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It is well known that the square, minimum phase plant $\dot{x} = Ax + Bu$, controlled by the control law $u = -ky$ will be stable for all sufficiently large values of the scalar k, provided that all eigenvalues of CB are in the (open) right half plane. This result is generalized: a one-parameter controller is given, capable of stabilizing any minimum phase plant with "positive instantaneous gain", with any "relative degree" up to a certain bound. By using a certain switching among the controllers that arize in this way, this gives rise to an adaptive algorithm. This algorithm will stabilize any minimum phase plant with positive instantaneous gain, if only a bound on the "relative degree" known.

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Abstract: It is well known that the square, minimum phase plant $\dot{x} = Ax + Bu$, controlled by the control law $u = -ky$ will be stable for all sufficiently large values of the scalar k, provided that all eigenvalues of CB are in the (open) right half plane. This result is generalized: a one-parameter controller is given, capable of stabilizing any minimum phase plant with "positive instantaneous gain", with any "relative degree" up to a certain bound. By using a certain switching among the controllers that arize in this way, this gives rise to an adaptive algorithm. This algorithm will stabilize any minimum phase plant with positive instantaneous gain, if only a bound on the "relative degree" known.

1. Introduction

Recently there has been a fair amount of effort devoted to adaptive stabilization of minimum-phase, scalar or multivariable linear systems. An algorithm which did not require knowledge of the "instantaneous gain" was first published in [Nussbaum]. This was generalized to scalar systems of arbitrary degree in [Willems-Byrnes], where the set of plants considered were scalar minimum phase plants of relative degree 1. In [Byrnes-Willems] an algorithm was outlined, capable of stabilizing multivariable minimum-phase plants of "relative degree one" (i.e. $\det CB \neq 0$).

An algorithm based on incorporating a sign-switching "Nussbaum-function" in a standard MRAC-scheme has been published in [Mudgett-Morse]. This algorithm required knowledge of the relative degree and a bound on the order of the plant.

[Saberi-Khalil] presented an algorithm for adaptive stabilization of scalar minimum phase plants with the relative degree arbitrary but known, and no bound on the order of the plant required.

[Mårtensson 1985,1986] presented algorithms capable of stabilizing any plant stabilizable by a fixed, linear, time-invariant controller of the same dimension as the linear dynamics in the adaptive controller.

The present paper returns to the high-gain based algorithms. The main contribution is a linear time-invariant one-parameter controller, which, given any plant with "positive instantaneous gain" and "relative degree" less than a certain limit r^*, will stabilize it for sufficiently large value of the parameter. (The concept of *positive instantaneous gain* and *relative degree* for square multivariable (or scalar) plants will be strictly defined in the sequel.) In the multivariable case a (generically satisfied, [Byrnes-Stevens]) condition has been imposed.

The emphasize of the paper is on the one-parameter controller. By invoking a powerful, general switching-function controller result proved in [Mårtensson 1986], it is easy to make an adaptive stabilizing algorithm.

High-gain based adaptive stabilization without the requirement of positive instantaneous gain is solved in [Mårtensson 1986].

Section 2 presents the one-parameter controller for scalar plants. In Section 3, this is generalized to square, multivariable plants. The following section ties this result together with the general switching function controller results to give an adaptive algorithm. Finally, Section 5 contains an example and a simulation of a resulting controller applied to a model of the ball-and-beam process.

2. High-Gain Stabilization of Scalar Plants of Arbitrary Relative degree

In this section we will give a one parameter controller, with the property that it will stabilize all minimum phase scalar plants with "positive instantaneous gain", with *any* relative degree up to a limit r^*.

Recall that a polynomial $p(s) \in \mathbb{R}[x]$ is called a *Hurwitz-polynomial* if $p(s) \neq 0$ for $\operatorname{Re} s \geq 0$. Consider the set of plants

$$\mathcal{G} = \Big\{ g(s) \in \mathbb{R}(x) : g(s) = \frac{n(s)}{d(s)}; \quad n, d \in \mathbb{R}[x]; \quad \deg d - \deg n \leq r^*; \quad d \text{ is monic};$$
$$n \text{ is Hurwitz; and the highest non-zero coefficient of } n \text{ is positive} \Big\}$$

For l a non-negative integer, introduce the one-parameter family of l'th order controllers

$$c(s,k) = k^{\alpha} \frac{h(s)}{r(s,k)}$$

where

$$r(s,k) = (s + k^{2^l}) \dots (s + k^4)(s + k^2)$$
$$\alpha = \sum_{j=1}^{l} 2^j + 1 = 2^{l+1} - 1$$

and $h(s)$ is an arbitrary monic Hurwitz polynomial of degree l. For $l = 0$, $c(s,k)$ reduces to k.

