ and $\alpha = 0$

$$\hat{\alpha}(t+1) = \hat{\alpha}(t) + \frac{y(t)y(t+1)}{\sum_1^t y(s)^2} = \hat{\alpha}(t) - \frac{(2+\delta)\hat{\alpha}(t)y(t)^2}{\sum_1^t y(s)^2} + \frac{y(t)e(t+1)}{\sum_1^t y(s)^2}$$

Suppose that $\hat{\alpha}(0) > \frac{2}{\delta}$. Neglect the noise term $y(t)e(t+1)$. Straightforward calculation shows that then $\hat{\alpha}(k)$ alternates between positive and negative values so that $|\hat{\alpha}(k)|$ tends to infinity. Consequently, this system has no stabilization property. □

Remark. Notice that the upper bound on $\beta_o/\hat{\beta}_o$ that is necessary to obtain a stable behaviour depends on several features of the regulator. For example, if $\{\gamma(t)\}$ is chosen as

$$\gamma(t) = \frac{1}{|x(t)|^2} \cdot t^{-s}$$

then the closed loop system will have the stability property (4.11) as soon as $\beta_o/\hat{\beta}_o$ is positive.

Also, if the open loop system is stable and the input to the system is limited, then clearly the overall system is stable and (4.11) holds trivially, irrespectively of $\beta_o/\hat{\beta}_o$.

5.2 Overall stability of STURE 1.

To give an idea of the stabilization property of STURE1, consider first Example 4.1. It was there shown that a white noise input signal with variance μ gives the following estimate of α

$$\hat{\alpha} = a - \frac{c\lambda(1-a^2)}{\lambda(1+c^2-2ac) + \mu}$$

Consequently, the bias depends on the signal to noise ratio. A large variance of the input gives small bias. In a similar manner, the bias depends on the regulator parameter if u(t) is formed as output feedback

$$u(t) = g\, y(t)$$

Then the estimate is

$$\hat{\alpha} = a - \frac{c(1-(a-g)^2)}{1+c^2-2(a-g)c}$$

If the closed loop is almost unstable, i.e. $|a-g|$ is close to 1, then the bias term is small.

The example suggests that unstable, or nearly unstable, closed loop systems give system parameter estimates with insignificant bias. This, in turn, gives a closed loop system with all poles close to the origin. Thus the closed loop system is stable, in the sense that $\frac{1}{N}\sum_{1}^{N} y(t)^2$ cannot increase without limit. This result is formally proved in the following theorem:

Theorem 5.1 Consider self-tuning regulator STURE1 applied to the system (2.3). The sequence $\{\varepsilon(t)\}$ in (2.3) may be any sequence such that

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{1}^{N} \varepsilon(t)^2 < \infty \quad \text{w.p.1} \tag{5.12}$$

Suppose that the time delay, k, is known and the system orders m' and n are not underestimated. β_0 is assumed to be known. Then

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{1}^{N} y(t)^2 < \infty \quad \text{w.p.1} \tag{5.13}$$

If the system is minimum phase, then also

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{1}^{N} u(t)^2 < \infty \quad \text{w.p.1} \tag{5.14}$$

<u>Proof</u>: Introduce $\tilde{\theta}(t)$ as in Section 5.1. Then

$$y(t+k+1) = \tilde{\theta}(t)^T x(t) + \varepsilon(t+k+1) \tag{5.15}$$

Denote

$$r_{yy}(t',t) = \frac{1}{t'-t} \sum_{t+1}^{t'} y(s)^2$$

Then

$$r_{yy}(t',0) = \frac{t}{t'} r_{yy}(t,0) + \frac{t'-t}{t'} r_{yy}(t',t)$$

Suppose that $\limsup_{t\to\infty} r_{yy}(t,0) = \infty$. Then it is possible for arbitrarily large K to find t,t' such that

$$\frac{1}{2} \leq \frac{t}{t'} \leq 1$$

$$K \leq r_{yy}(s,0) \leq c_1 K \qquad t < s < t' \qquad c_1 > 1$$

$$r_{yy}(t',0) \geq c_1 K$$

Then

$$\frac{t}{t'} r_{yy}(t,0) + \frac{t'-t}{t'} r_{yy}(t',t) \geq c_1 K$$

and $r_{yy}(t,0)$ is not a negligible part of $r_{yy}(t',0)$. Not all terms in $r_{yy}(t',t)$ must be larger than c_1K. However, y(t) cannot increase arbitrarily fast, and a number of terms must be larger than K. This number increases as c_1 increases. Choose c_1 such that at least $\hat{n} + \hat{m}$ terms are larger than K. Introduce as in (2.9)

$$R(t) = P^{-1}(t) = \frac{1}{t} \sum_{s=1}^{t} x(s-k-1)\, x(s-k-1)^T$$

The LS criterion (2.1) can then be written

$$V_t(\tilde{\theta}) = \tilde{\theta}^T R(t)\tilde{\theta} + 2\,\frac{1}{t} \sum_{s=1}^{t} \tilde{\theta}^T x(s-k-1)\,\varepsilon(s) + \frac{1}{t} \sum_{s=1}^{t} \varepsilon(s)^2$$

Since $\tilde{\theta} = 0$ is a possible choice, the optimal $\tilde{\theta}(t)$ must yield a value less or equal to

$$V_t(0) = \frac{1}{t} \sum_{s=1}^{t} \varepsilon(s)^2$$

This implies that

$$\tilde{\theta}(t)^T R(t)\tilde{\theta}(t) + 2/t \sum_{s=1}^{t} \tilde{\theta}(t)^T x(s-k-1)\varepsilon(s) \leq 0$$

Hence

$$[\tilde{\theta}(t)^T R(t)\tilde{\theta}(t)]^2 \leq 4/t^2 \left[\sum_{s=1}^{t} \tilde{\theta}(t)^T x(s-k-1)\varepsilon(s)\right]^2 \leq$$

$$\leq 4/t\left[\sum_{s=1}^{t} \tilde{\theta}(t)^T x(s-k-1)x(s-k-1)^T \theta(t)\right]\cdot\left[1/t \sum_{s=1}^{t} \varepsilon^2(s)\right]$$

(Schwarz inequality) or

$$\tilde{\theta}(t)^T \frac{R(t)}{r_{yy}(t,0)} \tilde{\theta}(t) \leq \frac{4}{t} \sum_{s=1}^{t} \varepsilon(s)^2/r_{yy}(t,0) \leq \frac{4K_1(\omega)}{r_{yy}(t,0)} \qquad (5.16)$$

Now take $K \gg K_1$ and choose t,t' as above. Eq. (5.16) then implies that $\tilde{\theta}(s)$ must lie arbitrarily close to the null space of

$$R(s)/r_{yy}(s,0) \text{ for } t \leq s \leq t' \tag{5.17}$$

Since

$$R(s) = \frac{t}{s} R(t) + \frac{s-t}{s} R(s,t) \quad \text{and } \frac{t}{s} \geq 1/2$$

we also have that all $\theta(s)$, $s = t, \ldots, t'$ lie arbitrarily close to the null space of $R(t)/r_{yy}(t,0)$.

From the assumption that (5.13) does not hold, it follows that $y(s)$ is "large" for a number of $s = t, \ldots, t'$. In view of (5.15) this means that $\tilde{\theta}(s)^T x(s)$ also is "large". Since $\tilde{\theta}(s)$ is arbitrarily close to the null space of $R(t)/r_{yy}(t,0)$, $x(s)$ cannot belong to the range space of $R(t)/r_{yy}(t,0)$. (Since the matrix is symmetric the null space and the range space are orthogonal). Consequently

$$\tilde{R}(t',t) = \frac{1}{t'-t} \sum_{t+1}^{t'} x(s)x(s)^T$$

gets a significant contribution from matrices with range space not belonging to the range space of $R(t)/r_{yy}(t,0)$. In other words, the rank of $R(t)/r_{yy}(t,0)$ is less than that of $R(t')/r_{yy}(t',0)$. Repeating the argument $\hat{n} + \hat{m}$ times, it follows that

$$R(t')/r_{yy}(t',0)$$

has full rank, yielding the only possible choice $\tilde{\theta} = 0$ (i.e. the true parameters). This gives $y(t) = \varepsilon(t)$, which contradicts the assumption that $r_{yy}(t,0)$ increases without limit.

If the system is minimum phase, the inverse system is stable. If the input of the inverse system, $y(t)$, satisfies (5.13), then the output, $u(t)$, must satisfy (5.14). □

This stabilization property of STURE1 is an important feature. It implies that the regulator stabilizes the system even if the parameters do not converge.

6. ANALYSIS OF THE SELF-TUNING REGULATOR "STURE".

Theorems 4.1 and 4.2 provide a tool for analysis of the class of self-tuning algorithms defined by (2.15). In this chapter the regulators STURE0 and STURE1 are considered. The basis of the analysis is the ODE defined in Theorem 4.1. In Section 6.1 the ODE's that correspond to the present regulators are determined. These equations are investigated in Section 6.2 under the assumption that the LS noise condition is satisfied.

One important result in Åström - Wittenmark (1973) is that the regulator seems to be equally well behaving also for more general noise structures. That this really might be the case is shown in Section 6.3. There it is proved that the regulator converges to the optimal one in a simple case when the LS noise condition is not satisfied.

In Section 6.4 the differential equations for the general case are linearized around the desired solution. Analysis of the linearized equations shows that they may be unstable, provided the noise has certain properties. Thus, in these cases the self-tuning algorithms will not converge.

