Perspective Area-Invariants

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Perspective area-invariants

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ABSTRACT

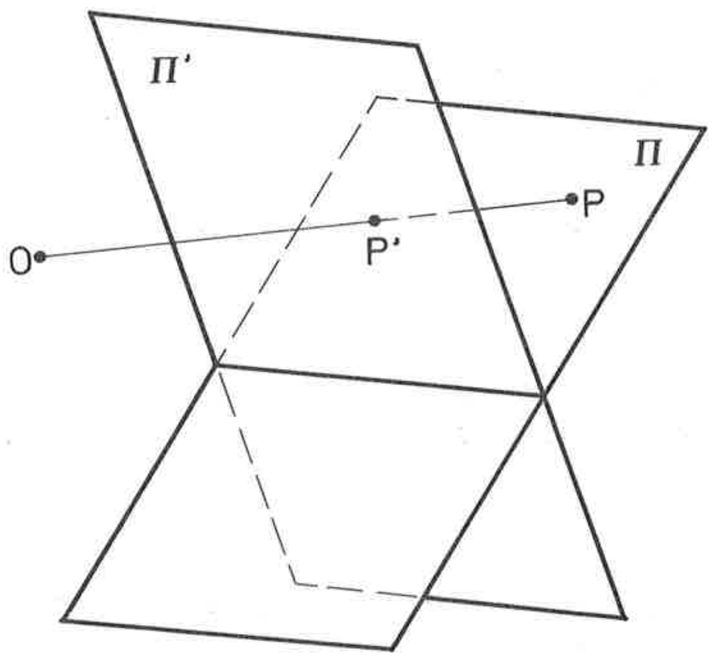
An area-relation which is invariant under perspective is derived for certain classes of objects.

1. Introduction

Let there be given two planes Π and Π' and one point $O \notin \Pi \cup \Pi'$ in Euclidean 3-space. Adjoining to the planes their respective points at infinity we know from projective geometry that perspective with center O defines a bijective mapping $\Pi \rightarrow \Pi'$. We agree to call Π the <u>object plane</u> and Π' the <u>image plane</u>.

It is well known that perspective have the property of mapping projective lines into projective lines, with preservation of cross-ratios (i.e. certain relations involving distances). The results of this article in a sense generalize for certain objects the concept of cross-ratio to a relation, involving areas, which is invariant under perspective. We thus equip the projective planes with the structure of the surrounding Euclidean 3-space.

The problem was formulated by the first author in connection with an application in visual servoing cf. [1] and Section 4 below. There the relations (4) were proved by parameterizing the mapping $\Pi \rightarrow \Pi'$ using a translation vector and



<u>Figure 1</u>

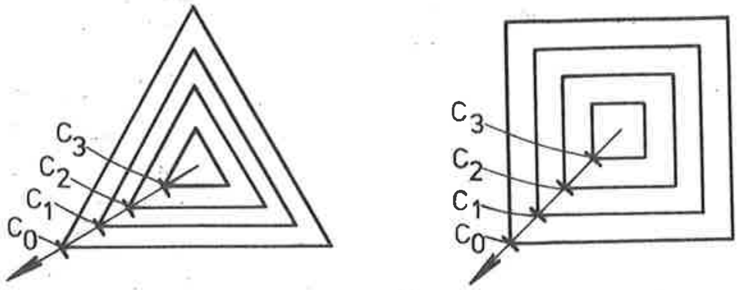


Figure 2

three Euler angles. The present proofs are due to the second author.

The problem is formulated in Section 2. The main results with some remarks are given in Section 3. Section 4 indicates possible applications.

2. Problem formulation

We consider here two kinds of objects which we call 3-symbols and 4-symbols. These consist respectively of systems of four triangles and four parallelograms concentrically spaced with respect to their centers. (By center we mean the point of intersection between their medians or diagonals). The m-symbols are illustrated in Figure 2.

