# Controllers for Bilinear Systems

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Prope mare erat tubulatur
Qui virginem ingrediebatur.
  Dessine ingressus
  Audivi progressus:
Est mihi inquit tubulatur.

# CONTROLLERS FOR BILINEAR SYSTEMS

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# 1. INTRODUCTION

The problem of controlling the bilinear system

$$\dot{x} = Ax + \sum_{i=1}^{m} B_i x u_i + B_0 u \tag{1.1}$$

evokes interest because of three reasons:

1. It is a "simple" class of non-linear systems, slightly more complex than the fully understood linear systems. Knowing how to control this class is a logical step in the development of control theory.

2. Some non-linear systems may be approximated by bilinear models, see Sussman (1976) and Mohler (1979). Especially a bilinear approximation might be an acceptable one in a greater region of the state space around the operating point, than a linear one.

3. Several real life control processes can be modelled as bilinear systems. Several examples are found in Mohler (1973), for instance the control of the neutron level in a fission reactor. España (1978) proposes a bilinear model for a distillation process. Bilinear models for biological systems can be found in for instance Mohler (1978).

General necessary and sufficient conditions for controllability and stabilizability of bilinear systems are lacking today. Consequently there is no complete theory on the control of bilinear processes. Most of the results in the literature cover special cases. Some of these are surveyed here.

The main effort is spent on investigating feedback controllers. Moylan (1973) and Jacobson (1977) present a feedback law which is comprehensive when A is a stability matrix. Landau (1979) contains a development of those papers. Here the results are extended to the case when A is unstable, under suitable conditions. New results, covering special cases, are also presented.

The case when additive and multiplicative controls act independently is treated briefly.

It is suggested that future work is undertaken along the following lines:

1. The controllability problem should be solved.
2. Specific bilinear systems, modelling real life processes, should be studied.
3. A slightly more general class of systems should be considered:

$$\dot{x} = p(x) + \sum_{i=1}^{m} B_i x u_i + B_0 u_i \quad , \tag{1.2}$$

   where the elements of p(x) are homogenous polynomials of degree n, and $u_i = u_i(x)$ is a polynomial of degree $\leq$ (n - 1). This class is closed under state feedback which is not the case with bilinear systems.

This report is organized as follows: the stabilizing properties of various controllers are investigated with respect to different assumptions on the system.

Section 2 contains definitions, the problem statement and a discussion of them.

Section 3 presents the "quadratic" feedback solution of Moylan (1973) and Jacobson (1977), shows its use when A is strictly stable and discusses when it might work when

A is not strictly stable.

Section 4 is devoted to other controllers and special systems, among them systems for which constant controls work, and for which the new Division Controller is applicable. Various other approaches are also discussed.

Section 5 treats briefly the case when additive and multiplicative controls are independent.

Section 6 contains the above mentioned example on neutron level control (Mohler (1973)), solved in two new ways.

Section 7, finally, contains a summary.

# 2. DEFINITIONS AND PROBLEM STATEMENT

## 2.1 Definition of a bilinear system

Definition: The bilinear system under consideration in all but section 5 is:

$$\dot{x} = Ax + \sum_{i=1}^{m} B_i x u_i + B_0 u \tag{2.1}$$

$x \in R^n$ ,

$u \in R^m$ ,

$A$, $B_i$ real constant matrices of appropriate dimensions.

Equ. (2.1) can be rewritten:

$$\dot{x} = Ax + G(x)u \tag{2.2}$$

where

$$G(x) = [B_1 x + b_{10} | B_2 x + b_{20} | \cdots\cdots | B_m x + b_{m0}], \tag{2.3}$$

where $b_{i0}$ is the i:th column of $B_0$. Note that the control u acts additively and multiplicatively simultaneously.

Another rewriting yields

$$\dot{x} = Ax + \sum_{i=1}^{m} (B_i x + b_{i0}) u_i \tag{2.4}$$

Remark 2.1: Equ. (2.1) can also be rewritten into

$$\dot{z} = Fz + \sum_{i=1}^{m} H_i z u_i \tag{2.5}$$

with $F = \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix}$ , $H_i = \begin{bmatrix} 0 & 0 \\ b_{i0} & B_i \end{bmatrix}$ , and $z = \begin{bmatrix} 1 \\ x \end{bmatrix}$

(Brockett (1972)). Equ. (2.5) is not in a controllable form. It is believed that the form (2.5) offers no advantage when trying to find a feedback control, and will not be used. When a system of the form

$$\dot{x} = Ax + \sum_{i=1}^{m} B_i x u_i \tag{2.6}$$

is given in the sequel, it will denote equ. (2.1) with $B_0 = 0$ and not equ. (2.5).

## 2.2 Problem statement

Problem: Find a stabilizing state feedback control.
This problem is of interest because:

1. a bilinear model of a process may cover adequately a larger region of the state space than a linear one, and thus larger disturbances may be taken care of.
2. in the same vein, a change of operating point might require a bilinear model description (see for instance España (1978)).
3. work on observers for bilinear systems is being undertaken, and thus output feedback could be covered by the above approach. An observer of the type

$$\dot{\hat{x}} = A\hat{x} + G(\hat{x})u + H(y - C\hat{x}) \tag{2.7}$$

   has been studied by Derese (1979).

Stability is as usual only a necessary property of a good controller. However, the results presented below permit the designer some freedom in selecting other properties of interest.

The relevant stabilizability concept is:
Definition: (Null) stabilizability: Given the system (2.1) with the initial condition $x(t_0) = x_0$ , $\Omega$ is said to be a *(null) stabilizable region* if for every $x_0 \in \Omega$ and for every neighbourhood $\omega$ of the origin, there exists a locally bounded function $u(t)$, $t \geq t_0$ , and a finite time interval T, such that the solution of (2.1) with $u(t)$ as input satisfies $x(t) \in \omega$ for all $t > t_0 + T$. If $\Omega = R^n$ then the system is said to be *stabilizable*.

There are no general criteria for necessary and sufficient conditions neither for stabilizability nor for any definition of controllability. Sufficient conditions (and necessary for the case of scalar u) are found in Mohler (1973). Hirschorn (1974) determines the reachable set for a large class of bilinear systems using Lie theory. Hermann (1977) has some results for non-linear systems, applicable to the bilinear case. Wei (June 1978) proves total controllability under very restrictive conditions. In the sequel, stabilizability will be assumed when not stated otherwise.

Stabilizability of the system (2.1) in general implies that the linear system $\dot{x} = Ax + B_0u$ is stabilizable, because near the origin, the terms $\sum_{i=1}^{m} B_i x u_i$ (with $u_i = o(x)$) are of smaller order than the term $B_0u$. One exception is the case when $B_0 = 0$ while the system might still be stabilizable.

## 2.3 Change of operating point

The above stabilizability concept can be applied to other stationary points in the state space. Let the system be given by equ. (2.1) and assume $u_e$ and $x_e$ such that

$$0 = Ax_e + \sum_{i=1}^{m} B_i x_e u_{ei} + B_0 u_e \qquad (2.8)$$

where $u_{ei}$ is the i:th component of $u_e$.
Then by introducing $\Delta x = x - x_e$ , $\Delta u = u - u_e$
(2.1) can be rewritten

$$\Delta \dot{x} = (A+\sum_{i=1}^{m} B_i u_{ei})\Delta x + \sum_{i=1}^{m} B_i \Delta x \Delta u_i + \sum_{i=1}^{m} (B_i x_e + b_{i0})\Delta u_i \qquad (2.9)$$

We note that (2.9) describes a bilinear system of the same structure as equ. (2.1) around the new operating point $x_e$. (With more general non-linear systems, the structure is not necessarily preserved when changing the operating point.)

Of course not all points in the state space are stationary points. In practice however, the purpose of control is often to keep the system at a stationary operating point, or to bring it to a new one.

## 2.4 A case of non-stabilizability

Stabilizability around one operating point does not imply stabilizability around another operating point:

Example 2.2 : Consider

$$\dot{x} = x + xu + u \tag{2.10}$$

$x \in R^1$ , $u \in R^1$ .

Equilibrium points:

$$0 = (1 + u_e)x_e + u_e \tag{2.11}$$

$$x_e = - \frac{u_e}{1 + u_e} \tag{2.12}$$

Note that $x_e \in \{x | x \neq -1, x \in R\}$.

The system equation around the new operating point is, using equ. (2.9):

$$\Delta \dot{x} = (1 + u_e)\Delta x + \Delta x \Delta u + (x_e + 1)\Delta u \tag{2.13}$$

For $x_e = 0$, $u_e = 0$ (i.e. equ. (2.10)) the system is not stabilizable. Rewrite equ. (2.8):

$$\dot{x} = x + (x + 1)u \tag{2.14}$$

and let the initial condition be $x(0) < -1$. Then no bounded $u(t)$ can take the state across the point -1, because

$$\lim_{x \to -1} \dot{x} = -1 \tag{2.15}$$

as the control action ceases for $x = -1$.

However for $x_e = -2$, $u_e = -2$, equ. (2.13) gives:

$$\Delta \dot{x} = -\Delta x + \Delta x \cdot \Delta u - \Delta u \tag{2.16}$$

The control $u = 0$ stabilizes (2.16).

Example 2.2 can be generalized:

*Theorem 2.3*: Given the system

$$\dot{x} = Ax + (Bx + b)u \ , \quad x \in R^n \ , \quad u \in R \tag{2.17}$$

Assume that $Bx + b = b(c^T x + 1)$. Let $d(x) = c^T x + 1$.
$d(x) = 0$ defines an $(n-1)$-dimensional hyperplane.
Divide the state space into the following sets:

$S_+ = \{x | d(x)>0\}$

$S_0 = \{x | d(x)=0\}$

$S_- = \{x | d(x)<0\}$

If all the trajectories of the autonomous system $\dot{x} = Ax$ in $S_0$ are directed into $S_0 \cup S_-$ then (2.17) is not stabilizable.

*Proof*: The origin belongs to $S_+$. Control action ceases in $S_0$. Therefore, the only way to pass from $S_-$ into $S_+$ is by force of the autonomous system $\dot{x} = Ax$. If no trajectory of the autonomous system leads from $S_0$ into $S_+$, stabilization is not possible.

The result of theorem 2.3 is extended in theorem 4.22. Other results on non-stabilizability appear in section 4.2.

# 3. A FEEDBACK CONTROL SOLUTION

## 3.1 The basic theorem

For conveniance equ. (2.2) is repeated:

$$\dot{x} = Ax + G(x)u$$

The following is adopted from Moylan (1973) and Jacobson (1977).

Theorem 3.1 (Jacobson): Suppose there exists a radially unbounded function $\Phi(x)>0$ , $\Phi:R^n \to R$ , which is once continuously differentiable such that $(\nabla_x\Phi)^T Ax \leq 0$. Suppose further that there is no non-zero $x \in R^n$ for which $(\nabla_x\Phi)^T Ax$ and $(\nabla_x\Phi)^T G(x)$ are both zero. Then

$$u^*(t) = -\frac{1}{2} G^T(x)\nabla_x\Phi(x) \tag{3.1}$$

globally asymptotically stabilizes equ. (2.2).

Remark 3.2 (Jacobson): Under the conditions of theorem 3.1, the control (3.1) minimizes the performance criterion

$$V(x(t_0),u(\cdot),t_0,\infty) = \int_{t_0}^{\infty}\{m(x(t)) + u^T(t)u(t)\}dt \tag{3.2}$$

in the class of stabilizing control functions. Here

$$m(x) = -(\nabla_x\Phi)^T Ax + \frac{1}{4}(\nabla_x\Phi)^T GG^T\nabla_x\Phi \tag{3.3}$$

The assumptions ensure $m(x) > 0$ , $x \neq 0$ .

Proof (outline): Consider the closed loop system:

$$\dot{x} = Ax + G(x)u^* = Ax - \frac{1}{2} G(x)G^T(x)\nabla_x\Phi(x) \qquad (3.4)$$

$\Phi(x)$ is a Lyapunov function for this system because:

1. $\Phi(x) > 0$ (3.5)

2. $\dot{\Phi}(x) = \nabla_x{}^T\Phi(x)\left[Ax - \frac{1}{2} G(x)G^T(x)\nabla_x\Phi(x)\right] =$

$$= \nabla_x{}^T\Phi(x)Ax - 2\cdot\left(\frac{1}{4}\nabla_x{}^T\Phi(x)G(x)G^T(x)\nabla_x\Phi(x)\right) =$$

$$= -\left\{m(x) + 2u^{*T}u^*\right\} < 0 , \; x \neq 0 \qquad (3.6)$$

□

Moylan (1973) proves that the optimal feedback control that minimizes (3.2) is given by (3.1), where $\Phi(x)$ is a positive definite solution of the non-linear stationary Riccatilike equation (3.3). The $\Phi$ assumed in the theorem is such a solution.

## 3.2 A strictly stable

When A is strictly stable there exists a $P = P^T > 0$ for which $\Phi(x) = x^T Px$ satisfies theorem 3.1:

1. $\Phi(x) > 0$

2. $\nabla_x\Phi Ax = 2x^T PAx = x^T(PA + A^T P)x$ (3.8)

When A is strictly stable there exists a $P = P^T > 0$ such that

$$PA + A^T P = - Q \; , \; Q > 0 \tag{3.9}$$

(see e.g. Lancaster (1969)). Thus

$$\nabla_x \Phi A x = -x^T Q x < 0 \qquad \text{for all } x \neq 0 \; . \tag{3.10}$$

3. $\nabla_x \Phi A x$ and $\nabla_x{}^T \Phi G(x)$ are not both zero for any non-zero x , because $\nabla_x \Phi A x < 0$ for all $x \neq 0$ .

So in this case

$$u_i^* = -(B_i x + b_{i0})^T P x \quad , \quad i = 1, \ldots\ldots, m \tag{3.11}$$

is a stabilizing control.

It is possible to pick a suitable Q in equ. (3.9) to get a suitable performance criterion (3.2).

Remark 3.3: Trivially when A is strictly stable, the control u = 0 stabilizes the system.

## 3.3 A has eigenvalues on the imaginary axis

When A has eigenvalues on the imaginary axis it is sometimes possible to find a $P = P^T > 0$ that causes the control (3.11) to stabilize the system.

For the sake of simplicity we first assume that all eigenvalues of A lie on the imaginary axis. In this case there does not exist a $P = P^T > 0$ such that $PA + A^T P < 0$. There does, however, exist a $P = P^T > 0$ such that $PA + A^T P = 0$. This case is treated in theorems 3.4, 3.7 and 3.8.

For all other $P = P^T > 0$, $PA + A^TP$ is indefinite. This case does not differ from the case when A has strictly unstable nodes and we must resort to the weaker results of section 3.4 (see below).

Theorem 3.4: Given the system

$$\dot{x} = Ax + \sum_{i=1}^{m} (B_i x + b_{i0}) u_i \tag{3.12}$$

where all eigenvalues of A are purely imaginary. If there exists a $P = P^T > 0$ such that $PA + A^TP = 0$, and there is no non-zero x for which

$$(B_i x + b_{i0})^T Px \quad , \; i = 1,2, \; \ldots \; ,m \tag{3.13}$$

are all zero, then the control

$$u_i^* = - (B_i x + b_{i0})^T Px \quad , \; i = 1,2, \; \ldots \; ,m \tag{3.14}$$

asymptotically stabilizes (3.12).

Proof: When all eigenvalues of A are purely imaginary there exists a $P = P^T > 0$ such that $PA + A^TP = 0$ (see Lancaster (1969)). Now a simple application of theorem 3.1 is needed: Let $\Phi = x^T Px$ be a Lyapunov function candidate:

$$\dot{\Phi} = \underbrace{x^T (PA + A^TP) x}_{0} - 2 \sum_{i=1}^{m} \left[ (B_i x + b_{i0})^T Px \right]^2 < 0$$

□

Theorem 3.4 gives only sufficient conditions which are very restrictive. In the single input case, for instance, we must demand that B is definite and b = 0 in order to satisfy the conditions of theorem 3.4. See section 3.5

in which the single input case is discussed.

When A has some strictly stable eigenvalues and some purely imaginary eigenvalues $P = P^T > 0$ can be chosen such that $PA + A^TP \leq 0$. This case will be treated in theorem 3.5 and remark 3.10. For other $P = P^T > 0$, we must resort to the weaker results of section 3.4.

Theorem 3.5: Given the system

$$\dot{x} = Ax + \sum_{i=1}^{m} (B_i x + b_{i0}) u_i \tag{3.15}$$

where all eigenvalues of A have non-positive real parts. If there exists a $P = P^T > 0$ such that $PA + A^TP \leq 0$ and such that

$$\begin{bmatrix} (B_1x+b_{10})^T Px \\ (B_2x+b_{20})^T Px \\ \vdots \\ (B_mx+b_{m0})^T Px \end{bmatrix} \neq 0 \quad \text{for } \{x \mid x\neq 0,\ x^T(PA+A^TP)x = 0\} \tag{3.16}$$

then the control

$$u_i^* = -(B_i x + b_{i0})^T Px \qquad , i = 1, \ldots , m \tag{3.17}$$

asymptotically stabilizes (3.15).

Proof: Let $\Phi = x^T Px$ be a Lyapunov function candidate.

$$\dot{\Phi} = \underbrace{x^T(PA + A^TP)x}_{=0} - 2 \sum_{i=1}^{m} \Big[ \underbrace{(B_i x + b_{i0})^T Px}_{\neq 0} \Big]^2 < 0 \tag{3.18}$$

□

Remark 3.6: A similar procedure to that in remark 3.17 can be used to find a suitable P.

Even if $P = P^T > 0$ is such that $PA + A^TP \leq 0$ but condition (3.16) fails, the control (3.17) might still stabilize the system. This is so because the trajectories of the autonomous system $\dot{x} = Ax$ , when $u = 0$ , might "assist" in the stabilization. Special cases when this is true for A having purely imaginary eigenvalues are presented in theorem 3.7 and theorem 3.8. Theorem 3.8 is extended to the case when A has some strictly stable eigenvalues and some purely imaginary eigenvalues in remark 3.10.