6.1 Derivation of the associated differential equations.

The regulators STURE0 and STURE1 are given by (2.15) with $u(t) = \theta(t)^T x(t)$

$$\theta(t+1) = \theta(t) + \gamma(t+1)S(t)x(t-\hat{k})[y(t+1) - \theta(t)^T x(t-\hat{k}) +$$

$$+ \theta(t-\hat{k})^T x(t-\hat{k})] \tag{6.1}$$

When computing to corresponding ODE, the two last terms cancel. For the algorithm of SA-type, STURE0, the ODE (4.16) is

$$\frac{d}{d\tau}\theta = \begin{bmatrix} -r_{yy}(\hat{k}+1;\theta) \\ \vdots \\ -r_{yy}(\hat{k}+\hat{n},\theta) \\ \hat{\beta}_o r_{uy}(\hat{k}+2,\theta) \\ \vdots \\ \hat{\beta}_o r_{uy}(\hat{k}+1+\hat{m},\theta) \end{bmatrix} \tag{6.2}$$

where $r_{yy}(i,\theta)$ is the autocorrelation for the stationary process defined by

$$y(t+k+1) = \left[\theta_o - \frac{\beta_o}{\hat{\beta}_o}\theta\right] x(t) + \varepsilon(t+k+1) \tag{6.3}$$

Here $\theta_o = (\alpha_1 \; ... \; \alpha_n, \; 0, \; ... \; 0, \; \frac{\beta_o}{\hat{\beta}_o}\beta_1, \; ... \; \frac{\beta_o}{\hat{\beta}_o}\beta_m, \; 0, \; ... \; 0,)$

Eq. (6.2) is suitable for numerical solution of the ODE. This is further discussed in Chapter 7.

For analysis it might give better insight to rewrite (6.1) using (6.3), provided $\hat{k} = k$

$$\theta(t+1) = \theta(t) + \gamma(t+1)S(t)x(t-k)\{x^T(t-k)[\theta_o - \frac{\beta_o}{\hat{\beta}_o}\theta(t-k)] -$$

$$- \theta(t) + \theta(t-k) + \varepsilon(t+1)\} \tag{6.4}$$

The ODE (4.16) corresponding to this form of the algorithm then is

$$\frac{d}{d\tau}\theta = G_1(\theta)\left(\theta_o - \frac{\beta_o}{\hat{\beta}_o}\theta\right) + E[x(t-k)\varepsilon(t+1)] \tag{6.5}$$

where $G_1(\theta) = E\, x(t)x(t)^T$, and where all expectations are evaluated, given that the feedback law is constant and expressed by $u(t) = -\frac{1}{\hat{\beta}_o}\theta^T x(t)$.

For the regulator STURE1 the ODE corresponding to (4.15) becomes

$$\frac{d}{d\tau}\theta = S(\tau)\, f(\theta(\tau))$$
$$\frac{d}{d\tau} S(\tau) = \ddot{S}(\tau) - S(\tau)G_1(\theta(\tau))S(\tau) \qquad (6.6)$$

where $G_1(\theta)$ is defined as above and $f(\theta)$ is the right hand side of (6.2). This ODE contains more variables than (6.2) and may be more difficult to analyse theoretically for this reason. In Ljung (1972) a reparametrization of $(\theta,P) \to c$ is made, so that the transformed ODE has the structure:

$$\frac{d}{d\tau} c = h_1[h_2(c)] - c \qquad (6.7)$$

where the range space of h_2 has the same dimension as θ. This structure can, as shown in Ljung (1972), be utilized for the analysis in some cases.

In a number of cases theoretical stability analysis is practically impossible. Then, numerical solution of the ODE's is a possibility to obtain detailed information about the stability that may suffice from a practical point of view. Eq. (6.6) can be used straightforwardly for numerical solution.

6.2 Convergence in case the LS noise condition is satisfied.

The case when the self-tuning regulator STURE0 is used is treated in this section. If the LS noise condition is satisfied, the noise is independent of x(t-k). Hence (6.5) is

$$\frac{d}{d\tau}\theta = G_1(\theta)(\theta_o - \frac{\beta_o}{\hat{\beta}_o}\theta) \qquad (6.8)$$

Now take $\gamma(t) = c_\gamma t^{-s}$; $s > 0$. In Section 5.1 it was shown that the stability condition (4.11) is satisfied in case $0 < \frac{\beta_o}{\hat{\beta}_o} < 2$. Suppose that $E|\varepsilon(t)|^{\beta} < C$, where $\beta > 4/\alpha$. Then all conditions of Theorem 4.1 are satisfied.

The analysis in Åström - Wittenmark (1973) implies that in case $\hat{m} = m'$ and $\hat{n} = n$, there is only one stationary point of (6.8), namely $\theta^* = \frac{\hat{\beta}_o}{\beta_o}\theta_o$.

Since $G_1(\theta)$ is a non-negative definite matrix, all solutions tend to this stationary point unless they tend to the boundary of the area where the closed loop system is stable and $G_1(\theta)$ defined.

It should therefore also be shown that the trajectories of (6.8), near the boundary of the area where the closed loop system is stable, point into this area. This conclusion is indicated in the analysis of Section 5.1, but no formal proof of it will be given here.

Assuming this result, it now follows from Theorem 4.1 that $\theta(N) \to \frac{\hat{\beta}_o}{\beta_o}\theta_o$ w.p.1 as $N\to\infty$, which gives the desired minimum variance controller.

In case the open loop system is stable and the output of the controller (i.e. the input to the system) is limited, the ODE (6.8) is defined everywhere. Then there are no problems with stability regions and asymptotic stability of (6.8) follows straightforwardly.

6.3 Analysis of a simple system.

So far, in Chapter 3 and in the previous section, convergence of the regulators STURE0 and STURE1 has been shown in case the LS noise condition is satisfied. Convergence for a simple system for which the LS noise condition is not satisfied is shown in this section. See also the analysis on page 32.

Example 6.1 Consider the system

$$y(t+1) + a\, y(t) = b\, u(t) + e(t+1) + c\, e(t) \quad ; |c| < 1$$

where the sequence {e(t)} is white noise with zero mean value and unit variance. The minimum variance control regulator for this system is

$$u(t) = (a-c)/b\; y(t)$$

The regulator parameter is estimated using the model

$$y(t+1) + \alpha y(t) = \beta u(t) + \varepsilon(t+1)$$

where β is a priori fixed but not necessarily equal to b. In this case with one regulator parameter the regulators STURE0 ans STURE1 are identical. The feedback law is

$$u(t) = \alpha(t)/\beta \quad y(t)$$

From (6.2) the corresponding ODE is

$$\dot{\alpha} = -r_{yy}(1)$$

where

$$r_{yy}(1) = \frac{(c-a+\alpha b/\beta)(1-c(a-\alpha b/\beta))}{1 - (a-\alpha b/\beta)^2}$$

The desired convergence point α^* is $\alpha^* = (a-c)\beta/b$

Introduce $z = \alpha - \alpha^*$. Then

$$\dot{z} = -\frac{b}{\beta} z \cdot \frac{(1-c(c-\frac{b}{\beta} z))}{1 - (c-z\, b/\beta)^2} \; ; \quad |c| < 1; \; |c-z\,\frac{b}{\beta}| < 1 \qquad (6.9)$$

It is easy to show that the last factor in (6.9) is positive where it is defined. Hence, (6.9) is globally asymptotically stable if b and β have the same sign.

The stability condition (4.11) is satisfied if $0 < \frac{b}{\beta} < 2$, as shown in Section 5.1. Hence Theorem 4.1 assures that $\alpha(t)$ tends to α^* w.p.1 as $t \to \infty$.

Summing up the results of this section and of Section 5.1 we have

- If $\frac{b}{\beta} < 0$, the regulator will not converge. The closed loop system becomes eventually unstable, whereafter the pole of the closed loop system is forced to infinity.
- If $0 < \frac{b}{\beta} < 2$, the regulator converges to the desired value w.p.1.
- If $\frac{b}{\beta} > 2$, the regulator converges to the desired value as long as the closed loop system is stable. However, there is a non-zero probability that the estimate tends to infinity.

6.4 Linearization of the differential equations.

Consider the system

$$y(t+1) + a_1 y(t) + \ldots + a_n y(t-n+1) = u(t) + \ldots + b_m u(t-m) +$$

$$+ e(t+1) + c_1 e(t) + \ldots + c_n e(t-n+1) \tag{6.10}$$

where $\{e(t)\}$ is a sequence of independent, random variables with zero mean values. Suppose that this system is controlled by the regulator STUREO. Then the corresponding ODE is (6.2) with $\hat{k} = 0$ or (6.5). Suppose that the correct model orders m and n have been chosen. Then the only stationary point of the ODE is the minimum variance control law, cf. Åström - Wittenmark (1973), given by

$$\theta^* = (a_1 - c_1, \ldots a_n - c_n,\ b_1 \ldots, b_m)$$

We will now linearize (6.5) around this solution. The result is formulated as a lemma.

Lemma 6.1: Consider the system (6.10) controlled by the regulator STUREO. Assume that the system is minimum phase. Linearization of the corresponding ODE (6.5) around θ^* gives with $\Delta\theta = \theta - \theta^*$

$$\dot{\Delta}\theta = M\Delta\theta \tag{6.11}$$

where

$$M = -E \begin{bmatrix} e(t) \\ \vdots \\ e(t-n+1) \\ u(t) \\ \vdots \\ u(t-m) \end{bmatrix} [\tilde{e}(t) \ \ldots \ \tilde{e}(t-n+1) \ \tilde{u}(t) \ \ldots \ \tilde{u}(t-m)]$$

where

$$\tilde{e}(t+1) + c_1\tilde{e}(t) + \ldots + c_n\tilde{e}(t-n+1) = e(t+1) \tag{6.12}$$

$$\tilde{u}(t+1) + c_1\tilde{u}(t) + \ldots + c_n\tilde{u}(t-n+1) = u(t+1) \tag{6.13}$$

and

$$u(t) + b_1u(t-1) + \ldots + b_mu(t-m) = (a_1-c_1)e(t) +$$

$$+ \ldots + (a_n-c_n)e(t-n+1) \tag{6.14}$$

Proof: Let

$$\theta = (\hat{a}_1, \ldots \hat{a}_n,\ \hat{b}_1, \ldots, \hat{b}_m)$$

where $\hat{a}_i$ is the estimate of a_i. Then $M = \frac{d}{d\theta} f(\theta)$. Denote the elements of M by m_{ij}. Then

$$m_{ji} = -\frac{\partial}{\partial \hat{a}_i} r_{yy}(j) \qquad i, j \leq n$$

$$m_{j(i+n)} = -\frac{\partial}{\partial \hat{b}_i} r_{yy}(j) \qquad i \leq m, \quad j \leq n$$

$$m_{(j+n)i} = \frac{\partial}{\partial \hat{a}_i} r_{uy}(j+1) \qquad i \leq n, \quad j \leq m$$

$$m_{(j+n)(i+n)} = \frac{\partial}{\partial \hat{b}_i} r_{uy}(j+1) \qquad i \leq m, \quad j \leq m$$

Now

$$\frac{\partial}{\partial \hat{a}_i} r_{yy}(j) = \frac{\partial}{\partial \hat{a}_i} E\, y(t)y(t+j) = E\left\{\left[\frac{\partial}{\partial \hat{a}_i} y(t)\right] y(t+j) + \right.$$

$$\left. + y(t) \frac{\partial}{\partial \hat{a}_i} y(t+j)\right\}$$

Since $y(t) = e(t)$ in the point in which the expression is evaluated, the first term in the RHS is zero.

