Adaptive Stabilization without High-Gain

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A very brief introduction to the concept of adaptive control is first given. The prototype problem of stabilizing an unstable, unknown plant is studied. The main result is the complete characterization of necessary and sufficient a priori knowledge needed for adaptive stabilization, namely knowledge of the order of any stabilizing controller. The concept of switching function controller is introduced, and some properties stated. 'The Turing Machine of Universal Controllers' is then presented. As the title indicates, this adaptive controller possessed the greatest stabilizing power a smooth, non-linear controller can have. The preceding works in this field have all dealt with variations on the theme of high-gain stabilization. This paper deals only with adaptive stabilization algorithms not requiring high-gain-stabilizability. Finally, the problem of stabilization to a possibly non-zero reference value is solved.

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1. Introduction

The discipline of *Control Theory* studies the problem of achieving "satisfactory performance" of a *plant*, i.e. a dynamical system to be controlled, by manipulating the input u in order to e.g. keep the output y close to 0, or to follow a reference signal r. The most general problem of control theory can in loose terms be described as the following: *Given a set $\mathcal{G}$ of plants, we are to find one controller K that achieves "satisfactory performance" (or optimal in some sense) to each one of the plants $G \in \mathcal{G}$.* Figure 1 illustrates the concept. The dependence of the input u of the output is exactly the concept of *feedback*.

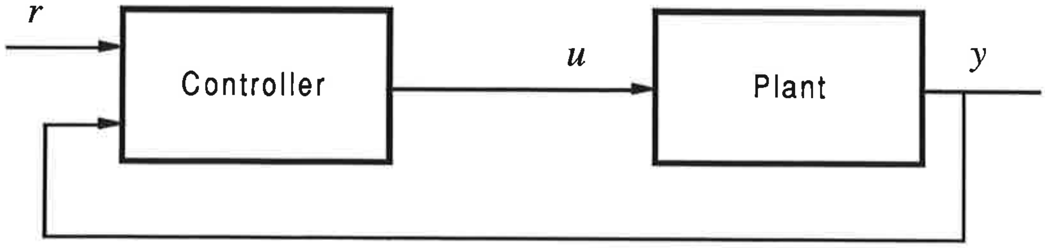

Figure 1. The Most General Control Configuration.

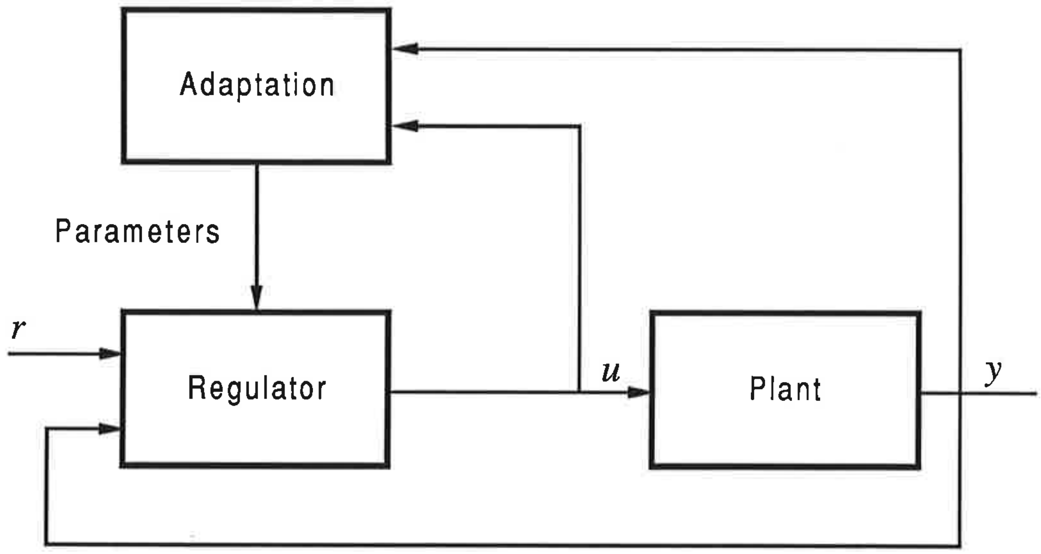

Figure 2. The General Adaptive Controller.

Adaptive Control is one—out of several other possible—approaches to solving this problem. It is an approach based on the concept of *learning*, i.e. the splitting of the 'true' state space of the non-linear system an adaptive controller constitutes in *parameters* and *states*. See Figure 2! The parameters reside in the "adaptation box", while the states reside in the "regulator box". The parameters are moving "slower" that the states, thereby motivating the values of the parameters as a state of knowledge on the dynamics of the plant.

Adaptive control is a vital subfield within control theory, with over 100 papers published every year. For an excellent overview of the field see [Åström].