Theorem 3.7: (Slemrod (1978)): Given the system

$$\dot{x} = Ax + Bxu \quad , x \in R^{2n} \quad , u \in R \tag{3.19}$$

where

$$A = \begin{bmatrix} 0 & I \\ -J & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ -H & 0 \end{bmatrix}$$

and I is the n/n dimensional unit matrix, J is a diagonalizable n/n matrix with positive eigenvalues, H is an n/n-matrix.

Let $P = P^T > 0$ be such that $PA + A^TP = 0$, and $W = \{x | x^TB^TPx = 0\}$. Let $S = \{x_0 \in W | e^{At}x_0 \in W, t \in (-\infty, \infty)\}$. If $S = \{0\}$ then $u_i^* = - x^TB^TPx$ yields global asymptotic stability.

□

Slemrod (1978) also presents methods how to determine S.

Theorem 3.8 (Brockett (1974)): Suppose A has purely imaginary eigenvalues $\lambda_i(A)$, and that $\lambda_i(A) + \lambda_j(A) \neq \lambda_k(A)$ for all i, j, k. Assume [A,b] a controllable pair. Then there is a matrix $P = P^T > 0$ such that $PA + A^TP = 0$ and $u^* = -(Bx + b)^TPx$ makes the null solution of $\dot{x} = Ax + (Bx + b)u$ asymptotically stable in the large.

Remark 3.9: Note that since A has real entries, A, $n \geq 2$, is not allowed to have an eigenvalue = 0 if the conditions of theorem 3.8 are to be satisfied.

Remark 3.10: If

$$A = \left[\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right]$$

where $A_1$ has purely imaginary eigenvalues and $A_2$ is strictly stable, then theorem 3.8 still holds.

We conclude with a few examples where theorem 3.4 is applicable:

Example 3.11 (Jacobson (1977)): Given

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} xu_1 + \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} xu_2 \qquad (3.20)$$

$$\Phi = \frac{1}{2}x^Tx \quad \Rightarrow \quad \begin{cases} u_1^* = -2x_1x_2 \\ u_2^* = x_1^2 - x_2^2 \end{cases} \qquad (3.21)$$

satisfies theorem 3.4, because $u_1^*$ and $u_2^*$ are never zero simultaneously. $u_1^*$ , $u_2^*$ stabilize (3.20).

Example 3.12: Here is shown a case when both theorem 3.5 and remark 3.10 are applicable:

$$\dot{x} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} xu \qquad (3.22)$$

Choose P = I which gives $x^T(PA + A^TP)x = -2x_2^{\ 2}$.

$$u^* = -x^T\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x = -x_1^{\ 2} \qquad (3.23)$$

satisfies theorem 3.5 because

$$u \neq 0 \text{ when } x^T(PA + A^TP)x = 0, \; x \neq 0 \;. \qquad (3.24)$$

Thus, the control (3.23) asymptotically stabilizes (3.22).

## 3.4 A strictly unstable

The previous sections showed that when A is strictly stable it is always possible to find a stabilizing feedback by the approach (3.11); and when A has imaginary eigenvalues, it is sometimes possible to do so.

When A has some strictly unstable eigenvalues, a $\Phi(x) = x^TPx$ is never a Lyapunov function that yields asymptotic stability by theorem 3.1. The reason is that there are some x in every neighbourhood of the origin for which $x^T(PA + A^TP)x > 0$ , and even if

$$u^{*T}u^* = \sum_{i=1}^{m} \left[ x^TP(B_ix + b_{i0}) \right]^2 \neq 0$$

in the set

$$\{x | x \neq 0, \; x^T(PA + A^TP)x \geq 0\}, \; \dot{\Phi} = x^T(PA + A^TP)x - 2u^{*T}u^* > 0$$

for some x in a sufficiently small neighbourhood of the origin.

However, as we are satisfied with stabilizability as defined in section 2.2, the above trail of thought leads to the following theorem which gives sufficient conditions for stabilizability:

Theorem 3.13: Given the system

$$\dot{x} = Ax + \sum_{i=1}^{m} (B_i x + b_{i0}) u_i \qquad (3.25)$$

where the real parts of the eigenvalues of A might be negative, zero or positive. If there exists a $P = P^T > 0$ such that

$$\begin{bmatrix} (B_1 x+b_{10})^T Px \\ (B_2 x+b_{20})^T Px \\ \vdots \\ (B_m x+b_{m0})^T Px \end{bmatrix} \neq 0 \text{ in the set } \{x | x\neq 0,\ x^T(PA+A^TP)x \geq 0\} \qquad (3.26)$$

then $\forall \varepsilon > 0$ there exists an $\alpha > 0$ such that the control

$$u_i^* = -\alpha (B_i x + b_{i0})^T Px \ , \quad i = 1, 2, \ldots , m \qquad (3.27)$$

causes the state of (3.25) to enter an ε-neighbourhood of the origin.

Proof: Let $\Phi = x^T Px$ be a Lyapunov function candidate.

$$\dot{\Phi} = x^T(PA + A^TP)x - 2\sum_{i=1}^{m} \left[x^T P(B_i x+b_{i0})\right]^2 \alpha < 0 \text{ for } ||x|| > \varepsilon \qquad (3.28)$$

if $\alpha$ is chosen $>$ constant $\cdot \frac{1}{\varepsilon^2}$ . (3.29)

□

Remark 3.14: Notice that theorem 3.13 shows stabilizability but not that the resulting system is asymptotically stable or stable! It is normally impossible to guarantee asymptotic stability by increasing $\alpha$ in (3.27). This fact is illustrated here.

Given the system (3.25) and a P that satisfies the conditions of theorem 3.13. Let $B_0 = \left[ b_{10} \vdots b_{20} \vdots \dots \vdots b_{m0} \right]$.
Even if

$$\tilde{A}(\alpha) \triangleq (A - B_0 B_0^T P\alpha) \quad (3.30)$$

is a stability matrix for all $\alpha \geq N > 0$, asymptotic stability of (3.25) with the control (3.27) with some constant $\alpha > N$ cannot be proved in general.

It is known from linear system theory (e.g. Egardt(1978).) that the eigenvalues of $\tilde{A}(\alpha)$ tend to $-\infty$ and to the zeros of $B_0^T P(sI - A)^{-1} B_0$ when $\alpha \to \infty$. (3.31)

Assume in the sequel that $\dot{x} = \tilde{A}(\alpha)$ is asymptotically stable for all $\alpha > N$. (3.32)

Apply the control (3.27) with a constant $\alpha > N$ on the process (3.25). The closed loop system becomes

$$\dot{x} = \tilde{A}(\alpha)x + g(x) \cdot \alpha \quad (3.33)$$

with

$$g(x) \triangleq - \sum_{i=1}^{m} (B_i x x^T B_i^T + B_i x b_{i0}^T + b_{i0} x^T B_i^T) Px \quad (3.34)$$

It is clear that $||g(x)|| < c_2 ||x||^2$ , some constant $c_2 > 0$ for x sufficiently small.

Because of the assumption (3.32), for every $W = W^T > 0$ we can find a $Q = Q^T > 0$ such that

$$Q\tilde{A}(\alpha) + \tilde{A}^T(\alpha)Q = -W < 0 \tag{3.35}$$

Choose a W, which generates a Q according to (3.35).
Let $V = x^TQx$ be a Lyapunov function candidate for (3.33).
Then

$$\dot{V} = -x^TWx + 2x^TQg(x)\cdot\alpha \tag{3.36}$$

It is clear that there exists an $\varepsilon_1 > 0$ such that $\dot{V} < 0$, when

$$x \in S = \{x| \ ||x|| < \varepsilon_1\} \tag{3.37}$$

because $W > 0$ and $||x^TQg(x)|| < c_3 ||x||^3$ for $||x||$ sufficiently small.

However there is no guarantee that the stability regions defined by (3.28) and (3.37) cover all of the state space. That is, $\varepsilon_1$ of (3.37) might very well be less than $\varepsilon$ of (3.28).

It is possible to decrease $\varepsilon$ of (3.28) by increasing $\alpha$, see (3.29). This will, however, also decrease $\varepsilon_1$ of (3.37) because of (3.29) and because of the way $\alpha$ figures in (3.36).

One might also attempt to increase $\varepsilon_1$ of (3.37) by choosing a $W_1 > W$. However the $Q_1$ resulting form (3.35) will be such that $Q_1 > Q$, because $\tilde{A}(\alpha)$ has finite eigenvalues for all $\alpha$, see (3.31). Therefore an increase of $\varepsilon_1$ of (3.37) cannot be guaranteed because of the way W and Q figure in (3.36).

□

Lemma 3.15: (Yasuda (1977)): Given the system (3.25). Let

$$B_0 = \left[ b_{10} \,\vdots\, b_{20} \,\vdots\, \ldots \,\vdots\, b_{m0} \right] \tag{3.38}$$

If $\left[A, B_0\right]$ is a stabilizable pair then there exists a linear feedback control

$$u = - K^T x \tag{3.39}$$

and an $\varepsilon > 0$ such that (3.25) is asymptotically stable using the control (3.39) for all initial conditions in the set

$$\{x| \quad \| x \| < \varepsilon \} \quad . \tag{3.40}$$

Proof: The proof follows easily from the Lyapunov-Poincaré theorem, see e.g. Hahn (1967).

□

Design 3.16: Given the system (3.25). Assume that there exists a $P = P^T > 0$ satisfying theorem 3.13. Assume that $\left[A, B_0\right]$ is a stabilizable pair. Then the following control scheme will make (3.25) asymptotically stable:

1. Choose a matrix K that makes $(A - B_0 K^T)$ an asymptotic stability matrix. Compute an $\varepsilon$, that defines the set $\{x| \quad \| x \| < \varepsilon\}$ in which $u = - K^T x$ asymptotically stabilizes (3.25) according to Lemma 3.15.

2. Choose an $\alpha$ so that the control (3.27) forces the state of (3.25) into the set $\{x | \ ||x|| < \varepsilon\}$ . This is possible according to theorem 3.13.

3. Start controlling (3.25) with the control (3.27). As soon as the state is in the set $\{x | \ ||x|| < \varepsilon\}$ , switch to the linear control $u = - K^T x$.

Remark 3.17: A sufficient criterion for the condition of theorem 3.13 is the following procedure which might be performed numerically:
Compute

$$w = \max_{P=P^T>0} \ \min_{x \in D \cap H_P} \ \sum_{i=1}^{m} \left[ (B_i x + b_{i0})^T P x \right]^2 , \qquad (3.41)$$

where D is the region of interest, and
$H_P = \{x | x \neq 0, \ x^T (PA + A^T P) x \geq 0\}$ , with the stopping criterion $w \geq \delta$ , for some $\delta > 0$.

The constraint $x \in H_P$ has a lot of structure. $H_P$ is a double cone, i.e. if $x \in H_P$ then $\alpha x \in H_P$ for all $\alpha \in R$.

An algorithm to solve (3.41) numerically can be found in e.g. Polak (1979).

Example 3.18: Here is a case when theorem 3.13 is applicable because $u^* \neq 0$ for nonzero x:

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x u_1 + \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} x u_2 \qquad (3.42)$$

$$\Phi = k\cdot\frac{1}{2}x^Tx \geq 0 \quad \Rightarrow \quad \begin{cases} u_1^* = -2x_1x_2k \\ u_2^* = (X_1^2 - x_2^2)\cdot k \end{cases} \tag{3.43}$$

$$\dot{\Phi} = k(x_1^2 + x_2^2)\left[1 - k(x_1^2 + x_2^2)\right] < 0 \tag{3.44}$$

when $kr^2 > 1$ , $r^2 = x_1^2 + x_2^2$ .

Set $k > \frac{1}{\varepsilon^2}$ . (3.45)

Note that design 3.16 is not applicable, because $B_0=0$.

Example 3.19: This is another case when theorem 3.13 is applicable. Given the system:

$$\dot{x} = \begin{pmatrix} \frac{1}{6} & 1 \\ 0 & \frac{1}{6} \end{pmatrix} x + \left[\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}x - \begin{pmatrix} 3 \\ 2 \end{pmatrix}\right]u_1 + \left[\begin{pmatrix} 4 & 5 \\ 5 & 4 \end{pmatrix}x + \begin{pmatrix} 2 \\ -2 \end{pmatrix}\right]u_2 \tag{3.46}$$

Select P = I. Then

$$\begin{bmatrix} \left[\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}x - \begin{pmatrix} 3 \\ 2 \end{pmatrix}\right]^T x \\ \left[\begin{pmatrix} 4 & 5 \\ 5 & 4 \end{pmatrix}x + \begin{pmatrix} 2 \\ -2 \end{pmatrix}\right]^T x \end{bmatrix} = 0 \text{ only for } x=0 \text{ and } x \approx \begin{bmatrix} -1.6 \\ 4.0 \end{bmatrix}$$

$$\text{But} \quad x^T(PA + A^TP)x\Big|_{x = \begin{bmatrix} -1.6 \\ 4.0 \end{bmatrix}} \approx -7 < 0 \tag{3.48}$$

Therefore the controls

$$u_1 = \left[\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}x - \begin{pmatrix} 3 \\ 2 \end{pmatrix}\right]^T x\cdot\alpha \tag{3.49}$$

$$u_2 = \left[\begin{pmatrix} 4 & 5 \\ 5 & 4 \end{pmatrix} x + \begin{pmatrix} 2 \\ -2 \end{pmatrix}\right]^T x \cdot \alpha \qquad (3.50)$$

will force the state of (3.46) arbitrarily close to the origin if $\alpha$ is chosen sufficiently large.

Note that $\left[\begin{pmatrix} \frac{1}{6} & 1 \\ 0 & \frac{1}{6} \end{pmatrix}, \begin{pmatrix} -3 & 2 \\ 2 & -2 \end{pmatrix}\right]$ is a stabilizable pair. Design 3.16 is thus applicable.

## 3.5 The single input case

Special attention will be given to the single input case:

$$\dot{x} = Ax + (Bx + b)u \qquad (3.51)$$

A strictly unstable, i.e. Re $\lambda_i(A) > 0$ for all i, and diagonalizable.

In this case there is no $P = P^T > 0$ such that there exist non-zero x for which $x^T(PA + A^TP)x$ is negative. To apply theorem 3.13 we must therefore demand that there exists a $P = P^T > 0$ such that $u^* = (Bx + b)^T Px \neq 0$ for all non-zero x. Consequently we investigate the solution of the equation:

$$x^T B^T Px = - b^T Px \qquad (3.52)$$

For which P does equ. (3.52) lack a non-zero solution x? The discussion that follows will also enhance the understanding of the condition of theorem 3.13 in the multi-input case.

**Lemma 3.20:** There exists a $P = P^T > 0$ such that $x^T B^T P x > 0$ when $x \neq 0$, iff all eigenvalues of B have positive real parts.

**Proof:** $x^T B^T P x = \frac{1}{2} x^T (B^T P + PB) x > 0$ for $x \neq 0$ iff $B^T P + PB = W > 0$.
Given $W > 0$, there exists a positive definite solution $P = P^T$ to the equation $B^T P + PB = W$ iff all eigenvalues of B have positive real parts (see Lancaster (1969)).

□

The consequence of Lemma 3.20 is that the mapping $x \to x^T B^T P x$, $R^n \to R$ is a

1. paraboloid in (n + 1)-space iff sign Re $\lambda_i(B) =$
   $=$ sign Re $\lambda_k(B) \neq 0$, all $i,k$, (3.53a)
   and P suitably chosen. (3.53b)

2. saddle in (n + 1)-space (which might be degenerate), otherwise. (3.53c)

The mapping

$$x \to b^T P x \, , \quad R^n \to R \tag{3.54}$$

is a plane in (n + 1)-space through the origin.

When $x \in R^2$ figures 3.1 a and 3.1 b illustrate the situation.

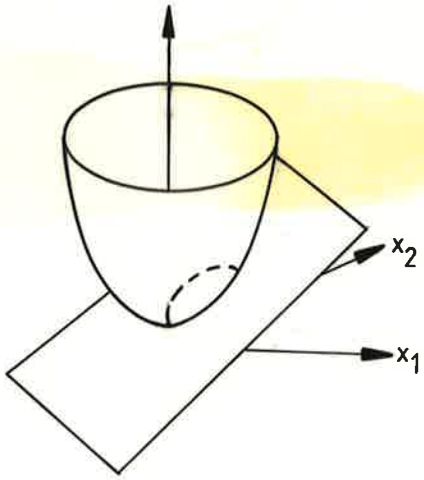


Fig. 3.1 a: $x \to x^T B^T P x$, $R^2 \to R$ is a paraboloid.

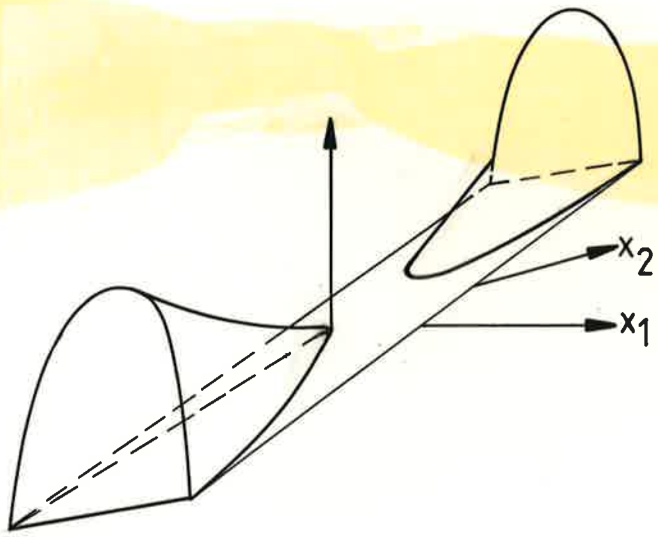


Fig. 3.1 b: $x \to x^T B^T P x$, $R^2 \to R$ is a saddle.