We always consider the m-symbols to be in Π. Possible perspective images in the plane Π' are illustrated in Figure 3. The primitive objects are not similar in the image, and parallel lines of the objects are not parallel in the image.

Let the relative sizes be represented by the vector

$$c = (c_0, c_1, c_2, c_3)$$

A certain composite object is now uniquely described by m and c. The image of the symbol may contain points at infinity in Π'. We neglect this case at the moment. (With suitable interpretations our results holds also for this case). Using the Euclidean structure in Π' we measure the areas within the primitive objects composing the image of the symbol, and define

$$a = (a_0, a_1, a_2, a_3)$$

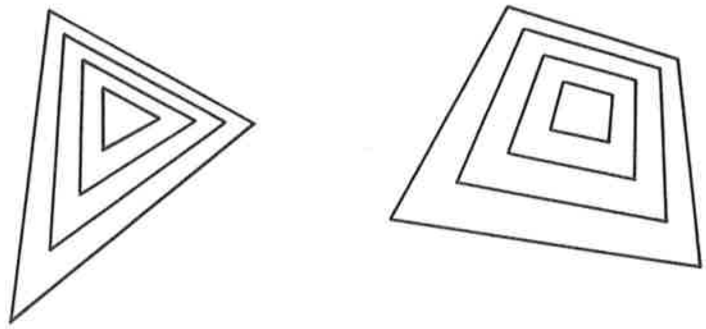


Figure 3

We pose the following problem: Find a relation, invariant under perspectives, involving only the object and image characteristics c and a. In other words, we search a relation which holds regardless of the positions of Π and Π' and the placement of the symbol within Π.

It is clear that any such relation must be homogeneous in c and a separately. The c and a vectors thus may be considered as object and image descriptions in homogeneous coordinates, where we use the first coordinate (c_0 respectively a_0) as reference.

3. Main Results

Put

$$q_k = a_k/(a_0 - a_k) \qquad k = 1,2,3$$

Introduce, for m=3,4, the functions

$$(1) \quad h(x_1,x_2,x_3) = \begin{vmatrix} x_1^2 - x_2^2 & x_2^2 x_1^{m-2} - x_2^m \\ x_1^2 - x_3^2 & x_3^2 x_1^{m-2} - x_3^m \end{vmatrix} =$$

$$= x_1^2 x_2^m - x_1^m x_2^2 + x_2^2 x_3^m - x_2^m x_3^2 + x_3^2 x_1^m - x_3^m x_1^2$$

and the cyclic expressions

$$\lambda_0 = c_0^2\, h(c_1,c_2,c_3)$$

$$\lambda_1 = c_1^2\, h(c_2,c_3,c_0)$$

$$\lambda_2 = c_2^2\, h(c_3,c_0,c_1)$$

$$\lambda_3 = c_3^2\, h(c_0,c_1,c_2)$$

Notice that

$$(2) \quad \lambda_0 - \lambda_1 + \lambda_2 - \lambda_3 = 0$$

Theorem The equation

$$(3) \quad \frac{\lambda_1}{q_1} - \frac{\lambda_2}{q_2} + \frac{\lambda_3}{q_3} = 0$$

or equivalently

$$(4) \quad \frac{\lambda_0}{a_0} - \frac{\lambda_1}{a_1} + \frac{\lambda_2}{a_2} - \frac{\lambda_3}{a_3} = 0$$

is invariant under perspectives of m-symbols, m=3,4.

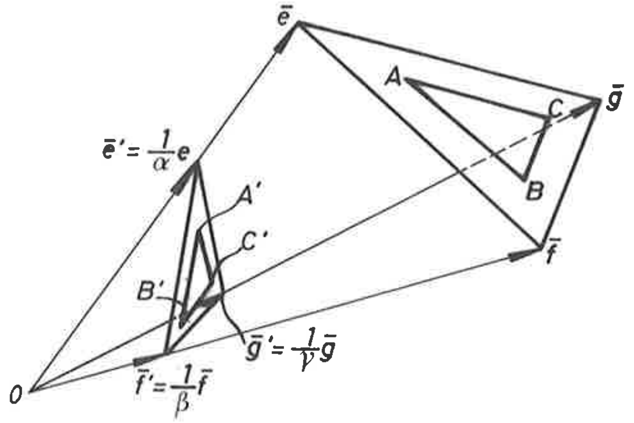


Figure 4

Proof:

Case m=3 Two bases

$\bar{e}, \bar{f}, \bar{g}$ and $\bar{e}', \bar{f}', \bar{g}'$

are used as indicated in Figure 4. The coordinates are transformed as

$$(5) \quad \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Let c stand for any c_k, k=1,..,3. The corresponding image area is denoted by a. Use further $q=a/(a_0-a)$.

The direction vectors OA, OB, and OC are

$(c_0+2c,\ c_0-c,\ c_0-c)$

$(c_0-c,\ c_0+2c,\ c_0-c)$

$(c_0-c,\ c_0-c,\ c_0+2c)$

with respect to the basis $\bar{e},\bar{f},\bar{g}$. The same direction vectors OA, OB, and OC are described in the basis $\bar{e}',\bar{f}',\bar{g}'$ as

$(\alpha(c_0+2c),\ \beta(c_0-c),\ \gamma(c_0-c))$

$(\alpha(c_0-c),\ \beta(c_0+2c),\ \gamma(c_0-c))$

$(\alpha(c_0-c),\ \beta(c_0-c),\ \gamma(c_0+2c))$

With respect to the coordinate system $O\bar{e}'\bar{f}'\bar{g}'$ the plane Π' has the equation $x'+y'+z'=1$. In that coordinate system we have the following expressions for the points A', B', and C':

$$A' : \frac{(\alpha(c_0+2c),\ \beta(c_0-c),\ \gamma(c_0-c))}{\Sigma_1} \quad \text{where}$$

$$\Sigma_1 = (c_0-c)(\alpha+\beta+\gamma) + 3c\alpha$$

$$B' : \frac{(\alpha(c_0-c),\ \beta(c_0+2c),\ \gamma(c_0-c))}{\Sigma_2} \quad \text{where}$$

$$\Sigma_2 = (c_0-c)(\alpha+\beta+\gamma) + 3c\beta$$

$$C' : \frac{(\alpha(c_0-c),\ \beta(c_0-c),\ \gamma(c_0+2c))}{\Sigma_3} \quad \text{where}$$

$$\Sigma_3 = (c_0-c)(\alpha+\beta+\gamma) + 3c\gamma$$

Comparing the volumes of the tetrahedron OA'B'C' and the tetrahedron spanned by the basis vectors $\bar{e}'$, $\bar{f}'$, and $\bar{g}'$ we obtain the area-relation

$$\frac{a}{a_0} = \det(OA', OB', OC') =$$

$$= \frac{\alpha\beta\gamma}{\Sigma_1\Sigma_2\Sigma_3} \begin{vmatrix} c_0+2c & c_0-c & c_0-c \\ c_0-c & c_0+2c & c_0-c \\ c_0-c & c_0-c & c_0+2c \end{vmatrix} =$$

$$= \frac{\alpha\beta\gamma}{\Sigma_1\Sigma_2\Sigma_3}\ 27c_0c^2$$

By means of q this can be written

$$(6) \quad (c_0-c)^2(c_0+2c)\ S_1^3 + 9c^2(c_0-c)\ S_1S_2 + 27c^2(c - c_0 - \frac{c_0}{q})\ S_3 = 0$$

where

$$S_1 = \alpha+\beta+\gamma \qquad S_2 = \alpha\beta+\beta\gamma+\gamma\alpha \qquad S_3 = \alpha\beta\gamma$$