PROPOSITION 2.1. *If $l \geq r^* - 1$, then for any plant in $\mathcal{G}$ there is a k_0 such that the closed loop system $g(s)/(1 + c(s,k)g(s))$ is stable for all $k \geq k_0$.*

Since we will make heavy use of this result in the sequel, we reformulate it in a more mathematical setting. It is easy to see that the latter formulation implies the former.

Notation. By writing $x \sim k^\gamma$ we shall mean that the quantity x is asymptotically proportional to k^γ as $k \to \infty$, i.e.

$$\lim_{k \to \infty} \frac{x}{k^\gamma} \neq 0, \infty$$

In particular, the limit exists. □

PROPOSITION 2.2. *Let α, $r(s,k)$, and $h(s)$ be as above, and $\mu \in \mathbb{C}^+$. Suppose that $n(s)$, $d(s) \in \mathbb{R}[x]$ are such that $n(s)$ is Hurwitz, $\deg d(s) = n$, the highest non-zero coefficients of d and n have the same sign, and that $1 \leq \deg d(s) - \deg n(s) = r \leq l+1$. Then the polynomial*

$$p_k(s) := r(s,k)d(s) + \mu k^\alpha h(s)n(s)$$

will have its zeros in the left half plane for all sufficiently large k. More precisely, denoting the zeros by $s_1, \ldots, s_{n+l}$, it holds that $s_1 \sim k^{2^l}, \ldots, s_{r-1} \sim k^{2^{l-r+2}}, s_r \sim k^{1+2+\ldots+2^{l-r+1}}$, while $s_{r+1}, \ldots, s_{n+l}$ will converge to the zeros of $h(s)n(s)$.

Remark 2.3. Note that s_r is slower than the rest of the unbounded zeros. □

Proof. Assume without restriction that $d(s)$ is monic. Since $\alpha > \sum_1^l 2^j$ it follows that the second term of $p_k(s)$ will dominate over the first on every bounded subset of $\mathbb{C}$. The claim on the finite zeros now follow e.g. from Rouché's theorem, see [Ahlfors]. For the infinite zeros, the claim follows from a consideration of the different powers of k in the coefficients of $s^{n+l-1}, \ldots, s^{n+l-r}$ in $p_k(s)$. By the use of the Newton-Puiseux diagram, see e.g. [Postlethwaite-MacFarlane], this follows straightforwardly. The proof is complete. ■

Remark 2.4. The purpose behind the unorthodox choice of the polynomial $r(s,k)$ is to separate all closed loop poles in different powers of k, thereby making it harmless to over-estimate the relative degree of the plant. A more naive approach, inspired e.g. by [Zames-Bensoussan], and used in [Saberi-Khalil], is the following: Let

$$c(s,k) = k\frac{h(s)}{(Ts+1)^l}$$

where $T \to 0$ as $k \to \infty$, in some sense fast compared to k. This approach does not work in general if we over-estimate the relative degree of $g(s)$, i.e. if $l > r - 1$. Say that $l = r$. Then the infinite branches are asymptotically determined by

$$(Ts+1)^l + k = 0$$

i.e.

$$s_i = \frac{-1 + \varepsilon_i \sqrt[l]{-k}}{T} + o(\sqrt[l]{k})$$

where $\varepsilon_1, \dots, \varepsilon_l$ are the primitive l-th roots of unity. For $l \geq 3$, some s_i must have a positive real part. □

3. Multivariable Extensions

In order to give multivariable extensions of the previous result, some new terminology is introduced. Consider the multivariable linear plant

$$\begin{aligned} \dot{x} &= Ax + Bu; \qquad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m \\ y &= Cx \qquad y \in \mathbb{R}^m \\ G(s) &= C(sI - A)^{-1}B \end{aligned} \tag{MIMO}$$

Definition 3.5. We shall say that $G(s)$ has ***relative degree r*** if all infinite branches of the root locus $s_i(k)$ is of order $\leq r$, and equality holds for some i.

If there is a $s_0 \in \mathbb{R}^+$ such that for some $\alpha > 0$ it holds that $\arg \lambda_i G(s) \in [-\frac{\pi}{2} + \alpha, \frac{\pi}{2} - \alpha]$ for $i = 1, \dots, m$ and all real $s \geq s_0$ we will call $G(s)$ a plant with *positive instantaneous gain.*

We shall say that (MIMO) satisfies *Assumption Simple Null Structure (SNS)* if all infinite branches $s_i(k)$ are of integer order in k. □

Note that these definitions agree with the usual single-input, single-output concepts. Also observe that (MIMO) will, by the initial value theorem for the Laplace transform, have positive instantaneous gain if and only if the impulse response matrix $H(t) := \mathcal{L}^{-1}G(s)$ has its eigenvalues in a fixed, closed sector in the right half plane for sufficiently small $t > 0$. This property accounts for the name.