Consider $\frac{\partial}{\partial \hat{a}_i} y(t+j)$. Introduce the polynomials

$$A(z) = 1 + a_1 z + \ldots + a_n z^n$$

$$B(z) = z + b_1 z^2 + \ldots + b_m z^{m+1},$$

$$\hat{A}(z) = 1 + \hat{a}_1 z + \ldots + \hat{a}_n z^n$$

$$\hat{B}(z) = z + \hat{b}_1 z^2 + \ldots + \hat{b}_m z^{m+1}$$

$$C(z) = 1 + c_1 z + \ldots + c_n z^n$$

Then the closed loop system is given by

$$[A(q^{-1})\hat{B}(q^{-1}) - B(q^{-1})\{\hat{A}(q^{-1}) - 1\}]y(t) = \hat{B}(q^{-1})C(q^{-1})e(t)$$

where q^{-1} is the backward shift operator.

Take the derivative with respect to $\hat{a}_i$:

$$[A(q^{-1})\hat{B}(q^{-1}) - B(q^{-1})\{\hat{A}(q^{-1}) - 1\}]\frac{\partial}{\partial \hat{a}_i} y(t) -$$

$$- B(q^{-1})\frac{\partial}{\partial \hat{a}_i} \hat{A}(q^{-1})y(t) = 0$$

The derivative is to be evaluated at $\theta = \theta^*$ i.e. $\hat{A} = A - C + 1$, $\hat{B} = B$. Then $y(t) = e(t)$ and

$$A\hat{B} - B(\hat{A}-1) = BC$$

Hence

$$B(q^{-1})C(q^{-1})\frac{\partial}{\partial \hat{a}_i} y(t) = B(q^{-1})\ q^{-i}\ e(t)$$

Introduce $\tilde{e}(t)$ by

$$C(q^{-1})\tilde{e}(t) = e(t)$$

Then

$$\frac{\partial}{\partial \hat{a}_i} y(t) = \tilde{e}(t-i)$$

and

$$\left.\frac{\partial}{\partial \hat{a}_i} r_{yy}(j)\right|_{\theta=\theta^*} = E\ e(t)\tilde{e}(t+j-i) = E\ e(t-j)\tilde{e}(t-i)$$

Consequently, the upper left block matrix of M is

$$-E \begin{bmatrix} e(t) \\ \vdots \\ e(t-n+1) \end{bmatrix} [\tilde{e}(t)\ \ldots\ldots\ \tilde{e}(t-n+1)]$$

The rest of the lemma is proved analogously. □

The properties of (6.11) will now be discussed. It is easily seen that the upper left block in the matrix M is triangular with -1 in the diagonal. This means that in case there are no b-parameters to estimate, the linearized equation is asymptotically stable. However, the diagonal elements in the lower right block are more interesting. They are, see Åström (1970), given by

$$-E\, u(t)\, \tilde{u}(t) = -\frac{1}{2\pi} \oint \frac{1}{C(z)} \phi_{uu}(z)\, dz$$

where

$$\phi_{uu}(z) = \frac{(A(z) - C(z))\,(A^*(z) - C^*(z))}{B(z)\, B^*(z)}$$

is the autospectrum of u. Here $A^*(z) = z^n A(z^{-1})$, etc.

The element $E\, u(t)\, \tilde{u}(t)$ can be made negative with arbitrary magnitude. To do so, choose the parameters $c_1, \ldots, c_n$ so that $C(\exp(i\omega))$ has negative real part for some ω. This is possible as soon as the degree of C(z) is greater than or equal to 2. Then choose A and B so that the system $[A(q^{-1}) - C(q^{-1})]/B(q^{-1})$ has a resonance for the frequency ω. In this way it is possible to make

$$\mathrm{tr}\, M = -[n + m\, E\, u(t)\, \tilde{u}(t)]$$

positive. This means that M must have at least one eigenvalue with positive real part.

An example of such a system is given below.

Example 6.2. Consider the following system

$$y(t+1) - 1.6\, y(t) + 0.75\, y(t-1) = u(t) + u(t-1) +$$

$$+ 0.9\, u(t-2) + e(t+1) + 1.5\, e(t-1) + 0.75\, e(t-2) \qquad (6.15)$$

For this system the C-polynomial has a negative real part on the unit circle for $1.78 < \omega < 2.48$. The system $1/B(z)$ has a sharp resonance for $\omega = 2.10$. This is sufficient to make (6.1ã), i.e. the differential equation linearized around $\theta = \theta^*$, have positive eigenvalues.

According to Åström - Wittenmark (1973) there is only one possible convergence point; θ^*. Since this has proved to be an unstable stationary point of the ODE (4.16), the self-tuning algorithm is not likely to converge. According to the results of Section 5.1 the estimates do not tend to infinity. Therefore they must vary in a bounded area without converging to any point.

The minimum variance regulator for (6.15) is given by

$$u(t) = -3.1\, y(t) - u(t-1) - 0.9\, u(t-2) \qquad (6.16)$$

In figure 6.1 the behaviour of STURE1 with $\hat{\beta}_0 = 1$, $\hat{n} = 2$, $\hat{m} = 2$ and $\hat{k} = 0$ is shown. The sequence $\{\gamma(t)\}$ was chosen such that it decreases very slowly in order to accentuate the behaviour of the system. The initial values are the values of the optimal regulator (6.16). The average loss per step from one simulation was about 2.90, while the optimal regulator gave the average loss 1.02 for the same noise sequence.

From figure 6.1 it is obvious that the algorithm tries to reach the optimal values. When the estimates come close to the optimal ones, they are thrown away. This behaviour is in good agreement with the results of the analysis.

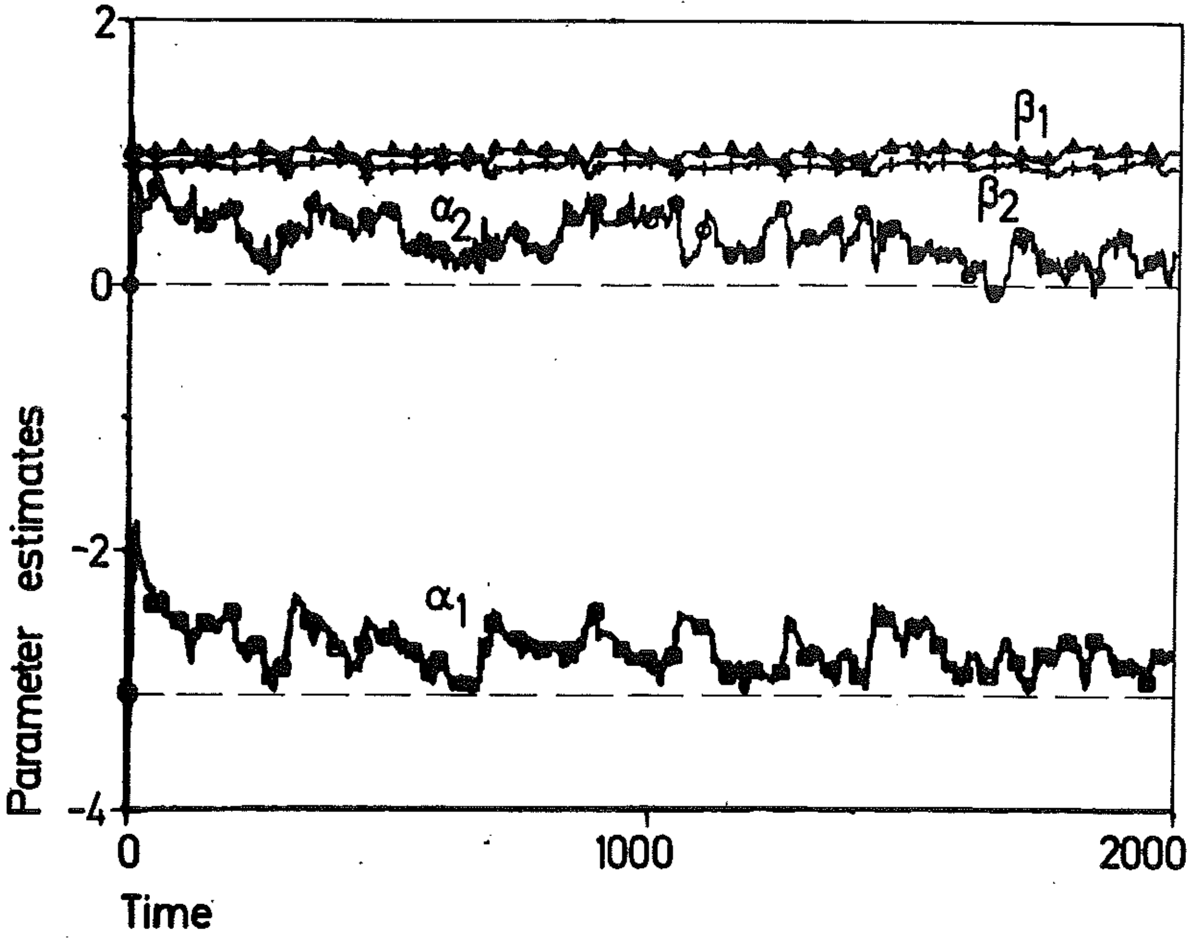


Fig. 6.1 Parameter estimates when the system (6.15) is controlled by STURE1. The dashed lines show the values corresponding to the optimal controller.

7. NUMERICAL EXAMPLES

In this chapter some of the theoretical results in the previous chapters will be illustrated through some numerical simulations.