The solution of (3.52) is given by the projection into the n-dimensional state space of the curve in (n + 1)-space formed by the intersection of the paraboloid or saddle (3.53) and the plane (3.54). From theorem 3.13, equ. (3.27) applied to the single input case, we note that $u^* = 0$ for some $x \neq 0$ iff equ. (3.52) is satisfied.

We realize that $u^* = 0$ for some $x \neq 0$ except when

$$x^T B^T P x \text{ is a paraboloid and } b = 0 \qquad (3.55)$$

which is shown in fig. 3.2. This corresponds to the system

$$\dot{x} = Ax + Bxu \qquad (3.56)$$

with B satisfying equ. (3.53a).

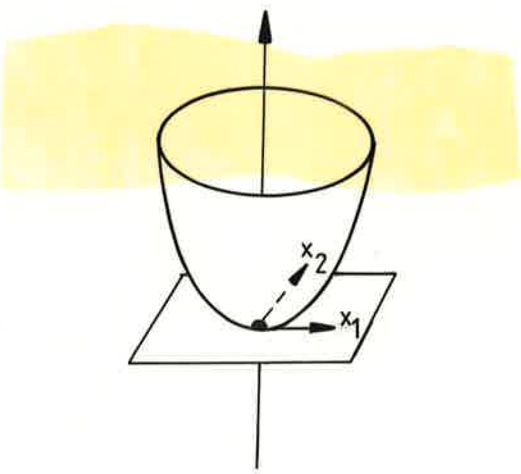


Fig. 3.2: $x \rightarrow x^T B^T P x$ is a paraboloid, and $x \rightarrow b^T P x = 0$, because $b = 0$.

Theorem 3.13 is applicable in this and only this single input case: there exists a $P = P^T > 0$ such that $u^* = x^T B^T Px \cdot \alpha$ stabilizes the system (3.56). However in this case there is also a constant control $u_c$ that makes the system (3.56) asymptotically stable. The latter fact is true because of condition (3.53), see section 4.2.

Example 3.21:

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} xu \tag{3.57}$$

is stabilized by

$$u^* = -x^T \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x \cdot k \quad , \; k > 0 \tag{3.58}$$

Remark 3.22: In the case $x^T B^T Px$ is a paraboloid and $b \neq 0$ there exists an $\alpha > 0$ such that $u^* = -\alpha(Bx + b)^T Px$ will cause the state of the system (3.51) to reach arbitrarily near or inside the ellipsoid

$$\{y \mid y^T B^T Py = -b^T Py\} \tag{3.59}$$

This is so because for the Lyapunov function $\Phi = x^T Px$ of the system (3.51) with the control $u^* = -\alpha(Bx+b)^T Px$,

$$\dot{\Phi} = x^T (PA + A^T P)x - 2\left[x^T P(Bx + b)\right]^2 \alpha < 0 \tag{3.60}$$

at least outside an arbitrarily small neighbourhood of the ellipsoid (3.59) provided $\alpha$ is chosen sufficiently large.

The controller $u^* = -\alpha(Bx + b)^T Px$ is thus quite good in case the ellipsoid (3.59) is close to zero.

Note that the point

$$x = -B^{-1}b \quad , \tag{3.61}$$

the point where any control action vanishes, lies on the ellipsoid (3.59).

The effect of constant controls on the system is commented upon in example 4.19b.

□

Consider the following special case of system (3.51):

$$\begin{cases} \dot{x} = Ax + (Bx + b)u & (3.62a) \\ B \text{ diagonalizable with all eigenvalues purely imaginary} & (3.62b) \end{cases}$$

Lemma 3.23: Condition (3.62b) is equivalent with the condition

There exists a $P = P^T > 0$ such that $PB + B^TP = 0$ (3.63)

Proof: ⇒ ) Assume (3.62b).
Then there exists an invertible matrix T, with real entries, such that

$$TBT^{-1} = \begin{bmatrix} \begin{matrix} 0 & \omega_1 \\ -\omega_1 & 0 \end{matrix} & & 0 \\ & \begin{matrix} 0 & \omega_2 \\ -\omega_2 & 0 \end{matrix} & \\ 0 & & \ddots \end{bmatrix} \tag{3.64}$$

It is easily realized that $P = T^T T$ satisfies (3.63).

$\Leftarrow$ ) Assume that $P = P^T > 0$ is such that $PB + B^T P = 0$. Consider

$$\dot{z} = Bz \tag{3.65}$$

Let $V = z^T Bz$ be a Lyapunov function candidate:

$$\dot{V} = z^T (PB + B^T P) z = 0 \tag{3.66}$$

Thus the linear system (3.65) is stable but not asymptotically stable. (3.62b) follows.

□

An attempt to stabilize the system (3.62) with the controller

$$\begin{cases} u^* = -\alpha(Bx + b)^T Px \\ PB + B^T P = 0 \end{cases} \Leftrightarrow \quad u^* = -\alpha b^T Px \tag{3.67}$$

will not necessarily work. The reason is that $u^* = 0$ in the hyperplane perpendicular to $b^T P$ (cf. equ. (3.52)). Thus theorem 3.13 is not necessarily applicable.

However, Yasuda (1977) has shown how to stabilize (3.62) under certain assumptions:

<u>Theorem 3.24</u> (Yasuda): If $\left[A,b\right]$ in the system (3.62) is a stabilizable pair and there exists a $P = P^T > 0$ together with a vector $\ell$ that satisfy
$P(A + b\ell^T) + (A + b\ell^T)^T P < 0$ and $PB + B^T P = 0$

then the feedback control $u = \ell^T x$ exponentially stabilizes the system.

Proof: Apply the feedback $u = \ell^T x$ to (3.62) and get the closed loop system

$$\dot{X} = Ax + (Bx + b)\ell^T x \quad . \qquad (3.68)$$

Let $\Phi = x^T Px > 0$ be a Lyapunov function.

$$\dot{\Phi} = 2x^T P(A\dot{x} + (Bx+b)\ell^T x) = 2x^T P(A + b\ell^T)x < 0$$

using the conditions of the theorem.

□

# 4. ALTERNATIVE CONTROLLERS

## 4.1 The non-quadratic Lyapunov function approach

The essence of the sections 3.4 and 3.5 is, quite obviously, that control action must be retained in order to control. The controllers presented above fail when there exist x such that $G(x)u^* = 0$. Note that we hitherto have used quadratic Lyapunov functions $\Phi(x) = x^T P x$ to generate the control $u^*$. The following ways to overcome the problem were attempted, particularly in the single input case.

4.1.1. The best way would be to solve analytically the non-linear Riccati equation (3.3) for $\Phi(x)$ given $m(x) > 0$. This does not seem to be possible.

4.1.2. The system equation (2.1) was expanded à la Brockett (1973), page 49. A new state vector is introduced;

$$z = \begin{bmatrix} x \\ x^{[2]} \\ \vdots \\ x^{[p]} \end{bmatrix} \triangleq h(x) \qquad (4.1)$$

where the elements of $x^{[p]}$ are

$$\sqrt{\binom{p}{p_1}\binom{p-p_1}{p_2}\cdots\binom{p-p_1-\cdots-p_{p-1}}{p_p}}\; x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n}, \qquad (4.2)$$

$$\sum_{i=1}^{n} p_i = p \quad , \quad p_i \geq 0$$

Example 4.1:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad , \; p = 3 \tag{4.3}$$

gives

$$x^{[p]} = [x_1^{\,3}, \sqrt{3}\, x_1^{\,2} x_2, \sqrt{3}\, x_1^{\,2} x_3, \sqrt{3}\, x_1 x_2^{\,2}, \sqrt{6}\, x_1 x_2 x_3,$$
$$\sqrt{3}\, x_1 x_3^{\,2}, x_2^{\,3}, \sqrt{3}\, x_2^{\,2} x_3, \sqrt{3}\, x_2 x_3^{\,2}, x_3^{\,3}] \tag{4.4}$$

Note that the mapping $z = h(x)$ is injective. It is easily shown that

$$\dot{z} = \bar{A}z + \sum_{i=1}^{m} \bar{B}_i z u_i + \bar{B}_0 u \triangleq \bar{A}z + \bar{G}(z)u \tag{4.5}$$

i.e. the bilinear structure is preserved. Now, design according to theorem 4.1 was attempted, i.e.

$$\Phi(z) = z^T \bar{P} z \; , \; \bar{P} = \bar{P}^T > 0 \quad ; \tag{4.6}$$

$$u_i^* = -(\bar{B}_i z + \bar{b}_{0i})^T \bar{P} z \quad . \tag{4.7}$$

This is quite a general way to construct a Lyapunov function as a sum of products of powers of $x_1, x_2, \ldots, x_n$ (the lowest degree of a term is 2). This author's attempt to find a $\bar{P} > 0$ that retained control action in all of the state space was unsuccessful, even in the case of simple examples.

□

Sandor (Feb 1977) and (Oct 1977) used the same method to construct a non-linear regulator for the linear system $\dot{x} = Ax + B_0u$. If, in this case, A is unstable, the linear system can first be stabilized by a conventional linear controller, so that the final control is a sum of a linear and a non-linear term:
$u(x) = u_{lin}(x) + u_{n.l.}(x)$ .

The analogue approach, i.e. first to stabilize A by a linear controller and then apply the control (4.7) (or (3.27)) was not successful in the bilinear case. The reason is that the introduction of a linear term in u increases the degree in x of the right hand side, which generally destabilizes the system.

Given the system (4.5). Assume that the system is such that $(\bar{A} + \bar{B}_0K)$ is strictly stable. Let $k^T = [k_1^T \mid \dots \mid k_m^T]^T$ and set $u_i = k_i^Tz + v_i$. The closed loop system becomes

$$\dot{z} = (\bar{A} + \bar{B}_oK)z + \sum_{i=1}^{m} \bar{B}_i z k_i^T z + \sum_{i=1}^{m} (\bar{B}_i z + \bar{b}_{oi})v_i \qquad (4.8)$$

The second term on the right hand side will often destabilize the autonomous system, except near z = 0, as in lemma 3.15.

<u>4.1.3.</u> Attempts were made to tailor-make a Lyapunov-function by constructing different Lyapunov functions for different parts of the state space. One partial success was achieved in the single input case:

Proposition 4.2: Consider the single input system

$$x = Ax + (Bx + b)u \tag{4.9}$$

Assume $x^TBx > 0$ , i.e. $(B + B^T) > 0$. Then

$$u = \begin{cases} -\alpha(Bx + b)^T(Bx + b) & b^Tx \geq 0 \\ -\alpha(Bx + b)^T(Bx - b) & b^Tx < 0 \end{cases} \tag{4.10}$$

for sufficiently large $\alpha > 0$ will force the state arbitrarily near or inside the ellipsoidical "disk" whose boundary is defined by

$$x^TB^TBx - b^Tb = 0 . \tag{4.11}$$

Proof: Let the Lyapunov function candidate be

$$\Phi = \alpha(x^TBx + 2|b^Tx|) \geq 0 \tag{4.12}$$

$\Phi$ is continuous. Divide the state space into 3 regions:

$$S_+ = \{x \mid b^Tx > 0\} \tag{4.13a}$$

$$S_0 = \{x \mid b^Tx = 0\} \tag{4.13b}$$

$$S_- = \{x \mid b^Tx < 0\} \tag{4.13c}$$

Let $\varphi_+(x) = 2\alpha(Bx + b)^TAx - 2\alpha^2[(Bx+b)^T(Bx+b)]^2$ (4.14a)

and $\varphi_-(x) = 2\alpha(Bx - b)^TAx - 2\alpha^2[x^TB^TBx-b^Tb]^2$ (4.14b)

Then, when

$$x \in S_+ : \quad \dot{\Phi}(x) = \varphi_+(x) \qquad (4.15a)$$

$$x \in S_- : \quad \dot{\Phi}(x) = \varphi_-(x) \qquad (4.16b)$$

$$x \in S_0 : \quad \limsup_{h \to 0} \frac{\Phi(t+h) - \Phi(t)}{h} \leq \max \{\varphi_+(x), \varphi_-(x)\} \qquad (4.16c)$$

We see that

$$\limsup_{h \to 0} \frac{\Phi(t+h) - \Phi(t)}{h} < 0 \qquad (4.17)$$

outside a neighbourhood of the ellipsoid (4.11). $\Phi$ is therefore a Lyapunov function outside this neighbourhood, see Hahn (1967). This neighbourhood can be made arbitrarily small by choosing a sufficiently large $\alpha$.

□

Remark 4.3: Note that the ellipsoid (4.11) is not equal to the ellipsoid (3.59) for any P.

Remark 4.4: The origin is interior to the ellipsoidical disk (4.11).

Remark 4.5:

$$x_0 = - B^{-1}b \qquad (4.18)$$

(where the control vanishes) is on the ellipsoid (4.11). $x_0$ gives a rough measure of how far from the origin (4.11) lies.

Remark 4.6: The effect of constant controls on the system (4.9) is commented upon in example 4.19b.

Remark 4.7: $x_0 \in \{x \mid b^T x \leq 0\}$ . See figure 4.1.

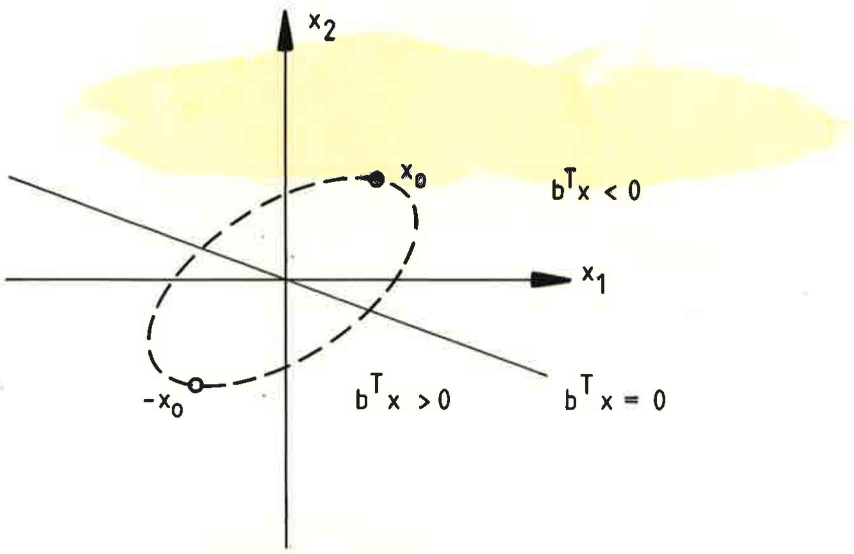


Fig. 4.1: The ellipsoid (4.11) in the two dimensional case.

Remark 4.8: The practicality of this method depends obviously on how near the ellipsoid (4.11) is to the origin, and how large a bias that is acceptable. A second order example is depicted in example 4.9.

Example 4.9: The system

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} xu + \begin{bmatrix} -2 \\ -2 \end{bmatrix} u \qquad (4.19)$$

satisfies the conditions of proposition 4.2. Apply the control given by equ. (4.10) with $\alpha = 100$:

$$u = \begin{cases} -100([2x_1+x_2-2]^2 + [x_1+3x_2-2]^2), & -2(x_1+x_2) \geq 0 \\ -100([2x_1+x_2]^2 + [x_1+3x_2]^2 - 8), & -2(x_1+x_2) < 0 \end{cases} \qquad (4.20)$$

A simulation result, generated by the SIMNON-program INDEV (see Appendix A) is found in figure 4.2.

We compute

$$b^T x = -2(x_1 + x_2) \qquad (4.21)$$

and (see remark 4.5)

$$x_o = (0.8, 0.4) \qquad (4.22)$$

Remark 4.10: Sometimes a locally stabilizing linear feedback (see Lemma 3.15) might be found whose stability region includes the equilibrium points achieved by the control (4.10). Such a linear feedback was not found for example 4.9.

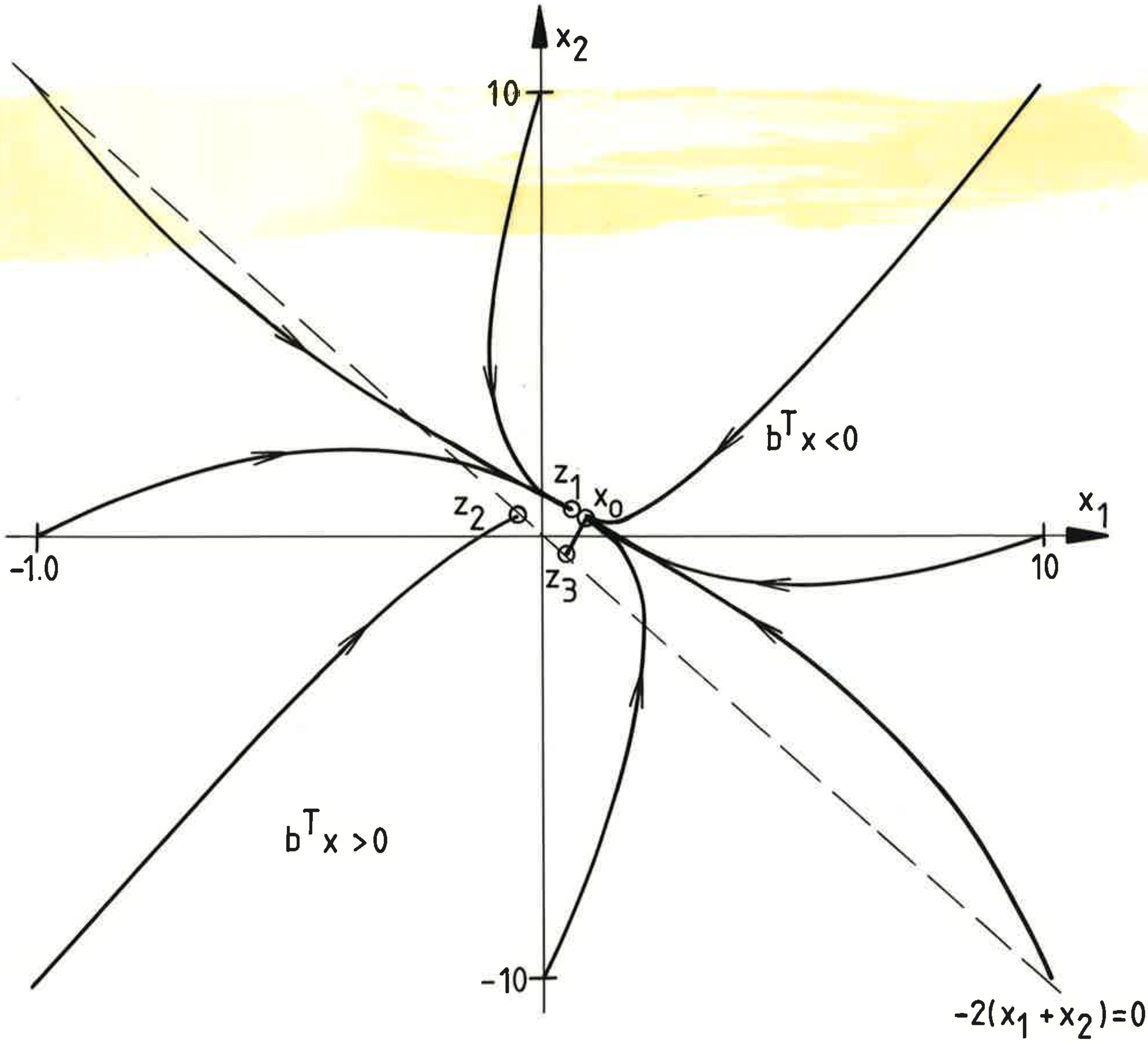


Fig. 4.2: Phase plane trajectories of a simulation of example 4.9.