For l a non-negative integer, we introduce the most natural multivariable generalization of the controller $c(s,k)$, namely

$$C(s,k) = c(s,k)I_m$$

The following result holds.

PROPOSITION 3.6. *If $l \geq r^* - 1$, then for any G in the set*

$$\mathcal{G} = \Big\{ G(s) \in \mathbb{R}^{m\times m}(x) : r \leq r^*; \det G(s) \neq 0 \textit{ for } \operatorname{Re} s \geq 0; \\ G(s) \textit{ has positive instantaneous gain}; \\ G(s) \textit{ satisfies assumption SNS} \Big\}$$

for sufficiently large, fixed k, the control law $u = -C(s,k)y$ will stabilize G.

Proof. The poles of the closed loop system are given as the zeros of $\det(I+c(s,k)G(s))$. These are by definition the s for which $-1/c(s,k)$ is an eigenvalue of $G(s)$.

By a standard argument based on e.g. Rouché's theorem, the finite endpoints of the root locus as $k \to \infty$ are exactly the zeros of $\det c(s,k)G(s)$, which by assumption resides in the left half plane.

Let $d(s)$ be the monic least common multiple of the denominators of the elements in $G(s)$, i.e. the characteristic polynomial of the plant. Since the eigenvalues λ_i of $G(s)$ are the zeros of the polynomial $q(s,\lambda) := \det(\lambda d(s)I - d(s)G(s))$ and $G(s)$ is strictly proper, the eigenvalues have an asymptotic expansion, see e.g. [Brockett-Byrnes],

$$\lambda_i = \mu_i s^{-p_i/q_i} + o(s^{-p_i/q_i}), \qquad p_i, q_i \in \mathbb{Z}^+, \qquad i = 1, \ldots, m$$

when $\mathbb{R} \ni s \to \infty$. By assumption SNS, the quantities p_i/q_i are all integers. Since $G(s)$ is assumed to have positive instantaneous gain, it follows that $\mu_i \in \mathbb{C}^+$. By the assumptions, $p_i/q_i \leq r$ for $i = 1, \ldots, m$. The infinite branches of the root locus are therefore given as the unbounded zeros of

$$-\frac{1}{c(s,k)} = -\frac{r(s,k)}{k^\alpha h(s)} = \lambda_i = \mu_i s^{-p_i/q_i} + o(s^{-p_i/q_i})$$

for $i = 1, \ldots, m$. By Proposition 2.2, these are in the the left half plane for k sufficiently large. This completes the proof. ∎

Remark 3.7. Other treatments of high-gain root locus can be found in e.g. [Byrnes-Stevens], [Brockett-Byrnes], and [Sastry-Desoer]. □

4. Adaptive Stabilization

The results from Sections 2 and 3 provided us with a possibility to stabilize a large class of plants by just turning up the parameter in a one-parameter controller. It is tempting to do this automatically, on-line—that is, to tie these ingredients together to make an adaptive stabilizing controller. This will be done in the present section. The section will be only a brief overview of some known results, with no proofs. For a fuller presentation, the reader is referred to [Mårtensson 1986].

The basic glue for sticking the parts together will be the formalism of switching function controllers, to which the following subsection will be devoted.

Switching Function Controllers

In this subsection we will deal with the following problem: We want to adaptively stabilize an unknown plant G of type (MIMO), for which we know that G belongs to a set $\mathcal{G}$. Here $\mathcal{G}$ is a set of plants for which there exists a finite or countable set of controllers $\mathcal{K}$, such that for any $G \in \mathcal{G}$, there is at least one controller $K \in \mathcal{K}$ such that the control law $u = Ky$ will stabilize G.

A heuristically appealing algorithm for stabilizing the unknown plant G would be to try each one of the K's for ε units of time, until we find one that stabilizes the system. It is shown in [Mårtensson 1986] that this is possible if and only if we know a bound on the McMillan degree of the plants belonging to $\mathcal{G}$. Instead we try each one of the controllers for some time, according to some criterion, in a way that will hopefully converge, and thus will switch among the controllers only a finite number of times. A *switching function* is a criterion of this type.

