Perspective Area-Invariants

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Abstract

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Perspective area-invariants

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1. Introduction

Image analysis deals with the situation where a scene is projected through a camera lens to form an image. The problem is to recognize objects in the scene by measurements in the image. The goal of the present work is to find certain classes of objects suited for recognition by computer vision. These objects can be used as sign posts or marking symbols in the applications. For reasons of error robustness and existing hardware, *area* measurements are preferable. The regions of interest must be identified in the image before the area measurements can be done. Different identification methods are available, but they are all more successful in images with high contrast. Especially well suited are marking symbols composed by regions with non-intersecting boundaries. (In particular such symbols may be colored by means of black and white only.)

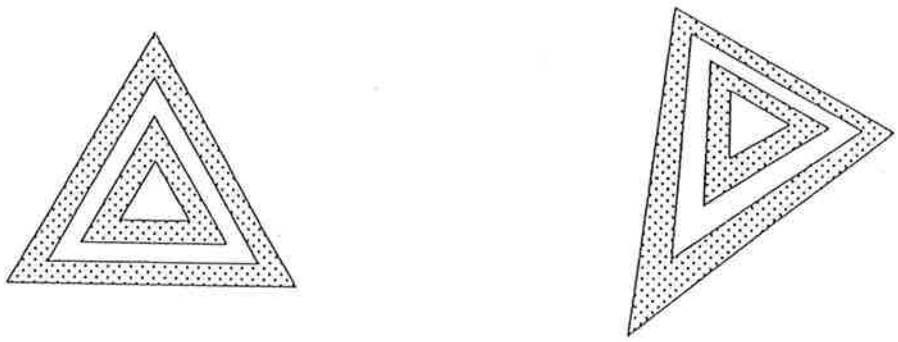

Figure 1. An example of a feasible object and a possible image of it

The study is restricted to planar objects, that viewed directly from the side reduces to a line. It is, however, possible to mark all planar sides of an object moving in three-dimensional space, so the restriction is immaterial. Figure 1 shows one feasible object and a possible image of it. This symbol, composed by inscribed triangles, and also symbols composed by inscribed rectangles, were studied first in (Nielsen, 1985) and later in (Nielsen and Sparr, 1985). The work has continued (Nielsen and Sparr, 1987), and the present paper includes a brief presentation of both new and old results. Throughout the paper details will be given only for the case of concentric triangles. In the growth of this article a particular role has been played by the symbolic manipulation program Macsyma. It was especially useful in the first discovery of area-invariants (Nielsen, 1985), and has also been used for experimental purposes thereafter.

2. Areas

In this section we derive the necessary tools for the calculation of areas.

Polygons

Let A_0, A_1, A_2 be three points in a plane π in the Euclidean space $\mathbb{E}^3$. The points are represented by their coordinates. Suppose that the origin $O \notin \pi$. Then

$\det(A_0, A_1, A_2)$ = the volume, with signs depending on the orientation, of the parallelepiped spanned by the vectors $\overline{OA_0}, \overline{OA_1}, \overline{OA_2}$ =

6·(the volume of the tetrahedron with vertices in (O, A_0, A_1, A_2))=

3·(the area of the triangle with vertices (A_0, A_1, A_2)) · (the distance between O and π).

In other words, we have

LEMMA 1 Apart from a factor, $\delta(A_0, A_1, A_2) = \det(A_0, A_1, A_2)$ measures the area of the triangle in π, having vertices in A_0, A_1, A_2. □

Remark 1. Our results will only contain relations between areas where the common factor can be omitted. □

Remark 2. The area of an arbitrary polygon is obtained by dividing it into triangles. □

It is well known that under parallel projections of π all areas are changed by a common factor, the determinant of the transformation matrix. One thus arrives at the following invariant: For any two triangles (A_0, A_1, A_2) and (B_0, B_1, B_2) in π, the quotient between their areas $\delta(A_0, A_1, A_2)/\delta(B_0, B_1, B_2)$ is invariant under parallel projection. In this paper we aim at extending this type of result to perspective projection.