Initial conditions: (10,0), (10,10), (0,10), (-10,10), (-10,0), (-10,-10), (0,-10), (10,-10).

Final equilibrium points: $z_1$ = (0.6,0.544), $z_2$ = = (-0.464,0.464), $z_3$ = (0.434,-0.434).

## 4.2 Constant controls

It is apparent that system (2.6) is asymptotically stabilized by constant controls iff there exist constants $u_{ic}$ , $i = 1, \ldots ,m$ such that $(A + \sum_{i=1}^{m} B_i u_{ic})$ is stable. However, (2.6) is not always stabilizable with constant controls, see for instance example 4.16 below.

Constant controls have evoked some interest: Wei (Dec. 1978) and Krener (1978) consider (2.6) where each pair in the set $\{A, B_1, \ldots , B_m\}$ commutes. Then constant controls stabilize the system, while optimizing a minimum energy criterion. Note that the assumption on commutativity is very restrictive.

The effect of applying a constant control $u_{ic}$ on the system (2.4) is that, even if $(A + B_i u_{ic})$ is stable, $b_{i0} u_{ic}$ places a bias on the closed loop system so that it will not converge to $x = 0$. Therefore the other controls must be used to compensate for this bias. Necessary and sufficient conditions for a constant control to stabilize (2.4) are given in the following theorem:

Theorem 4.11: Consider the system (2.4) which is repeated for conveniance:

$$\dot{x} = Ax + \sum_{i=1}^{m} (B_i x + b_{i0}) u_i \quad , \tag{4.23}$$

$$B_0 = [b_{10} \,\vdots\, b_{20} \,\vdots\, \ldots \,\vdots\, b_{m0}] \quad .$$

The constant control $u_c = (u_{1c}\ u_{2c}\ \dots\ u_{mc})^T$ asymptotically stabilizes (4.23) iff

$$u_c \in \text{Ker } B_0 \quad , \quad \text{and} \tag{4.24}$$

$$(A + \sum_{i=1}^{m} B_i u_{ic}) \text{ is a strictly stable matrix.} \tag{4.25}$$

Proof: Apply the constant control $u_c$ on (4.23). The closed loop system becomes

$$\dot{x} = (A + \sum_{i=1}^{m} B_i u_{ic})x + B_0 u_c \tag{4.26}$$

Sufficiency: Assume that $u_c$ satisfies (4.24) and (4.25). Then (4.26) becomes an asymptotically stable linear system:

$$\dot{x} = \underbrace{(A + \sum_{i=1}^{m} B_i u_{ic})}_{\text{strictly stable}}x + \underbrace{B_0 u_c}_{=\,0} \tag{4.27}$$

Necessity: A necessary condition for stability is that the origin is a stationary point of the closed loop system, i.e.

$$x = 0 \Rightarrow \dot{x} = 0 \quad \text{for (4.26):} \tag{4.28}$$

$$0 = 0 + B_0 u_c \tag{4.29}$$

which gives (4.24).

Given (4.24), (4.26) becomes the linear system

$$\dot{x} = (A + \sum_{i=1}^{m} B_i u_{ic})x \tag{4.30}$$

A necessary condition for stability of (4.30) is the condition (4.25).

□

Remark 4.12: Naturally there exists a control

$$u = u_c + u_q \tag{4.31}$$

which stabilizes the system (2.4), where

$u_c$ satisfies conditions (4.24) and (4.25), (4.32)

and $u_q$ is of the structure

$$u_q = (u_{1q}, u_{2q}, \ldots, u_{mq})^T \tag{4.33}$$

$$u_{iq} = -(B_i x + b_{i0})^T Px \quad , \tag{4.34}$$

investigated in chapter 3.

Remark 4.13: The conditions of theorem 4.11 are restrictive.

In the following theorem a special case is treated. A class of single input systems is displayed for which stabilizability is equivalent to stabilizability with a constant control.

Theorem 4.14: Given the system

$$\dot{x} = Bxu \quad , \tag{4.35}$$

B is a real n/n matrix, $x \in R^n$ , $u \in R$ .

Let $\lambda_i$ be the i:th eigenvalue of B.

For the system (4.35) stabilizability is equivalent to stabilizability with a constant control iff

$$\text{sign Re } \lambda_i = \text{sign Re } \lambda_k \neq 0 \quad \text{all } i,k. \tag{4.36}$$

Proof: Jordanize B, thereby transforming (4.35):

$$\dot{z} = Jzu \quad , \tag{4.37}$$

$$J = \bigoplus_{i=1}^{m} J_i \quad , \quad J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix}$$ is a $n_i/n_i$ matrix.

Let $z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \end{bmatrix}$ and $z_i = \begin{bmatrix} z_{i,1} \\ \vdots \\ z_{i,n_i} \end{bmatrix}$ $i = 1, \ldots , m.$

a. Assume (4.36). Then the constant control $u = -\text{sign Re } \lambda_i$ stabilizes (4.35).

b. Assume that $\text{Re } \lambda_i = 0$ for some i. Then stabilization is not possible.

c. Assume that $\text{sign Re } \lambda_i = - \text{sign Re } \lambda_k \neq 0$ for some pair i,k. Then the differential equations for the states corresponding to the last rows in the Jordan blocks $J_i$ and $J_k$ are

$$\dot{z}_{j,n_j}(t) = \lambda_j u(t) z_{j,n_j}(t) \quad , \tag{4.38}$$

j = i and k respectively. This gives

$$z_{j,n_j}(t) = e^{\lambda_j \int_0^t u(s)ds} z_{j,n_j}(0) \quad j = i,k \quad . \tag{4.39}$$

It is then apparent that

$$|z_{i,n_i}(t)|^{|Re\lambda_k|} \cdot |z_{k,n_k}(t)|^{|Re\lambda_i|} = \text{constant} \tag{4.40}$$

independent of u(t). Stabilization is not possible.

From a, b and c the statement of the theorem follows.

□

We conclude this section by showing a few examples where constant controls can be used, and some examples where they cannot.

<u>Example 4.15:</u> Given the system

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x + \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} xu_1 + \begin{bmatrix} -2 \\ -2 \end{bmatrix} u_1 + \begin{bmatrix} -1.5 & -0.5 \\ 1 & -2.5 \end{bmatrix} xu_2 + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_2 \tag{4.41}$$

We find that Ker $B_0$ is spanned by $(1\ 2)^T$, and that

$$\begin{cases} u_1 = 1 \\ u_2 = 2 \end{cases} \tag{4.42}$$

stabilize the system.

Example 4.16: Given the system of example 3.11 respectively 3.18. Even though Ker $B_0=R^2$ it is impossible to find a constant control $u_c$ that satisfies theorem 4.11. No constant control will stabilize these systems.

Example 4.17: The system of example 3.12 is stabilized by any constant $u_c < 0$.

Example 4.18: For the system given in example 3.19, Ker $B_0 = 0$. Thus no constant control will stabilize (4.23).

Example 4.19a: Example 3.21 displays a case when a constant control, e.g. $u_c = -3$, is superior to the control $u^* = -x^T B^T P x \cdot \alpha$.

Example 4.19b: Consider the single input system of remark 3.22 and remark 4.6. A constant control $u_c$ will force the system to the equilibrium point

$$x_c = (A + Bu_c)^{-1} bu_c = -\left(\frac{B^{-1}A}{u_c} + I\right)^{-1} B^{-1} b \qquad (4.43a)$$

if $(A + Bu_c)$ is a stability matrix. Note that

$$x_c \to -B^{-1}b \text{ , when } |u_c| \to \infty \text{ .} \qquad (4.43b)$$

$-B^{-1}b$ is the point where all control action vanishes.

□

A case of constant control is also commented upon in remark 3.3.

## 4.3 Phase plane methods. Dyadic bilinear systems.

It is obvious that a linear control $u = k^T x$ exists that locally asymptotically stabilizes the system (2.1) in a sufficiently small neighbourhood around the origin, provided $(A,B_0)$ is a stabilizable pair, see lemma 3.15. In such a case a possible way to stabilize the system would be to have this locally stabilizable neighbourhood of the origin as a target set for other controls. When the state enters the target set, the linear control might be applied.

Some ways to generate suitable controls via phase plane analysis are found in the literature. As all phase plane methods they tend to become cumbersome when the dimension of the state increases beyond 2.

Mohler (1973) studies the trajectories of constant controls, and generates a bang-bang control. The switching times are precomputed, and thus it is an open loop method.

Utkin (1977) refers to results that might be applicable to certain bilinear systems. The state space is divided into regions, and on each region a separate feedback control $u_k(x)$ is applied, so that a sliding (chattering) mode is generated on the boundary between the regions. Gutman (1979) also has applicable results, taking disturbances into account.

Mehra (1979) analyzes the stability regions of a non-linear system by investigating the location and character of equilibrium points as a function of the constant control u. Bifurcation and catastrophy theory provides some conclusions and suggestions how to steer u(t) in order to reach a desired region of the state space.

It is believed that the above three methods might provide valuable insight and ideas when facing the problem of controlling a specific bilinear system. However they do not seem suited for generating a general theory of feedback control.

There is a special class of bilinear systems of practical interest that is sometimes suited for "phase plane carpentrying". This class is defined here:

Definition 4.20: A bilinear system (defined by equ. (2.4) and repeated here for convenience)

$$\dot{x} = Ax + \sum_{i=1}^{m} (B_i x + b_{i0}) u_i$$

is called dyadic of order d, if for $i = i_1, \dots, i_d$, $d \in \{1, \dots, m\}$

$$B_i x + b_{i0} = b_{i0}(c_i^T x + 1). \tag{4.44}$$

The following example illustrates that dyadic bilinear systems is a quite common class of systems:

Example 4.21: Consider the following flow system which models, for instance, the effect of drugs on the transfer of some dissolved living matter in the human body:

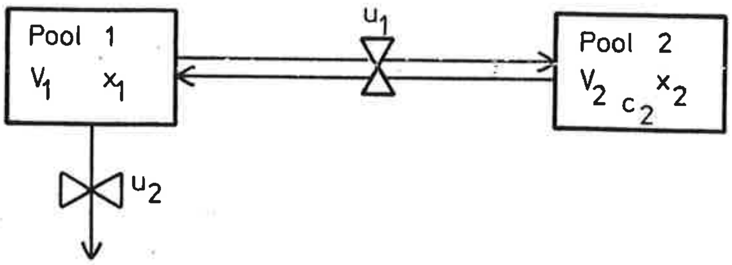


Fig. 4.3: Model of flow system in the human body.

The equations governing the flow system are:

$$\begin{cases} V_1\dot{x}_1 = u_1(x_1 - x_2) + x_1u_2 & (4.45a) \\ V_2\dot{x}_2 = -u_1(x_1 - x_2) + c_2x_2 & (4.45b) \end{cases}$$

with the constraints $x_1, x_2 \geq 0;\ c_2 > 0;\ u_1, u_2 \leq 0$ (4.46)

where $V_1$ = the volume of pool 1 $(m^3)$

$x_1$ = the concentration in pool 1 $(kg/m^3)$

$V_2$ = the volume of pool 2 $(m^3)$

$x_2$ = the concentration in pool 2 $(kg/m^3)$

$c_2$ = the growth rate in pool 2 $(s^{-1})$

$u_1$ = the transfer rate between pool 1 and pool 2 $(s^{-1})$

$u_2$ = the transfer rate out of pool 1 $(s^{-1})$

Rewritten in standard fashion the systems equations become:

$$\dot{x} = Ax + B_1 x u_1 + B_2 x u_2 \tag{4.47a}$$

where

$$A = \begin{bmatrix} 0 & 0 \\ 0 & \frac{c_2}{V_2} \end{bmatrix} \tag{4.47b}$$

$$B_1 = b_1 d_1^T = \begin{bmatrix} \frac{1}{V_1} \\ -\frac{1}{V_2} \end{bmatrix} [1 \quad -1] = \begin{bmatrix} \frac{1}{V_1} & -\frac{1}{V_1} \\ -\frac{1}{V_2} & \frac{1}{V_2} \end{bmatrix} , \tag{4.47c}$$

$$B_2 = b_2 d_2^T = \begin{bmatrix} \frac{1}{V_1} \\ 0 \end{bmatrix} [1 \quad 0] = \begin{bmatrix} \frac{1}{V_1} & 0 \\ 0 & 0 \end{bmatrix} . \tag{4.47d}$$

We note that $B_1$ and $B_2$ are dyads.

A desired equilibrium point $x_e = (x_{1e}, x_{2e})^T$ may be chosen, such that $x_{2e} > x_{1e} > 0$. Setting the left hand sides in (4.45) equal to zero gives the corresponding equilibrium input $u_e = (u_{1e}, u_{2e})^T$ :

$$\begin{cases} u_{1e} = \dfrac{c_2 x_{2e}}{x_{1e} - x_{2e}} & (4.48) \\[2em] u_{2e} = \dfrac{-c_2 x_{2e}}{x_{1e}} & (4.49) \end{cases}$$

The method of section 2.3 gives the following bilinear system with $x_e$ as the origin in the state space:

$$\Delta\dot{x} = \tilde{A}\Delta x + [b_1 d_1^T \Delta x + b_1(x_{1e} - x_{2e})]\Delta u_1 +$$

$$+ [b_2 d_2^T \Delta x + b_2 x_{1e}]\Delta u_2 \qquad (4.50a)$$

where

$$\Delta x = x - x_e \quad , \quad \Delta u = u - u_e \quad , \text{ and} \qquad (4.50b)$$

$$\tilde{A} = \begin{bmatrix} \frac{c_2 x_{2e}^2}{V_1(x_{1e}-x_{2e})x_{1e}} & \frac{-c_2 x_{2e}}{V_1(x_{1e}-x_{2e})} \\ \frac{-c_2 x_{2e}}{V_2(x_{1e}-x_{2e})} & \frac{c_2 x_{1e}}{V_2(x_{1e}-x_{2e})} \end{bmatrix} \qquad (4.50c)$$

Define

$$\tilde{b}_1 = (x_{1e} - x_{2e})b_1 \qquad (4.51a)$$

$$\tilde{d}_1 = \frac{d_1}{(x_{1e}-x_{2e})} \qquad (4.51b)$$

$$\tilde{b}_2 = x_{1e} b_2 \qquad (4.51c)$$

$$\tilde{d}_2 = \frac{d_2}{x_{1e}} \qquad (4.51d)$$

Then (4.50) can be rewritten as:

$$\Delta\dot{x} = \tilde{A}\Delta x + \tilde{b}_1(\tilde{d}_1^T \Delta x + 1)\Delta u_1 + \tilde{b}_2(\tilde{d}_2^T \Delta x + 1)\Delta u_2 \qquad (4.52)$$

Consequently (4.50) is dyadic of order 2.

□

Conditions for stabilizability are given in theorem 4.22 and remark 4.25. For the sake of simplicity we will mainly treat bilinear systems that are dyadic of order 1. A new controller algorithm, the Division Controller, that can sometimes stabilize dyadic systems is presented in Design 4.26. As with all phase plane methods this design is hard to apply on systems of higher order than 2.

Consider the single input dyadic system

$$\begin{cases} \dot{x} = Ax + (Bx + b)u \quad , \quad x \in R^n \; , \quad u \in R \\ (Bx + b) = b(c^T x + 1) \end{cases} \tag{4.53}$$

As in theorem 2.3,

$$\text{let} \quad d(x) = c^T x + 1 \tag{4.54}$$

$d(x) = 0$ defines an (n-1)-dimensional hyperplane.

Divide the state space into the following sets:

$$S_+ = \{x \mid d(x) > 0\} \tag{4.55}$$

$$S_0 = \{x \mid d(x) = 0\} \tag{4.56}$$

$$S_- = \{x \mid d(x) < 0\} \tag{4.57}$$

Let $\varphi(x_0, u(\cdot), t)$ be the solution of (4.53) at time t when $x(0) = x_0$ and $u(s)$ , $s \in [0,t]$ , is the control input.