The concept of switching function was first introduced in [Willems-Byrnes], where the set of plants $\mathcal{G}$ under consideration was single-input, single-output, minimum phase plants of relative degree one. In [Byrnes-Willems] this was generalized to multivariable plants satisfying analogous conditions.

Definitions

Definition 4.8. Let $s(k)$ be a function of a real variable, and $\{\tau_i\}_{i=0}^{\infty}$ a sequence of increasing real numbers. For $r = 2, 3, \ldots, \aleph_0$, we shall say that $s(k)$ is a *switching function of rank* r with *associated switching points* $\{\tau_i\}$, if $s(k)$ is constant for $k \in [\tau_i, \tau_{i+1})$, $i = 1, 2, \ldots$ and, for all $a \in \mathbb{R}$, $s(\{k \geq a\}) = \{1, \ldots, r\}$. □

Remark 4.9. Note that it follows from the definition that infinity is the only limit point of the sequence $\{\tau_i\}$. □

By switching function controller we shall mean the following.

Definition 4.10. For $r = 2, 3, \ldots, \aleph_0$, let $\mathcal{K} = \{K_1, \ldots, K_r\}$ be a set of linear, time invariant controllers, with card $\mathcal{K} = r$. Let $s(k)$ be a switching law of rank r. A controller of the type

$$\begin{aligned} u &= K_{s(k)} y \\ \dot{k} &= f(y, u, k, t) \end{aligned} \tag{SFC}$$

will be called a *switching function controller.* □

Remark 4.11. Note that in general the control law $u = K_i y$ must be interpreted in an operator-theoretic way, not as matrix multiplication. □

Remark 4.12. The way (SFC) is written requires all the controllers $K_1, \ldots, K_r$ to be simultaneously connected to the output of the plant, while the switching law chooses which controller's output to connect to the plants input, at least if the K_i's contain dynamics. For r large or infinite, this is clearly not a practical way of implementing a controller. However, if all the controllers have a (not necessarily minimal) realization on a state space of a certain dimension, then this difficulty can be circumvented by considering the augmented plant as in [Mårtensson 1985] or [Mårtensson 1986], and considering the controllers as static controllers. □

We shall say that *a set of controllers $\mathcal{K}$ is stabilizing for a set of plants $\mathcal{G}$* if for any $G \in \mathcal{G}$ there is a controller $K \in \mathcal{K}$ such that K stabilizes G. We can now formulate the following results on switching function controllers.

THEOREM 4.13. *Suppose that the set of controllers $\mathcal{K}$ is stabilizing for the set of plants $\mathcal{G}$. Then there is a sequence $\sigma = \{\tau_i\}$, e.g. defined by $\tau_1 = 2, \tau_{i+1} = \tau_i^2$, such that for $s(k)$ any switching function of rank equal to card $\mathcal{K}$, with associated switching points $\{\tau_i\}$, the control law (SFC) will stabilize any plant $G \in \mathcal{G}$ in the sense that for all $x(0), k(0)$, it holds that $\|x(t)\| \to 0$ as $t \to \infty$, while k converges to a finite limit.*

Summing up, we have the following result, containing almost all results of this paper:

THEOREM 4.14. *Let $\mathcal{G}$ and $c(s,k)$ be as before, and $l \geq r^* - 1$. Then there is an integer M and a sequence of switching points $\{\tau_i\}$ such that for any switching function $s(k)$ of rank $\aleph_0$ with associated switching points $\{\tau_i\}$, the switching function controller*

$$u = -c(s, s(k))y$$
$$\dot{k} = \|y\|^2 + \|u\|^2$$

will stabilize any plant $G \in \mathcal{G}$.

Remark 4.15. In [Mårtensson 1986] it is shown constructively how to modify a discontinuous controller of this sort to a C^∞-controller, without losing its stabilizing power.

□

5. An Example

In the laboratory at the Department of Automatic Control in Lund there is a process called the *ball and beam*. This consists on an electrical motor whose shaft is connected to an approximately one meter long beam. There is a slot in the beam in which a steel ball is rolling. With the use of a resistance wire, a measurement of the position of the ball is available. Also the angle of the beam is available as output. The input is the current to the motor.

