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Abstract

In system identification experiments, the input signal has to be designed be the user. A very convenient choice is to use discrete time sinusoids generated by oscillators, easily implemented on a computer. Due to round-off errors etc. in the computer, the amplitude of the sinusoids may be varying, and hence have to be stabilized.
The amplitude stability of the oscillator is examined with Lyapunov theory. A class of nonlinear, stabilizing feedbacks is characterized. A special choice of feedback turns out to be the time-optimal (dead-beat) solution. This solution is intuitively appealing, and very simple to implement on computers.

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NONLINEAR_AMPLITUDE_STABILIZATION_OF_A DISCRETE_TIME_OSCILLATOR

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Contents

1. INTRODUCTION

The area of system identification has expanded widely in the last two decades. Since micro-computers now are becoming more powerful, it is time to start implementing various algorithms for identification and control. These algorithms, however, ought to be simple and robust, since the micro-computer after all is rather slow and has limited numerical precision.
In an identification experiment, there is always the trouble of choosing a suitable input signal. One attractive choice is to use discrete time sinusoids, which are easy to generate in a small computer by recursive equations.

The first reason for writing this paper is:
Some years ago, Per Molander and I tried to make the mini-computer of the department , a PDP-15, into a music-machine. We implemented the recursive equation of an oscillator, and by changing one parameter and initial conditions it was possible to change frequency and amplitude of the "music-instrument" output.
We observed, however, that often the amplitude of the oscillation was either increasing or decreasing with time. This of course was the effect of numerical troubles, such as round-off errors etc.

Hence, if for identification purposes or other reasons, one wants to implement an algorithm for an oscillator on a computer, it is suitable to incorporate some amplitude stabilizing device.

The second reason for writing this, is that there is very little material published on nonlinear discrete time systems. These systems are generally hard to analyze, but they nevertheless will appear more and more frequently as computers will take over the control of industrial processes.
This paper presents one example where analysis is succesful.

The paper is organized as follows: in Section 2 the problem is formulated. In Section 3 it is attacked with a Lyapunov-type approach, and certain types of stabilizing controls are characterized. The choice of one certain solution leads to an intuitively appealing implementation, which is elaborated in Section 4. Section 5 is a conclusion, and references are contained in Section 6.

2. THE PROBLEM

A state-space realization of a discrete time oscillator is

$$\begin{bmatrix} x_1(t+h) \\ x_2(t+h) \end{bmatrix} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \qquad (2.1)$$

$$\alpha = \cos(\omega h) \qquad \beta = \sin(\omega h)$$

where h is the sample interval and ω is the oscillation angular frequency.

The state trajectory $x(t)$ will for all times lie on a circle with radius

$$r = [\, x_1(0)^2 + x_2(0)^2]^{1/2} = \|x(0)\| \qquad (2.2)$$

Since the system is on the stability boundary, small disturbances can cause the state vector to move away from its nominal trajectory. In a practical implementation of (2.1) such disturbances typically are round-off errors in the matrix multiplication and limited capacity to represent numbers.

The problem thus is: Find (nonlinear) control functions u_1 and u_2 such that the system

$$\begin{cases} x_1(t+h) = \alpha x_1(t) - \beta x_2(t) + u_1(x_1(t), x_2(t)) \\ x_2(t+h) = \beta x_1(t) + \alpha x_2(t) + u_2(x_1(t), x_2(t)) \end{cases} \qquad (2.3)$$

is trajectory stable in the sense that

$$\lim_{t \to \infty} \|x(t)\| = 1 \qquad (2.4)$$

irrespectively of initial conditions $x(0)$.

The choice of "reference amplitude" to unity is of course arbitrary.

A feedback solution to the problem is desired since it then can handle disturbances affecting the system at any time instant.

3. ANALYSIS

In this section the problem will be analyzed with the second method of Lyapunov. Characterization of some possible stabilizing controls will be made.
The idea of using a Lyapunov approach has been borrowed from Aström [1].

The system equations are

$$\begin{cases} x_1(t+h) = \alpha x_1(t) - \beta x_2(t) + u_1 \\ x_2(t+h) = \beta x_1(t) + \alpha x_2(t) + u_2 \end{cases} \quad (3.1)$$

A possible candidate to Lyapunov function is

$$V(t) = [\|x(t)\|^2 - 1]^2 \quad (3.2)$$

Insertion of equation (3.1) gives

$$V(t+h) =$$

$$= [(u_1+\alpha x_1(t)-\beta x_2(t))^2 + (u_2+\beta x_1(t)+\alpha x_2(t))^2 - 1]^2 \quad (3.3)$$

Now, for the sake of simple analysis, u_1 and u_2 are restricted to be of the form

$$\begin{cases} u_1 = (\alpha x_1(t) - \beta x_2(t))f(\|x(t)\|) \\ u_2 = (\beta x_1(t) + \alpha x_2(t))f(\|x(t)\|) \end{cases} \quad (3.4)$$

The equation (3.3) for V(t+h) then reduces to

$$V(t+h) = [(1+f)^2\|x(t)\|^2 - 1]^2 \quad (3.5)$$

where it has been used that

$$\alpha^2 + \beta^2 = 1 \quad (3.6)$$

From well known stability results, see e.g. LaSalle [2], the system (3.1) will now be asymptotically trajectory stable in the sense that

$$\lim_{t\to\infty} \|x(t)\| = 1 \qquad (3.7)$$

if

$$\nabla V(t) = V(t+h) - V(t) =$$

$$[(1+f)^2\|x(t)\|^2 - 1]^2 - [\|x(t)\|^2 - 1]^2 \le 0 \qquad (3.8)$$

with equality only for $\|x(t)\| = 1$.

