Perspective Invariant Markings

A Geometric Approach

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Perspective Invariant Markings — A Geometric Approach

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Abstract.
The question of perspective invariant markings, i.e. markings with some property that is preserved while the marking is moved in space and then seen through a camera lens, is raised in a recent PhD-thesis [Nielsen]. The present paper investigates existence and uniqueness questions for such objects. The main result is that if a dimensionality requirement is satisfied, then almost all "nice" markings have (locally) properties of this kind almost everywhere. As an example, a perspective invariant marking system for e.g. automatically guided vehicles moving in a plane is given. This marking consists of just two points.

1. Introduction

In a recent PhD-thesis [Nielsen] the question of so called perspective invariant markings is discussed. A marking of this type is, loosely speaking, a kind of symbol that, when seen through a camera lens, the image have some property that is invariant while the marking is moved around in space. The objective is to be able to distinguish between a set of different markings, despite the fact that they are moved around in space and seen through a camera lens. The case of the perspective invariant marking being four concentric regular m-gons is depicted in Figure 1, for $m = 3, 4, 5, 6$. In [Nielsen] area-based invariants for the first two of these are given explicitly.

The aim of the present paper is to pose the problem in a slightly more general setting, and to investigate the question of existence and uniqueness of perspective invariant markings under different conditions, and some of their properties. Essentially, only the local question is considered. Also, this is a very unfinished paper with lots of open ends and details to work out.

In the next section, notation is introduced and the problem formulated. Section 3 contains some general results. The main theorem tells us that if a dimensionality requirement is satisfied, then almost all "nice" functions are locally perspective invariant markings almost everywhere! The following section contains a very simply application of these results to the practically important case of marking objects confined to a known 2-dimensional plane (e.g. automatically guided vehicles on a factory floor). Section 5