In the end of the seventies and the beginning of the eighties, proof for convergence and stability of the commonly used adaptive schemes appeared. These proofs all required some variant of the following assumptions:

(i) A bound n^* on the order of the transfer function $g(s) = n(s)/d(s)$ is known.

(ii) The relative degree $r = \deg d(s) - \deg n(s)$ is known exactly.

(iii) The plant is minimum phase.

(iv) The sign of the 'instantaneous gain', i.e. the leading coefficient of $n(s)$*, is known.

This work is concerned with the fundamental limitations and possibilities of adaptive control, regardless of the particular algorithm used. In particular—are the four assumptions (i)–(iv) really necessary? To this end, what is believed to be the most fundamental problem is studied, namely the stabilization of an unstable plant. It can be argued that this is the "prototype problem", if we can do this there is hope for more achievements, and vice versa. It is also a very clean, quantitative problem.

We next give some more precise definitions for the sequel.

Definitions

Consider Figure 2! In general, with fixed values of the parameters, the dynamics in the states are assumed to be linear. Under this condition, we make the following definition.

---

* We assume that $d(s)$ is monic

Definition 1.1. Let the set of plants $\mathcal{G}$, its times $\mathcal{T}$, its input space $\mathcal{U}$, its output space $\mathcal{Y}$, and its space of reference signals $\mathcal{R}$ are given. Let l be a non-negative integer and $\mathcal{X}$ a vector bundle of rank l over the C^∞-manifold $\mathcal{M}$. We shall call the mapping

$$S : \mathcal{Y} \times \mathcal{R} \times \mathcal{X} \longrightarrow \mathcal{U}$$

a *linear adaptive controller* with *state space* $\mathbb{R}^l$ and *parameter space* $\mathcal{M}$ if it is smooth in the sense of a control system, [Brockett], and for fixed $k \in \mathcal{M}$ the mapping $S_k : \mathcal{Y} \times \mathcal{R} \times \mathbb{R}^l \longrightarrow \mathcal{U}$ is linear. That is, it can locally be written as

$$\begin{aligned} \dot{z} &= F(t,k)z + G(t,k)y \qquad x \in \mathbb{R}^l \\ u &= H(t,k)z + K(t,k)y \\ \dot{k} &= f(y,r,t,z,k) \end{aligned}$$

where F, G, H, K, and f are locally defined C^∞-functions. Here $x = (z^T, k^T)^T$ is a decomposition of the state of the controller corresponding to the local decomposition of $\mathcal{X}$ in $\mathbb{R}^l$ and $\mathcal{M}$. □

For a global, coordinate free description of a non-linear control system as a section of a certain pull-back bundle, see [Brockett].

With this definition, what makes a nonlinear controller into a linear adaptive controller is the (local) decomposition of the state space into a vector space times a manifold, together with linearity for fixed values of the parameters.

This definition covers the traditional approaches to adaptive control, namely model reference adaptive control and the self tuning regulator. Compare Figure 2!

Convergence of Adaptive Control

We will next make precise what we mean by convergence of a certain adaptive controller, controlling a certain plant. Only the stabilization problem, i.e. when $r \equiv 0$, will be considered. We restrict our attention to stabilization of strictly proper, time-invariant, linear plants described by finite dimensional differential equations, with vector spaces as their state space. That is, plants that can be written on state space form as

$$\begin{aligned} \dot{x} &= Ax + Bu, \qquad x \in \mathbb{R}^n, \qquad u \in \mathbb{R}^m \\ y &= Cx, \qquad y \in \mathbb{R}^p \end{aligned} \tag{MIMO}$$

Definition 1.2. We shall say that the linear adaptive controller $K \not\equiv 0$, controlling the plant G, whose state space is $\mathbb{R}^n$, converges, if, as $t \to \infty$, $\mathcal{M} \ni k$ converges to a finite value k_∞, while $\mathbb{R}^l \ni z \to 0$, and $\mathbb{R}^n \ni x \to 0$ as $t \to \infty$. □

Adaptive Control Problems

Finally, this is what shall be meant by an *adaptive control problem.*

Definition 1.3. We shall call the following an *adaptive control problem*: Let $\mathcal{G}$ be a set of plants. The adaptive control problem consists of finding a linear adaptive controller K, such that for any plant $G \in \mathcal{G}$, the controller K, controlling G, converges in the sense above. □

The 'size' of $\mathcal{G}$ can be considered as a measure of the uncertainty of the plant.

2. Necessary and Sufficient Conditions for Adaptive Stabilization

This section contains the complete characterization of the a priori knowledge needed to adaptively stabilize an unknown plant, namely the order of *any* fixed linear controller capable of stabilizing the plant. The necessity was proved in [Byrnes-Helmke-Morse], while the sufficiency was proved in [Mårtensson 1985]. A new proof of the sufficiency part is given, based on the results on switching functions presented in Section 5.

