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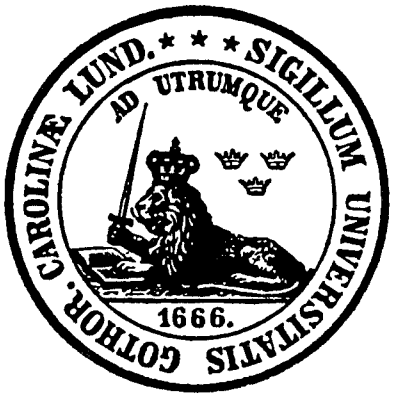

Stochastic Convergence of Algorithms for Identification and Adaptive Control

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To Ann-Kristin

INTRODUCTION.

Most of the established control theory requires that the process to be controlled and its environment are described by a known mathematical model. For many processes such models are not available and cannot be constructed from theoretical considerations. The reason may be that the process is too complex, or that the basic knowledge of process mechanisms and process parameters is lacking.

Two approaches for solving this problem have been suggested. One is to use experimental input-output data from the process to select a model that adequately describes the measured data. This procedure is known as identification. Another approach is to use a regulator with adjustable parameters and to tune the parameters until a satisfactory performance is achieved. In the majority of cases the parameters are adjusted manually like in most industrial PI regulators. It is also possible to have some device which automatically adjusts the parameters. Such an approach is called adaptive control.

Identification algorithms, as well as algorithms for adaptive control are supplied with input-output data from the process and they produce a set of parameters that describes the model or the regulator respectively. As the number of data supplied increases it is naturally desirable that the produced results should improve. The problem of convergence of the algorithms concerns the question of what happens to the parameters as the number of data tends to infinity.

In many cases the measured input-output data from the process cannot be exactly reproduced if an experiment is repeated. This is due to a number of unmeasurable variables, called disturbances, that influence the process. Therefore, the parameter values that are produced by the algorithms as well as their convergence properties may vary from experiment to ex-

periment. It is customary to describe the disturbances acting on the process as stochastic variables. Then also the parameters become stochastic variables, and the convergence problem must be considered in a stochastic setting.

This thesis deals with the problem of convergence of algorithms for identification and adaptive control. The stochastic convergence concept used is "convergence with probability one".

CONVERGENCE AND CONSISTENCY.

Two different kinds of algorithms can be distinguished. In recursive or on-line algorithms, input-output data from the process are treated sequentially, directly after they have been measured. In off-line algorithms data collected during previous experiments are treated at the same time.

Recursive algorithms can be regarded as time varying stochastic systems. Their convergence therefore could be considered within the general framework of stochastic stability theory, as treated by e.g. Kushner (1967). However, these concepts do not seem to be very suitable for this specific application, and the various algorithms suggested have instead been analysed by individual methods. Stochastic approximation algorithms, which form an important subclass of the algorithms treated in this thesis, have been considered by e.g. Blum (1954). Convergence of recursive identification methods can in some cases be investigated as consistency analysis of the corresponding off-line method; this has been done for the least squares method in e.g. Åström-Eykhoff (1971). Many algorithms for adaptive control have been studied mainly by means of simulation. In this thesis a unified approach to the problem of convergence of recursive algorithms is suggested.

Off-line algorithms are naturally of interest only for identification. For such methods it is of great importance to know if the parameter estimates are consistent, i.e. if they approach the true values as the number of data used increases to infinity. A large number of identification methods exist, see e.g. Åström-Eykhoff (1971). A method that seems to have wide applicability is the maximum likelihood method. It was introduced originally by Fisher (1912) and applied to system identification by Åström-Bohlin (1965). The consistency properties of the original method have been analysed e.g. by Wald (1949). When applied to system identification, the consistency problem for the maximum likelihood method is more difficult, and it has been studied by several authors under varying assumptions. In this thesis consistency of a class of identification methods that, under certain conditions on the noise, includes the maximum likelihood method is considered. The results obtained are believed to have general applicability, since they are not associated with any restrictive assumptions about noise characteristics, model structure, time invariance (ergodicity) or input signal generation.

The thesis consists of three parts,

I. Convergence of Recursive Stochastic Algorithms. Report 7403, Div. of Automatic Control, Lund Institute of Technology, Lund, Sweden (1974).

II. Asymptotic Properties of Self-Tuning Regulators. Report 7404, Div. of Automatic Control, Lund Institute of Technology, Lund, Sweden (1974). (Coauthor B. Wittenmark)

III. On Consistency for Prediction Error Identification Methods. Report 7405, Div. of Automatic Control, Lund Institute of Technology, Lund, Sweden (1974).

each of which will be briefly reviewed below.

CONVERGENCE OF RECURSIVE STOCHASTIC ALGORITHMS.

For a recursive algorithm, the data that is collected from the process, is condensed into a vector x_n, which represents the current knowledge about the process. This vector is modified each time a new observation φ_n is obtained. The procedure can be written as

$$x_n = x_{n-1} + \gamma_n Q_n(x_{n-1}, \varphi_n) \tag{1}$$

Algorithms of this type thus represent a general recursive algorithm. The observation φ_n as mentioned above is a stochastic variable, which may depend on previous vectors x_k, $k \leq n-1$ in an arbitrary manner. In part I of the thesis, a subclass of algorithms (1) is studied; where φ_n depends only on x_{n-1} and not on all previous x_k. This class of recursive stochastic algorithms contains so-called stochastic approximation algorithms like the schemes suggested by Robbins and Monro (1951) and Kiefer and Wolfowitz (1952). Therefore, it also contains all forms of algorithms for identification, adaptive control and pattern recognition, that are based on stochastic approximation, see e.g. Tsypkin (1973), Fu (1969). Several other algorithms, like e.g. the real time least squares algorithm belong to the class under consideration. In the thesis the problem of convergence of (1) is treated. It is shown that the question of convergence can be separated into two problems:

1) Stability of the ordinary differential equation

$$\frac{d}{dt} x = f(x) = \lim_{n\to\infty} E\{Q_n(x,\varphi) | x\} \tag{2}$$

where $E\{Y|x\}$ denotes the conditional expectation of Y given x.

and

2) consistency of a simple mean value estimator, connected with (1).

With this approach, convergence of stochastic approximation algorithms can be shown under conditions that are weaker than previously reported.

The ODE (2) can be used not only for the convergence analysis, but also to investigate convergence rate, choice of $\{\gamma_n\}$ and other asymptotic properties of (1).

ASYMPTOTIC PROPERTIES OF SELF-TUNING REGULATORS.

An adaptive controller adjusts its regulator parameters in accordance with the process dynamics. It can often be formed as a real time identification algorithm connected to a regulator whose parameters are determined using the current estimates from the identification. Such adaptive controllers have been discussed by many authors. The case where the identification algorithm is an equation error algorithm (least squares algorithm) has received special interest, see Kalman (1958), Peterka (1970) and Åström-Wittenmark (1973). This class of adaptive controllers has been called "Self Optimizing Control Systems" or "Self-Tuning Regulators". Specific controllers belonging to this class have successfully been used in practice, but convergence of them has so far been studied only with simulation and heuristic arguments. Part II of the thesis presents tools for the theoretical convergence analysis of self-tuning regulators.

The regulators can be expressed in the form (1) where x consists of the process parameter estimates. Then φ_n consists of

the latest recorded inputs and outputs and it is a complex function of all previous x_k, $k < n$. However, the same convergence result applies as above. Hence the question of convergence relies upon the stability of the ODE (2). This ODE has been studied for some specific controllers and convergence with probability one has been established in some cases. By careful examination of the ODE, examples of processes have been constructed for which the regulator proposed in Åström-Wittenmark (1973) does not converge. No such process was known before the analysis, and, in fact, extensive simulations had indicated general convergence. It is also shown that these self-tuning regulators possess a stabilization property, which assures overall stability even if the regulator does not converge.

ON CONSISTENCY FOR PREDICTION ERROR IDENTIFICATION METHODS.

Many identification methods are based on minimization of some discrepancy, or error, between the model and the measured data. For output error methods the difference between the model output and the measured output is minimized, while for equation error methods the error in the equation describing the model is considered. These methods do not model the dynamic behaviour of the disturbances that cause the discrepancy.

Another approach is to predict the next output of the process based on previous measurements, taking into account the characteristics of the disturbances. In prediction error identification methods some function of the difference between the predicted output and the measured output is minimized. If the disturbances are gaussian, the maximum likelihood method is a special case of prediction error methods.

In part III of the thesis consistency of such methods is discussed. Special interest is paid to the problem of feedback terms in the input sequence.

The traditional way of studying the consistency problem of the maximum likelihood method, see e.g. Åström-Bohlin (1965), Rissanen-Caines (1974), is to consider the limit of the log-likelihood function. To do this, ergodic theory is used, and restrictive assumptions on the input must be imposed. This makes the results inapplicable to a number of situations in practice, in particular when the system is governed by an adaptive controller.

A different technique, partly based on martingale theory and partly inspired by Wald's (1949) proof, is used in the thesis. This approach allows a general input sequence as well as a general system description.

IMPLICATIONS OF THE RESULTS.

In the thesis the results are applied to some specific examples. The general implications and the possible application areas of the results will now be briefly discussed.

Numerical solution of differential equations as a tool for the analysis of recursive algorithms.

The theoretical convergence analysis of a given recursive algorithm is in many cases very difficult. In particular, this is true for algorithms for adaptive control. According to the results of parts I and II of this thesis convergence can be proved by showing stability of the differential equation (2). This differential equation may be quite complex and cumbersome to handle analytically. However, some insight into the stability properties of (2) can be obtained by numerical solution with different initial conditions. Such analysis can often be made relatively easily. The question then asked is:

What is the advantage of simulating (2) over simulating (1)? The main feature is that, loosely speaking, the stochastic element is removed. This means that it is immediately clear whether a given effect is something peculiar to the algorithm or just a random influence. Also, by solving the ODE (2), the area of feasible parameter values can be searched systematically for local, undesirable attraction points with small domains of attraction. If such points exist, there is a non-zero probability that the estimates produced by (1) are trapped there. The probability may, however, be quite small and the point might never be revealed by simulation of (1). This illustrates a basic limitation with simulation as an instrument for investigation of the algorithm (1). The numerical solution of the associated ODE has thus proved to be a valuable complement to simulation. It has been shown that the trajectories of (2) are also directly related to the paths of (1). Bounds for the probability that the estimates produced by (1) deviate with more than a given distance from the corresponding trajectory of (2) can be calculated. Such a result supports the use of numerical solution of (2) as a tool for the analysis of (1).

Off-line and on-line identification algorithms.

On-line, i.e. recursive, identification algorithms can be investigated using the ODE discussed above. An example of its application to the recursive least squares method is given in part I. The technique seems to be applicable also to identification algorithms for which convergence has not been shown previously, like recursive approximate maximum likelihood. For identification methods that are based on a criterion minimization it is relatively easy to find Lyapunov functions that assure stability of the associated ODE: the criterion itself can be chosen. For other methods, numerical solution of the ODE may be useful.

Also for off-line identification methods it is, of course, important to have convergence results that are valid under general conditions. In many cases it is necessary to use data for identification that are obtained when the system is operating in closed loop. For industrial processes, e.g., production or security reasons may prevent open loop identification experiments. Other processes, like economic systems, may often inherently be in closed loop. The consistency results of part III are valid under such general feedback situations. This facilitates a systematic analysis of the identifiability properties for systems operating in closed loop, see Gustavsson-Ljung-Söderström (1974).

Towards a theory of adaptive control.

The field of adaptive control of dynamical systems has received much interest during the past fifteen years. Many different schemes have been proposed and in some cases also tested in practice. It seems as though each method has been analysed by individual methods, and many studies have been carried out on the basis of simulation of the proposed scheme.

Much of the discussion of this thesis is indirectly focused on the problem of how to produce techniques with some generality for the theoretical analysis of adaptive systems.

It seems that many adaptive schemes can be expressed in the form (1), which makes the ODE (2) available as a tool for analysis. The important condition is that the observation $\varphi(t)$ in (1) depends on old estimates $x(s)$, $s \ll t$, to an extent that is rapidly decreasing. The forcing term in the system need not be a stochastic process; it can equally well be a deterministic signal such that certain variables are Cesaro summable. Therefore this technique is applicable to a broad class of adaptive regulators, and not only to the class dis-

cussed in part II of this thesis.

A different way of considering adaptive systems is to single out a part that is monitoring the system dynamics. This part can be understood as a real time identifier. It passes on information to a controller which determines the input to the process on the basis of the information received. Such a decomposition is possible for many adaptive schemes, in particular for the self-tuning regulators discussed in part II, where the identifier is a real time least squares algorithm. The limiting behaviour of the algorithm can in such cases be studied as the consistency properties of the identifier. A main concern of the discussion in part II is to obtain consistency results that are valid under general feedback conditions, including various types of adaptive controllers. Such results are necessary to understand how identification algorithms behave when they are part of an adaptive controller.

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ASYMPTOTIC PROPERTIES OF SELF-TUNING REGULATORS

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Asymptotic Properties of Self-Tuning Regulators

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Asymptotic Properties of Self-Tuning Regulators

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ABSTRACT

Adaptive controllers of a certain structure are considered in this report. The parameters in a difference equation model of the process are estimated on-line using the least squares method. The current parameter estimates are used to calculate the parameters of the feedback law that governs the process. The resulting adaptive controller is called a self-tuning algorithm. It is shown that the convergence properties of such algorithms can be investigated by analysing an associated ordinary differential equation. The analysis is applied to specific examples of self-tuning algorithms.

TABLE OF CONTENTS

1. SELF-TUNING REGULATORS

Adaptive control of dynamic systems has been extensively discussed during the last ten years. An important special case is when the process parameters are known to be time invariant, but the values are unknown. For this case the control algorithms should be such that they converge to the optimal control algorithms that could be derived if the system characteristics were known. Such an algorithm is called a self-tuning regulator.

Most suggested adaptive and self-tuning regulators are based on the assumption that the real time estimation of the parameters of the process can be separated from the determination of the control signal. See fig. 1.1. In many cases the control signal is determined without taking into consideration that the estimates of the parameters are uncertain. A more sophisticated type of controllers are those which

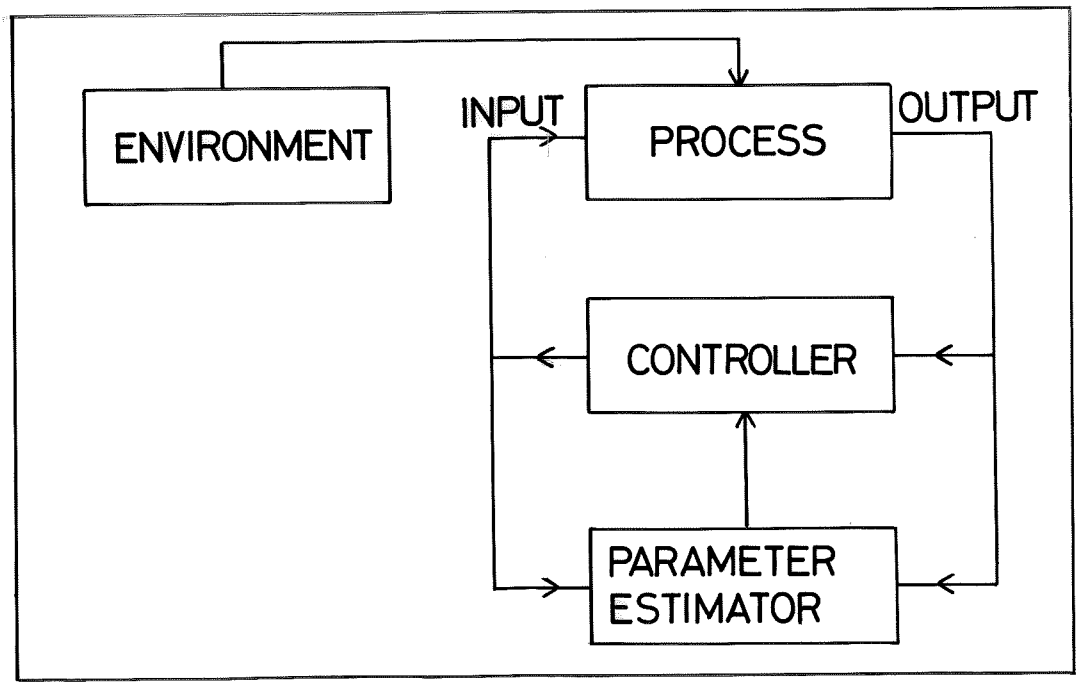

Fig. 1.1 - Schematic block diagram for adaptive regulators.

consider the parameter uncertainties when the control signal is determined, but do not make anything to obtain better estimates. The dual controllers, see Feldbaum (1960, 1961), represent a further degree of sophistication where the input signal is chosen to increase the accuracy of the parameters in the same time as the controller tries to make as good control as possible. The two activities of the dual controller are mutually contradictory and there must be a compromise between the identification and control activities of the controller. The dual controllers have attractive features, but it is very difficult to get practical solutions to the dual control problem.

In this report the behaviour of self-tuning regulators with a structure as in fig. 1.1 is discussed in the case when the identifier is based on the least squares (LS) method. However, the techniques put forward are applicable to more general identification schemes.

Special attention will be paid to the case when the controller is a minimum variance control law based on current estimates. This self-tuning regulator is discussed in Åström - Wittenmark (1973).

The central question for the analysis of such a regulator is of course: Will the regulator converge to the desired one? Techniques and basic theorems to answer this question are presented in this report. Specific examples of self-tuning regulators are analysed using these tools.

In Chapter 2 the LS method is defined, and known results are repeated for easy reference. A class of self-tuning regulators is strictly defined. The main problem in the analysis of the regulators is that the feedback is time varying. This makes the input and output sequences non-stationary and the usual consistency results for the LS method are inapplicable.

A theorem on consistency that is valid in the present case is proved in Chapter 3.

In general the result of the identification depends on the feedback law. This introduces essential non-linearities into the identification process. To handle this problem an ordinary differential equation (ODE), which is connected with the regulator, is derived in Chapter 4. Stability of this ODE is shown to be equivalent to convergence of the regulator. Also, the paths of the ODE define "expected behaviour" of the regulator.

In Chapters 3 and 4 the controller is not specified. In Chapters 5, 6, and 7 most of the analysis is concerned with the self-tuning regulators discussed in Wittenmark (1973). In Chapter 5 the behaviour near or outside the stability boundary of the closed loop system is discussed. The regulators are shown to stabilize the closed loop system even if the model noise does not agree with the true noise characteristics.

In Chapter 6 the ODE defined in Chapter 4 is investigated for some self-tuning regulators. It is shown that the regulators do not converge for general noise structures. Actually, it was indicated by extensive simulations, Åström - Wittenmark (1973), that the regulators converge in general. Only after using the analysis of Chapter 6 could examples be constructed for which the regulators do not converge.

In Chapter 7 the ODE is solved numerically for a number of cases of interest.

In Appendix A the proof of Lemma 4.1 is given. The results of the report hold for several different model structures. In Appendix B it is shown how other model structures can be handled, and the modifications of the results shown are also given there.

2. PRELIMINARIES

The class of self-tuning regulators to be treated is formally defined in this chapter. In Section 2.1 some different models for least squares identification are discussed. Off-line and on-line algorithms for least squares identification are given in Section 2.2. In Section 2.3 algorithms of stochastic approximation type are introduced. The class of self-tuning regulators is defined in Section 2.4, and in Section 2.5 the special algorithms "STURE0" and "STURE1" (self-tuning regulator) are introduced.

2.1 Models.

Assume that the system can be described by the difference equation

$$y(t+1) + a_1 y(t) + \ldots + a_n y(t+1-n) = \\ = b_0 u(t-k) + \ldots + b_m u(t-k-m) + v(t+1) \qquad (2.1)$$

where $k \geq 0$, $\{v(t)\}$ is a sequence of random variables and where $\{y(t)\}$ is the output and $\{u(t)\}$ the input of the system. The usual model for least squares identification has the same structure as (2.1) and in general all the constants $a_1,\ldots,a_n$, $b_0,\ldots,b_m$ are estimated. In connection with self-tuning regulators, cf. Åström - Wittenmark (1973), it is in some cases meaningful not to estimate b_0.

It is possible to rewrite (2.1) as

$$y(t+k+1) + \alpha_1 y(t) + \ldots + \alpha_n y(t-n+1) = \beta_0 u(t) + \ldots + \\ + \beta_{m'}' u(t-m') + \varepsilon(t+k+1) \qquad (2.2)$$

where $\{\varepsilon(t)\}$ is a process formed as a moving average from $v(t),\ldots,v(t-k)$. The variable m' equals $m+k$, see Åström - Wittenmark (1973). This modification proves to be of great

value when control laws are synthesized as shown in Section 2.5.

Also in this case it is suitable to consider β_o as an a priori known constant. Then, by introducing $\beta_i = \beta_i'/\beta_o$, (2.2) can be written as

$$y(t+k+1) + \alpha_1 y(t) + \ldots + \alpha_n y(t-n+1) = \beta_o[u(t) + \beta_1 u(t-1) + \\ + \ldots + \beta_{m'} u(t-m')] + \varepsilon(t+k+1) \qquad (2.3)$$

The model of the system then is

$$y(t+\hat{k}+1) + \hat{\alpha}_1 y(t) + \ldots + \hat{\alpha}_{\hat{n}} y(t-\hat{n}+1) = \hat{\beta}_o[u(t) + \ldots + \\ + \hat{\beta}_{\hat{m}} u(t-\hat{m})] + \hat{\varepsilon}(t+\hat{k}+1) \qquad (2.4)$$

Here $\hat{\alpha}_i$ is the estimate of α_i and $\hat{\beta}_i$ the estimate of β_i. β_o is regarded as a priori known and is not estimated. It is, however, not realistic to assume that the value of β_o is exactly known. Therefore, some of the analysis in this report will deal with the case when β_o is assumed to have the value $\hat{\beta}_o$, which may be different from β_o. The orders $\hat{n}$, $\hat{m}$ and the time delay $\hat{k}$ may not be the same as the true ones.

In this report (2.4) is used as the basic model, since this structure is used in the self-tuning algorithms STURE0 and STURE1, defined in Section 2.5. However, the other model structures can be treated formally in exactly the same way. This is shown in Appendix B.

2.2 Least squares identification.

Introduce

$$\theta = (\hat{\alpha}_1, \ldots \hat{\alpha}_{\hat{n}}, \quad \hat{\beta}_1, \ldots \hat{\beta}_{\hat{m}})^T$$

and

$$x(t) = (-y(t), \ldots, -y(t-\hat{n}+1), \hat{\beta}_o u(t-1), \ldots, \hat{\beta}_o u(t-\hat{m}))^T$$

Then (2.4) can be written

$$y(t+\hat{k}+1) = \theta^T x(t) + \hat{\beta}_o u(t) + \hat{\varepsilon}(t+\hat{k}+1) \qquad (2.5)$$

The LS criterion for this model is (initial value effects are neglected):

$$V_t(\theta) = \frac{1}{t} \sum_{s=1}^{t} [y(s) - \hat{\beta}_o u(s-\hat{k}-1) - \theta^T x(s-\hat{k}-1)]^2 \qquad (2.6)$$

This function is minimized by

$$\theta(t) = P(t)h(t) \qquad (2.7)$$

where

$$P^{-1}(t) = \frac{1}{t} \sum_{s=1}^{t} x(s-\hat{k}-1)\, x(s-\hat{k}-1)^T \qquad (2.8)$$

and

$$h(t) = \frac{1}{t} \sum_{s=1}^{t} [y(s) - \hat{\beta}_o u(s-\hat{k}-1)]\, x(s-\hat{k}-1) \qquad (2.9)$$

If $\{v(t)\}$ in (2.1) is a sequence of <u>independent</u> random variables, then the <u>LS noise condition</u> is said to be satisfied. Then $\varepsilon(t)$ and $\varepsilon(s)$ are independent for $|t-s| > k$. If the LS noise condition is satisfied, $\hat{n} \geq n$, $\hat{m} \geq m$, $\hat{k} = k$ and $\hat{\beta}_o = \beta_o$, then it can be shown that $\theta(t)$ tends to the true value $(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m)^T$ w.p.1 (with probability one) as t tends to infinity, see Åström (1968). This question is further discussed in Chapter 3.

The solution (2.7) can be written recursively as

$$\theta(t+1) = \theta(t) + \frac{1}{t+1} P(t)\, x(t-\hat{k})[y(t+1) - \hat{\beta}_o u(t-\hat{k}) - \theta(t)^T x(t-\hat{k})] \cdot$$

$$\frac{t+1}{t+x(t-k)^T P(t) x(t-\hat{k})} \qquad (2.10)$$

$$P^{-1}(t+1) = P^{-1}(t) + \frac{1}{t+1}\,[x(t-\hat{k})\, x(t-\hat{k})^T - P^{-1}(t)]$$

In asymptotic analysis the factor $\dfrac{t+1}{t+x(t-k)^T P(t) x(t-k)}$ will be replaced by 1. Recursive formulas can also be given for P(t) directly, Åström (1968).

