On the Model Reduction Problem

Gutman, Per-Olof; Mannerfelt, Carl-Fredrik; Molander, Per

1980

Document Version:
Publisher's PDF, also known as Version of record

Link to publication

Citation for published version (APA):
Gutman, P.-O., Mannerfelt, C.-F., & Molander, P. (1980). *On the Model Reduction Problem*. (Technical Reports TFRT-7211). Technical Reports.

Total number of authors:
3

LUND UNIVERSITY

PO Box 117
221 00 Lund
+46 46-222 00 00

CODEN: LUTFD2/(TFRT-7211)/1-012/(1980)

ON THE MODEL REDUCTION PROBLEM

PER-OLOF GUTMAN
CARL-FREDRIK MANNERFELT
PER MOLANDER

DEPARTMENT OF AUTOMATIC CONTROL
LUND INSTITUTE OF TECHNOLOGY
DECEMBER 1980

Organization
LUND INSTITUTE OF TECHNOLOGY

Department of Automatic Control
Box 725
S-220 07 Lund 7 Sweden

Document name
Internal report

Date of issue
December 1980

CODEN:
LUTFD2/(TFRT-7211)/1-012/(1980)

Author(s)
P-O Gutman, C. F. Mannerfelt, P. Molander

Sponsoring organization

Title and subtitle
On the model reduction problem

Abstract

A new method of model reduction is introduced based on the differentiation of polynomials. The reciprocals of the numerator and denominator polynomials of the high-order transfer function are differentiated suitably many times to yield the coefficients of the reduced order transfer function. An error analysis shows the accuracy of the method, and an eighth order example illustrates it. The method is computationally very simple and is equally applicable to unstable and non-minimum phase systems.

Key words
Linear Systems, Model reduction, transfer function

Classification system and/or index terms (if any)

Supplementary bibliographical information

Language
English

ISSN and key title

ISBN

Recipient's notes

Number of pages
12

Price

Security classification

Distribution by (name and address)

ON THE MODEL REDUCTION PROBLEM

by

Per-Olof Gutman
Carl Fredrik Mannerfelt
Per Molander

Abstract:

A new method of model reduction is introduced based on the differentiation of polynomials. The reciprocals of the numerator and denominator polynomials of the high-order transfer function are differentiated suitably many times to yield the coefficients of the reduced order transfer function. An error analysis shows the accuracy of the method, and an eighth order example illustrates it. The method is computationally very simple and is equally applicable to unstable and non-minimum phase systems.

Department of Automatic Control
Lund Institute of Technology
Box 725
S-220 07 Lund, Sweden

1. Introduction

It is with some hesitation that we mingle into what seems to be an internal scientific affair ([1], [2], [3]), but it is our feeling that the model reduction problem is an important one which has received far too little attention in the literature. All contributions to the field, small or great, should therefore be equally welcome.

Recall that the method suggested by Krishnamurthy and Seshadri consists in separately reducing the degrees of the numerator and denominator polynomials of a given transfer function. Lower order polynomials are generated from the Routh stability array.

Serious criticism on the suggested method has been exerted by Rao, Lamba, and Rao in [2] and by Singh in [3]. Singh's critique brings forward the so-called "non-uniqueness" of the algorithm, whereby it is meant that different transfer functions give rise to the same model. Since any model reduction technique necessarily contains a non-injective mapping, this point is somewhat hard to understand, however.

As to the discussion in [2] and the companion Author's Reply, it is not our intention to judge on the theoretical justification or the applicability of the suggested method. Instead, we wish to point out that there is a great variety of ways to reduce the order of a given polynomial. Of these we have chosen to study differentiation, an operation well-known to all persons with even a superficial contact with mathematical analysis.

2. The algorithm

We thus propose to use differentiation as a means to reduce the order of a given polynomial. The question naturally arises how well a polynomial is approximated by its derivative. A partial answer is given in the following lemma.

Lemma: Assume that the polynomial

$$p(s) = \prod_{i=1}^{n} (s - z_i)$$

has all its zeros in the left half plane. Then $p'(s)$ also has all its zeros in the left half plane.

Proof: Form the logarithmic derivative

$$\frac{d}{ds} \ln p(s) = \frac{p'(s)}{p(s)} = \sum_{i=1}^{n} \frac{1}{s - z_i} .$$

Let Γ be the contour encircling the right half plane in the positive direction. It is easily realized that, because

$$\operatorname{Re} z_i < 0, \; i = 1, 2, \ldots , n ,$$

for every s on Γ,

$$\operatorname{Re} \left[\sum \frac{1}{s - z_i} \right] > 0 .$$

Thus the variation of the argument of

$$\sum \frac{1}{s - z_i}$$

when the right half plane is encircled is equal to zero. Equivalently

$$\frac{1}{2\pi} \Delta_\Gamma \arg \frac{p'(s)}{p(s)} = 0 . \qquad (2.1)$$

Using the Principle of the variation of the argument on $p'(s)/p(s)$, i.e.

$$\frac{1}{2\pi} \Delta_\Gamma \arg \frac{p'(s)}{p(s)} =$$

= the number of zeros of $p'(s)/p(s)$ in the right half plane − the number of poles of $p'(s)/p(s)$ in the right half plane, together with Equ. (2.1) and the assumption that $p(s)$ lacks zeros in the right half plane, gives the result.