Define:

$$V = \{x_0 \in S_+ \mid \exists\, u_V \text{ such that } \varphi(x_0,u_V,t) \in S_+ \;\forall\, t \text{ , and } \varphi(x_0,u_V,t) \to 0, \quad t \to \infty\} \tag{4.58}$$

$$Y = \{x_0 \in S_0 \mid e^{At}x_0 \in V \text{ for some } t > 0\} \tag{4.59}$$

$$W = \{x_0 \mid \exists\, u_Y \text{ such that } \varphi(x_0,u_Y,t) \in Y \text{ for some } t > 0\} \tag{4.60}$$

<u>Theorem 4.22:</u>

A necessary and sufficient condition for stabilizing (4.53) is that

1. Y is nonempty, and (4.61)

2. $V \cup Y \cup W = R^n$ (4.62)

<u>Proof</u>: Note that the origin is interior to $S_+$ and belongs to V. (Note also that V is non-empty because the origin belongs to V.)

<u>Sufficiency:</u>

If $x_0 \in V$ the origin may be reached. If $x_0 \in Y$ the trajectory may end in V, from where it may continue to the origin. If $x_0 \in W$ the trajectory may end in Y, etc. The sufficiency is proved because $V \cup Y \cup W = R^n$.

<u>Necessity</u>:

1. Control action ceases in $S_0$. The only way to pass from $S_0 \cup S_-$ to $S_+$ is by power of the autonomous system $\dot{x} = Ax$. In order to reach the origin at least one of the trajectories of the autonomous system emanating from $S_0 \cup S_-$ must end in V. Therefore Y must be non-empty.

2. By definition a trajectory emanating from the set

$$\complement\,(V \cup Y \cup W) \tag{4.63}$$

will remain in this set. Therefore it must be empty.

□

Remark 4.23: Theorem 4.22 is an extension of theorem 2.3.

Remark 4.24: The case

$$\dot{x} = Ax + Bxu \quad , \quad B = bc^T \tag{4.64}$$

can be treated along lines similar to but more complicated than those of theorem 4.22. This will not be done here.

Remark 4.25: The sufficiency part of theorem 4.22 is applicable to the multi-input case (4.43) if the system is dyadic of order 1 or more.

Theorem 4.22 is weak. Its use stems from the fact that the system (4.53) might be analyzed as a set of linear systems when suitable controls are applied. Then the control problem is transformed to one of finding controls that satisfy theorem 4.22. This leads to the following:

Design 4.26: (The Division Controller):
Consider the system (4.53), (4.54). Define the sets:

$$S'_0 = \{x \mid -\varepsilon_1 \leq d(x) \leq \varepsilon_2\} \tag{4.65}$$

where $\varepsilon_1 \geq 0$ , $\varepsilon_2 \geq 0$ are control parameters in the sense that the control designer may let them vary depending on time, the actual state, etc.

$$S'_+ = S_+ \setminus S'_0 \tag{4.66}$$

$$S_-' = S_- \setminus S_0' \tag{4.67}$$

Apply

$$u_+ = \frac{k_+^T x + u_{ref}(x,t)}{d(x)} \quad , \quad x \in S_+' \tag{4.68}$$

$$u_0 = 0 \quad , \quad x \in S_0' \tag{4.69}$$

$$u_- = \frac{k_-^T x + u_{ref}(x,t)}{d(x)} \quad , \quad x \in S_-' \tag{4.70}$$

where $k_+ , k_- \in R^n$.

The closed loop system now becomes

$$\dot{x} = (A + bk_+^T)x + bu_{ref} \quad , \quad x \in S_+' \tag{4.71}$$

$$\dot{x} = Ax \quad , \quad x \in S_0' \tag{4.72}$$

$$\dot{x} = (A + bk_-^T)x + bu_{ref} \quad , \quad x \in S_-' \tag{4.73}$$

Select, if possible, $k_+$ , $k_-$ , $u_{ref}$ , $\varepsilon_1$ , and $\varepsilon_2$ such that the conditions of theorem 4.22 are satisfied.

We see that the problem of controlling (4.53) is reduced to combining suitable state space trajectory portraits (4.71), (4.72), (4.73) into a stable whole. $u_{ref}$ is used to (temporarily) change the equilibrium point in order to change for instance the set Y. $\varepsilon_1$, $\varepsilon_2$ are used as controls (for instance as hystereses) in order to avoid stationary points or limit cycles on or around the boundaries of $\acute{S}_0$.

Remark 4.27: Variations of design 4.26 can also be constructed. See for instance example 4.30.

An attempt was made to find necessary and sufficient conditions for $k_+$, $k_-$, $u_{ref}$, $\varepsilon_1$, and $\varepsilon_2$ when applying design 4.26 to construct a stabilizing control law for a general single input dyadic bilinear system (4.53). This attempt was not successful. However, special cases can be analyzed, see examples 4.28 and 4.29. It is felt that the analysis and control design is most profitably done in each specific numerical case.

Example 4.28: Consider the following single input second order dyadic bilinear system:

$$\dot{w} = Aw + b(g^T w + 1)u \tag{4.74}$$

Assume that [A,b] is a controllable pair. Standard linear theory allows us to transform (4.73) into

$$\dot{x} = \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} d(x)u \tag{4.75}$$

where $d(x) = g^T Tx + 1$ , $w = Tx$ , and T is a suitable invertible 2×2 matrix.

The question we ask is: Under what conditions is $V = S_+$? Therefore we restrict our attention to $\acute{S}_+$, which can be made to approximate $S_+$ arbitrarily well by choosing $\varepsilon_2$

sufficiently small. Moreover, assume that $\varepsilon_2$ is a constant.

Let $u = u_+$ be given by (4.68). The closed loop system (4.71) will be in controllable canonical form:

$$\dot{x} = \begin{bmatrix} -a_1-k_{1+} & -a_2-k_{2+} \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_{ref} \quad , \; x \in S_+ \qquad (4.76)$$

The characteristic polynomial is

$$s^2 + (a_1 + k_{1+})s + (a_2 + k_{2+}) \qquad (4.77)$$

Assume stable real eigenvalues $\lambda_1 \leq \lambda_2 < 0$ ; (4.78)

$$(s-\lambda_1)(s-\lambda_2) \equiv s^2 + (a_1+k_{1+})s + (a_2+k_{2+}) \qquad (4.79)$$

which gives

$$\begin{cases} a_1 + k_{1+} = -(\lambda_1 + \lambda_2) & (4.80) \\ a_2 + k_{2+} = \lambda_1\lambda_2 & (4.81) \end{cases}$$

The fast eigenvector $e_1$, associated with $\lambda_1$, $= \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$ (4.82a)

The slow eigenvector $e_2$, associated with $\lambda_2$, $= \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$ (4.82b)

$e_1$ and $e_2$ can be selected freely in the second quadrant according to assumption (4.78).

We now state that if $S_0$ is such that $e_1 = e_2$ can be selected parallell with $S_0$, then $V = S_+$. (4.83)

This is proved as follows: If $k_+$ is selected such that $e_1 = e_2$ and $u_{ref} = 0$ then a stable one-tangent node with the origin as equilibrium is achieved. See Åström (1969). If the eigenvector $e_1 = e_2$ is parallell to $S_0$, all trajectories at the boundary of $S_0$ and $\bar{S}_+$ ($S_+$) point into $\bar{S}_+$ ($S_+$). This follows easily from elementary properties of the stable one-tangent node. See figure 4.4. The statement follows.

Another analysis has to be done for the case $e_1$ and $e_2$ cannot be selected parallell with $S_0$. To establish stability, the sets Y and Z must also be determined. This depends on the properties of the autonomous system. As noted above, the analysis and design is probably most profitably done in each specific numerical case.

□

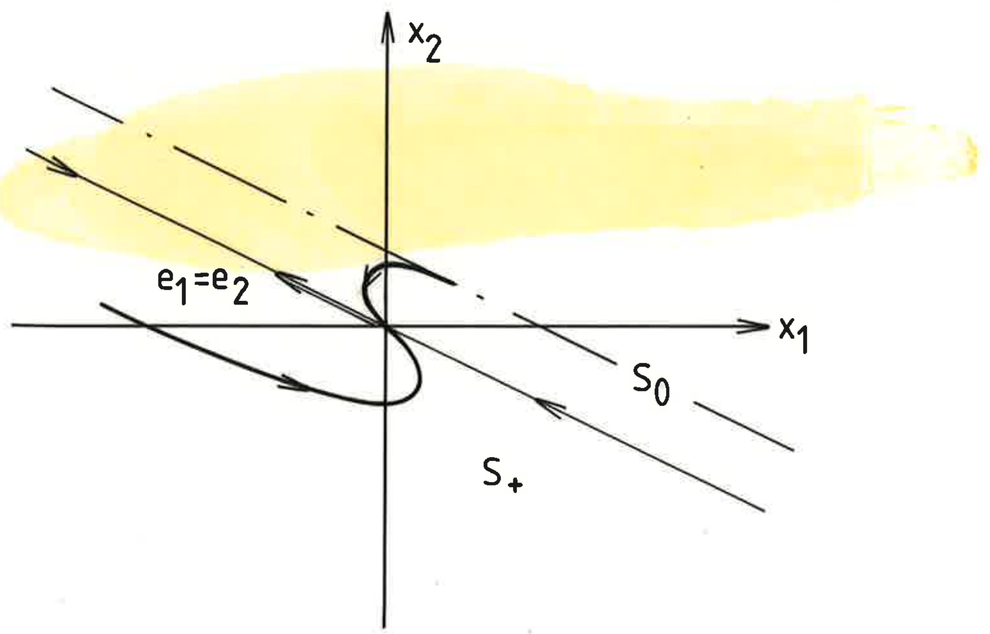


Fig. 4.4: Illustration to example 4.28.

Example 4.29: The analysis of example 4.28 can be done in a more general way. Consider the following single--input dyadic bilinear system:

$$\dot{x} = Gx + r(q^T x + 1)u \quad , \quad x \in R^n \tag{4.84}$$

As in example 4.28 we ask: Under what conditions is $V = S_+$ ? We assume that $x \in \bar{S}_+$ for all t, i.e.:

$$q^T x + 1 \geq \varepsilon \tag{4.85}$$

By applying the control

$$u = \frac{v}{q^T x + 1} \quad , \quad x \in \bar{S}_+ \; , \tag{4.86}$$

and assuming that $b_1 \triangleq q^T r \neq 0$ (4.87)

we can transform (4.84) into

$$\dot{z} = Az + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} v \quad , \tag{4.88}$$

$$b_1 z_1 + 1 \geq \varepsilon \tag{4.89}$$

Rewrite

$$v = -(a_{11} \; a_{12} \; \dots \; a_{1n})z + s \tag{4.90}$$

which gives

$$\begin{cases} \dot{z}_1 = s & (4.91) \\ \begin{bmatrix} \dot{z}_2 \\ z_3 \\ \vdots \\ z_n \end{bmatrix} = F \begin{bmatrix} z_2 \\ z_3 \\ \vdots \\ z_n \end{bmatrix} + f z_1 \quad , & (4.92) \\ z_1 \geq \frac{\varepsilon - 1}{b_1} & (4.93) \end{cases}$$

$$\text{where} \quad F = \begin{bmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{n2} & \cdots & a_{nn} \end{bmatrix} \tag{4.94}$$

$$f = [a_{21} \; a_{31} \cdots a_{n1}]^T \tag{4.95}$$

To control the system (4.91), (4.92) with s is equivalent to controlling the system (4.92) with $z_1$. Thus the problem is reduced to controlling a linear system with the control constrained to a semicompact interval. Lee (1967) page 92 treats this problem:

The system (4.92) is nullcontrollable iff [F,f] is a controllable pair and every eigenvalue $\lambda_i$ of F satisfies Re $\lambda_i(F) \leq 0$. (4.96)

Consequently these are precisely the necessary and sufficient conditions to ensure that $V = S_+$. (Note, however, that the case $b_1 = 0$ (see equ. (4.87)) is not treated here.)

Let us now apply these results to the second order system (4.75) when

$$\begin{cases} d(x) \triangleq (\alpha\ \beta)x + 1 \quad , & (4.97) \\ \alpha \neq 0,\ \beta \neq 0 & (4.98) \end{cases}$$

Let the transformation be given by

$$z = \begin{bmatrix} 1 & \frac{\beta}{\alpha} \\ 0 & 1 \end{bmatrix} x \qquad (4.99)$$

The system corresponding to (4.91), (4.92), (4.93) becomes

$$\begin{cases} \dot{z}_1 = s & (4.100) \\ \dot{z}_2 = -\frac{\beta}{\alpha} z_2 + z_1 & (4.101) \\ z_1 \geq \frac{-1+\varepsilon}{\alpha} & (4.102) \end{cases}$$

The conditions (4.96) result in:

$\left[-\frac{\beta}{\alpha}\ ,\ 1\right]$ is a controllable pair; and (4.103a)

$$-\frac{\beta}{\alpha} < 0 \quad . \qquad (4.103b)$$

(4.103a) is always true, while (4.103b) is true iff

$$\text{sign } \alpha = \text{sign } \beta \qquad (4.104)$$

Comparing (4.99), the definition of $S_0$ (equ. (4.56)), and (4.78), (4.82), (4.83) we realize that (4.104) represents the same condition on $S_0$ to ensure that $V = S_+$

as the condition (4.83) in example 4.28. In fact, the result of this example is stronger because here the condition has been shown to be necessary and sufficient, while in example 4.28 only sufficiency was proved.

This exposure on the Division Controller is concluded by two numerical examples. It should be noted that it is not trivial to find a design that works.

Example 4.30: Consider the following system

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x + \begin{bmatrix} x_1 - x_2 - 2 \\ x_1 - x_2 - 2 \end{bmatrix} u \qquad (4.105)$$

It is apparent that

$$\begin{bmatrix} x_1 - x_2 - 2 \\ x_1 - x_2 - 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left( x_1 - x_2 - 2 \right) = \begin{bmatrix} -2 \\ -2 \end{bmatrix} \left( - \frac{x_1}{2} + \frac{x_2}{2} + 1 \right) \qquad (4.106)$$

A comparison with equ. (4.52) and (4.53) gives

$$b = \begin{bmatrix} -2 \\ -2 \end{bmatrix} \qquad (4.107)$$

$$c = \begin{bmatrix} -0.5 \\ 0.5 \end{bmatrix} \qquad (4.108)$$

It is easily shown that [A,b] is a controllable pair. The following control strategy is selected:

$$u = \begin{cases} \dfrac{1}{x_1-x_2-2}\,(4x_1-9x_2)\;, & |x_1-x_2-2| > 0.001 \qquad (4.109) \\ \\ \dfrac{1}{0.001\ \mathrm{sign}(x_1-x_2-2)}\,(4x_1-9x_2)\;, & |x_1-x_2-2| \le 0.001 \qquad (4.110) \end{cases}$$

where $\mathrm{sign}\ (x) = \begin{cases} 1\;, & x \ge 0 \\ -1\;, & x < 0 \end{cases}$ .

This control is a slight modification of the control (4.68), (4.69), (4.70). The simulation result can be seen in figures 4.5a and 4.5b. The simulation was governed by the SIMNON-program PARAB, see Appendix B.

The figures indicate that $S_0 = \{x \mid x_1 - x_2 - 2 = 0\}$ is easily traversed with the control (4.109), (4.110). A close look at the trajectories in and around $\tilde{S}_0$ shows that $k_+$ and $k_-$ are chosen so that the trajectories in $\tilde{S}_+$, $\tilde{S}_0$, and $\tilde{S}_-$ "cooperate". No limit cycles or stationary points outside the origin are found. The controlled system is globally asymptotically stable (the proof is not complicated and is not given here).

□

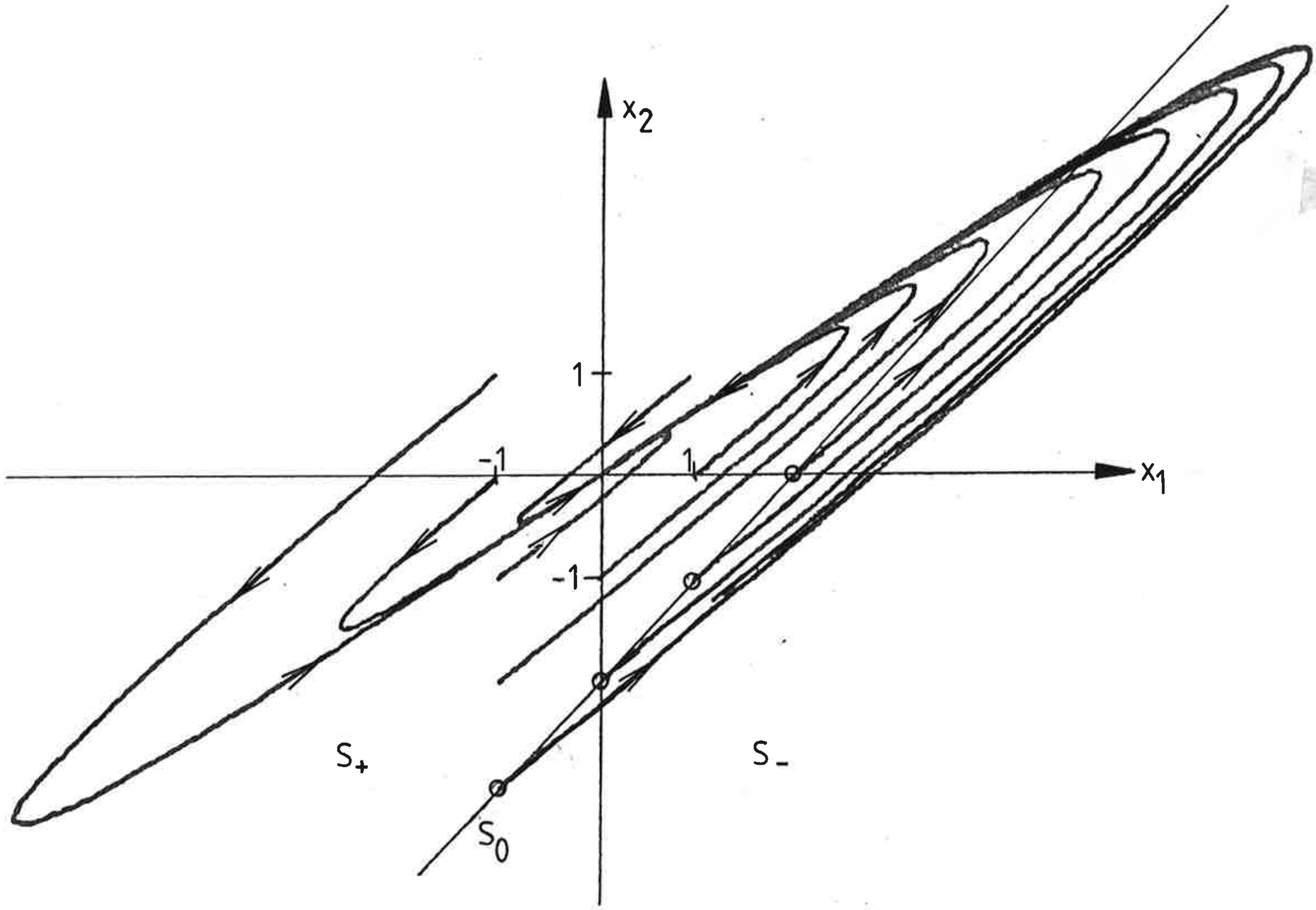


Fig. 4.5a: Phase plane trajectories of a simulation of example 4.30.