2.3 Identification using stochastic approximation.

A suitable identification criterion for model (2.4) is

$$J(\theta) = E[y(t+1) - \hat{\beta}_o u(t-\hat{k}) - \theta^T x(t-\hat{k})]^2 \qquad (2.11)$$

where the expectation is taken with respect to $\varepsilon(t+1)$, $\varepsilon(t)$, ... Naturally (2.11) cannot be computed when only a finite number of data y(t), u(t) are known. One approach is to replace (2.11) with the estimated mean value (2.6). This has been discussed in Section 2.2. Another approach is to apply the Robbins-Monro scheme, see e.g. Tsypkin (1973) to the derivative of (2.11). This gives

$$\theta(t+1) = \theta(t) + \gamma(t+1)\, x(t-\hat{k})[y(t+1) - \hat{\beta}_o u(t-\hat{k}) - x(t-\hat{k})^T \theta(t)] \qquad (2.12)$$

The sequence of scalars $\{\gamma(t)\}$ must satisfy certain conditions, Tsypkin (1973), which are further discussed in Chapter 4. Common choices of the sequence are

$$\gamma(t) = \frac{1}{t} \qquad (2.13a)$$

$$\gamma(t) = \left[\sum_{s=1}^{t-\hat{k}} |x(s)|^2\right]^{-1} \qquad (2.13b)$$

$$\gamma(t) = \frac{1}{t}\,(x(t-\hat{k})^T\, x(t-\hat{k}))^{-1} \qquad (2.13c)$$

Algorithm (2.12) is clearly quite similar to (2.10). The latter one requires more computation and more memory storage than (2.12). In return it converges more rapidly.

2.4 Self-tuning regulators.

Suppose that the input to the process, u(t), is determined as a feedback from old inputs and outputs. Suppose also that the coefficients of the feedback law are calculated from the current LS-estimates of the process parameters:

$$u(t) = f(\theta(t), x(t)) \tag{2.14}$$

This type of adaptive controllers, which are based on a straightforward separation between identification and control, is discussed by e.g. Kalman (1958), Peterka (1970), Åström - Wittenmark (1971), (1973) and Peterka - Åström (1973).

The equations (2.10), (2.12) and (2.14) thus define a class of self-tuning regulators. They have the form:

$$\theta(t+1) = \theta(t) + \gamma(t+1)S(t)x(t-\hat{k})[y(t+1) - \hat{\beta}_o u(t-\hat{k}) - \theta(t)^T x(t-\hat{k})] \tag{2.15a}$$

$$u(t) = f(\theta(t), x(t)) \tag{2.15b}$$

Linear regulators

$$u(t) = F(\theta(t))\, x(t) \tag{2.15c}$$

form an important subclass of the algorithm. Two choices of S(t) will be considered. Let P(t) be defined by

$$P^{-1}(t+1) = P^{-1}(t) + \gamma(t+1)[x(t-\hat{k})x(t-\hat{k})^T - P^{-1}(t)] \tag{2.15d}$$

Then S(t) is taken either as

$$S(t) = \frac{P(t)}{1+\gamma(t+1)\ [x(t-\hat{k})^T P(t) x(t-\hat{k})-1]} \qquad (2.15e)$$

or

$$S(t) = \frac{1}{\mathrm{tr}\ P^{-1}(t+1)} \qquad (2.15f)$$

The sequence $\{\gamma(t)\}$ is a sequence of deterministic scalars. If $\gamma(t) = 1/t$, the choice (2.15e) gives the recursive least squares algorithm, and (2.15f) gives the stochastic approximation algorithm (2.12) with (2.13b).

For future reference, the regulators with S(t) as in (2.15e) will be called <u>self-tuning regulators of LS type</u>. Correspondingly, with S(t) as in (2.15f) they will be called <u>self-tuning regulators of SA type</u>. Equation (2.15) will be used as a basic reference.

Various conditions on the noise sequence $\{\varepsilon(t)\}$ ($\{v(t)\}$) will be considered. In Chapter 3 and in Section 6.2 the <u>LS noise condition</u> is assumed to be satisfied. Then $\varepsilon(t+\hat{k}+1)$ and $x(t)$ are independent if $\hat{k} \geq k$. In the rest of the analysis the conditions are much less restrictive. They will be defined for each case, cf. (4.12) and (5.3).

2.5 Minimum variance control and self-tuning regulators.

Consider the system (2.3). Suppose that the LS noise condition is satisfied. If the input is chosen as

$$u(t) = -\frac{1}{\beta_o}\ [-\alpha_1 y(t) - \ldots - \alpha_n y(t-n+1) + \beta_1 u(t-1) + \ldots +$$

$$+ \beta_{m'} u(t-m')] \qquad (2.16)$$

then the output is

$$y(t) = \varepsilon(t)$$

Obviously, no other control law can yield lower variance of the output. The feedback law (2.16) is therefore called the minimum variance controller. It is discussed at length in Åström (1970).

If the parameters of model (2.3) are unknown, the minimum variance control law (2.16) cannot be computed. It is suggested in Åström - Wittenmark (1973) that the coefficients of (2.16) should be chosen as the LS-estimates of the system parameters. This means that $f(\theta(t), x(t))$ in (2.15) is chosen as

$$f(\theta(t), x(t)) = -\frac{1}{\hat{\beta}_0}\,\theta(t)^T x(t) \qquad (2.17)$$

The self-tuning regulator of SA type with this feedback law is called STURE0 (self-tuning regulator). The corresponding algorithm of LS-type is called STURE1. These regulators are discussed in e.g. Peterka (1970) and Wittenmark (1973). They have also been applied to industrial processes, see e.g. Cegrell - Hedqvist (1973) and Borisson - Wittenmark (1973).

Example 2.1. The behaviour of STURE1 will be illustrated on the system

$$y(t+1) + a\,y(t) = b\,u(t-1) + v(t+1) \qquad (2.18)$$

where $a = -0.9$, $b = 1$ and $v(t) = e(t) - 0.4\,e(t-1)$, $\{e(t)\}$ being a sequence of independent, random variables with distribution $N(0,1)$. The LS noise condition is thus not satisfied for (2.18). The minimum variance controller for the system (2.18) is given by

$$u(t) = -0.45\, y(t) - 0.5\, u(t-1) \qquad (2.19)$$

This controller gives in each step an expected loss

$$E\, y(t)^2 = 1.25$$

Without any control the loss is 2.32 per step. When using the self-tuning regulator the parameters $\hat{\alpha}$ and $\hat{\beta}$ are estimated from the model

$$y(t+2) + \hat{\alpha} y(t) = \hat{\beta}_o [u(t) + \hat{\beta} u(t-1)] + \hat{\varepsilon}(t+2)$$

using a stochastic approximation method, STURE0, or the least squares method, STURE1. Hence, in this case $\hat{m} = m'$, $\hat{n} = n$. The control law is then simply

$$u(t) = \frac{1}{\hat{\beta}_o} [\hat{\alpha} y(t) - \hat{\beta} u(t-1)]$$

The system has been simulated with STURE1 using $\hat{\beta}_o = 1$. The sequence $\{\gamma(t)\}$ was chosen as $1/t$ and the initial covariance matrix $P(0)$ was 0.1 times a unit matrix. The parameter estimates from one simulation are given in figure 2.1. Very quickly the estimates are quite close to the optimal ones. The quality of the control can be determined from the accumulated loss $\sum_{s=1}^{t} y(s)^2$. The accumulated loss at $t = 2000$ when using STURE1 is 2447, while when using the optimal regulator (2.19) the loss is 2430. The accumulated loss is shown in figure 2.2.

The good behaviour is somewhat unexpected. The LS-identification gives biased estimates in case the variables $\{v(t)\}$ are dependent, Åström - Eykhoff (1971). Also, for dependent noise the minimum variance control law is not given by (2.16). These two effects, however, compensate each other. It is shown in Åström - Wittenmark (1973) that if the regulator converges it must converge to the minimum variance control regulator. One of the main questions treated in this report is whether the algorithm actually converges.

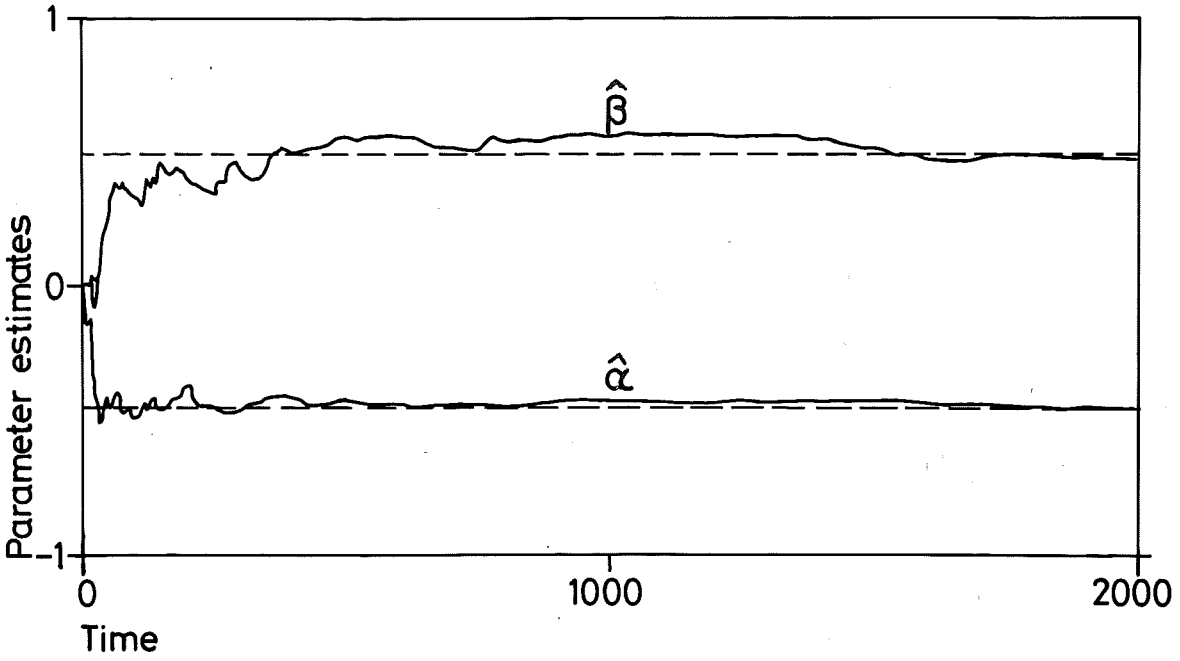

Fig. 2.1 Parameter estimates when the system (2.18) is controlled with STURE1. The dashed lines show the values corresponding to the optimal regulator (2.19).

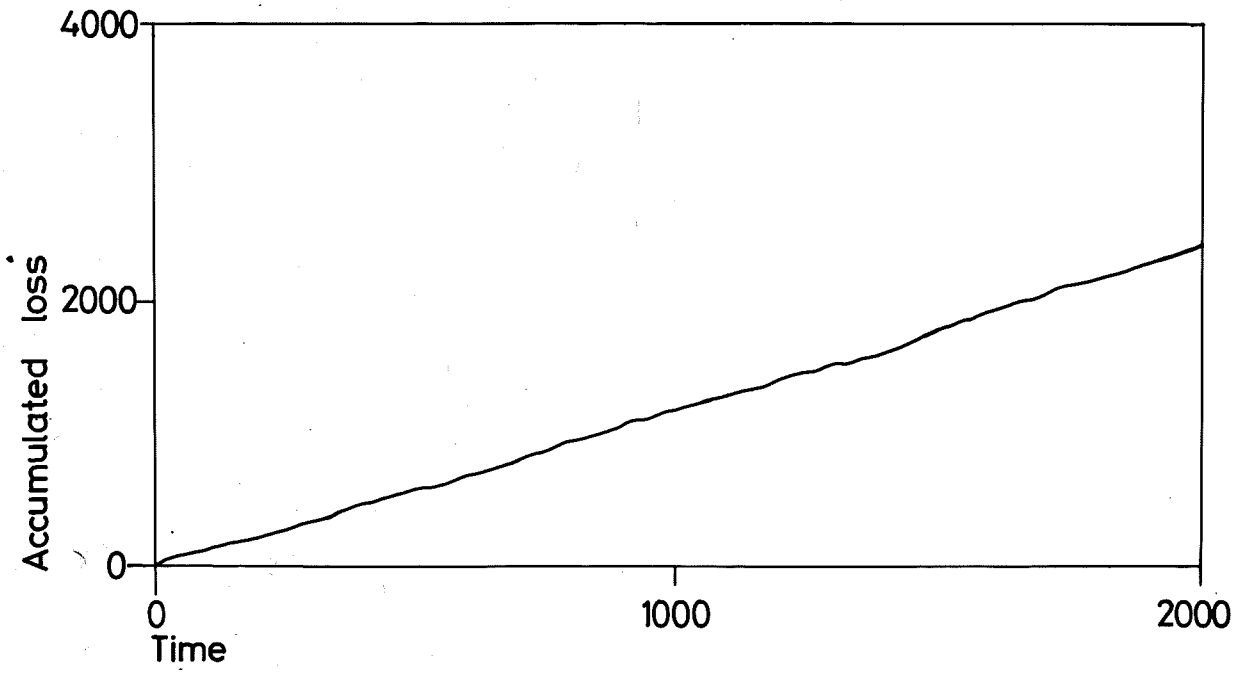

Fig. 2.2 The accumulated loss when the system (2.18) is controlled by STURE1.

3. LEAST SQUARES IDENTIFICATION OF CLOSED LOOP SYSTEMS

In Section 2.2 least squares identification of parameters in different equations was described. The convergence properties of the method are well known in case the input is persistently exciting and independent of the noise, Åström - Eykhoff (1971). However, for adaptive regulators, the input is determined as output feedback and will consequently be correlated with the noise. Moreover, the coefficients in the feedback law are time varying and depend in a complex way on previous input and output. The convergence under such conditions is treated in Section 3.1.

In Section 3.2 the results are applied to the self-tuning regulator STURE1. This analysis concerns basically the convergence of the regulator parameters.

It should be emphasized that the results of this chapter are valid only if the assumptions made about the model structure are true. This means that the LS noise condition is assumed to be satisfied. Furthermore, β_o and the time delay k must be known, and the model orders must not be underestimated (i.e. $\hat{n} \geq n$, $\hat{m} \geq m'$). The convergence properties when these assumptions no longer are true are discussed in the following chapters.

3.1 Consistency of least squares estimates.

Consider a system that is described by (2.3):

$$y(t+k+1) + \alpha_1 y(t) + \ldots + \alpha_n y(t-n+1) = \beta_o[u(t) + \beta_1 u(t-1) + \ldots + \beta_{m'} u(t-m')] + \varepsilon(t+k+1) \qquad (3.1)$$

Suppose that k and β_o are known constants, and that upper bounds $\hat{n}$ and $\hat{m}$ respectively for n and m' are known. Then

the model of (3.1) is

$$y(t+k+1) + \hat{\alpha}_1 y(t) + \ldots + \hat{\alpha}_{\hat{n}} y(t-\hat{n}+1) = \hat{\beta}_o [u(t) + \ldots +$$

$$+ \hat{\beta}_{\hat{m}} u(t-\hat{m})] + \hat{\varepsilon}(t+k+1) \qquad (3.2)$$

Introduce

$\theta_o = (\alpha_1, \ldots \alpha_n, 0,\ldots,0, \beta_1 \ldots \beta_{m'}, 0,\ldots,0)$; $\hat{n}$ - n and $\hat{m}$ - m' zeros respectively.

Then (3.1) and (3.2) can be written

$$y(t+k+1) = \theta_o^T x(t) + \beta_o u(t) + \varepsilon(t+k+1) \qquad (3.4)$$

$$y(t+k+1) = \theta^T x(t) + \beta_o u(t) + \hat{\varepsilon}(t+k+1) \qquad (3.5)$$

To show consistency (with probability one, w.p.1) of θ the following assumptions are usually made (see e.g. Åström - Eykhoff (1971)):

- o $\{\varepsilon(t)\}$ is a stationary sequence of random variables with zero mean values and bounded fourth moments, such that $\varepsilon(s)$ and $\varepsilon(t)$ are independent for $|t - s| > k$. (3.6a)
- o The polynomial (3.6b)
  $z^{n+k} + \alpha_1 z^{n-1} + \ldots + \alpha_n$
  is stable, i.e. all roots have magnitude less than one.
- o The input sequence $\{u(t)\}$ is independent of $\{\varepsilon(t)\}$ (3.6c)
- o The input sequence is persistently exciting of order $\hat{m}$. (3.6d)

Condition (3.6c) excludes systems with feedback. To show consistency also for closed loop systems the following assumptions are introduced:

- $\{\varepsilon(t)\}$ is a sequence (not necessarily stationary) of random variables with uniformly bounded second moments and zero mean values, such that $\varepsilon(t)$ and $\varepsilon(s)$ are independent for $|s - t| > k$. (3.7a)

- $\limsup_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} y^2(t) < \infty$ w.p.1 (3.7b)

  $\limsup_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} u^2(t) < \infty$ w.p.1

- $u(t)$ is independent of $\varepsilon(s)$ $s > t + k$ (3.7c)

- $\{\theta \mid \liminf_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} [\theta^T x(t)]^2 = 0\} = \{0\}$ w.p.1 (3.7d)

Clearly, (3.7abc) corresponds to (3.6abc) and are weaker conditions. Condition (3.7d) deserves some discussion. Let

$$\underline{u}(t) = [u(t-1) \quad u(t-2) \ \ldots \ u(t-\hat{m})]^T$$

$$\underline{y}(t) = [y(t) \quad y(t-1) \ \ldots \ y(t-\hat{n}+1)]^T$$

Thus

$$x(t) = \begin{bmatrix} -\underline{y}(t) \\ \hat{\beta}_o \underline{u}(t) \end{bmatrix}$$

Then condition (3.6d) can be written

$\lim_{N\to\infty} \frac{1}{N} \sum_{1}^{N} [\theta_u{}^T \underline{u}(t)]^2$ exists w.p.1 and is strictly positive for any non-zero vector θ_u of dimension $\hat{m}$.

The relationship between (3.6d) and (3.7d) is now quite clear. We have:

$$\theta^T x(t) = \theta_y{}^T \underline{y}(t) + \hat{\beta}_o \, \theta_u{}^T \underline{u}(t)$$

If the system is open loop, i.e. if (3.5c) is satisfied, it is easy to show that

$$\lim_{N\to\infty} \inf \frac{1}{N} \sum_{1}^{N} [\theta^T x(t)]^2 = 0 \quad \text{implies w.p.1 that } \theta_y = 0.$$ Thus,

in this case (3.7d) means

$$\lim_{N\to\infty} \inf \frac{1}{N} \sum_{1}^{N} [\theta_u^T \underline{u}(t)]^2 = 0 \Rightarrow \theta_u = 0$$

which essentially means that {u(t)} must be persistently exciting. It is, however, somewhat weaker, since it does not require that the limit exists. In case the system operates under output feedback, relationships between u and y exist. Condition (3.7d) states that these relations must not be of a certain kind. For example, if $\hat{n} = 2$, $\hat{m} = 1$ and the feedback is $u(t) = -y(t)$, then (3.7d) is not satisfied. The vector θ can be chosen as $(0\ 1\ 1)^T$ and all terms are identically zero.

The condition is formulated for limit inferior, since it is not known that the limit of

$$\frac{1}{N} \sum_{1}^{N} [\theta^T x(t)]^2$$

exists. This would require stationarity of the closed loop system, and might not be true in a number of applications.

Of the conditions (3.7) only (3.7d) is restrictive in practice. It can be interpreted as an identifiability condition for systems operating in closed loop. Such problems are discussed in a more general context in Gustavsson-Ljung-Söderström (1974). Conditions that are sufficient for (3.7d) to be satisfied are given there.

A theorem on consistency for LS estimates can now be formulated. The main tool to overcome the difficulties with non-

stationary processes {y(t)} and {u(t)} is the convergence theorem for martingales, see Doob (1953).

Theorem 3.1 Consider the system (3.1). The system parameters θ_o are estimated using an ordinary least squares criterion (cf. Section 2.2). Suppose that (3.7) is satisfied. Then the estimates $\theta(t)$ converge with probability one to their true values as the number of data tends to infinity.

Proof: Let k = 0 for convenience. The LS criterion to be minimized with respect to θ at step N is

$$V_N(\theta) = \frac{1}{N} \sum_{1}^{N} [y(t+1) - \beta_o u(t) - \theta^T x(t)]^2 =$$

$$= \frac{1}{N} \sum_{1}^{N} [(\theta_o-\theta)^T x(t) + \varepsilon(t+1)]^2 =$$

$$= \frac{1}{N} \sum_{1}^{N} \varepsilon(t+1)^2 + 2\frac{1}{N} \sum_{1}^{N} \varepsilon(t+1)\,(\theta_o-\theta)^T x(t) + \frac{1}{N} \sum_{1}^{N} [(\theta_o-\theta)^T x(t)]^2 \hat{=}$$

$$\hat{=} V_N^{(1)} + 2\, V_N^{(2)}(\theta) + V_N^{(3)}(\theta)$$

Let the minimizing θ be denoted by $\theta(N)$. Let F_t be the σ-algebra generated by $\{\varepsilon(0),\varepsilon(1),\ldots,\varepsilon(t)\}$. It is no loss of generality to assume that $E\varepsilon(t)^2 = 1$. Then

$$E\{[\varepsilon(t+1)y(t)]^2 | F_t\} = y(t)^2$$

Let

$$s(t) = \sum_{r=1}^{t} y(r)^2 + 1 \qquad s(0) = 1$$

and consider

$$z(t+1) = \sum_{r=1}^{t} \varepsilon(r+1)y(r)/s(r), \qquad z(1) = 0$$

The sequence $[z(t), F_t]$ is a martingale, since

$$E[z(t+1)|F_t] = z(t) + E[y(t)\varepsilon(t+1)/s(t)|F_t] =$$

$$= z(t) + \frac{y(t)}{s(t)} \; E(\varepsilon(t+1)|F_t) = z(t)$$

Consider

$$E\, z(N)^2 = \sum_{r=2}^{N} E(z(r)^2 - z(r-1)^2) = E \sum_{r=2}^{N} E[z(r)^2 - z(r-1)^2|F_{r-1}] =$$

$$= E \sum_{r=2}^{N} E[(z(r) - z(r-1))^2|F_{r-1}] = E \sum_{1}^{N-1} y(r)^2 / s(r)^2$$

$$\leq E \sum_{1}^{N-1} [s(r) - s(r-1)]/s(r)s(r-1) = E \sum_{1}^{N-1} \left[\frac{1}{s(r-1)} - \frac{1}{s(r)}\right] \leq 1$$

Hence z(N) converges with probability one due to the martingale convergence theorem. Kronecker's lemma (see e.g. Chung (1968)) now gives that

$$\frac{1}{s(N)} \sum_{1}^{N} \varepsilon(t+1)\, y(t) \to 0 \quad \text{w.p.1 as} \quad N \to \infty.$$

Since

$$\frac{N}{s(N)} > \delta \quad \text{for } N > N_0 \quad \text{from (3.7b)}$$

this means that

$$\frac{1}{N} \sum_{1}^{N} \varepsilon(t+1)\, y(t) \to 0 \quad \text{w.p.1} \quad \text{as } N \to \infty.$$

This is the first element in the column vector

$$\frac{1}{N} \sum_{1}^{N} \varepsilon(t+1)\, x(t)$$

Repeating the argument, it can be shown that also the other elements of the vector tend to zero w.p.1. Hence, the term

$$V_N^{(2)}(\theta) = \frac{1}{N}[\theta_o - \theta]^T \sum_1^N \varepsilon(t+1)\, x(t)$$

tends to zero uniformly in θ, w.p.1.

As the second step in the proof it will now be shown that

$\forall \varepsilon$, $\exists$ N_o (that depends on the realization) and δ_1 such that if $N > N_o(\omega)$, then $|\theta - \theta_o| > \varepsilon \Rightarrow V_N^{(3)}(\theta) > \delta_1$ (3.8)

If (3.8) is true, then

$V_N(\theta) > V_N(\theta_o) + \delta_1/2$ (where $V_N(\theta_o) = V_N^{(1)}$) for $|\theta - \theta_o| > \varepsilon$ and $N > N(\omega)$. Since $\theta(N)$ minimizes $V_N(\theta)$, this implies that $\theta(N) \to \theta_o$ w.p.1 as $N \to \infty$, i.e. the assertion of the theorem.

Suppose that (3.8) is not true. Then there exists a sequence $\{\tilde{\theta}_N\}$ such that

$$C > |\tilde{\theta}_N| > \delta$$

and

$$V_{N_i}(\theta_o + \tilde{\theta}_{N_i}) = \frac{1}{N_i} \sum_1^{N_i} [\tilde{\theta}_{N_i}^T x(t)]^2 \to 0 \text{ as } i \to \infty \text{ for a subsequence } \{\tilde{\theta}_{N_i}\}.$$

Let $\bar{\theta}$ be a cluster point to this subsequence. Then with

$$R_N = \frac{1}{N} \sum_1^N x(t)x(t)^T$$

$$V_{N_i}^{(3)}(\theta_o + \tilde{\theta}_{N_i}) = \bar{\theta}^T R_{N_i} \bar{\theta} - 2(\bar{\theta} - \tilde{\theta}_{N_i})^T R_{N_i} \bar{\theta} + (\bar{\theta} - \tilde{\theta}_{N_i})^T R_{N_i} (\bar{\theta} - \tilde{\theta}_{N_i})$$

But R_N is bounded according to (3.7b), $|\theta - \tilde{\theta}_{N_i}|$ tends to zero along a subsequence and $\bar{\theta}^T R_{N_i} \bar{\theta} > \delta_3$ according to (3.7d).