Relation (3.8) can be separated into three cases:

$$\begin{cases} \|x(t)\|^2-1 < (1+f)^2\|x(t)\|^2-1 < 1-\|x(t)\|^2 & \text{if } \|x(t)\|<1 \\ (1+f)^2\|x(t)\|^2 - 1 = \|x(t)\|^2 - 1 & \text{if } \|x(t)\| = 1 \\ 1-\|x(t)\|^2 < (1+f)^2\|x(t)\|^2-1 < \|x(t)\|^2-1 & \text{if } \|x(t)\|>1 \end{cases} \qquad (3.9)$$

or equivalently, that

$$\begin{cases} 1 < (1+f)^2 < 2\|x(t)\|^{-2} - 1 & \text{if } \|x(t)\| < 1 \\ (1+f)^2 = 1 & \text{if } \|x(t)\| = 1 \\ 2\|x(t)\|^{-2} - 1 < (1+f)^2 < 1 & \text{if } \|x(t)\| > 1 \end{cases} \qquad (3.10)$$

Additionally, of course

$$(1+f)^2 \ge 0 \qquad (3.11)$$

This leads to the picture in Fig.3:1

4. A SPECIFIC SOLUTION

In the previous section, a class of stabilizing controls in feedback form for the system (3.1) was characterized.

Now a specific choice of the function $f(\cdot)$ will be made:

$$f(\|x(t)\|) = \|x(t)\|^{-1} - 1 \tag{4.1}$$

It is easily veryfied that this $f(\cdot)$ is admissible.

The nonlinear system then can be written with the aid of equations (3.1), (3.4) and (4.1):

$$\begin{cases} x_1(t+h) = [\ \alpha x_1(t) - \beta x_2(t)\ ]/\|x(t)\| \\ x_2(t+h) = [\ \beta x_1(t) + \alpha x_2(t)\ ]/\|x(t)\| \end{cases} \tag{4.2}$$

or in matrix form

$$x(t+h) = Ax(t)/\|x(t)\| \tag{4.3}$$

where

$$A = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \tag{4.4}$$

It is now evident what the control with f as in (4.1) does to the system: When recursion is performed, the state vector is normalized to unity.

This can also be seen in the Lyapunov funcion (3.5)

$$V(t+h) = [(1 + \|x(t)\|^{-1} - 1)^2\|x(t)\|^2 - 1]^2 = 0 \tag{4.5}$$

An error is totally compensated for in one step. Since this is the fastest possible convergence to the stationary amplitude, the control with $f(\cdot)$ as in (4.1) is time-optimal.

In Åström [1], the corresponding continuous time problem is solved. It is interesting to see that the convergence properties are superior in the discrete time case.

A straightforward extension of the stable system (4.3) for the state vector to converge to any chosen amplitude M, is

$$x(t+h) = Ax(t)M/\|x(t)\| \qquad (4.6)$$

which will have the same stability properties as (4.3).

Here it is appropriate to point out that the system (4.6) will be trajectory stable also if there are errors in the coefficients α and β. The trajectory then converges to a circle with radius

$$r = [\,\alpha^2 + \beta^2]^{1/2} M \qquad (4.7)$$

The number of arithmetic operations required in a straightforward implementation of (4.3) are

3 additions
8 multiplications
1 evaluation of $y = 1/\sqrt{a}$

The last function can be solved in a few iterations of the Newton-Raphson algorithm

$$y_{n+1} = y_n [\,3 - a y_n^2]/2 \qquad (4.8)$$

The final result, system (4.6) is intuitively appealing for its simplicity. It is also very simple to implement in a computer.

5. CONCLUSIONS

The problem of nonlinear amplitude stabilization of a discrete time oscillator has been attacked by a Lyapunov method approach. A class of stabilizing controls has been characterized. A specific choice of control turned out to be time-optimal, in that an error in amplitude is totally eliminated in one recursion step.
This solution is intuitively appealing - when known, it is almost trivial!
The final equations for the amplitude stable oscillator are easily implemented in e.g. a micro-computer.

6. REFERENCES

[1] Åström K.J. Olinjära System. Lecture notes in Automatic Control. LUND 1968.

[2] LaSalle J.P. The Stability of Dynamical Systems. Brown University, 1976.