Conics

We consider quadratic functions $q(x) = \frac{1}{2}x^T Q x + a^T x + b$, $x \in \mathbb{R}^2$. Let Q be symmetric and non-singular, and let x^* be defined as the solution of $Qx^* + a = 0$ Then

$$q(x) = \frac{1}{2}(x - x^*)^T Q (x - x^*) - \frac{1}{2}x^{*T} Q x^* + b = \frac{1}{2}(x - x^*)^T Q (x - x^*) - \frac{1}{2}a^T Q^{-1} a + b$$

Suppose that Q is positive definite with eigenvalues λ_1, λ_2. Then $\lambda_1\lambda_2 = \det Q$. Diagonalization of the quadratic form $(x - x^*)^T Q (x - x^*)$ yields

LEMMA 2 The area of the region $q(x) \leq 0$ in the Euclidean plane is

$$\alpha(q) = \pi \frac{a^T Q^{-1} a - 2b}{\sqrt{\lambda_1 \lambda_2}} = \pi \frac{a^T Q^{-1} a - 2b}{\sqrt{\det Q}}$$

3. Invariants

This section is devoted to invariant relations between the areas of certain triangles, quadrangles, and conics in the plane.

Triangles

Let π, π' be two distinct planes and $O \notin \pi \cup \pi'$ a point in the three-dimensional Euclidean space, cf. Figure 2. Here π' is the object plane, and π is the image plane. Let P', A', B', C', be four points in π' with A', B', C' non-colinear. Under the perspectivity from π' to π with center O, the points P', A', B', C' are mapped on P, A, B, C respectively.

Our construction of objects is based on certain transformations on π', namely dilations. A *dilation* on π' with center P' and scale t is defined by $X' \to X'_t$ where $\overline{P'X'_t} = t\overline{P'X'}$. We now define a dilated triangle A'_t, B'_t, C'_t by

$$\overline{OA'_t} = (1-t)\overline{OP'} + t\overline{OA'}, \quad \overline{OB'_t} = (1-t)\overline{OP'} + t\overline{OB'}, \quad \overline{OC'_t} = (1-t)\overline{OP'} + t\overline{OC'}$$

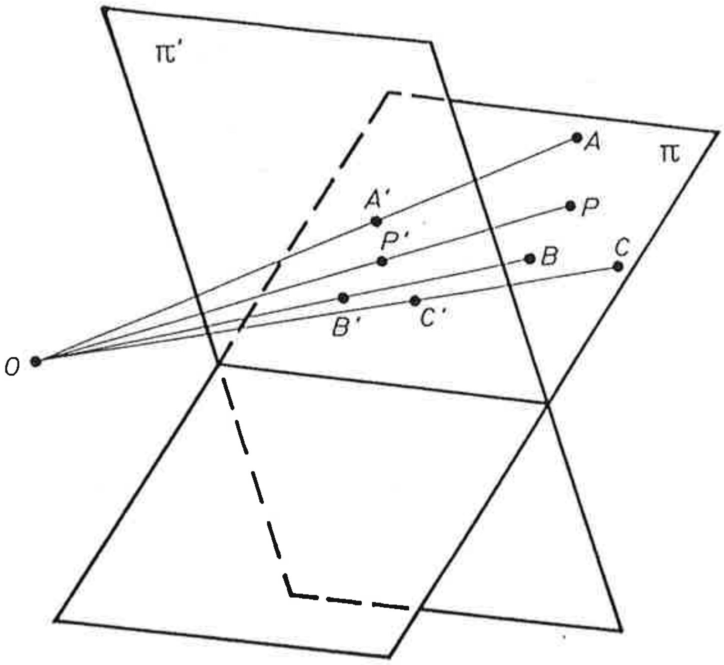

Figure 2.