Hence $V_{N_i}^{(3)}(\theta_o+\theta_{N_i})$ cannot tend to zero and (3.8) follows.

This concludes the proof of the theorem. □

Remark: In case $k > 0$, the sum $V_N^{(2)}$ has to be split up into k sums:

$$V_N^{(2)}(\theta) = \frac{1}{N} \sum_{r=0}^{k-1} \sum_{t=1}^{N} \varepsilon(kt+r+1)[\theta_o-\theta]^T x(kt+r)$$

in order to apply the martingale theorem.

In case (3.7d) does not hold, the estimates may converge to several different limits depending on the realization. The set of possible convergence points is characterized in the following corollary, which is obtained by a slight modification of the proof of the theorem.

Corollary: Suppose (3.7abc) holds. Define the set

$$D_I = \{\theta \mid \lim_{N\to\infty} \inf \frac{1}{N} \sum_{1}^{N} [(\theta-\theta_o)^T x(t)]^2 = 0\}$$

(which depends on the realization ω).

Then $\theta(N) \to D_I$ w.p.1.†

Furthermore,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{1}^{N} [\theta(N)^T x(t)]^2 = 0 \quad \text{w.p.1}$$

□

Remark. The corollary can be seen as a special case of Theorem 5.1 in Ljung (1974b).

---

†By this it is meant that $\inf_{\theta\in D_I(\omega)} |\theta(N,\omega) - \theta| \to 0$ as $N \to \infty$ a.e. ω.

3.2 Self-tuning regulators.

In this section Theorem 3.1 is applied to the self-tuning regulator STURE1, described in Section 2.5. According to the corollary of this theorem,

$\theta(N) \to D_I$ w.p.1 as $N \to \infty$

It will be shown that all elements in D_I actually give the desired minimum variance control law. That is, if $\tilde{\theta} \in D_I$, then the feedback law (let $\beta_o = 1$)

$$u(t) = -\tilde{\theta}^T x(t)$$

gives the output

$$y(t) = \varepsilon(t)$$

Since

$$y(t+1) = \theta_o^T x(t-k) + u(t-k) + \varepsilon(t+1)$$

this implies that

$$(\theta_o - \tilde{\theta})^T x(t) = 0 \quad \text{all } t.$$

Consider a feedback law

$$F(q^{-1})\, u(t) = G(q^{-1})\, y(t)$$

where

$$F(z) = 1 + f_1 z + \ldots + f_{\hat{m}} z^{\hat{m}}$$

$$G(z) = g_1 + g_2 z + \ldots + g_{\hat{n}} z^{\hat{n}}$$

and q^{-1} is the backward shift operator.

If the polynomials F(z) and G(z) have common factors, this feedback law will generate x(t)-vectors that lie in a certain subspace of $R^{\hat{m}+\hat{n}}$. Let the subspace corresponding to the minimum variance control law be denoted by H. The dimension of this subspace is $\hat{m} - m' + \hat{n} - n$.

We will by a somewhat heuristic argument show that

$$\tilde{\theta} \in D_I \Rightarrow (\theta_0 - \tilde{\theta}) \perp H$$

This implies that all elements in D_I give the minimum variance control law, since

$$u(t) = \tilde{\theta}^T x(t) = \theta_0^T x(t) + (\tilde{\theta} - \theta_0)^T x(t) = \theta_0^T x(t)$$

and

$$y(t) = \varepsilon(t)$$

for $x(t) \in H$.

Suppose first that $\theta(N)$ gives the minimum variance control law for some N. The produced x(t) then belong to H and the obtained estimates $\theta(k)$, k=N,... must then satisfy

$$(\theta(k) - \theta_0) \perp H$$

since, according to the corollary of Theorem 3.1

$$\frac{1}{k} \sum_{t=1}^{k} [\theta(k)^T x(t)]^2 \to 0 \text{ as } k \to \infty$$

These estimates consequently also give the minimum variance control law, which shows that minimum variance control will continue if it once has started.

Suppose now that the estimates $\theta(N_k)$ tend to a point θ^* that does not give minimum variance control. Then the produced $x(t)$ are not orthogonal to $\theta_0 - \theta^*$, and the obtained new estimates will move away from θ^*. When doing so, the produced $x(t)$ will eventually span either the whole space or H. In both cases an estimate that gives minimum variance control results, and according to the discussion above, it will continue thereafter.

Hence, if $\hat{n} \geq n$, $\hat{m} \geq m'$, D_I consists of parameters which give the desired minimum variance control law, so even if the estimates do not converge to their true values, they will still give the desired controller. It also follows that if $\hat{n} = n$, $\hat{m} = m'$, the estimates will actually converge to the true parameter θ_0.

4. TOOLS FOR CONVERGENCE ANALYSIS

4.1 Background.

In Chapter 3 the self-tuning regulator STURE1 was analysed in case the assumptions about orders and noise characteristics were true. It was remarked in example 2.1 that this self-tuning regulator has desired behaviour also in some cases when the assumptions are not satisfied. The analysis of these cases cannot be formulated as consistency questions for the identification. This is clear, since there no longer are any "true" parameter values and no consistent estimates.

When the LS noise condition is not satisfied, the estimates will in general be biased. The bias depends on the feedback law. The effect is clearly seen from the following example:

Example 4.1 Consider the system

$$y(t+1) + ay(t) = u(t) + e(t+1) + ce(t) \tag{4.1}$$

where $\{e(t)\}$ is white noise with variance λ and $|c| < 1$. The following model is assumed:

$$y(t+1) + \hat{a}y(t) = u(t) + \varepsilon(t+1) \tag{4.2}$$

It is straightforward to show that if $u(t)$ is chosen as white noise with variance μ, independent of e, then the LS estimate of $\hat{a}$ tends to

$$\hat{a} = a - \frac{c\lambda(1-a^2)}{\lambda(1+c^2-2ac)+\mu} \tag{4.3}$$

On the other hand, if $u(t)$ is determined as output feedback,

$$u(t) = g\, y(t)$$

then the LS estimate of $\hat{\alpha}$ tends to

$$\hat{\alpha} = a - \frac{c\,(1 - (a-g)^2)}{1 + c^2 - 2(a-g)\,c} \qquad (4.4)$$

For the self-tuning regulator, STURE1, the feedback coefficient g at time t is chosen as the current estimate of α. It is thus time varying, which makes it difficult to analyse the behaviour of the algorithm. In Åström - Wittenmark (1973) an attempt is made to heuristically analyse it when applied to the system (4.1). The feedback coefficient is assumed to be fixed $= \hat{\alpha}_k$ over a long time period. During this period the estimated of α converges to

$$\hat{\alpha}_{k+1} = a - \frac{c(1 - (a-\hat{\alpha}_k)^2)}{1 + c^2 - 2(a-\hat{\alpha}_k)\,c} \qquad (4.5)$$

which is taken as the next feedback coefficient, etc. It is then argued that if (4.5) converges to the desired regulator ($\hat{\alpha} = a-c$), this should be taken as an indication of convergence of the self-tuning regulator.

In this heuristic analysis the important feature, that the feedback coefficient actually changes in every step, is neglected. To include it, consider the change of the estimate over just one step, instead of over a very long time period. The estimate of α at time t, $\hat{\alpha}_t$ is given by (2.15).

$$\hat{\alpha}_{t+1} = \hat{\alpha}_t + \frac{y(t)}{\sum_1^t y^2(k)} \{y(t+1) + \hat{\alpha}_t y(t) - u(t)\}$$

Since u(t) is chosen as $\hat{\alpha}_t y(t)$ we have

$$\hat{\alpha}_{t+1} - \hat{\alpha}_t = \frac{y(t)y(t+1)}{\sum_1^t y^2(k)} = \frac{1}{\frac{1}{t}\sum_1^t y^2(k)} \cdot \frac{1}{t}\, y(t)y(t+1) \qquad (4.6)$$

The first factor is disregarded in this intuitive discussion. Consider $Ey(t)y(t+1)$. This value of the covariance function depends on the feedback coefficient, since the closed loop behaviour is affected by the feedback. The expectation exists only if the closed loop system is stable. Let the feedback be

$$u(t) = \alpha y(t)$$

and denote

$$E\, y(t)y(t+1) = f(\alpha)$$

In (4.6) the difference $\hat{\alpha}_{t+1} - \hat{\alpha}_t$ tends to zero as t tends to infinity. Hence an increasing number of sample points, say T, are required to change α a given small distance. Let $\Delta\tau = \sum_{t}^{t+T} 1/k$. The change is caused by a large number of random variables $y(t)y(t+1)$, which all have approximately the mean value $f(\hat{\alpha}_t)$. It is reasonable to assume that due to some "law of large numbers," the change is proportional to $f(\hat{\alpha}_t)$:

$$\hat{\alpha}_{t+T} = \hat{\alpha}_t + \Delta\tau\, f(\hat{\alpha}_t)$$

This scheme can be seen as an approximation to the ordinary differential equation (ODE):

$$\frac{d}{d\tau}\hat{\alpha} = f(\hat{\alpha}) \qquad (4.7)$$

□

From the example it seems plausible that the trajectories of (4.7) in some sense describe the sequence of estimates. In fact, in Section 4.2 it is shown that stability of (4.7) implies convergence of the algorithm. In Section 4.3 it is shown that the trajectories of (4.7) actually can be interpreted as "expected paths" for the sequence of estimates. The results are shown for the general linear self-tuning regulator (2.15) with (2.15c). The regulator is treated as a

general recursive algorithm:

$$\theta(t+1) = \theta(t) + \gamma(t)\, Q(t, \theta(t), \ldots, \theta(0), e(t+1)) \qquad (4.8)$$

Similar convergence results for such algorithms are shown in Ljung (1974).

4.2 Convergence.

Consider the class of self-tuning algorithms (2.15) with linear feedback (2.15c). Some additional assumptions about the noise, the gain sequence $\{\gamma(t)\}$ and the closed loop behaviour are first introduced.

Introduce

$$f(\theta) = E\, x(t)\, [y(t+\hat{k}+1) - \theta^T x(t) - \hat{\beta}_o u(t)] \qquad (4.9)$$

$$G_1(\theta) = E\, x(t)x(t)^T$$

$$G_2(\theta) = E\, x(t)^T x(t) \qquad (4.10)$$

where the expectation shall be taken, assuming that the system is regulated by the time invariant feedback law

$$u(t) = F(\theta)\, x(t)$$

It shall be assumed that the input and output sequences have reached stationarity, i.e. effects of initial values are neglected. This statement deserves some discussion. In the algorithm (2.15) strict stationarity for the input and output sequences is never achieved, mainly due to the time varying feedback. The expected change in the variables actually depends on all previous feedback laws. It would be quite impossible to calculate the expectation values in (4.9), (4.10) taking such dependences into account. It is therefore

a significant result if these effects can be neglected. The functions f and G_i are simple functions of certain covariances.

Stationarity can be achieved, and hence the functions f and G_i defined, only if the closed loop system obtained with $u(t) = F(\theta)\, x(t)$ is stable. Therefore a condition that assures that the closed loop system is not unstable all the time must be introduced:

> The feedback regulator is such that there w.p.1 exists a subsequence N_k, (which may depend on the realization) such that $\theta(N_k)$ belongs to a closed subset of the area that gives stable closed loop systems, and such that $|x(N_k)|$ is bounded. If the area which gives stable closed loop systems is unbounded, it is assumed that the estimates are prevented from tending to infinity by some suitable projection algorithm, cf. Ljung (1974) Chapter 5. (4.11)

The convergence of the algorithm (2.15) also depends on the sequence $\{\gamma(t)\}$ and on the noise $\{\varepsilon(t)\}$. Assume that $\varepsilon(t)$ is obtained as filtered white noise:

$$\varepsilon(t) = \frac{C(q^{-1})}{D(q^{-1})}\, e(t) \tag{4.12}$$

where $\{e(t)\}$ is white noise, C(z) and D(z) are polynomials and q^{-1} is the backward shift operator. The polynomial D(z) is assumed to have all zeroes outside the unit circle.

Further assume that

$$E|\varepsilon(t)|^{4p} \leq C_1 \qquad p \text{ integer} \tag{4.13}$$

The sequence $\{\gamma(t)\}$ is taken as

$$\gamma(t) = c_\gamma t^{-s} \qquad 1/p < s \leq 1 \qquad c_\gamma > 0, \tag{4.14}$$

Theorem 4.1 Consider the algorithm (2.15) with linear feedback (2.15c). Suppose that the feedback law $F(\theta)$ is Lipschitz continuous and such that the stability condition (4.11) is satisfied. Let (4.12) - (4.14) hold for the noise and for the gain sequences. Let $f(\theta)$, $G_1(\theta)$ and $G_2(\theta)$ be defined by (4.9), (4.10). Consider the ODE

$$\frac{d}{d\tau}\,\theta(\tau) = S_i(\tau)f(\theta(\tau)) \tag{4.15a}$$

$$\frac{d}{d\tau}\,S_i(\tau) = S_i(\tau) - S_i(\tau)G_i(\theta(\tau))S_i(\tau) \tag{4.15b}$$

where for the LS algorithm (2.15e) $i = 1$, and for the SA algorithm (2.15f) $i = 2$. Assume that it has a stationary point (θ^*, S^*) that is globally asymptotically stable.† Then the solution of (2.15), $\theta(t)$, tends to θ^* w.p.1 as t tends to infinity.

Remark: Notice that for the SA algorithm, S_2 is a positive scalar. Then, instead of (4.15), it is sufficient to require that the ODE

$$\frac{d}{d\tau}\,\theta(\tau) = f(\theta(\tau)) \tag{4.16}$$

is asymptotically stable. Notice also that (4.15b) can be written

$$\frac{d}{d\tau}\,S_i^{-1}(\tau) = G_i(\theta(\tau)) - S_i^{-1}(\tau)$$

Proof: The theorem is proved quite analogously to theorems 3.1 and 4.1 in Ljung (1974).

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†In the region where S_i is strictly positive (definite).

The proof is technically involved. The basic idea is however, simple. The idea is that the sequence $\{\theta(t), S_i(t)\}$ behaves like solutions to the ODE (4.15). In the proof the intuitive arguments on page 26 are formalized. The following lemma characterizes the local behaviour of the estimate sequence and a connection with the ODE (4.15) is established.

Lemma 4.1 Suppose $\theta(n)$ and $\bar{\theta}$ belong to the area where f and G_i are defined. Let $m(n,\Delta\tau)$ satisfy

$$\sum_{n}^{m(n,\Delta\tau)} \gamma(k) \to \Delta\tau \quad \text{as } n\to\infty$$

Suppose that $|x(n)| < C$ (C may depend on the realization). Then for sufficiently small $\Delta\tau$ and $(\theta(n),S_i(n))$ sufficiently close to $(\bar{\theta},\bar{S}_i)$

$$\theta(m(n,\Delta\tau)) = \theta(n) + \Delta\tau\bar{S}_i f(\bar{\theta}) + R_1(n,\Delta\tau,\bar{\theta},\bar{S}_i) + R_2(n,\Delta\tau,\bar{\theta},\bar{S}_i) \tag{4.17}$$

$$S_i(m(n,\Delta\tau)) = S_i(n) + \Delta\tau[-\bar{S}_i G_i(\bar{\theta})\bar{S}_i + \bar{S}_i] + R_1'(n,\Delta\tau,\bar{\theta},\bar{S}_i) +$$

$$+ R_2'(n,\Delta\tau,\bar{\theta},\bar{S}_i)$$

where

$$|R_1^{(')}(n,\Delta\tau,\bar{\theta},\bar{S}_i)| \leq \Delta\tau\cdot K\{|\theta(n) - \bar{\theta}| + |S_i(n) - \bar{S}_i|\} + A(\Delta\tau)^2$$

and

$$R_2^{(')}(n,\Delta\tau,\bar{\theta},\bar{S}_i) \to 0 \text{ w.p.1 as } n\to\infty. \qquad \square$$

The lemma is proved in appendix A. It implies that for large n, $\theta(n)$ will follow the ODE (4.15) locally. Consider from now on the SA algorithm and assume that (4.16) is asymptotically stable. The LS case is treated analogously.

From Krasovskij (1963) the existence of a Lyapunov function $V(\theta)$ for the ODE (4.16) is inferred. The function V is infinitely differentiable, positive definite and has a negative definite time derivative along solutions to (4.16). It is readily shown that (4.17) implies that, for sufficiently small $\Delta\tau$ and large n, and $\theta(n)$, $S_2(n)$ sufficiently close to $\bar{\theta}$, $\bar{S}_2$ we have

$$V[\theta(m(n,\Delta\tau))] < V(\bar{\theta}) - \Delta\tau\bar{S}_2\delta/2 \qquad (4.18)$$

where

$$-\delta = \frac{d}{d\tau} V(\theta(\tau))\Big|_{\theta=\bar{\theta}} = V'(\bar{\theta})\, f(\bar{\theta})$$

Consider from now on a fixed realization ω. In order to use eq. (4.17) a sequence $\{\theta(n_k,(\omega)\}$ tending to $\bar{\theta}(\omega)$ as n_k tends to infinity will be considered. The existence of such a sequence follows from (4.11). The argument ω is suppressed in the sequel. Now, applying (4.17) and (4.18) to the sequence $\{\theta(n_k)\}$ gives, cf. Ljung (1974)

$$V(\theta(m(n_k,\Delta\tau))) < V(\bar{\theta}) - S_2(n_k)\Delta\tau\delta/64$$

It is quite clear that $S_2(n_k)$ is, possibly after extraction of a new subsequence n_k, bounded from below by a positive constant. Hence

$$V[\theta(m(n_k,\Delta\tau))] < V(\bar{\theta}) - \Delta\tau\cdot\delta'$$

In lemmas A.2 and A.3 in Ljung (1974) it is formally shown that the inequality above, which holds for any clusterpoint $\bar{\theta} \neq \theta^*$, and sufficiently large n_k implies that $\theta(n) \to \theta^*$ as $n\to\infty$ for the chosen realization. This holds for almost every realization (cf. Ljung (1974)) and the theorem follows.

□

Let us apply the theorem to the system in Example 4.1, governed by the self-tuning regulator STURE1. Clearly, $F(\theta) = -\theta$ is Lipschitz continuous. It is quite straightforward to show that (4.11) is satisfied, see Section 5.1. Let the noise $\{\varepsilon(t)\}$ be normally distributed and take $\gamma(t) = t^{-1}$. Then (4.12), (4.14) are satisfied. Let the feedback be $u(t) = \alpha y(t)$. The function

$$f(\alpha) = E\, y(t)y(t+1)$$

is then easily calculated and the ODE (4.16) is

$$\dot{\alpha} = -\frac{(c-a+\alpha)\,(1-c(a-\alpha))}{1-(a-\alpha)^2} \qquad \text{defined for } |\alpha-a| < 1 \qquad (4.19)$$

With $z = \alpha - a + c$

$$\dot{z} = -z\,\frac{1-c(c-z)}{1-(c-z)^2} \qquad \text{defined for } |z-c| < 1$$

Clearly, the solution $z^* = 0$ is globally asymptotically stable. It now follows from the theorem that

$$\hat{\alpha}(t) \to a-c \quad \text{as } t\to\infty$$

which gives the minimum variance control law $u(t) = (a-c)\,y(t)$. In Chapter 6 more general systems are analysed.

The theorem is formulated for linear feedback laws. In a number of applications the input is limited. If, in such a case, the open loop system is stable, then (4.11) is trivially satisfied and the ODE's are defined everywhere. This kind of nonlinear regulator requires some minor modifications of the proof of Theorem 4.1. Limitation of the input signals naturally affects the function $f(\theta)$. Therefore limitation may affect both convergence and limit point.

It is relevant to ask what connection Theorem 4.1 has to stochastic Lyapunov functions. These have been discussed e.g. by Kushner (1967). A stochastic Lyapunov function is a positive supermartingale. It is assumed that the process for which convergence shall be shown (i.e. $\{\theta(t)\}$ in the present case) is a Markov process. However, for the present application $\{\theta(t)\}$ is not a Markov process. Also, a Lyapunov function for (4.15) is generally not a stochastic Lyapunov function for the sequence of estimates. This means that it is more difficult to find a stochastic Lyapunov function for the sequence of estimates than to show stability for (4.15). This is illustrated in the following simple example.

Example 4.2 Consider the algorithm

$$c_{n+1} = c_n + \gamma_n(e_n - c_n); \quad \gamma_n = 1/n$$

where $\{e_n\}$ is a sequence of independent, random variables with zero mean values and unit variances. This example is much simpler than the algorithms considered in Theorem 4.1. It is readily shown that the corresponding ODE is

$$\dot{c} = -c$$

the stability of which easily is shown, e.g. by means of the Lyapunov function $V(c) = 1/2\ c^2$. However, $V(c)$ is not a stochastic Lyapunov function for c_n since

$$E\{V(c_{n+1}) - V(c_n) \mid c_n, \ldots, c_o\} = -1/n\ c_n^2 + \frac{1}{n^2}(c_n^2+1)/2$$

The RHS is greater than zero for $c_n = 0$.

4.3 Behaviour of the algorithm.

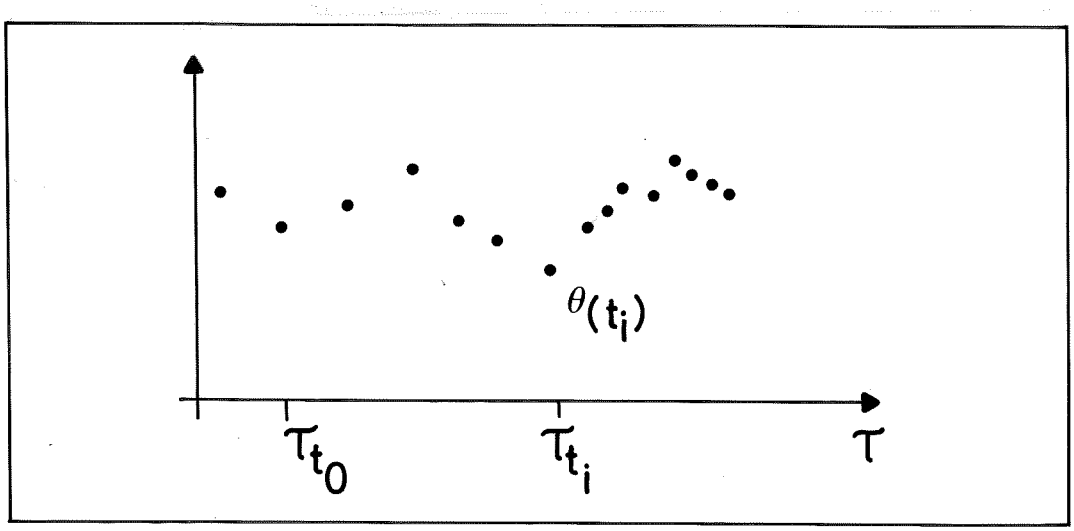
The ODE (4.15) is important not only for the question of convergence. It can, in fact, be shown that the trajectories of (4.15) also govern the behaviour of the sequence of estimates

$\{\theta(t)\}$ obtained from (2.15).