Example 7.1 The first example will show the significance of the expected trajectories obtained through the differential equations described in Chapter 4. Consider the system

$$y(t) + a\, y(t-1) = b\, u(t-1) + e(t) + c\, e(t-1) \tag{7.1}$$

where a = -0.95, b = 1 and c = -0.45. The sequence $\{e(t)\}$ is white noise. The optimal control law is given by

$$u(t) = \frac{a - c}{b}\, y(t) = -0.5\, y(t)$$

Let the model be

$$y(t+1) + \hat{\alpha} y(t) = u(t) + \varepsilon(t+1)$$

With $G_1(\theta) = r_{yy}(0)$, the differential equations (6.6) for the self-tuning algorithm STURE1 will be

$$\begin{aligned} \dot{\alpha} &= -S \cdot r_{yy}(1) = -S \cdot \frac{(c-a-b\alpha)(1-c(a-b\alpha))}{1-(a-b\alpha)^2} \\ \dot{S} &= S - S^2 \cdot r_{yy}(0) = S - S^2 \cdot \left[1 + \frac{(c-a+b\alpha)^2}{1-(a-b\alpha)^2}\right] \end{aligned} \tag{7.2}$$

The equations (7.2) are simulated using a program package, SIMNON, for simulation of nonlinear differential equations available at the Division of Automatic Control in Lund, see Elmqvist (1972).

Figure 7.1 shows the trajectories for different initial values α when S(0) = 5. The system (7.1) is also controlled using

STURE1, with

$$\gamma(t) = \frac{c_\gamma}{t^s}$$

The values used in the algorithm were $c_\gamma = 0.002$ and $s = 0.0645$. According to Theorem 4.2 the time in the differential equation (7.2) is related to the number of samples, N, through

$$\tau_N = \sum_{t=1}^{N} \gamma(i) = c_\gamma \sum_{i=1}^{N} \frac{1}{i^s}$$

The value of s was chosen rather small in order to get a reasonable value of N. With the chosen values, 5 time units correspond to 4000 steps. Figure 7.2 shows the parameter estimates for different starting values of the parameter α. The initial value of S was $S(0) = 5$. The parameter estimates correspond well with the trajectories of the differential equation.

If c_γ or $S(0)$ are increased, the estimates will vary more in the beginning, but after a short period of time the estimates will behave as in figure 7.2. If α is outside the stability boundary of the closed loop system, then Eq. (7.2) is not valid. The self-tuning regulator has, however, a stabilization property, cf. Chapter 5, and will rapidly give an estimate which makes the closed loop system stable. □

<u>Example 7.2</u> Consider the system

$$y(t) + a\,y(t-1) = u(t-1) + b\,u(t-2) + e(t) + c\,e(t-1) \qquad (7.3)$$

with $a = -0.99$, $b = 0.5$ and $c = -0.7$. The optimal control law is

$$u(t) = \frac{a - c}{1 + bq^{-1}}\, y(t) = \frac{-0.29}{1 + 0.5q^{-1}}\, y(t)$$

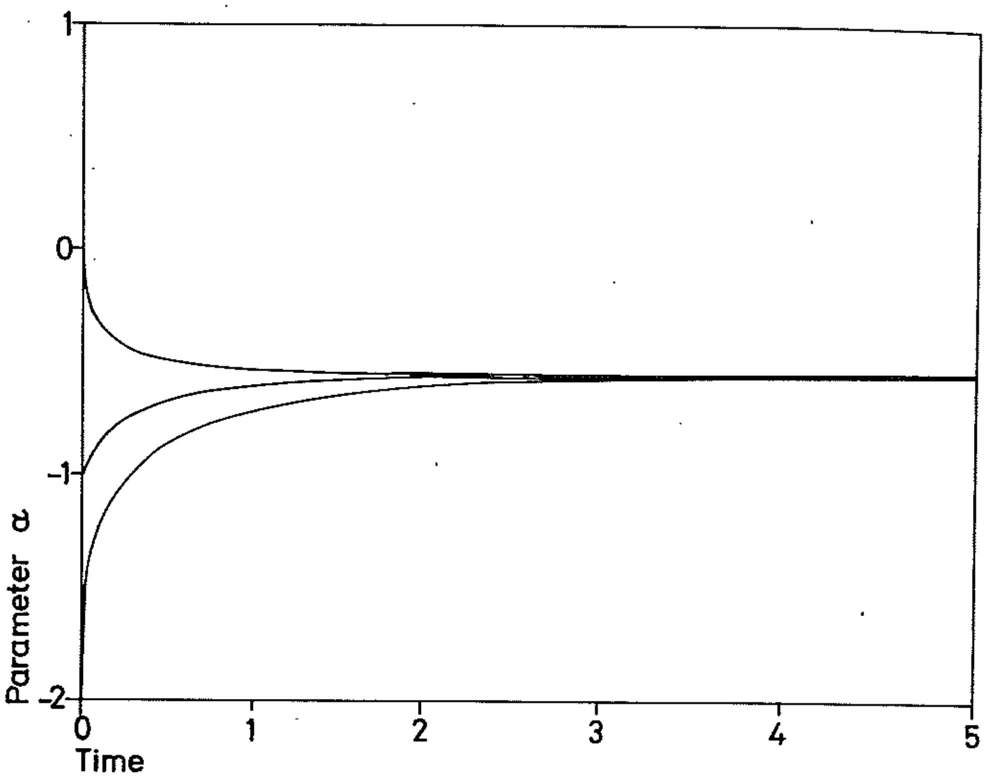


Fig. 7.1 Trajectories for the initial values $\hat{\alpha}(0) = 0$, -1 and -1.9 respectively of the equation (7.2) and where $S(0) = 5$.

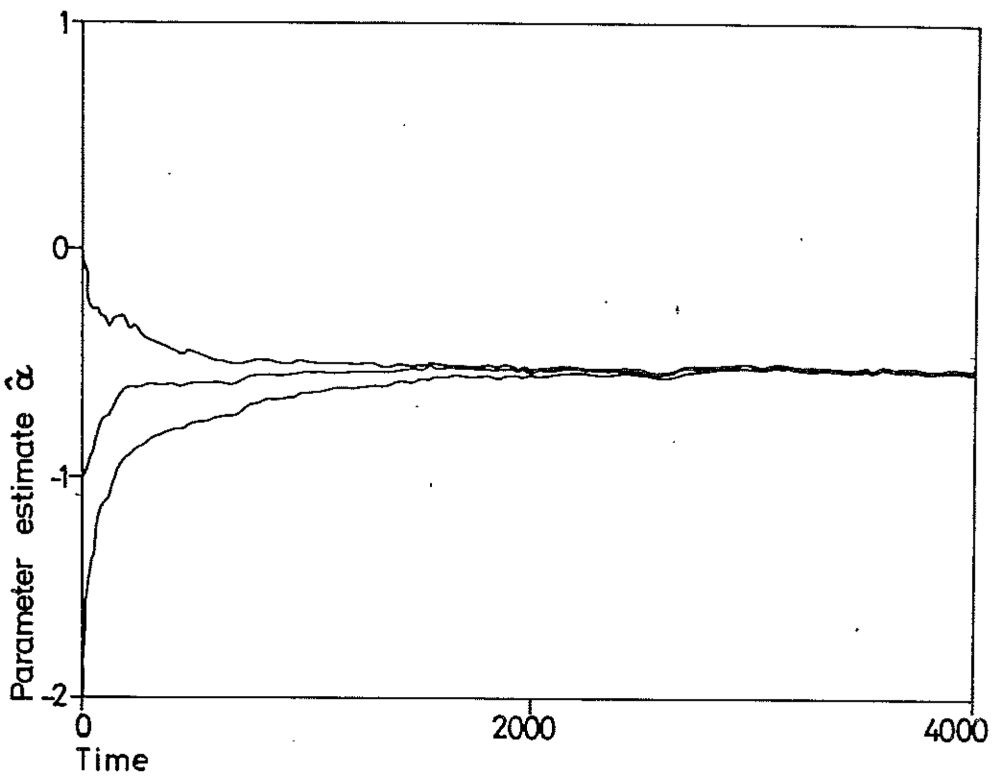


Fig. 7.2 The parameter estimates for different starting values when the self-tuning algorithm is used with $S(0) = 5$, $c_{\gamma} = 0.002$ and $s = 0.0645$ on the system (7.1).

In the self-tuning algorithm STURE0 the parameters have been estimated from the model

$$y(t+1) + \hat{\alpha}y(t) = u(t) + \hat{\beta}u(t-1) + \varepsilon(t+1)$$

The differential equations are in this case

$$\begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = S \begin{bmatrix} -r_{yy}(1) \\ r_{uy}(2) \end{bmatrix}$$
$$\dot{S}^{-1} = -S^{-1} + r_{yy}(0) + r_{uu}(0) \qquad (7.4)$$

The equations for α and β are difficult to analyse. The equations have been simulated with different starting values of the parameters α and β and with $S(0) = 10$. The phase plane is shown in figure 7.3. At the beginning parts of the trajectories every 2nd time unit is indicated. From the starting point $\alpha(0) = -1.5$ and $\alpha(0) = -0.1$ it takes about 9.7 time units before the estimates are within a distance of 0.1 from the convergence point. Corresponding curves are shown in figure 7.4 for one realization when the system is controlled by the self-tuning regulator STURE0. The values $c_\gamma = 0.002$ and $s = 0.1$ were used. The value of c_γ is much smaller than one would use in a practical case. This value was, however, chosen in order to better see the agreement between the parameter estimates and the trajectories of the differential equations (7.4).