A standard simple linearized model for the transfer function an electric motor from current to position of the shaft is $b/s(s+a)$, where $a > 0$. The dynamics from the angle of the beam to the position of the ball is, for small angles, clearly a double integrator. Thus, the transfer function from current to position of the ball is given by

$$g(s) = \frac{b}{s^3(s+a)}$$

where we consider a and b as unknown. Theorem 4.14 can easily be modified to take into account the possibility of also the sign of b unknown. Then it gives e.g. the controller

$$c(s,k) = \sigma(s(k))s(k)^{15}\frac{(s+1)^3}{(s+s(k)^8)(s+s(k)^4)(s+s(k)^2)}$$
$$\dot{k} = y^2 + u^2$$

where $s : \mathbb{R}^+ \longrightarrow \mathbb{Z}^+$ is a suitable switching function, and $\sigma : \mathbb{Z}^+ \longrightarrow \{-1, 1\}$ a sign-switching function. This will stabilize $g(s)$ for all a and $b \neq 0$. When trying to simulate this process in Simnon, severe numerical problems occured, because of the size of the coefficients, and since the system turned out to be extremely stiff, i.e. to have a very large difference in the time scale of the different states. Instead, the controller

$$c(s,k) = \sigma(k)s(k)^7 \frac{(s+1)^3}{(s+s(k)^2)^3}$$

$$(\sigma, s) = \begin{cases} (1,1) & k < 2 \\ (-1,1) & 2 \le k < 4 \\ (1,2) & 4 \le k < 16 \\ (-1,2) & 16 \le k < 256 \\ (1,3) & 256 \le k < 65536 \\ \ldots \end{cases}$$

$$\dot{k} = y^2$$

was simulated. It can easily be shown in the same way as before that this controller is stabilizing for single-input, single-output minimum phase plants of relative degree four. A simulation where $a = b = 1$ is shown in Figure 1. The upper diagram shows the logarithms of $|y|$ (dashed line) and $|u|$ (solid line). (Actually, for practical reasons $\log(|y|+1)$ and $\log(|u|+1)$ is shown instead.) The lower diagram shows s and $\sigma(s)$. Note the wild, but fairly short excursions of u when s switches.

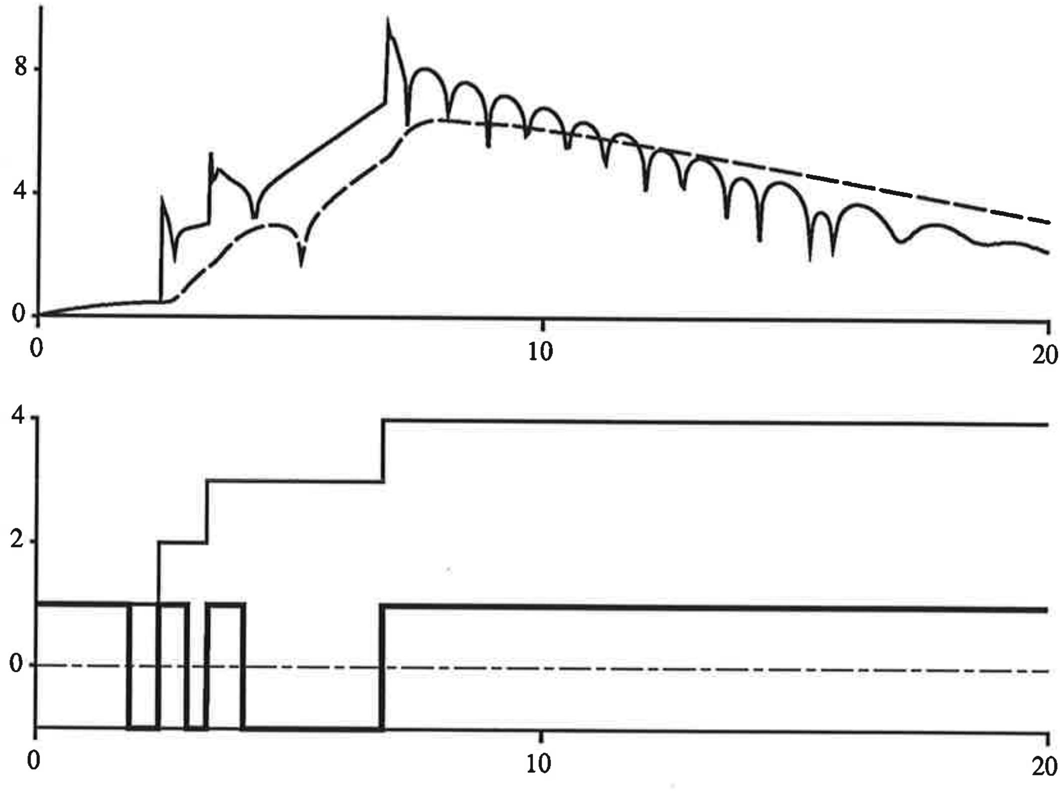

Figure 1. Simulation of adaptive stabilization of the ball and beam process.

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