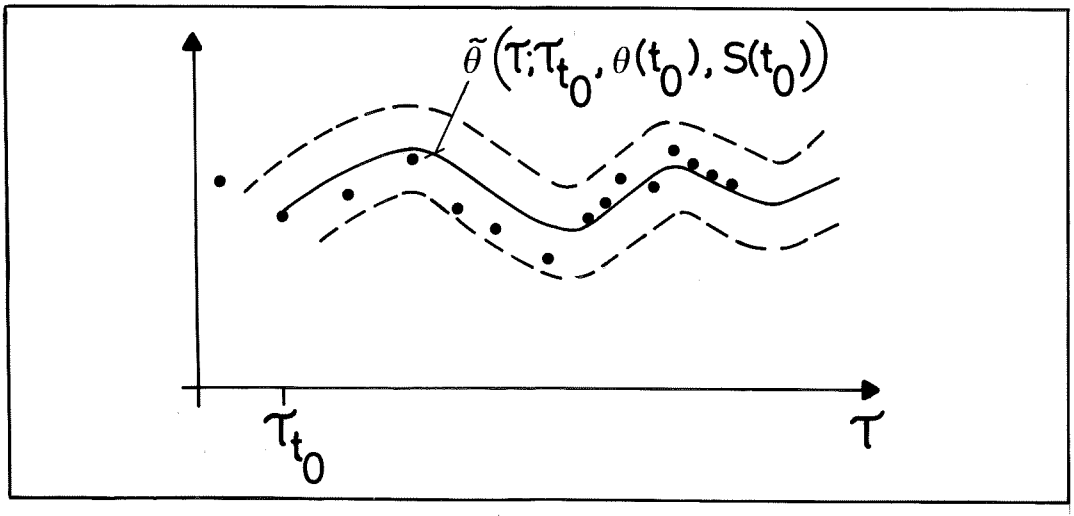
The result is formulated as follows. Let $\{\theta(t)\}$ be generated by (2.15). Introduce a fictitious time τ by

$$\tau_n = \sum_{1}^{n} \gamma(t)$$

Suppose that the estimates $\theta(t)$ are plotted against τ_t:



Let $\tilde{\theta}(\tau;\ \tau_{t_o},\ \theta(t_o),\ S(t_o))$ be the solution of (4.15) with initial value $\theta(t_o)$, $S(t_o)$ at time τ_{t_o}. Plot also this solution in the same diagram:



Let I be a set of integers. The probability that all points $\theta(t)$; $t\in I$ simultaneously are within a certain distance ε from the trajectory is estimated in the following theorem:

Theorem 4.2 Consider algorithm (2.15) with the same conditions on $\{e(k)\}$ and $\{\gamma(k)\}$ as in Theorem 4.1. Denote $\tau_t = \sum_1^t \gamma(s)$. Assume that the right hand side of (4.15) is continously differentiable. Denote the solution of (4.15) with initial condition $\theta(t_o)$, $S(t_o)$ at $\tau = \tau_{t_o}$ by $\tilde{\theta}(\tau; \tau_{t_o}, \theta(t_o), S(t_o))$. Consider the ODE (4.15) linearized around this solution. Suppose that there exists a quadratic Lyapunov function for this linear, time varying ODE. (See e.g. Brockett (1970)). Let I be a set of integers, such that $\inf |\tau_i - \tau_j| = D > 0$ where $i\neq j$ and $i,j\in I$. Then there exists a K, δ_o and ε_o, such that for $\varepsilon < \varepsilon_o$, $|\gamma(t_o)x(t_o)| < \delta_o$.

$$P\{\sup_{\substack{t\in I\\ t>t_o}} |\theta(t) - \tilde{\theta}(\tau_t; \tau_{t_o}, \theta(t_o), S(t_o))| > \varepsilon\} \leq$$

$$\leq \frac{K^r}{\varepsilon^{4r}} \sum_{j=t_o}^{N} \gamma(j)^r \qquad r \leq p \tag{4.20}$$

where $N = \sup_{t\in I} t$, which may be ∞. □

The proof is based on Lemma 4.1 and follows from this lemma in exactly the same way as Theorem 6.1 in Ljung (1974) is proved.

Since the sum

$$\sum_1^\infty \gamma(j)^p$$

is convergent, the RHS of (4.20) can, for fixed ε, be chosen arbitrarily small by taking t_o sufficiently large. Thus, the theorem states that the trajectories of the ODE (4.15) arbitrarily well describe the behaviour of the algorithm (2.15)

for sufficiently large time points.

It should be remarked that, although the proof of Theorem 4.2 provides an estimate of K, it is not practically feasible to use the theorem to obtain numerical bounds for the probability. The estimates are too crude. The main value of the theorem is that a basic relationship between the trajectories and the algorithm is established.

To summarize, Theorems 4.1 and 4.2 state that analysis of the time invariant, deterministic ODE (4.15) gives valuable insight into the behaviour of the time varying, nonlinear stochastic difference equation (2.15). In Chapter 6 the ODE's that correspond to the self-tuning algorithms, STURE0 and STURE1, are derived and analysed.

5. STABILIZATION PROPERTIES

The self-tuning regulators STURE1 and STURE0, were defined in Section 2.5. They were originally, Wieslander - Wittenmark (1971), designed for control of system (2.1) when the LS noise condition is satisfied. The analysis in Chapter 3 shows that the regulators have desired behaviour in this case. If the noise has a more general structure, the parameter estimates will be biased. The bias depends on the control law and this in turn depends on the current estimates. This makes it quite difficult to follow the estimation process. The performance outside the stability region of the closed loop system is considered in this chapter.

To apply Theorem 4.1 it is required that the stability condition (4.11) is satisfied. In Section 5.1 it is shown that both self-tuning regulators have this desired property for quite general noise sequences. The time delay k must be known, and the orders of the system must not be underestimated.

In Section 5.2 a stronger result is shown for STURE1. It is shown that

$$\frac{1}{N}\sum_{k=1}^{N} y(k)^2$$

is uniformly bounded in N w.p.1. This result ensures a stable behaviour of the closed loop system. This holds also if the open loop system is unstable. Thus STURE1 stabilizes any system, provided the time delay is known.

5.1 A general stability property.

The stability properties of the self-tuning regulators, STURE1 and STURE0, are investigated in this section. It is shown that these regulators satisfy the condition (4.11). To make

the discussion easier to follow, some of the arguments are kept on a somewhat heuristic level. It should, however, meet no difficulties to convert the discussion into a formal proof.

Consider the system (2.3) and the model (3.4). Assume that $\hat{k} = k$, and that $\hat{m} \geq m'$, $\hat{n} \geq n$. For simplicity in the formal treatment, the time delay k is supposed to be zero in this section.

Form the following vector from the parameter values α_i, β_i and the estimates $\hat{\alpha}_i(t)$, $\hat{\beta}_i(t)$

$$\tilde{\theta}(t) = [\alpha_1 - \frac{\beta_o}{\hat{\beta}_o}\hat{\alpha}_1(t), \ldots, \alpha_n - \frac{\beta_o}{\hat{\beta}_o}\hat{\alpha}_n(t), \ldots, -\frac{\beta_o}{\hat{\beta}_o}\hat{\alpha}_{\hat{n}}(t),$$

$$, \frac{\beta_1}{\hat{\beta}_o} - \frac{\beta_o}{\hat{\beta}_o}\hat{\beta}_1(t), \ldots, \frac{\beta_{m'}}{\hat{\beta}_o} - \frac{\beta_o}{\hat{\beta}_o}\beta_{m'}(t), \ldots, -\frac{\beta_o}{\hat{\beta}_o}\hat{\beta}_{\hat{m}}(t)]^T$$

Then, with $u(t) = -\frac{1}{\hat{\beta}_o}\theta(t)^T x(t)$, (2.15a) and (2.3) can be rewritten as

$$\tilde{\theta}(t+1) = \tilde{\theta}(t) - \gamma(t+1)\, S(t)\, x(t)[x(t)^T \frac{\beta_o}{\hat{\beta}_o}\tilde{\theta}(t) - \varepsilon(t+1)] \quad (5.1)$$

$$y(t+1) = \tilde{\theta}(t)^T x(t) + \varepsilon(t+1) \quad (5.2)$$

The sequence $\varepsilon(t)$ is supposed to satisfy

$$\frac{1}{N}\sum_{t=1}^{N}\varepsilon(t)^2 < K \qquad N > N_o \quad (5.3)$$

where K may depend on the realization. For STURE1

$$S(t) = \frac{P(t)}{1 + \gamma(t+1)[x(t)^T P(t) x(t) - 1]}$$

where P(t) is given by (2.15d). For STURE0

$$S(t) = \frac{1}{\sum_{s=1}^{t} \xi_s^t x(s)^T x(s)}$$

where

$$\xi_s^t = \gamma(s) \prod_{j=s+1}^{t} (1-\gamma(j)); \quad \xi_t^t = \gamma(t)$$

Decompose $\tilde{\theta}(t)$ into one component parallell to x(t), denoted by $\tilde{\theta}_{\|}(t)$ and one orthogonal to x(t), $\theta_{\perp}(t)$:

$$\tilde{\theta}(t) = \tilde{\theta}_{\|}(t) + \tilde{\theta}_{\perp}(t) \tag{5.4}$$

The symmetrical matrix $x(t)\, x(t)^T$ has all eigenvalues but one equal to zero. The non-zero eigenvalue is $x(t)^T\, x(t)$, and the corresponding eigenvector is x(t). Hence

$$x(t)\, x(t)^T\, \tilde{\theta}(t) = x(t)^T\, x(t)\, \tilde{\theta}_{\|}(t)$$

Introduce also

$$\lambda(t) = \frac{\gamma(t+1)\, x(t)^T\, x(t)}{\sum_{1}^{t} \xi_s^t x(s)^T\, x(s)} \tag{5.5}$$

Clearly

$$0 \leq \lambda(t) \leq 1 \tag{5.6}$$

Eq. (5.1) can now be written

$$\tilde{\theta}(t+1) = \tilde{\theta}(t) - \lambda(t) \frac{\beta_0}{\hat{\beta}_0} \tilde{\theta}_{\|}(t) + \gamma(t+1) S(t) x(t) \varepsilon(t+1) \tag{5.7}$$

When $|x(t)|$ is sufficiently large, say $\geq K'$, the last term in (5.7) can be neglected, and the following relation is obtained:

$$|\tilde{\theta}(t+1)|^2 \simeq |\tilde{\theta}(t)|^2 - 2\lambda(t)\frac{\beta_o}{\hat{\beta}_o}|\tilde{\theta}_{\parallel}(t)|^2 + \lambda(t)^2\left(\frac{\beta_o}{\hat{\beta}_o}\right)^2|\tilde{\theta}_{\parallel}(t)|^2 \tag{5.8}$$

If $0 < \frac{\beta_o}{\hat{\beta}_o} < 2$, it follows from (5.8) and (5.5) that

$$|\tilde{\theta}(t+1)|^2 \leq |\tilde{\theta}(t)|^2 - c\lambda(t)|\tilde{\theta}_{\parallel}(t)|^2 \qquad c > 0 \tag{5.9}$$

From (5.2) follows that for large $|x(t)|$

$$|y(t+1)|^2 \simeq |\tilde{\theta}_{\parallel}(t)|^2|x(t)|^2 \leq |\tilde{\theta}_{\parallel}(t)|^2 \cdot \max_{t\leq s\leq t-\hat{n}}(y(s)^2, u(s)^2)\cdot 4\hat{n}^2 \tag{5.10}$$

Assume that the system is minimum phase, i.e. that the polynomial

$$B(z) = \beta_o + \beta_1 z + \beta_2 z^2 + \ldots + \beta_{m'} z^{m'}$$

has all its zeroes outside the unit circle. Then $|u(t)|$ is not significantly much larger than $|y(t)|$.

Assume now that (4.11) does not hold, i.e. that the feedback

$$u(t) = -\frac{1}{\hat{\beta}_o}\theta(t)^T x(t)$$

gives an unstable, closed loop system for all $t > N$. We will lead this assumption into contradiction. Distinguish between the following two cases:

Case a.

$$\limsup_{t\to\infty} |\tilde{\theta}(t+1) - \tilde{\theta}(t)| = 0$$

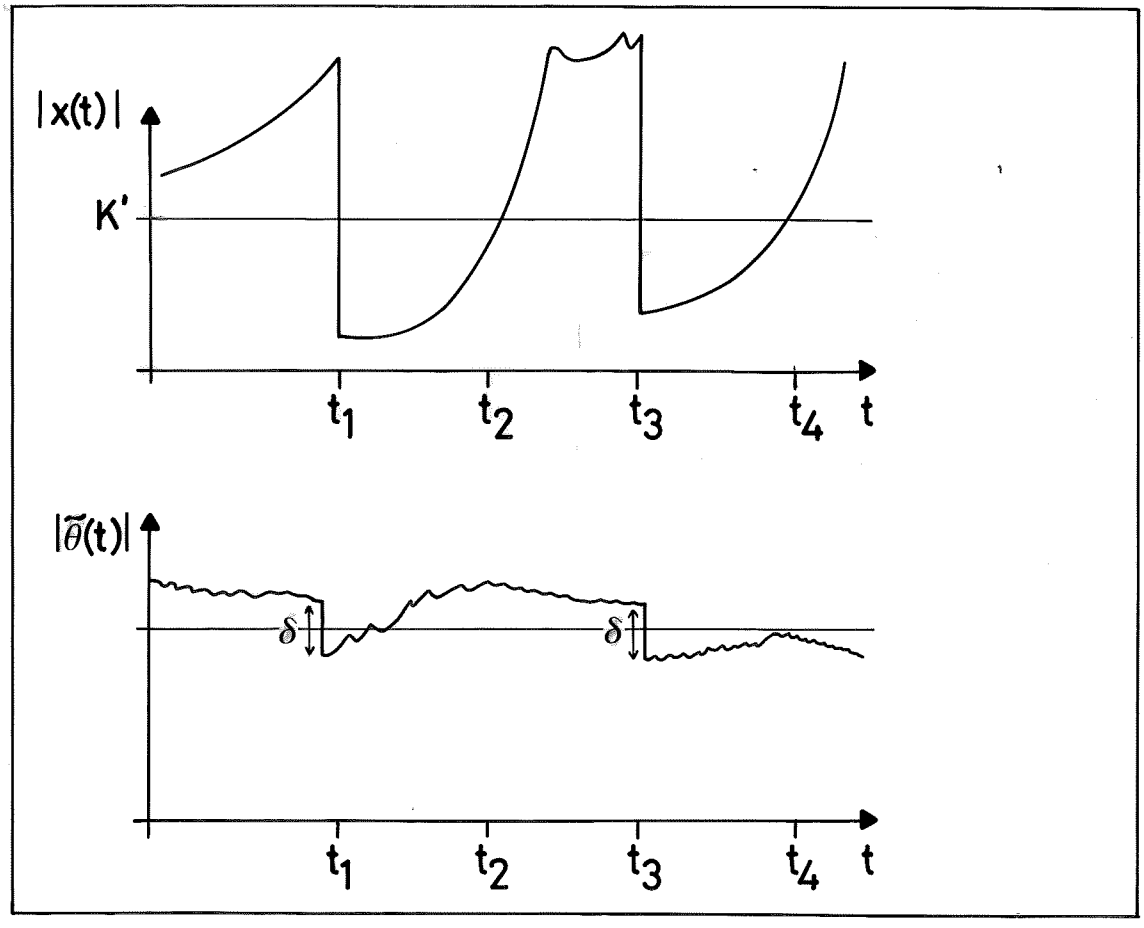
In this case the feedback law and hence the parameters of the closed loop system change arbitrarily slowly as t increases. Consequently $|x(t)|$ increases exponentially for

sufficiently large t. This implies that (5.9) holds, and also that $\lambda(t)$ is bounded from below by a strictly positive number. From (5.9) now follows that $|\tilde{\theta}_{\|}(t)| \to 0$, which from (5.10) implies that $|y(t)|$ and $|x(t)|$ start to decrease when $|\tilde{\theta}_{\|}(t)|$ is sufficiently small. This contradicts the exponential increase of $|x(t)|$.

Case b.

$$\limsup_{t\to\infty} |\tilde{\theta}(t+1) - \tilde{\theta}(t)| = \delta > 0$$

According to (5.7) such "jumps" in the estimates are possible only if $|x(t)|$ assumes arbitrarily large values. Therefore (5.9) is valid as soon as the estimate jumps. A jump may cause $|x(t)|$ to decrease drastically. In fact, it is the only way for $|x(t)|$ to decrease if the closed loop system is unstable. Consider the following figure



The only possibility for $|\tilde{\theta}(t)|$ to increase is when (5.9) is not valid, i.e. when $|x(t)| < K'$. Now, each such period must be preceded by a jump; $|\tilde{\theta}(t+1)| \leq |\tilde{\theta}(t)| - \delta$. Also, the length of a period when $|x(t)| < K'$ is essentially bounded by a fixed length. The unstable modes are excited by the noise terms and $|x(t)|$ quickly starts to increase. During such a period $|\tilde{\theta}(t+1) - \tilde{\theta}(t)|$ is arbitrarily small, and the possible increase in $|\tilde{\theta}(t)|$ becomes eventually less than $\delta/2$. Hence, it follows that $|\tilde{\theta}(t)|$ decreases with the net amount of at least $\delta/2$ infinitely many times which, of course, is impossible.

Consequently the assumption that $\theta(t)$ belongs to the area which gives unstable, closed loop systems for all $t > N$ is contradicted and (4.11) follows.

The self-tuning regulator STURE1 is treated analogously.

To summarize, the regulators STURE0 and STURE1 satisfy condition (4.11) in case the time delay, k, is known (only the case $k = 0$ has been treated in this section), and in case the system orders are not underestimated. The estimate $\hat{\beta}_o$ must be so good that

$$0 < \frac{\beta_o}{\hat{\beta}_o} < 2 \qquad (5.11)$$

The process has been assumed to be minimum phase. The noise $\varepsilon(t)$ may be quite general as long as

$$\lim_{N\to\infty} \sup \frac{1}{N} \sum_{1}^{N} \varepsilon(t)^2 < \infty \quad \text{w.p.1}$$

It has so far only been shown that (5.11) is a sufficient condition for the stability condition to hold. The following simple example shows that (5.11) is in fact also necessary.

Example 5.1 Consider the system

$$y(t+1) + \alpha y(t) = u(t) + e(t+1)$$

and the model

$$y(t+1) + \hat{\alpha} y(t) = \hat{\beta} u(t) + \varepsilon(t+1)$$

where

$$\hat{\beta} = \frac{1}{2+\delta}$$

Then, with $\gamma(t) = 1/t$ and $\alpha = 0$

$$\hat{\alpha}(t+1) = \hat{\alpha}(t) + \frac{y(t)y(t+1)}{\sum_{1}^{t} y(s)^2} = \hat{\alpha}(t) - \frac{(2+\delta)\hat{\alpha}(t)y(t)^2}{\sum_{1}^{t} y(s)^2} + \frac{y(t)e(t+1)}{\sum_{1}^{t} y(s)^2}$$

Suppose that $\hat{\alpha}(0) > \frac{2}{\delta}$. Neglect the noise term $y(t)e(t+1)$. Straightforward calculation shows that then $\hat{\alpha}(k)$ alternates between positive and negative values so that $|\hat{\alpha}(k)|$ tends to infinity. Consequently, this system has no stabilization property. □

Remark. Notice that the upper bound on $\beta_o/\hat{\beta}_o$ that is necessary to obtain a stable behaviour depends on several features of the regulator. For example, if $\{\gamma(t)\}$ is chosen as

$$\gamma(t) = \frac{1}{|x(t)|^2} \cdot t^{-s}$$

then the closed loop system will have the stability property (4.11) as soon as $\beta_o/\hat{\beta}_o$ is positive.

Also, if the open loop system is stable and the input to the system is limited, then clearly the overall system is stable and (4.11) holds trivially, irrespectively of $\beta_o/\hat{\beta}_o$.

5.2 Overall stability of STURE 1.

To give an idea of the stabilization property of STURE1, consider first Example 4.1. It was there shown that a white noise input signal with variance μ gives the following estimate of α

$$\hat{\alpha} = a - \frac{c\lambda(1-a^2)}{\lambda(1+c^2-2ac) + \mu}$$

Consequently, the bias depends on the signal to noise ratio. A large variance of the input gives small bias. In a similar manner, the bias depends on the regulator parameter if u(t) is formed as output feedback

$$u(t) = g\ y(t)$$

Then the estimate is

$$\hat{\alpha} = a - \frac{c(1-(a-g)^2)}{1+c^2-2(a-g)c}$$

If the closed loop is almost unstable, i.e. $|a-g|$ is close to 1, then the bias term is small.

The example suggests that unstable, or nearly unstable, closed loop systems give system parameter estimates with insignificant bias. This, in turn, gives a closed loop system with all poles close to the origin. Thus the closed loop system is stable, in the sense that $\frac{1}{N}\sum_{1}^{N} y(t)^2$ cannot increase without limit. This result is formally proved in the following theorem:

Theorem 5.1 Consider self-tuning regulator STURE1 applied to the system (2.3). The sequence $\{\varepsilon(t)\}$ in (2.3) may be any sequence such that

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{1}^{N} \varepsilon(t)^2 < \infty \quad \text{w.p.1} \tag{5.12}$$

Suppose that the time delay, k, is known and the system orders m' and n are not underestimated. β_o is assumed to be known. Then

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{1}^{N} y(t)^2 < \infty \quad \text{w.p.1} \tag{5.13}$$

If the system is minimum phase, then also

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{1}^{N} u(t)^2 < \infty \quad \text{w.p.1} \tag{5.14}$$

Proof: Introduce $\tilde{\theta}(t)$ as in Section 5.1. Then

$$y(t+k+1) = \tilde{\theta}(t)^T x(t) + \varepsilon(t+k+1) \tag{5.15}$$

Denote

$$r_{yy}(t',t) = \frac{1}{t'-t} \sum_{t+1}^{t'} y(s)^2$$

Then

$$r_{yy}(t',0) = \frac{t}{t'} r_{yy}(t,0) + \frac{t'-t}{t'} r_{yy}(t',t)$$

Suppose that $\limsup_{t\to\infty} r_{yy}(t,0) = \infty$. Then it is possible for arbitrarily large K to find t,t' such that

$$\frac{1}{2} \leq \frac{t}{t'} \leq 1$$

$$K \leq r_{yy}(s,0) \leq c_1 K \qquad t < s < t' \qquad c_1 > 1$$

$$r_{yy}(t',0) \geq c_1 K$$

Then

$$\frac{t}{t'} r_{yy}(t,0) + \frac{t'-t}{t'} r_{yy}(t',t) \geq c_1 K$$

and $r_{yy}(t,0)$ is not a negligible part of $r_{yy}(t',0)$. Not all terms in $r_{yy}(t',t)$ must be larger than c_1K. However, $y(t)$ cannot increase arbitrarily fast, and a number of terms must be larger than K. This number increases as c_1 increases. Choose c_1 such that at least $\hat{n} + \hat{m}$ terms are larger than K. Introduce as in (2.9)

$$R(t) = P^{-1}(t) = \frac{1}{t} \sum_{s=1}^{t} x(s-k-1)\, x(s-k-1)^T$$

The LS criterion (2.1) can then be written

$$V_t(\tilde{\theta}) = \tilde{\theta}^T R(t) \tilde{\theta} + 2 \frac{1}{t} \sum_{s=1}^{t} \tilde{\theta}^T x(s-k-1)\, \varepsilon(s) + \frac{1}{t} \sum_{s=1}^{t} \varepsilon(s)^2$$

Since $\tilde{\theta} = 0$ is a possible choice, the optimal $\tilde{\theta}(t)$ must yield a value less or equal to

$$V_t(0) = \frac{1}{t} \sum_{s=1}^{t} \varepsilon(s)^2$$

This implies that

$$\tilde{\theta}(t)^T R(t) \tilde{\theta}(t) + 2/t \sum_{s=1}^{t} \tilde{\theta}(t)^T x(s-k-1)\varepsilon(s) \leq 0$$

Hence

$$[\tilde{\theta}(t)^T R(t) \tilde{\theta}(t)]^2 \leq 4/t^2 \left[\sum_{s=1}^{t} \tilde{\theta}(t)^T x(s-k-1)\varepsilon(s) \right]^2 \leq$$

$$\leq 4/t \left[\sum_{s=1}^{t} \tilde{\theta}(t)^T x(s-k-1)x(s-k-1)^T \theta(t) \right] \cdot \left[1/t \sum_{s=1}^{t} \varepsilon^2(s) \right]$$

(Schwarz inequality) or

$$\tilde{\theta}(t)^T \frac{R(t)}{r_{yy}(t,0)} \tilde{\theta}(t) \leq \frac{4}{t} \sum_{s=1}^{t} \varepsilon(s)^2 / r_{yy}(t,0) \leq \frac{4K_1(\omega)}{r_{yy}(t,0)} \qquad (5.16)$$

Now take $K >> K_1$ and choose t,t' as above. Eq. (5.16) then implies that $\tilde{\theta}(s)$ must lie arbitrarily close to the null space of

$$R(s)/r_{yy}(s,0) \text{ for } t \leq s \leq t' \qquad (5.17)$$

Since

$$R(s) = \frac{t}{s} R(t) + \frac{s - t}{s} R(s,t) \quad \text{and} \quad \frac{t}{s} \geq 1/2$$

we also have that all $\theta(s)$, $s = t, \ldots, t'$ lie arbitrarily close to the null space of $R(t)/r_{yy}(t,0)$.

From the assumption that (5.13) does not hold, it follows that $y(s)$ is "large" for a number of $s = t, \ldots, t'$. In view of (5.15) this means that $\tilde{\theta}(s)^T x(s)$ also is "large". Since $\tilde{\theta}(s)$ is arbitrarily close to the null space of $R(t)/r_{yy}(t,0)$, $x(s)$ cannot belong to the range space of $R(t)/r_{yy}(t,0)$. (Since the matrix is symmetric the null space and the range space are orthogonal). Consequently

$$\tilde{R}(t',t) = \frac{1}{t' - t} \sum_{t+1}^{t'} x(s)x(s)^T$$

gets a significant contribution from matrices with range space not belonging to the range space of $R(t)/r_{yy}(t,0)$. In other words, the rank of $R(t)/r_{yy}(t,0)$ is less than that of $R(t')/r_{yy}(t',0)$. Repeating the argument $\hat{n} + \hat{m}$ times, it follows that

$$R(t')/r_{yy}(t',0)$$

has full rank, yielding the only possible choice $\tilde{\theta} = 0$ (i.e. the true parameters). This gives $y(t) = \varepsilon(t)$, which contradicts the assumption that $r_{yy}(t,0)$ increases without limit.