When the self-tuning regulator STURE1 is used, the differential equations will be

$$\begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = S \begin{bmatrix} -r_{yy}(1) \\ r_{yu}(2) \end{bmatrix}$$
$$\dot{S}^{-1} = -S^{-1} + \begin{bmatrix} r_{yy}(0) & -r_{yu}(1) \\ -r_{yu}(1) & r_{uu}(0) \end{bmatrix} \qquad (7.5)$$

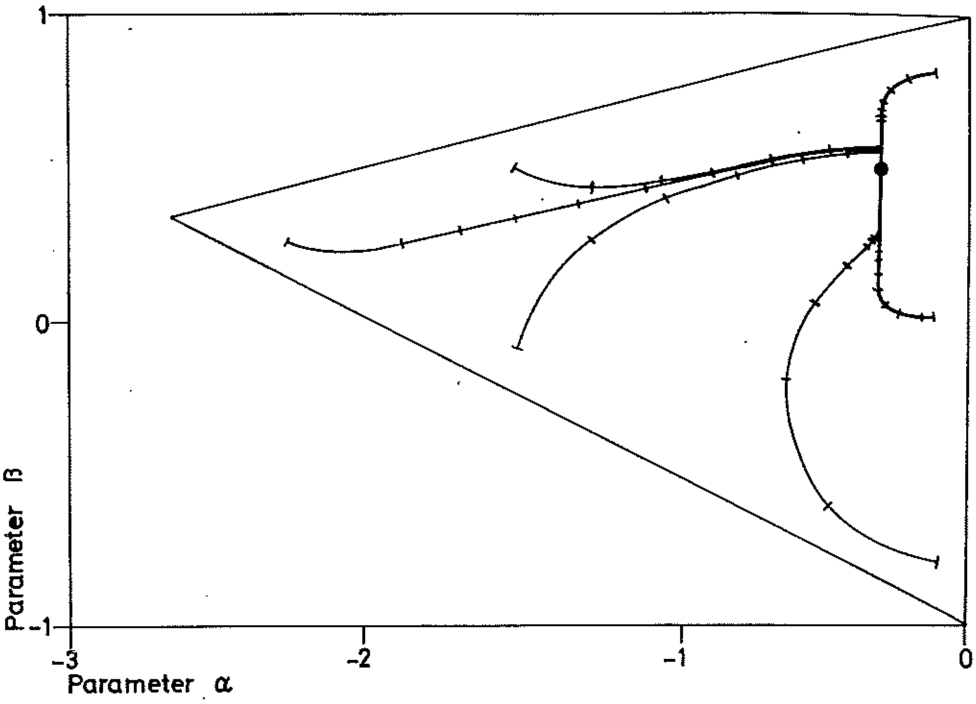


Fig. 7.3 Phase plane of the differential equations (7.4) for different starting values when S(0) = 10. The parameter values corresponding to the optimal regulator are indicated by a dot. The triangel shows the stability boundary of the equations (7.4). At the first parts of the trajectories every 2nd time unit is marked.

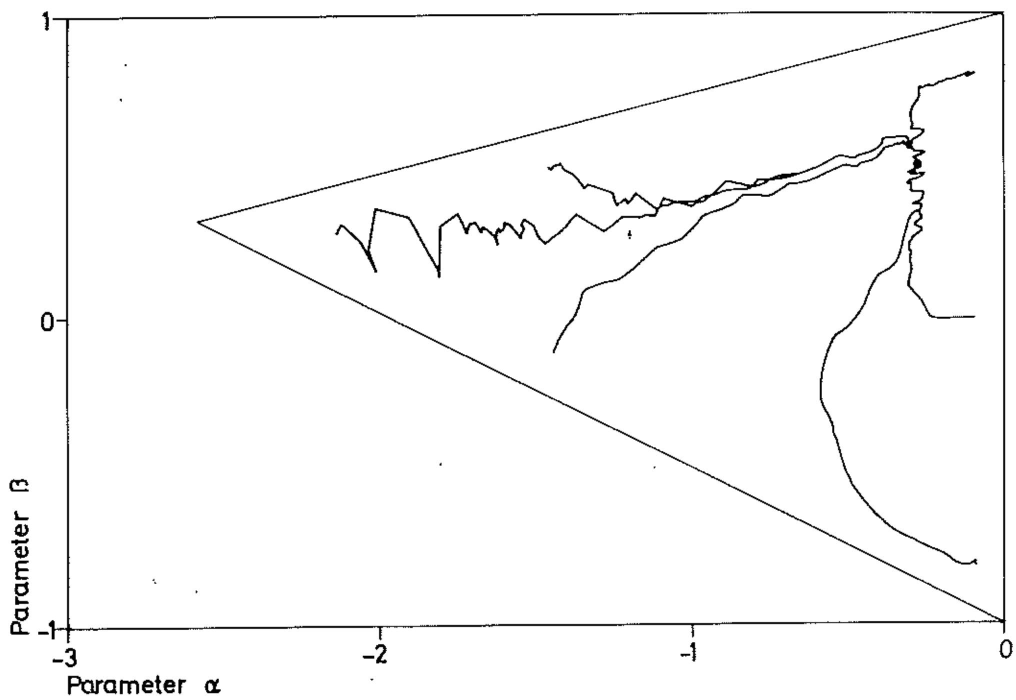


Fig. 7.4 Phase plane of the parameter estimates when the system (7.3) is controlled by the self-tuning regulator STURE0 with c_{γ} = 0.002, s = 0.1 and S(0) = 10.

The phase plane of (7.5) is shown in figure 7.5. Compared with STURE0 the trajectories in this case are leading more directly to the optimal point. The starting directions of the parameter estimates are determined by the initial value of S, which in this case was 10•I. The convergence time is also shorter in this case. For the starting values $\alpha(0) = -1.5$ and $\beta(0) = -0.1$ the convergence time is about 5.7 time units.

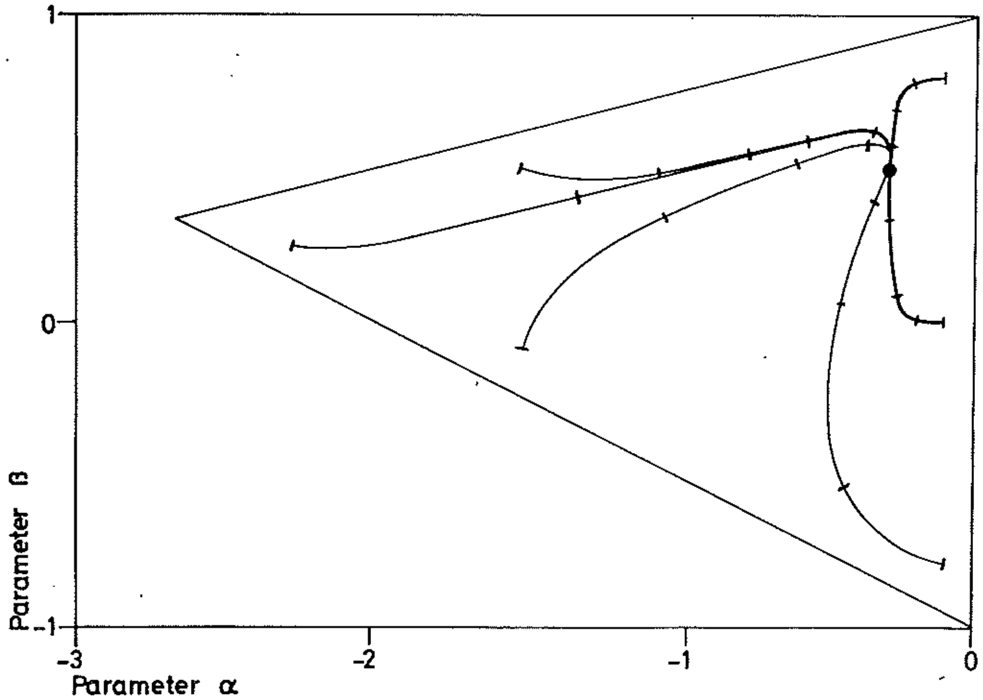


Fig. 7.5 Phase plane of the differential equations (7.5) for different starting values. The initial value of S was 10•I. Every second time unit is indicated in the beginning of the trajectories.

Example 7.3 In this example the differential equations of a more complex self-tuning regulator than STURE0 and STURE1 are simulated. In the previous examples the control law was linear in the parameter estimates, but in this case the control law is determined in a more complex way.

Consider the system

$$y(t) + a\ y(t-1) = b\ u(t-3) + e(t) \qquad (7.6)$$

where a = -0.9 and b = 1. The parameters a and b are estimated using (2.15f) with the model (B.4), (see appendix B). The minimum variance regulator based on the estimates is then computed as

$$u(t) = \frac{\hat{a}^3/\hat{b}}{1 - \hat{a}q^{-1} + \hat{a}^2 q^{-1}}\ y(t) \qquad (7.7)$$

The corresponding differential equations are given by (4.16):

$$\begin{aligned} \dot{\hat{a}} &= -r_{yy}(1) - \hat{a}\ r_{yy}(0) + \hat{b}\ r_{yu}(2) \\ \dot{\hat{b}} &= r_{yu}(3) + \hat{a}\ r_{yu}(2) - \hat{b}\ r_{uu}(0) \end{aligned} \qquad (7.8)$$

Trajectories of (7.8) are shown in figure 7.6. Since the LS noise condition is satisfied, the estimates converge to the true parameter values. The covariance function of the closed loop system, $r_{yy}(0)$, corresponding to the estimates in figure 7.6 is given in figure 7.7. It is interesting to notice that the expected variance of the output actually increases for some values of $\hat{a}$, $\hat{b}$.

