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CURRENT TOPICS IN STABILITY THEORY

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Abstract

The report contains lecture notes written in connection with a graduate course on stability problems. Topics treated are frequency domain criteria for multivariable systems, Lyapunov functions, stability in infinite-dimensional and stochastic systems, and structural stability.

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Current Topics in Stability Theory

Per Molander

Preface

These are lecture notes that were written in connection with a short course on stability problems given in the spring of 1980. They were produced under some time pressure, which gives an excuse for the sometimes sloppy notation ("stable" and "asymptotically stable" are used indiscriminately etc.). The object of the course was to give an introduction to various ramifications of the standard stability theorems. The reader is thus assumed to be acquainted with the basics of stability theory (frequency domain criteria, Lyapunov functions etc.).

Eva Dagnegård has typed the manuscript with customary professionalism.

Lund in June 1980
Per Molander

Contents

CHAPTER 1. FREQUENCY DOMAIN STABILITY CRITERIA

The frequency domain stability criteria developed in the early sixties (notably the Popov criterion, the circle criterion) soon gained popularity due to their simplicity and the relative ease with which they could be fitted into the framework of classical control theory. However, in their original form they apply only to nonlinear systems which can be separated into a linear part (normally characterised by its transfer function G(s)) and one single nonlinearity. The generic configuration is shown in Fig. 1.1.

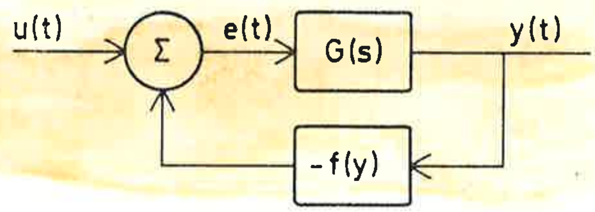

Fig. 1.1 - The generic configuration.

By contrast, the functional analysis based methods used originally by Sandberg and Zames lend themselves as easily to more complicated operators. The drawback of this approach is the difficulties that arise as the general results (as formulated in the small gain theorem or the passivity theorem) are to be adapted to special situations. Consider for instance the configuration of Fig. 1.1 but with a feedback loop consisting of several (=p) nonlinearities. The small-gain theorem guarantees stability for this set-up if the loop gain, i.e. the product of the gains of the linear and the nonlinear operators, is less than one. However, these gains depend on the norm chosen on $\mathbb{R}^p$. There remains thus the as yet unresolved problem of optimizing the choice of this basis.

The multivariable generalisations of the Nyquist criterion form a natural point of departure for an overview of the frequency-domain criteria for stability of systems with multiple nonlinearities. The chapter therefore starts with a short account of the characteristic gain approach, which is the true generalization of the Nyquist criterion to multivariable feedback systems, and the diagonal-dominance based criteria, which are only sufficient and intended to form a tool for synthesis of multivariable systems.

1.1 Linear systems

1.1.1 The generalised Nyquist criterion

Consider the linear feedback system given by

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \\ u = -ky \end{cases} \qquad (1.1)$$

As usual, $x \in \mathbb{R}^n$, u and $y \in \mathbb{R}^p$, A, B, C, and D are matrices of appropriate dimensions, and k is a scalar. To facilitate the discussion, D will be set equal to zero in the sequel. Normally, the number of inputs will of course be exceeded the number of outputs (as obtained from direct measurements or via observers), but before closing the loop, the system must in any case be "squared down". Notice also that more complicated feedback gain matrices than $k \cdot I$ can be included in C.

The problem is to study how the stability of the closed loop system varies as a function of the feedback gain k. Let Γ_R denote the negatively oriented curve in the complex plane given by (R large)

$$\Gamma_R : \begin{cases} i\omega & -R \leqslant \omega \leqslant R \\ Re^{i\theta} & -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2} . \end{cases}$$

Let $G(s) = C(sI-A)^{-1}B$ be the transfer function of the open loop system. The Nyquist criterion for the case $p=1$ can then be stated as follows:

<u>Theorem 1.1.</u> (Nyquist criterion) The system given by Eq. (1.1) is closed-loop stable if and only if

i) uncontrollable and/or unobservable modes are stable, and

ii) the number of anti-clockwise encirclements of the point $(-\frac{1}{k}, 0)$ by the image of Γ_R under $G(\cdot)$ equals the number of right half-plane poles of G. ■

The proof is a straightforward application of the principle of the variation of the argument.

In the multivariable case, G(s) is a matrix, and it turns out that stability can be decided from the behaviour of the eigenvalues of this matrix, normally referred to as the <u>characteristic gains</u>. Introduce $F(s) = I_p + k \cdot G(s)$, the so-called <u>return-difference matrix</u>. For the determinant of F(s) one obtains using Schur's formula ([1]):

$$\det(F(s)) = \det\,(I_p + k\cdot C(sI_n - A)^{-1}B) =$$

$$= \det \begin{pmatrix} sI_n - A & B \\ -kC & I_p \end{pmatrix} / \det(sI_n - A) =$$

$$= \det \begin{pmatrix} I_n & -B \\ 0 & I_p \end{pmatrix} \det \begin{pmatrix} sI_n-A & B \\ -kC & I_p \end{pmatrix} / \det(sI_n-A)$$

$$= \det \begin{pmatrix} sI_n-A+BkC & 0 \\ -kC & I_p \end{pmatrix} / \det(sI_n-A)$$

$$= \frac{\det\ (sI_n - A + BkC)}{\det\ (sI_n - A)} .$$

But $\det(sI_n-A+BkC)$ and $\det(sI_n-A)$ are the closed-loop and open-loop characteristic polynomials (CLCP(s) and OLCP(s)) respectively, and the following relationship, originally due to Desoer and Chan ([2]), has thus been established:

$$\det(F(s)) = \frac{CLCP(s)}{OLCP(s)} . \tag{1.2}$$

From (1.2) the stability of the closed-loop system can be determined from a study of the scalar quantity $\det(F(s))$. To eliminate the dependence on k, notice that

$$\det(F(s)) = \prod_{i=1}^{p} \lambda_i\ (F(s)) = k^p \prod_{i=1}^{p} \lambda_i\ [k^{-1} \cdot I_p + G(s)] =$$

$$= k^p \prod_{i=1}^{p} [k^{-1} + \lambda_i(G(s))].$$

The variation of the argument as Γ_R is mapped under $\det(F(s))$ can thus be obtained as the algebraic sum of the variation of each of the eigenvalues in the above product. As in the ordinary Nyquist criterion, the loci can be drawn for G(s), and the stability problem for various k-values can be resolved by moving the point $(-\frac{1}{k}, 0)$. The multivariable form of the Nyquist criterion thus reads:

Theorem 1.2. (Nyquist criterion, multivariable form) The system given by Eq. (1.1) is closed-loop stable if and only if

i) uncontrollable and/or unobservable modes are stable, and

ii) the algebraic sum of anti-clockwise encirclements of the point $(-\frac{1}{k}, 0)$ by the set of characteristic gain loci equals the number of right half-plane poles of G. ■

For a more detailed proof based on the theory of algebraic functions, see MacFarlane and Postlethwaite ([3]).

Example 1.1. ([4]) Let

$$A = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0.6 \\ 1 & 0.5 \end{pmatrix}, \quad \text{and} \quad C = \begin{pmatrix} 2.4 & -1.6 \\ 4.8 & -4.8 \end{pmatrix}.$$

This yields $G(s) = \dfrac{1}{1.25\ (s+1)(s+2)} \begin{pmatrix} s-1 & s \\ -6 & s-2 \end{pmatrix}.$

The characteristic gain loci, i.e. the eigenvalues of G(s) as s traverses Γ_R are shown in Fig. 1.2. (Actually these loci live on a two-sheeted Riemann surface, but they have been projected into the ordinary complex plane.)

The following stability results are obtained:

i) For $-\infty \leq -\frac{1}{k} < -0.8$ there are no encirclements of $(-\frac{1}{k}, 0)$, and the closed-loop system is consequently stable for $0 \leq k < 1.25$.

ii) For $-0.8 < -\frac{1}{k} < -0.4$ there is one clockwise encirclement of $(-\frac{1}{k}, 0)$. The closed-loop system is thus unstable for $1.25 < k < 2.5$.

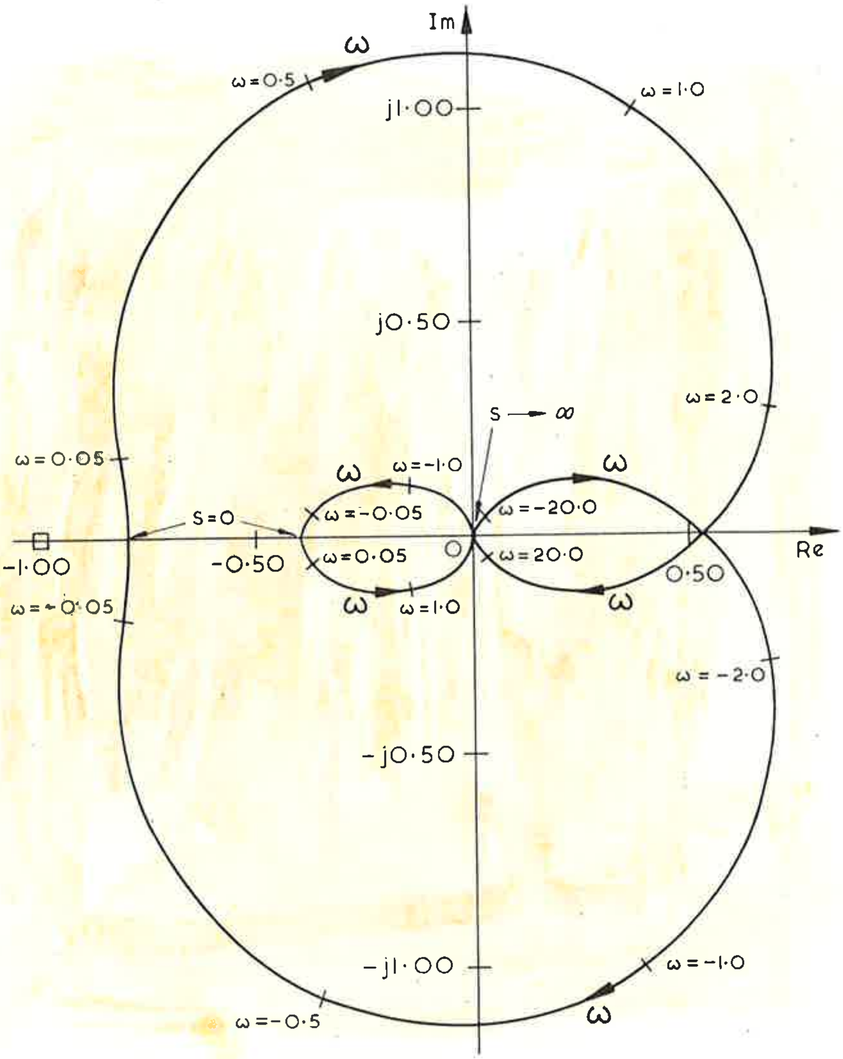

Fig. 1.2 - The characteristic gain loci of Ex. 1.1.

iii) For $-0.4 < -\frac{1}{k} < 0$ there are again no encirclements. The closed-loop system is stable in the region $2.5 < k < \infty$.

iv) For $0 < -\frac{1}{k} < 0.533$ there are two clockwise encirclements. The closed-loop system is unstable for $-\infty < k < -1.875$.

v) For $0.533 < -\frac{1}{k} \leqslant \infty$ there are no encirclements. Stability obtains for $-1.875 < k \lesssim 0$.

Notice that cases iv) and v) correspond to positive feed-back. ■

1.1.2 Stability criteria based on diagonal dominance

In some cases it is possible to conclude stability without actually computing the eigenvalues of G(s). The procedure is based on a result by Geršgorin.

Lemma 1.1. Let $A = \{a_{ij}\}$ be a complex matrix and define

$$\Omega_i = \left\{ z \in \mathbb{C};\ \ |z - a_{ii}| \leq \sum_{j \neq i} |a_{ij}| \right\}.$$

Then the spectrum of A is contained in $\bigcup_i \Omega_i$. If the Ω_i's are disjoint, there is precisely one eigenvalue in each Ω_i.

■

The lemma is of course efficient only if the off-diagonal elements are small compared to the diagonal elements. In this case one speaks of "diagonal dominance" in a qualitative sense. Diagonal dominance also has a precise meaning:

Definition 1.1. A is said to be diagonally row dominant if

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|$$

and diagonally column dominant if

$$|a_{ii}| > \sum_{j \neq i} |a_{ji}|.$$

It is diagonally dominant if it is row or column dominant or both. ■

If the diagonal elements of G(s) are plotted for all s surrounded by a circle of radius $r_i(s) = \sum_{j \neq i} |g_{ij}(s)|$ (or $\bar{r}_i(s) = \sum_{j \neq i} |g_{ji}(s)|$), this will generate p bands in the complex plane, referred to as Geršgorin bands. For each s one may select either r_i or $\bar{r}_i$. The following result follows from the generalised Nyquist criterion.

Theorem 1.3. ([5]) Let each of the bands swept out by g_{ii}, $i = 1, 2, \ldots, p$, exclude the point $(-\frac{1}{k}, 0)$. Then the closed-loop system given by Eq. (1.1) is stable if and only if

i) uncontrollable and/or unobservable modes are stable, and

ii) the algebraic sum of anti-clockwise encirclements of $(-\frac{1}{k}, 0)$ by the Geršgorin bands equals the number of right half-plane poles of G(s). ■

For reasons peculiar to certain authors, the Geršgorin bands are sometimes drawn for $G^{-1}(s)$ instead of G(s), and the resulting graph is called the inverse Nyquist array. The appropriate stability theorem is the generalised inverse Nyquist criterion.

Theorem 1.4. ([4]) The closed-loop system given by Eq. (1.1) is stable if and only if

i) uncontrollable and/or unobservable modes are stable, and

ii) the algebraic sum of anti-clockwise encirclements of (-k, 0) by the set of inverse characteristic gain loci, minus the algebraic sum of anti-clockwise encirclements of the origin by the gain loci, equals the number of right half-plane poles of G(s). ■

Theorem 1.3 may be modified accordingly.

Example 1.2. The inverse Nyquist array of a (2×2)-system is shown in Fig. 1.3.

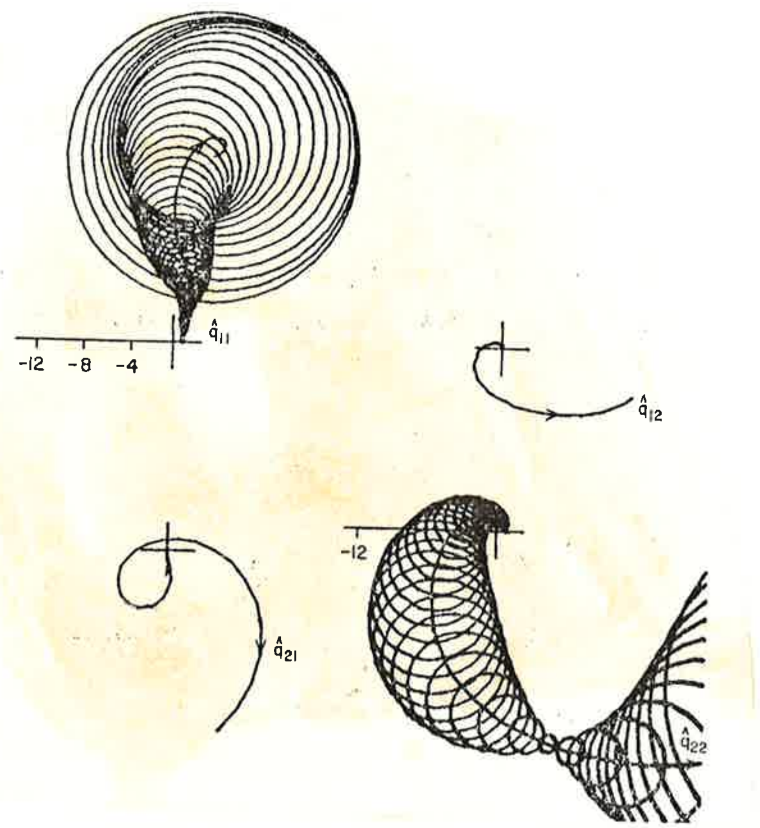

Fig. 1.3 - An inverse Nyquist array. The Geršgorin bands are based on the rows.

1.2 Nonlinear systems

Consider now the configuration of Fig. 1.1 but with several nonlinear feedback loops:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \\ u = -\Phi(y) \end{cases} \tag{1.3}$$

As before, $x \in \mathbb{R}^n$, u and $y \in \mathbb{R}^p$, A, B, and C are matrices of appropriate dimensions, and $\Phi(\cdot)$ is nonlinear, possibly timevarying function from $\mathbb{R}^p$ to $\mathbb{R}^p$ subject to

$$\|\Phi(\sigma_1) - \Phi(\sigma_2)\| \leq K \|\sigma_1 - \sigma_2\| \quad \text{for all } \sigma \in \mathbb{R}^p. \tag{1.4}$$

$\|\cdot\|$ denotes any norm on $\mathbb{R}^p$, and in fact the stability results obtained from a straightforward application of the standard small gain theorem may often be sharpened by a judicious choice of this norm. The case when $\Phi(\cdot)$ is not centered may be taken care of by a loop transformation (see [6], p. 204).

The derivation of the small gain theorem will not be repeated here (see [7], ch. III). In short, the trivial solution of (1.3), (1.4) is globally asymptotically stable if the product of the gains of the linear and the nonlinear links is less than one. The gain of $\Phi(\cdot)$ is K, and a short computation shows that the gain of the linear part is

$$\sup_{\omega \geq 0} \left\{ \max_i \lambda_i \left[[G^T(-i\omega)\ G(i\omega)]^{\frac{1}{2}} \right] \right\}.$$

$G^T(-i\omega)G(i\omega)$ is a positive-semidefinite matrix, and the eigenvalues λ_i are called the <u>singular values</u> of $G(i\omega)$. To check the stability condition, it is thus sufficient to plot the singular values as functions of ω and see if they are contained in the interval $[0, \frac{1}{K} - \varepsilon]$ for some $\varepsilon > 0$. As was pointed out above, the singular values depend on the norm (or, equivalently, on the basis). This will be capitalised on in § 1.2.3.

Referring to the theorems given in § 1.1, it is natural to ask if anything can be said on the stability of the nonlinear feedback system from a plot of the eigenvalues of $G(i\omega)$. The answer is, broadly speaking, no, as can be seen from the following example, which is adapted from Rosenbrock and Cook ([8]).