For the 3-symbol it follows

$$\begin{cases} (c_0 - c_1)^2 (c_0 + 2c_1)\, S_1^3 + 9c_1^2 (c_0 - c_1)\, S_1 S_2 + 27c_1^2 \left(c_1 - c_0 - \frac{c_0}{q_1}\right) S_3 = 0 \\ (c_0 - c_2)^2 (c_0 + 2c_2)\, S_1^3 + 9c_2^2 (c_0 - c_2)\, S_1 S_2 + 27c_2^2 \left(c_2 - c_0 - \frac{c_0}{q_2}\right) S_3 = 0 \\ (c_0 - c_3)^2 (c_0 + 2c_3)\, S_1^3 + 9c_3^2 (c_0 - c_3)\, S_1 S_2 + 27c_3^2 \left(c_3 - c_0 - \frac{c_0}{q_3}\right) S_3 = 0 \end{cases}$$

This is a homogeneous linear system in the variables S_1^3, S_1S_2, and S_3. We know that it has a non-trivial solution, expressed by α, β, and γ from the geometrical construction. Hence, the system determinant is zero, which after column operations simplifies to

$$\begin{vmatrix} (c_0^2 - c_1^2) & c_1^2 (c_0 - c_1) & \frac{c_1^2}{q_1} \\ (c_0^2 - c_2^2) & c_2^2 (c_0 - c_2) & \frac{c_2^2}{q_2} \\ (c_0^2 - c_3^2) & c_3^2 (c_0 - c_3) & \frac{c_3^2}{q_3} \end{vmatrix} = 0$$

Expanding the determinant after the third column directly gives (3) for m=3. The formula (4) is a consequence of (2).

<u>Case m=4</u> With loss of symmetry we select three corners E, F, and G as basis points. See Figure 5. The fourth corner is then characterized by

$$OH = -\bar{e} + \bar{f} + \bar{g}$$

The corners A, B, C, and D of the inscribed quadrangle are expressed by the direction vectors OA, OB, OC, and OD, which are

$$(\ 2c,\ c_0 - c,\ c_0 - c)$$

$$(\ 0,\ c_0 + c,\ c_0 - c)$$

$$(\ 0,\ c_0 - c,\ c_0 + c)$$

$$(-2c,\ c_0 + c,\ c_0 + c)$$

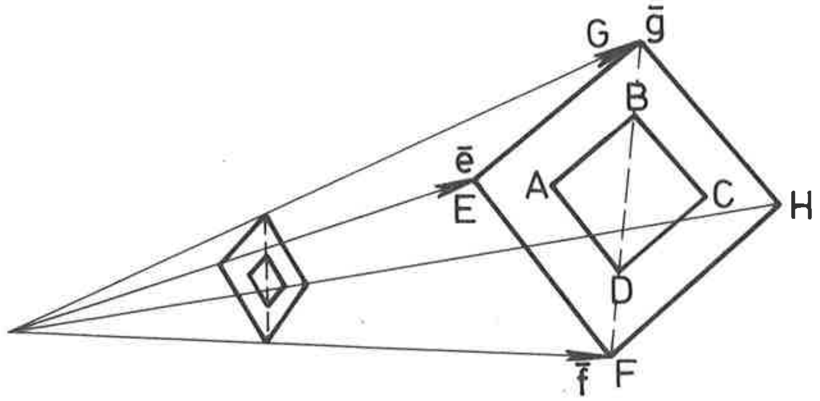


Figure 5

with respect to the basis $\bar{e},\bar{f},\bar{g}$. Let A', B', C', and D' be the images of A, B, C, and D. Arguing as in the case m=3, by the same change of coordinates (5), one obtains the coordinates of A', B', C', and D' with respect to the basis $\bar{e}',\bar{f}',\bar{g}'$. Using the area of the triangle E'F'G' as unit, one finds the area of the quadrangle E'F'H'G' as

$$\det(OE',OF',OG') - \det(OH',OF',OG') = \frac{\beta+\gamma}{\beta+\gamma-\alpha}$$