Let P' in π' be the center of A', B', C' i.e. $\overline{OP'} = (\overline{OA'} + \overline{OB'} + \overline{OC'})/3$. Then A', B', C' and A'_t, B'_t, C'_t are concentric, and

$$\begin{aligned}
\overline{OA'_t} &= ((1-t)/3 + t)\,\overline{OA'} + (1-t)/3\,\overline{OB'} + (1-t)/3\,\overline{OC'} \\
\overline{OB'_t} &= (1-t)/3\,\overline{OA'} + ((1-t)/3 + t)\,\overline{OB'} + (1-t)/3\,\overline{OC'} \\
\overline{OC'_t} &= (1-t)/3\,\overline{OA'} + (1-t)/3\,\overline{OB'} + ((1-t)/3 + t)\,\overline{OC'}
\end{aligned}$$

Let A_t, B_t, C_t be the points in π corresponding to A'_t, B'_t, C'_t in π'. Fix the coordinate system

$$O, \quad \overline{OA} = \frac{1}{\alpha}\overline{OA'}, \quad \overline{OB} = \frac{1}{\beta}\overline{OB'}, \quad \overline{OC} = \frac{1}{\gamma}\overline{OC'}$$

for the image plane π. (The perspective mapping is then given by α, β, γ.) The plane π has the equation $x + y + z = 1$, and the points A_t, B_t, C_t have (after the perspective mapping α, β, γ and normalization to be points in π) the coordinates

$$\begin{aligned}
3A_t &= ((1+2t)\,\alpha, (1-t)\,\beta, (1-t)\,\gamma)/\sigma_{A_t} \\
3B_t &= ((1-t)\,\alpha, (1+2t)\,\beta, (1-t)\,\gamma)/\sigma_{B_t} \\
3C_t &= ((1-t)\,\alpha, (1-t)\,\beta, (1+2t)\,\gamma)/\sigma_{C_t}
\end{aligned}$$

where the normalization is $\sigma_{A_t} = (1+2t)\,\alpha + (1-t)\,\beta + (1-t)\,\gamma$, and where $\sigma_{B_t}, \sigma_{C_t}$ are defined in the same way.

By the considerations of Section 2, Lemma 1, computation of $\delta(A_t, B_t, C_t)$ gives the area of A_t, B_t, C_t relative to the unit $\delta(A, B, C)$. Recall that $(A_1, B_1, C_1) = (A, B, C)$. To achieve homogeneity in t, we change the t-scale by a factor of t_0 i.e. substitute t/t_0 for t, and replace A_1, B_1, C_1 with $A_{t_0}, B_{t_0}, C_{t_0}$. We obtain

$$((t_0 - t)^2(t_0 + 2t)S_1^3 + 9t^2(t_0 - t)S_1S_2 + 27t^3S_3) \cdot \delta(A_t, B_t, C_t) = t^2 t_0 S_3 \delta(A_{t_0}, B_{t_0}, C_{t_0}) \quad (1)$$

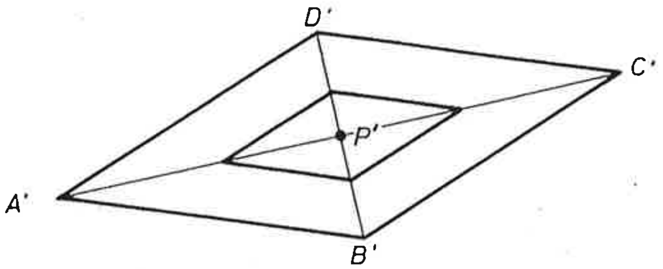

Figure 3.

where

$$S_1 = \alpha + \beta + \gamma, \quad S_2 = \alpha\beta + \beta\gamma + \gamma\alpha, \quad S_3 = \alpha\beta\gamma \tag{2}$$

Introduce the notation

$$\begin{aligned} h_3(t_1, t_2, t_3) &= t_1^2 t_2^3 - t_1^3 t_2^2 + t_2^2 t_3^3 - t_2^3 t_3^2 + t_3^2 t_1^3 - t_3^3 t_1^2 \\ &= (t_1 - t_2)(t_2 - t_3)(t_3 - t_1)(t_1 t_2 + t_2 t_3 + t_3 t_1) \end{aligned}$$