□

Corollary: By linear transformations of the complex plane, it can be shown with the same method as in the lemma that the zeros of $p'(s)$ lie inside the convex hull of the zeros of $p(s)$.

□

The corollary implies that by differentiating the numerator- and denominator polynomial of a linear system transfer function, the poles and zeros of the reduced order model will lie inside the convex hull of the poles and zeros, respectively, of the original system.

A drawback of straightforward differentiation is that zeros with a large modulus tend to be better approximated than those with a small modulus. This problem is remedied e.g. by differentiating the reciprocal polynomial.

Algorithm 1: Given the polynomial

$$p_n(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0 .$$

Compute the reciprocal polynomial:

$$p_n^*(z) \triangleq z^n \cdot p_n\left(\frac{1}{z}\right) = a_n + a_{n-1} z + \dots + a_1 z^{n-1} + a_0 z^n .$$

Differentiate:

$$p_{n-1}^*(z) \triangleq \frac{dP_n^*(z)}{dz} = a_{n-1} + 2a_{n-2} z + \dots + na_0 z^{n-1} .$$

Reciprocate back and normalize:

$$p_{n-1}(s) \triangleq \frac{1}{n} \cdot s^{n-1} \cdot p_{n-1}^*\left(\frac{1}{s}\right) = \frac{1}{n} \cdot a_{n-1} s^{n-1} + \dots + a_0 .$$

The result can also be expressed in terms of the original polynomial and its derivative:

$$p_{n-1}(s) = p_n(s) - \frac{s}{n} \cdot p_n'(s) . \qquad (2.2)$$

$p_{n-1}(s)$ constitutes our reduced order polynomial.

□

Although the zeros of $p^*_{n-1}(s)$ lie inside the convex hull of the zeros of $p^*_n(s)$ according to the corollary, this fact may not be true for the zeros of $p_{n-1}(s)$ with respect to the zeros of $p_n(s)$. However, for polynomials normally occurring in linear systems transfer functions, the zeros of $p_{n-1}(s)$ lie close to or inside the convex hull of the zeros of $p_n(s)$. For instance, if all zeros of $p_n(s)$ are real (imaginary) then all the zeros of $p_{n-1}(s)$ are real (imaginary) and inside the convex hull. If all the zeros of $p_n(s)$ have negative (positive) real part then the same is true for the zeros of $p_{n-1}(s)$.

A natural modification of the given algorithm is the following:

<u>Algorithm 2</u>: Let the transfer function be

$$H(s) = \frac{q(s)}{p(s)} .$$

Factorize q(s) and p(s)

$$\begin{cases} q(s) = \tilde{q}(s) \cdot \hat{q}(s) \\ p(s) = \tilde{p}(s) \cdot \hat{p}(s) \end{cases}$$

where $\hat{q}(s)$ and $\hat{p}(s)$ include those zeros and poles of H(s) that you want to retain in the reduced order transfer function $H_{red}(s)$.

Reduce the order of $\tilde{q}(s)$ k_q times, each time following Algorithm 1. Let the resulting polynomial be $\tilde{q}_{red}(s)$. Reduce the order of $\tilde{p}(s)$ k_p times, each time according to Algorithm 1. Let the resulting polynomial be $\tilde{p}_{red}(s)$.

Construct the reduced order transfer function:

$$H_{red}(s) = C \cdot \frac{\tilde{q}_{red}(s) \cdot \hat{q}(s)}{\tilde{p}_{red}(s) \cdot \hat{p}(s)}$$

where C is a real constant.

$\hat{q}(s)$ might e.g. include the zeros whose real parts are non-negative; $\hat{p}(s)$ might e.g. include the unstable poles, the purely imaginary poles, the badly damped high-frequency poles, and the control poles. k_q and k_p are non-negative, not necessarily equal integers chosen e.g. such that the pole-zero excess of $H_{red}(s)$ is equal to the pole-zero excess of H(s). C is adjusted to give the best approximation in the relevant frequency range.

□

3. Error Analysis

How well a reduced order model approximates the original system depends on how many times the numerator and denominator polynomials respectively are reduced. For instance, the high frequency behaviour will only be preserved if the original and reduced order models have the same pole-zero excess.