If the system is minimum phase, the inverse system is stable. If the input of the inverse system, $y(t)$, satisfies (5.13), then the output, $u(t)$, must satisfy (5.14). □

This stabilization property of STUREl is an important feature. It implies that the regulator stabilizes the system even if the parameters do not converge.

6. ANALYSIS OF THE SELF-TUNING REGULATOR "STURE".

Theorems 4.1 and 4.2 provide a tool for analysis of the class of self-tuning algorithms defined by (2.15). In this chapter the regulators STURE0 and STURE1 are considered. The basis of the analysis is the ODE defined in Theorem 4.1. In Section 6.1 the ODE's that correspond to the present regulators are determined. These equations are investigated in Section 6.2 under the assumption that the LS noise condition is satisfied.

One important result in Åström - Wittenmark (1973) is that the regulator seems to be equally well behaving also for more general noise structures. That this really might be the case is shown in Section 6.3. There it is proved that the regulator converges to the optimal one in a simple case when the LS noise condition is not satisfied.

In Section 6.4 the differential equations for the general case are linearized around the desired solution. Analysis of the linearized equations shows that they may be unstable, provided the noise has certain properties. Thus, in these cases the self-tuning algorithms will not converge.

6.1 Derivation of the associated differential equations.

The regulators STURE0 and STURE1 are given by (2.15) with $u(t) = \theta(t)^T x(t)$

$$\theta(t+1) = \theta(t) + \gamma(t+1)S(t)x(t-\hat{k})[y(t+1) - \theta(t)^T x(t-\hat{k}) +$$

$$+ \theta(t-\hat{k})^T x(t-\hat{k})] \qquad (6.1)$$

When computing to corresponding ODE, the two last terms cancel. For the algorithm of SA-type, STURE0, the ODE (4.16) is

$$\frac{d}{d\tau}\theta = \begin{bmatrix} -r_{yy}(\hat{k}+1,\theta) \\ \vdots \\ -r_{yy}(\hat{k}+\hat{n},\theta) \\ \hat{\beta}_o r_{uy}(\hat{k}+2,\theta) \\ \vdots \\ \hat{\beta}_o r_{uy}(\hat{k}+1+\hat{m},\theta) \end{bmatrix} \tag{6.2}$$

where $r_{yy}(i,\theta)$ is the autocorrelation for the stationary process defined by

$$y(t+k+1) = \left[\theta_o - \frac{\beta_o}{\hat{\beta}_o}\theta\right] x(t) + \varepsilon(t+k+1) \tag{6.3}$$

Here $\theta_o = (\alpha_1 \ldots \alpha_n, 0, \ldots 0, \frac{\beta_o}{\hat{\beta}_o}\beta_1, \ldots \frac{\beta_o}{\hat{\beta}_o}\beta_m, 0, \ldots 0,)$

Eq. (6.2) is suitable for numerical solution of the ODE. This is further discussed in Chapter 7.

For analysis it might give better insight to rewrite (6.1) using (6.3), provided $\hat{k} = k$

$$\theta(t+1) = \theta(t) + \gamma(t+1)S(t)x(t-k)\{x^T(t-k)[\theta_o - \frac{\beta_o}{\hat{\beta}_o}\theta(t-k)] -$$

$$- \theta(t) + \theta(t-k) + \varepsilon(t+1)\} \tag{6.4}$$

The ODE (4.16) corresponding to this form of the algorithm then is

$$\frac{d}{d\tau}\theta = G_1(\theta)\left(\theta_o - \frac{\beta_o}{\hat{\beta}_o}\theta\right) + E[x(t-k)\varepsilon(t+1)] \tag{6.5}$$

where $G_1(\theta) = E\, x(t)x(t)^T$, and where all expectations are evaluated, given that the feedback law is constant and expressed by $u(t) = -\frac{1}{\hat{\beta}_o}\theta^T x(t)$.

For the regulator STURE1 the ODE corresponding to (4.15) becomes

$$\frac{d}{d\tau}\theta = S(\tau)\, f(\theta(\tau))$$
$$\frac{d}{d\tau} S(\tau) = S(\tau) - S(\tau)G_1(\theta(\tau))S(\tau) \qquad (6.6)$$

where $G_1(\theta)$ is defined as above and $f(\theta)$ is the right hand side of (6.2). This ODE contains more variables than (6.2) and may be more difficult to analyse theoretically for this reason. In Ljung (1972) a reparametrization of $(\theta,P) \to c$ is made, so that the transformed ODE has the structure:

$$\frac{d}{d\tau} c = h_1[h_2(c)] - c \qquad (6.7)$$

where the range space of h_2 has the same dimension as θ. This structure can, as shown in Ljung (1972), be utilized for the analysis in some cases.

In a number of cases theoretical stability analysis is practically impossible. Then, numerical solution of the ODE's is a possibility to obtain detailed information about the stability that may suffice from a practical point of view. Eq. (6.6) can be used straightforwardly for numerical solution.

6.2 Convergence in case the LS noise condition is satisfied.

The case when the self-tuning regulator STURE0 is used is treated in this section. If the LS noise condition is satisfied, the noise is independent of x(t-k). Hence (6.5) is

$$\frac{d}{d\tau}\theta = G_1(\theta)\left(\theta_o - \frac{\beta_o}{\hat{\beta}_o}\theta\right) \qquad (6.8)$$

Now take $\gamma(t) = c_\gamma t^{-s}$; $s > 0$. In Section 5.1 it was shown

that the stability condition (4.11) is satisfied in case $0 < \frac{\beta_o}{\hat{\beta}_o} < 2$. Suppose that $E|\varepsilon(t)|^\beta < C$, where $\beta > 4/\alpha$. Then all conditions of Theorem 4.1 are satisfied.

The analysis in Åström - Wittenmark (1973) implies that in case $\hat{m} = m'$ and $\hat{n} = n$, there is only one stationary point of (6.8), namely $\theta^* = \frac{\hat{\beta}_o}{\beta_o} \theta_o$.

Since $G_1(\theta)$ is a non-negative definite matrix, all solutions tend to this stationary point unless they tend to the boundary of the area where the closed loop system is stable and $G_1(\theta)$ defined.

It should therefore also be shown that the trajectories of (6.8), near the boundary of the area where the closed loop system is stable, point into this area. This conclusion is indicated in the analysis of Section 5.1, but no formal proof of it will be given here.

Assuming this result, it now follows from Theorem 4.1 that $\theta(N) \to \frac{\hat{\beta}_o}{\beta_o} \theta_o$ w.p.1 as $N \to \infty$, which gives the desired minimum variance controller.

In case the open loop system is stable and the output of the controller (i.e. the input to the system) is limited, the ODE (6.8) is defined everywhere. Then there are no problems with stability regions and asymptotic stability of (6.8) follows straightforwardly.

6.3 Analysis of a simple system.

So far, in Chapter 3 and in the previous section, convergence of the regulators STURE0 and STURE1 has been shown in case the LS noise condition is satisfied. Convergence for a simple system for which the LS noise condition is not satisfied is shown in this section. See also the analysis on page 32.

Example 6.1 Consider the system

$$y(t+1) + a\,y(t) = b\,u(t) + e(t+1) + c\,e(t) \quad ; |c| < 1$$

where the sequence $\{e(t)\}$ is white noise with zero mean value and unit variance. The minimum variance control regulator for this system is

$$u(t) = (a-c)/b\; y(t)$$

The regulator parameter is estimated using the model

$$y(t+1) + \alpha y(t) = \beta u(t) + \varepsilon(t+1)$$

where β is a priori fixed but not necessarily equal to b. In this case with one regulator parameter the regulators STURE0 ans STURE1 are identical. The feedback law is

$$u(t) = \alpha(t)/\beta \;\; y(t)$$

From (6.2) the corresponding ODE is

$$\dot{\alpha} = -r_{yy}(1)$$

where

$$r_{yy}(1) = \frac{(c-a+\alpha b/\beta)(1-c(a-\alpha b/\beta))}{1 - (a-\alpha b/\beta)^2}$$

The desired convergence point α^* is $\alpha^* = (a-c)\beta/b$

Introduce $z = \alpha - \alpha^*$. Then

$$\dot{z} = -\frac{b}{\beta}\, z \cdot \frac{(1-c(c-\frac{b}{\beta}\, z))}{1 - (c-z\; b/\beta)^2} \quad ; \quad |c| < 1;\ |c-z\,\frac{b}{\beta}| < 1 \qquad (6.9)$$

It is easy to show that the last factor in (6.9) is positive

where it is defined. Hence, (6.9) is globally asymptotically stable if b and β have the same sign.

The stability condition (4.11) is satisfied if $0 < \frac{b}{\beta} < 2$, as shown in Section 5.1. Hence Theorem 4.1 assures that $\alpha(t)$ tends to α^* w.p.1 as $t \to \infty$.

Summing up the results of this section and of Section 5.1 we have

- If $\frac{b}{\beta} < 0$, the regulator will not converge. The closed loop system becomes eventually unstable, whereafter the pole of the closed loop system is forced to infinity.

- If $0 < \frac{b}{\beta} < 2$, the regulator converges to the desired value w.p.1.

- If $\frac{b}{\beta} > 2$, the regulator converges to the desired value as long as the closed loop system is stable. However, there is a non-zero probability that the estimate tends to infinity.

6.4 Linearization of the differential equations.

Consider the system

$$y(t+1) + a_1 y(t) + \ldots + a_n y(t-n+1) = u(t) + \ldots + b_m u(t-m) +$$

$$+ e(t+1) + c_1 e(t) + \ldots + c_n e(t-n+1) \qquad (6.10)$$

where $\{e(t)\}$ is a sequence of independent, random variables with zero mean values. Suppose that this system is controlled by the regulator STURE0. Then the corresponding ODE is (6.2) with $\hat{k} = 0$ or (6.5). Suppose that the correct model orders m and n have been chosen. Then the only stationary point of the ODE is the minimum variance control law, cf. Åström - Wittenmark (1973), given by

$$\theta^* = (a_1-c_1,\ldots a_n-c_n,\ b_1\ldots,b_m)$$

We will now linearize (6.5) around this solution. The result is formulated as a lemma.

Lemma 6.1: Consider the system (6.10) controlled by the regulator STURE0. Assume that the system is minimum phase. Linearization of the corresponding ODE (6.5) around θ^* gives with $\Delta\theta = \theta - \theta^*$

$$\dot{\Delta\theta} = M\Delta\theta \tag{6.11}$$

where

$$M = -E \begin{bmatrix} e(t) \\ \vdots \\ e(t-n+1) \\ u(t) \\ \vdots \\ u(t-m) \end{bmatrix} [\tilde{e}(t) \ \ldots \ \tilde{e}(t-n+1)\ \tilde{u}(t) \ \ldots \ \tilde{u}(t-m)]$$

where

$$\tilde{e}(t+1) + c_1\tilde{e}(t) + \ldots + c_n\tilde{e}(t-n+1) = e(t+1) \tag{6.12}$$

$$\tilde{u}(t+1) + c_1\tilde{u}(t) + \ldots + c_n\tilde{u}(t-n+1) = u(t+1) \tag{6.13}$$

and

$$u(t) + b_1u(t-1) + \ldots + b_mu(t-m) = (a_1-c_1)e(t) +$$

$$+ \ldots + (a_n-c_n)e(t-n+1) \tag{6.14}$$

Proof: Let

$$\theta = (\hat{a}_1,\ldots\hat{a}_n,\ \hat{b}_1,\ldots,\hat{b}_m)$$

where $\hat{a}_i$ is the estimate of a_i. Then $M = \frac{d}{d\theta} f(\theta)$. Denote the elements of M by m_{ij}. Then

$$m_{ji} = -\frac{\partial}{\partial \hat{a}_i} r_{yy}(j) \qquad i, j \leq n$$

$$m_{j(i+n)} = -\frac{\partial}{\partial \hat{b}_i} r_{yy}(j) \qquad i \leq m, \quad j \leq n$$

$$m_{(j+n)i} = \frac{\partial}{\partial \hat{a}_i} r_{uy}(j+1) \qquad i \leq n, \quad j \leq m$$

$$m_{(j+n)(i+n)} = \frac{\partial}{\partial \hat{b}_i} r_{uy}(j+1) \qquad i \leq m, \quad j \leq m$$

Now

$$\frac{\partial}{\partial \hat{a}_i} r_{yy}(j) = \frac{\partial}{\partial \hat{a}_i} E\, y(t)y(t+j) = E\left\{\left[\frac{\partial}{\partial \hat{a}_i} y(t)\right] y(t+j) + y(t) \frac{\partial}{\partial \hat{a}_i} y(t+j)\right\}$$

Since $y(t) = e(t)$ in the point in which the expression is evaluated, the first term in the RHS is zero.

Consider $\frac{\partial}{\partial \hat{a}_i} y(t+j)$. Introduce the polynomials

$$A(z) = 1 + a_1 z + \ldots + a_n z^n$$

$$B(z) = z + b_1 z^2 + \ldots + b_m z^{m+1}$$

$$\hat{A}(z) = 1 + \hat{a}_1 z + \ldots + \hat{a}_n z^n$$

$$\hat{B}(z) = z + \hat{b}_1 z^2 + \ldots + \hat{b}_m z^{m+1}$$

$$C(z) = 1 + c_1 z + \ldots + c_n z^n$$

Then the closed loop system is given by

$$[A(q^{-1})\hat{B}(q^{-1}) - B(q^{-1})\{\hat{A}(q^{-1}) - 1\}]y(t) = \hat{B}(q^{-1})C(q^{-1})e(t)$$

where q^{-1} is the backward shift operator.

Take the derivative with respect to $\hat{a}_i$:

$$[A(q^{-1})\hat{B}(q^{-1}) - B(q^{-1})\{\hat{A}(q^{-1}) - 1\}]\frac{\partial}{\partial \hat{a}_i} y(t) -$$

$$- B(q^{-1})\frac{\partial}{\partial \hat{a}_i} \hat{A}(q^{-1})y(t) = 0$$

The derivative is to be evaluated at $\theta = \theta^*$ i.e. $\hat{A} = A - C + 1$, $\hat{B} = B$. Then $y(t) = e(t)$ and

$$A\hat{B} - B(\hat{A}-1) = BC$$

Hence

$$B(q^{-1})C(q^{-1})\frac{\partial}{\partial \hat{a}_i} y(t) = B(q^{-1})\, q^{-i}\, e(t)$$

Introduce $\tilde{e}(t)$ by

$$C(q^{-1})\tilde{e}(t) = e(t)$$

Then

$$\frac{\partial}{\partial \hat{a}_i} y(t) = \tilde{e}(t-i)$$

and

$$\frac{\partial}{\partial \hat{a}_i} r_{yy}(j)\Big|_{\theta=\theta^*} = E\, e(t)\tilde{e}(t+j-i) = E\, e(t-j)\tilde{e}(t-i)$$

Consequently, the upper left block matrix of M is

$$-E \begin{bmatrix} e(t) \\ \vdots \\ e(t-n+1) \end{bmatrix} [\tilde{e}(t) \ldots\ldots \tilde{e}(t-n+1)]$$

The rest of the lemma is proved analogously. □

The properties of (6.11) will now be discussed. It is easily seen that the upper left block in the matrix M is triangular with -1 in the diagonal. This means that in case there are no b-parameters to estimate, the linearized equation is asymptotically stable. However, the diagonal elements in the lower right block are more interesting. They are, see Åström (1970), given by

$$-E\ u(t)\ \tilde{u}(t) = -\frac{1}{2\pi} \oint \frac{1}{C(z)}\ \phi_{uu}(z)\ dz$$

where

$$\phi_{uu}(z) = \frac{(A(z) - C(z))\ (A^*(z) - C^*(z))}{B(z)\ B^*(z)}$$

is the autospectrum of u. Here $A^*(z) = z^n\ A(z^{-1})$, etc.

The element $E\ u(t)\ \tilde{u}(t)$ can be made negative with arbitrary magnitude. To do so, choose the parameters $c_1,\ldots,c_n$ so that $C(\exp(i\omega))$ has negative real part for some ω. This is possible as soon as the degree of C(z) is greater than or equal to 2. Then choose A and B so that the system $[A(q^{-1}) - C(q^{-1})]/B(q^{-1})$ has a resonance for the frequency ω. In this way it is possible to make

$$\text{tr } M = -[n + m\ E\ u(t)\ \tilde{u}(t)]$$

positive. This means that M must have at least one eigenvalue with positive real part.

An example of such a system is given below.

Example 6.2. Consider the following system

$$y(t+1) - 1.6\, y(t) + 0.75\, y(t-1) = u(t) + u(t-1) +$$

$$+ 0.9\, u(t-2) + e(t+1) + 1.5\, e(t-1) + 0.75\, e(t-2) \qquad (6.15)$$

For this system the C-polynomial has a negative real part on the unit circle for $1.78 < \omega < 2.48$. The system $1/B(z)$ has a sharp resonance for $\omega = 2.10$. This is sufficient to make (6.15), i.e. the differential equation linearized around $\theta = \theta^*$, have positive eigenvalues.

According to Åström - Wittenmark (1973) there is only one possible convergence point; θ^*. Since this has proved to be an unstable stationary point of the ODE (4.16), the self-tuning algorithm is not likely to converge. According to the results of Section 5.1 the estimates do not tend to infinity. Therefore they must vary in a bounded area without converging to any point.

The minimum variance regulator for (6.15) is given by

$$u(t) = -3.1\, y(t) - u(t-1) - 0.9\, u(t-2) \qquad (6.16)$$

In figure 6.1 the behaviour of STURE1 with $\hat{\beta}_o = 1$, $\hat{n} = 2$, $\hat{m} = 2$ and $\hat{k} = 0$ is shown. The sequence $\{\gamma(t)\}$ was chosen such that it decreases very slowly in order to accentuate the behaviour of the system. The initial values are the values of the optimal regulator (6.16). The average loss per step from one simulation was about 2.90, while the optimal regulator gave the average loss 1.02 for the same noise sequence.

From figure 6.1 it is obvious that the algorithm tries to reach the optimal values. When the estimates come close to the optimal ones, they are thrown away. This behaviour is in good agreement with the results of the analysis.

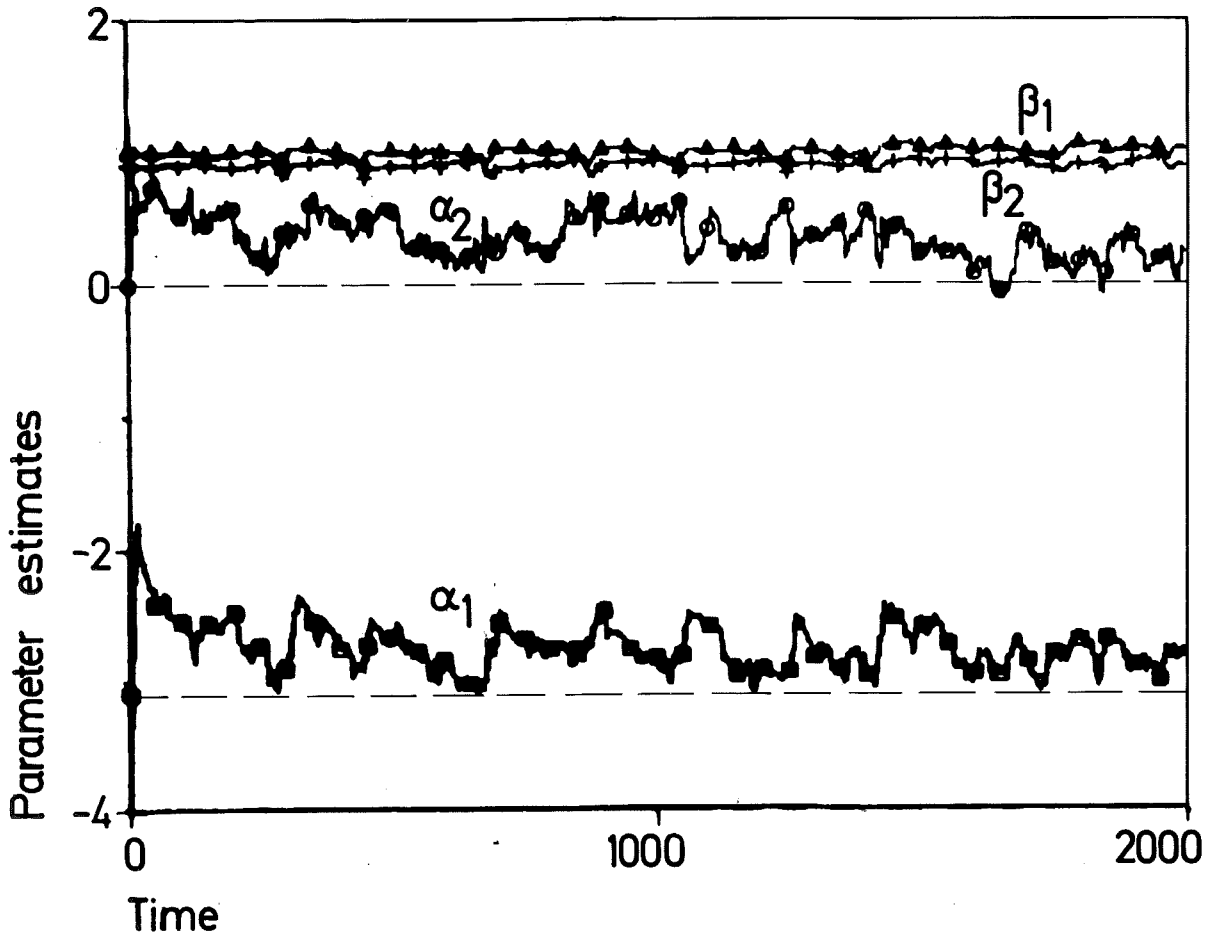


Fig. 6.1 Parameter estimates when the system (6.15) is controlled by STURE1. The dashed lines show the values corresponding to the optimal controller.

7. NUMERICAL EXAMPLES

In this chapter some of the theoretical results in the previous chapters will be illustrated through some numerical simulations.

Example 7.1 The first example will show the significance of the expected trajectories obtained through the differential equations described in Chapter 4. Consider the system

$$y(t) + a\,y(t-1) = b\,u(t-1) + e(t) + c\,e(t-1) \tag{7.1}$$

where $a = -0.95$, $b = 1$ and $c = -0.45$. The sequence $\{e(t)\}$ is white noise. The optimal control law is given by

$$u(t) = \frac{a - c}{b}\,y(t) = -0.5\,y(t)$$

Let the model be

$$y(t+1) + \hat{\alpha}y(t) = u(t) + \varepsilon(t+1)$$

With $G_1(\theta) = r_{yy}(0)$, the differential equations (6.6) for the self-tuning algorithm STURE1 will be

$$\begin{aligned}\dot{\alpha} &= -S \cdot r_{yy}(1) = -S \cdot \frac{(c-a-b\alpha)(1-c(a-b\alpha))}{1-(a-b\alpha)^2} \\ \dot{S} &= S - S^2 \cdot r_{yy}(0) = S - S^2\cdot\left[1 + \frac{(c-a+b\alpha)^2}{1-(a-b\alpha)^2}\right]\end{aligned} \tag{7.2}$$

The equations (7.2) are simulated using a program package, SIMNON, for simulation of nonlinear differential equations available at the Division of Automatic Control in Lund, see Elmqvist (1972).

Figure 7.1 shows the trajectories for different initial values α when $S(0) = 5$. The system (7.1) is also controlled using

STURE1, with

$$\gamma(t) = \frac{c_\gamma}{t^s}$$

The values used in the algorithm were $c_\gamma = 0.002$ and $s = 0.0645$. According to Theorem 4.2 the time in the differential equation (7.2) is related to the number of samples, N, through

$$\tau_N = \sum_{t=1}^{N} \gamma(i) = c_\gamma \sum_{i=1}^{N} \frac{1}{i^s}$$

The value of s was chosen rather small in order to get a reasonable value of N. With the chosen values, 5 time units correspond to 4000 steps. Figure 7.2 shows the parameter estimates for different starting values of the parameter α. The initial value of S was $S(0) = 5$. The parameter estimates correspond well with the trajectories of the differential equation.