□

The examples in this chapter show that the differential equations defined by (4.15) and (4.16) are very useful for the analysis of the different self-tuning algorithms. It is possible to determine the transient behaviour, as well as to investigate the convergence properties. The differential equations also have the advantage that the stochastic part is removed from the analysis. It is, however, in most cases difficult to analyse the differential equations. When only one parameter is estimated, it is possible to carry through the analysis. One example is given in Wittenmark (1973). If the system contains two cr more para-

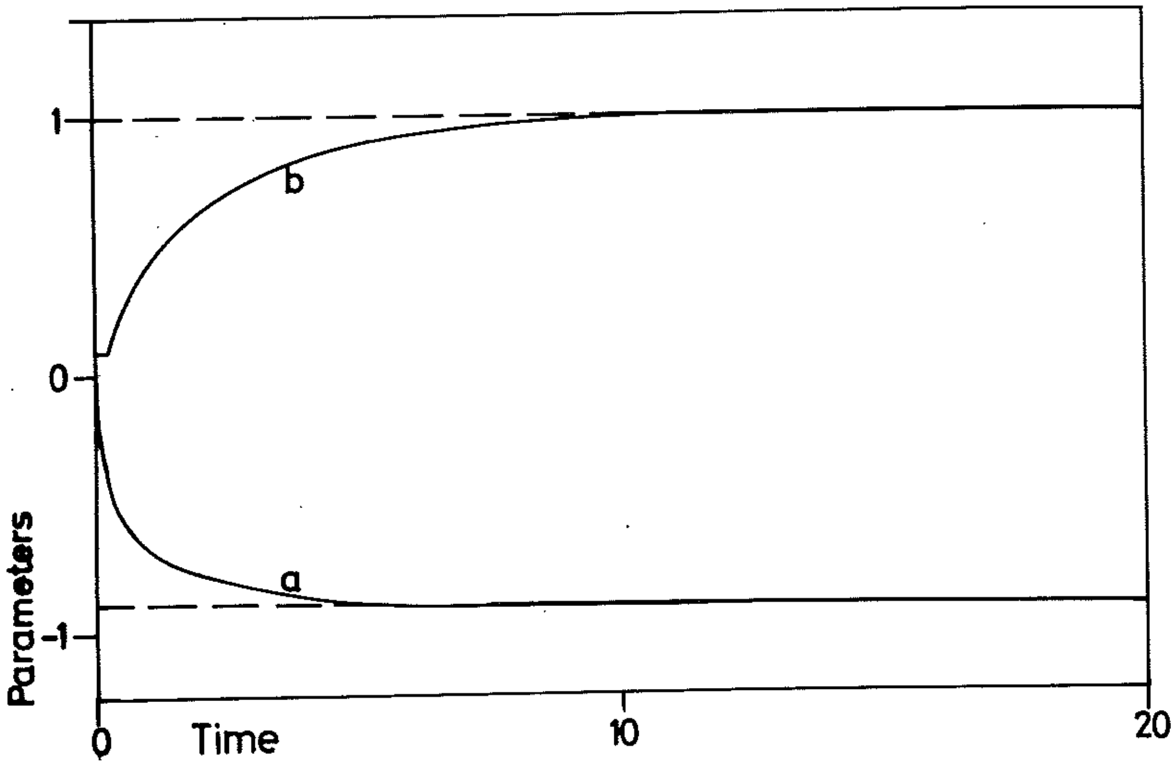


Fig. 7.6 Trajectories of (7.8) corresponding to the estimates of the system (7.6) when the control law is (7.7). The initial values were $\hat{a}(0) = -0.1$ and $\hat{b}(0) = 0.1$

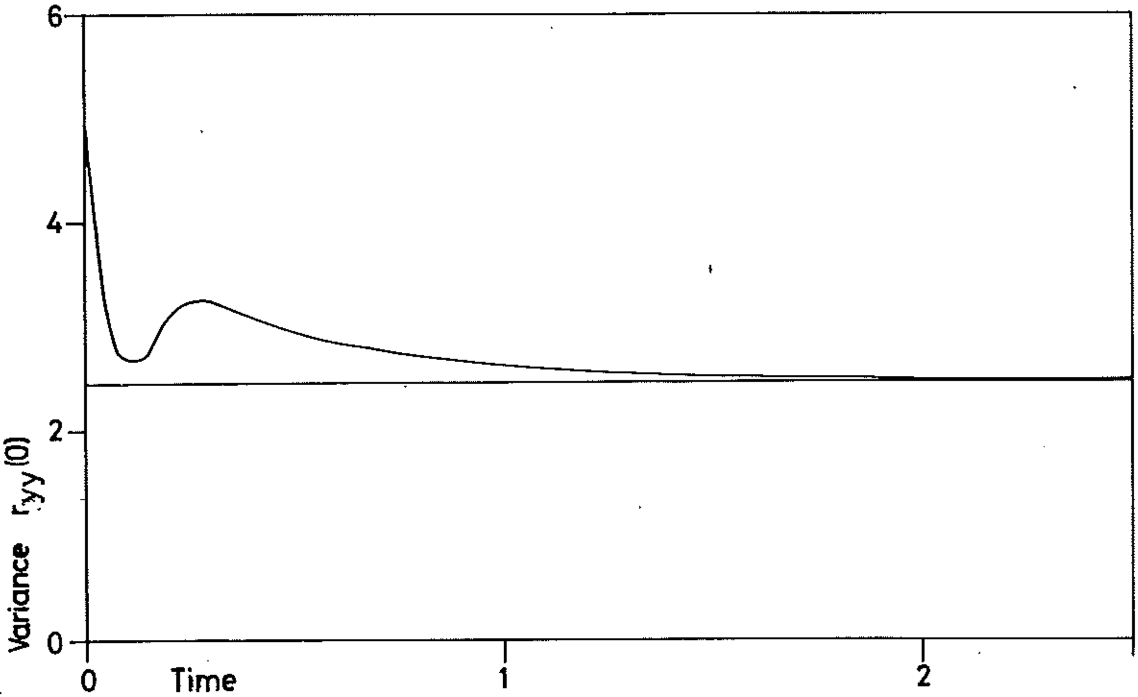


Fig. 7.7 The variance function $r_{yy}(0)$ corresponding to the estimates given in figure 7.6.

meters, it is difficult to investigate for instance the stability. The differential equations can, however, be simulated and much insight can be gained in this way. The computations will be rather extensive even for systems of low order.

The self-tuning regulators can without any difficulties be simulated with many parameters, but many simulations have to be done in order to investigate the convergence properties. The self-tuning algorithms are also more timeconsuming than the differential equations since the time in the differential equations is related to the number of steps through $\tau = \sum_{1}^{N} \gamma(t)$, and the number of steps per time unit is rapidly increasing since $\gamma(t)$ is decreasing.

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APPENDIX A. PROOF OF LEMMA 4.1

Lemma 4.1 Suppose $\theta(n)$ and $\bar{\theta}$ belong to the area where f and G_i are defined. Let $m(n,\Delta\tau)$ satisfy

$$\sum_{n}^{m(n,\Delta\tau)} \gamma(k) \to \Delta\tau \quad \text{as } n\to\infty$$

Suppose that $|x(n)| < C$ (C may depend on the realization). Then for sufficiently small $\Delta\tau$ and $(\theta(n),S_i(n))$ sufficiently close to $(\bar{\theta},\bar{S}_i)$

$$\theta(m(n,\Delta\tau)) = \theta(n) + \Delta\tau\bar{S}_i f(\bar{\theta}) + R_1(n,\Delta\tau,\bar{\theta},\bar{S}_i) + R_2(n,\Delta\tau,\bar{\theta},\bar{S}_i) \tag{A.1}$$

$$S_i(m(n,\Delta\tau)) = S_i(n) + \Delta\tau[-\bar{S}_i\ G_i(\bar{\theta})\ \bar{S}_i+\bar{S}_i] + R_1'(n,\Delta\tau,\bar{\theta},\bar{S}_i) + R_2'(n,\Delta\tau,\bar{\theta},\bar{S}_i) \tag{A.2}$$

where

$$|R_1^{(')}(n,\Delta\tau,\bar{\theta},\bar{S}_i)| \leq \Delta\tau\cdot K\{|\theta(n) - \bar{\theta}| + |S_i(n) - \bar{S}_i|\} + A(\Delta\tau)^2$$

and

$$R_2^{(')}(n,\Delta\tau,\theta,S_i) \to 0 \quad \text{w.p.1 as } n\to\infty. \qquad \square$$

Proof: To abbreviate notation the term

$$\theta(t)^T\ x(t) - \hat{\beta}_o u(t-\hat{k}) = x(t)^T[\theta(t) - \hat{\beta}_o F(\theta(t-\hat{k}))^T]$$

in (2.15) will be omitted, since it is treated in the same way as $x(t)\ y(t+\hat{k}+1)$. The variable $\hat{k}$ will be taken as zero.

The analysis will be carried out for a given, fixed realization ω. Many of the variables below depend on ω, but this argument is suppressed. The technical problem with non-countable unions of null sets can be treated in the same way as in Ljung (1974), appendix A, and is not explicitly dealt with here.

Consider first

$$\sum_{n}^{m(n,\Delta\tau)} \gamma(t)x(t)x(t)^T \qquad (A.3)$$

It will be shown that, if $|\theta(n) - \bar{\theta}|$ is sufficiently small,

$$\sum_{n}^{m(n,\Delta\tau)} \gamma(t)x(t)x(t)^T = \Delta\tau G_1(\bar{\theta}) + R_3(n,\Delta\tau,\bar{\theta}) + R_4(n) \qquad (A.4)$$

where

$$|R_3(n,\Delta\tau,\bar{\theta})| \leq \Delta\tau \cdot C_2\{|\theta(n) - \bar{\theta}|\} + C_3 \cdot (\Delta\tau)^2$$

and

$R_4(n) \to 0$ w.p.1 as $n \to \infty$.