The Main Theorem

The following theorem is the most general result on adaptive stabilization.

THEOREM 2.1. *Let $\mathcal{G}$ be a set of plants of the type* (MIMO). *The necessary and sufficient a priori knowledge for adaptive stabilization is knowledge of an integer l such that for any plant $G \in \mathcal{G}$ there exists a fixed linear controller of order l stabilizing G.*

Proof of Necessity. See [Byrnes-Helmke-Morse]. ∎

The original proof of the sufficiency of this a priori information is the controller given in Section 6. The result can also be obtained by the method of switching functions introduced in Section 5.

We will devote the next sections to the development of some tools for proving this result.

3. A Viewpoint on Dynamic Feedback

In this section it is shown that, from a certain point of view, dynamic feedback can conceptually be replaced by static feedback.

The idea is very simple: the plant is augmented by a box of integrators, each with its own input and output. Static feedback is then applied to the augmented plant, i.e. the plant together with the integrators. The situation is depicted in Figure 3.

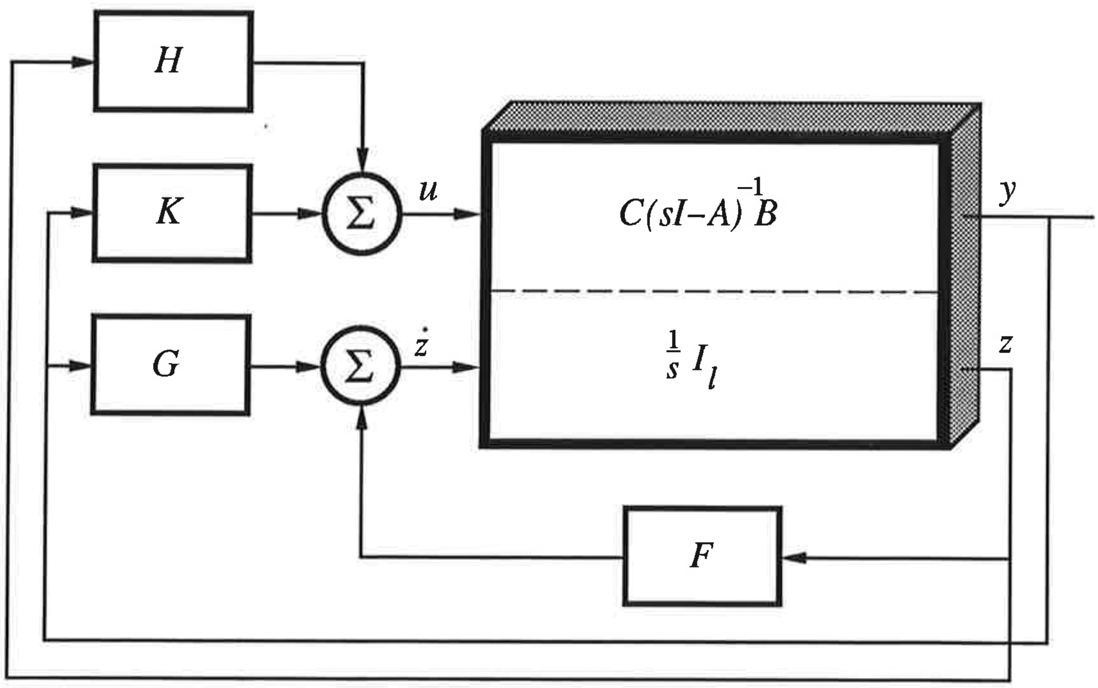

Figure 3. Dynamic feedback considered as static feedback.

More formally: Consider the following dynamic feedback problem: Given the plant

$$\begin{aligned} \dot{x} &= Ax + Bu, \qquad x \in \mathbb{R}^n, \qquad u \in \mathbb{R}^m \\ y &= Cx, \qquad y \in \mathbb{R}^p \end{aligned} \tag{MIMO}$$

and the controller

$$\begin{aligned} \dot{z} &= Fz + Gy, \qquad z \in \mathbb{R}^l \\ u &= Hz + Ky \end{aligned}$$