<u>Example 1.3.</u> Consider

$$G(s) = \begin{pmatrix} s+3 & \frac{4}{\varepsilon}(s+1) \\ 0 & s+3 \end{pmatrix} \frac{2}{(s+3)^2}$$

with the nonlinear feedback

$$\Phi(\sigma) = \begin{bmatrix} 0 & 0 \\ -\varepsilon & 0 \end{bmatrix} \sigma.$$

The norm of $\Phi(\cdot)$ is ε. The eigenvalues of $G(s)$ are both $\frac{2}{s+3}$, so the characteristic gain loci are contained in a ball of radius 2/3. However, the transfer matrix of the closed-loop system is

$$\tilde{G}(s) = \begin{bmatrix} s+3 & \frac{4}{\varepsilon}(s+1) \\ 2\varepsilon & s+3 \end{bmatrix} \frac{2}{(s-1)^2},$$

which is unstable. ■

1.2.1 Normal transfer functions

Under certain conditions, the eigenvalues of $G(s)$ yield all the information necessary for the solution of the stability problem. This happens when $G(s)$ is normal, i.e. commutes with its adjoint.

Theorem 1.5. Given the nonlinear feedback system (1.3), (1.4). Then the origin is stable if $G(s)$ is normal and the characteristic gain loci of G are contained in a ball of radius $(\frac{1}{K} - \varepsilon)$ for some $\varepsilon > 0$.

Proof. The theorem was proved in a very general setting by Freedman, Falb and Zames in [9]. In the context studied here, it is sufficient to notice that normality of $G(s)$ implies that the maximal singular value equals the modulus of the maximal eigenvalue. ■

Unfortunately, the assumption on normality is a restrictive one.

1.2.2 Diagonal dominance criteria

Since the characteristic gain loci contain too little information, they must be supported by something that in some sense measures the deviation from normality. Diagonal dominance is such a measure.

Theorem 1.6. Consider the nonlinear feedback system (1.3), (1.4). Let $G(s) = \{g_{ij}(s)\}$ be the transfer function of the linear part and assume that $\Phi(\cdot)$ is diagonal, i.e.

$$\Phi(\cdot) = \{\mathrm{diag}\ (\Phi_i(\cdot))\}.$$

Then the trivial solution is stable if there are numbers θ_i, $i = 1, 2, \ldots, p$, subject to

$$\sum_{i=1}^{p} \theta_i^{-2} \leq 1,$$

such that

$$\sup_{\omega \geq 0} \left\{ |g_{ii}(i\omega)| + \sum_{j \neq i} |g_{ij}(i\omega)| \right\} < \frac{1}{\theta_i K} \qquad (1.5)$$

$$\left(\text{or} \quad \sup_{\omega \geq 0} \left\{ |g_{ii}(i\omega) + \sum_{j \neq i} |g_{ji}(i\omega)| \right\} < \frac{1}{\theta_i K} \right).$$

Proof. See Rosenbrock [10]. ■

Condition (1.5) means that the Geršgorin bands should be contained in circles that have been shrunk by a factor θ_i compared to the single-variable case. The dependence on the parameters θ_i is somewhat awkward, and as p grows, the criterion becomes more and more conservative. This can be avoided by modifying somewhat the Geršgorin bands.

Theorem 1.7. Given the conditions of Thm. 1.6 but with inequality (1.5) replaced by

$$\sup_{\omega \geq 0} \left\{ |g_{ii}(i\omega)| + \sum_{j \neq i} \frac{|g_{ij}(i\omega)| + |g_{ji}(i\omega)|}{2} \right\} < \frac{1}{K}. \qquad (1.7)$$

Then the trivial solution is globally asymptotically stable.

Proof. See Cook [11]. ∎

According to (1.7), the row or column based Geršgorin bands should be replaced by the Geršgorin mean bands, generated from circles whose radii equal the arithmetic mean of the row and column based circles. Theorem 1.7 has also been proved for operators in Hilbert space (see § 3.1).

1.2.3 Other criteria

It was pointed out above that the estimates obtained from the singular values plot may be improved by a change of basis. This has been exploited by Mees and Rapp ([12]). Assume that $G(i\omega)$ can be diagonalised by the transformation $T(i\omega)$:

$$T^{-1}(i\omega)\; G(i\omega)\; T(i\omega) = \Lambda(i\omega) = \mathrm{diag}\,(\lambda_i(i\omega)).$$

A new norm on $\mathbb{R}^p$ may then be defined by

$$\| A \|_T = \| TAT^{-1} \| \quad .$$

The maximal singular value induced by this norm equals the spectral radius. The problem is that T varies with ω. However, the singular values may be estimated using the inequality

$$\| A \| = \| T \| \; \| \Lambda \| \; \| T^{-1} \| = \max_i |\lambda_i| \cdot \rho,$$

where ρ is the condition number of T. The stability condition is thus that the loci $\lambda_i(i\omega) \cdot \rho(i\omega)$ be contained in the usual circle of radius $1/K - \varepsilon$.

$\rho(i\omega)$ measures the deviation of $G(i\omega)$ from normality. When $G(i\omega)$ is normal for all ω, $\rho(i\omega) \equiv 1$, and Theorem 1.5 is regained.

A problem arises if $G(i\omega)$ becomes singular or near singular for some ω-value. In this case $\rho(i\omega) \to \infty$, and the criterion is useless. One way to overcome this difficulty is to close the loop with a feedback matrix and then to study the effect of additional feedback, linear or nonlinear.

A systematic way of studying the effect of variations of the norm on $\mathbb{R}^p$ has been devised by Araki ([13]). Consider first the following example.

Example 1.4. ([13]) Given the standard configuration with

$$G(s) = \begin{pmatrix} \dfrac{0.6}{(1+s)(1+2s)} & 0 \\ 0 & \dfrac{0.6}{(1+3s)(1+0.5s)} \end{pmatrix}$$

and

$$\Phi(\sigma_1, \sigma_2) = (\sigma_1 + 2.5\,\varphi(\sigma_2),\ 0.1\,\varphi(\sigma_1) + \sigma_2),$$

where $\varphi(\cdot)$ is subject to $|\varphi(\sigma)| \leqslant |\sigma|$. Using the ordinary norm on $\mathbb{R}^2$, $||z||^2 = z_1^2 + z_2^2$, yields

$$\begin{cases} ||G|| = 0.6 \\ ||\Phi|| \approx 2.86, \end{cases}$$

so the condition of the small gain theorem is not satisfied. If the norm on $\mathbb{R}^2$ is changed into $||z||^2_{(1,5)} = z_1^2 + 25\, z_2^2$, the induced norms of G and Φ become

$$\begin{cases} ||G||_{(1,5)} = 0.6 \quad \text{(unchanged)} \\ ||\Phi||_{(1,5)} \approx 1.5, \end{cases}$$

so stability can be deduced from the small gain theorem.

■

Araki's result is very general and particularly suited for composite systems (see further § 2.2). Here only two special cases will be considered, namely when G and Φ are diagonal, respectively.

Definition 1.2. (cf. [14]) A square matrix is said to be an M-matrix if

i) the off-diagonal elements are non-positive, and

ii) the principal minors are all positive. ■

Theorem 1.8. Given the standard configuration with $G(s) = \{g_{ij}(s)\}$ and

$$\Phi(\sigma_1, \sigma_2, \ldots, \sigma_p) = (\varphi_1(\sigma_1, \sigma_2, \ldots, \sigma_p), \varphi_2(\sigma_1, \sigma_2, \ldots, \sigma_p), \ldots, \varphi_p(\sigma_1, \sigma_2, \ldots, \sigma_p)).$$

i) Assume that G is diagonal with the characteristic loci contained in a ball of radius $1/K - \varepsilon$. Assume further that Φ satisfies

$$|\varphi_i(\sigma_1, \sigma_2, \ldots, \sigma_p)| \leq \sum_{j=1}^{p} \beta_{ij} |\sigma_j|, \qquad i = 1, 2, \ldots, p.$$

Define the matrix $A = \{a_{ij}\}$ by

$$\begin{cases} a_{ii} = K - \beta_{ii} & \\ a_{ij} = -\beta_{ij} & i \neq j. \end{cases}$$

Then the closed-loop system is stable if A is an M-matrix.

ii) Let $G(s) = \{g_{ij}(s)\}$ be arbitrary but assume that Φ is diagonal (i.e. φ_i depends only on σ_i) and is contained only in the sector $[-K, K]$. Define

$$\begin{cases} a_{ii} = \frac{1}{K} - \sup\limits_{\omega \geq 0} |g_{ii}(i\omega)| \\ a_{ij} = - \sup\limits_{\omega \geq 0} |g_{ij}(i\omega)| \qquad i \neq j. \end{cases}$$

Then the closed-loop system is stable if A is an M-matrix.

Proof. See [13]. ■

Remark. The M-matrix condition covers all that can be done using $\mathbb{R}^p$-norms of the form

$$||z||_d = \sum_{i=1}^{p} d_i \; z_i^2, \quad \text{all } d_i > 0.$$

■

References

[1] Gantmacher, F.R. (1959): The Theory of Matrices, Vol. 1. Chelsea, New York.

[2] Desoer, C.A., Chan, W.S. (1975): "The feedback interconnection of lumped linear time-invariant systems", J. Franklin Inst. 300, pp. 335-351.

[3] MacFarlane, A.G.J., Postlethwaite, I. (1977): "The generalized Nyquist stability criterion and multivariable root-loci", Int. J. Control 25, pp. 81-127.

[4] Postlethwaite, I., MacFarlane, A.G.J. (1979): A Complex Variable Approach to the Analysis of Linear Multivariable Feedback Systems. Springer Lecture Notes in Control and Information Sciences 12.

[5] Rosenbrock, H.H. (1974): Computer-Aided Control System Design. AP, London etc.

[6] Vidyasagar, M. (1978): Nonlinear Systems Analysis. Prentice-Hall, Englewood Cliffs, N.J.

[7] Desoer, C.A., Vidyasagar, M. (1975): Feedback Systems - Input-output Properties. AP, New York etc.

[8] Rosenbrock, H.H., Cook, P.A. (1975): "Stability and the eigenvalues of G(s)", Int. J. Control 21, pp. 99-104.

[9] Freedman, M.I., Falb, P.L., Zames, G. (1969): "A Hilbert space stability theory over locally compact Abelian groups", SIAM J. Control 7, pp. 479-495.

[10] Rosenbrock, H.H. (1973): "Multivariable circle theorems", in Bell (ed.): Recent Mathematical Developments in Control. AP, London etc.

[11] Cook, P.A. (1973): "Modified multivariable circle theorems", in Bell op. cit.

[12] Mees, A.I., Rapp, P.E. (1977): "Stability criteria for multiple-loop nonlinear feedback systems", Proc. 4th IFAC Int. Symp. Multivariable Technological Systems, Fredericton, Canada.

[13] Araki, M. (1978): "Stability of large-scale nonlinear systems - quadratic-order theory of composite-system method using M-matrices", IEEE Trans. AC-23, pp. 129--142.

[14] Araki, M. (1974): M-matrices. Publ. 74/19, Dept. of Computing and Control, Imperial College, London.

CHAPTER 2. LYAPUNOV FUNCTIONS REVISITED

The pros and cons of the input-output approach as compared to the classical Lyapunov function methods have been the object of much debate. In short, the I-O view often provides simpler methods of proof, particularly for equations not obtained from ordinary differential operators. The Lyapunov function, on the other hand, has its merits in cases where global stability does not obtain, since it provides a means to estimate the domain of attraction.

Consider the equations

$$\dot{x} = f(x,t) \tag{2.1}$$

and

$$\dot{x} = f(x,t) + g(x,t). \tag{2.2}$$

$x = 0$ is supposed to be the equilibrium point under study. Suppose that this point is asymptotically stable. What can be said about the solutions of Eq. (2.2)?

This question is classical and has led to concepts such as <u>total stability</u> and <u>integral stability</u>, depending on whether bounds on $g(\cdot,t)$ are given in the supremum norm or some L^p-norm. Asymptotic stability of the equilibrium can in fact be shown to imply total stability. The proof consists in generating a Lyapunov function for the free system (2.1), which can then be used to provide bounds for the solutions of the perturbed system (2.2). The reader is referred to Hahn [1], § 56, for the details.

The reverse problem, namely that of studying Lyapunov stability given I-O-stability, is considered in § 2.1. The general result is fairly natural, and the interesting part is rather to see how the general Lyapunov functions specialise in various standard configurations. As a by-product, a simple proof of the Yakubovich-Kalman lemma is obtained.

§ 2.2 is devoted to the study of composite systems.

2.1 Generation of Lyapunov functions for input-output stable systems

This section follows Willems [2] closely. It is well-known that for linear systems, input-output stability holds if and only if the A-matrix has all its eigenvalues in the open left half-plane, provided that the system contains no unstable uncontrollable and/or unobservable modes. It is obvious that some minimality condition must be satisfied also in the general nonlinear case.

First a few definitions. P_T denotes the truncation operator:

$$(P_T f)(t) = \begin{cases} f(t) & t \leq T \\ 0 & t > T, \end{cases}$$

and $Q_T = 1 - P_T$. The solution of a differential equation subject to the input $u(\cdot)$ that has initial condition x_0 at time t_0 will be denoted by $x(t;\, x_0, t_0, u)$. Recall that a function $V(x,t)$ is <u>decrescent</u> if there exists a function $\beta\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\beta(0) = 0$ and $V(x,t) \leq \beta(\|x\|)$ for all x and t. It is <u>positive definite</u> if there exists an $\alpha\colon \mathbb{R}^+ \to \mathbb{R}^+$ which is monotone increasing, and $V(x,t) \geq \alpha(\|x\|)$. It is <u>radially unbounded</u> if further

$$\lim_{\sigma\to\infty} \alpha(\sigma) = +\infty.$$

(For the application of these concepts in Lyapunov theory, see Vidyasagar [3], Chapter 5.)

<u>Definition 2.1.</u> The state space of a dynamical system is said to be <u>uniformly reachable</u> if for any x_0 and t_0 there exists a T and a u with

$$\| P_{t_0} u \|^2 \leqslant \beta(\|x_0\|)$$

subject to $x(t_0;\ 0,\ t_0-T,\ u) = x_0$. <u>Uniform observability</u> holds if for all x_0 and t_0 there exists a T such that

$$\| P_{t_0+T}\ Q_{t_0}\ y(\cdot,\ x(\cdot;x_0,\ t_0, 0),\ 0) \|^2$$

is positive definite and radially unbounded in x_0. ∎

As Lyapunov function candidates the following two functions will be considered:

i) $\quad V_R(x,t) = \inf \| P_t\ Q_{t_0}\ u \|^2,$

where the infimum is taken over all $t > t_0$ and u with $x(t;0,\ t_0, u) = x$.

ii) $\quad V_O(x,t) = \| Q_t\ y(\cdot,\ x(\cdot;x,\ t, 0),\ 0) \|^2$

V_R is defined if the state-space is reachable, and V_O is defined if further the dynamical system is I-O-stable. From

$$V_O(x,t) = \int_t^{\infty} \| y(\tau;\ x(\tau;x,t,0),\ 0) \|^2\ d\tau$$

it is clear that V_O is monotone nonincreasing along the solutions of the free system. To see that this holds also for V_R, consider $V_R(x(t_1;x,\ t,0),\ t_1)$ where $t_1 > t$, and let $x(t_1;\ x,\ t,\ 0) = x_1$. By definition,

$$V_R(x_1,t_1) = \inf \| P_{t_1}\ Q_{t_0}\ u \|^2.$$

The state can be taken to x_1 by first taking it from 0 at time t_0 and then applying zero control till time t_1. This is a suboptimal control for taking the state from 0 at t_0 to x_1 at t_1, which shows the inequality. Notice also that the assumptions on uniform reachability and observability together with a finite-gain condition imply the inequalities

$$V_O \leqslant K_1 V_r \leqslant K_2 V_O.$$

Finally, it is clear that $V_O \to 0$ as $t \to \infty$. The following result has thus been shown.

<u>Theorem 2.1</u>. Consider a uniformly observable realisation of a finite-gain I-O-stable system, and assume that the state space is uniformly reachable. Then the system is globally asymptotically stable, and V_R and V_O are radially unbounded, decrescent Lyapunov functions for it. □

<u>Example 2.1</u>. Consider the linear system

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \ , \end{cases} \tag{2.3}$$

where A is strictly Hurwitz, and (A,B) and (C,A) are controllable and observable pairs respectively. To get an expression for V_R, a fixed-point problem must be solved: Find

$$\inf \int_{-T}^{0} u^T(s)\ u(s)\ ds$$

subject to $x(-T) = 0$, $x(0) = x$. The controllability condition implies that the Gramian $W(-T,0)$ is nonsingular, and the solution is consequently

$$x^T\ e^{-A^T T}\ W^{-1}(-T,0)\ e^{-AT}\ x$$

(see Brockett [4], p. 137). Let

$$K_T = e^{AT}\ W(-T,0)\ e^{A^T T}.$$

Using the differential equation satisfied by $W(-T,0)$ (cf. [4], p. 78), one obtains

$$K_T = \int_0^T e^{As} BB^T e^{A^T s} \, ds.$$

Let

$$K = \lim_{T\to\infty} K_T.$$

Then

$$V_R(x,t) = x^T K^{-1} x,$$

where K is the positive definite solution of the Lyapunov equation

$$AK + KA^T = - BB^T.$$

It is easy to show that

$$V_O(x,t) = x^T L x,$$

where L solves

$$A^T L + LA = - C^T C.$$

Of course, $x^T K^{-1} x$ and $x^T L x$ are only two out of infinitely many possibly Lyapunov functions for Equation (2.3). ■

The functions V_R and V_O may be modified somewhat. Let the system operator be G, and pick any positive constant $\kappa > \|G\|$. Then the following two functions exist:

$$V_{R,\kappa}(x,t) = \inf \left(\|P_t u\|^2 - \frac{1}{\kappa^2} \|P_t G u\|^2 \right),$$

where the infimum is taken over all u with $x(t; 0, t_0, u) = x$,

and

$$V_{O,\kappa}(x,t) = - \inf_u \left(\|Q_t u\|^2 - \frac{1}{\kappa^2} \|Q_t G u\|^2 \right).$$

Example 2.2. Consider again the system given by Equation (2.3), and let $G(s) = C(sI-A)^{-1}B$. The variational problem pertaining to $V_{O,\kappa}$ is

$$- \inf_u \int_t^\infty (u^T(s)u(s) - \frac{1}{\kappa^2} y^T(s)y(s))\, ds,$$

and the corresponding Riccati equation is

$$A^TP + PA - \frac{1}{\kappa^2} C^TC - PBB^TP = 0. \qquad (2.4)$$

It can be inferred that Eq. (2.4) has a negative definite solution if A is strictly Hurwitz and

$$\sup G^T(-i\omega)\, G(i\omega) < \kappa^2,$$

and

$$V_{O,\kappa} = - x^TPx.$$

(The reader may consult [4], § 25, on Eq. (2.4).)

The expression for $V_{R,\kappa}$ is slightly more complicated. ■

This section will be closed with a study of passive systems. Recall that an operator G is passive if

$$< P_tu,\ P_tGu > \ \geq 0,$$

where <·,·> denotes the inner product, and the system is assumed to be in equilibrium at $t = -\infty$. It is strictly passive if $G - \varepsilon I$ is passive for some $\varepsilon > 0$.