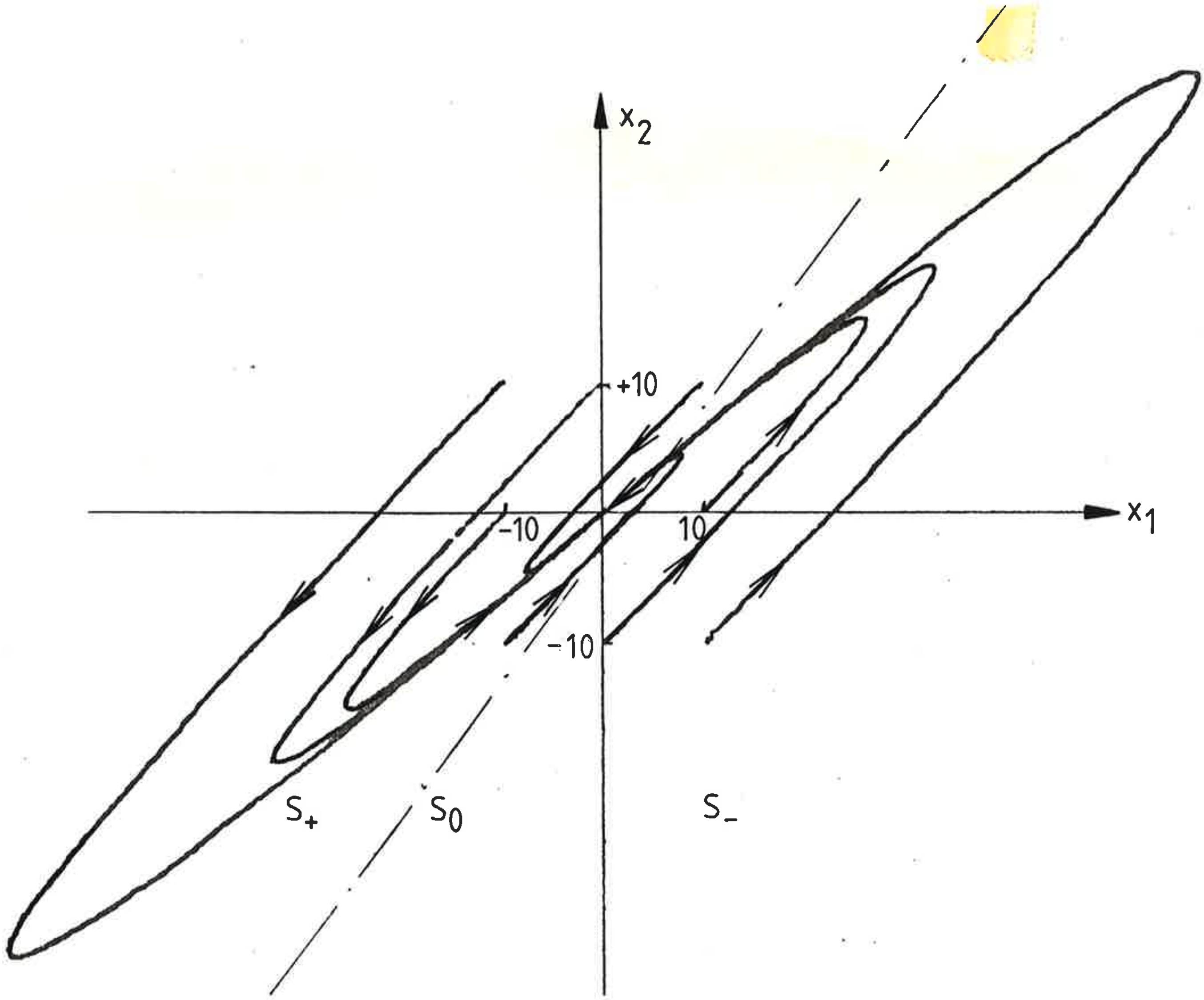


Fig. 4.5b: Phase plane trajectories of a simulation of example 4.30.

Example 4.31: Consider the following system:

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} x_1 - x_2 - 2 \\ x_1 - x_2 - 2 \end{bmatrix} u \qquad (4.111)$$

It is apparent that equ. (4.106), (4.107), (4.108) remain valid. A straightforward control of the type (4.109), (4.110) does not seem to work, because a limit cycle occurs around roughly the point (1,-1).

To overcome this problem, hysteresis was introduced, see Design 4.26. The state space was divided according to figure 4.6.

Introduce

$$MU \triangleq x_1 - x_2 - 2 \qquad (4.112)$$

EPS is defined by the hysteresis function, see figure 4.8.

Now, the control is constructed as follows:

$$u = \frac{-2x_2}{x_1 - x_2 - 2} \quad \text{when} \quad MU < -0.1 \qquad (4.113)$$

$$u = 0 \quad \text{when} \quad -0.1 \leq MU < 0.05 \qquad (4.114)$$

$$u = 0 \quad \text{when} \quad 0.05 < MU \leq 0.1 \text{ and } EPS<0 \qquad (4.115)$$

$$u = \frac{-2x_2}{0.1} \quad \text{when} \quad 0.05 < MU \leq 0.1 \text{ and } EPS>0 \qquad (4.116)$$

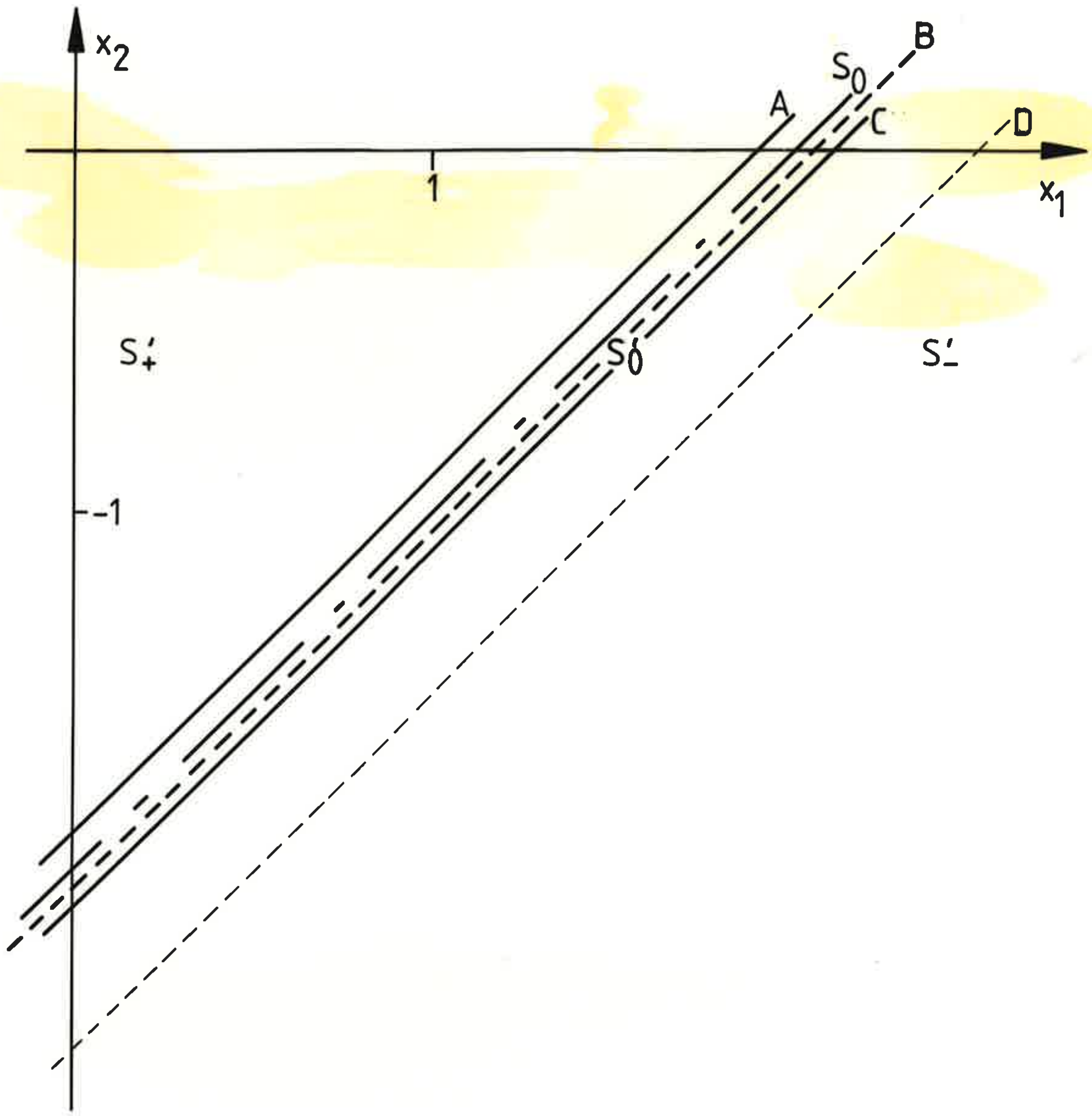


Fig. 4.6: The division of the state space in example 4.31.

Set $MU \triangleq x_1 - x_2 - 2$

The lines are:

A : $MU = -0.1$

$S_0$: $MU = 0$

B : $MU = 0.05$

C : $MU = 0.1$

D : $MU = 0.5$

The sets are:

$S_+' = \{x \mid MU < -0.1\}$

$S_-' = \{x \mid MU > 0.5\}$

$S_0' = R^n \setminus \{S_+' \cup S_-'\}$

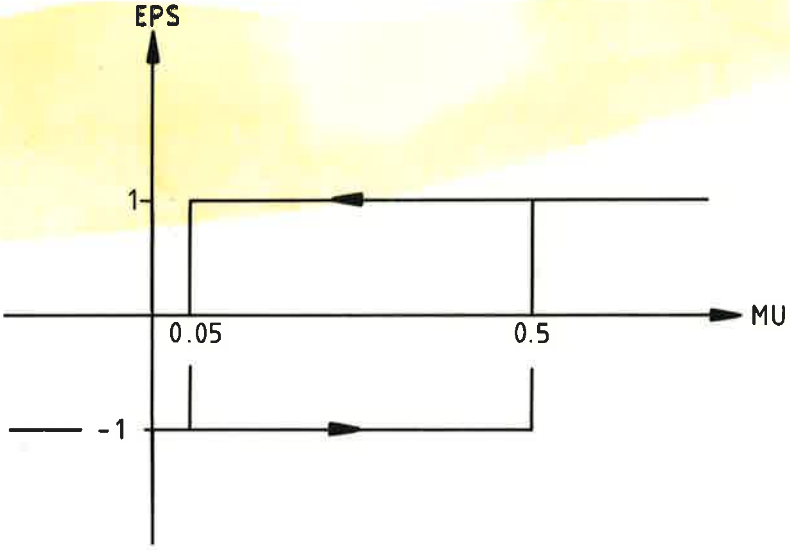


Fig. 4.7: The hysteresis used in example 4.31.

$$u = 0 \qquad \text{when } 0.1<MU\leq 0.5 \quad \text{and } EPS<0 \tag{4.117}$$

$$u = \frac{-2x_2}{x_1-x_2-2} \qquad \text{when } 0.1<MU\leq 0.5 \quad \text{and } EPS>0 \tag{4.118}$$

$$u = \frac{-2x_2}{x_1-x_2-2} \qquad \text{when } 0.5<MU \quad . \tag{4.119}$$

The simulations were done with the SIMNON-program BEST in appendix C.

Some of the simulation results are displayed in figures 4.8a and 4.8b. In addition to these results extensive simulations were done. For all initial conditions tried the trajectories tended asymptotically to the origin. However no analytical proof of stability will be given.

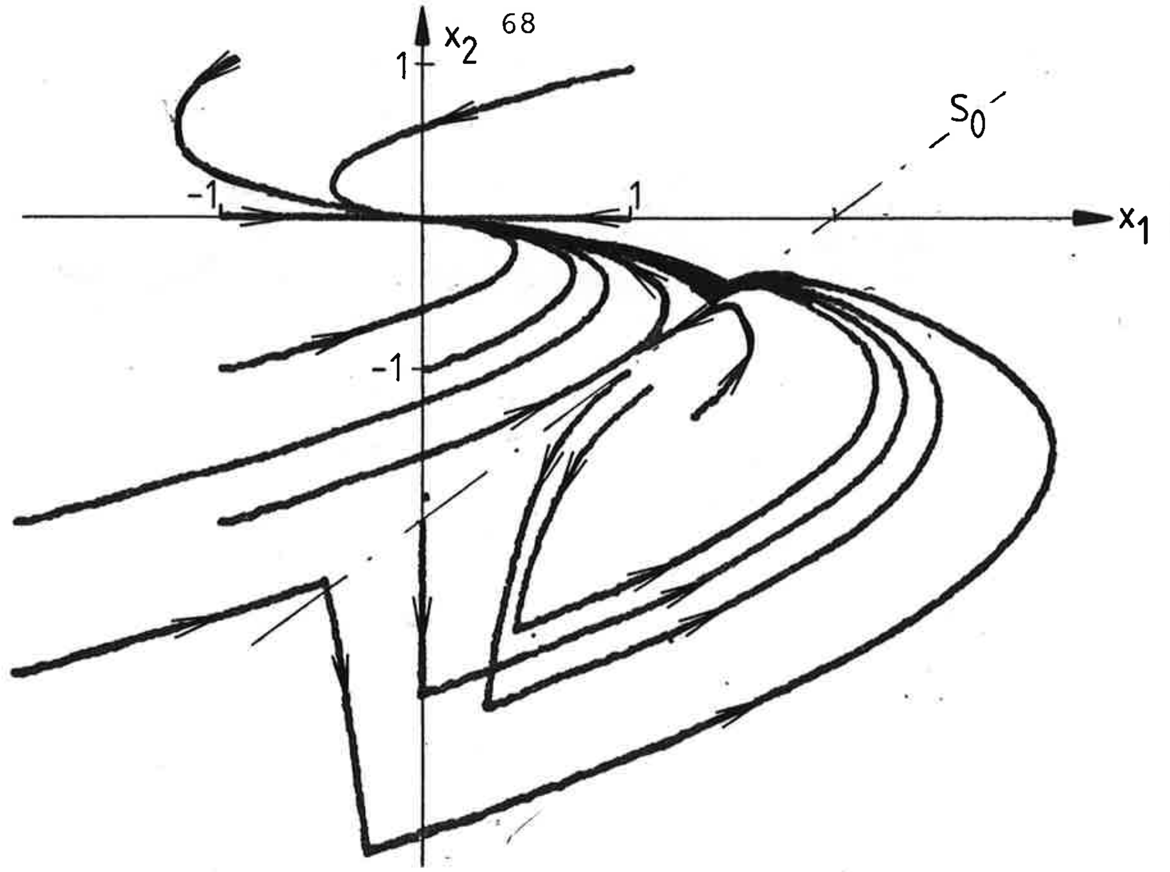


Fig. 4.8a: Phase plane trajectories of a simulation of example 4.31.

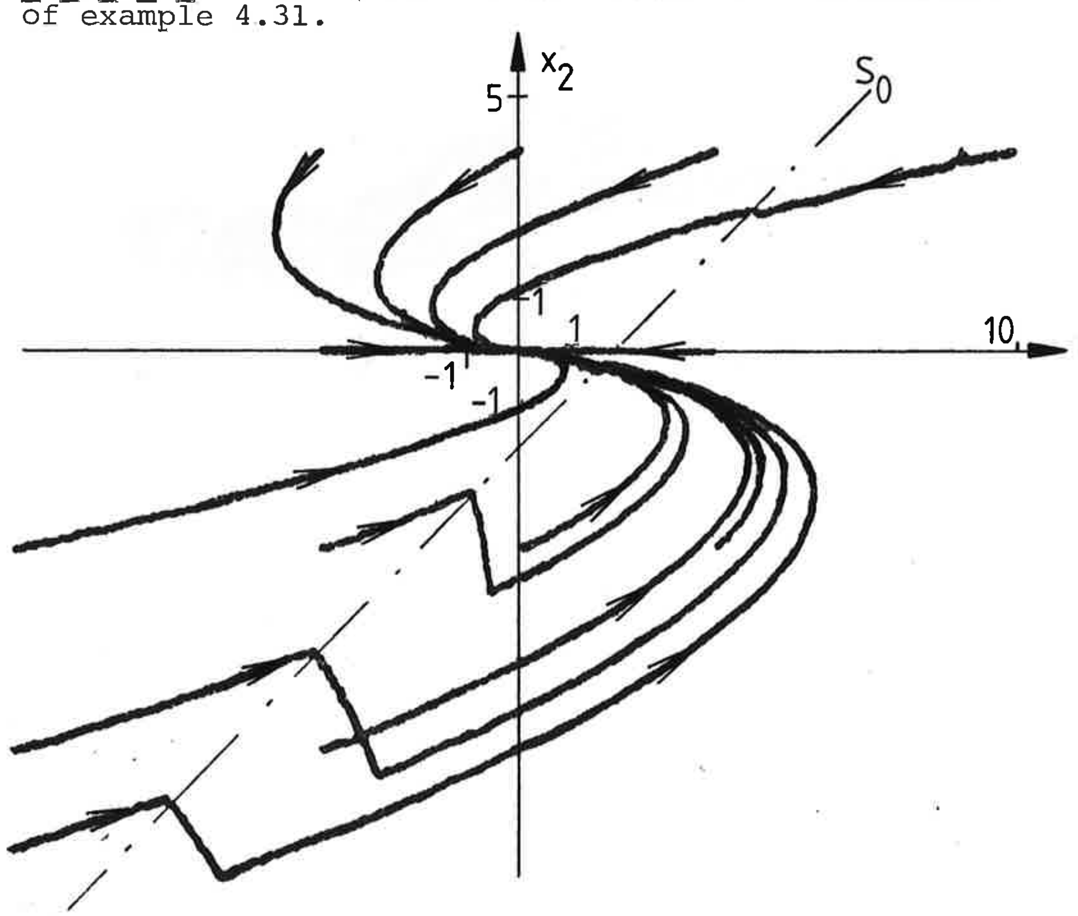


Fig. 4.8b: Phase plane trajectories of a simulation of example 4.31. (Note that the scales are different in figures 4.8a and 4.8b.)