It is easy to see that this is equivalent to the static feedback problem

$$\begin{aligned} \dot{\tilde{x}} &= \tilde{A}\tilde{x} + \tilde{B}\tilde{u} \\ \tilde{y} &= \tilde{C}\tilde{x} \\ \tilde{u} &= \tilde{K}\tilde{y} \end{aligned} \tag{MIMOA}$$

where

$$\tilde{x} = \begin{pmatrix} x \\ z \end{pmatrix} \qquad \tilde{u} = \begin{pmatrix} u \\ \dot{z} \end{pmatrix} \qquad \tilde{y} = \begin{pmatrix} y \\ z \end{pmatrix}$$

and

$$\tilde{A} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \qquad \tilde{B} = \begin{pmatrix} B & 0 \\ 0 & I \end{pmatrix} \qquad \tilde{C} = \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix} \qquad \tilde{K} = \begin{pmatrix} K & H \\ G & F \end{pmatrix}$$

Remark 3.1. This observation might seem very powerful at least at first sight, but note the highly non-generic nature of $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$. This means e.g. that results on generic pole placement by static output feedback, see [Brockett-Byrnes], [Byrnes], do not translate at all. □

4. Estimation of the Norm of the State

In this section a lemma is proven, which gives an estimate of the norm of the state x of (MIMO), expressed in the L^2 norm of y and u. The lemma has a simple corollary, which implies that, under mild conditions, to show that an adaptive algorithm converges and stabilizes the plant, it is enough to show that the controller stays bounded. First we give the continuous time version.

LEMMA 4.1. *Assume that the linear system* (MIMO) *is observable. Then:*

(i) *For all $x(0)$, there are constants c_0 and c_1 such that*

$$\|x(t)\|^2 \leq c_0 + c_1 \left(\int_0^t \|y(\tau)\|^2 \, d\tau + \int_0^t \|u(\tau)\|^2 \, d\tau \right)$$

for all $u(.)$, and $t \geq 0$. Here c_0 does not depend on t or u; and c_1 does not depend on t, $u(.)$ or $x(0)$.

(ii) *For $T > 0$, c_1 can be taken so*

$$\|x(t)\|^2 \leq c_1 \left(\int_{t-T}^t \|y(\tau)\|^2 \, d\tau + \int_{t-T}^t \|u(\tau)\|^2 \, d\tau \right)$$

for all t, $u(.)$, and $x(t-T)$.

Remark 4.2. In (ii) we can consider c_1 as a function of T. This function can clearly be taken continuous and decreasing. □

Remark 4.3. Note that, for t bounded from below (i) follows trivially from (ii). Also note that the c_0-term is necessary if and only if we allow arbitrary small $t > 0$. □

Remark 4.4. It is not possible to improve the result by deleting the integral of u. A simple counter-example can be constructed by letting (MIMO) be an integrator, the initial state $x(0) = 0$, and the input $u(\tau) = \delta\,(\tau - (t - \varepsilon))$, for some small $\varepsilon > 0$. Choose coordinates in the state space so that $x = y$. Then clearly $x(t) = 1$, and $\int y^2 d\tau = \varepsilon$, so by letting $\varepsilon \to 0$, we arrive at a contradiction. The lemma is true without the u-dependent term if and only if $G(s)$ has a proper left-inverse. □

Proof. In an obvious operator notation we have

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau =: L_1^t x(0) + L_2^t u(\,.\,) \qquad (\smile)$$

$$y(\,.\,) = L_3 x(0) + L_4 u(\,.\,) \qquad (\frown)$$

where L_1^t, L_2^t, L_3, and L_4 are bounded linear operators between suitable Hilbert spaces. We first prove (ii). Let $T > 0$ be given. By using time invariance, it is enough to show (ii) for $t = T$. From observability, $(\frown)$ can be solved with respect to $x(0)$, i.e. $x(0)$ is the image of $y(\,.\,)$ and $u(\,.\,)$ under a continuous linear mapping. Inserted into $(\smile)$, this proves (ii).

By Remark 4.3, it only remains to show (i) for small t, say $t \leq 1$. For this, note that the operators $\mathcal{L}_1 = \{L_1^t : 0 \leq t \leq 1\}$ and $\mathcal{L}_2 = \{L_2^t : 0 \leq t \leq 1\}$ are uniformly bounded by, say, k_1 and k_2. From these observations, (i) follows (for $t \leq 1$) from $(\smile)$, since $\int_0^t \|u\|^2\,d\tau \leq \int_0^t \left(\|u\|^2 + \|y\|^2\right)\,d\tau$. The proof is finished. ■

A Useful Corollary

The lemma has the following immediate corollary, which will be used in the connection with adaptive stabilizers. We make the following definition:

Definition 4.5. A function $f : \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ will be called *L^2-compatible* if it is satisfies a Lipschitz-condition and there exists a constant $c > 0$ such that $f(y, u, k, t) \geq c\big(\|y\|^2 + \|u\|^2\big)$ for all k and all t. □

The name is motivated by the fact that for f being an L^2-compatible function, we can estimate the L^2-norm of (y, u) by the integral of f, as will be done in the proof of the following corollary.