To illustrate with an example, let the original model be

$$H(s) = \frac{q_m(s)}{p_n(s)}$$

where

$$q_m(s) = s^m + b_{m-1} s^{m-1} + \dots + b_0 = \prod_{i=1}^{m} (s - z_i)$$

$$p_n(s) = s^n + a_{n-1} s^{n-1} + \dots + a_0 = \prod_{i=1}^{n} (s - p_i)$$

and suppose we want to reduce the degrees of both the denominator and the numerator polynomials by one. The reduced order model is then

$$H_{red}(s) = \frac{q_{m-1}(s)}{p_{n-1}(s)} .$$

Using Equ. (2.2), the relative error between H(s) and $H_{red}(s)$ asymptotically satisfies:

$$\lim_{s \to 0} \frac{H(s) - H_{red}(s)}{H(s)} = 0$$

$$\lim_{s \to \infty} \frac{H(s) - H_{red}(s)}{H(s)} = 1 - \frac{n \cdot b_{m-1}}{m \cdot a_{n-1}}$$

where

$$b_{m-1} = - \sum_{i=1}^{m} z_i \quad \text{and} \quad a_{n-1} = - \sum_{i=1}^{n} p_i \quad .$$

The high frequency behaviour of the reduced order model thus will closely follow that of the original system if the mean values of the zeros and poles are approximately equal.

This is typically the case for a high order system whose behaviour resembles that of a lower order. This is exactly the type of system whose order you would like to reduce.

4. A numerical example

The numerical example to be studied is the same one as in [1]. The system has order eight and its transfer function is

$$H(s) = \frac{q(s)}{p(s)}$$

$$q(s) = 35s^7 + 1086s^6 + 13285s^5 + 82402s^4 + 278376s^3 + \\ + 511812s^2 + 482964s + 194480$$

$$p(s) = s^8 + 33s^7 + 437s^6 + 3017s^5 + 11870s^4 + \\ + 27470s^3 + 37492s^2 + 28880s + 9600 \ .$$

When applying Algorithm 2 we proceed as follows. As the system is stable and minimum phase, we let $\hat{p}(s) = 1$ and $\hat{q}(s) = 1$. We set $k_p = k_q = k$ in order to get approximants $H_r(s)$ that are comparable to those of [1]. This means that the pole-zero excess is kept equal to one and that the high frequency slope of the Bode plot is retained. C is chosen such that the low frequency gain of the approximant $H_r(s)$ is equal to the low frequency gain of H(s), i.e. C = 1. (In [1] the low frequency gain is also retained.)

The reduction is performed for k = 1, 2, ..., 7; i.e. the reduced order systems $H_r(s)$ are of order r = 8 − k = 7, 6, ... , 1. For r = 5 and 2, the transfer functions are

$$H_5(s) = \frac{8}{5} \cdot \frac{q_4(s)}{p_5(s)}$$

$$q_4(s) = 494412s^4 + 6681024s^3 + 30708720s^2 + $$
$$+ 57955680s + 40840800$$

$$p_5(s) = 18102s^5 + 284880s^4 + 1648200s^3 + 4499040s^2 + $$
$$+ 6064800s + 3225600$$

and

$$H_2(s) = 4 \cdot \frac{347734080s + 980179200}{26994240s^2 + 145555200s + 193536000} .$$

The pole-zero locations are tabulated in Table 1. It is apparent how well the poles and zeros of the reduced order systems approximate those of the original system. Note especially that the number of complex poles and zeros of the approximants never exceed the number of complex poles and zeros of the original transfer function. This is indeed not the case in [1].

As may be seen from Fig. 1, the frequency responce pattern is excellently preserved in the reduced systems, even for low order approximants. Judging from this example, the method suggested here seems to be superior to that proposed in [1].

Table 1

Order of approximant	Poles	Zeros
r = 8 (exact)	−1±i, −1, −3, −4, −5, −8, −10	−1.03±0.631i, −2.64, −3.83, −4.90, −7.80, −9.78 (†)
r = 7	−1.12, −1.19±1.06i, −3.28, −4.41, −6.24, −9.05	−1.20±0.668i, −2.93, −4.25, −6.06, −8.83
r = 6	−1.27, −1.45±1.10i, −3.65, −5.18, −7.72	−1.42±0.696i, −3.32, −4.97,−7.49
r = 5	−1.48, −1.80±1.09i, −4.21, −6.45	−1.71±0.698i, −3.89, −6.19
r = 4	−1.76, −2.29±0.948i, −5.23	−2.15±0.619i, −4.90
r = 3	−2.18, −2.79, −3.22	−2.65, −3.02
r = 2	−2.38, −3.01	−2.82
r = 1	−2.66	---

(†) In [1], the zeros are incorrect.