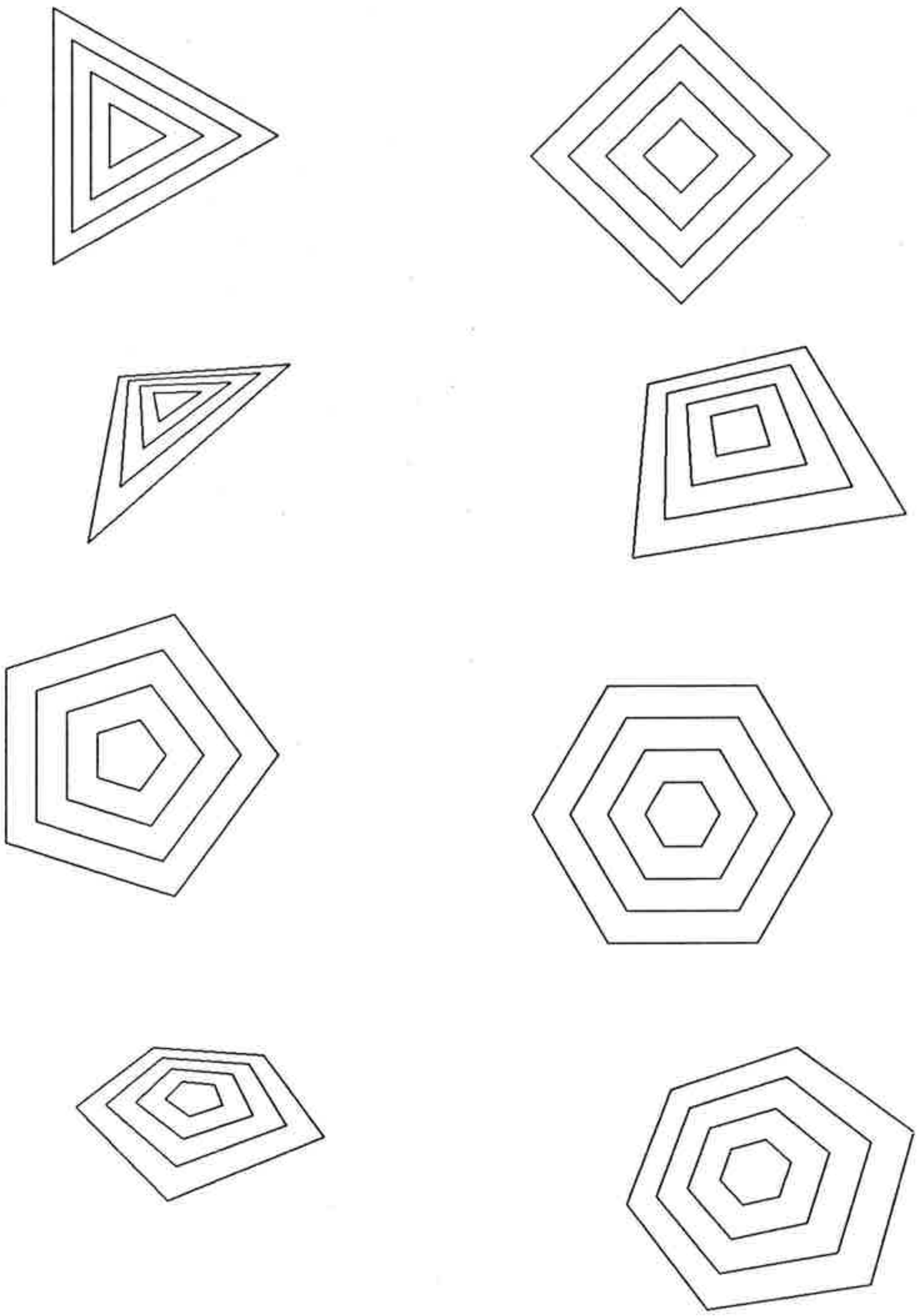

Figure 1. Some perspective invariant markings based on regular, concentric m-gons. Original image and image after transformation shown.

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is ties in the result in [Nielsen]. Finally, the last section contains some exercises to challenge the reader.

I would like to thank Lars Nielsen for giving me the problem, and for many stimulating discussions. The figures on page 2 has been generated by some functions written in the "matrix manipulation language" CTRL-C. These functions are described in the report

[Mårtensson]. This report has been typeset by TEX, and the figures directly included in the TEX file.

2. Definitions and Problem Statement

First of all, what do we mean by a marking? From a practical viewpoint, there is never a need to distinguish between more than a finite number of objects, so the problem can always be solved with some kind of licence plate system. Clearly, we can even obtain a countable infinite set of markings in this fashion. In order to prohibit these solutions, and to pose the problem in a mathematically interesting way, we require the marking system to be able to distinguish between an uncountable set of markings. This is (hopefully) not just mathematic paranoia, since a solution based on these ideas *might* turn out to me more efficient to compute with today's or tomorrow's hardware.

We now strictly introduce all the different mappings that make up the composite mapping from the marking plane to the image plane.

Our objective is to identify an open set of $\mathbb{R}^\ell$ with a set of markings, such that the marking function, composed by the image transformations, when the motion is restricted to a certain, given subset of the Euclidean group, still is an injective function.

In the sequel, we will use the following "meta-notation": Let f be a function with domain $\mathcal{X}$ and codomain $\mathcal{Y}$, i.e.

$$f : \mathcal{X} \longrightarrow \mathcal{Y}$$

By f^m we mean the component-wise application of f to a m-dimensional vector of elements in $\mathcal{X}$, i.e.

$$f^m : \mathcal{X}^m \longrightarrow \mathcal{Y}^m$$

$$f^m : (x_1, \ldots, x_m) \longmapsto (y_1, \ldots, y_m) = (f(x_1), \ldots, f(x_m))$$

Sometimes $\mathcal{X}$ or $\mathcal{Y}$ will be modulo a permutation group $\subset \mathrm{S}^m$, but this will be clear from the context.

We denote the marking plane by $\mathbb{M}$ and the image plane by $\mathbb{I}$. Clearly $\mathbb{M} \cong \mathbb{I} \cong \mathbb{R}^2$, but we will keep the distinction for emphasis. m points in $\mathbb{M}$ ($\mathbb{I}$) will be denoted by $\mathbb{M}^m$ ($\mathbb{I}^m$). These are clearly $2m$-dimensional spaces.

The marking function is really a function $\sigma \in C^k(U, \mathbb{M}^m)$, $\quad k = 1, \ldots, \infty, \omega$ (C^ω denotes the analytic functions), where U is an open subset of $\mathbb{R}^\ell$.