If c_γ or $S(0)$ are increased, the estimates will vary more in the beginning, but after a short period of time the estimates will behave as in figure 7.2. If α is outside the stability boundary of the closed loop system, then Eq. (7.2) is not valid. The self-tuning regulator has, however, a stabilization property, cf. Chapter 5, and will rapidly give an estimate which makes the closed loop system stable. □

Example 7.2 Consider the system

$$y(t) + a\,y(t-1) = u(t-1) + b\,u(t-2) + e(t) + c\,e(t-1) \qquad (7.3)$$

with $a = -0.99$, $b = 0.5$ and $c = -0.7$. The optimal control law is

$$u(t) = \frac{a - c}{1 + bq^{-1}}\,y(t) = \frac{-0.29}{1 + 0.5q^{-1}}\,y(t)$$

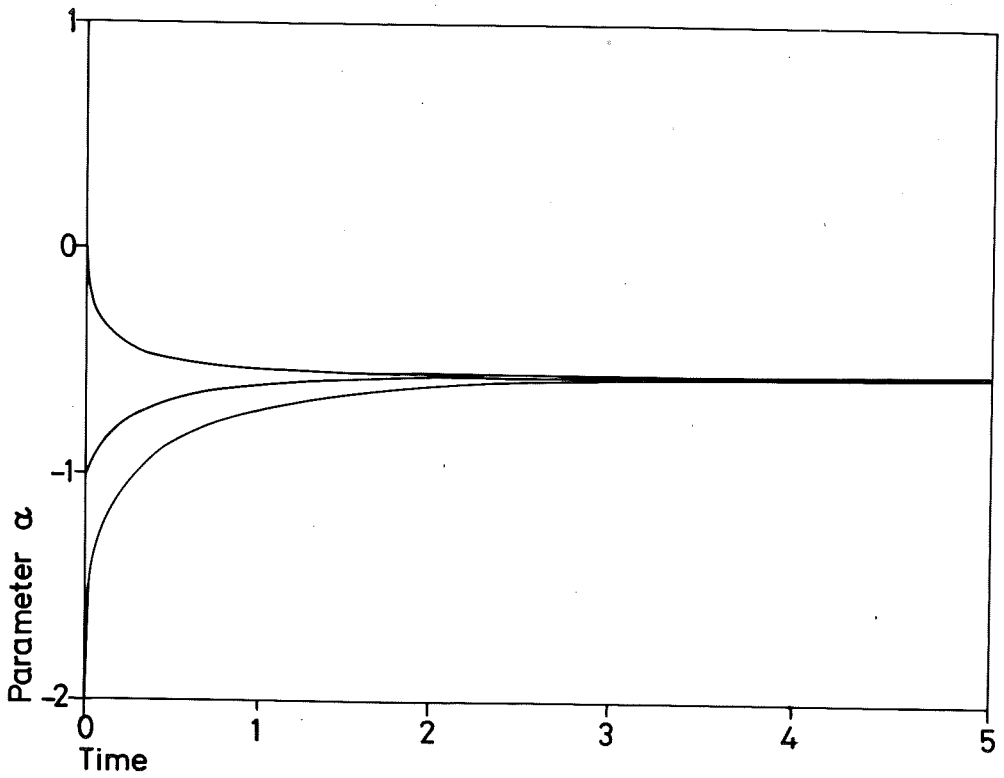


Fig. 7.1 Trajectories for the initial values $\hat{\alpha}(0) = 0, -1$ and -1.9 respectively of the equation (7.2) and where $S(0) = 5$.

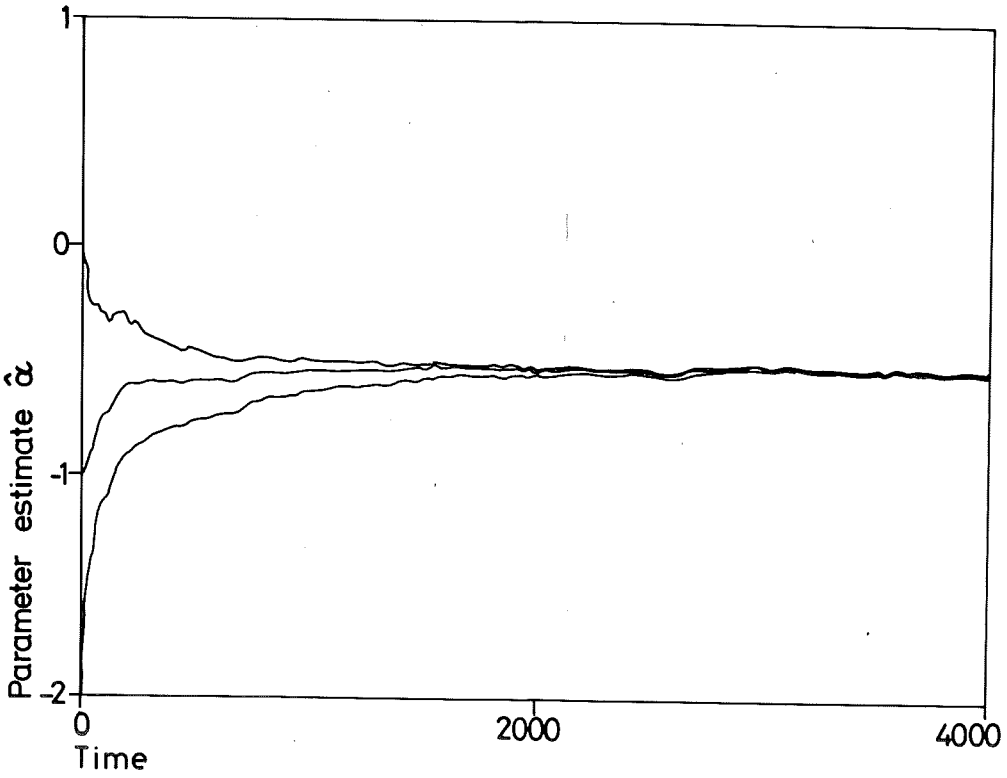


Fig. 7.2 The parameter estimates for different starting values when the self-tuning algorithm is used with $S(0) = 5$, $c_\gamma = 0.002$ and $s = 0.0645$ on the system (7.1).

In the self-tuning algorithm STURE0 the parameters have been estimated from the model

$$y(t+1) + \hat{\alpha}y(t) = u(t) + \hat{\beta}u(t-1) + \varepsilon(t+1)$$

The differential equations are in this case

$$\begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = S \begin{bmatrix} -r_{yy}(1) \\ r_{uy}(2) \end{bmatrix}$$

$$\dot{S}^{-1} = -S^{-1} + r_{yy}(0) + r_{uu}(0) \qquad (7.4)$$

The equations for α and β are difficult to analyse. The equations have been simulated with different starting values of the parameters α and β and with $S(0) = 10$. The phase plane is shown in figure 7.3. At the beginning parts of the trajectories every 2nd time unit is indicated. From the starting point $\alpha(0) = -1.5$ and $\alpha(0) = -0.1$ it takes about 9.7 time units before the estimates are within a distance of 0.1 from the convergence point. Corresponding curves are shown in figure 7.4 for one realization when the system is controlled by the self-tuning regulator STURE0. The values $c_\gamma = 0.002$ and $s = 0.1$ were used. The value of c_γ is much smaller than one would use in a practical case. This value was, however, chosen in order to better see the agreement between the parameter estimates and the trajectories of the differential equations (7.4).

When the self-tuning regulator STURE1 is used, the differential equations will be

$$\begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = S \begin{bmatrix} -r_{yy}(1) \\ r_{yu}(2) \end{bmatrix}$$

$$\dot{S}^{-1} = -S^{-1} + \begin{bmatrix} r_{yy}(0) & -r_{yu}(1) \\ -r_{yu}(1) & r_{uu}(0) \end{bmatrix} \qquad (7.5)$$

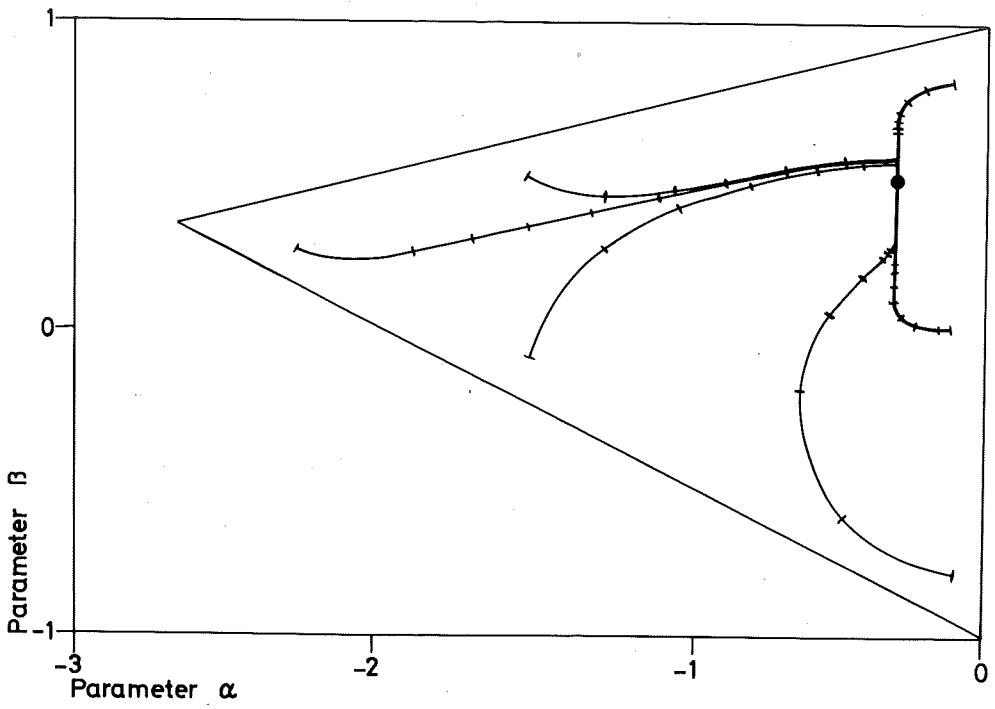


Fig. 7.3 Phase plane of the differential equations (7.4) for different starting values when S(0) = 10. The parameter values corresponding to the optimal regulator are indicated by a dot. The triangel shows the stability boundary of the equations (7.4). At the first parts of the trajectories every 2nd time unit is marked.

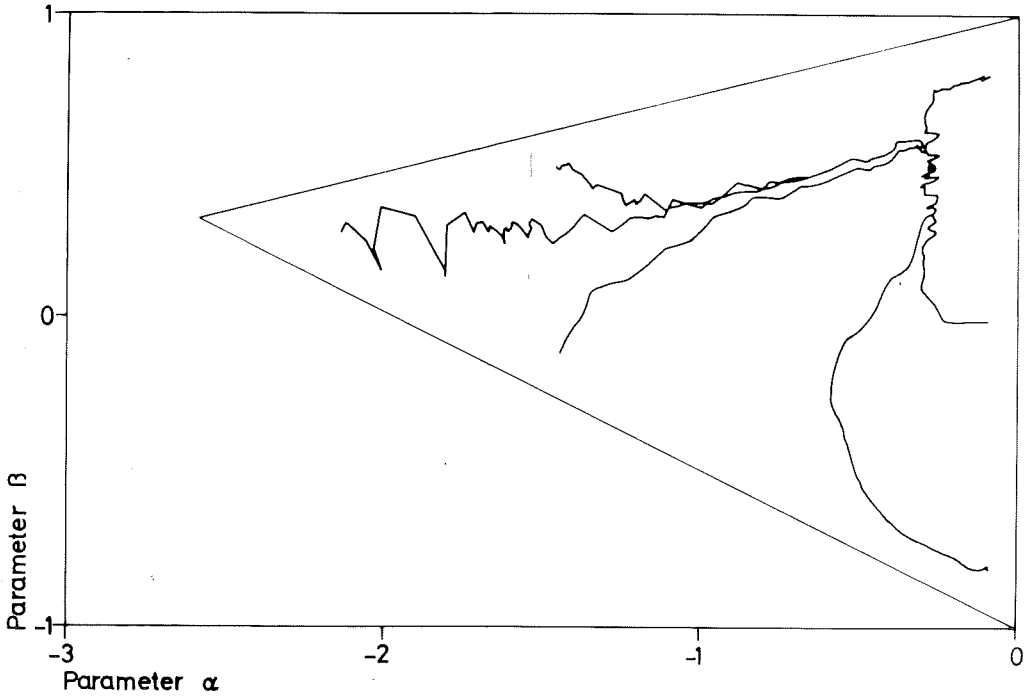


Fig. 7.4 Phase plane of the parameter estimates when the system (7.3) is controlled by the self-tuning regulator STURE0 with c_γ = 0.002, s = 0.1 and S(0) = 10.

The phase plane of (7.5) is shown in figure 7.5. Compared with STURE0 the trajectories in this case are leading more directly to the optimal point. The starting directions of the parameter estimates are determined by the initial value of S, which in this case was 10·I. The convergence time is also shorter in this case. For the starting values $\alpha(0) = -1.5$ and $\beta(0) = -0.1$ the convergence time is about 5.7 time units.

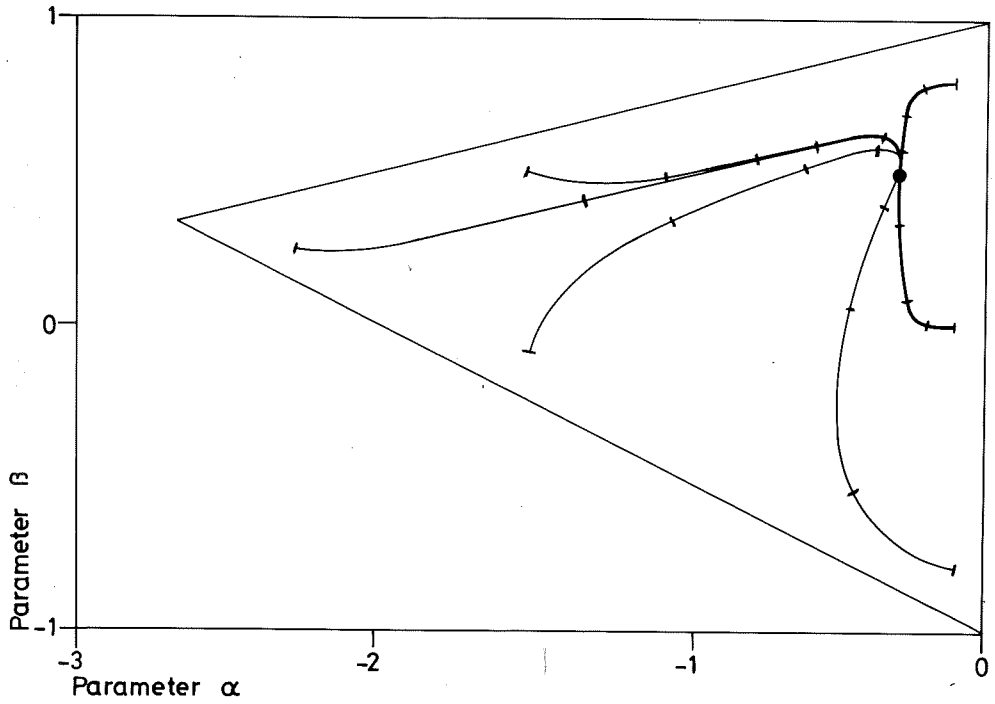


Fig. 7.5 Phase plane of the differential equations (7.5) for different starting values. The initial value of S was 10·I. Every second time unit is indicated in the beginning of the trajectories.

Example 7.3 In this example the differential equations of a more complex self-tuning regulator than STURE0 and STURE1 are simulated. In the previous examples the control law was linear in the parameter estimates, but in this case the control law is determined in a more complex way.

Consider the system

$$y(t) + a\, y(t-1) = b\, u(t-3) + e(t) \tag{7.6}$$

where $a = -0.9$ and $b = 1$. The parameters a and b are estimated using (2.15f) with the model (B.4), (see appendix B). The minimum variance regulator based on the estimates is then computed as

$$u(t) = \frac{\hat{a}^3/\hat{b}}{1 - \hat{a}q^{-1} + \hat{a}^2 q^{-1}}\, y(t) \tag{7.7}$$

The corresponding differential equations are given by (4.16):

$$\begin{aligned} \dot{\hat{a}} &= -r_{yy}(1) - \hat{a}\, r_{yy}(0) + \hat{b}\, r_{yu}(2) \\ \dot{\hat{b}} &= r_{yu}(3) + \hat{a}\, r_{yu}(2) - \hat{b}\, r_{uu}(0) \end{aligned} \tag{7.8}$$

Trajectories of (7.8) are shown in figure 7.6. Since the LS noise condition is satisfied, the estimates converge to the true parameter values. The covariance function of the closed loop system, $r_{yy}(0)$, corresponding to the estimates in figure 7.6 is given in figure 7.7. It is interesting to notice that the expected variance of the output actually increases for some values of $\hat{a}$, $\hat{b}$.

□

The examples in this chapter show that the differential equations defined by (4.15) and (4.16) are very useful for the analysis of the different self-tuning algorithms. It is possible to determine the transient behaviour, as well as to investigate the convergence properties. The differential equations also have the advantage that the stochastic part is removed from the analysis. It is, however, in most cases difficult to analyse the differential equations. When only one parameter is estimated, it is possible to carry through the analysis. One example is given in Wittenmark (1973). If the system contains two or more para-

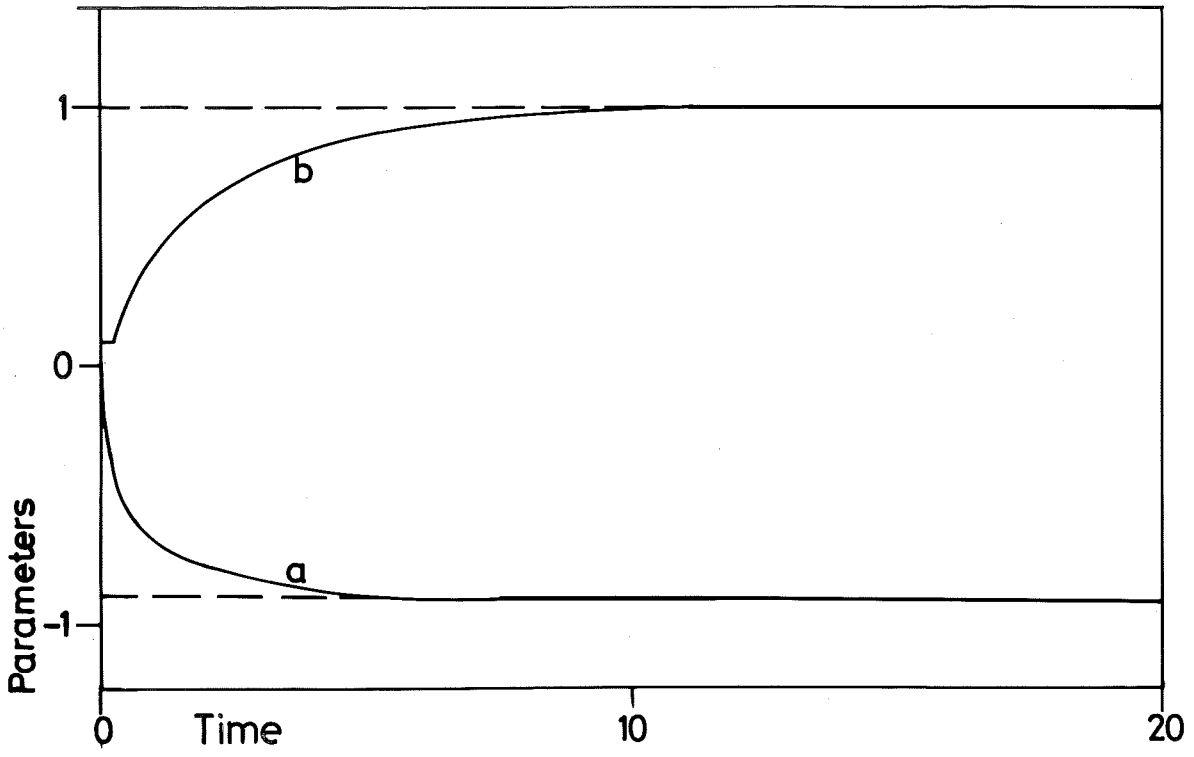


Fig. 7.6 Trajectories of (7.8) corresponding to the estimates of the system (7.6) when the control law is (7.7). The initial values were $\hat{a}(0) = -0.1$ and $\hat{b}(0) = 0.1$

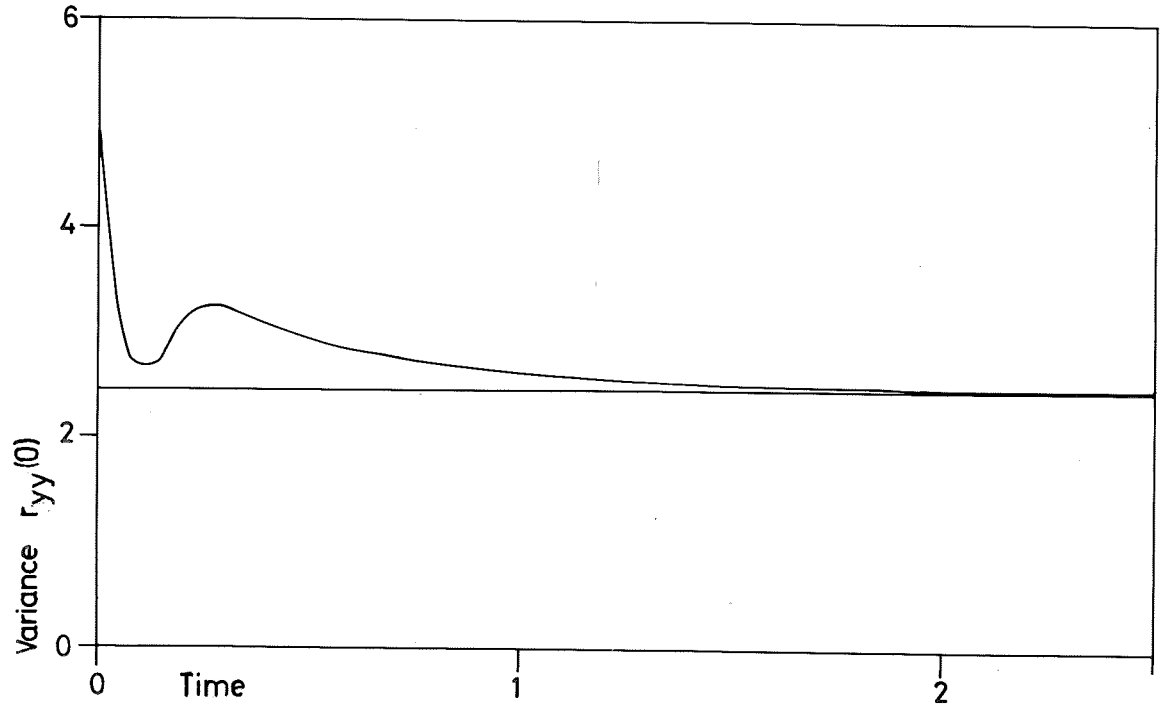


Fig. 7.7 The variance function $r_{yy}(0)$ corresponding to the estimates given in figure 7.6.

meters, it is difficult to investigate for instance the stability. The differential equations can, however, be simulated and much insight can be gained in this way. The computations will be rather extensive even for systems of low order.

The self-tuning regulators can without any difficulties be simulated with many parameters, but many simulations have to be done in order to investigate the convergence properties. The self-tuning algorithms are also more timeconsuming than the differential equations since the time in the differential equations is related to the number of steps through $\tau = \sum_{1}^{N} \gamma(t)$, and the number of steps per time unit is rapidly increasing since $\gamma(t)$ is decreasing.

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APPENDIX A. PROOF OF LEMMA 4.1

Lemma 4.1 Suppose $\theta(n)$ and $\bar{\theta}$ belong to the area where f and G_i are defined. Let $m(n,\Delta\tau)$ satisfy

$$\sum_{n}^{m(n,\Delta\tau)} \gamma(k) \to \Delta\tau \quad \text{as } n\to\infty$$

Suppose that $|x(n)| < C$ (C may depend on the realization). Then for sufficiently small $\Delta\tau$ and $(\theta(n),S_i(n))$ sufficiently close to $(\bar{\theta},\bar{S}_i)$

$$\theta(m(n,\Delta\tau)) = \theta(n) + \Delta\tau\bar{S}_i f(\bar{\theta}) + R_1(n,\Delta\tau,\bar{\theta},\bar{S}_i) + R_2(n,\Delta\tau,\bar{\theta},\bar{S}_i) \tag{A.1}$$

$$S_i(m(n,\Delta\tau)) = S_i(n) + \Delta\tau[-\bar{S}_i\, G_i(\bar{\theta})\, \bar{S}_i + \bar{S}_i] + R_1'(n,\Delta\tau,\bar{\theta},\bar{S}_i) +$$

$$+ R_2'(n,\Delta\tau,\bar{\theta},\bar{S}_i) \tag{A.2}$$

where

$$|R_1^{(')}(n,\Delta\tau,\bar{\theta},\bar{S}_i)| \leq \Delta\tau\cdot K\{|\theta(n) - \bar{\theta}| + |S_i(n) - \bar{S}_i|\} + A(\Delta\tau)^2$$

and

$$R_2^{(')}(n,\Delta\tau,\theta,S_i) \to 0 \quad \text{w.p.1 as } n\to\infty. \qquad \square$$

Proof: To abbreviate notation the term

$$\theta(t)^T x(t) - \hat{\beta}_o u(t-\hat{k}) = x(t)^T[\theta(t) - \hat{\beta}_o F(\theta(t-\hat{k}))^T]$$

in (2.15) will be omitted, since it is treated in the same way as $x(t)\, y(t+\hat{k}+1)$. The variable $\hat{k}$ will be taken as zero.