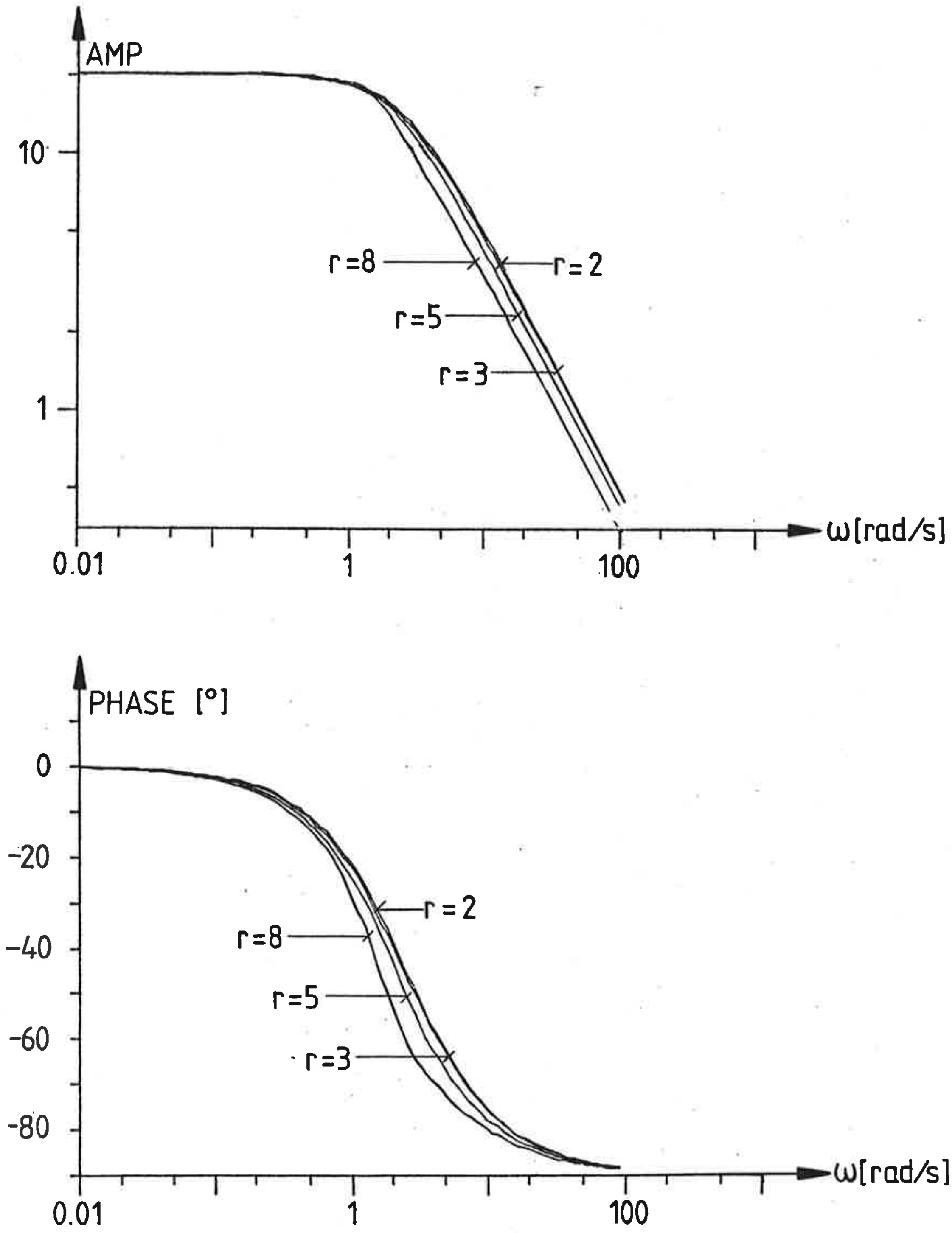

Figure 1: Bode plots for H(s), $H_5(s)$, $H_3(s)$, $H_2(s)$.

5. References

[1] V. Krishnamurthy and V. Seshadri, "Model Reduction Using the Routh Stability Criterion", IEEE Trans. Automat. Contr., (Tech. Notes and Corresp.), vol. AC-23, pp. 729-731, Aug. 1978.

[2] A.S. Rao, S.S. Lamba, and S.V. Rao, "Comments on 'Model Reduction Using the Routh Stability Criterion'", IEEE Trans. Automat. Contr., (Tech. Notes and Corresp.), vol. AC-24, p. 518, June 1979.

[3] V. Singh, "Nonuniqueness of Model Reduction Using the Routh Approach", IEEE Trans. Automat. Contr., (Tech. Notes and Corresp.), vol. AC-24, pp. 650-651, Aug. 1979.