The function e is the embedding of $\mathbb{M}$ in $\mathbb{R}^3$, i.e.

$$e : \mathbb{M} \longrightarrow \mathbb{R}^3$$

$$e : \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

Let $\mathcal{F} \subset \mathcal{E} = \mathcal{E}(3)$, the Euclidean group of motions of 3-dimensional space. $\mathcal{E}$ can be written as the direct sum of subgroups, namely as $\mathcal{E} = \mathcal{E}^R \oplus \mathcal{E}^T$, where $\mathcal{E}^R \cong \mathcal{SO}(3)$ is the rotational component, and $\mathcal{E}^T$ is the translational part. We will pose the problem of perspective invariant markings for $\mathcal{F}$, not only for the full $\mathcal{E}$ as in [Nielsen]. In order to be able to differentiate, we also require $\mathcal{F}$ to have the structure of a C^k-manifold of positive dimension. For a suitable proper algebraic subset $\mathcal{E}_0 \subset \mathcal{E}$, there exists coordinates on $\mathcal{E}\backslash\mathcal{E}_0$. Thus it is natural to denote the typical element of $\mathcal{F}$ by M_ξ, and think of ξ as coordinates. We will here use the Euler angles on $\mathcal{E}^R$, and the Cartesian coordinates on $\mathcal{E}^T$. The Euler angles, considered as a bijection from an open, dense subset of $\mathbb{T}^3$ to an open dense subset of $\mathcal{SO}(3)$, can extended by continuity to a surjective mapping from $\mathbb{T}^3$ to $\mathcal{SO}(3)$.

For a fixed element $M_\xi \in \mathcal{F}$ we have the mapping

$$M_\xi^m : (\mathbb{R}^3)^m \longrightarrow (\mathbb{R}^3)^m / \mathsf{P}^m$$

where P^m is a subgroup of the symmetric group S^m of permutations of m letters. It is natural to consider the space $(\mathbb{R}^3)^m/\mathsf{P}^m$ instead of $(\mathbb{R}^3)^m$, since there might be much symmetry in the image, as embedded in three-space.

Finally, consider the image projection $\imath$

$$\imath^m : (\mathbb{R}^3)^m \longrightarrow \mathbb{I}^m / \mathsf{P}^m$$

We select a length scale equal to the focal distance. It is shown in [Nielsen] that $\imath$ can be described as

$$\imath : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} \dfrac{x}{z+1} \\ \dfrac{y}{z+1} \end{pmatrix}$$

Because of available hardware and insensitivity to measurement noise, invariants based only on area measurements might be prefarable. This is the case [Nielsen] treats. I.e. we only want to consider the enclosed areas formed by suitable subsets of the m points, which can be represented by a function $a : \mathbb{I}/\mathsf{P}^m \longrightarrow \mathbb{R}^n$, for a suitable n.

To conclude, we have the following composition of functions

$$\mathbb{R}^\ell \xrightarrow{\sigma} \mathbb{M}^m \xrightarrow{e^m} (\mathbb{R}^3)^m \xrightarrow{M_\xi^m} (\mathbb{R}^3)^m/\mathsf{P}^m \xrightarrow{\imath^m} \mathbb{I}^m/\mathsf{P}^m \xrightarrow{a} \mathbb{R}^n$$

We will denote $\imath^m \circ M_\xi^m \circ e^m$ by f_ξ^m. Clearly $f^m \in C^\omega(\mathbb{M}^m, \mathbb{I}^m/\mathsf{P}^m)$, except when some z-coordinate of $M_\xi^m(e^m)$ is equal to -1.

Next we define the concept of a projective invariant marking, PIM and APIM.

Definition. Let U be an open subset of $\mathbb{R}^\ell$, $\ell \leq 1$, and $\sigma \in C^k(U, \mathbb{M}^m)$. We say that $\sigma \in \mathrm{PIM}(\ell, m, \mathcal{F}, U)$ if there is an open neighborhood $V \subset \mathbb{I}^m/\mathsf{P}^m$ and a local 'coordinate system' $\{\alpha_i\}_{i=1}^{2m}$ on V; $\alpha_i : V \rightarrow \mathbb{R}$ such that $\alpha_1 \circ f_\xi^m \circ \sigma, \ldots, \alpha_\ell \circ f_\xi^m \circ \sigma$ are independent of M_ξ, and $\{\alpha_i \circ f_\xi^m \circ \sigma\}_{i=1}^\ell$ is an injective mapping from U to V. Analogously, we say that $\sigma \in \mathrm{APIM}(\ell, n, \mathcal{F}, U)$ if there is an open neighborhood $V \subset \mathbb{R}^n$ and a local 'coordinate system' $\{\alpha_i\}_{i=1}^n$ on V; $\alpha_i : V \rightarrow \mathbb{R}$ such that $\alpha_1 \circ f_\xi^m \circ \sigma, \ldots, \alpha_\ell \circ f_\xi^m \circ \sigma$ are independent of M_ξ, and $\{\alpha_i \circ f_\xi^m \circ \sigma\}_{i=1}^\ell$ is an injective mapping from U to V. $\alpha_1, \ldots, \alpha_\ell$ are called invariants, and the set $\{\alpha_1, \ldots, \alpha_\ell\}$ a complete set of invariants.