COROLLARY 4.6. *Consider the plant* (MIMO), *and let* $u(\,.\,)$ *be a continuous time-function. Let* k *satisfy*

$$\dot{k} = f(y, u, k, t), \quad k(0) = k_0$$

where f is an L^2-compatible function. Then, if k converges to a finite limit k_∞ as $t \to \infty$, it holds that $\|x(t)\| \to 0$ as $t \to \infty$.

Proof. Clearly

$$\int_0^\infty \left(\|y\|^2 + \|u\|^2\right) dt \leq \frac{1}{c}\int_0^\infty f(y, u, k, t)\,dt = \frac{1}{c}\,(k_\infty - k_0) < \infty$$

Thus, for any $T > 0$, the right hand side of (ii) in Lemma 4.1 approaches zero when t approaches infinity. The corollary follows. ■

Remark 4.7. In previous 'universal' stabilizing algorithms, the step of showing that $x(t) \to 0$ as $t \to \infty$ has involved a minimum phase argument. This is not required here. □

5. Switching Function Controllers

In this section we will deal with the following problem: We want to adaptively stabilize an unknown plant G of type (MIMO), for which we know that G belongs to a set $\mathcal{G}$. Here $\mathcal{G}$ is a set of plants for which there exists a finite or countable set of controllers $\mathcal{K}$, such that for any $G \in \mathcal{G}$, there is at least one controller $K \in \mathcal{K}$ such that the control law $u = Ky$ will stabilize G.

A heuristically appealing algorithm for stabilizing the unknown plant G would be to try each one of the K's for ε units of time, until we find one that stabilizes the system. It is shown in [Mårtensson 1986] that this is possible if and only if we know a bound on the McMillan degree of the plants belonging to $\mathcal{G}$. Instead we try each one of the controllers for some time, according to some criterion, in a way that will hopefully converge, and thus will switch among the controllers only a finite number of times. A *switching function* is a criterion of this type.

The concept of switching function was first introduced in [Willems-Byrnes], where the set of plants $\mathcal{G}$ under consideration was single-input, single-output, minimum phase plants of relative degree one. In [Byrnes-Willems] this was generalized to multivariable plants satisfying analogous conditions.

In the remainder of this section, we introduce the pertinent concepts formally, and give a result on switching function based adaptive stabilization.

Definitions

Definition 5.1. Let $s(k)$ be a function of a real variable, and $\{\tau_i\}_{i=0}^{\infty}$ a sequence of increasing real numbers. For $r = 2, 3, \ldots, \aleph_0$, we shall say that $s(k)$ is a *switching function of rank r* with *associated switching points* $\{\tau_i\}$, if $s(k)$ is constant for $k \notin \{\tau_i\}$, and, for all $a \in \mathbb{R}$, $s\big(\{k \geq a\}\big) = \{1, \ldots, r\}$. Further, just as a notational convenience, we require a switching function to be right continuous. □

Remark 5.2. Note that it follows from the definition that infinity is the only limit point of the sequence $\{\tau_i\}$. □

By switching function controller we shall mean the following.

Definition 5.3. For $r = 2, 3, \ldots, \aleph_0$, let $\mathcal{K} = \{K_1, \ldots, K_r\}$ be a set of controllers, with card $\mathcal{K} = r$. Let f be a Lipschitz-continuous function and $s(k)$ a switching law of rank r. A controller of the type

$$\begin{aligned} u &= K_{s(k)} y \\ \dot{k} &= f(y, u, k, t) \end{aligned} \qquad \text{(SFC)}$$

will be called a *switching function controller.* □

Remark 5.4. Note that in general the control law $u = K_i y$ must be interpreted in an operator-theoretic way, not as a matrix multiplication. □

Remark 5.5. The way (SFC) is written requires all the controllers $K_1, \ldots, K_r$ to be simultaneously connected to the output of the plant, while the switching law chooses which controller's output to connect to the plants input, at least if the K_i's contain dynamics. For r large or infinite, this is clearly not a practical way of implementing a controller. However, if all the controllers have a (not necessarily minimal) realization on a state space of a certain dimension, then this difficulty can be circumvented by considering the augmented plant as in Section 3, and considering the controllers as static controllers. □

For further reference, we shall make clear what we mean by a set of controllers stabilizing a set of plants in some sense.