## 4.4 The Lie theory approach.

Several authors, for instance Brockett (1972) and (1973), and Hirschorn (1974) have treated bilinear systems from a Lie theory point of view. This approach does not seem to have yielded feedback control laws. Baillieul (1977) presents an open loop optimal control with a criterion of minimum energy type.

# 5. INDEPENDENT ADDITIVE AND MULTIPLICATIVE CONTROLS

In this section we will briefly treat the case when some additive and multiplicative controls are independent. Consider the system:

$$\dot{x} = Ax + \sum_{i=1}^{m} (B_i x + b_{i0}) u_i + \sum_{j=1}^{p} C_j x v_j + \sum_{k=1}^{q} d_k w_k \qquad (5.1)$$

where

$x \in R^n$

$u_i$ , $i = 1, \ldots , m$ are the dependent controls,

$v_j$ , $j = 1, \ldots , p$ are the independent multiplicative controls,

$w_k$ , $k = 1, \ldots , q$ are the independent additive controls,

and $A$, $B_i$, $C_j$, $d_k$ are matrices of appropriate dimensions.

This type of system has been treated in Mohler (1973), Ionescu (1975), and Graselli (1979). Graselli takes into consideration additive unknown disturbances, and his solution is based on periodic linear systems techniques.

A system described by (5.1) should in general be easier to control than one described by (2.1). Here follow some simple sufficient conditions for stabilizability:

Theorem 5.1: Consider equ. (5.1). Define

$$B_0 = \left[ b_{10} \,\vdots\, b_{20} \,\vdots\, \cdots \,\vdots\, b_{m0} \right] \quad , \text{ and } u = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} \quad . \qquad (5.2)$$

If there exist a vector $r \in \text{Ker } B_0$ , and vectors $f_k \in R^n$, $k \in J_k \subseteq \{1, \ldots , q\}$ and constants $g_j \in R$, $j \in J_j \subseteq \{1, \ldots , p\}$ such that the matrix

$$\tilde{A} \triangleq (A + \sum_{i=1}^{m} B_i r_i + \sum_{j \in J_j} C_j g_j + \sum_{k \in J_k} d_k f_k^T) \qquad (5.3)$$

is strictly stable, then the control

$$\begin{cases} u = r + s \\ w_k = f_k^T \cdot x \ , \ k \in J_k \ , \\ v_j = g_j \quad , \ j \in J_j \ ; \text{ and} \\ w_k \ , \ k \notin J_k \ , \ v_j \ , \ j \notin J_j \ , \text{ and } s \text{ are all zero, or computed} \\ \text{according to the method in section 3.2} \end{cases}$$

asymptotically stabilizes the system.

Proof: Applying the controls (5.4) gives the closed loop system

$$\dot{x} = \tilde{A}x + \sum_{i=1}^{m} (B_i x + b_{i0}) s_i + \sum_{j \notin J_j} c_j x v_j + \sum_{k \in J_k} d_k w_k \qquad (5.4)$$

According to the conditions of the theorem the matrix $\tilde{A}$ is strictly stable, and consequently controls according to section 3.2 will asymptotically stabilize the system.

# 6. AN EXAMPLE: NEUTRON CONTROL IN A FISSION REACTOR

## 6.1 Preliminaries

As a realistic example of bilinear control, we choose the neutron level control problem in a fission reactor, described in Mohler (1973), pp. 112-119.

The neutron population is described by a second order model:

$$\begin{cases} \dot{n} = \dfrac{u - \beta}{\ell} \cdot n + \lambda c \\ \dot{c} = \dfrac{\beta}{\ell} \cdot n - \lambda c \end{cases} \tag{6.1}$$

with the state constraints $n > 0$ , $c > 0$ (6.2a)

where

$n$ = neutron population (6.2b)

$c$ = precursor population (6.2c)

$u$ = reactivity, which is the control input. (6.2d)

Typical values of the constants are:

$\ell = 10^{-5}$ (6.3a)

$\beta = 0.0065$ (6.3b)

$\lambda = 0.4$ (6.3c)

A control constraint might be assumed:

$$|u| \leq 10^{-3} \tag{6.4}$$

The control objective is to stabilize (6.1) to a chosen neutron population equilibrium level, $n_e$. With $n_e$ chosen, the precursor population level, $c_e$, follows from (6.1):

$$c_e = \frac{\beta}{\ell\lambda} n_e \tag{6.5}$$

Now, $(n_e, c_e)$ is taken as the origin in the transformed state space. The new state space variables are chosen dimensionless:

$$\begin{cases} x_1 = \dfrac{n - n_e}{n_e} \\ \\ x_2 = \dfrac{c - c_e}{c_e} \end{cases} \tag{6.6}$$

with the state constraint (6.2) transformed into

$$x_1 > -1 \ , \quad x_2 > -1 \quad . \tag{6.7}$$

In the new state space, (6.1) becomes:

$$\dot{x} = Ax + bd(x)w \tag{6.8}$$

where

$$A \triangleq \begin{bmatrix} -\frac{\beta}{\ell} & \frac{\beta}{\ell} \\ \lambda & -\lambda \end{bmatrix} \tag{6.9}$$

$$d(x) \triangleq \left(\frac{1}{\ell}x_1 + \frac{1}{\ell}\right) \quad , \tag{6.10}$$

and

$$b \triangleq \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{6.11}$$

Note that u is defined by (6.2d).

The eigenvalues of A are 0 and $-(\lambda + \frac{\beta}{\ell})$. (6.12)

A has thus an eigenvalue on the imaginary axis, and according to section 3.3, there might or might not exist a $P = P^T > 0$ such that the control

$$u = -[d(x)\ \ 0]Px \tag{6.13}$$

stabilizes (6.8). The possibility of such a control will be investigated in section 6.2, where it will be called Quadratic Control.

We also note that equ. (6.8) and the state space constraint (6.7) reveal that the system is dyadic and satisfies the conditions of theorem 4.22. The dyadicity stems from the change of equilibrium point, compare example 4.21. The situation is extremely favourable because the allowable state space (defined by (6.7)) is a subset of $S_+$. Thus $S_0$ ( $= \{x \mid d(x) = 0\}$ ) does not belong to the allowable state space. Moreover [A,b] is a controllable pair. Using example 4.29 we have thus proved that the control

$$u = \frac{k^T x}{d(x)} \tag{6.14}$$

will asymptotically stabilize the system for a suitable choice of k. The Division Control will be displayed in section 6.3. Section 6.4 contains a brief description of the bang-bang controller by which Mohler (1973) solved the control problem.

## 6.2 Quadratic Control

Set $P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} > 0 \; , \; p_i \in R \; , \; i = 1,2,3.$ (6.15)

The quadratic control (6.13) is computed:

$$u = - [d(x) \;\; 0] \, Px$$

$$u = - \begin{bmatrix} \left(\frac{x_1}{\ell} + \frac{1}{\ell}\right) & 0 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{6.16}$$

$$u = - \frac{(x_1+1)}{\ell} (p_1 x_1 + p_2 x_2) \tag{6.17}$$

Theorem 3.5 is used to find conditions on P. Equ. (3.16) gives that

$u = - \frac{x_1+1}{\ell} (p_1 x_1 + p_2 x_2)$ must be non-zero on

$$M = \{x \mid x^T(PA + A^T P)x = 0 \; , \; x \neq 0\} \tag{6.18}$$

Let $\Phi = x^T P x$ be a Lyapunov function. Compute $\dot{\Phi}$ as a function of P and select $p_1$ and $p_2$ that stabilize the system. One suitable choice of P is

$$\begin{cases} p_1 = 40 \cdot 10^{-10} \\ p_2 = 20 \cdot 10^{-10} \\ p_3 > \frac{p_2^{\,2}}{p_1} \end{cases} \tag{6.19}$$

The SIMNON-program NUKE was written to simulate system (6.7), (6.8) together with the controller (6.18), (6.19). See appendix D.

Simulations were performed without and with the control constraint (6.4), see figures 6.1a and 6.2. The unconstrained u(t) with the initial condition (2.1) is plotted in figure 6.1a. We note that u(t) does not hit the bound (6.4) in this case.

We conclude from figures 6.1 and 6.2 that the chosen quadratic control stabilizes the system, although the convergence is rather slow.

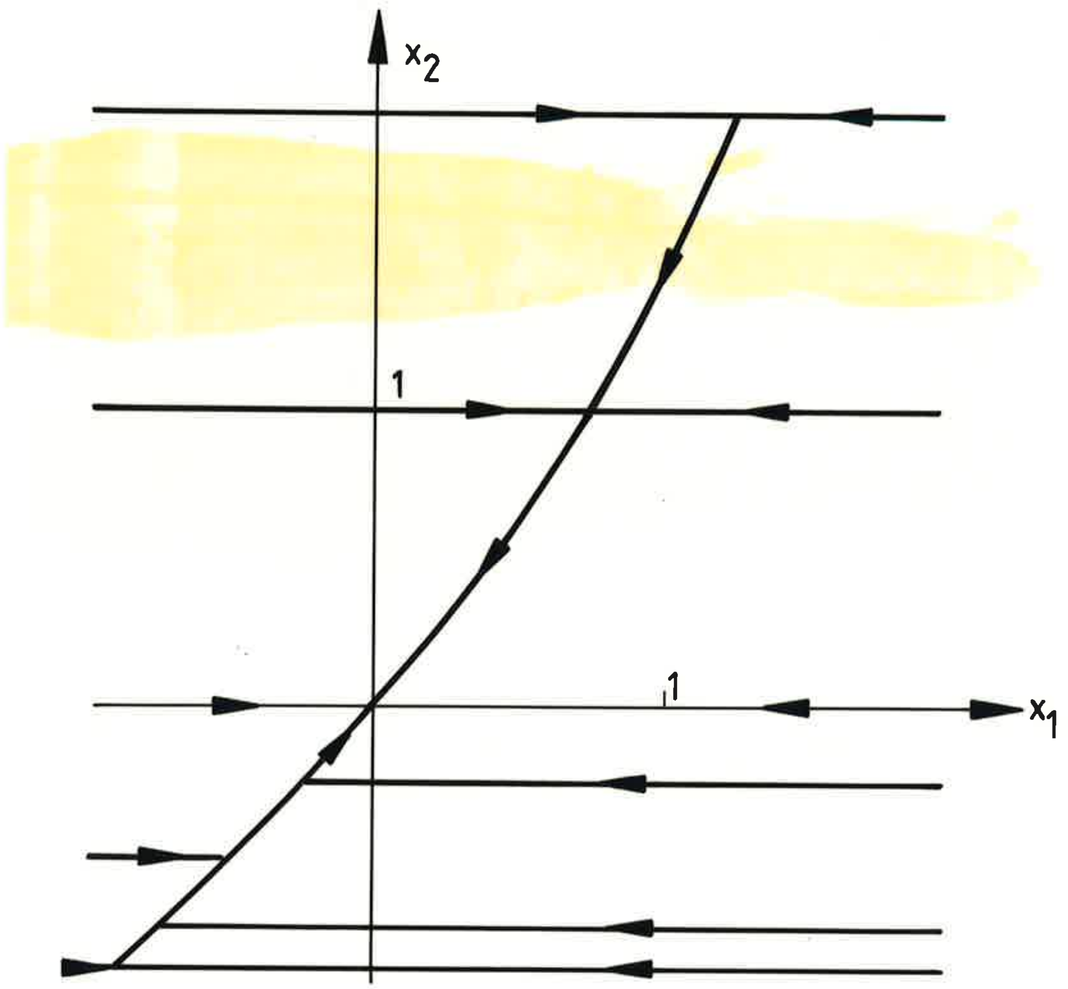


Fig. 6.1a - Phase plane trajectories for simulations with the unconstrained Quadratic Controller (6.18), (6.19).

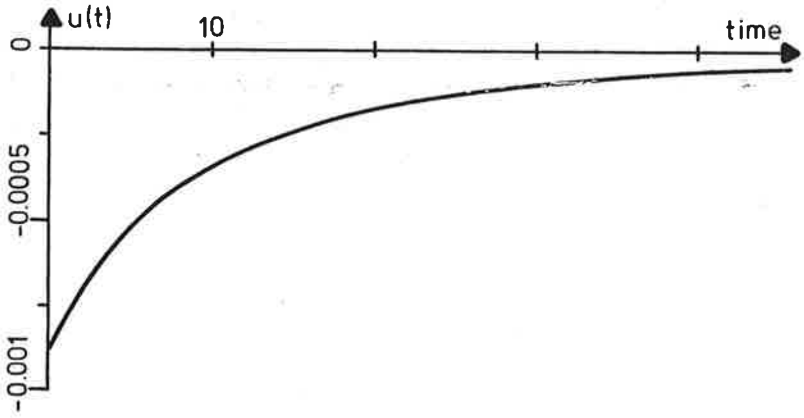


Fig. 6.1b - The control input u(t) for a simulation with the unconstrained Quadratic Controller, with the initial condition (2,1).

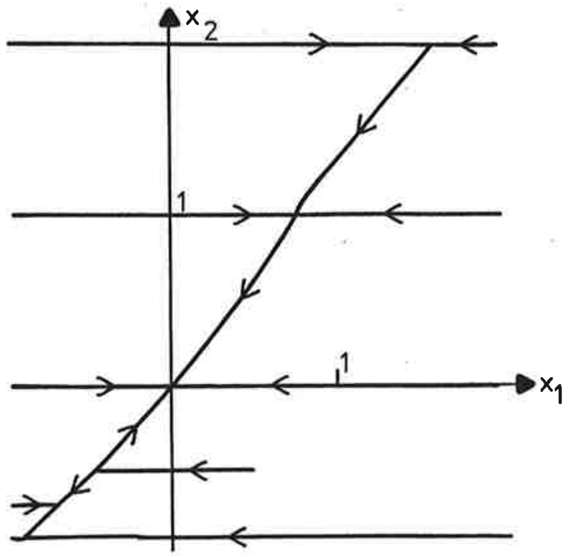


Fig. 6.2 - Phase plane trajectories for simulations with the Quadratic Controller (6.18), (6.19) subject to the constraint (6.4).

## 6.3 Division Control

In the division algorithm (6.16)

$$u = \frac{k^T x}{x_1+1} \cdot \ell \tag{6.20}$$

the vector $k^T$ is chosen as if the linear system

$$\dot{x} = Ax + bu \tag{6.21}$$

is to be controlled by the linear controller

$$u = k^T x \tag{6.22}$$

The choice can be done for instance through the use of linear optimal control with state constraints, see Mårtensson (1972). Here the choice was done to get a stable one-tangent node:

$$k^T = (648.4, -650.9) \tag{6.23}$$

Moreover, in the simulation program (the SIMNON-program NUKE, appendix D) a lower bound on the denominator in (6.20) was set to avoid numerical overflow. The control input in the simulation was thus:

$$u = \frac{648.4\ x_1 - 650.9\ x_2}{\max\ (x_1+1, 0.01)} \cdot 10^{-5} \tag{6.24}$$

Simulations were performed without and with the control constraint (6.4), see figures 6.3a and 6.4a. The control input u(t), with the initial condition (2,1), is plotted without and with the control constraint (6.4) in figures 6.3b and 6.4b respectively.