(Here the subscript 3 refers to triangles, cf. quadrangles below)

THEOREM 1 *With the notation introduced above, for any perspective image of the configuration in π' holds the relation*

$$\frac{t_0^2 h_3(t_1, t_2, t_3)}{\delta(A_{t_0}, B_{t_0}, C_{t_0})} - \frac{t_1^2 h_3(t_2, t_3, t_0)}{\delta(A_{t_1}, B_{t_1}, C_{t_1})} + \frac{t_2^2 h_3(t_3, t_0, t_1)}{\delta(A_{t_2}, B_{t_2}, C_{t_2})} - \frac{t_3^2 h_3(t_0, t_1, t_2)}{\delta(A_{t_3}, B_{t_3}, C_{t_3})} = 0 \tag{3}$$

□

Proof: The proof relies on (1). Since the perspectivity is uniquely determined by the non-zero numbers α, β, γ, then "invariants under perspectives" must be independent of α, β, γ.

Put

$$m_t = \delta(A_t, B_t, C_t) / \delta(A_{t_0}, B_{t_0}, C_{t_0})$$

Introduced in (1) it yields

$$(t_0 - t)^2 (t_0 + 2t) S_1^3 + 9t^2 (t_0 - t) S_1 S_2 + 27t^2 (t - \frac{t_0}{m_t}) S_3 = 0 \tag{4}$$

Putting together three such equations, corresponding to $t = t_1, t = t_2$, and $t = t_3$, one gets a homogeneous system of linear equations in the unknowns $S_1^3, S_1 S_2, S_3$ (that in turn are functions of α, β, γ). This system is known to have a nontrivial solution, determined by the geometrical construction above. Hence the determinant of the system is zero, i.e.

$$\begin{vmatrix} (t_0 - t_1)^2 (t_0 + 2t_1) & t_1^2 (t_0 - t_1) & t_1^2 (t_1 - \frac{t_0}{m_{t_1}}) \\ (t_0 - t_2)^2 (t_0 + 2t_2) & t_2^2 (t_0 - t_2) & t_2^2 (t_2 - \frac{t_0}{m_{t_2}}) \\ (t_0 - t_3)^2 (t_0 + 2t_3) & t_3^2 (t_0 - t_3) & t_3^2 (t_3 - \frac{t_0}{m_{t_3}}) \end{vmatrix} = 0$$

Expansion of the determinant (after the third column) directly gives (3). □

Quadrangles

We obtain the area of a quadrangle A, B, C, D by dividing it into two triangles i.e. by

$$\Delta(A, B, C, D) = \delta(A, B, C) + \delta(A, C, D)$$

Starting as usual in an affine plane π', let A', B', C', D' be a parallelogram and let P' be the intersection of its diagonals (cf. Figure 3). By a dilation with center P', scale t, the points A'_t, B'_t, C'_t, D'_t are constructed. Let A_t, B_t, C_t, D_t be the images of A'_t, B'_t, C'_t, D'_t.

After computation of the two determinants we obtain the analogue of (1):

$$((1-t^2)^2 s_1^4 + 4t^2(1-t^2)s_1^2 s_2 + 16t^4 s_4)\Delta(A_t, B_t, C_t, D_t) = 16t^2 s_4 \Delta(A, B, C, D) \tag{5}$$

where $s_1 = \beta + \gamma$, $s_2 = \alpha\beta + \beta\gamma + \gamma\alpha - \alpha^2$, $s_4 = \alpha\beta\gamma(\beta + \gamma - \alpha)$. In analogy with h_3 above we define

$$\begin{aligned} h_4(t_1, t_2, t_3) &= t_1^2 t_2^4 - t_1^4 t_2^2 + t_2^2 t_3^4 - t_2^4 t_3^2 + t_3^2 t_1^4 - t_3^4 t_1^2 \\ &= (t_1^2 - t_2^2)(t_2^2 - t_3^2)(t_3^2 - t_1^2) \end{aligned}$$