They represent some property of the image $f_\xi^m \circ \sigma$ that does not change as M_ξ varies over $\mathcal{F}$.

Remark. This definition is quite different from the one in [Nielsen], who only consider the case $\ell = 1$. In [Nielsen], the invariant is allowed to depend explicitly on σ. The invariant for a particular σ_0 is zero if and only if $\sigma = \sigma_0$, and in this case, this holds for all $M_\xi \in \mathcal{E}$. A practical drawback with this weaker definition is that we need one test per symbol involved in order to find which σ that generated the image.

Remark. Note that although we call the α_i's 'coordinates', we actually do not require any regularity property of them, only injectivity. It will be shown in the next section that we without loss of generality can assume that $\alpha_i \in C^k$.

Remark. It follows immediately from the definition that $\{\alpha_1, \ldots, \alpha_\ell\}$ are unique in the sense that given another complete set of invariants $\{\beta_1, \ldots, \beta_\ell\}$, there is a bijection between $\{\alpha_1, \ldots, \alpha_\ell\}$ and $\{\beta_1, \ldots, \beta_\ell\}$, with the same regularity properties as the least regular of $\{\alpha_1, \ldots, \alpha_\ell\}$ and $\{\beta_1, \ldots, \beta_\ell\}$.

Now we can formulate our path: Examine the properties of the mappings above, and the possible existence of PIM and APIM for different ℓ, m, n, and $\mathcal{F}$. Describe, as explicitly as possible, a set of invariants for some nice σ.

Essentially, we will only consider the local question. Everywhere it is needed, it is assumed that the figure is in front of the camera, i.e. that all the z-coordinates of $M_\xi^m \circ e^m \circ \sigma$ are greater than -1. This translates to a fairly complicated restriction of the allowed elements of $\mathbb{R}^\ell \times \mathcal{F}$.

3. Some General Results

In this section we collect a few general results pertaining to the general problem as posed in the preceding section.

One projective invariant marking can be immediately deduced from classical projective geometry:

PROPOSITION. *Let $\sigma(\gamma)$ be four points on a line, three of which are fixed, and the fourth is moved with γ, in an injective fashion. Then $\sigma \in \mathrm{PIM}(1, 4, \mathcal{E}, \mathbb{R})$. An invariant is given by the crossratio between the four points.*

Proof. This follows from the fact that the crossratio between the four points are a projective invariant.

PROPOSITION. *Let $\sigma(\mathbb{R}^\ell) \subset \{$Regular, concentric n-gons$\}$ for some n. Then $\mathrm{P}^m \cong \mathrm{S}^n$.*

Proof. Clearly P^m contains a subgroup isomorphic to S^n. Since the object is known to be in front of the camera, ordering of points on a straight line is preserved [Nielsen]. If we consider the lines drawn through the corresponding corners of the polygons, we see that P^m cannot be larger than S^n. This completes the proof.

The next proposition shows us that there is nothing to be gained by looking for e.g. discontinuous invariants.

PROPOSITION. *Assume that $\alpha_1, \dots, \alpha_\ell$ are a complete set of invariants for $\sigma \in$ PIM$(\ell, m, \mathcal{F}, U)$ or APIM$(\ell, n, \mathcal{F}, U)$, as defined above. Then there are a complete set of invariants $\beta_1, \dots, \beta_\ell$ for σ belonging to C^k. These can be taken so that the α_i's are functions of the β_i's.*

Proof. We prove the proposition only for PIM$(\ell, m, \mathcal{F}, U)$. The proof for APIM$(\ell, n, \mathcal{F}, U)$ differs only in notation. Denote $\alpha = (\alpha_1, \dots, \alpha_\ell)^T$, and similarly for β. $\alpha \circ f_\xi \circ \sigma$ is, by assumption, an injection from $U \subset \mathbb{R}^\ell$ to $\mathbb{R}^\ell$. Put $\beta = \left(\alpha \circ f_\xi^m \circ \sigma\right)^{-1} \circ \alpha$ as a candidate. This is well defined since the inverse exists on Codomainα. By assumption, β does not depend on ξ. Clearly $\beta \circ f_\xi^m \circ \sigma = id$, the identity transformation. Thus β is an invariant $\in C^k$, and $\{\beta_1, \dots, \beta_\ell\}$ a complete set of invariants. The last claim follows from the second remark in Section 2.