Definition 2.2. The required energy $E_r(x,t)$ is defined by

$$E_r(x,t) = \inf < P_tu,\ P_tGu > ,$$

where the inf is taken over all $t > t_0$ and u with $x(t;\ 0,\ t_0,\ u) = x$. The available energy E_a is given by

$$E_a(x,t) =$$

$$= \sup_{\substack{u \\ t_1 > t}} - < P_{t_1}Q_tu,\ P_{t_1}Q_ty\ (\cdot,\ x(\cdot;x,t,u),u)> .$$

The cycle energy $E_c(x,t)$ is defined as

$$E_c(x,t) = E_r(x,t) - E_a(x,t).$$

■

The required energy is the energy needed to take a system from the equilibrium to a set of initial conditions, and the available energy is the maximum amount of energy that can be extracted from the system. The cycle energy is the amount of energy necessary to take the state from the equilibrium to x and back again, i.e. the loss. An alternative formula for $E_c(x,t)$ is

$$E_c(x,t) = \inf < P_{t_1} u,\ P_{t_1} Gu > ,$$

where the infimum is taken over all t_0 and t_1, $t_0 < t < t_1$, subject to $x(t;0,t_0,u) = x$.

It is clear that E_r, E_a, and E_c all exist for reachable, passive systems. Further, $E_a + E_c = E_r$, and $E_a \leqslant E_r$.

Theorem 2.2. Consider a uniformly observable realisation of a passive dynamical system, and assume that the state space is uniformly reachable. Then E_r and E_a are radially unbounded, decrescent Lyapunov functions for the equilibrium solution. ■

The proof follows the same lines as those of Thm 2.1. See [2] for details.

Example 2.3. Consider the linear system $S(A,B,C,D)$, and let $G(s) = C(sI-A)^{-1} B + D$. Assume for simplicity that G(s) has all its poles in the open left half-plane. Now, by Parseval's formula,

$$\int u^T(s)\ y(s)\ ds = \frac{1}{2\pi} \int \hat{u}^T(-i\omega)\ \hat{y}(i\omega)\ d\omega =$$

$$= \frac{1}{2\pi} \int \hat{u}^T(-i\omega)\ G(i\omega)\ \hat{u}(i\omega)\ d\omega$$

$$= \frac{1}{4\pi} \int \hat{u}^T(-i\omega)(G^T(-i\omega) + G(i\omega))\ \hat{u}(i\omega)\, d\omega,$$

so that passivity of S is equivalent to G(s) being positive real, i.e.

$$G^T(-i\omega) + G(i\omega) \geq 0, \qquad \forall\ \omega \in \mathbb{R}.$$

It will also be assumed that $D + D^T > 0$, so that the system is in fact strictly passive.

The available energy E_a is obtained from

$$E_a(x,t) = - \inf_{u \in L_2(t_0,\infty)} \int_{t_0}^{\infty} u^T(s)\ Y(s)\ ds$$

subject to

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \\ x(t_0) = x. \end{cases}$$

Now

$$\int_{t_0}^{\infty} u^T(s)\ y(s)\ ds =$$

$$= \frac{1}{2} \int_{t_0}^{\infty} [u^T(s)(Cx(s) + Du(s)) + (Cx(s) + Du(s))^T u(s)]\ ds =$$

$$= \frac{1}{2} \int_{t_0}^{\infty} [(u(s) + (D+D^T)^{-1}\ Cx(s))(D+D^T)\cdot(u(s) + (D+D^T)^{-1}Cx(s)) - \\ - x(s)^T\ C^T\ (D+D^T)^{-1}\ Cx(s)]\ ds.$$

Introduce $\tilde{u}(s) = u(s) + (D+D^T)^{-1}\ Cx(s)$. Then the system equation becomes

$$\dot{x} = (A - B(D+D^T)^{-1}C)\ x + B\tilde{u},$$

and the performance index is of the type considered in Example 2.2. The infimum is $x^T P x$, where P is the (unique) negative definite solution of

$$(A - B(D+D^T)^{-1}\ C)^T\ P\ +\ P(A - B(D+D^T)^{-1}\ C)\ -$$

$$-\ C^T(D+D^T)^{-1}\ C\ -\ PB(D+D^T)^{-1}\ B^T P\ =\ 0$$

or

$$A^T P + PA - (PB+C^T)(D+D^T)^{-1}(PB+C^T)\ =\ 0. \qquad (2.4)$$

One obtains

$$E_a(x,t) = -\ x^T P x.$$

To proceed, let $(D+D^T)$ be factored as $W^T W$, and set $PB + C^T = LW$. With $S = -P > 0$ we have thus established (the Yakubovich-Kalman lemma):

If $G(s) = C(sI-A)^{-1}B + D$ is positive real, there exist matrices $S > 0$, L, and W such that

$$\begin{cases} A^T S + SA = -\ LL^T \\ SB = C^T + LW \\ WW^T = D + D^T. \end{cases}$$

(The assumption $(D+D^T) > 0$ is normally no restriction in applications.)

The computation of $E_r(x,t)$ is somewhat more laborious and yields

$$E_r(x,t) = x^T(-P + K^{-1})\ x,$$

where P is the solution of Eq. (2.4), and K solves

$$[A - B(D+D^T)^{-1}(C + B^TP)]^T K + K[A - B(D+D^T)^{-1}(C+B^TP)] =$$

$$= - B(D+D^T)^{-1} B^T.$$

Notice that

$$E_c(x,t) = E_r(x,t) - E_a(x,t) = x^T K^{-1} x > 0,$$

so that energy is dispersed in the system. □

For a survey of the relations between variational problems, Riccati equations, frequency domain inequalities etc., see Willems [5].

2.2 Composite systems

So-called large-scale systems have been the subject of much effort in the latest years. By "large-scale" is implied that ordinary methods of analysis are not applicable due to the size of the system; rather, analysis must be based on decomposition into subsystems. A typical problem is then to deduce some property (for instance stability) of the overall system from a study of the subsystems and their interconnections. Since this is a question of structure rather than of size, the term "composite systems" is more appropriate than "large-scale systems".

In stability problems, the normal course of action is

i) to investigate the stability of the subsystems, and to express this stability in the form of Lyapunov functions (state-space formulation) or gains (I-0 formulation);

ii) to use the results from i) in order to produce bounds on the strength of the interconnections that guarantee stability of the composite system.

A problem is that the interconnection may very well improve the stability properties, whereas the above approach requires that the interconnections be small in some sense. On the other hand, if there are strong interconnections that support stability, the decomposition should be redone.

A concept that has turned out to be very useful is unifying the state-space and I-0-descriptions is that of dissipative systems (for a survey, see Willems [6]). The presentation here follows Moylan and Hill [7], since their results seem to incorporate most of what has been achieved in the area of composite-system stability.

Definition 2.3. Given a system with input $u \in \mathbb{R}^n$ and output $y \in \mathbb{R}^p$. Let $Q = Q^T$, $R = R^T$, and S be $(p\times p)$, $(m\times m)$, and $(p\times m)$ matrices respectively. Then the system is said to be (Q,S,R)-dissipative if

$$< P_t y, QP_t y > + 2 < P_t y, SP_t u > + < P_t u, RP_t u > \geq 0$$

for all $t \in \mathbb{R}$. ■

The dissipativeness concept specialises to various other known concepts for a proper choice of the matrices Q, S, and R. $Q = -I$, $S = 0$, and $R = \kappa \cdot I$ yields finite-gain stability with gain κ. $Q = R = 0$ and $S = I$ $(m = p)$ yields passivity.

Suppose now that we have given N subsystems, linearly interconnected as described by

$$u_i = (u_i)_e - \sum_{j=1}^{N} H_{ij}\, y_j \qquad i = 1, 2, \ldots, N$$

or, in more compact notation,

$$u = (u)_e - Hy. \tag{2.5}$$

Here, $(u)_e$ denotes exterior inputs.

Theorem 2.3. Consider an interconnection of N systems described by Eq. (2.5). Let subsystem i be (Q_i, S_i, R_i)--dissipative, and set $Q = \text{diag}(Q_i)$, $S = \text{diag}(S_i)$ and $R = \text{diag}(R_i)$. With $\hat{Q} = H^T S^T + SH - H^T RH - Q$, the composite system is stable if $\hat{Q}$ is positive definite.

Proof. It follows from the assumptions that the composite system with u as input is (Q,S,R)-dissipative. This implies (set $< P_t f, P_t g > = < f, g >_t$)

$$< y, \hat{Q}y >_t - 2 < y, \hat{Q}^{1/2} \hat{S}u_e >_t \leq < u_e, Ru_e >_t$$

with $\hat{S} = \hat{Q}^{-1/2}(S-H^T R)$. Pick an $\alpha > 0$ such that $R + \hat{S}^T\hat{S} \leq \alpha^2 I$.

$$\| Q^{1/2}y - \hat{S}u_e \|_t \leq \alpha \|u_e\|_t,$$

whence

$$\| y \|_t \leq \| \hat{Q}^{-1/2} \| (\alpha + \|\hat{S}\|) \|u_e\|_t,$$

so that the composite system with u_e as the input and y as the output is indeed finite-gain I-O stable. ■

Theorem 2.3 can be formulated in Lyapunov function terminology. Using the same arguments as in the definition of E_a and E_r in the preceding section, it can be shown that (Q,S,R)-dissipativeness implies the existence of a scalar function $V(x) \geq 0$ which satisfies the dissipation inequality

$$V(x(t_0)) + \int_{t_0}^{t_1} w(s)\, ds \geq V(x(t_1))$$

with respect to the supply rate

$$w(t) = y(t)Qy(t) + 2y(t)Su(t) + u(t)Ru(t).$$

The triple (Q_i, S_i, R_i) defines Lyapunov functions V_i for each subsystem, and the condition of Theorem 2.3 then guarantees that

$$V = \sum_{i=1}^{N} V_i$$

is a Lyapunov function for the composite system.

Corollary 2.1. (passivity) Suppose that all subsystems are passive and have only one output. Then a sufficient condition for stability is that there exists a positive definite matrix $P = \text{diag}(p_i)$ such that $H^T P + PH$ is positive definite. ∎

Corollary 2.2. (small-gain) Suppose that subsystem i has finite gain κ_i and let $K = \text{diag}(\kappa_i)$. Then a sufficient condition for stability is that there exists a positive definite matrix $P = \text{diag}(p_i)$ such that $P - (KH)^T P(KH)$ is positive definite. ∎

The conditions of Cors. 2.1 and 2.2 are closely related to the M-matrix condition of Araki [8]. In fact, let $H = \{h_{ij}\}$ and define $\hat{H} = \{\hat{h}_{ij}\}$ via

$$\hat{h}_{ij} = \begin{cases} h_{ij} & i = j \\ -|h_{ij}| & i \neq j. \end{cases}$$

Then a sufficient condition for the existence of the P in Cor. 2.1 is that $\hat{H}$ be an M-matrix. Likewise, let $\bar{H}$ be defined as

$$\bar{h}_{ij} = \begin{cases} 1 - |(KH)_{ij}| & i = j \\ - |(KH)_{ij}| & i \neq j. \end{cases}$$

Then a P for Cor. 2.2 exists if $\bar{H}$ is an M-matrix. These conditions are not necessary, however.

References

[1] Hahn, W. (1967): Stability of Motion. Springer, Berlin etc.

[2] Willems, J.C. (1969): "The generation of Lyapunov functions for input-output stable systems. SIAM J. Control 9, No. 1, pp. 105-134.

[3] Vidyasagar, M. (1978): Nonlinear Systems Analysis. Prentice-Hall, Englewood Cliffs, N.J.

[4] Brockett, R. (1970): Finite Dimensional Linear Systems. J Wiley & Sons, N.Y. etc.

[5] Willems, J.C. (1971): "Least squares stationary optimal control and the algebraic Riccati equation. IEEE Trans. AC-16, No. 6, pp. 621-634.

[6] Willems, J.C. (1976): "Mechanisms for the stability and instability in feedback systems", Proc. IEEE 64, No. 1, pp. 24-35.

[7] Moylan, P.J., Hill, D.J. (1978): "Stability criteria for large-scale systems", IEEE Trans. AC-23, No. 2, pp. 143-149.

[8] Araki, M., op. cit.

CHAPTER 3. INFINITE-DIMENSIONAL SYSTEMS

Infinite-dimensional systems arise in control problems for instance due to time-delays or in processes guided by partial differential equations. The linear theory has acquired a fairly closed form. Well-known concepts from the infinite-dimensional theory such as stability, observability, and controllability have got their infinite-dimensional counterparts. Also many of the methods and criteria from finite-dimensional stability theory can be transferred with minor modifications. A major problem, however, is that the state, taking its values in an infinite-dimensional space, can never be measured. A typical problem, such as control of the temperature in a rod or a slab from the boundary, will have to be based on point measurements and point sources. This calls for unbounded operators in the modelling and creates problems.

The chapter starts with a short account of the semigroups-based theory for infinite-dimensional systems developed for instance in Curtain and Pritchard [1]. The following subsections exemplify the use of Lyapunov functions and frequency domain methods. Section 2 is devoted to control problems.

3.1 Free systems

3.1.1 Semigroup theory

Consider the linear equation

$$\begin{cases} \dot{z} = Az \\ z(0) = z_0 \end{cases} \tag{3.1}$$

where A is an operator on a Banach space Z. Eq. (3.1) is autonomous, and we can write

$$z(t+h) = T(h)\, z(t) \qquad h \geq 0,$$

where T(h) is some linear mapping. Clearly,

$$\begin{cases} T(0) = \mathrm{id} & (3.2) \\ T(h_1+h_2) = T(h_1)\, T(h_2). & (3.3) \end{cases}$$

<u>Definition 3.1</u>. <u>A strongly continuous semigroup</u> is a family of mappings $\{T(h);\ h\geq 0\}$ that satisfies (3.2), (3.3) and

$$\lim_{h\downarrow 0} \| T(h)z - z \|_Z = 0, \qquad z \in Z.$$

■

In this case, $T(\cdot)$ was defined from A. It is also possible to reconstruct the operator that generates the semigroup from the semigroup itself.

<u>Definition 3.2</u>. The <u>infinitesimal generator</u> A of a strongly continuous semigroup T(h) is defined by

$$Az = \lim_{h\downarrow 0} \frac{1}{h} (T(h)z - z),$$

the domain of A being those $z \in Z$ for which the limit exists.

■

The operators on Z that generate strongly continuous semigroups (SCS) are characterised in the following theorem.

<u>Theorem 3.1</u>. (Hille-Yosida) A necessary and sufficient condition for a closed densely defined operator A to generate a SCS is that there are real numbers ω and M such that, for every $\lambda > \omega$, $(\lambda I-A)$ is invertible and

$$\| (\lambda I-A)^{-i} \| \leq \frac{M}{(\lambda-\omega)^i}, \qquad i = 1, 2, \ldots$$

<u>Proof</u>. See [1], Ch. 2. ■

It can be shown that T(t) always satisfies an estimate $\| T(t) \| \leq \overline{M}e^{\omega t}$ for some sufficiently large ω.

In Hilbert spaces the condition of Thm. 3.1 takes the form

$$\begin{cases} \mathrm{Re} < Az, z > \ \leq \omega \|z\|^2 \\ \mathrm{Re} < A^* z, z > \ \leq \omega \|z\|^2 . \end{cases} \quad (3.4)$$

Consider now the inhomogeneous equation

$$\begin{cases} \dot{z} = Az + f(t) \\ z(0) = z_0 . \end{cases} \quad (3.5)$$

Then, in analogy with what is done for ordinary differential equations, we may propose

$$z(t) = T(t)\, z_0 + \int_0^t T(t-s)\, f(s)\, ds \quad (3.6)$$

as a solution of (3.4). For $f \in C^1$ and $z_0 \in \mathcal{D}(A)$, this is in fact a solution (everywhere). This condition is too restrictive for applications, however. It is therefore customary to work with (3.6) as a solution for an arbitrary $f \in L^p$, $p \geq 1$. This is referred to as the mild solution of Eq. (3.5). It is in fact equivalent to the concept of weak solutions known from distribution theory.

Example 3.1. Let z be an $\mathbb{R}^n$-vector and consider

$$\dot{z} = Az.$$

The semigroup generated from this equation is

$$T(t) = e^{At}.$$

The ω of Hille-Yosida's theorem may be taken as $\max_i \mathrm{Re}\,\lambda_i(A)$. Condition (3.4) is satisfied with

$$< z_1, z_2 > \ = z_1^T P z_2$$

for a suitably chosen P. ■

Example 3.2. Consider the heat equation on the unit interval [0,1]:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad t > 0, \quad 0 < x < 1,$$

with boundary conditions

$$u(0,t) = u(1,t) = 0$$

and initial condition

$$u(x,0) = u_0(x).$$

This can be written as

$$\frac{du}{dt} = Au \qquad t > 0$$

where

$$\begin{cases} Av = \dfrac{\partial^2 v}{\partial x^2} & 0 < x < 1 \\ v(0) = v(1) = 0. \end{cases}$$

The scalar product is defined by

$$< u,v > = \int_0^1 u(x)\ v(x)\ dx.$$

Using partial integration we get

$$< u, Au > = \int_0^1 u \frac{\partial^2 u}{\partial x^2} dx = \Big|_0^1 u \frac{\partial u}{\partial x} - \int_0^1 \left(\frac{\partial u}{\partial x}\right)^2 dx =$$

$$= - \int_0^1 \left(\frac{\partial u}{\partial x}\right)^2 dx \leq - \pi^2 \int_0^1 u^2\, dx$$

by Rayleigh's inequality. Consequently,

$$< u, Au > \leq - \pi^2 ||u||^2,$$

so the condition of Hille-Yosida's theorem is satisfied with $\omega = - \pi^2$ (notice that A is self-adjoint). In this case, an explicit formula for the semigroup can be derived. Let $\sqrt{2}\, \sin(n \cdot \pi x)$, n = 1, 2, ..., be the (complete) system of orthogonal eigenfunctions of A, and expand $u_0(x)$ in a

Fourier series

$$u_0(x) = \sum_{i=1}^{\infty} c_i \cdot \sqrt{2}\,\sin(i\pi x),$$

where

$$c_i = \int_0^1 \sqrt{2}\,\sin(i\pi x)\; u_0(x)\; dx.$$

The complete solution of the equation is then

$$u(x,t) = \sum_{i=1}^{\infty} c_i\; e^{-(i\pi)^2 t}\; \sqrt{2}\,\sin(i\pi x).$$

Otherwise expressed,

$$(T(t)u_0)(x) = \int_0^1 G(x,y,t)\; u_0(y)\; dy,$$

where

$$G(x,y,t) = \sum_{i=1}^{\infty} e^{-(i\pi)^2 t} \cdot 2 \cdot \sin(i\pi x)\; \sin(i\pi y)$$

is the Green's function of the problem. ■

Example 3.3. Consider the functional differential equation

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{N} A_i\; x(t+\Theta_i) + \int_{-b}^{0} \bar{A}(\Theta)\; x(t+\Theta)\; d\Theta,$$

where $x(t) = h(t)$, $-b \leqslant t \leqslant 0$, is given and, $-b \leqslant \Theta_1 \leqslant \dots < \Theta_N < 0$. The scalar product chosen is

$$< x, y > \; = \; < x(0), y(0) >_H + \int_{-b}^{0} < x(t), y(t) >_H dt,$$

where $< , >_H$ is the scalar product of the Hilbert space H in which x takes its values. The abstract formulation of the above equation is

$$\begin{cases} \dot{z} = Az \\ z(0) = h, \end{cases}$$

where A is defined by

$$(Af)(\Theta) = \begin{cases} A_0 f(0) + \sum_{i=1}^{N} A_i f(\Theta_i) + \int_{-b}^{0} \bar{A}(\Theta) f(\Theta) d\Theta, & \Theta = 0 \\ \frac{df}{d\Theta}, & \Theta \neq 0. \end{cases}$$

■

For stability considerations, the number ω of the estimate $\|T(t)\| \leq Me^{\omega t}$ is of course crucial. Let $\sigma(A)$ be the spectrum of A, and set

$$\underline{\omega}(A) = \sup \{\text{Re } \lambda; \quad \lambda \in \sigma(A)\}$$

$$\overline{\omega}(A) = \inf \{\omega; \quad \|T(t)\| \leq Me^{\omega t}, \text{ some M and all } t \geq 0\}.$$

Then

$$\underline{\omega}(A) \leq \overline{\omega}(A), \tag{3.7}$$

so exponential stability implies $\underline{\omega}(A) < 0$. The inequality in (3.7) may be strict; the spectrum of A may even be empty. Equality holds for instance for compact operators or self-adjoint operators that are bounded from above.