We can now formulate

THEOREM 2 *For any perspective image of the configuration in π' holds the relation*

$$\frac{t_0^2 h_4(t_1, t_2, t_3)}{\Delta_{t_0}} - \frac{t_1^2 h_4(t_2, t_3, t_0)}{\Delta_{t_1}} + \frac{t_2^2 h_4(t_3, t_0, t_1)}{\Delta_{t_2}} - \frac{t_3^2 h_4(t_0, t_1, t_2)}{\Delta_{t_3}} = 0 \tag{6}$$

□

Proof: The proof is similar to the proof of Theorem 1. □

Other polygons

Comparing (3) and (6) , one notes at least two common features. First, the number of figures needed were in both cases four, and second, the coefficients h_3 and h_4 in the invariant formula have the same structure. The natural question arises whether this can be generalized to polygons with k vertices. The answer is *no*, at least in the sense that the number of figures needed depends on k. This number is highly dependent on the symmetry properties of the figure. Calculations with a symbolic manipulation program have showed that for regular pentagons, $k = 5$, one needs nine and for regular hexagons, $k = 6$, six t-values (i.e. dilations of the reference polygon).

Conics

Now consider a non-degenerate cone in the three-dimensional space. Let O be its vertex and let π, π' be two planes with $O \notin \pi \cup \pi'$, $\pi \neq \pi'$. Let $\ell = \pi \cap \pi'$. Two conics C and C' are defined by the intersections of the cone with π and π' respectively. Suppose that C' is a circle. It is possible to choose the coordinate systems in π' and π, so that one obtains tractable expressions both for the circles in π' and the ellipses in π (Nielsen and Sparr, 1987). The use of α in Lemma 3 together with an elimination of the perspectivity parameter yields

THEOREM 3 *With the notation introduced above, for any perspectivity holds*

$$\frac{t_1^2 - t_2^2}{(\frac{\alpha(q_0)}{t_0^2})^{2/3}} + \frac{t_2^2 - t_0^2}{(\frac{\alpha(q_1)}{t_1^2})^{2/3}} + \frac{t_0^2 - t_1^2}{(\frac{\alpha(q_2)}{t_2^2})^{2/3}} = 0 \tag{7}$$

□

Remark. It is noteworthy that the number of terms in (7) is three, while it earlier in the plane has been four in the Theorems 1 and 2. □

Projective area-invariants

Above we have restricted ourselves to perspectivities of concentric figures. However, the results raise mathematical questions that are beyond the scope of the present paper, but they have been treated in (Nielsen and Sparr, 1987). There it is proved that the relations above are invariant for arbitrary projectivities, not only for perspectivities. Also general dilations and translations are considered there. A general reference on these aspects is (Coxeter, 1974).