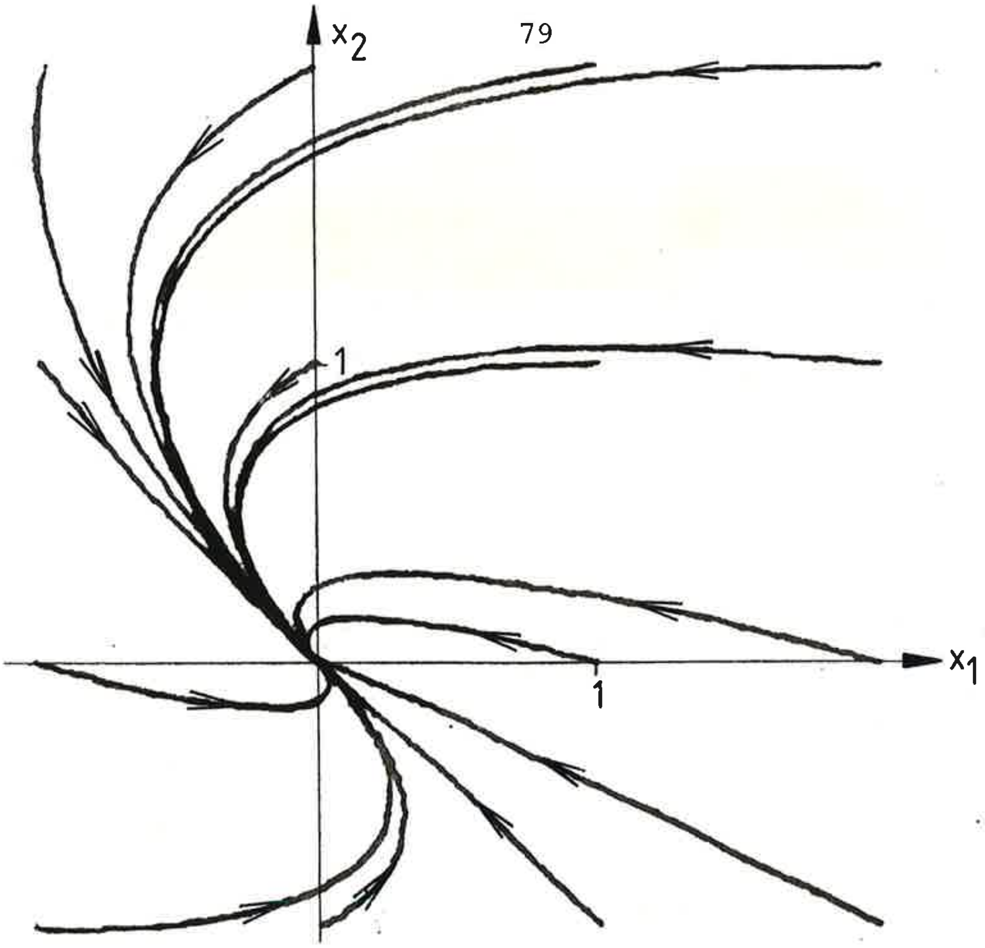


Fig. 6.3a - Phase plane trajectories for simulations with the unconstrained Division Controller (6.24).

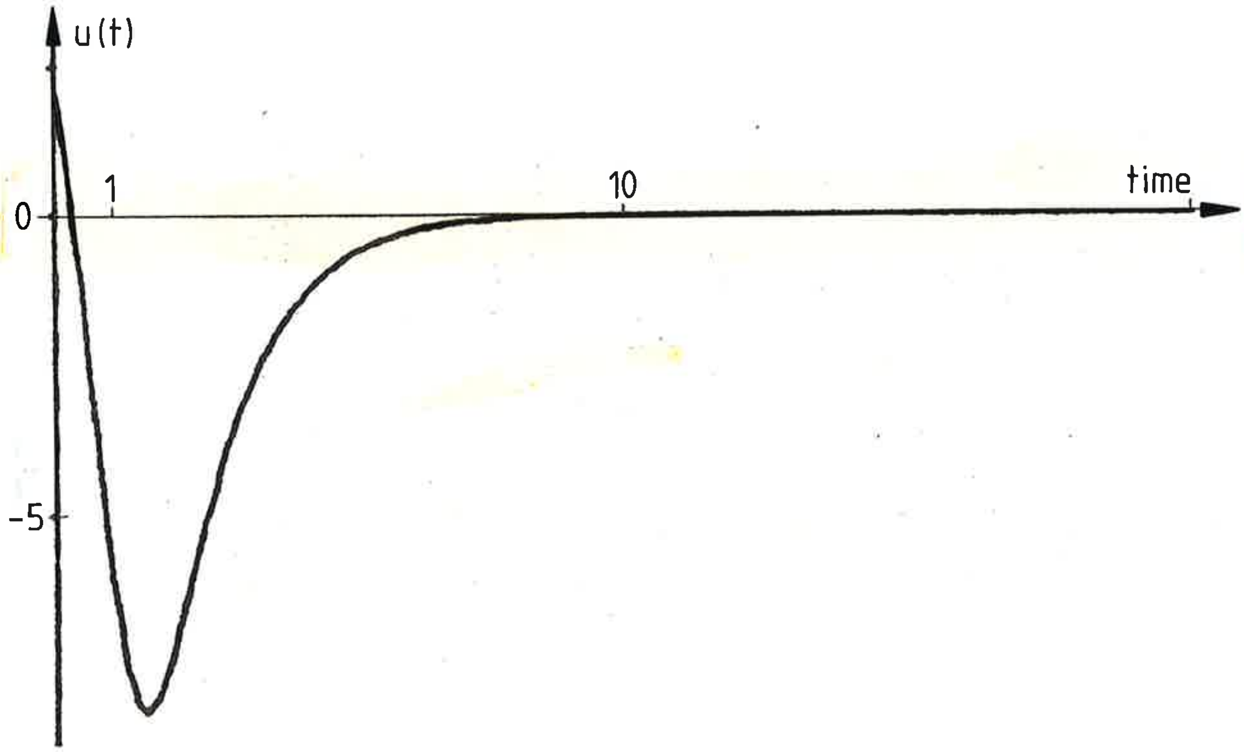


Fig. 6.3b - The control input u(t) for a simulation with the unconstrained Division Controller (6.24) with the initial condition (2,1).

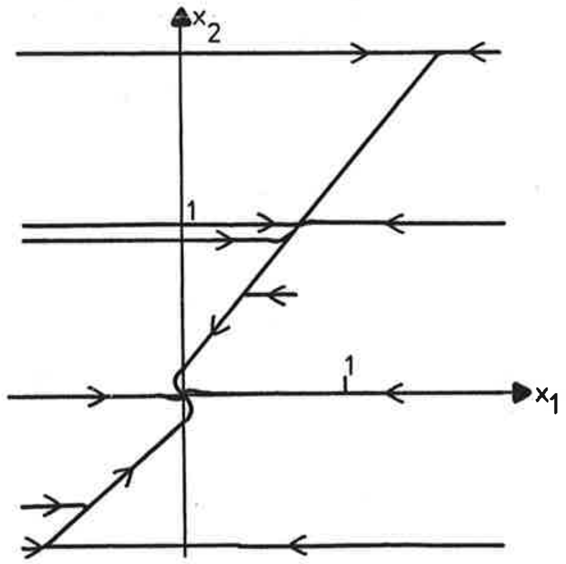


Fig. 6.4a - Phase plane trajectories for simulations with the Division Controller (6.24) subject to the constraint (6.4).

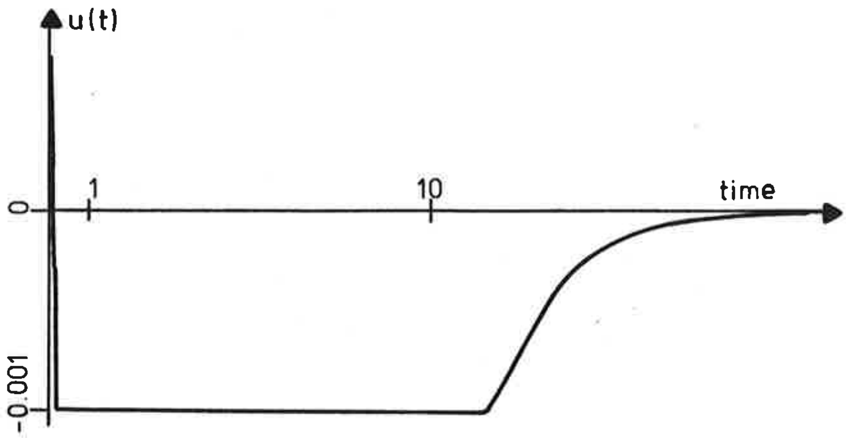


Fig. 6.4b - The control input u(t) for a simulation with the Division Controller (6.24) subject to the constraint (6.4). Initial condition: (2,1).

The Division Controllers seem very attractive. They are easy to design, and they provide good local and global control with the same algorithm.

## 6.4 Bang-bang control

Mohler (1973) solves the problem of stabilizing equ. (6.8) by a time optimal bang-bang controller. See figure 6.5. This control mode is not suitable when the state is near the origin, therefore Mohler (1973) suggests a PI-controller for local control.

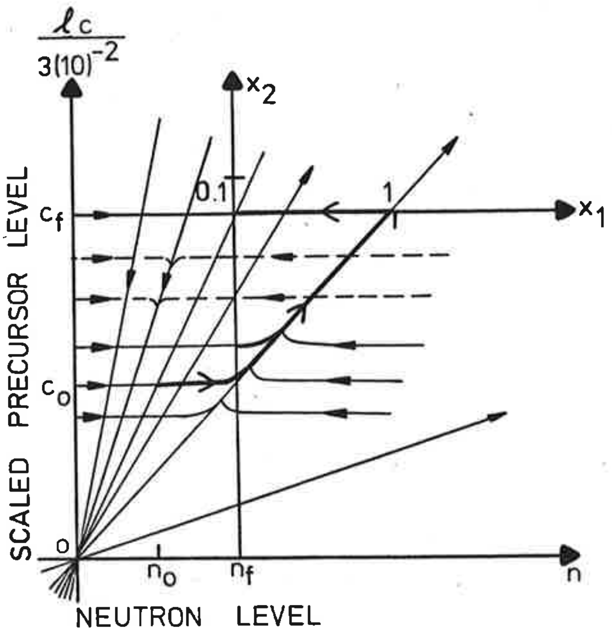


Fig. 6.5 - Adopted from Mohler (1973), fig. 4.1. Phase plane trajectories for the Mohler (1973) bang-bang control. The trajectory from the initial condition $(n_0, c_0)$ to the desired equilibrium point $(n_f, c_f)$ is heavily drawn. $(n_f, c_f)$ corresponds to the origin in the $x_1$-$x_2$ space.

## 6.5 Conclusions

When comparing the Quadratic Controller with control constraint, the Division Controller with control constraint, and the bang-bang controller, it seems as if the Division Controller offers most advantages. It is very easy to design, the other two are more complicated. Different control objectives can be taken into consideration, including time optimality. The bang-bang control might give an endpoint error, which must be compensated for by another controller, for instance a PI-controller. The Division Controller provides good local as well as global control with the same algorithm.

# 7. SUMMARY, DISCUSSION AND CONCLUSION

The problem of finding a feedback regulator for the bilinear system

$$\dot{x} = Ax + \sum_{i=1}^{m} (B_i x + b_{i0}) u_i \qquad (7.1)$$

in the case when the A-matrix is stable, was solved satisfactorily by Jacobson (1977). He used a quadratic Lyapunov function $\phi = x^T P x$ to generate the control $u_i = -(B_i x + b_{i0})^T P x$, which makes the system faster than the trivial control $u = 0$.

The case when A is unstable is still unsolved in general. Neither do all encompassing stabilizability results exist. This report surveys the literature for the solved special cases. It contains some new contributions of solved special cases:

1. an extension of Jacobsons result, subject to the condition that there exists a $P = P^T > 0$ such that

   $$\{x | (B_i x + b_{i0})^T P x = 0 \text{ all } i\} \cap \{x | x \neq 0,\ x^T (PA + A^T P) x \geq 0\}$$

   is empty (theorem 3.13).

2. sufficient conditions for constant controls to work (theorem 4.11).

3. a special class of bilinear systems is defined; the dyadic bilinear systems which are characterized by $(B_i x + b_{i0}) = b_{i0}(c^T x + 1)$ for some i (Definition 4.20).

For these systems a design method is proposed that reduces the bilinear problem to a problem of concatenating linear control systems in three regions in the state space. Conditions for stabilizability are found in theorem 4.22 and the method, the Division Controller, is presented in Design 4.26.

The report contains numerous examples. Note especially example 2.2 (a case of non-stabilizability), examples 3.18 and 3.19 (applications of theorem 3.13), theorem 4.14 (stabilizability equivalent to stabilizability with a constant control), example 4.21 (a real-life dyadic system), examples 4.28 and 4.29 (applications of theorem 4.22), example 4.31 (application of Design 4.26), and the whole of chapter 6, where the neutron level control problem in a fission reactor is solved in three ways. The report discusses various ideas how to attack the problem when A is unstable.

A short note is included on the case when additive and multiplicative controls are independent of each other. Some results are mentioned including

4. sufficient conditions for linear and constant stabilizing controls (theorem 5.1).

As stated above very little is known how to control (7.1) when A is unstable. A general theory would be desirable. In my opinion the route towards this goal should be:

1. the stabilizability problem should be solved before an attempt to find a general feedback control law. It is like staggering in a dark room trying to find control laws not knowing if the system is possible to stabilize at all. During work on stabilizability properties, control ideas might well be born.

2. Real life processes, modelled bilinearly, should be tackled. There is still a lack of such models. The relevancy of bilinear systems theory can only be tested against reality. If there are few bilinear processes in real life that can be subjected to control inputs determined externally, then it might not be worth while spending the effort. The physiological bilinear systems reported in e.g. Mohler (1979) can often not be controlled externally. Moreover, specific bilinear systems might display special features that make them easier to control. One such case, examplified by the neutron level control problem in chapter 6, is the dyadic system.

3. A more general class of non-linear systems which is closed under state feedback should be investigated. It would be nice to imbed the bilinear system into a slightly more general class that has desirable properties. One possible class would be:

$$\dot{x} = p(x) + \sum_{i=1}^{m} (B_i x + b_{i0}) x u_i$$

where the elements of $p(x)$ are homogenous polynomials of degree n, and $u_i(x)$ is a polynomial of degree (n-1).

Other classes might certainly be of interest. Maybe a class could be found where the non-linear Riccati-equation, corresponding to equ. (3.3) could be solved exactly. The Moylan (1973) paper is a possible point of departure.

# 8. ACKNOWLEDGEMENT

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# APPENDIX A : The SIMNON-program INDEV

```
CONTINUOUS SYSTEM INDEV
STATE X1 X2
DER DX1 DX2
B1 = 2*X1 + X2
B2 = X1 + 3*X2
UH1 = -((B1-2)*(B1-2) + (B2-2)*(B2-2))*K
UH2 = -((B1-2)*(B1+2) + (B2-2)*(B2+2))*K
"FIS = K*((B1+2)*X1 - (B2+2)*X2) + 2*UH2*UH2 "TEST QUANTITY NOT USED
UH = IF -2*(X1+X2)>0 THEN UH1 ELSE UH2
UL = K1*X1 + K2*X2
U = IF LIN>0.5 THEN UL ELSE UH
DX1 = -X1 + (B1-2)*U
DX2 = X2 + (B2-2)*U
K:100
K1:50
K2:100
LIN:0
END
```

Refer to example 4.9.

APPENDIX B : The SIMNON-program PARAB

```
CONTINUOUS SYSTEM PARAB
STATE X1 X2
DER DX1 DX2
D = X1 - X2 - 2
S = IF SIGN(D)<0 THEN -1 ELSE 1
DEN = IF ABS(D)>EPS THEN D ELSE ETA*S
U = (L1*X1 + L2*X2)/DEN
DX1 = X1 + D*U
DX2 = 2*X2 + D*U
EPS:0.001
ETA:0.001
L1:4
L2:-9
END
```

Refer to example 4.30.

APPENDIX C : The SIMNON-program BEST

```
CONTINUOUS SYSTEM BEST
STATE X1 X2
DER DX1 DX2
INPUT EPS
OUTPUT MU
MU=B11*X1+B12*X2+B1
DIV = IF ABS(MU)>G THEN MU ELSE G
UH = (K1*X1 + K2*X2)/DIV
U0 = 0
U = IF MU<-G OR EPS>0 THEN UH ELSE U0
DX1 = A11*X1+A12*X2 +(B11*X1 +B12*X2 +B1)*U
DX2 = A21*X1+A22*X2 +(B21*X1 +B22*X2 +B2)*U
A11:-1
A12:0
A21:0
A22:1
B11:1
B12:-1
B21:1
B22:-1
B1:-2
B2:-2
K1:0
K2:-2
G:0.1
END
- - - - - - - - - - - - - - - - - - - - - - - - - - - - -
CONTINUOUS SYSTEM HYS
STATE OUT
DER DOUT
INPUT E
OUTPUT R
INITIAL
DU:1
L0:0.05
H1:0.5
C:1
OUTPUT
R=SIGN(OUT)*DU
DYNAMICS
C1=IF OUT<1 THEN C*(1.5-OUT) ELSE 0
C2=IF OUT>-1 THEN -C*(1.5+OUT) ELSE 0
DOUT= IF E>H1 OR OUT>0 AND E>L0 THEN C1 ELSE C2
END
- - - - - - - - - - - - - - - - - - - - - - - - - - - - -
CONNECTING SYSTEM CONI
E[HYS]=MU[BEST]
EPS[BEST]=R[HYS]
END
```

Refer to example 4.31.

APPENDIX D : The SIMNON-program NUKE

```
CONTINUOUS SYSTEM NUKE
STATE X1 X2
DER DX1 DX2
"
D = MAX((X1+1),EPS)
U1 = (K1/(K2*D))*(L1*X1 +L2*X2)      "DIVISION CONTROLLER
U2 = -(X1+1)*(P1*X1 + P2*X2)*L      "QUADRATIC CONTROLLER
E = MIN((X1+1),(X2+1))
"
UH = IF DIV>0.5 THEN U1 ELSE U2     "CONTROL SELECTOR
U = IF UH<-G THEN -G ELSE IF UH<G THEN UH ELSE G "LIMITER
H1 = (-BETA/L)*(X1 - X2) + (X1+1)*(U/L)
DX1 = IF X1>-1 THEN H1 ELSE MAX(0,H1)
H2 = LA*(X1 - X2)
DX2 = IF X2>-1 THEN H2 ELSE MAX(0,H2)
"
EPS:0.01
K1:1
K2:100000
L1:648.4
L2:-650.9
L:1.0E-5
BETA:0.0065
LA:0.4
P1:40
P2:20
G:0.001    "WILL BE SET = 1.E5 WHEN NO LIMITER IS DESIRED
DIV:1      "WILL BE SET = 0 WHEN QUADRATIC CONTROL IS DESIRED
END
```

Remark: See chapter 6.
The variable E is not used. Precautions are taken in the program so that the state will not get outside the allowed region (6.7).