Definition 5.6. Let f an L^2-compatible function, $\mathcal{G}$ a set of plants of the type (MIMO), all of which having the same number of inputs and outputs, and $\mathcal{K}$ a set of controllers of compatible dimensions. For $k_0 \in \mathbb{R}$, let k be the unique solution to $\dot{k} = f(y, u, k, t)$, $k(0) = k_0$. We shall say that $\mathcal{K}$ *is stabilizing for* $\mathcal{G}$ *with respect to* f (or *is f-stabilizing for* $\mathcal{G}$) if the following holds: For any plant $G \in \mathcal{G}$ there is a controller $K \in \mathcal{K}$ and constants c, T such that the control law $u = Ky$ will stabilize G in the sense that

$$\int_{t_0}^{\infty} f(y, u, k, t)\, dt \leq c\|x(t_0)\|^2$$

for all $x(0) \in \mathbb{R}^n$ and for all $k_0 \in \mathbb{R}, t_0 \geq T$. □

Remark 5.7. In particular, the left hand side stays finite, so it follows from Corollary 4.6 that $x(t) \to 0$ as $t \to \infty$. It also follows that the solution to the differential equation is indeed globally defined. □

Remark 5.8. By considering singleton sets in the definition, it is clear what we shall mean by the statement *the controller K stabilizes the plant G with respect to f.* □

The Main Result on Switching Functions

With the machinery developed so far, we can now easily prove the following results on switching function controllers.

THEOREM 5.9. *Suppose that f is an L^2-compatible function, and that the set of controllers $\mathcal{K}$ is f-stabilizing for the set of plants $\mathcal{G}$. Then there is a sequence $\sigma = \{\tau_i\}$ such that for $s(k)$ any switching function of rank equal to* card $\mathcal{K}$, *with associated switching points $\{\tau_i\}$, the control law* (SFC) *will stabilize any plant $G \in \mathcal{G}$ in the sense that for all $x(0), k(0)$, it holds that $\|x(t)\| \to 0$ as $t \to \infty$, while k converges to a finite limit. Further, there is a 'universal' switching point sequence σ, independent of the individual set $\mathcal{G}$.*

Proof. The steps in the proof are the following: To say that the theorem is false is to say that for all switching sequences, there is a switching function with the stated properties such that stabilization does not take place. It will be shown that, if stabilization does not take place, the sequence $\{\tau_i\}$ has to satisfy a certain requirement, depending on $\mathcal{G}$, namely ($\mathcal{L}$) below. A sequence σ is given, with the property that for all allowed $\mathcal{G}$, the requirement is violated. We conclude that with this very sequence stabilization takes place, which will establish the theorem.

From Corollary 4.6, and since k is increasing, it follows that in order to show stabilization it is enough to show that k is bounded. By the definition of switching function, this is equivalent to the statement that s, considered as a function of time, only switches a finite number of times. So we assume that this is not the case, and investigate the implications of this assumption.

Consider an arbitrary, but fixed, $G \in \mathcal{G}$. Say that controller K_i is f-stabilizing for G, and that the controller K_i is used with start at time t_0. That is, $k(t_0) = \tau_j$, where $s(\tau_j) = i$. By the assumptions, this will happen for arbitrarily large k and t. Therefore, with T as in Definition 5.6, we shall make the assumption that $t_0 \geq T$.

The assumption that s switches an infinite number of times implies that we will reach the next switching point τ_{j+1} after a finite time. But this is exactly the statement that

$$\int_{t_0}^{\infty} f(y,u,k,t)\,dt \geq \tau_{j+1} - \tau_j \qquad (¿)$$

where the left hand side, by assumption finite, is evaluated as if the controller K_i was used forever. We will show that the sequence $\{\tau_i\}$ can be taken in a way so that (¿) cannot be satisfied for j sufficiently large, which will prove the theorem.

By definition of f being L^2-compatible, there is a c, so that the left hand side of (¿) can be estimated as

$$\int_{t_0}^{\infty} f(y,u,k,t)\,dt \leq c\|x(t_0)\|^2$$

Using the same argument as in the proof of Corollary 4.6, it follows from Lemma 4.1, part(i), that for all $x(0)$, there exist constants c_0 and c_1 such that

$$\|x(t)\|^2 \leq c_0 + c_1 k(t)$$

for all t. Substituting $t = t_0$, $k = \tau_j$, and combining the last two estimates, we see that a necessary condition for (¿) to be satisfied, is that

$$\tau_{j+1} - \tau_j \leq cc_0 + cc_1\tau_j \qquad (£)$$

But there are sequences $\{\tau_i\}$ such that, for any c, c_0, c_1, the statement (£) will be false for all sufficiently large j. This is the case e.g. for the sequence defined by

$$\tau_{j+1} = \tau_j^2, \qquad i = 2, 3, \ldots$$
$$\tau_1 = 2$$

Therefore, with a switching sequence like this chosen, the assumption of s to switch infinitely many times leads to a contradiction. Since G was arbitrary, the proof is complete. ∎