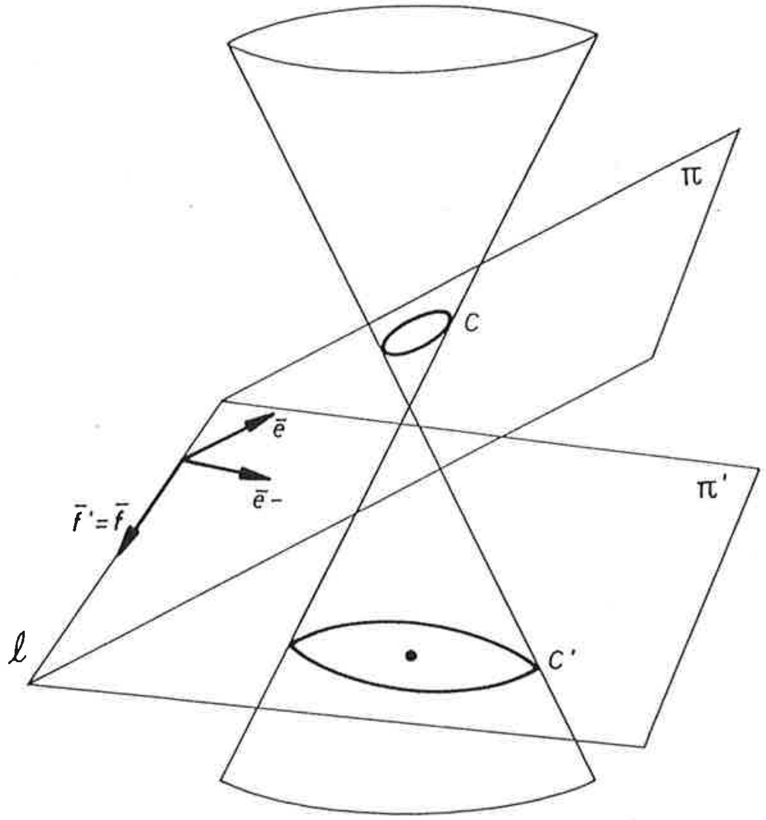

Figure 4.

4. Identification and Localization

The invariants derived in Theorems 1–3, will now be utilized for identification and localization. Before doing so we will write Eqs. (3), (6), and (7) in a common form. They are all three a homogeneous relation in inverted areas. Let a be a common notation for δ, Δ, and $\alpha^{2/3}$. Then our invariants are

$$\sum_{0}^{m} \frac{\lambda_i}{a_i} = 0 \tag{8}$$

where $m = 3$ for triangles and quadrangles, and $m = 2$ for conics. The coefficients are for triangles and quadrangles $\lambda_0 = t_0^2 h(t_1, t_2, t_3)$ with cyclic permutation of the indices for $\lambda_1, \lambda_2, \lambda_3$, and the coefficients for conics are $\lambda_0 = t_0^{2/3}(t_1^2 - t_2^2)$, also here with cyclic permutation of indices to obtain λ_1, λ_2. We note here that for triangles and quadrangles we have

$$\lambda_0 - \lambda_1 + \lambda_2 - \lambda_3 = 0 \tag{9}$$

whereas for conics we do *not* have such a linear relation between the coefficients.

Due to (9) we can, by the introduction of $\hat{a}_k = a_k/(a_0 - a_k)$, $k = 1, 2, 3$, write (8) as

$$\frac{\lambda_1}{\hat{a}_1} - \frac{\lambda_2}{\hat{a}_2} + \frac{\lambda_3}{\hat{a}_3} = 0 \tag{10}$$

in the case of triangles and quadrangles.

Symbols

The objects we study are constructed as concentric triangles, parallelograms (in fact we use quadrats), or circles. The relative size of the concentric figures are given by the t-values. We also note that the coefficients λ are functions of t only. The relations between t and λ are given by a system of nonlinear equations. We will here not go into detail, but since the equations are homogeneous there exist tractable methods (Hodge and Pedoe, 1947; Collins, 1967; Ku and Adler, 1969). We will here only consider as a fact that we may describe a symbol either by t or by λ, and that it is possible to pick t-values to get distinct λ-values even for a set of symbols.

Observations

Consider two images of the same symbol. If the symbol is in the same position in the two images they contain the same information. We need some notion of independent observations.

DEFINITION 1 An *observation*, ω_i, is defined as $\omega_i = \left(\frac{1}{\hat{a}_{1i}}, \frac{1}{\hat{a}_{2i}}, \frac{1}{\hat{a}_{3i}}\right)$ for triangles and quadrangles, and as $\omega_i = \left(\frac{1}{a_{0i}}, \frac{1}{a_{1i}}, \frac{1}{a_{2i}}\right)$ for conics. □

DEFINITION 2 Two observations, ω_1 and ω_2, are said to be *independent* if ω_1 and ω_2 are linearly independent vectors. □

Identification

Consider an unknown symbol, i.e., unknown λ-values. For conics we have three unknown λ-values, and for triangles and quadrangles four unknown. However, due to (9) effectively only three. We have

THEOREM 4 *An unknown symbol can, apart from a common factor, be recognized from two independent observations.* □