3.1.2 Lyapunov functions

Lyapunov's theorem has the following infinite-dimensional counterpart.

Theorem 3.2. Let Z be Hilbert space. Then the system

$$\begin{cases} \dot{z} = Az \\ z(0) = z_0 \end{cases}$$

is exponentially stable if and only if there exists a non-negative definite operator P satisfying

$$\langle Az, Pz \rangle + \langle Pz, Az \rangle = - \langle z, z \rangle . \tag{3.8}$$

If T(t) is the corresponding semigroup, an equivalent condition is

$$\int_0^\infty \|T(t)z\|^2 \, dt < \infty \qquad \text{for all } z \in \mathcal{Z}.$$

Proof: See Datko [2]. □

Example 3.4. ([3]) Consider the second-order system

$$\begin{cases} \ddot{x} + \alpha\dot{x} - Ax = 0 \\ \dot{x}(0) \in \mathcal{H}, \quad x(0) \in \mathcal{D}((-A)^{1/2}), \end{cases} \tag{3.9}$$

where A is a self-adjoint, non-positive, not necessarily bounded operator, and α is > 0. Obviously, the system (3.9) will not be exponentially stable if $\underline{\omega}(A) = 0$, so assume $\underline{\omega}(A) < 0$ and equip $\mathcal{D}((-A)^{1/2})$ with the norm $\| (-A)^{1/2} z \|$. Let $\mathcal{Z} = \mathcal{D}((-a)^{1/2}) \times \mathcal{H}$ and set $\left(z = \begin{pmatrix} x \\ y \end{pmatrix}\right)$

$$\|z\|_{\mathcal{Z}}^2 = \| (-A)^{1/2} x \|^2 + \|y\|^2.$$

An equivalent form of Eq. (3.9) is

$$\frac{dz}{dt} = \bar{A}z$$

with

$$\bar{A} = \begin{pmatrix} 0 & I \\ A & -\alpha I \end{pmatrix}.$$

It is possible to solve the formal Lyapunov equation (3.8) as in [3], but instead a physical argument will be used. Assume for instance that

$$Au = \begin{cases} \dfrac{\partial^2 u}{\partial x^2} \\ u(0,t) = u(1,t) = 0, \end{cases}$$

in which case (3.9) is a damped wave equation describing the vibrations of string fixed at $x = 0$ and $x = 1$. The energy E(t) stored in the string at time t is given by

$$E(t) = \int_0^1 \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2\right] dx. \tag{3.10}$$

Hence

$$\frac{dE(t)}{dt} = \int_0^1 \left(2\,\frac{\partial u}{\partial x}\cdot\frac{\partial^2 u}{\partial x\partial t} + 2\,\frac{\partial u}{\partial t}\cdot\frac{\partial^2 u}{\partial t^2}\right) dx =$$

$$= \int_0^1 \left[-2\,\frac{\partial^2 u}{\partial x^2}\cdot\frac{\partial u}{\partial t} + 2\,\frac{\partial u}{\partial t}\left(-\alpha\,\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2}\right)\right] dx =$$

$$= -\,2\alpha \int_0^1 \left(\frac{\partial u}{\partial t}\right)^2 dx.$$

Thus

$$\lim_{t\to\infty} \frac{\partial u}{\partial t} = 0$$

Using again Equation (3.9) we get

$$\lim_{t\to\infty} \frac{\partial^2 u}{\partial x^2} = 0.$$

The limit $u(x,\infty)$ is thus linear and identically zero due to the boundary conditions.

The general Lyapunov function corresponding to (3.10) is

$$\| (-A)^{1/2} x \|^2 + \|y\|^2,$$

i.e.

$$\|z\|_Z^2.$$

■

So far our only concern has been with linear equations. Nonlinear problems have been treated by Pritchard and Zabczyk in [3]. The theory becomes very technical, since not even existence problems are trivial in nonlinear PDE's. Below an example is given to illustrate the use of Lyapunov functions.

<u>Example 3.5</u>. Consider the nonlinear diffusion equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + u^3 & 0 < x < 1, \quad t < 0 \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = u_0(x). \end{cases}$$

$V = \int_0^1 u^2\,dx$ is a Lyapunov function for the linear equation, so there is reason to believe that it will work at least locally for the full nonlinear equation. We have

$$\frac{dV}{dt} = \int_0^1 2u \frac{\partial u}{\partial t}\,dx = \int_0^1 2u \left(\frac{\partial^2 u}{\partial x^2} + u^3\right) dx =$$

$$= -2 \int_0^1 \left(\frac{\partial u}{\partial x}\right)^2 dx + 2 \int_0^1 u^4\,dx.$$

Now

$$u(x,t) = \int_0^x \frac{\partial u}{\partial x}(x,t)\,dx$$

implies

$$\sup_{x \in [0,1]} |u(x,t)|^2 \leqslant \int_0^1 \left(\frac{\partial u}{\partial x}(x,t)\right)^2 dx$$

by Schwarz' inequality. Further,

$$\int_0^1 u(x,t)^4 dx \leqslant \sup_{x \in [0,1]} |u(x,t)|^2 \int_0^1 (u(x,t))^2\,dx,$$

so, provided that

$$\int_0^1 u_0(x)^2 dx \leqslant 1,$$

one obtains

$$\int_0^1 (u(x,t_2))^2 dx \leqslant \int_0^1 (u(x,t_1))^2 dx, \quad \text{for } t_2 > t_1. \tag{3.11}$$

With somewhat more labour, it may be shown that

$$\int_0^1 u_0^2(x)\ dx \leqslant 4$$

is sufficient to ensure (3.11) and thus stability. ∎

The generation of Lyapunov functions for functional differential equations may be cumbersome. The dependence of $\frac{dx}{dt}(t)$ upon <u>all</u> $x(t-\Theta)$, $\Theta \in [0,\Theta_0]$, implies that the Lyapunov function should contain all these values.

<u>Example 3.6</u>. Consider

$$\begin{cases} \dot{x}(t) = ax^3(t) + bx^3(t-r), & r > 0 \\ x(t) = h(t), & t \in [-r,0]. \end{cases}$$

Let

$$V(x(t)) = cx^4(t) + \int_{t-r}^{t} x^6(s)\ ds.$$

Then

$$\frac{dV}{dt} = 4cx^3(t)(ax^3(t) + bx^3(t-r)) + x^6(t) - x^6(t-r).$$

With $c = -1/2a$, one gets

$$\frac{dV}{dt} = -(x^6(t) + \frac{2b}{a}x^3(t)\ x^3(t-r) + x^6(t-r)).$$

From this we conclude:

i) If $a < 0$, $|b| < a$, $V > 0$, and $\dot{V} \leqslant 0$ with equality iff $x(t) = x(t-r) = 0$. This implies stability.

ii) If $a < 0$, $b = a$, $\dot{V} = 0$ iff $x(t) = -x(t-r)$ i.e. $\dot{x} = 0$, $x = C$. Since $x(t) = -x(t-r)$, $C = 0$. Again we conclude stability.

iii) If $a < 0$, $b = -a$, each solution will approach a constant (not necessarily 0).

iv) If $a > 0$, $|b| < a$, $\dot{V} \leq 0$, and V takes negative values in any neighbourhood of the origin. This implies instability of the null solution. ■

The above argument was based on a functional. In some instances, it may in fact be possible to use a function, provided that certain resctrictions are imposed on the initial data. The idea, which is due to Razumikhin, is best illustrated by an example.

Example 3.7. Consider the equation

$$\dot{x}(t) = -a(t)x(t) - b(t)\, x(t-r(t)),$$

where $a(\cdot)$, $b(\cdot)$, and $r(\cdot)$ are bounded continuous functions and $0 \leq r(t) \leq r_0$, Let $V(x) = x^2/2$. Then

$$\dot{V}\, x(t) = -a(t)x^2(t) - b(t)\, x(t)(x(t-r(t))) \leq$$
$$\leq -(a(t) - |b(t)|)\, x^2(t),$$

provided that $x(t-r(t)) \leq x(t)$. So if $|b(t)| \leq a(t)$, $\dot{V}(x(t)) \leq 0$ provided that $V(x(t)) \geq V(x(t-r(t)))$. This implies stability. If further $a(t) \geq a_0$ and there is a $k < 1$ such that $|b(t)| \leq ka_0$, uniform asymptotic stability obtains. ■

For a proof of Razumikhin's result, the reader is referred to Hale [4], which gives a comprehensive treatment of stability theory for functional differential equations.

3.1.3 Frequency domain criteria

Several of the frequency-domain criteria treated in Ch. 1 may be extended to Hilbert or Banach spaces. In principle, these criteria are intended as a tool of analysis for feedback systems, but since any practical control system necessarily works with a finite number of inputs, they are of limited interest in this respect. The example below illustrates the use of Cook's criterion (Thm. 1.7) in a free system.

Example 3.8. ([5]) Consider the quasilinear damped wave equation

$$\begin{cases} \dfrac{\partial^2 z}{\partial t^2} + \lambda \dfrac{\partial t}{\partial t} - \mu \dfrac{\partial^2 z}{\partial x^2} + (1 + 2\delta \cos(\pi x))\, f(z) = 0, \\ \qquad\qquad\qquad\qquad\qquad 0 < x < 1, \quad t > 0 \\ z(0,t) = z(1,t) = 0. \end{cases}$$

Here $f(\cdot)$ is a possibly timevarying nonlinear function satisfying the usual conicity condition

$$az^2 \lesssim zf(z) \lesssim bz^2 \quad \text{for all } z.$$

An alternative formulation is

$$\begin{cases} \dfrac{\partial^2 z}{\partial t^2} + \lambda \dfrac{\partial z}{\partial t} - \mu \dfrac{\partial^2 z}{\partial x^2} = -\,(1 + 2\delta \cos(\pi x))\, u \\ u = f(z), \end{cases}$$

so that the equation formally describes a feedback system composed of a linear link in the forward path and a nonlinear feedback. The eigenfunctions of $\frac{\partial^2}{\partial x^2}$ subject to the given boundary conditions are $\sqrt{2}\sin(k\pi x)$, $k = 1, 2, \ldots$. Expanding z and u in Fourier series and taking Laplace transforms yields

$$(s^2 + \lambda s + \mu \cdot k^2)\ \hat{z}_k(s) = \hat{u}_k(s) + \delta(\hat{u}_{k-1}(s) + \hat{u}_{k+1}(s)).$$

The transfer matrix $G(s) = \{g_{jk}(s)\}$ is thus given by

$$g_{jk}(s) = \begin{cases} \dfrac{1}{s^2 + \lambda s + \mu j^2} & k = j \\[2ex] \dfrac{\delta}{s^2 + \lambda s + \mu j^2} & k = j \pm 1 \\[2ex] 0 & \text{otherwise.} \end{cases}$$

Stability can now be deduced from a plot of the mean Geršgorin bands and the standard circle of radius $\frac{1}{2}\left|\frac{1}{a}-\frac{1}{b}\right|$ centered at $\left(\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right), 0\right)$. ■

3.2 Controlled systems

It is possible to develop a theory for control of infinite--dimensional systems based on the semigroup approach sketched in the previous section. The theory resembles the standard finite-dimensional theory in many respects. For instance, the linear-quadratic optimal-control problem leads to an operator Riccati equation. Provided that the system is stabilisable, this equation has a steady-state solution, which furnishes the solution of the infinite--horizon problem.

The stabilisation problem requires some care, since there are several counterparts to finite-dimensional controllability. The major general result in this direction that does not lay restrictions on the semigroup and its infinitesimal generator is that exact null controllability with L^2-controls implies exponential stabilisability (in Hilbert spaces). Other conditions that guarantee stabilisability can be found in [1], Ch. 3.

3.2.1 Partial differential equations

An established way to produce a stable feedback regulator for linear systems is to solve an infinite-time quadratic regulator problem. This will be done now for the special case of controlling the temperature of a rod.

Example 3.9. ([1]) Consider the controlled heat equation

$$\begin{cases} \dfrac{\partial z}{\partial t} = \dfrac{\partial^2 z}{\partial x^2} + u(x,t) & 0 < x < 1, \quad t > 0 \\ z(0,t) = z(1,t) = 0 \\ z(x,0) = z_0(x). \end{cases} \tag{3.12}$$

Suppose we want to minimize

$$J(u) = \int_0^1 z(x,t_1)^2\,dx + \int_0^{t_1}\int_0^1 (qz(x,t)^2 + u(x,t)^2)\,dx\,dt. \tag{3.13}$$

(3.12), (3.13) is a special case of the following general LQOC problem:

$$\min_u J(u) = \langle z(t_1), Q_1 z(t_1)\rangle + \\ + \int_0^{t_1} (\langle z(t), Qz(t)\rangle + \langle u(t), Ru(t)\rangle)\,dt,$$

where z and u are related via

$$z(t) = z_0 + \int_0^t T(t-s)\,Bu(s)\,ds, \qquad 0 \leq t \leq t_1.$$

z and u take their values in Hilbert spaces $\mathcal{H}$ and $\mathcal{U}$ respectively. Let A be the infinitesimal generator of the semigroup T(t). Then the optimal input can be obtained as

$$u(t) = -R^{-1} B^* P(t)\, z(t),$$

where P(t) is the (unique) solution of the operator Riccati equation

$$\begin{cases} -\dfrac{d}{dt}\langle P(t)h,h\rangle = \langle P(t)h,Ah\rangle + \langle Ah,P(t)h\rangle + \langle Qh,h\rangle - \\ \qquad - \langle P(t)BR^{-1}B^*P(t)h,h\rangle; \\ P(t_1) = Q_1. \end{cases} \tag{3.14}$$

Here h is a typical element of $\mathcal{D}(A)$.

Let $\{\varphi_i\}_{i=0}^{\infty}$ be the orthonormal set of eigenfunctions of A. Assume that P(t) can be expanded as

$$P(t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} P_{ij}(t) < \cdot, \varphi_i > \varphi_j .$$

Then (3.14) takes the form

$$- \dot{p}_{ij} = (\lambda_i + \lambda_j)\, p_{ij} + < Q\varphi_i, \varphi_j > -$$

$$- \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} p_{ik}\, p_{j\ell} < R^{-1}B^*\varphi_k, B^*\varphi_\ell > ,$$

λ_i being the eigenvalues of A. In our case $\varphi_i = \sin(i\pi x)$, $i = 1, 2, \ldots,$ and $\lambda_i = (i\pi)^2$. This yields

$$\begin{cases} \dot{p}_{ij} = \pi^2(i^2 + j^2)\, p_{ij} - q \cdot \delta_{ij} + \sum_{k=0}^{\infty} p_{ik}\, p_{jk} \\ p_{ij}(t_1) = \delta_{ij} \end{cases}$$

($\delta_{ij} = 0,\ i \neq j,\ = 1,\ i = j$). The solution is

$$\begin{cases} p_{ij} \equiv 0 & i \neq j \\ p_{ii}(t) = \dfrac{a_i(1-b_1) - b_i(1-a_i)\, e^{-\alpha_i(t-t_1)}}{(1-b_1) - (1-a_i)\, e^{-\alpha_i(t-t_1)}} \end{cases}$$

where

$$\begin{cases} \alpha_i = 2\sqrt{\pi^4 i^4 + q} \\ a_i = -\pi^2 i^2 - \sqrt{\pi^4 i^4 + q} \\ b_i = -\pi^2 i^2 + \sqrt{\pi^4 i^4 + q} . \end{cases}$$

It turns out that the expansion of P(t) is in fact well-defined. With

$$u(t) = \sum_{i=0}^{\infty} u_i(t)\, \varphi_i, \qquad z(t) = \sum_{i=0}^{\infty} z_i(t)\, \varphi_i,$$

the optimal input is obtained as

$$u_i(t) = p_{ii}(t)\; z_i(t).$$

As $t_1 \to \infty$, $p_{ii} \to b_i$. Notice that

$$b_i = \frac{q}{\pi^2 i^2 + \sqrt{\pi^4 i^4 + q}},$$

so that b_i converges rapidly to zero. ■

As in finite-dimensional problems, a realisation of the optimal feedback regulator requires knowledge of the state. Even if $z(x,t)$ is known for all t, a computation of the infinite member of coefficients $z_i(t)$ is required. In practice, however, only a few coefficients need to be computed. The problem is rather that Equation (3.12) does not model a realistic situation, since the control is distributed over the entire interval $]0,1[$. In a typical application, the temperature or the heat flow at the boundary is controlled. In such cases, the model will contain unbounded operators.

Example 3.9, cont'd. Consider again the problem of controlling the temperature of a rod, but this time from the boundary:

$$\begin{cases} \dfrac{\partial z}{\partial t} = \dfrac{\partial^2 z}{\partial x^2} \\ -\Delta \dfrac{\partial z}{\partial x}(0,t) = u(t); \quad z(1,t) = 0 \\ z(x,0) = z_0(x). \end{cases} \tag{3.15}$$

$\Delta\frac{\partial z}{\partial x}(0,t)$ denotes the jump of the derivative at $x=0$, and the physical meaning is that the heat flow is controlled. By a standard trick, (3.15) can be brought into

$$\begin{cases} \dfrac{\partial z}{\partial t} = \dfrac{\partial^2 z}{\partial x^2} + u\delta(x) \\ z(0,t) = z(1,t) = 0 \\ z(x,0) = z_0(x), \end{cases}$$

$\delta(x)$ being the Dirac distribution. If a solution is sought in terms of an eigenfunction expansion as in the previous calculation, this leads to an infinite system of coupled ODE's. Alternatively, $P(t)z$ may be expressed using a kernel:

$$(P(t)z)(x) = \int_0^1 K(x,y,t)\, z(y)\, dy.$$

This leads to a PDE in the kernel (see [3], p. 243). The optimal input is again a function of the whole state.