The vector $x(t)$ can be seen as a state vector for the system (2.3) with feedback $u(t) = F(\theta(t))\, x(t)$. The closed loop system can be written on a state space form:

$$x(t+1) = A(\theta(t))\, x(t) + B\varepsilon(t+1)$$

Let $\bar{x}(t)$ $t = n,\ldots$ denote the sequence of vectors that are obtained by

$$\bar{x}(t+1) = A(\bar{\theta})\, \bar{x}(t) + B\varepsilon(t+1); \quad \bar{x}(n) = x(n)$$

Let $\tilde{x}(t) = x(t) - \bar{x}(t)$

Then (A.3) can be written as

$$\sum_{n}^{m(n,\Delta\tau)} \gamma(t)\bar{x}(t)\bar{x}(t)^T + \sum_{n}^{m(n,\Delta\tau)} \gamma(t)[\bar{x}(t)\tilde{x}(t)^T + \tilde{x}(t)\bar{x}(t)^T] \qquad (A.5)$$

Consider first the second term of (A.5). We have

$$\bar{x}(t) = \bar{A}^{t-n}\, x(n) + \sum_{k=n}^{t} \bar{A}^{t-k} B\varepsilon(k); \quad \bar{A} = A(\bar{\theta}) \tag{A.6}$$

and

$$\tilde{x}(t) = \sum_{k=n}^{t} \left\{ \left[\prod_{k+1}^{t} A_j \right] - \bar{A}^{t-k} \right\} B\varepsilon(k); \quad A_j = A(\theta(j)) \tag{A.7}$$

Since $F(\theta)$ is Lipschitz continous we have

$$|A(\theta(t)) - A(\bar{\theta})| < C_4 |\theta(t) - \bar{\theta}|$$

Since $\bar{\theta}$ gives a stable closed loop system

$$|\bar{A}^{t-k}| \leq C_5\, \lambda_1^{t-k} \qquad \lambda_1 < 1$$

If $\max_{n<t<k} |\theta(t) - \bar{\theta}|$ is sufficiently small, say less than $\bar{\delta}$, it follows that

$$\left|\prod_{k+1}^{t} A_j\right| \leq C_6\, \lambda_2^{t-k} \qquad \lambda_2 < 1$$

Then we have

$$\left|\prod_{k+1}^{t} A_j - \bar{A}^{t-k}\right| \leq \max_{n \leq i \leq t} |\theta(i) - \bar{\theta}| \cdot C_7 \cdot \lambda_3^{t-k} \qquad \lambda_3 < 1$$

for $t \leq K$. Introduce $q(n,t) = \max_{n \leq i \leq t} |\theta(i) - \bar{\theta}|$. Assume that

$|\theta(n) - \bar{\theta}| < \bar{\delta}$ and $|S_i(n) - \bar{S}| < \bar{\delta}$ and denote by $K(n)$ the

first number $\geq n$ such that

$$|\theta(K(n)) - \bar{\theta}| \geq \bar{\delta} \qquad \text{or}$$

$$|S_i(K(n)) - \bar{S}| \geq \bar{\delta}$$

Introduce

$$k(n,\Delta\tau) = \min(m(n,\Delta\tau),\ K(n))$$

Then

$$|\tilde{x}(t)| \leq q(n,t)\cdot v(t) \qquad t \leq k(n,\Delta\tau)$$

where

$$v(t) = \sum_{k=n}^{t} C_7 \cdot \lambda_3^{t-k} |B\varepsilon(k)|$$

Similarly

$$|\bar{x}(t)| \leq v(t) + C_5 \lambda_1^{t-n} |x(n)|$$

Hence the second term of (A.5) satisfies

$$\left| \sum_{n}^{k(n,\Delta\tau)} \gamma(t)\{\tilde{x}(t)\bar{x}(t)^T + \bar{x}(t)\tilde{x}(t)^T\} \right| \leq$$

$$\leq 2q(n,k(n,\Delta\tau))\cdot \sum_{n}^{k(n,\Delta\tau)} \gamma(t)v(t)^2 + C_8 \sum_{n}^{k(n,\Delta\tau)} \gamma(t)\lambda_1^{t-n} \qquad \text{(A.7)}$$

The first term of (A.5) is quite similar to the first term of (A.7): both are formed as sums of products of stochastic variables ($\bar{x}(t)$ and $v(t)$ respectively) that are obtained from white noise ($\{e(t)\}$) through exponentially stable filters (giving first $\varepsilon(t)$ and then $\bar{x}(t)$ and $v(t)$). The convergence of such sums is considered in the following lemma:

<u>Lemma A.1</u> Let the random variables $f_1(t)$ and $f_2(t)$ be generated from white noise $\{e(t)\}$ with zero mean value and unit variance:

$$f_i(t) = \sum_{k=0}^{\infty} g_i(t,k)\ e(t-k) \qquad i = 1,2$$

where $|g_i(t,k)| < C\lambda^k \qquad \lambda < 1 \qquad i = 1,2$

Suppose $E|e(t)|^{4p} < C$ and that the sequence $\{\gamma(t)\}$ satisfies (4.14).

Then

$$\sum_{t=n_k}^{n_{k+1}} \gamma(t)[f_1(t)\cdot f_2(t) - Ef_1(t)\cdot f_2(t)] \to 0 \quad \text{w.p.1}$$

as $k\to\infty$, where the subsequence $\{n_k\}$ satisfies

$$\limsup_{k\to\infty} \sum_{n_k}^{n_{k+1}} \gamma(t) = L < \frac{1}{2}$$

Proof of Lemma A.1: For simplicity denote $n_k = n$ and $n_{k+1} = m$ and $f_1(j)\cdot f_2(j) - Ef_1(j)\cdot f_2(j) = f(j)$ and consider

$$\left|E\left(\sum_n^m \gamma(k)f(k)\right)^{2p}\right| =$$

$$= \left|E \sum_{j_1=n}^{m} \cdots \sum_{j_{2p}=n}^{m} \gamma(j_1)\cdot\ldots\cdot\gamma(j_{2p})f(j_1)\cdot\ldots\cdot f(j_{2p})\right| \leq$$

$$\leq \gamma(n)^{2p} \sum_{j_1=n}^{m} \cdots \sum_{j_{2p}=n}^{m} |Ef(j_1)\cdot\ldots\cdot f(j_{2p})| \leq$$

$$\leq \gamma(n)^{2p} \sum_{j_1=n}^{m} \cdots \sum_{j_{2p}=n}^{m} \sum_{k_1=0}^{\infty} \sum_{\ell_1=0}^{\infty} \cdots \sum_{k_{2p}=0}^{\infty} \sum_{\ell_{2p}=0}^{\infty} \Big\{ K^{2p} \lambda^{\sum_1^{2p}(k_i+\ell_i)}$$

$$|E[e(j_1-k_1)e(j_1-\ell_1) - \delta_{k_1\ell_1}]\cdot\ldots\cdot[e(j_{2p}-k_{2p})e(j_{2p}-\ell_{2p}) -$$

$$- \delta_{k_{2p}\ell_{2p}}]|\Big\}$$

Since $e(j)$ and $e(k)$ are independent for $j = k$, the expectation in the above sum is zero unless for each r

$$j_r - k_r = \begin{cases} j_s - k_s \\ \text{or} \\ j_s - \ell_s \end{cases} \quad \underline{\text{and}} \quad j_r - \ell_r = \begin{cases} j_{s'} - k_{s'} \\ \text{or} \\ j_{s'} - \ell_{s'} \end{cases} \tag{A.8}$$

for some $s, s' \neq r$.

Regard k_i, ℓ_i and $j_1,\dots,j_p$ as fixed. Then the other j_{p+1}, $\dots,j_{2p}$ are determined by (A.8) (up to permutations, the number of which depends on p and <u>not</u> on m-n). Hence p of the outer summing indices can be eliminated. Summing first over k_i and ℓ_i gives a finite result $C(\lambda)$ depending only on λ.

Thus

$$\left| E\left(\sum_n^m \gamma(k)f(k)\right)^{2p} \right| \leq \gamma(n)^{2p} \sum_{j_1=n}^m \dots \sum_{j_p=n}^m C(\lambda) \leq \tag{A.9}$$

$$\leq \gamma(n)^{2p}\cdot(m-n)^p\cdot C(\lambda) \leq \gamma(n)^{2p}\cdot\gamma(m)^{-p}\cdot C_1(\gamma) \leq \gamma(n)^p C_2(\gamma)$$

The second inequality follows from

$$(m-n)\gamma(m) \leq \sum_n^m \gamma(t) < 2L \tag{A.10}$$

From (A.10) we also have

$$\left[\frac{m}{n} - 1\right]\cdot m^{-s} \leq 2L/n$$

or

$$\frac{m}{n} \leq 1 + 2L\,\frac{m}{n}^{+s} \leq 1 + 2L\,\frac{m}{n} \quad \text{since } s \leq 1$$

which gives

$$\frac{m}{n} \leq \frac{1}{2L - 1} \leq \text{Const} \tag{A.11}$$

This inequality implies the last inequality in (A.9).

From Chebysjev's inequality it follows that

$$P(|\sum_{j=n_k}^{n_{k+1}} \gamma(j)f(j)| > \varepsilon) \leq \frac{E|\sum_{n_k}^{n_{k+1}} (j)f(j)|^{2p}}{\varepsilon^{2p}} \leq$$

$$\leq \gamma(n_k)^p \cdot C_3/\varepsilon^{2p} \tag{A.12}$$

Now

$$\sum_{k=1}^{\infty} \gamma(n_k)^p \leq \sum_{t=1}^{\infty} \gamma(t)^p = \sum_{t=1}^{\infty} C_\gamma \, t^{-sp} < \infty$$

since $sp > 1$. Application of the Borel - Cantelli lemma yields, in view of (A.12) and (A.13) that

$$\sum_{n_k}^{n_{k+1}} \gamma(t)f(t) \to 0 \quad \text{w.p.1 as } k \to \infty$$

and Lemma A.1 is proved. □

With this lemma applied to (A.5) and (A.7) we obtain

$$\sum_{n}^{k(n,\Delta\tau)} \gamma(t)x(t)x(t)^T =$$

$$= \sum_{n}^{k(n,\Delta\tau)} \gamma(t)E[\bar{x}(t)\bar{x}(t)^T] + R_5(n) + R_6(n,\Delta\tau,\bar{\theta}) \tag{A.14}$$

where $R_5(n) \to 0$ w.p.1 as $n \to \infty$, and where

$$|R_6(n,\Delta\tau,\bar{\theta}) \leq$$

$$\leq |q(n,k(n,\Delta\tau))\cdot \sum_{n}^{k(n,\Delta\tau)} \gamma(t)v(t)^2 + C_9 \sum_{n}^{k(n,\Delta\tau)} \gamma(t)\lambda^{t-n}| \leq$$

$$\leq q(n,k(n,\Delta\tau))\cdot[E\ v(t)^2]\cdot \sum_{n}^{m(n,\Delta\tau)} \gamma(t) + R_7(n) \leq$$

$$\leq C_{10}\ q(n,k(n,\Delta\tau))\cdot\Delta\tau + R_7(n)$$

with $R_7(n) \to 0$ as $n \to \infty$. The same result naturally holds for

$$\sum_{n}^{k(n,\Delta\tau)} \gamma(t)x(t)x(t+1)^T$$

Consider now $(y(t) = D^T x(t))$

$$|\theta(k(n,\Delta\tau)) - \theta(n)| = |\sum_{n}^{k(n,\Delta\tau)} \gamma(t)\ S_i(t-1)\ x(t)y(t+1)| \leq$$

$$\leq |\bar{S}_i \sum_{n}^{k(n,\Delta\tau)} \gamma(t)\ x(t)\ x(t+1)^T\ D| + \max_{n\leq t\leq k} |\bar{S}_i - S_i(t)|\cdot$$

$$|\sum_{n}^{k(n,\Delta\tau)} \gamma(t)x(t)x(t+1)^T\ D| \leq (|\bar{S}_i|+\bar{\delta})(C_{11}\cdot\Delta\tau+R_8(n)) \qquad \text{(A.15)}$$

where $R_8(n) \to 0$ w.p.1 as $n\to\infty$

$$|S_i^{-1}(k(n,\Delta\tau)) - S_i^{-1}(n)| = |\sum_{n}^{k(n,\Delta\tau)} \gamma(t)[x(t)x(t)^T\ S_i^{-1}(t)]| \leq$$

$$\leq \Delta\tau\cdot C_{12} + R_9(n); \quad \text{where } R_9(n) \to 0 \text{ w.p.1 as } n\to\infty$$