Proof of Sufficiency in Theorem 2.1

The proof is a fairly straightforward application of Theorem 5.9. Consider a controller in the spirit of Section 3, namely as a constant $M \times P$-matrix, where $M := m + l$, and $P := p + l$. The set of controllers $\mathcal{K}$ is taken to be all such with rational coefficients, i.e. $\mathcal{K} := \mathbb{Q}^{M \times P}$. Let f be defined as $f(y,u,k,t) = \|y\|^2 + \|u\|^2$. This is an L^2-compatible function. A stabilizing controller places the closed loop poles in the open left half plane. The poles depend continuously of the parameters in the controller. Since $\mathcal{K}$ is dense in the space of all controllers of order l, i.e. $\mathbb{R}^{M \times P}$, $\mathcal{K}$ is thus f-stabilizing for $\mathcal{G}$. Theorem 5.9 establishes the existence of a switching function such that the corresponding switching function controller (SFC) stabilizes any plant in $\mathcal{G}$. This completes the proof. ∎

Remark 5.10. By some additional effort, an explicit algorithm based on the ideas in the proof can be constructed. □

In [Mårtensson 1986], it is shown that the controller can also be taken to be continuous by 'smoothing-out' the discontinuities. Another approach is presented in the next section.

6. "The Turing Machine" of Universal Stabilizers

In this section we will consider the problem of adaptively stabilizing the plant (MIMO), given only the a priori information that an integer l is known, such that there exists a fixed linear time-invariant controller of order l that will stabilize the system. An explicit algorithm for this will be given. This will be given only very briefly, without proof. A more detailed discussion, including a discrete time version, is given in [Mårtensson 1986]. The proof is also given in [Mårtensson 1985].

As shown in Section 3, it suffices to consider adaptive control based on static feedback. A (fixed) controller is then nothing but a matrix in $\mathbb{R}^{M\times P}$, where M and P denotes the number of inputs and outputs to the augmented plant (MIMOA). For the sequel, we assume that this augmentation has been done, and therefore we only consider static feedback. Since a (fixed) controller achieving internal stability to the closed loop system places all the eigenvalues in the open left-half plane, (or the open unit disc) and these depend continuously on the parameters of the controller, there is an open set in parameter space yielding a stable system. Equip $\mathbb{R}^{M\times P}$ with the norm

$$\|A\|^2 = \sum_{i,j} (A)_{ij}^2$$

Thus we identify $\mathbb{R}^{M\times P}$, as a normed space, with $\mathbb{R}^{MP}$, equipped with the Euclidean norm. For the rest of this section, we let $\|\,.\,\|$ denote the this vector norm, or the corresponding induced matrix norm. Partition $\mathbb{R}^{M\times P} = \mathbb{R}^+ \times S^{MP-1}$ in a natural way, namely by dividing out the norm of every non-zero matrix. S^{MP-1} is now the unit sphere in a normed space of controllers. Let the controller be

$$\tilde{u} = g(h(k))N(h(k))\tilde{y} \tag{1}$$

$$\dot{k} = \|\tilde{y}\|^2 + \|\tilde{u}\|^2 \tag{2}$$

where

$$N(h) \text{ is `almost periodic' and dense on } S^{MP-1} \tag{3}$$

while h and g are continuous, scalar functions satisfying

$$h(k) \nearrow \infty, \qquad k \to \infty \tag{4}$$

$$\text{There exists an } a \text{ such that } \left|\frac{dg}{dh}\right| < a \tag{5}$$

$$g(\{\alpha\nu + (\beta,\gamma)\}_{\nu=n}^{\infty}) = \mathbb{R}^+ \quad \text{for } n \in \mathbb{Z}, \quad \alpha \neq 0, \quad \gamma > \beta \tag{6}$$

$$kg\big(h(k)\big)\frac{dh}{dk} \to 0, \qquad k \to \infty \tag{7}$$

THEOREM 6.1. *Consider the minimal plant* (MIMO). *Assume that l is chosen so that there exists a fixed linear stabilizing controller, and that the augmentation to (MIMOA) has beed done. The controller (1) – (2), subject to (3) – (7), will then stabilize the system in the sense that*

$$\big(x(t), z(t), k(t)\big) \to (0, 0, k_\infty) \qquad \text{as } t \to \infty$$

where $k_\infty < \infty$.

One set of functions satisfying (4) – (7) is

$$h(k) = \sqrt{\log k}, \qquad k \geq 1$$

$$g(h) = \sqrt{h}\left(\sin\sqrt{h} + 1\right)$$

The construction of the function $N(h)$ is a standard exercise in calculus on manifolds. One such is given explicitly in the references cited above.