If the temperature at the endpoint is controlled, the boundary conditions are

$$z(0,t) = u(t); \quad z(1,t) = 0,$$

and the equivalent equation is

$$\begin{cases} \dfrac{\partial z}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} - u\delta'(x) \\ z(0,t) = z(1,t) = 0 \\ z(x,0) = z_0(x). \end{cases}$$

Also in this case a Riccati equation results, and a solution exists if the cost functional is a pure integral. ■

This section will be closed by a brief study of what is perhaps the most common set-up, namely the case when the temperature is measured at one point in the interior, and, for instance, the temperature at one end-point is the input. The system is thus SISO. What makes it a non-standard problem is that the transfer function is transcendental.

<u>Example 3.9, cont'd.</u> Consider

$$\begin{cases} \dfrac{\partial z}{\partial t} = \dfrac{\partial^2 z}{\partial x^2} \\ z(0,t) = u(t); \; z(1,t) = 0 \\ z(x,0) = z_0(x) \\ y(t) = z(x_0,t). \qquad 0 < x_0 < 1 \end{cases}$$

Let $\hat{z}(x,s)$ be the Laplace transform with respect to time. Then (forgetting about initial conditions)

$$\begin{cases} \dfrac{d^2\hat{z}(x,s)}{dx^2} - s\hat{z}(x,s) = 0 \\ \hat{z}(0,s) = \hat{u}(s); \quad \hat{z}(1,s) = 0, \end{cases}$$

so that

$$\hat{y}(s) = \hat{z}(x_0,s) = \frac{\sin h(1-x_0)\sqrt{s}}{\sin h\sqrt{s}}\;\hat{u}(s)$$

with a suitable definition of the root function. The open--loop system has poles at $s = -n^2 \cdot \pi^2$, $n = 0, 1, 2, \ldots,$ of which at least the one at the origin is also a zero. The Nyquist locus of G(s) is shown in Fig. 3.1 for the case $x_0 = 0.5$.

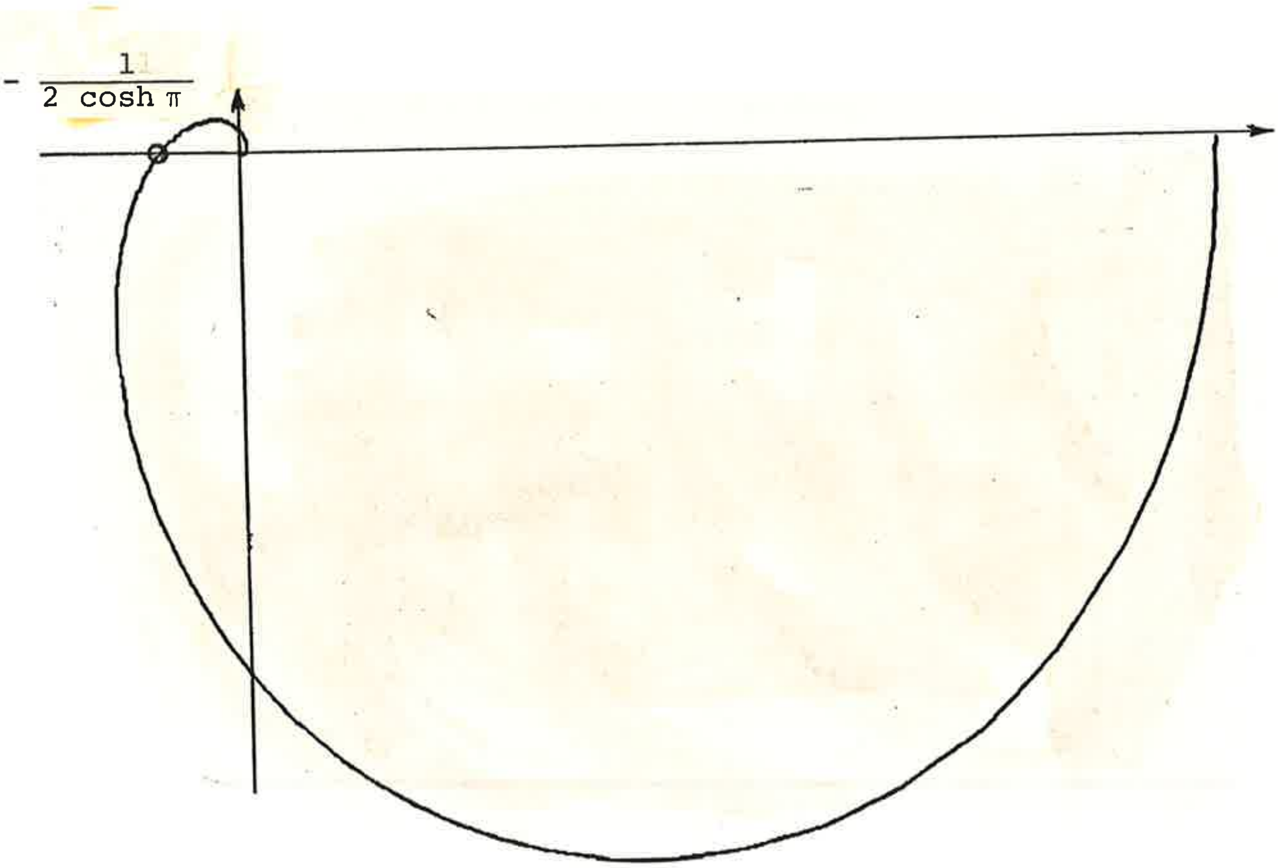

Fig. 3.1 - Nyquist locus of $G(s) = \dfrac{\sin h\,0.5\sqrt{s}}{\sin h\sqrt{s}}$.

The Hurwitz range is $[0, 2 \cosh \pi] \approx [0, 23.2]$. For non-linear feedback, information can be obtained from the Popov criterion or the circle criterion. □

3.2.2 Delay equations

Synthesis in control systems governed by delay equations may be treated within the same framework as the PDE case. Ref. [1], Ch. 1, contains an example and further references.

As in the PDE case, practical control problems often lend themselves to descriptions based on standard transfer matrices with transcendental elements. In such cases, the Nyquist criterion and its ramifications remain an efficient tool for stability analysis.

References

[1] Curtain, R.F., Pritchard, A.J. (1978): Infinite Dimensional Linear Systems Theory. Lecture Notes in Control and Information Sciences 8. Springer, Berlin etc.

[2] Datko, R. (1970): "Extending a theorem of A.M. Liapunov to Hilbert space", J. Math. Anal. Appl. 32, pp. 610--616.

[3] Pritchard, A.J., Zabczyk, J. (1977): "Stability and stabilizability of infinite dimensional systems", Control Theory Centre Report No. 70, Univ. of Warwick.

[4] Hale, J. (1977): Theory of Functional Differential Equations. Springer, New York etc.

[5] Cook, P.A. (1975): "Circle criteria for stability in Hilbert space", SIAM J. Control 13, No. 3, pp. 593--610.

CHAPTER 4. STOCHASTIC STABILITY

The present chapter deals with stability in random-parameter systems. Section 1 contains the basic stability definitions and a brief account of stochastic Lyapunov functions. The following section gives criteria for moment stability and almost-sure stability in free systems. The chapter is closed with a discussion of the linear-quadratic optimal control problem in white-noise parameter systems and its implication for the stabilisability problem.

4.1 Preliminaries

4.1.1 Basic definitions

There are several convergence modes to choose between when adapting the Lyapunov stability concepts to a stochastic framework. Almost-sure convergence and mean-square convergence seem to be the most relevant ones. The definitions given suppose that the null solution is being studied. The time variable may be continuous or discrete.

Definition 4.1. The null solution is said to be almost surely stable (or stable with probability one) if for all $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ there exists a $\delta > 0$ such that

$$P\left(\sup_{\|x_0\| < \delta} \; \sup_{t \geq t_0} \| x(t;\, x_0, t_0) \| > \varepsilon_1 \right) < \varepsilon_2 .$$

■

Alternatively,

$$P\left(\lim_{\|x_0\| \to 0} \; \sup_{t \geq t_0} \| x(t;\, x_0, t_0) \| = 0 \right) = 1,$$

which means that the deterministic stability definition holds w.p. 1 (i.e. for all sample paths).

Definition 4.2. The null solution is almost surely attractive if there exists a $\delta > 0$ such that $\|x_0\| \leq \delta$ implies

$$P(\lim_{t\to\infty} \| x(t; x_0, t_0) \| = 0) = 1.$$

■

Definition 4.3. The equilibrium is almost surely asymptotically stable if it is stable and attractive w.p. 1. ■

Definition 4.4. The null solution is p-th mean asymptotically stable if the function

$$E \{ \| x(t; x_0, t_0) \|_p^p \}$$

is asymptotically stable. It is p-th mean exponentially stable if there is a C and an $\alpha > 0$ such that

$$E \{ \| x(t; x_0, t_0) \|_p^p \} \leq C \, \| x_0 \|_p^p \cdot \exp(-\alpha(t-t_0)).$$

■

In general, there are no implications between these modes of convergence (see Chung [1], Ch. 4). However, if the L^p-moments converge sufficiently fast, p-th mean stability implies almost-sure stability (see the following paragraph). For linear Itô-equation, there is an even more precise result: almost-sure stability is equivalent to p-th mean stability as p tends to zero (see Kozin and Sugimoto [2]).

4.1.2 Stochastic Lyapunov functions

The idea of using Lyapunov functions in the study of random-parameter systems dates back to the fifties (Bertram and Sarachik [3], Kats and Krasovskii [4]). Recall that a

Lyapunov function for a deterministic system is a function which is decreasing along the trajectories of the system. In stochastic systems, one calculates the expected rate of change of the Lyapunov function, conditioned with respect to the present state. If this is negative, stability can be inferred from a supermartingale theorem. No proof will be given here; the reader is referred to Kushner [5] for an extensive discussion.

Theorem 4.1. Let $x(t)$, $t \geq t_0$, be a discrete-time Markov process, $V(x)$ a non-negative function and set $\Omega_M = \{x;\ V(x) \leq M\}$. Assume that in Ω_M,

$$E\{V(x(t+1))\,|\,x(t)=x\} - V(x) = -W(x) \leq 0. \tag{4.1}$$

Then

i) $P\{\sup_{t \geq t_0} V(x(t)) \geq M \,|\, x(t_0)=x\} \leq V(x)/M,$

and

ii) $\lim_{t\to\infty} W(x(t)) = 0$ with probability $\geq (1 - V(x)/M)$.

■

In continuous-parameter processes, the condition becomes somewhat more technical.

Definition 4.5. The function φ is said to be in the domain of the weak infinitesimal operator L if

i) $$\lim_{\delta\to 0} \frac{E\{\varphi(x(t+\delta))\,|\,x(t)=x\} - \varphi(x)}{\delta} = (L\varphi)(x) \text{ exists } = \psi(x)$$

and

ii) $$\lim_{\delta\to 0} E\{\psi(x(t+\delta))\,|\,x(t)=x\} = \psi(x).$$

■

The continuous-time version of Thm. 4.1 simply replaces condition (4.1) by

$$(LV)(x) = -W(x) \leq 0. \tag{4.2}$$

As in deterministic systems, the results obtained depend critically on the Lyapunov function candidate chosen. The following theorem establishes a link between moment stability and the existence of Lyapunov functions.

Theorem 4.2.

i) If the equilibrium is p-th mean exponentially stable for some $p > 0$, there exists a time-invariant function V solving

$$E\{V(x(t+1)) \mid x(t)=x\} - V(x) = -W(x) \tag{4.1}$$

with V and W subject to

$$\begin{cases} \alpha_1 \|x\|_p^p \leq V(x) \leq \alpha_2 \|x\|_p^p \\ \alpha_3 \|x\|_p^p \leq W(x) \leq \alpha_4 \|x\|_p^p, \end{cases} \tag{4.3}$$

all $\alpha_i > 0$. In particular, p-th mean exponential stability for some $p > 0$ implies stability w.p. 1.

ii) If Eq. (4.1) has a solution V with V and W subject to (4.3), the equilibrium is p-th mean exponentially stable.

Proof: The proof uses standard arguments and can be found for instance in Molander [6]. ∎

4.2 Free systems

4.2.1 Mean-square stability

The moment stability problem will be studied only for $p=2$. One condition for mean-square stability in linear systems with white-noise parameter disturbances was found by

Willems and Blankenship [7]. The proof will be sketched here using a Lyapunov function argument.

Consider thus the system

$$x(t+1) = Ax(t) - f(t)\ bc^T x(t), \qquad (4.4)$$

where $x(\cdot)$, b, and c are $\mathbb{R}^n$ vectors, A is $n\times n$, and $f(\cdot)$ is a white-noise sequence with zero mean and variance σ. To determine the stability limit, choose the quadratic Lyapunov function candidate $V(x) = x^T Px$, where P solves

$$A^T PA - A = - cc^T.$$

This solution exists and is positive definite if A is stable and (c^T, A) is an observable pair. One gets

$$E\ \{V(x(t+1))\ |\ x(t) = x\} - V(x) =$$

$$= x^T (A^T PA - P + \sigma^2 b^T Pb\ cc^T)\ x = - x^T cc^T x(1 - \sigma^2 b^T Pb).$$

This implies mean-square stability (cf. Thm. 4.2) if

i) The noise-free system is asymptotically stable, and

ii) $\sigma^2 < (b^T Pb)^{-1}$.

Condition ii) can be made more explicit. In fact,

$$P = \sum_{k=0}^{\infty} (A^T)^k\ cc^T\ A^k$$

so that

$$b^T Pb = \sum_{k=0}^{\infty} b^T (A^T)^k\ cc^T\ A^k\ b = \frac{1}{2\pi i} \oint_{|z|=1} G(z)\ G(z^{-1})\ \frac{dz}{z}. \qquad (4.5)$$

The condition ii) is consequently a small-gain condition, where the gain of the forward loop is obtained from the energy integral (4.5), and the gain of the feedback noise is determined by the variance. An analogous formula holds in continuous-time. The condition can be shown to be necessary and sufficient.

In case the system equation is

$$x(t+1) = Ax(t) - \sum_{i=0}^{m} f_i(t)\, b_i\, c_i^T\, x(t),$$

where the f_i's are mutually uncorrelated with variance σ^2, the energy integral (4.5) will yield a matrix. The necessary and sufficient condition for mean square stability is then that this matrix have no eigenvalues greater than σ^{-2}.

4.2.2 Almost-sure stability

The almost-sure stability problem is considerably more difficult. Only in exceptional cases explicit conditions can be given.

Consider the Markov chain x(t) given by

$$x(t+1) = A(t)\, x(t), \tag{4.6}$$

where $A(\cdot)$ are independent identically distributed matrices. The problem is to determine under what conditions $x(t) \to 0$ w.p. 1 as $t \to \infty$. Let $z(t) = x(t) / |x(t)|$. Then z(t) will be a Markov chain on the unit sphere. Assume that it is ergodic and let the invariant measure be denoted by $dP(\cdot)$. Clearly one has

$$\log |x(t)| = \log |x(t-1)| + \log |A(t-1)\, z(t-1)| =$$

$$= \log |x_0| + \sum_{i=0}^{t-1} |A(i)\, z(i)|.$$

If the conditions of the strong law of large numbers are satisfied, the sum on the right-hand side will behave like its mean value, or, more precisely,

$$\log |x(t)| \to + \text{ or } - \infty$$

according as

$$E\{\log|Az|\} > \text{ or } < 0. \tag{4.7}$$

Here, the expectation is with respect to the ergodic measure dP(·) on the unit sphere and the original distribution of A.

Example 4.1. For the scalar equation

$$x(t+1) = a(t)\, x(t),$$

one obtains simply

$$E\{\log|a|\} < 0$$

as a necessary and sufficient condition for stability. ■

In general, it is not possible to obtain an explicit formula for the invariant measure involved in (4.7). Two-dimensional Itô equations form an exception. The reason is that the z-process is also an Itô process in this case. Since it is one-dimensional, the invariant measure can be obtained from the steady-state Fokker-Planck equation, which is an ODE in this case. The reader is referred to articles by Khasminskii [8] and Kozin and Prodromou [9] for further details. An example is given below (Ex. 4.2).

Normally, one is referred to conditions that are only necessary. The following theorem is due to Infante [10].

Theorem 4.3. Consider the equation

$$\dot{x}(t) = A(t)\, x(t), \tag{4.8}$$

where the entries of A are assumed to be ergodic. If there exists a matrix $P > 0$ and an $\varepsilon > 0$ such that

$$E\{\lambda_{max}(A^T + PAP^{-1})\} \leq -\varepsilon,$$

then the null solution of Eq. (4.8) is almost surely asymptotically stable in the large.

Proof: Choose $V(x) = x^T Px$ as a Lyapunov function candidate. Alone the trajectories of (4.8), one has

$$\frac{\dot{V}(x)}{V(x)} = \frac{x^T(A^TP + PA)x}{x^TPx} \leq \lambda_{max}(A^T + PAP^{-1})$$

(see Gantmacher [11], Ch. 10). Consequently,

$$V(x(t)) \leq V(x(t_0)) \exp\left(\int_{t_0}^{t} \lambda_{max}(A^T + PAP^{-1})\, dt\right) =$$
$$= V(x(t_0)) \exp\left((t-t_0) \cdot \frac{1}{t-t_0} \int_{t_0}^{t} \lambda_{max}\, dt\right).$$

If $E\{\lambda_{max}\} \leq -\varepsilon$, then

$$\frac{1}{t-t_0} \int_{t_0}^{t} \lambda_{max}\, dt \leq -\frac{\varepsilon}{2}$$

for $t >$ some T. This shows that $V(x(t)) \to 0$ as $t \to \infty$, and stability follows. ■

Example 4.2. Consider the equation

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -2\zeta x_2 - (1 + f(t))\, x_1, \end{cases} \tag{4.9}$$

where $f(t)$ is assumed to be ergodic. Suppose that the stability boundary is sought in terms of $E\{f^2\}$. A lengthy computation then yields

$$E\{f^2\} < 4\zeta^2$$

as a sufficient condition for stability. This result may be sharpened if further assumptions on the noise are introduced, for instance that f is Gaussian or periodic.

For the corresponding Itô equation,

$$\begin{cases} dx_1 = x_2\, dt \\ dx_2 = -2\zeta\, x_2\, dt - x_1\, dt - \sigma x_1\, dw, \end{cases} \tag{4.9'}$$

where dw is the unit Wiener process, the exact stability boundary can be computed from Khasminskii's results. The results are displayed in Fig. 4.1.