The analysis will be carried out for a given, fixed realization ω. Many of the variables below depend on ω, but this argument is suppressed. The technical problem with non-countable unions of null sets can be treated in the same way

as in Ljung (1974), appendix A, and is not explicitly dealt with here.

Consider first

$$\sum_{n}^{m(n,\Delta\tau)} \gamma(t)x(t)x(t)^T \qquad (A.3)$$

It will be shown that, if $|\theta(n) - \bar{\theta}|$ is sufficiently small,

$$\sum_{n}^{m(n,\Delta\tau)} \gamma(t)x(t)x(t)^T = \Delta\tau G_1(\bar{\theta}) + R_3(n,\Delta\tau,\bar{\theta}) + R_4(n) \qquad (A.4)$$

where

$$|R_3(n,\Delta\tau,\bar{\theta})| \leq \Delta\tau \cdot C_2\{|\theta(n) - \bar{\theta}|\} + C_3 \cdot (\Delta\tau)^2$$

and

$R_4(n) \to 0$ w.p.1 as $n \to \infty$.

The vector $x(t)$ can be seen as a state vector for the system (2.3) with feedback $u(t) = F(\theta(t))\, x(t)$. The closed loop system can be written on a state space form:

$$x(t+1) = A(\theta(t))\, x(t) + B\varepsilon(t+1)$$

Let $\bar{x}(t)$ $t = n,\ldots$ denote the sequence of vectors that are obtained by

$$\bar{x}(t+1) = A(\bar{\theta})\, \bar{x}(t) + B\varepsilon(t+1); \quad \bar{x}(n) = x(n)$$

Let $\tilde{x}(t) = x(t) - \bar{x}(t)$

Then (A.3) can be written as

$$\sum_{n}^{m(n,\Delta\tau)} \gamma(t)\bar{x}(t)\bar{x}(t)^T + \sum_{n}^{m(n,\Delta\tau)} \gamma(t)[\bar{x}(t)\tilde{x}(t)^T + \tilde{x}(t)\bar{x}(t)^T] \qquad (A.5)$$

Consider first the second term of (A.5). We have

$$\bar{x}(t) = \bar{A}^{t-n} x(n) + \sum_{k=n}^{t} \bar{A}^{t-k} B\varepsilon(k); \quad \bar{A} = A(\bar{\theta}) \tag{A.6}$$

and

$$\tilde{x}(t) = \sum_{k=n}^{t} \left\{ \left[\prod_{k+1}^{t} A_j \right] - \bar{A}^{t-k} \right\} B\varepsilon(k); \quad A_j = A(\theta(j)) \tag{A.7}$$

Since $F(\theta)$ is Lipschitz continous we have

$$|A(\theta(t)) - A(\bar{\theta})| < C_4 |\theta(t) - \bar{\theta}|$$

Since $\bar{\theta}$ gives a stable closed loop system

$$|\bar{A}^{t-k}| \leq C_5 \lambda_1^{t-k} \qquad \lambda_1 < 1$$

If $\max_{n<t<k} |\theta(t) - \bar{\theta}|$ is sufficiently small, say less than $\bar{\delta}$, it follows that

$$\left| \prod_{k+1}^{t} A_j \right| \leq C_6 \lambda_2^{t-k} \qquad \lambda_2 < 1$$

Then we have

$$\left| \prod_{k+1}^{t} A_j - \bar{A}^{t-k} \right| \leq \max_{n \leq i \leq t} |\theta(i) - \bar{\theta}| \cdot C_7 \cdot \lambda_3^{t-k} \qquad \lambda_3 < 1$$

for $t \leq K$. Introduce $q(n,t) = \max_{n \leq i \leq t} |\theta(i) - \bar{\theta}|$. Assume that

$|\theta(n) - \bar{\theta}| < \bar{\delta}$ and $|S_i(n) - \bar{S}| < \bar{\delta}$ and denote by $K(n)$ the

first number $\geq n$ such that

$|\theta(K(n)) - \bar{\theta}| \geq \bar{\delta}$ or

$|S_i(K(n)) - \bar{S}| \geq \bar{\delta}$

Introduce

$$k(n,\Delta\tau) = \min(m(n,\Delta\tau),\ K(n))$$

Then

$$|\tilde{x}(t)| \leq q(n,t)\cdot v(t) \qquad t \leq k(n,\Delta\tau)$$

where

$$v(t) = \sum_{k=n}^{t} C_7 \cdot \lambda_3^{t-k} |B\varepsilon(k)|$$

Similarly

$$|\bar{x}(t)| \leq v(t) + C_5 \lambda_1^{t-n} |x(n)|$$

Hence the second term of (A.5) satisfies

$$\left| \sum_{n}^{k(n,\Delta\tau)} \gamma(t)\{\tilde{x}(t)\bar{x}(t)^T + \bar{x}(t)\tilde{x}(t)^T\} \right| \leq$$

$$\leq 2q(n,k(n,\Delta\tau))\cdot \sum_{n}^{k(n,\Delta\tau)} \gamma(t)v(t)^2 + C_8 \sum_{n}^{k(n,\Delta\tau)} \gamma(t)\lambda_1^{t-n} \qquad (A.7)$$

The first term of (A.5) is quite similar to the first term of (A.7): both are formed as sums of products of stochastic variables ($\bar{x}(t)$ and $v(t)$ respectively) that are obtained from white noise ($\{e(t)\}$) through exponentially stable filters (giving first $\varepsilon(t)$ and then $\bar{x}(t)$ and $v(t)$). The convergence of such sums is considered in the following lemma:

Lemma A.1 Let the random variables $f_1(t)$ and $f_2(t)$ be generated from white noise $\{e(t)\}$ with zero mean value and unit variance:

$$f_i(t) = \sum_{k=0}^{\infty} g_i(t,k)\ e(t-k) \qquad i = 1,2$$

where $|g_i(t,k)| < C\lambda^k \qquad \lambda < 1 \qquad i = 1,2$

Suppose $E|e(t)|^{4p} < C$ and that the sequence $\{\gamma(t)\}$ satisfies (4.14).

Then

$$\sum_{t=n_k}^{n_{k+1}} \gamma(t)[f_1(t)\cdot f_2(t) - Ef_1(t)\cdot f_2(t)] \to 0 \quad \text{w.p.1}$$

as $k\to\infty$, where the subsequence $\{n_k\}$ satisfies

$$\limsup_{k\to\infty} \sum_{n_k}^{n_{k+1}} \gamma(t) = L < \frac{1}{2}$$

Proof of Lemma A.1: For simplicity denote $n_k = n$ and $n_{k+1} = m$ and $f_1(j)\cdot f_2(j) - Ef_1(j)\cdot f_2(j) = f(j)$ and consider

$$|E\left(\sum_{n}^{m} \gamma(k)f(k)\right)^{2p}| =$$

$$= |E \sum_{j_1=n}^{m} \cdots \sum_{j_{2p}=n}^{m} \gamma(j_1)\cdot\ldots\cdot\gamma(j_{2p})f(j_1)\cdot\ldots\cdot f(j_{2p})| \leq$$

$$\leq \gamma(n)^{2p} \sum_{j_1=n}^{m} \cdots \sum_{j_{2p}=n}^{m} |Ef(j_1)\cdot\ldots\cdot f(j_{2p})| \leq$$

$$\leq \gamma(n)^{2p} \sum_{j_1=n}^{m} \cdots \sum_{j_{2p}=n}^{m} \sum_{k_1=0}^{\infty} \sum_{\ell_1=0}^{\infty} \cdots \sum_{k_{2p}=0}^{\infty} \sum_{\ell_{2p}=0}^{\infty} \Big\{ K^{2p} \lambda^{\sum_1^{2p}(k_i+\ell_i)}$$

$$|E[e(j_1-k_1)e(j_1-\ell_1) - \delta_{k_1\ell_1}]\cdot\ldots\cdot[e(j_{2p}-k_{2p})e(j_{2p}-\ell_{2p}) -$$

$$- \delta_{k_{2p}\ell_{2p}}]| \Big\}$$

Since $e(j)$ and $e(k)$ are independent for $j = k$, the expectation in the above sum is zero unless for each r

$$j_r - k_r = \begin{cases} j_s - k_s \\ \text{or} \\ j_s - \ell_s \end{cases} \quad \underline{\text{and}} \quad j_r - \ell_r = \begin{cases} j_{s'} - k_{s'} \\ \text{or} \\ j_{s'} - \ell_{s'} \end{cases} \tag{A.8}$$

for some $s, s' \neq r$.

Regard k_i, ℓ_i and $j_1,\ldots,j_p$ as fixed. Then the other $j_{p+1}, \ldots,j_{2p}$ are determined by (A.8) (up to permutations, the number of which depends on p and not on m-n). Hence p of the outer summing indices can be eliminated. Summing first over k_i and ℓ_i gives a finite result $C(\lambda)$ depending only on λ.

Thus

$$\left|E\left(\sum_{n}^{m} \gamma(k)f(k)\right)^{2p}\right| \leq \gamma(n)^{2p} \sum_{j_1=n}^{m} \cdots \sum_{j_p=n}^{m} C(\lambda) \leq \tag{A.9}$$

$$\leq \gamma(n)^{2p}\cdot(m-n)^{p}\cdot C(\lambda) \leq \gamma(n)^{2p}\cdot\gamma(m)^{-p}\cdot C_1(\gamma) \leq \gamma(n)^{p}C_2(\gamma)$$

The second inequality follows from

$$(m-n)\gamma(m) \leq \sum_{n}^{m} \gamma(t) < 2L \tag{A.10}$$

From (A.10) we also have

$$\left[\frac{m}{n} - 1\right]\cdot m^{-s} \leq 2L/n$$

or

$$\frac{m}{n} \leq 1 + 2L\,\frac{m^{+s}}{n} \leq 1 + 2L\,\frac{m}{n} \quad \text{since } s \leq 1$$

which gives

$$\frac{m}{n} \leq \frac{1}{2L - 1} \leq \text{Const} \tag{A.11}$$

This inequality implies the last inequality in (A.9).

From Chebysjev's inequality it follows that

$$P\Big(\Big|\sum_{j=n_k}^{n_{k+1}} \gamma(j)f(j)\Big| > \varepsilon\Big) \leq \frac{E\Big|\sum_{n_k}^{n_{k+1}} (j)f(j)\Big|^{2p}}{\varepsilon^{2p}} \leq$$

$$\leq \gamma(n_k)^p \cdot C_3/\varepsilon^{2p} \qquad \text{(A.12)}$$

Now

$$\sum_{k=1}^{\infty} \gamma(n_k)^p \leq \sum_{t=1}^{\infty} \gamma(t)^p = \sum_{t=1}^{\infty} C_\gamma\, t^{-sp} < \infty$$

since sp > 1. Application of the Borel - Cantelli lemma yields, in view of (A.12) and (A.13) that

$$\sum_{n_k}^{n_{k+1}} \gamma(t)f(t) \to 0 \quad \text{w.p.1 as } k \to \infty$$

and Lemma A.1 is proved. □

With this lemma applied to (A.5) and (A.7) we obtain

$$\sum_{n}^{k(n,\Delta\tau)} \gamma(t)x(t)x(t)^T =$$

$$= \sum_{n}^{k(n,\Delta\tau)} \gamma(t)E[\bar{x}(t)\bar{x}(t)^T] + R_5(n) + R_6(n,\Delta\tau,\bar{\theta}) \qquad \text{(A.14)}$$

where $R_5(n) \to 0$ w.p.1 as $n \to \infty$, and where

$$|R_6(n,\Delta\tau,\bar{\theta}) \leq$$

$$\leq |q(n,k(n,\Delta\tau))\cdot \sum_{n}^{k(n,\Delta\tau)} \gamma(t)v(t)^2 + C_9 \sum_{n}^{k(n,\Delta\tau)} \gamma(t)\lambda^{t-n}| \leq$$

$$\leq q(n,k(n,\Delta\tau))\cdot[E\ v(t)^2]\cdot \sum_{n}^{m(n,\Delta\tau)} \gamma(t) + R_7(n) \leq$$

$$\leq C_{10}\ q(n,k(n,\Delta\tau))\cdot\Delta\tau + R_7(n)$$

with $R_7(n) \to 0$ as $n \to \infty$. The same result naturally holds for

$$\sum_{n}^{k(n,\Delta\tau)} \gamma(t)x(t)x(t+1)^T$$

Consider now $(y(t) = D^T x(t))$

$$|\theta(k(n,\Delta\tau)) - \theta(n)| = \Big|\sum_{n}^{k(n,\Delta\tau)} \gamma(t)\ S_i(t-1)\ x(t)y(t+1)\Big| \leq$$

$$\leq |\bar{S}_i \sum_{n}^{k(n,\Delta\tau)} \gamma(t)\ x(t)\ x(t+1)^T\ D| + \max_{n\leq t\leq k} |\bar{S}_i - S_i(t)|\cdot$$

$$\Big|\sum_{n}^{k(n,\Delta\tau)} \gamma(t)x(t)x(t+1)^T\ D\Big| \leq (|\bar{S}_i|+\bar{\delta})(C_{11}\cdot\Delta\tau+R_8(n)) \qquad (A.15)$$

where $R_8(n) \to 0$ w.p.1 as $n\to\infty$

$$|S_i^{-1}(k(n,\Delta\tau)) - S_i^{-1}(n)| = \Big|\sum_{n}^{k(n,\Delta\tau)} \gamma(t)[x(t)x(t)^T\ S_i^{-1}(t)]\Big| \leq$$

$$\leq \Delta\tau\cdot C_{12} + R_9(n);\quad \text{where } R_9(n) \to 0 \text{ w.p.1 as } n\to\infty$$

It follows from (A.15) that $\theta(k)$ and $S_i(k)$ can be made to differ arbitrarily little from $\theta(n)$ and $S_i(n)$ for large n. This means that, for sufficiently small $\Delta\tau$, $k(n,\Delta\tau) = m(n,\Delta\tau)$ for sufficiently large n. It also follows that

$$\sup_{n\le t\le m} |\theta(n) - \theta(t)| \le C_{13}\cdot\Delta\tau + R_{10}(n);$$

$$R_{10}(n) \to 0 \quad \text{w.p.1 as } n \to \infty$$

(A.16)

$$\sup_{n\le t\le m} |S(n) - S(t)| \le C_{14}\cdot\Delta\tau + R_{11}(n);$$

$$R_{11}(n) \to 0 \quad \text{w.p.1 as } n \to \infty$$

Combining (A.14), (A.15) and (A.16) we obtain for sufficiently small $\Delta\tau$

$$|\theta(m(n,\Delta\tau)) - \theta(n) - \bar{S}_i[E\,\bar{x}(t)\bar{x}(t+1)D]\cdot\sum_{n}^{m}\gamma(t)| \le$$

$$\le |\bar{S}_i|\cdot R_5(n) + R_6(n) + \max_{n\le t\le m}|\bar{S}_i - S_i(t)|\cdot[|\bar{S}|\,R_5(n) + R_6(n)] \le$$

$$\le |\bar{S}_i|\cdot R_5(n)[1 + C_{14}\cdot\Delta\tau + R_{11}(n) + |\bar{S}_i - S_i(n)|] +$$

$$+ [q(n,m(n,\Delta\tau))\cdot C_{10}\cdot\Delta\tau + R_7(n)]\,[1 + C_{14}\cdot\Delta\tau\,R_{11}(n) +$$

$$+ |\bar{S}_i - S_i(n)|] \qquad \text{(A.17)}$$

Now using that

$$E\,\bar{x}(t)\bar{x}(t+1)D = f(\bar{\theta})$$

$$\sum_{n}^{m(n,\Delta\tau)}\gamma(t) \to \Delta\tau \quad \text{as } n\to\infty$$

and

$$q(n,m(n,\Delta\tau)) \le C_{13}\cdot\Delta\tau + R_{10}(n) + |\theta(n) - \bar{\theta}|$$

and rearranging the terms of (A.17) gives the desired relation (A.1). Eq. (A.2) is obtained analogously. □

APPENDIX B. RESULTS FOR OTHER MODEL STRUCTURES

In Section 2.1 four different model structures were briefly mentioned. One of them, (2.4), was chosen as the basic model and has been used throughout the report. In this appendix it is shown how the other model structures can formally be treated in exactly the same way as (2.4).

B.1 Model structures.

The chosen model (2.4),

$$y(t+\hat{k}+1) + \hat{\alpha}_1 y(t) + \ldots + \hat{\alpha}_{\hat{n}} y(t+1-\hat{n}) = \hat{\beta}_0[u(t) + \hat{\beta}_1 u(t-1) + \ldots + \hat{\beta}_{\hat{m}} u(t-\hat{m})] + \hat{\varepsilon}(t+\hat{k}+1) \tag{B.1}$$

will be referred to as model A.

In case the variable β_0 is estimated, a more natural model structure is, cf. (2.2)

$$y(t+\hat{k}+1) + \hat{\alpha}_1 y(t) + \ldots + \hat{\alpha}_{\hat{n}} y(t+1-\hat{n}) = \hat{\beta}_0 u(t) + \hat{\beta}_1 u(t-1) + \ldots + \hat{\beta}_{\hat{m}} u(t-\hat{m}) + \hat{\varepsilon}(t+\hat{k}+1) \tag{B.2}$$

This model will be referred to as "model B."

In the models A and B, the system (2.1) is written on "predictor form," which is suitable for the self-tuning regulators STURE0 and STURE1. More straightforward models are

$$y(t+1) + \hat{a}_1 y(t) + \ldots + \hat{a}_{\hat{n}} y(t+1-\hat{n}) = \hat{b}_0[u(t-\hat{k}) + \ldots + \hat{b}_{\hat{m}} u(t-\hat{k}-\hat{m})] + \hat{e}(t+1) \tag{B.3}$$

and

$$y(t+1) + \hat{a}_1 y(t) + \ldots + \hat{a}_{\hat{n}} y(t+1-\hat{n}) = \hat{b}_o u(t-\hat{k}) + \ldots + \hat{b}_{\hat{m}} u(t-\hat{k}-\hat{m}) + \hat{e}(t+1) \tag{B.4}$$

These models will be called "model C" and "model D" respectively. Clearly, if $\hat{k} = 0$, models A and C and models B and D, respectively, are identical.

B.2 Modifications of the results of Chapter 2.

Introduce

$$\begin{aligned}
\theta_A &= (\hat{\alpha}_1, \ldots, \hat{\alpha}_{\hat{n}}, \hat{\beta}_1, \ldots, \hat{\beta}_{\hat{m}})^T \\
\theta_B &= (\hat{\alpha}_1, \ldots, \hat{\alpha}_{\hat{n}}, \hat{\beta}_o, \ldots, \hat{\beta}_{\hat{n}})^T \\
\theta_C &= (\hat{a}_1, \ldots, \hat{a}_{\hat{n}}, \hat{b}_1, \ldots, \hat{b}_{\hat{m}})^T \\
\theta_D &= (\hat{a}_1, \ldots, \hat{a}_{\hat{n}}, \hat{b}_o, \hat{b}_1, \ldots, \hat{b}_{\hat{m}})^T
\end{aligned} \tag{B.5}$$

$$\theta^o_A = (\alpha_1, \ldots, \alpha_n, 0, \ldots, 0, \beta_1/\hat{\beta}_o, \ldots, \beta_{m'}/\hat{\beta}_o, 0, \ldots, 0)^T$$

($\hat{n} - n$ and $\hat{m} - m'$ zeroes respectively) (B.6)

$$\theta^o_B = (\alpha_1, \ldots, \alpha_n, 0, \ldots, 0, \beta_o, \beta_1, \ldots, \beta_{m'}, 0, \ldots, 0)^T$$

$$\theta^o_C = (a_1, \ldots, a_n, 0, \ldots, 0, b_1 b_o/\hat{b}_o, \ldots, b_m b_o/\hat{b}_o, 0, \ldots, 0)^T$$

$$\theta^o_D = (a_1, \ldots, a_n, 0, \ldots, 0, b_o, b_1, \ldots, b_m, 0, \ldots, 0)^T$$

$$x_A(t) = [-y(t), \ldots, -y(y+1-\hat{n}), \hat{\beta}_o u(t-1), \ldots, \hat{\beta}_o u(t-\hat{m})]^T$$

$$x_B(t) = [-y(t), \ldots, -y(t+1-\hat{n}), u(t), u(t-1), \ldots, u(t-\hat{m})]^T \qquad (B.7)$$

$$x_C(t-\hat{k}) = [-y(t), \ldots, y(t+1-\hat{n}), \hat{b}_o u(t-1-\hat{k}), \ldots, \hat{b}_o u(t-\hat{k}-\hat{m})]^T$$

$$x_D(t-\hat{k}) = [-y(t), \ldots, y(t+1-\hat{n}), u(t-\hat{k}), u(t-\hat{k}-1), \ldots, u(t-\hat{k}-\hat{m})]^T$$

Then formulas (2.5) to (2.15) are valid for any subscript A, B, C, or D, provided all explicit $\hat{\beta}_o$ are set to zero for models B and D. For model C, $\hat{\beta}_o$ should be replaced by $\hat{b}_o$.

The minimum variance control law (2.17) has no direct counterpart for models C and D. For model B, u(t) should be chosen as the solution of

$$\theta_B(t)^T x_B(t) = 0 \qquad (B.8)$$

B.3 Modifications of the results of Chapter 3.

Theorem 3.1 and its corollary are valid for all model structures if in (3.7d) the corresponding x-vector according to (B.7) is chosen. The discussion in Section 3.2 can be carried out for model B if

" $(\theta_o - \tilde{\theta})^T x(t) = 0$ "

is replaced by

" $\tilde{\theta}_B\, x(t) = 0$ "

and the corresponding modifications in the following are made.

B.4 Modifications of the results of Chapter 4.

The only modifications necessary are directly implied by the modified form of (2.15).

B.5 Modifications of the results of Chapter 5.

Here only models A and B are relevant, since the discussion is concerned with the special regulators STURE0 and STURE1. The discussion in Section 5.1 and the proof of Theorem 5.1 are based on the variable

$$\tilde{\theta}(t) = \tilde{\theta}_A(t) = \theta_A^o - \theta_A(t)$$

If this is replaced by

$$\tilde{\theta}_B(t) = \theta_B^o - \theta_B(t)$$

and $\beta_o/\hat{\beta}_o$ is replaced by 1, the discussion remains unaltered.

Consequently the results of this chapter, like those of Section 3.2 are valid for STURE with model B (i.e. the variable β_o is estimated) without any restrictions on β_o and $\hat{\beta}_o$.

Convergence of Recursive Stochastic Algorithms

Lennart Ljung

ABSTRACT

Convergence with probability one for a class of recursive, stochastic algorithms is considered. The class contains stochastic approximation algorithms like the Robbins - Monro scheme and the Kiefer - Wolfowitz procedure. It also contains other estimation and control algorithms that are common in stochastic control theory.

It is shown that the problem of convergence can be separated into a deterministic problem and a stochastic one. The analysis of the deterministic problem reduces to stability analysis of an ordinary differential equation (ODE). For the stochastic problem it is sufficient to show consistency for a simple algorithm that estimates the mean value of a random variable.

Using this technique, the usual conditions for convergence can be extended. Correlated observations can be treated and the conditions on the gain sequence can be traded off against conditions on the moments of the noise.

The behaviour of the algorithm can also be described using the ODE that is associated with the convergence problem. Based on the trajectories of the ODE future values generated by the algorithm can be predicted. Numerical solution of the ODE therefore is a valuable tool to analyse the asymptotic properties of the recursive stochastic algorithm.

TABLE OF CONTENTS.

1. ALGORITHMS.

Recursive algorithms where stochastic observations enter, occur in many fields of applications, such as estimation, filtering and control theory. In this report convergence of a certain class of recursive algorithms will be considered. The class includes stochastic approximation algorithms and also other algorithms that are common in control applications. The obtained results are more general than earlier reported ones.

In this chapter some examples of recursive, stochastic algorithms that fit in the formulation are given. Chapter 2 contains a short survey and classification of previously reported convergence criteria. A theorem that separates the considered problem of convergence into a deterministic problem and a purely stochastic one is shown in Chapter 3. The stochastic part is further discussed in Chapter 4. In Chapter 5 conditions are given that assure bounded estimates. In Chapter 6 the question of convergence rate is discussed and a theorem is shown, which connects the behaviour of the recursive algorithm to that of a deterministic ordinary differential equation. Finally, in Chapter 7, the results are summarized and discussed.

1.1. A General Recursive Algorithm.

A general version of a recursive algorithm can be written

$$x_n = x_{n-1} + H_n(x_{n-1}, \varphi_n) \qquad x_0 = x^0 \qquad (1.1)$$

where $\{x_n\}$ is a sequence of vectors. These vectors will be called estimates and it is supposed that they are estimates of some desired or optimal value x^*, which is in-

dependent of n. It is then desirable that the sequence $\{x_n\}$ tends to x^* as n tends to infinity.

The correction $H_n(x_{n-1},\varphi_n)$ is a function of the previous estimate x_{n-1} and of an observation φ_n obtained at time n.