It follows from (A.15) that $\theta(k)$ and $S_i(k)$ can be made to differ arbitrarily little from $\theta(n)$ and $S_i(n)$ for large n. This means that, for sufficiently small $\Delta\tau$, $k(n,\Delta\tau) = m(n,\Delta\tau)$ for sufficiently large n. It also follows that

$$\sup_{n \leq t \leq m} |\theta(n) - \theta(t)| \leq C_{13} \cdot \Delta\tau + R_{10}(n);$$

$$R_{10}(n) \to 0 \quad \text{w.p.1 as } n \to \infty \tag{A.16}$$

$$\sup_{n \leq t \leq m} |S(n) - S(t)| \leq C_{14} \cdot \Delta\tau + R_{11}(n);$$

$$R_{11}(n) \to 0 \quad \text{w.p.1 as } n \to \infty$$

Combining (A.14), (A.15) and (A.16) we obtain for sufficiently small $\Delta\tau$

$$|\theta(m(n,\Delta\tau)) - \theta(n) - \bar{S}_i[E\ \bar{x}(t)\bar{x}(t+1)D] \cdot \sum_n^m \gamma(t)| \leq$$

$$\leq |\bar{S}_i| \cdot R_5(n) + R_6(n) + \max_{n \leq t \leq m} |\bar{S}_i - S_i(t)| \cdot [|\bar{S}|\ R_5(n) + R_6(n)] \leq$$

$$\leq |\bar{S}_i| \cdot R_5(n)[1 + C_{14} \cdot \Delta\tau + R_{11}(n) + |\bar{S}_i - S_i(n)|] +$$

$$+ [q(n,m(n,\Delta\tau)) \cdot C_{10} \cdot \Delta\tau + R_7(n)]\ [1 + C_{14} \cdot \Delta\tau\ R_{11}(n) +$$

$$+ |\bar{S}_i - S_i(n)|] \tag{A.17}$$

Now using that

$$E\ \bar{x}(t)\bar{x}(t+1)D = f(\bar{\theta})$$

$$\sum_n^{m(n,\Delta\tau)} \gamma(t) \to \Delta\tau \quad \text{as } n \to \infty$$

and

$$q(n,m(n,\Delta\tau)) \leq C_{13} \cdot \Delta\tau + R_{10}(n) + |\theta(n) - \bar{\theta}|$$

and rearranging the terms of (A.17) gives the desired relation (A.1). Eq. (A.2) is obtained analogously. □

APPENDIX B. RESULTS FOR OTHER MODEL STRUCTURES

In Section 2.1 four different model structures were briefly mentioned. One of them, (2.4), was chosen as the basic model and has been used throughout the report. In this appendix it is shown how the other model structures can formally be treated in exactly the same way as (2.4).

B.1 Model structures.

The chosen model (2.4),

$$y(t+\hat{k}+1) + \hat{\alpha}_1 y(t) + \ldots + \hat{\alpha}_{\hat{n}} y(t+1-\hat{n}) = \hat{\beta}_0 [u(t) + \hat{\beta}_1 u(t-1) + \ldots + \hat{\beta}_{\hat{m}} u(t-\hat{m})] + \hat{\varepsilon}(t+\hat{k}+1) \tag{B.1}$$

will be referred to as model A.

In case the variable β_0 is estimated, a more natural model structure is, cf. (2.2)

$$y(t+\hat{k}+1) + \hat{\alpha}_1 y(t) + \ldots + \hat{\alpha}_{\hat{n}} y(t+1-\hat{n}) = \hat{\beta}_0 u(t) + \hat{\beta}_1 u(t-1) + \ldots + \hat{\beta}_{\hat{m}} u(t-\hat{m}) + \hat{\varepsilon}(t+\hat{k}+1) \tag{B.2}$$

This model will be referred to as "model B."

In the models A and B, the system (2.1) is written on "predictor form," which is suitable for the self-tuning regulators STUREO and STURE1. More straightforward models are

$$y(t+1) + \hat{a}_1 y(t) + \ldots + \hat{a}_{\hat{n}} y(t+1-\hat{n}) = \hat{b}_0 [u(t-\hat{k}) + \ldots + \hat{b}_{\hat{m}} u(t-\hat{k}-\hat{m})] + \hat{e}(t+1) \tag{B.3}$$

and

$$y(t+1) + \hat{a}_1 y(t) + \ldots + \hat{a}_{\hat{n}} y(t+1-\hat{n}) = \hat{b}_o u(t-\hat{k}) + \ldots +$$
$$+ \hat{b}_{\hat{m}} u(t-\hat{k}-\hat{m}) + \hat{e}(t+1) \tag{B.4}$$

These models will be called "model C" and "model D" respectively. Clearly, if $\hat{k} = 0$, models A and C and models B and D, respectively, are identical.

B.2 Modifications of the results of Chapter 2.

Introduce

$$\theta_A = (\hat{\alpha}_1, \ldots, \hat{\alpha}_{\hat{n}}, \hat{\beta}_1, \ldots, \hat{\beta}_{\hat{m}})^T$$
$$\theta_B = (\hat{\alpha}_1, \ldots, \hat{\alpha}_{\hat{n}}, \hat{\beta}_o, \ldots, \hat{\beta}_{\hat{n}})^T$$
$$\theta_C = (\hat{a}_1, \ldots, \hat{a}_{\hat{n}}, \hat{b}_1, \ldots, \hat{b}_{\hat{m}})^T \tag{B.5}$$
$$\theta_D = (\hat{a}_1, \ldots, \hat{a}_{\hat{n}}, \hat{b}_o, \hat{b}_1, \ldots, \hat{b}_{\hat{m}})^T$$

$$\theta^o_A = (\alpha_1, \ldots, \alpha_n, 0, \ldots, 0, \beta_1/\hat{\beta}_o, \ldots, \beta_{m'}/\hat{\beta}_o, 0, \ldots, 0)^T$$

($\hat{n} - n$ and $\hat{m} - m'$ zeroes respectively) (B.6)

$$\theta^o_B = (\alpha_1, \ldots, \alpha_n, 0, \ldots, 0, \beta_o, \beta_1, \ldots, \beta_{m'}, 0, \ldots, 0)^T$$
$$\theta^o_C = (a_1, \ldots, a_n, 0, \ldots, 0, b_1 b_o/\hat{b}_o, \ldots, b_m b_o/\hat{b}_o, 0, \ldots, 0)^T$$
$$\theta^o_D = (a_1, \ldots, a_n, 0, \ldots, 0, b_o, b_1, \ldots, b_m, 0, \ldots, 0)^T$$

$$x_A(t) = [-y(t), \ldots, -y(y+1-\hat{n}), \hat{\beta}_o u(t-1), \ldots, \hat{\beta}_o u(t-\hat{m})]^T$$

$$x_B(t) = [-y(t), \ldots, -y(t+1-\hat{n}), u(t), u(t-1), \ldots, u(t-\hat{m})]^T$$

$$x_C(t-\hat{k}) = [-y(t), \ldots, y(t+1-\hat{n}), \hat{b}_o u(t-1-\hat{k}), \ldots, \hat{b}_o u(t-\hat{k}-\hat{m})]^T \quad (B.7)$$

$$x_D(t-\hat{k}) = [-y(t), \ldots, y(t+1-\hat{n}), u(t-\hat{k}), u(t-\hat{k}-1), \ldots, u(t-\hat{k}-\hat{m})]^T$$

Then formulas (2.5) to (2.15) are valid for any subscript A, B, C, or D, provided all explicit $\hat{\beta}_o$ are set to zero for models B and D. For model C, $\hat{\beta}_o$ should be replaced by $\hat{b}_o$.

The minimum variance control law (2.17) has no direct counterpart for models C and D. For model B, u(t) should be chosen as the solution of

$$\theta_B(t)^T \, x_B(t) = 0 \quad (B.8)$$

B.3 Modifications of the results of Chapter 3.

Theorem 3.1 and its corollary are valid for all model structures if in (3.7d) the corresponding x-vector according to (B.7) is chosen. The discussion in Section 3.2 can be carried out for model B if

" $(\theta_o-\tilde{\theta})^T \, x(t) = 0$ "

is replaced by

" $\tilde{\theta}_B \, x(t) = 0$ "

and the corresponding modifications in the following are made.

B.4 Modifications of the results of Chapter 4.

The only modifications necessary are directly implied by the modified form of (2.15).

B.5 Modifications of the results of Chapter 5.

Here only models A and B are relevant, since the discussion is concerned with the special regulators STURE0 and STURE1. The discussion in Section 5.1 and the proof of Theorem 5.1 are based on the variable

$$\tilde{\theta}(t) = \tilde{\theta}_A(t) = \theta_A^o - \theta_A(t)$$

If this is replaced by

$$\tilde{\theta}_B(t) = \theta_B^o - \theta_B(t)$$

and $\beta_o/\hat{\beta}_o$ is replaced by 1, the discussion remains unaltered.

Consequently the results of this chapter, like those of Section 3.2 are valid for STURE with model B (i.e. the variable β_o is estimated) without any restrictions on β_o and $\hat{\beta}_o$.