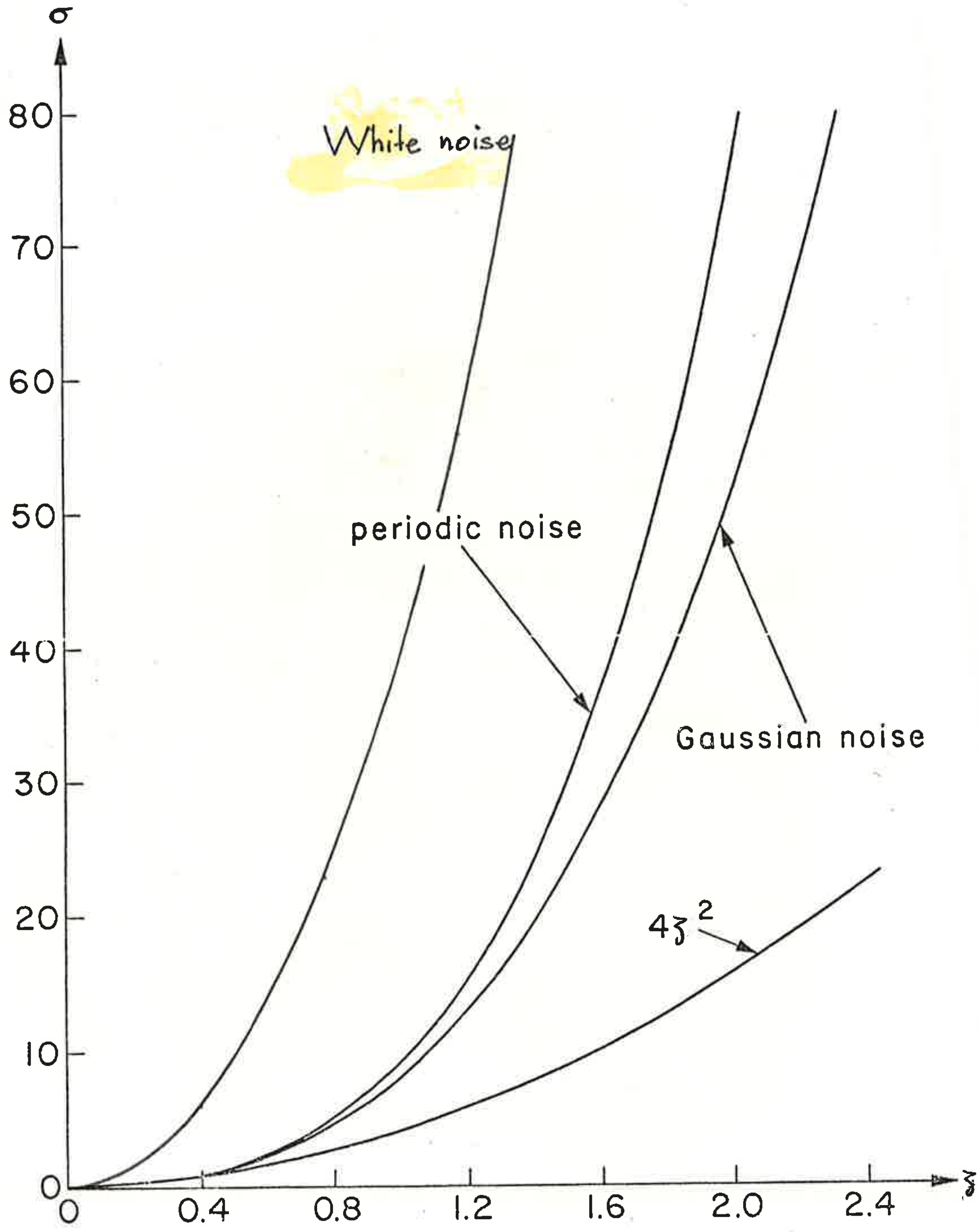

Fig. 4.1 - Stability boundaries for Eqs. (4.9), (4.9') with various assumptions on the noise.

To illustrate the use of stochastic Lyapunov functions, this section will be concluded with a nonlinear Itô equation.

Example 4.3. [5] Consider the Itô equation

$$\begin{cases} dx_1 = x_2\, dt \\ dx_2 = -\, g(x_1)\, dt - a x_2\, dt - x_2\, \sigma\, dw, \end{cases} \tag{4.10}$$

where

$$\begin{cases} dw \text{ is the unit Wiener process} \\ \int_0^x g(y)\, dy \to \infty \quad \text{as} \quad x \to \pm\infty \\ x g(x) > 0, \quad x \neq 0 \\ g(0) = 0. \end{cases}$$

In order to use Thm. 4.1, we must compute $(LV)(x)$. For Itô equations, the infinitesimal operator L is given by the Kolmogorov backward operator $\mathcal{L}$ (see Åström [12], pp. 72, 74). More precisely, if

$$dx = f(x)\, dt + \sigma(x)\, dw, \quad x \in \mathbb{R}^n,$$

$$(LV)(x) = (\mathcal{L}V)(x) = \sum_{i=1}^{n} f_i(x) \frac{\partial V}{\partial x_i} + $$

$$+ \frac{1}{2} \sum_{i,j=1}^{n} \sigma_i(x)\, \sigma_j(x) \frac{\partial^2 V}{\partial x_i \partial x_j}.$$

For Equation (4.10), try

$$V(x_1, x_2) = x_2^2 + 2 \int_0^{x_1} g(y)\, dy$$

as a candidate (this is the Lure function used in the analysis of the noise-free system). One obtains

$$(\mathcal{L}V)(x) = x_2 \cdot 2g(x_1) + (-g(x_1) - a x_2) \cdot 2x_2 + \frac{1}{2} \cdot \sigma^2 x_2^2 \cdot 2 =$$

$$= (-2a + \sigma^2)\, x_2^2 .$$

From Thm. 4.1, $x_2 \to 0$ w.p. 1 as $t \to \infty$ if $2a > \sigma^2$. With some more labour it can be shown that also $x_1 \to 0$ w.p. 1. ∎

4.3 The stabilisation problem

The linear-quadratic optimal control problem for linear systems with white-noise disturbances in the parameters was first treated in full generality by Wonham [13]. Wonham considered the continuous-time problem, but the discrete-time version, being conceptually and computationally simpler, will be treated here.

Consider thus the system described by the difference equation

$$x(t+1) = (A + \sum_{i=1}^{k} \kappa_i(t)\, A_i)\, x(t) + (B + \sum_{i=1}^{\ell} \lambda_i(t)\, B_i)\, u(t). \qquad (4.11)$$

Here κ_i and λ_i are independent, unit white-noise sequences. The object is to minimise, with respect to u, the performance index

$$J_T = E\left\{ \sum_{s=t_0}^{T} (x(s)^T Q x(s)) + u(s)^T R u(s) \right\}.$$

As usual, the state is assumed accessible to measurement.

Using dynamic programming, it can be shown that the optimal input is

$$u^*(t) = -\left[R + B^T P(t+1)\, B + \sum_{i=1}^{\ell} B_i^T\, P(t+1)\, B_i\right]^{-1} B^T P(t+1) A \cdot x(t),$$

where P solves the Riccati-type equation

$$\begin{cases} P(t) = A^T P(t+1) A + Q + \sum_{i=1}^{k} A_i^T P(t+1) A_i - \\ \quad - A^T P(t+1) B \left[R + B^T P(t+1) B + \sum_{i=1}^{\ell} B_i^T P(t+1) B_i \right]^{-1} \cdot \\ \quad \cdot B^T P(t+1) A \\ P(T) = 0 \end{cases} \tag{4.12}$$

The optimal loss is $x(t_0)^T P(t_0) x(t_0)$.

The consequences for the stabilisation problem are immediate from Thm. 4.2. A solution of the infinite-horizon problem ($T = \infty$) will exist only if the system is mean-square stabilisable. If stabilisability holds, a solution will exist for some Q and R. In this case, a steady state solution of Eq. (4.12) will exist, and

$$P = \lim_{t \to -\infty} P(t)$$

will yield a stochastic Lyapunov function $V(x) = x^T P x$ for the closed-loop system. This requires stabilisability of the noise-free system, and normally involves a condition on the noise intensities. Conditions for stabilisability for arbitrary noise intensities can be found in Willems & Willems [14] and Molander [6].

References

[1] Chung, K.L. (1974): A Course in Probability Theory. Academic Press, New York and London.

[2] Kozin, F., Sugimoto, S. (1977): "Relations between sample and moment stability for linear stochastic differential equations", in Proc. Conf. Stoch. Diff. Eqns, and Appls. (ed. J.D. Mason). Academic Press, New York etc.

[3] Bertram, J.E., Sarachik, P.E. (1959): "Stability of circuits with randomly time-varying parameters", Trans. IRE-PGIT 5, p. 260.

[4] Kats, I.I., Krasovskii, N.N. (1960): "On the stability of systems with random parameters", Prikl. Mat. i Makh. 24, pp. 809-823.

[5] Kushner, H. (1967): Stochastic Stability and Control. Academic Press, New York etc.

[6] Molander, P. (1979): Stabilisation of Uncertain Systems. PhD Thesis, Dept. of Automatic Control, Lund Inst. of Technology, Sweden.

[7] Willems, J.C., Blankenship, G.L. (1971): "Frequency domain stability criteria for stochastic systems", IEEE Trans. AC-16, No. 4, pp. 292-299.

[8] Khasminskii, R.Z. (1967): "Necessary and sufficient conditions for the asymptotic stability of linear stochastic systems", Th. Prob. Appls. 1, pp. 144-147.

[9] Kozin, F. Prodromou, S. (1971): "Necessary and sufficient conditions for almost sure sample stability of linear Itô equations", SIAM J. Appl. Math. 21, pp. 413-424.

[10] Infante, E.F. (1968): "On the stability of some linear nonautonomous random systems", ASME J. Appl. Mech., March 1968, pp. 7-12.

[11] Gantmacher, F.R., op. cit.

[12] Åström, K.J. (1970): Introduction to Stochastic Control Theory. Academic Press, New York etc.

[13] Wonham, W.M. (1970): "Random differential equations in control theory", in Prob. Methods in Appl. Mathematics, 2 (ed. A.T. Bharucha-Reid). Academic Press, New York etc.

[14] Willems, J.L., Willems, J.C. (1976): "Feedback stabilisability for stochastic systems with state and control dependent noise", Automatica 12, pp. 277-283.

CHAPTER 5. STRUCTURAL STABILITY

5.1 Introduction

All stability problems discussed so far refer to perturbations of the state. It is tacitly assumed that the right--hand sides of the equations governing the process remain constant or at least constrained to lie in certain regions in parameter space which are bounded away from "dangerous sets". In practice, however, parameters will normally drift as a result of exogenous forces. At certain critical values, together forming the bifurcation set, the topological picture of the flow may change brusquely. Equilibrium points split or coalesce, limit cycles are born or collapse, turbulent phenomena appear etc.

Consider the following

Tentative definition: An object (a quality etc.) is structurally stable if nearby objects are alike. ∎

To get a working definition, one must specify the meaning of "nearby" and "alike". The former notion refers to distances and calls for a topology. For the latter, an equivalence relation is needed. The choice is far from trivial in practical situations, and a major difficulty is to decide how many of the results depend on technically motivated choices, and how many tell something about "reality".

The two main problems of structure stability theory can be succinctly formulated in the following way:

i) Give conditions on a flow that guarantee structural stability, i.e., loosely speaking, that the qualitative behaviour of the flow does not change if the parameters of the system are modified slightly. Determine whether structural stability is a typical quality in the family of flows studied.

ii) If bifurcation occurs, i.e. small parameter changes do provoke a qualitatively different behaviour, determine what changes are possible, or typical. Preferably, instabilities should appear in a stable way.

Stability and bifurcation analysis was inaugurated by Poincaré in the 1880's, even if similar ideas appeared earlier for instance in Euler's analysis of the buckling beam. After Poincaré's work, nothing happened until Andronov and co-workers introduced and studied the concept of coarse systems (grubye sistemy) in the 1930's. The term "structural stability" is due to Lefschetz. Other key names in the development are Peixoto, Smale, and Thom , the founder of catastrophe theory. The bifurcation analysis was taken up by Hopf, whose paper on branching to periodic solutions appeared in 1942. Hopf considered systems of ordinary differential equations with analytic right-hand sides, but the results have later been extended to much more general motions.

The chapter starts with Andronov's main theorem on structural stability of motions in the plane. Section 5.3 deals with the Hopf bifurcation. Several examples are given, the most important of which is the Lorenz system. The following section contains applications to the problem of turbulence, and the chapter closes with a brief account of the main ideas in catastrophe theory.

5.2 Structural stability in the plane

Consider two dynamical systems given by

$$\begin{cases} \dot{x}_1 = P_1(x_1,x_2) \\ \dot{x}_2 = Q_1(x_1,x_2) \end{cases} \tag{5.1}$$

and

$$\begin{cases} \dot{x}_1 = P_2(x_1,x_2) \\ \dot{x}_2 = Q_2(x_1,x_2) \end{cases} \tag{5.2}$$

defined in some bounded domain Ω in $\mathbb{R}^2$ (typically the unit circle $x_1^2 + x_2^2 \leq 1$). The vectors are assumed to be nowhere tangent to the boundary $\partial\Omega$ and to point to the interior of Ω.

Definition 5.1. The equation (5.1) is said to be structurally stable if there is a $\delta > 0$ such that whenever

$$\begin{cases} \sup_{\Omega} |P_1 - P_2| < \delta \\ \sup_{\Omega} |Q_1 - Q_2| < \delta \\ \sup_{\Omega} \left|\frac{\partial P_1}{\partial x_i} - \frac{\partial P_2}{\partial x_i}\right| < \delta \\ \sup_{\Omega} \left|\frac{\partial Q_1}{\partial x_i} - \frac{\partial Q_2}{\partial x_i}\right| < \delta, \end{cases} \quad i = 1, 2 \tag{5.3}$$

there is a continuous bijection $\Omega \to \Omega$, which maps the trajectories of (5.1) onto those of (5.2) and preserves orientation. ■

Definition 5.2. A singularity x^* of Eq. (5.1) is said to be hyperbolic if the eigenvalues of the Jacobian matrix

$$J = \begin{pmatrix} \frac{\partial P_1}{\partial x_1} & \frac{\partial P_1}{\partial x_2} \\ \frac{\partial Q_1}{\partial x_1} & \frac{\partial Q_1}{\partial x_2} \end{pmatrix}$$

have nonzero real parts at x^*. □

A singular point is hyperbolic iff it is a node, a focus, or a saddle point.

The third definition refers to closed orbits. Consider a point p on a closed orbit γ of the differential equation. Through p is drawn a line segment σ, transversal to γ. For an arbitrary point x on σ close to p, the mapping $\pi: \sigma \to \sigma$ is defined as the point where the trajectory from x cuts σ the following time (see Fig. 5.1). π is called the Poincaré map.

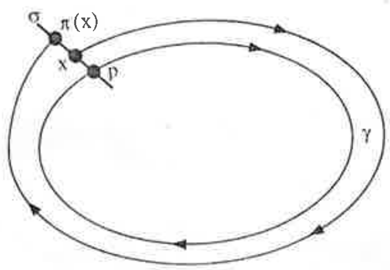

Fig. 5.1 - Defining the Poincaré map.

Definition 5.3. The closed orbit γ is said to be hyperbolic if $d\pi/dx \neq 1$ at p. ■

Theorem 5.1. (Andronov et al. [1]) Equation (5.1) is structurally stable if and only if

i) all singularities are hyperbolic,

ii) all closed orbits are hyperbolic, and

iii) no trajectory connects saddle points. ∎

Example 5.1. (Lotka-Volterra) Consider the system of equations

$$\begin{cases} \frac{dv}{dt} = v(a - bp) \\ \frac{dp}{dt} = p(cv - d), \end{cases} \tag{5.4}$$

where a, b, c, and d are positive constants, and v and p are prey and predator populations, respectively. But for the origin, the only equilibrium point is $v = d/c$, $p = a/b$. The eigenvalues of the Jacobian matrix are $\pm i\sqrt{ad}$. Since the real parts are zero, the system is not structurally stable.

In fact, the singularity is a centre, and there is an infinite family of closed orbits around it. Notice, however, that any set-up like (5.4) with different values of a, b, c, and d will behave the same way. The structural stability concept refers to more general perturbations than these. ∎

The conditions of Thm. 5.1 seem to imply that structurally unstable systems are exceptional. This is in fact true.

Definition 5.4. A quality is said to be generic in a family $\mathcal{F}$ if the set of its carriers contains an open, dense subset of $\mathcal{F}$. ∎

Theorem 5.2. (Peixoto [2]) In two dimensions, structural stability is generic in analytical dynamical systems. ∎

Theorem 5.2 does not generalise to higher dimensions. A class of structurally stable systems in $\mathbb{R}^k$ for general k are the so-called Morse-Smale systems. There are others, but together they are not dense in the space of dynamical systems.

5.3 The Hopf bifurcation

The monograph [1] by Andronov et al. contains not only a theory for structural stability but also a list of the bifurcations that may occur in two-dimensional systems. This is possible only because there are natural limits to the complexity that two-dimensional flows may exhibit. For instance, Poincaré-Bendixson's theorem (see e.g. Coddington--Levinson [3], Ch. 16) states that if the trajectories are bounded, they will ultimately approach either a singular point or a limit cycle. This is no longer true in three dimensions, and strange things may occur (see Ex. 5.4).

One of the few phenomena that do generalise to arbitrary dimensions is the Hopf bifurcation. Assume that a flow in $\mathbb{R}^n$ depends on some parameter μ, and that $x^*(\mu)$ is an equilibrium point. A classical theorem of Poincaré states that if the local linearisation at x^* asymptotically stable, this is true also for the nonlinear system. If there are eigenvalues in the open right halfplane, x^* is unstable also for the nonlinear system. In case there are eigenvalues on the imaginary axis with nonzero imaginary part, the linearisation will have a periodic solution. Hopf's theorem gives conditions for this to hold also for the nonlinear system.

Theorem 5.3. (Hopf [4]) Let $f(x,\mu)$ be a sufficiently differentiable vector field on $\mathbb{R}^n$ depending on the scalar parameter μ. Let $x^*(\mu)$ be an equilibrium point. Assume that

i) the Jacobian matrix $(Jf)(x^*)$ has all eigenvalues in the open LHP for $\mu < 0$;

ii) There are two distinct complex conjugate eigenvalues $\lambda(\mu)$, $\overline{\lambda}(\mu)$ such that

$$\begin{cases} \text{Re } \lambda(0) = 0 \\ \text{Re } \lambda(\mu) > 0, \quad \mu > 0 \\ \dfrac{d \text{ Re } \lambda(\mu)}{d\mu} > 0, \quad \mu = 0 \ ; \end{cases}$$

iii) the rest of the spectrum remains in the LHP for sufficiently small μ .