Remark. A natural question is weather something similar can be said about when σ is just required to be an injective function. This is not the case, since there exists bijections from $\mathbb{R}^\ell \longrightarrow \mathbb{R}$ for any $\ell \geq 1$ (of course not in C^1). Put in another way, given $\sigma \in$ PIM$(1, m, \mathcal{F}, U)$, then for *any* $\ell \geq 1$ we can construct $\tilde{\sigma} \in$ PIM$(\ell, m, \mathcal{F}, U)$!

LEMMA 1. *For fixed $\sigma \in \mathbb{M}^m$ in general position we have*

i) If $m = 1$ then $\dim\{f_\xi^m(\sigma) : M_\xi \in \mathcal{F}\} \leq \min(3, \dim \mathcal{F})$
ii) If $m = 2$ then $\dim\{f_\xi^m(\sigma) : M_\xi \in \mathcal{F}\} \leq \min(5, \dim \mathcal{F})$
iii) If $m \geq 3$ then $\dim\{f_\xi^m(\sigma) : M_\xi \in \mathcal{F}\} = \dim \mathcal{F}$

In case i) and ii), equality holds if $T\mathcal{F} \subset T\mathcal{E}$ does not nontrivially intersect a certain, of σ dependent proper subspace $\subset T\mathcal{E}$.

Proof. It is enough to show that rank $d_\xi f_\xi^m(\sigma)$ is 3, 5, or 6, in cases i), ii), and iii) respectivelly. Here d_ξ denotes the differential with respect to M_ξ, σ considered as constant. This is a linear mapping from $T\mathcal{F} \longrightarrow T(\mathbb{I}^m/\mathrm{P}^m) \cong \mathbb{R}^{2m}$, and can be described as

$$di^m \circ d_\xi M_\xi^m(e^m(\sigma))$$

By some book-keeping, it can be shown that, in case i), rank = 3; in case ii) rank = 5 (provided σ contains two distinct points); in case iii) rank = 6 (if e.g σ contains two linearly independent vectors). It only remains to show that ker di^m generically does not intersect im $d_\xi M_\xi^m$ in a nontrivial way. But since

$$\ker di = \mathrm{span}\left\{\begin{pmatrix} x \\ y \\ z+1 \end{pmatrix}\right\}$$

where $(x, y, z)^T$ denotes $M_\xi \circ e(\sigma_i)$, by chosing σ, ker di^m can be taken to be any m-dimensional subspace not containing $\{z_i = 0\}$ for $i = 1, \dots, m$. This completes the proof.

Remark. This result can and should be sharpened in the sense of finding out exactly what "general position" means. The motivation for this is given in the second remark following the theorem. It is believed that no $\mathcal{F}$ and $\sigma \in \mathrm{PIM}(\ell, m, \mathcal{F}, U)$ exist, for which Lemma 1 is false in an open set of points $\{\sigma(\gamma) : \gamma \in U\}$.

PROPOSITION AND DEFINITION. *If σ and $\mathcal{F}$ are such that Lemma 1 holds with equality at least at one point in U, then a necessary condition for $\mathrm{PIM}(\ell, m, \mathcal{F}, U)$ to be non-empty is that $\ell + \dim \mathcal{F} \leq 2m$. We call this property $\heartsuit$.*

Proof. $f_\xi^m \circ \sigma$ is C^k, so counting dimensions together with Lemma 1 yields the result.

Definition. We let $\mathcal{F}'$ denote all $M_\xi \in \mathcal{F}$ such that $d_\sigma f_\xi^m$ is nonsingular, where d_σ means the differential with respect to σ, M_ξ considered as constant.

Notation. For $M_\xi \in \mathcal{E}$ we use the decomposition $M_\xi r = Q_\xi r + r_\xi$ of M_ξ in a rotational part and a translational part, corresponding to the decomposition $\mathcal{E} = \mathcal{E}^R \oplus \mathcal{E}^T$. Furthermore, we will use the notation $[q_\xi^1, q_\xi^2, q_\xi^3] = Q_\xi$, and $r'_\xi = r_\xi - (0,0,-1)^T$.