7. Setpoint Stabilization

In this section it will be shown how to introduce integrators in the loop, thereby being able to track a constant reference signal with error approaching zero asymptotically. The problem is as follows: Let $\mathcal{G}$ be a set of plants as before, and $r \in \mathbb{R}^p$ a given constant (a reference value). We want to find a controller K such that for all $G \in \mathcal{G}$ it holds that

$$\begin{aligned} x &\to \hat{x} \qquad (= \text{constant}) \\ y &\to r \\ z &\to \hat{z} \\ k &\to k_\infty \end{aligned}$$

as $t \to \infty$.

Tracking with Zero Error Asymptotically

Every engineer knows that you cannot track a constant reference signal with zero error asymptotically without having integrators in the loop*. The analogous statement of course applies to multivariable plants. Conversely, with integrators in every loop, the asymptotic tracking error is zero, provided the closed loop system is stable. This shall mean that every fixed linear combination of rows or columns of the matrix $G(s)$ has a pole at the origin.

The construction for adaptively stabilizing a plant, with a constant reference signal $r(t) \equiv r_0$ is very simple: We just put the diagonal 'precompensator' $\bar{K} = s^{-1}I_m$ in front of the plant. For the sequel, consider the problem of adaptively stabilizing the 'plant' $\tilde{G}(s) := G(s)\bar{K}(s)$ instead. This is depicted in Figure 4.

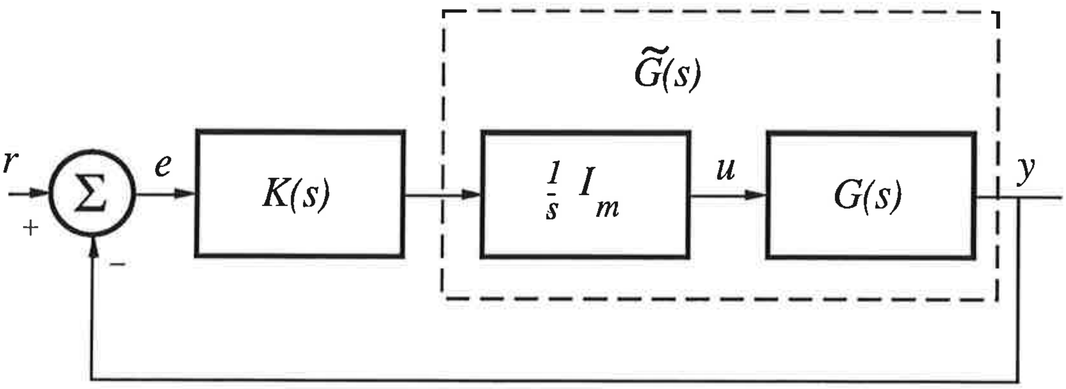

Figure 4. Setpoint Stabilization by Introducing Integrators.

More precisely, we have the following result.

* Quick and *dirty* proof: $y(\infty) = r(\infty) \Longleftrightarrow g(0)/(1+g(0)) = 1 \Longleftrightarrow g(0) = \infty$ ∎

THEOREM 7.1. *Assume that the controller K stabilizes the plant G in the usual sense. Let $r \in \mathbb{R}^p$ be given. Suppose that there exists a unique $\hat{x}$ such that*

$$0 = A\hat{x}$$
$$r = C\hat{x}$$

Let K operate on $-e := y - r$ instead of y. Then, as $t \to \infty$ it holds that $y \to r$, $x \to \hat{x}$, and $k \to k_\infty$.

Remark 7.2. The uniqueness follows automatically from observability. □

Proof. We can write

$$\frac{d}{dt}(x - \hat{x}) = A(x - \hat{x}) + Bu$$
$$y - r = C(x - \hat{x})$$

So, assuming we have a proof of a theorem saying that the assumptions are satisfyed, we only have to substitute all occurances of x by $x - \hat{x}$, and all occurances of y by $y - r$ in order to construct a proof of the above theorem for the case in question. So Theorem 7.1 is really a meta-theorem on adaptive stabilization. ■

The most natural use of Theorem 7.1 is in the form of the following corollary:

COROLLARY 7.3. *Assume K stabilizes $G(s) = \frac{1}{s}G_1(s)$, where $\det G \neq 0$. Then with error feedback K will also do set-point stabilization for any $r \in \mathbb{R}^p$.*

Extensions and Comments

Everyone with experience of practical control engineering knows that plants of high relative degree are very hard to control manually, but often fairly simple to control with simple controllers, such as standard PID-controllers. Something similar is true about adaptive control. We need some extra dynamics in our controllers, that is all. By preceding the plant by integrators as in the construction above, the minimal order of a stabilizing controller might increase. A classical control engineer would say that we do this at the expense of a decrease of the phase by 90°, and thus need some extra phase advancing to stabilize the plant.

The same argument may be used to introduce multiple integrators in the loop, thus being able to track ramps of higher order.

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