Then, for small μ, there is a periodic orbit close to the eigenspace of $\lambda(0)$, $\bar{\lambda}(0)$ with

i) period approximately equal to $2\pi / |\lambda(0)|$ and

ii) radius growing like $\sqrt{\mu}$, $\mu > 0$. □

Marsden and McCracken have developed a computable test for the stability of the emerging closed orbit ([5], Section 4). It assumes that the linearised system equalisation is written in the standard form

$$\begin{cases} \dfrac{dx_1}{dt} = \qquad\qquad |\lambda(0)| x_2 + a_{13} x_3 \\ \dfrac{dx_2}{dt} = - |\lambda(0)| x_1 \qquad\qquad + a_{23} x_3 \\ \dfrac{dx_3}{dt} \qquad\qquad\qquad\qquad a_{33} x_3 \end{cases} \tag{5.5}$$

Here a_{13}, a_{23}, and a_{33} are matrices of appropriate dimensions, and a_{33} is stable by assumption.

Introduce the displacement mapping

$$V(x_1) = \pi(x_1) - x_1.$$

Notice that a zero of the displacement mapping is equivalent to the existence of a periodic orbit through x_1. The first and second derivatives are always zero at $x_1 = 0$.

Theorem 5.4. The closed orbit appearing in Thm. 5.3 is stable if $V'''(0) < 0$ and unstable if $V'''(0) > 0$. ∎

For $V'''(0)$, we have the following formula. Let the vector field be written as (f_1, f_2, f_3) with the notation corresponding to (5.5). Then

$$V'''(0) = \frac{3\pi}{4|\lambda(0)|}\left(\frac{\partial^3 f_1}{\partial x_1{}^3} + \frac{\partial^3 f_1}{\partial x_1 \partial x_2^2} + \frac{\partial^3 f_2}{\partial x_1^2 \partial x_2} + \frac{\partial^3 f_2}{\partial x_2{}^3}\right) +$$

$$+ \frac{3\pi}{4|\lambda(0)|^2}\left(-\frac{\partial^2 f_1}{\partial x_1{}^2}\cdot\frac{\partial^2 f_1}{\partial x_1 \partial x_2} + \frac{\partial^2 f_2}{\partial x_2{}^2}\cdot\frac{\partial^2 f_2}{\partial x_1 \partial x_2} + \right.$$

$$+ \frac{\partial^2 f_2}{\partial x_1{}^2}\cdot\frac{\partial^2 f_2}{\partial x_1 \partial x_2} - \frac{\partial^2 f_1}{\partial x_2{}^2}\cdot\frac{\partial^2 f}{\partial x_1 \partial x_2} + \frac{\partial^2 f_1}{\partial x_1{}^2}\cdot\frac{\partial^2 f_2}{\partial x_2{}^2} -$$

$$\left. - \frac{\partial^2 f_1}{\partial x_2{}^2}\cdot\frac{\partial^2 f_2}{\partial x_2{}^2}\right), \qquad (5.6)$$

where all derivatives are evaluated at the equilibrium.

The bifurcation is called supercritical if the closed orbits are stable, otherwise subcritical.

Example 5.2. Consider the Liénard equation

$$\frac{d^2x}{dt^2} - \mu\frac{dx}{dt} + \left[\frac{dx}{dt}\right]^3 + x = 0.$$

In state space form,

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = \mu y - y^3 - x. \end{cases}$$

The Jacobian matrix at $x = y = 0$ is

$$J(0,\ 0,\ \mu) = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix}$$

with eigenvalues $\lambda_{1,2}(\mu) = \frac{1}{2}(\mu \pm i\sqrt{4-\mu^2})$. Re $\lambda_i(\mu) = 0$ for $\mu = 0$, and for this value, Im $\lambda_i = \pm 1$, $\frac{d}{d\mu}$ Re $\lambda_i = 1/2$. The conditions of Hopf's theorem are thus satisfied, and the system will sustain a periodic motion at least for small positive μ. To test for stability, notice that the only non-vanishing term in (5.6) is

$$\frac{\partial^3 f_2}{\partial x_2{}^3} = -6.$$

Thus

$$V'''(0) = \frac{3\pi}{4}(-6) < 0,$$

so the orbit is stable. ∎

The following example of a Hopf bifurcation is the celebrated Belusov-Žabotinskii reaction.

Example 5.3. The components of the reaction are cerium sulphate, sodium bromate, malonic acid, and sulphuric acid. Ferroine is used as a redox indicator. The reaction scheme is:

$$BrO_3^- + Br^- + 2H^+ \rightarrow HBrO_2 + HOBr$$
$$HBrO_2 + Br^- + H^+ \rightarrow 2HOBr$$
$$BrO_3^- + HBrO_2 + H^+ \rightarrow 2BRO_2 + H_2O$$
$$Ce^{3+} + BrO_2 + H^+ \rightarrow Ce^{4+} + HBrO_2$$
$$2HBrO_2 \rightarrow BrO_3^- + HOBr + H^+$$
$$nCe^{4+} + BrCH(COOH)_2 \rightarrow nCe^{3+} + Br^- + \text{oxidized products.}$$

The critical reactants are $HBrO_2$, Br^-, and Ce^{4+}. Let x, y, and z be their concentrations, respectively. The differential equation governing the reaction is then

$$\begin{cases} \dot{x} = x(x - xy + y - qx^2) \\ \dot{y} = \frac{1}{s}(-y - xy + fz) \\ \dot{z} = w(x - z), \end{cases}$$

where s, q, f, and w are parameters that are controlled exogenously. Let f be the bifurcation parameter. Then a Hopf bifurcation occurs at the critical value f_c solving the equation

$$2q(2 + 3f_c) = (2f_c + q - 1) \cdot$$
$$\cdot \left\{ 1 - f_c - q + \left[(1 - f_c - q)^2 + 4q(1 + f_c) \right]^{1/2} \right\}.$$

Depending on the parameters s, q, and w, the bifurcation may be supercritical or subcritical. A recipe giving oscillations at room temperature can be found in Glansdorff and Prigogine [6]. The malonic acid is consumed in the reaction and has to be replaced in order to sustain the oscillation. For a thorough analysis of the reaction, see Hastings and Murray [7]. ■

The final example is the Lorenz system, one of the first examples of a strange attractor to be discovered. Lorenz' paper [8] appeared in 1963, but it took some time before it became known to mathematicians. More recent investigations of the Lorenz system can be found in Ruelle [9] and in [5].

Example 5.4. The physical background of the Lorenz system is the so-called Bénard problem. A fluid or gas layer is heated from below, and the resulting convection is studied. This leads to a complicated partial differential equation. If the solution is expanded in a Fourier series where only the first three terms are kept, the following system results:

$$\begin{cases} \dot{x} = -\sigma x + \sigma y \\ \dot{y} = -xz + rx - y \\ \dot{z} = xy - bz. \end{cases} \tag{5.7}$$

x is proportional to the intensity of the convective motion, y is proportional to the temperature difference of the ascending and descending currents, and z measures the deviation from a linear vertical temperature profile. σ is the Prandtl number, and r, the Rayleigh number, is the bifurcation parameter.

For $r \leq 1$, the origin is the only equilibrium point of (5.7). For $r > 1$, two new points occur at $x = y = \pm \sqrt{b(r-1)}$, $z = r-1$. The Jacobian matrix at $x = y = \sqrt{b(r-1)}$, $z = r-1$, is

$$J = \begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & -\sqrt{b(r-1)} \\ \sqrt{b(r-1)} & \sqrt{b(r-1)} & -b \end{pmatrix},$$

which has purely imaginary eigenvalues for

$$r = \frac{\sigma(\sigma + b + 3)}{(\sigma - b - 1)},$$

assuming $\sigma > b + 1$. The condition of Hopf's theorem are satisfied, and a periodic orbit occurs. The stability of this orbit depends on the parameters b and σ. A lengthy calculation yields the result shown in Fig. 5.2 ([5], p. 147).

Lorenz' values were $\sigma = 10$, $b = 2.67$, and $r = 28$, which yields unstable periodic orbits. For these values of the parameters, the system exhibits a strange behaviour, jumping erratically between two butterfly wings (see Fig. 5.3). The dependence on the initial conditions is extremely sensitive. If the Lorenz system is a good model of atmospheric motions (which remains to be shown), this puts an end to all dreams about reliable long-term weather forecasts. □

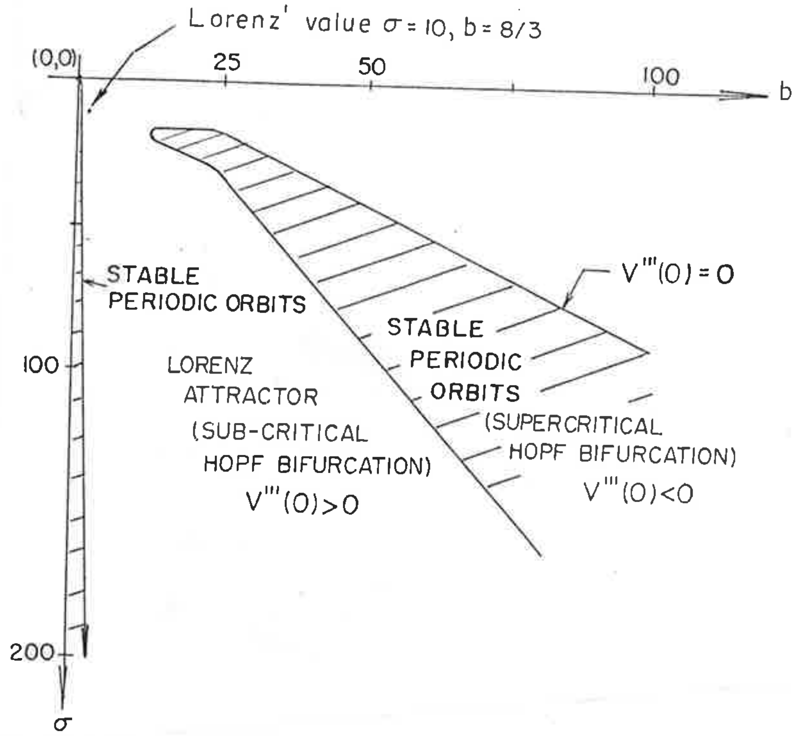

Fig. 5.2 - Stability of the periodic orbits of the Lorenz system as a function of the parameters b and σ.

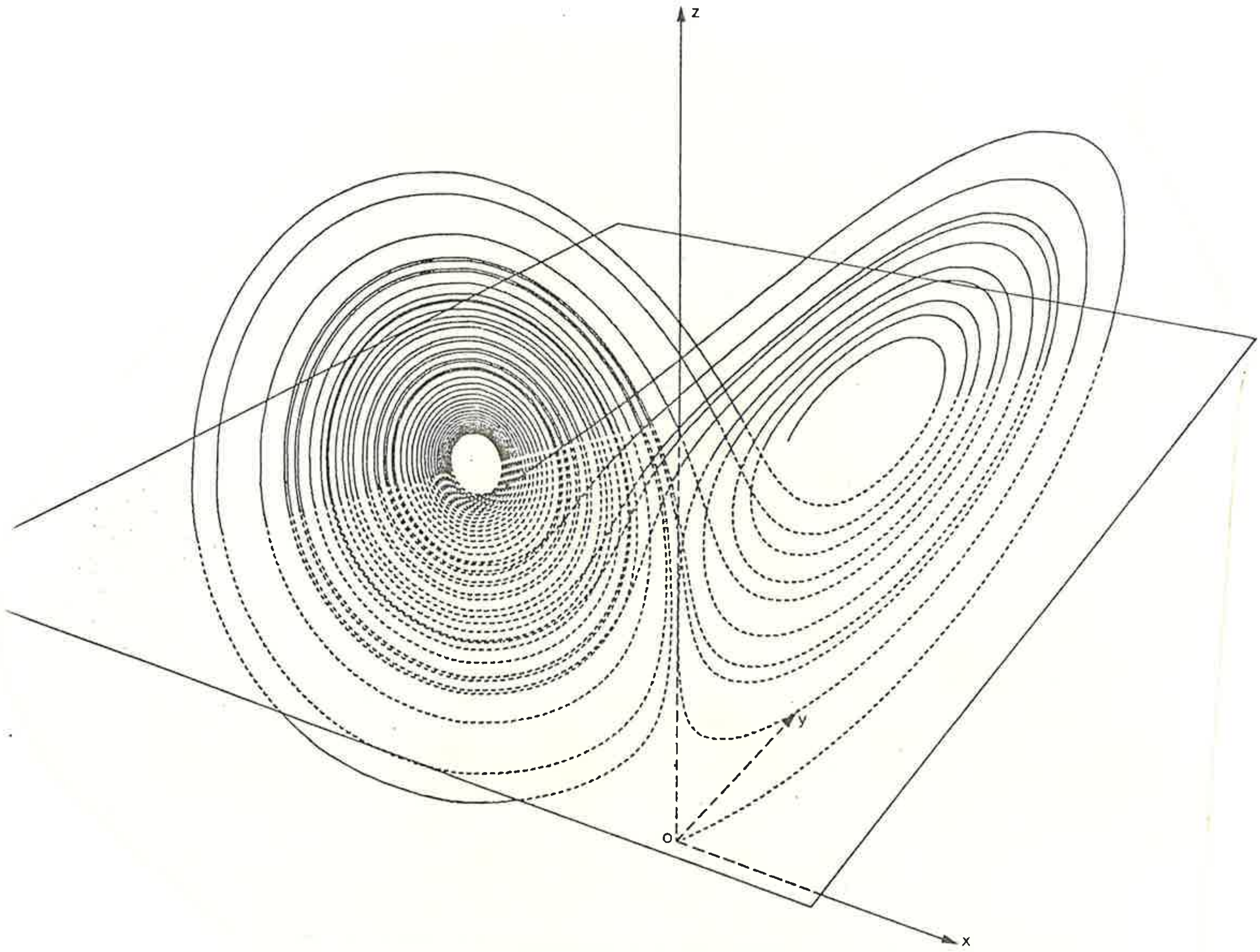

Fig. 5.3 - Trajectories of the Lorenz system for $\sigma = 10$, $b = 2.67$, and $r = 28$.

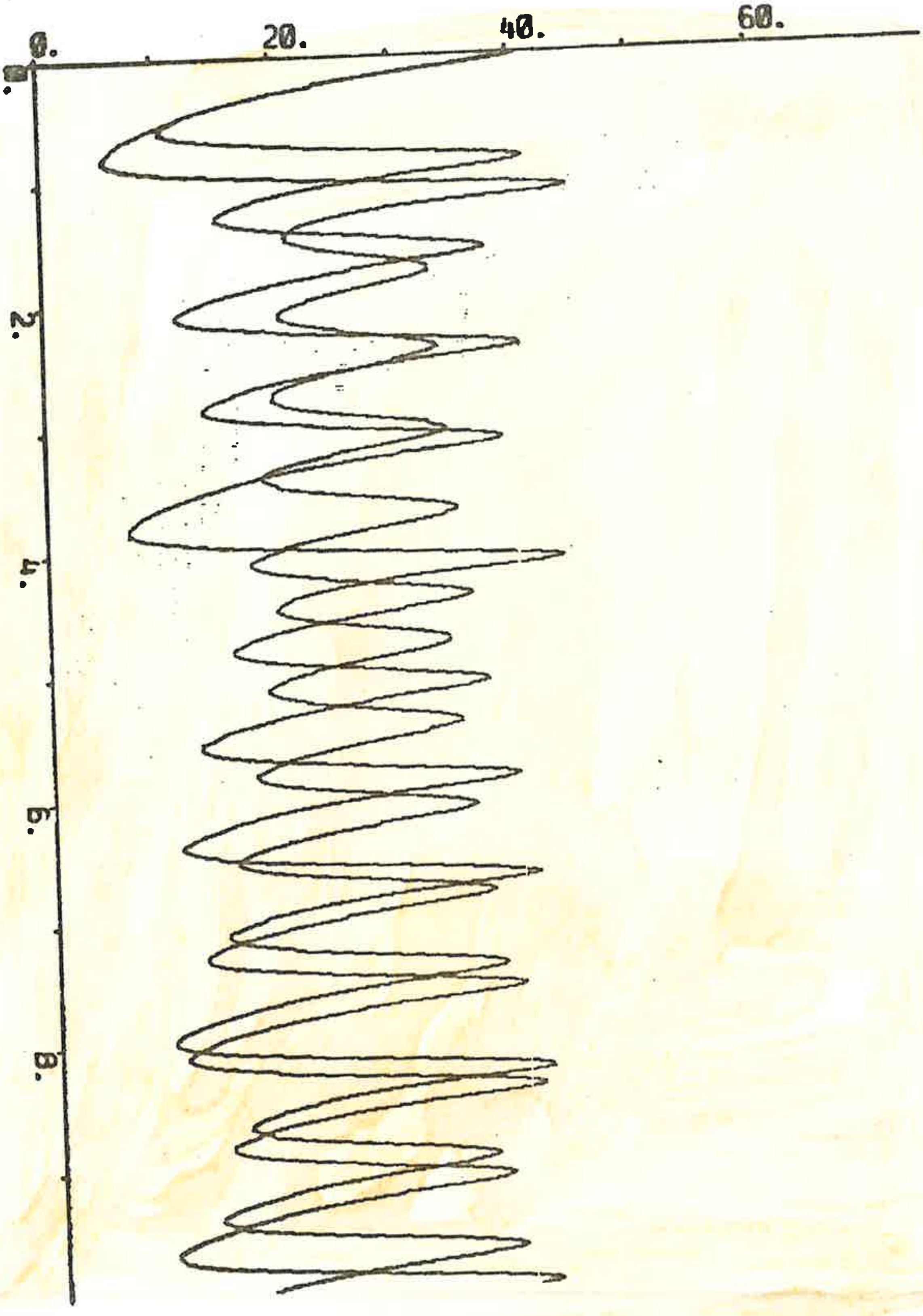

Fig. 5.4 - A requiem for weather forecasts? Two z-trajectories of the Lorenz system with close initial conditions.

5.4 Turbulence

The problem of describing the turbulence phenomenon in mathematical language is closely related to the bifurcations described in the previous section. Applications to such problems require some generalizations of theory, however. Firstly, ordinary differential equations are normally too blunt a tool for describing fluid motions: the Hopf theorem must be proved for evolution equations based on partial differential operators or functional differential operators. This can be done if the semigroups generated satisfy some smoothness conditions (see [5], Section 8, and Hale [10] for the FDE case).

Secondly, the interesting things normally happen not when the first pair of eigenvalues crosses the imaginary axis but when they are followed successively by others, for higher values of the bifurcation parameter. When the second pair moves across to the RHP, the plane periodic orbit normally bifurcates into a motion on a torus, and it is easy to imagine how further bifurcations will generate motions on higher-dimensional tori with a very complex flow picture.

The generally accepted tool for describing the motion of a homogeneous, incompressible fluid is the Navier-Stokes equations:

$$\begin{cases} \left.\begin{aligned} &\frac{\partial v}{\partial t} + (v\cdot\nabla)v - \nu\Delta v = -\nabla p + F \\ &\operatorname{div} v = 0 \end{aligned}\right\} \text{ in } \Omega \\ v = 0 \qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega. \end{cases} \tag{5.8}$$

Typically, Ω is an open set in $\mathbb{R}^3$. p is the pressure, F is an external force, and ν is the viscosity. For non-viscous flow, the appropriate equation is the Euler equation:

$$\begin{cases} \frac{\partial v}{\partial t} + (v \cdot \nabla)v = - \text{grad } p + F \\ \text{div } v = 0 \\ v \text{ tangent to the boundary.} \end{cases} \tag{5.9}$$

The transition from (5.8) to (5.9) is far from trivial, mainly because the highest-order derivatives disappear.