$\mathcal{F}'$ can now be further characterized:

LEMMA 2. *Assume that $2m \geq \ell$ and that no z-component of $M_\xi^m(e^m(\sigma))$ is equal to -1. Then $d_\sigma f_\xi^m$ is nonsingular, except when $q_\xi^3 \perp r'_\xi$, in which case rank $d_\sigma f_\xi^m = m$. In particular, this means that $\mathcal{F}'$ is an open, dense subset of $\mathcal{F}$ and an imbedded submanifold (together with the inclusion map).*

Remark. Physically, $q_\xi^3 \perp r'_\xi$, means that the marking $M_\xi^m \circ e^m \circ \sigma(U)$ contains a segment of a line through the focal point $(0,0,-1)^T$. It is thus physically obvious that the differential cannot have full rank. Cf. [Nielsen] page 77.

Proof. If $d_\sigma f_\xi^m$ is singular there is a $(a,b) \in T\mathrm{M}\backslash(0,0)$ such that

$$Q_\xi \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} = aq_\xi^1 + bq_\xi^2 \parallel Q_\xi \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} + r'_\xi = xq_\xi^1 + yq_\xi^2 + r'_\xi$$

i.e. r'_ξ is in the linear span of q_ξ^1 and q_ξ^2, and thus orthogonal to q_ξ^3. This proves the first statement. Note that this property is independent of σ. By the nature of f_ξ^m, the rank thus drop simultaneously in all components, and since $d_\xi f_\xi^m$ will not be identically zero, the next statement follows. The last sentence is an immediate consequence of the preceding ones.

Now we can prove the main theorem of this paper. Together with Sard's theorem, it tells us that if the dimension requirement $\heartsuit$ is fulfilled, then "almost any" σ which is a differentiable function will be a (locally) projective invariant marking almost everywhere.

THEOREM. *Let U be an open subset of $\mathbb{R}^\ell$ and $\sigma \in C^k(U, \mathbb{M}^m)$. Assume that no z-component of $M_\xi^m(e^m(\sigma(U)))$ is equal to -1, that property ♡ is fulfilled, and that* rank $d\sigma(\gamma) = \ell$ *for $\gamma \in U$. If a non-generic intersection of subspaces does not occur, then $\sigma \in$ PIM$(\ell, m, \mathcal{F}', U)$.*

Proof. Let K denote a matrix with columns spanning $\ker di^m$. It is enough to show that

$$\text{rank } \left[K \quad d_\gamma\left(M_\xi^m \circ e^m \circ \sigma(\gamma)\right) \quad d_\xi\left(M_\xi^m\right)\right] = m + \ell + \dim \mathcal{F}$$

for $\gamma \in U$ and $M_\xi \in \mathcal{F}'$. By the chain rule, this can then be rewritten as

$$\text{rank } \left[K \quad Q_\xi^m \circ e^m \circ d\sigma \quad d_\xi M_\xi^m(e^m(\sigma(\gamma)))\right] = m + \ell + \dim \mathcal{F}$$

To show that this is generically true, it is enough to show that is sometimes true. By choice of Q_ξ and $d\sigma$, the second and third part can be made to span $\ell + \dim \mathcal{F}$ dimensions. It was shown in the proof of lemma 1 that $\{\text{span}K : \sigma \in \mathbb{M}^m\}$ is an open dense subset of Grass$(m, 3m)$. Together, this proves the theorem.

Remark. It may happen that $\sigma \notin$ PIM$(\ell, m, \mathcal{F}, U)$, despite that property ♡ is fulfilled: Let $\sigma(\gamma)$ be the four vertices of a square with side γ, centered around the origin, and let $\mathcal{F}$ contain translations along the z-axis. Then σ is not a perspective invariant marking. We leave it to the reader to check the details.

Remark. The requirements on $\mathcal{F}$ and σ to be "generic" is not as innocent as it might seem. The proper question might very well be this: Given a particular $\mathcal{F}$ and a particular σ, is this a projective invariant marking? The theorem does not answer that question. It is believed that the theorem can be sharpened in this direction with a moderate amount of effort.