Proof: Each observation yields a linear equation $\omega_i \cdot (\lambda_1, \lambda_2, \lambda_3) = 0$. By means of two linearly independent ω_i:s, we have together with (9)

$$\begin{cases} \lambda_1 - \lambda_2 + \lambda_3 = \lambda_0 \\ \dfrac{\lambda_1}{\hat{a}_{11}} - \dfrac{\lambda_2}{\hat{a}_{21}} + \dfrac{\lambda_3}{\hat{a}_{31}} = 0 \\ \dfrac{\lambda_1}{\hat{a}_{12}} - \dfrac{\lambda_2}{\hat{a}_{22}} + \dfrac{\lambda_3}{\hat{a}_{32}} = 0 \end{cases}$$

Since the t-values are distinct we have $\lambda_0 \neq 0$. The system is inhomogeneous and we have

$$\lambda_1 : \lambda_2 : \lambda_3 = \begin{vmatrix} \frac{1}{\hat{a}_{21}} & \frac{1}{\hat{a}_{31}} \\ \frac{1}{\hat{a}_{22}} & \frac{1}{\hat{a}_{32}} \end{vmatrix} : - \begin{vmatrix} \frac{1}{\hat{a}_{31}} & \frac{1}{\hat{a}_{11}} \\ \frac{1}{\hat{a}_{32}} & \frac{1}{\hat{a}_{12}} \end{vmatrix} : \begin{vmatrix} \frac{1}{\hat{a}_{11}} & \frac{1}{\hat{a}_{21}} \\ \frac{1}{\hat{a}_{12}} & \frac{1}{\hat{a}_{22}} \end{vmatrix}$$

We obtain λ_0 by (9). □

A Family of Symbols

We can create a *continuous family* of symbols by letting $t_0 > t_1 > t_2 > t_3$ and making t_0, t_2, t_3 fix. Different symbols are then obtained by variation of t_1. We treat triangles and quadrangles. There has to be restrictions on α, β, γ corresponding to the natural situation that the object is in front of the camera, to obtain the following result

THEOREM 5 *A symbol belonging to a continuous family as above can be recognized from one single observation.* □

Proof: We have areas $a_i(t_0, t_1, t_2, t_3, \alpha, \beta, \gamma)$ obtained from an unknown symbol in the family. We also have the coefficients $\lambda_i(t_0, x, t_2, t_3)$ with x unknown. The invariant (3) gives an equation for x

$$\sum_0^3 \frac{\lambda_i(t_0, x, t_2, t_3)}{a_i(t_0, t_1, t_2, t_3, \alpha, \beta, \gamma)} = 0$$

This equation has in fact the unique solution $x = t_1$ (Nielsen, 1985). □

Localization

Assume now that we have a known symbol, either a priori known or identified, and that we from an observation wants to find its position relative to the camera. The problem is thus to find $\bar{\alpha} = (\alpha, \beta, \gamma)$ given $\bar{t}$ and ω. If we neglect the problem that permutations of $\bar{\alpha}$ is also a possible solution, then we can state

THEOREM 6 *A given symbol can be localized from one observation.* □

Proof: By means of three equations (4), where now $\bar{t}$ and $\bar{m}_t$ are known, we can determine S_1^3, S_1S_2, S_3, apart from a factor of proportionality. Knowing the absolute values of an area a_I in the image and the corresponding object area a_O, then S_3 is obtained as a_O/a_I. By this S_1, S_2, S_3 are fully determined. The relation between roots and coefficients (cf. (2)) then gives α, β, γ as the solutions of

$$x^3 - S_1x^2 + S_2x - S_3 = 0$$

□

5. Conclusions

Fundamental questions arise if the image analysis is restricted to use measurements of area only. Area measurements are attractive due to error robustness and existing hardware. In this paper we have studied figures composed of concentric triangles, quadrates, and circles. Area-relations that are invariant under perspective transformations have been derived for these figures. Solutions to the problems of identification and localization, still based only on area measurements, have been treated. We conclude that the presented figures can be used as marking symbols and signposts.

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