Let U be a typical velocity of the flow, such as the mainstream velocity, and L a typical length, such as the dimension of a pipe. The Reynolds number Re is defined by

$$\text{Re} = \frac{UL}{\nu} .$$

It is dimensionless, and collects some useful information about the flow. If x/L is chosen as length unit, t/T as time unit (T = L/U), and $p \cdot L^2/U^2$ as pressure unit, the only external parameter appearing in Eq. (5.9) is the Reynolds number (this is sometimes referred to as the law of similarity; see Hughes and Marsden [11], § 17). This makes it natural to choose Re as the bifurcation parameter.

Consider the fluid motion around a sphere, Fig. 5.5. For low Re, the motion is stationary. As Re increases, periodic patterns, known as Karman vortices, form. For even higher Reynolds numbers (≈ 2 000, says the thumb rule), turbulence occurs.

Leray [12] advanced the hypothesis in the 1930's that the onset of turbulence is related to loss of smoothness of the solutions to Navier-Stokes' equations. This was to some extent supported by Hopf, who proved global existence of weak solutions, but not uniqueness nor regularity. The global existence for all t and smoothness of the solutions is still an unsolved problem.

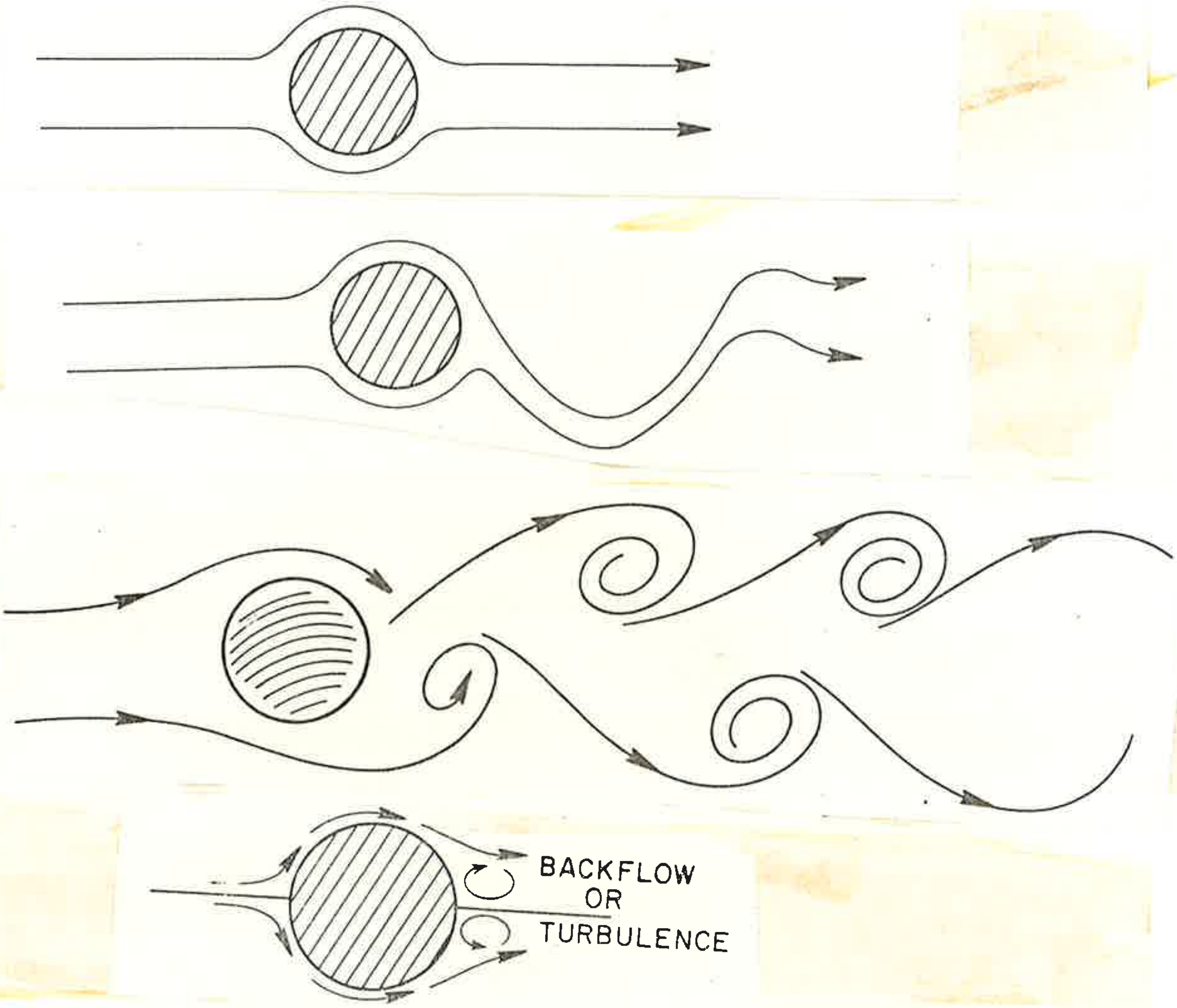

Fig. 5.5 - Flow pattern as a function of the Reynolds number.

A slightly different approach was taken by Landau (see Landau and Lifschitz [13]). He suggested that turbulence should be modelled as successive bifurcations along the lines sketched at the beginning of this section. The velocity field will then be described by an almost-periodic function, composed from an increasing number of harmonic oscillations with non-rationally related frequencies.

This was the prevailing theory for turbulence till the early 70's. It can be criticised on two different levels. Experimentally, the spectrum of a fluid motion does not exhibit a continuous increase in characteristic frequencies, but rather an initial increase and then an abrupt change into a continuous spectrum (see Fig. 5.6).

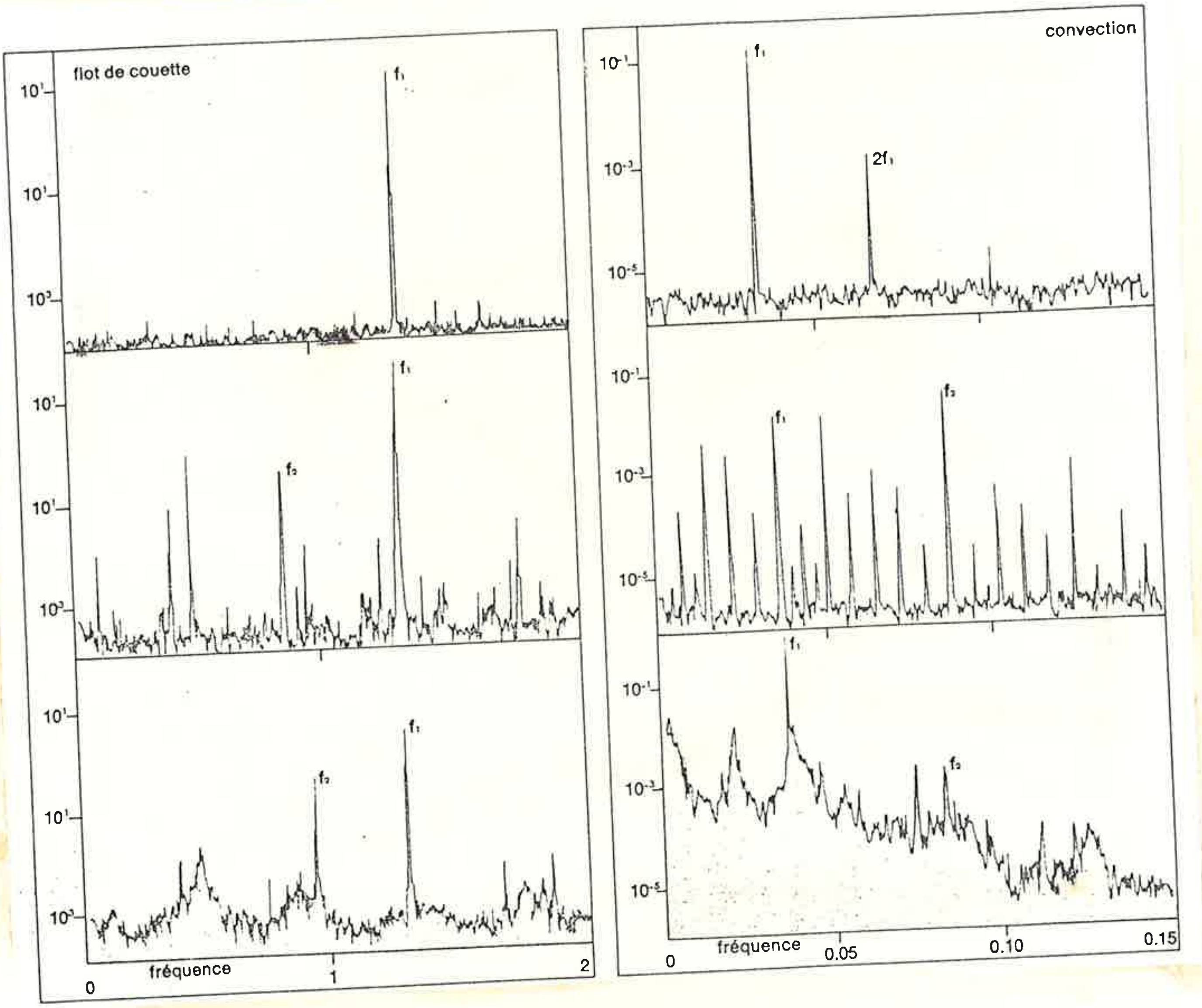

Fig. 5.6 - Experimentally measured velocity spectra for two turbulent phenomena. To the left is shown the spectrum of the velocity at a fixed point in a so-called Couette flow between two cylinders, one of which is rotating. The rotation speed is the bifurcation parameter. - To the right is shown a spectrum for the Bénard problem (cf. Ex. 5.4). The bifurcation parameter in this case is the intensity with which the lower surface is heated.

Ruelle and Takens attacked the Landau picture from a theoretical point of view in their seminal paper [14] from 1971. The essence of their critique is that turbulent phenomena should be described by functions that are generic in a suitable sense. However, already for a 2-torus, Peixoto's theorem shows that almost-periodic functions are in the complement of an open, dense subset of sufficiently differentiable vector fields. By contrast, for higher--dimensional tori, in the neighbourhood of every vector field there is an open set of vector fields with a strange attractor. Ruelle and Takens therefore proposed strange attractors as the appropriate model for turbulent phenomena.

In this sense, the strange behaviour of the Lorenz system is typical and not exceptional. An appealing feature about this model is its relative simplicity and the fact that no loss of regularity or uniqueness is assumed apriori.

5.5 Catastrophe theory

It is difficult to concentrate in a few pages the many diverse ideas that are collected under the name of catastrophe theory. The basic problems are those given in the introduction, namely to give conditions for the structural stability of dynamical systems, and to identify the various forms of instability that may occur.

To fix the ideas, let X and Y be two smooth manifolds, and assume that a vector field F is given on X. (X,F) thus constitutes a dynamical system. Together with this system is given an infinitely differentiable read-out map f: X → Y. Y is consequently the output space. The stability definition requires an equivalence concept and a topology.

Definition 5.5. Two functions $f_1, f_2 \in C^\infty(X,Y)$ are said to be equivalent if there are C^∞-diffeomorphisms g,h such that

$$f_1 = g\, f_2\, h.$$

■

The topology is somewhat more delicate to choose. The standard choice is the so-called Whitney C^∞-topology. The precise definition will not be given; suffice it to say that if X is compact, convergence in the Whitney topology is equivalent to uniform convergence of a function together with all its derivatives, and if X is not compact, the Whitney topology is stronger.

A general stability theory for dynamical systems is still lacking. The motions that have been studied most extensively are those for which F is the gradient of a potential. Since the state will ultimately approach a minimum of the potential function, the stability problem is reduced to a static stability problem, namely that of studying singularities of smooth mappings $X \to \mathbb{R}$.

As a preliminary, let $F \equiv 0$, and consider simply the C^∞--mappings from X to Y. When is structural stability generic? The answer was given by Mather and turns out to be dependent on the dimensions of X and Y in a rather intricate fashion. Fig. 5.7 illustrates the result.

Turning now to the problem of unstable mappings, consider first the following simple example.

Example 5.5. Let X(p,n) be the space of all p×n matrices. Two matrices X_1 and X_2 will be said to be equivalent if there are invertible matrices G and H such that

$$X_2 = G\, X_1\, H.$$

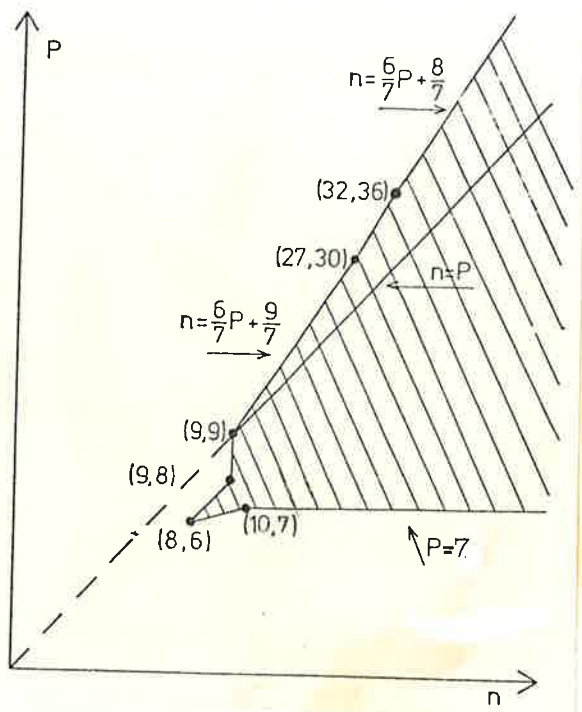

Fig. 5.7 - Mather's result on C^{∞}-mappings. n = dim X and p = dim Y. Stable mappings are dense in the non-hatched region.

This is a natural definition and corresponds to a change of basis in $\mathbb{R}^p$ and $\mathbb{R}^n$. A standard result says that X_1 and X_2 are equivalent in this sense iff they have the same rank. This implies that X(p,n) is split into a number of equivalence classes $X(p,n)_K$ with a varying degree of degeneracy:

$$\begin{cases} X(p,n)_K = \{X \in X(p,n);\ \text{rank } X = K\}; \\ X(p,n) = \bigcup_K X(p,n)_K \\ 0 \leqslant K \leqslant \min(p,n). \end{cases}$$

It is also clear that $X(p,n)_K$, $0 \leqslant K \leqslant \min(p,n) - 1$, will be manifolds in the parameter space.

Closeness may be defined using any metric on $\mathbb{R}^{(p\times n)}$. It is then obvious that

i) the stable matrices are precisely those that have full rank,

and

ii) stable matrices are dense. ■

Turning now to the problem of unstable C^∞-mappings $X \to \mathbb{R}$, one would like to have the same division into subclasses corresponding to an increasing degree of degeneracy. It turns out that this is to some extent possible. What Thom has done is to imbed the unstable mappings in families depending on a varying number K of parameters, or control variables. This imbedding procedure is called an unfolding of the singularity (if the mapping is unstable, there will always be a singularity). What is remarkable about Thom's solution is that if $K \leqslant 5$, there is a finite list of normal forms for the local behaviour of the potential function at the singularity. These are the celebrated elementary catastrophes. (A description is given for instance in Poston and Stewart [15].) Rephrasing, one might say that if a motion can be derived from a potential function which depends on a few external parameters or control variables, there are not so many "qualitatively different" possible ways of behaving near a critical point of the potential function. "Qualitatively different" is a very coarse notion, however, since C^∞-diffeomorphisms are supple objects.

We will not linger on the more or less bizarre "applications" of catastrophe theory that have appeared since Thom 's [16] and Zeeman's [17] early publications. The survey article by Golubitsky [18] contains a few (moderate) examples and further references. One point deserves to be stressed, though. The results have been derived under a few rather special assumptions:

i) the motion is obtainable from a potential function;

ii) the manifolds and mappings involved are smooth;

iii) equivalence is defined modulo C^{∞}-diffeomorphisms;

iv) closeness is defined with the aid of the Whitney topology.

A minimum requirement is thus that these assumptions be justified in a given application.

The implications (if any) for management and control problems remain to be sorted out. The monograph by Casti [19] contains a few examples where structural stability concepts provide some guidance.

References

[1] Andronov, A.A. et al. (1971): Theory of Bifurcations of Dynamic Systems in a Plane. Israel Program for Scientific Translations, Jerusalem.

[2] Peixoto, M.M. (1959): "On structural stability", Ann. Math. 69, No. 2, pp. 199-222.

[3] Coddington, E.A., Levinson, N. (1955): Theory of Ordinary Differential Equations. McGraw-Hill, New York etc.

[4] Hopf, E. (1942): "Abzweigung einer periodischen Lösung ...", translated in [5].

[5] Marsden, J.E., McCracken, M. (1976): The Hopf Bifurcation and its Applications. Springer, New York etc.

[6] Glansdorff, P., Prigogine, I. (1971): Thermodynamic Theory of Structure, Stability and Fluctuations. Wiley-Interscience, London etc.

[7] Hastings, S.P., Murray, J.D. (1975): "The existence of oscillatory solutions...", SIAM J. Appl. Math. 28, No. 3, pp. 678-688.

[8] Lorenz, E. (1963): "Deterministic nonperiodic flow", J. Atmos. Sci. 20, pp. 130-141.

[9] Ruelle, D. (1977): Statistical Mathematics and Dynamical Systems. Duke Univ. Math. Series III, Durham (USA).

[10] Hale, J., op. cit.

[11] Hughes, T.J.R., Marsden, J.E. (1976): A Short Course in Fluid Dynamics. Publish or Perish, Inc., Boston.

[12] Leray, J. (1934): "Sur le mouvement...", Acta Mathematica 63, pp. 193-248.

[13] Landau, L.D., Lifshitz, E.M. (1959): Fluid Mechanisms. Pergamon, Oxford.

[14] Ruelle, D., Takens, F. (1971): "On the nature of turbulence", Commun. Math. Phys. 20, pp. 167-192.

[15] Poston, T., Stewart, I.N. (1976): Taylor Expansions and Catastrophes. Pitman Publishing, London etc.

[16] Thom, R. (1975): Structural Stability and Morphogenesis. Benjamin, Reading.

[17] Zeeman, E.C. (1976): "Catastrophe theory", Scientific American, April.

[18] Golubitsky, M. (1978): "An introduction to catastrophe theory and its applications", SIAM Review <u>20</u>, No. 2, pp. 352-387.

[19] Casti, J. (1979): Connectivity, Complexity and Catastrophe in Large-Scale Systems. IIASA-Wiley-Interscience, Chichester etc.