The area of the quadrangle A'B'C'D' is

$$\det(OA',OB',OC') - \det(OD',OB',OC') =$$

$$= \frac{16\, c^2 c_0^2\, \alpha\beta\gamma(\beta+\gamma)}{((c_0-c)(\beta+\gamma)+2c\alpha)\ ((c_0+c)(\beta+\gamma)-2c\alpha)\ ((c_0-c)\beta+(c_0+c)\gamma)\ ((c_0+c)\beta+(c_0-c)\gamma)}$$

The analogue of (6) is

$$(c_0-c)^2(\beta+\gamma)^4 + 4c^2(c_0-c)^2(\beta+\gamma)^2(\alpha\beta+\beta\gamma+\gamma\alpha-\alpha^2) + 16c^2\left(c^2-c_0^2-\frac{c_0^2}{q}\right)\alpha\beta\gamma(\beta+\gamma-\alpha) = 0$$

By means of three such equations we get a linear system. Again (3) or equivalently (4) expresses the fact that its determinant is zero. The theorem is now proved.

Corollary Let Q_k, k=1,2,3 denote the values of q_k, k=1,2,3 corresponding to the symbol itself. Then

$$(7)\quad \lambda_1 : \lambda_2 : \lambda_3 = \begin{vmatrix} \frac{1}{q_2} & \frac{1}{q_3} \\ \frac{1}{Q_2} & \frac{1}{Q_3} \end{vmatrix} : - \begin{vmatrix} \frac{1}{q_3} & \frac{1}{q_1} \\ \frac{1}{Q_3} & \frac{1}{Q_1} \end{vmatrix} : \begin{vmatrix} \frac{1}{q_1} & \frac{1}{q_2} \\ \frac{1}{Q_1} & \frac{1}{Q_2} \end{vmatrix}$$

Proof: Formula (3) of course holds also in the limiting case when Π and Π' coincide. By means of (2) and (3) we thus have a linear equation system in λ_k:

$$\begin{cases} \lambda_1 - \lambda_2 + \lambda_3 = \lambda_0 \\ \dfrac{\lambda_1}{q_1} - \dfrac{\lambda_2}{q_2} + \dfrac{\lambda_3}{q_3} = 0 \\ \dfrac{\lambda_1}{Q_1} - \dfrac{\lambda_2}{Q_2} + \dfrac{\lambda_3}{Q_3} = 0 \end{cases}$$

Here $\lambda_0 \neq 0$ since the c_k:s are distinct, so the system is inhomogeneous. Solving this we get (7).

Remark In Figure 2 the symbols are equilateral. We see from the proof that this assumption is not necessary.

4. Application

Invariants are useful for recognition in computer vision. In applications with a camera looking down on a conveyor belt the invariants of two-dimensional translation and rotation, e.g. area, perimeter and shape factor (area/(perimeter)^2), are used. There are a number of robot vision applications where it is of interest to use a set of marking symbols either to mark the robots themselves or to mark the environment [1]. The marking symbols should be identified from image data, and they can be arbitrary positioned relative to the camera. The size and shape of the image of the symbols thus vary.

Existing computer vision systems are normally based on extraction of features and properties at different levels. One approach that has already been implemented in special purpose hardware is to extract the areas within closed contours in the image. A further advantage of using area-measurements is the robustness to noise. Here the marking problem consist of constructing symbols with descriptions based on perspective area-invariants. It is well-known that the quotient between two areas of a planar figure is invariant under parallel projection. This is independent of the shape of the areas. Extensions of this result has been the main motivation for initiating the research presented here. The restriction to study symbols composed of triangles or squares has the advantage that there is a potential of easy verification of the shape of such simple objects by a vision system.

In Nielsen (1985) it is also shown that if either the c_1 or the c_2 component of the shape vector c is continuously varied, then the continuous set of symbols so obtained can be uniquely identified by using the invariant (3) or (4) in our main theorem.

References

[1] L. Nielsen (1985). Simplifications in visual servoing. Ph D thesis CODEN: LUTFD2/(TFRT-1027), Department of Automatic Control, Lund Institute of Technology, Sweden.