CONJECTURE. *Replacing m by n, the analogous statements for* APIM$(\ell, n, \mathcal{F}, U)$ *are true.*

4. A Simple Example

In this section we will explicitly describe a perspective invariant marking for a practically important example, namely when the markings are confined to lie in a known 2-dimensional subspace. This is the case for e.g. automatically guided vehicles moving on a factory floor, each one having a marking attached to the top.

Say that the interesting plane is defined by $\{(x, y, z)^T \in \mathbb{R}^3 : x = 1\}$. (Taking the "natural" choice $\{x = 0\}$ does not work since it violates the condition in Lemma 1.) It is easy to see that $\mathcal{F} \cong \mathcal{E}(2)$, and can be described as follows:

$$\mathcal{F} =$$

$$\left\{ M_\xi \in \mathcal{E}(3) : Q_\xi = \begin{pmatrix} 0 & 0 & -1 \\ -\sin(\alpha) & \cos(\alpha) & 0 \\ \cos(\alpha) & \sin(\alpha) & 0 \end{pmatrix} ; r_\xi = \begin{pmatrix} 1 \\ y \\ z \end{pmatrix} ; \alpha \in [0, 2\pi); x, y \in \mathbb{R} \right\}$$

Note that $\mathcal{F}' = \mathcal{F}$. The proposed marking is the following: Choose $\ell = 1$. Place one point in $(0,0)^T \in \mathbb{M}$, and the other point in $(1,\gamma)^T$. This is clearly an analytic function from $\mathbb{R}$ to $\mathbb{M}^2$, its differential being of full rank everywhere. The preceding results thus tell us that there is a (local) invariant, and this is unique in the sense above. We will now describe this explicitly, and determine what 'local' means.

We have the mappings

$$\left\{\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}1\\\gamma\end{pmatrix}\right\} \overset{f_\ell^2}{\longmapsto} \left\{\begin{pmatrix}\frac{1}{z+1}\\\frac{y}{z+1}\end{pmatrix},\begin{pmatrix}\frac{1}{\cos(\alpha)+\gamma\sin(\alpha)+z+1}\\\frac{-\sin(\alpha)+\gamma\cos(\alpha)+y}{\cos(\alpha)+\gamma\sin(\alpha)+z+1}\end{pmatrix}\right\} = \left\{\begin{pmatrix}a\\b\end{pmatrix},\begin{pmatrix}c\\d\end{pmatrix}\right\}$$

where a, b, c, and d are defined by the last equality. Calculation shows that

$$\gamma^2 = \left(\frac{1}{c} - \frac{1}{a}\right)^2 + \left(\frac{d}{c} - \frac{b}{a}\right)^2 - 1$$

and thus we can locally solve for γ. In fact, the solution is unique if we know that $\gamma \geq 0$. The solution is not unique if $\gamma \in \mathbb{R}$, since e.g. $\alpha = \frac{\pi}{4}$, $\gamma = 1$ will give the same image as $\alpha = -\frac{\pi}{4}$, $\gamma = -1$, as the reader might check. Also observe that the expression given above is invariant under exchange of (a,b) with (c,d), so for this identification we do not have to distinguish between the two different image points.

5. Lars Nielsen's results

In [Nielsen] the case of the area-based perspective invariant marking being n concentric regular m-gons with their centers in the origin is treated. For $m = 3$ and 4, it is shown that for $n \leq 3$ no invariants exists. Invariants are explicitly given for $m = 3$ and 4. These have been commented upon in a remark in section 2.

A marking of this type can be described by a n-vector $k = (k_1, \ldots, k_n)$, with the i-th m-gon having a vertex in $(k_i, 0)^T$. Since only the full Euclidean group $\mathcal{E}(3)$ is considered, section 3 yields that $2n \geq \ell + \dim \mathcal{E} = 1 + 6 = 7$, i.e. $n \geq 4$, in order for invariants to exist.

[Nielsen] then suggests the following $\sigma : \mathbb{R} \longrightarrow \mathbb{M}^{nm}$, namely to fix all k_i's but one, say k_j, which we allow to vary as, say, γ.

[Mårtensson] contains several useful CTRLC-functions for manipulation and plotting of these objects. Figure 1 was generated using these.

6. Exercises

Exercise 1. Consider the case with a ν-dimensional marking plane $\mathbb{M}$, a μ-dimensional image plane $\mathbb{I}$, and a κ-dimensional embedding space, and do the same analysis again.

Exercise 2. Discuss the practical implications of an invariant which takes on its values in the Cantor set.

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