The observations are in general functions

$$\varphi_n = \varphi_n(e_n,e_{n-1},\ldots,e_0;x_{n-1},\ldots,x_0) \tag{1.2}$$

of the previous estimates $\{x_i\}$ and of a sequence of vector valued random variables $\{e_i\}$ that is independent of $\{x_i\}$. In case the experimenter has some test signal at his disposal, this can be included in the sequence $\{e_i\}$. In many cases the observations do not depend on previous estimates. Then the sequence $\{e_i\}$ can be taken as the observations themselves:

$$\varphi_n = e_n \tag{1.3}$$

Another common special case is that the observation depends only on x_{n-1}, i.e.

$$\varphi_n = \varphi_n(x_{n-1},e_n) \tag{1.4}$$

Since it is desired that $\{x_i\}$ converges, the correction $H_n(x_{n-1},\varphi_n)$ caused by a single stochastic observation φ_n must tend to zero as n tends to infinity, i.e.

$$H_n(x_{n-1},\varphi_n) = \gamma_n\tilde{H}_n(x_{n-1},\varphi_n)$$

where $\{\gamma_n\}$ is a sequence of positive scalars tending to zero. The variables may either be predetermined scalars, which gives the algorithm

$$x_n = x_{n-1} + \gamma_n \tilde{H}_n(x_{n-1},\varphi_n) \tag{1.5}$$

or functions of the observations, giving

$$x_n = x_{n-1} + \gamma_n(\varphi_n,\ldots,\varphi_0)\tilde{H}_n(x_{n-1},\varphi_n) \tag{1.6}$$

In some cases it is convenient to specially treat the case when $\tilde{H}_n$ is multiplied by a matrix $S_n(\varphi_n,\ldots,\varphi_0)$:

$$x_n = x_{n-1} + \gamma_n S_n(\varphi_n,\ldots,\varphi_0)\tilde{H}_n(x_{n-1},\varphi_n) \tag{1.7}$$

In (1.6) and (1.7) γ_n and S_n respectively have to be updated recursively. Therefore, it is always possible to rewrite (1.6) and (1.7) in the form (1.5) by extending the estimate vector x_n. However, in some applications it is more favourable to directly consider (1.6) and (1.7).

The results of this report are mainly concerned with the cases (1.3) and (1.4). Then $\tilde{H}_n(x_{n-1},\varphi_n)$ can be written explicitely as a function of x_{n-1} and e_n

$$\tilde{H}_n\left(x_{n-1},\varphi_n(x_{n-1},e_n)\right) = Q_n(x_{n-1},e_n)$$

Hence the basic algorithm to be considered here is

$$x_n = x_{n-1} + \gamma_n Q_n(x_{n-1},e_n) \tag{1.8}$$

The techniques that are used also apply for the general case (1.2) if the influence of old estimates $\{x_i\}$ on φ_n decreases sufficiently fast. Certain algorithms with φ_n as in (1.2) are treated in Ljung-Wittenmark (1974).

In Sections 1.2 and 1.3 several examples of algorithm (1.1) are given. These examples will, hopefully, clari-

fy the classification of H_n, and show that a number of control theory applications fit in the structures (1.1) - - (1.8).

1.2. Stochastic Approximation Algorithms.

> "Stochastic approximation is concerned with schemes converging to some sought value when, due to the stochastic nature of the problem, the observations involve errors." (Dvoretzky (1956))

The original scheme was devised by Robbins and Monro (1951).

Example 1.1 - The Robbins-Monro (RM) scheme.

Consider the problem to solve

$$E_e Q(x,e) = 0 \qquad (1.9)$$ [1)]

for x. Let the solution be x^*. Observations $\varphi_n = Q(x,e_n)$, n=1,... are available for any x. Robbins-Monro proved that under certain conditions the scheme

$$x_n = x_{n-1} + \gamma_n Q(x_{n-1},e_n) \qquad (1.10)$$

converges to x^*. These results were extended by Blum (1954b) to the multidimensional case. The conditions for convergence are discussed in Chapter 2. Let us just remark that among the necessary conditions we have

---

1) "E_e" denotes expectation with respect to e, while the vector x is considered as a fixed parameter

$$\sum_{1}^{\infty} \gamma_n = \infty \qquad (1.11a)$$

$$\sum_{1}^{\infty} \gamma_n^2 < \infty \qquad (1.11b)$$

In this report it is shown that (1.11b) can be replaced by a weaker condition.

Comparing with the formalism of the previous section the observation φ_n has the form (1.4), and (1.10) is thus a simple special case of (1.8). The sequence $\{\gamma_n\}$ is deterministic as in (1.5). In some applications, see Section 1.3, $Q(x_{n-1},e_n)$ is itself not a primary observation, but formed from observations and from x_{n-1}.

As a simple example of the RM scheme, consider the problem to find the mean value of a stochastic variable e. Let $Ee = x^*$. This value x^* can be found as the solution of

$$E_e(e-x) = 0$$

Now take in (1.9) $Q(x,e) = e - x$ and apply the RM scheme:

$$x_n = x_{n-1} + \gamma_n(e_n - x_{n-1}) \qquad (1.12)$$

With the choice $\gamma_n = 1/n$, which clearly satisfies (1.11) we obtain

$$x_n = \frac{1}{n} \sum_{k=1}^{n} e_k$$

which is an efficient estimate of x^*.

□

In many applications it is interesting to minimize a function

$$E_v J(x,v) = P(x) \tag{1.13}$$

with respect to x. If the derivative $\frac{\partial}{\partial x} J(x,v)$ can be calculated the stationary points of (1.13) can be found as solutions of

$$E_v\left(-\frac{\partial}{\partial x} J(x,v)\right) = 0$$

This is a problem that can be solved using the RM scheme. If the derivative cannot be calculated, it seems natural to replace it with some difference approximation. This was suggested by Kiefer and Wolfowitz (1952):

Example 1.2 - The Kiefer-Wolfowitz (KW) procedure.

Consider the problem to minimize (1.13) with respect to x. Observations $J(x,v_n)$, $n = 1,\ldots,$ of the criterion are available for each chosen x. The distribution of v_n is independent of x. Kiefer-Wolfowitz (1952) and Blum (1954b) suggested that the minimizing point x^* should be estimated recursively.

$$x_n = x_{n-1} + \gamma_n \bar{J}(x_{n-1},a_n,\bar{v}_n)/a_n \tag{1.14}$$

where

$$\bar{J}(x,a,\bar{v}) = \left(J(x-au_{i_1},v_{j_1}) - J(x,v_i), \ldots,\right.$$

$$\left. J(x-au_{i_m},v_{j_m}) - J(x,v_i)\right)$$

m is here the dimension of the vector x and $\{u_i\}$ are the unit vectors in R^m. Consequently, to advance one step with (1.14), m+1 measurements are made and m+1 outcomes of the noise v enter.

Blum (1954b) has shown convergence with probability one for (1.14) under certain conditions. The conditions on the sequences $\{\gamma_n\}$ and $\{a_n\}$ are

$$\lim_{n\to\infty} a_n = 0; \quad \sum_1^\infty \gamma_n = \infty, \quad \sum_1^\infty a_n\gamma_n < \infty, \quad \sum_1^\infty (\gamma_n/a_n)^2 < \infty$$

In this procedure the observation φ_n consists of $J(x_{n-1}+ +au_i, v_{j_i})$ and is of the type (1.4). The noise vector e_n in (1.4) must be chosen to include m+1 values of v. Clearly (1.14) is a special case of the basic algorithm (1.8). Notice that $Q_n(x_{n-1},e_n) = J(x_{n-1},a_n,v_n)/a_n$ actually is time varying in this case.

□

In the KW procedure the minimization is performed with a steepest descent method. Recently, Kushner (1972), (1973), and Kushner-Gavin (1973) have considered more general minimization routines. This approach seems to promise better numerical behaviour of the algorithms.

Dvoretzky (1956), considers a version of stochastic approximation algorithms, which includes both the RM and KW schemes. He writes the recursion

$$x_{n+1} = T_n(x_n,\ldots,x_1) + e_{n+1} \tag{1.15}$$

where

$$E(e_{n+1} \mid x_1,\ldots,x_n) = 0$$

Under certain conditions on the transformations T_n, (1.15) converges with probability one. Dvoretzky considers primarily the case with scalar valued x. Extensions to the multidimensional case has been given by Derman-Sacks (1959). Also (1.15) fits in the formulation (1.1) with general observations (1.2).

The term "stochastic approximation algorithms" will be used frequently in this report. By this term will be meant the algorithms discussed in this section, in particular the RM and KW procedures.

1.3. Applications to Control Theory.

Learning systems.

Tsypkin (e.g. 1968, 1971, 1973) has applied stochastic approximation techniques to a broad variety of problems in control theory. The approach is known as "learning systems". In short, the "goal of learning" is defined as to minimize some criterion P(x) with respect to the vector x. However, only noisy observations J(x,v) of the criterion are available, where

$$E_v J(x,v) = P(x) \tag{1.16}$$

If the derivative $\frac{\partial}{\partial x} J(x,v)$ can be formed the RM scheme can be applied to

$$E_v\left[-\frac{\partial}{\partial x} J(x,v)\right] = 0$$

If not, the KW procedure can be applied to (1.16).

In this framework estimation and identification problems ("learning models"), adaptive systems, supervised and unsupervised pattern recognition ("learning pattern recognition systems" and "Self-learning systems of classification") etc. can be treated. Similar approaches are considered by several other authors, see e.g. Fu (1969), Saridis et al (1969), Sakrison (1967).

Basically, the learning algorithms rely upon the RM scheme (or KW scheme) and convergence criteria for these can be applied. However, it may happen that the usual criteria are not applicable in a given case.

An approach that is related to stochastic approximation is suggested by Aizerman, Braverman and Rozonoer (e.g. 1964ab, 1970). Their "Potential Function Method" can be applied to various problems in machine learning. The connection with the RM scheme is discussed in Aizerman et al (1965).

Estimation.

A common problem is to estimate the coefficients in a difference equation

$$y(t+1) + a_1 y(t) + \ldots + a_m y(t-m+1) =$$

$$= b_0 u(t) + \ldots + b_m u(t-m) + v(t+1) \qquad (1.17)$$

where $\{v(t)\}$ is a sequence of independent, random variables with zero mean values. The variables y, u and e are scalars. Measurements of the input u and of the output y are available.

Let x^* denote the vector of true values:

$$x^* = (a_1,\ldots,a_m,b_0,\ldots,b_m)^T$$

and let

$$\xi_t = \left(-y(t),\ldots,-y(t-m+1),u(t),\ldots,u(t-m)\right)^T$$

denote the vector of observations. Then (1.17) can be written

$$y(t+1) - \xi_t^T x^* = v(t+1) \tag{1.18}$$

A reasonable "goal of learning" for this estimation problem is to find the vector x that minimizes the criterion

$$P(x) = E\left(y(t+1) - \xi_t^T x\right)^2 \tag{1.19}$$

This function can be minimized by taking the derivative and applying the RM scheme:

$$-P'(x)=E_e Q(x,e) = 0 \tag{1.20}$$

where

$$Q(x,e_{t+1}) = \xi_t y(t+1) - \xi_t \xi_t^T x \quad ; \quad e_{t+1} = \left(\xi_t, y(t+1)\right)$$

Then

$$x_{n+1} = x_n + \gamma_{n+1}\left\{\xi_n y(n+1) - \xi_n \xi_n^T x_n\right\} \tag{1.21}$$

The variables γ_n can be chosen in several ways. Let it first be a sequence of scalars. This stochastic approximation version of least squares estimation is treated e.g. by Wieslander (1969), Tsypkin (1973) and Mendel (1973),

where also other variants are discussed.

For normalization reasons γ_n is often chosen as

$$\gamma_{n+1} = \frac{1}{n}\left(\xi_n^T \xi_n\right)^{-1}$$

or

$$\gamma_{n+1} = \left(\sum_{k=0}^{n} \xi_k^T \xi_k\right)^{-1}$$

Comparing with Section 1.1, the observation $\varphi_{n+1} = (\xi_n, y(n+1))$ does not depend on $\{x_i\}$ and is consequently of the type (1.3). The algorithm (1.21) with the discussed choices of $\{\gamma_n\}$ is a special case of (1.6).

Generally speaking, when we are faced with a problem (1.9), to solve

$$E_e Q(x,e) = 0$$

for x, it seems desirable to solve

$$\frac{1}{n} \sum_{k=1}^{n} Q(x,e_k) = 0 \qquad (1.22)$$

for x at time n. This is likely to give a good estimate x_n. In the case (1.4) the basic observations $\varphi(x_{k-1},e_k)$, from which $\tilde{H}(x_{k-1},\varphi_k) = Q(x_{k-1},e_k)$ is formed, depend on x_{k-1}. It is then not clear how the function $Q(x,e_k)$ could be formed from $\varphi(x_{k-1},e_k)$. Hence, it is not possible to

solve (1.22).

In the present estimation problem, however, (1.22) can be solved, since the function $Q(x,e_{k+1}) = (\xi_k y(k+1) - \xi_k \xi_k^T x)$ can be formed for any x, as soon as y(k+1) and ξ_k are known. Furthermore, in this special case Q(x,e) is linear in x. Then it is actually possible to solve (1.22) recursively with a special (matrix) choice of γ_n in (1.21):

Example 1.3 - Real time least squares.

For the estimation problem (1.20) Eq. (1.22) is

$$\frac{1}{n} \sum_{k=1}^{n} Q(x,e_k) = \frac{1}{n} \sum_{k=1}^{n} \xi_k [y(k+1) - \xi_k^T x] = 0 \qquad (1.23)$$

which can be solved for x if only

$$\frac{1}{n+1} \sum_{k=0}^{n} \xi_k \xi_k^T = R_n \qquad (2m \times 2m \text{ matrix})$$

is known. The solution of (1.23) can be written recursively (Åström (1968)):

$$x_{n+1} = x_n + \frac{1}{n+1} R_n^{-1} \left[\xi_n y(n+1) - \xi_n \xi_n^T x_n \right] \qquad (1.24)\ ^{1)}$$

In Eq. (1.24) the correction can be written

$$H_n = \gamma_n S_n(e_n,\ldots,e_0) \cdot Q(x_{n-1},e_n)$$

where $S_n = R_n^{-1}$ and $\gamma_n = 1/n$, which is of type (1.7).

---

1) This expression holds only asymptotically as n tends to infinity.

Notice that if R_n is included in the estimate vector x_n, (1.24) can be written on the form (1.8).

Remark: A recursive solution of (1.22) can be obtained as soon as $\varphi_n = e_n$ and $Q(x,e) = g_1(x)g_2(e) + g_3(e)$. □

Example 1.4 - General recursive estimation algorithms.

Algorithms (1.21) and (1.24) give biased parameter estimates in case $\{v(t)\}$ is not a sequence of independent variables. A more general model is

$$y(t+1) + a_1 y(t) + \ldots + a_m y(t-m+1) =$$

$$= b_0 u(t) + \ldots + b_m u(t-m) +$$

$$+ e(t+1) + c_1 e(t) + \ldots + c_m e(t-m+1) \qquad (1.25)$$

where $\{e(t)\}$ is a sequence of independent random variables. Let the vector of true parameters be

$$x^* = (a_1,\ldots,a_m,b_0,\ldots,b_m,c_1,\ldots,c_m)^T$$

and let an estimate be

$$x = (\hat{a}_1,\ldots,\hat{a}_m,\hat{b}_0,\ldots,\hat{b}_m,\hat{c}_1,\ldots,\hat{c}_m)^T$$

Then a suitable criterion is to minimize

$$E[y(t+1) - \hat{y}(t+1|t,x)]^2 = P(x) \qquad (1.26)$$

where $\hat{y}(t+1|t,x)$ is the predicted value of $y(t+1)$ given $\{u(t),\ldots,u(0),y(t),\ldots,y(0)\}$ and given that the system

parameters are assumed to be x. This value can be obtained as the solution of

$$\hat{y}(s+1|s,x) + \hat{c}_1\hat{y}(s|s-1,x) + \ldots + \hat{c}_m\hat{y}(s-m+1|s-m,x) =$$

$$= (\hat{c}_1-\hat{a}_1)y(s) + \ldots + (\hat{c}_m-\hat{a}_m)y(s-m+1) +$$

$$+ \hat{b}_0u(s) + \ldots + \hat{b}_mu(s-m) \qquad (1.27)$$

where $s = 0,\ldots,t$. Suitable initial values must be chosen. Eq. (1.26) corresponds to minimization of the prediction error, and clearly (1.19) is a special case ($c_i = 0$) of (1.26).

One way to determine the estimate x at time n is to minimize

$$\frac{1}{n}\sum_{1}^{n}[y(t) - y(t|t-1,x)]^2 \qquad (1.28)$$

with respect to x. This is possible to do, using (1.27) if the observations $\{u(t),\ldots,u(0),y(t),\ldots,y(0)\}$ are available. However, in the general case it is not possible to write the sequence of estimates that minimize (1.28) recursively as in Example 1.3. This is due to the fact that the criterion is a more complex function of x in this case.

The RM scheme can be applied to the derivative of (1.26). This gives

$$x_{t+1} = x_t - \gamma_{t+1}\left\{\frac{\partial}{\partial x}\hat{y}(t+1|t,x_t)[\hat{y}(t+1|t,x_t) - y(t+1)]\right\} \qquad (1.29)$$

It is important to notice that it is possible to exactly calculate $\hat{y}(t+1|t,x_t)$ according to (1.27) only if infinitely many old y(s) and u(s) are known. This means that

(1.29) is no recursive algorithm.

For an on line algorithm only a relatively small number (say r) of old values can be kept in memory. Hence $\hat{y}(t+1|t,x)$ can be replaced by some approximation $\hat{y}(t,x_t,x_{t-1},\ldots,x_0)$ that is obtained from (1.27) starting with s=t-r. The initial values are determined from previous predictions. The same holds for $\frac{\partial}{\partial x}\hat{y}(t+1|t,x_t)$.

With the terminology of Section 1.1 this means that the observation φ_n consists of y(n-s), u(n-s), s = 0,...,r, and the initial values y(n-r-s), s = 0,...,m. The observation φ_n is of the general type (1.2) since it depends on all previous estimates via old y(t). Consequently the resulting approximate algorithm (1.29) is of type (1.5) or (1.6) with φ_n as in (1.2).

The sequence $\{\gamma_n\}$ can be multiplied by a sequence of matrices as in (1.24). The properties of approximations of (1.29) with various choices of γ_n and matrices are discussed e.g. by Söderström (1973). No convergence results for these algorithms are available.

□

Adaptive systems.

Adaptive systems concern the control of processes with unknown dynamics. An adaptive system may contain a feedback regulator of some given structure, the parameters of which are automatically determined. A performance index P(x) for the behaviour of the adaptive system can be defined as a function of the regulator parameters x. Usually, only a stochastic function of the performance index J(x,v) can be observed or formed from the observations. It is supposed that (1.13) holds. If the derivative $\frac{\partial}{\partial x}J(x,v)$ can be calculated, the RM scheme can be applied to minimize P(x). How-

ever, for adaptive systems, calculation of this derivative requires that the characteristics of the system are known. Therefore this approach cannot be applied straightforwardly. The KW procedure can be used instead.

A different approach is to estimate the system dynamics and use the estimates for design of the regulator. The regulator is updated recursively. In Ljung (1972) is discussed how such adaptive structures fit in the formulation (1.1).

Example 1.5 - A class of self-tuning regulators.

Suppose algorithm (1.21) or (1.24) is used to estimate the system dynamics. The estimate at time t can be used to determine the next input from old input output data:

$$u(t+1) = h(x_t, \xi_t) \tag{1.30}$$

Åström-Wittenmark (1973) consider a minimum variance control law, which gives several nice features to the resulting adaptive (or self-tuning) regulator.

It is important to notice that since u(t) depends on x_t, also the observations ξ_t and y(t+1) will depend on x_t, $x_{t-1},\dots,x_0$. This means that φ has the general form (1.2) and Q defined in (1.20)

$$Q(x_t, e_{t+1}) = \tilde{H}(x_t, \varphi_{t+1})\xi_t(x_t, x_{t-1},\dots,x_0) \cdot$$

$$\cdot \left[y(t+1; x_t,\dots,x_0) - \xi_t^T(x_t,\dots,x_0)x_t\right]$$

is no longer a linear function of x_t. In particular (1.24) no longer gives the estimates that are solutions to (1.22).

The convergence properties of the class of self-tuning regulators under consideration are treated in detail in Ljung-Wittenmark (1974).

□

It is of interest to compare Examples 1.3, 1.4 and 1.5. In all these cases the objective is to solve

$$E_e Q(x,e) = 0$$

for x. In Examples 1.3 and 1.4 the basic observations {y(t)} and {u(t)} do not depend on x. From these observations the function

$$Q_t(x,e_t) \quad \text{or} \quad Q_t(x,e_t,\dots,e_0)$$

can be formed for any x. It is then possible to solve

$$\frac{1}{n}\sum_{1}^{n} Q(x,e_t) = 0 \tag{1.31}$$

for x, which gives the estimate x_n.

In Example 1.3, because of the linear dependence of x in Q(x,e), it is possible to calculate the sequence of solutions to (1.31) recursively as in (1.24). In Example 1.4 Q depends on x in a more complex way, and the solution to (1.31) cannot be written as recursions containing a fixed and finite number of observations. In Example 1.5 also the basic observations y(t) and u(t) depend on x_t. The estimate at time t is used for a decision that affects future observations, which is a typical feature of an adaptive system. It is then not possible to calculate $Q(x,e_t)$ for arbitrary x and (1.31) cannot be solved.

An automatic (self learning)classifier is an example of an adaptive way to estimate mean values of two stochastic variables.

Example 1.6 - Self learning classification (Unsupervised pattern recognition).

A classifier receives scalar valued signals e_n, which may belong to either of two a priori unknown classes A and B. The classifier must find a classification rule, i.e. a number c_n such that e is classified as A if $e_n \leq c_n$ and B otherwise. The number c_n can e.g. be determined as follows

$$c_n = (x_n^A + x_n^B)/2$$

where

$$x_n^A = \begin{cases} x_{n-1}^A + \gamma_n(e_n - x_{n-1}^A) & \text{if } e_n \text{ is classified as A} \\ x_{n-1}^A & \text{otherwise} \end{cases} \qquad (1.32)$$

x_n^B is defined analogously. Clearly, x_n^A is the mean value of the outcomes classified as A. This scheme is proposed e.g. by Tsypkin (1968).

Algorithm (1.32) can be considered as an RM scheme to solve

$$x^A = E\{e|e\in A(x)\} = E\{e|e \leq \tfrac{1}{2}(x^A+x^B)\} = E_e Q^A(x^A,x^B,e) = 0$$

$$x^B = E\{e|e\in B(x)\} = E\{e|e > \tfrac{1}{2}(x^A+x^B)\} = E_e Q^B(x^A,x^B,e) = 0$$

where

$$Q^A(x^A,x^B,e) = \begin{cases} e & \text{if } e \leq \frac{1}{2}(x^A+x^B) \\ & \\ x^A & \text{if } e > \frac{1}{2}(x^A+x^B) \end{cases} \quad \text{and } Q^B \text{ analogously}$$

Since the classification $e \in A$ or $e \in B$ depends on x, the right hand side depends on x. For the simple algorithm (1.12), where the mean value of a stochastic variable is estimated, this is not the case. As in Example 1.5, the adaptive nature of the algorithm makes Q a more complex function of x.

□

To summarize, we have in this chapter seen examples of a variety of algorithms in control theory, that have a structure given by (1.1). In this report the convergence properties of such algorithms are treated. The results will also imply new convergence criteria for stochastic approximation algorithms. The results are summarized in Chapter 7.

2. CLASSIFICATION OF CONVERGENCE CRITERIA.

The convergence properties of some of the algorithms discussed in Chapter 1 have been treated by many authors. However, there does not seem to exist a unified approach.

Convergence of stochastic approximation algorithms is treated in a number of papers. A selection of these follows below. However, most of the given results cannot be applied to cases with correlated observations.

The potential function method is extensively treated in Aizerman, Braverman and Rozonoer (1970).

Convergence of the real time least squares method (Example 1.3) follows from the consistency of least squares estimation, see e.g. Åström and Eykhoff (1971). Convergence for the more complex cases of general recursive identification schemes (Example 1.4) and self-tuning regulators (Example 1.5) does not seem to be treated in the literature.

The objective of this chapter is to illustrate what types of conditions that are usually imposed to assure convergence with probability one (w.p.1). The discussion here is basically confined to the Robbins-Monro scheme. This procedure and variants thereof are extensively treated in the literature. The convergence criteria can be classified into three classes. This conclusion is supported by several examples. The chapter can therefore also be read as a short and incomplete survey of previous results. As such it is not essential for the rest of this report, but it serves as a background to the new results presented here.

2.1. Problem Formulation.

In Example 1.1 we have introduced the RM scheme as a recursive method to solve

$$E_e Q(x,e) = f(x) = 0 \tag{2.1}$$

by

$$x_n = x_{n-1} + \gamma_n Q(x_{n-1}, e_n) \tag{2.2}$$

Many papers and books deal with convergence w.p.1 of (2.2) to the desired point x^*, see e.g. Blum (1954ab), Dvoretzky (1956), Burkholder (1956), Derman and Sacks (1959), Gladysjev (1965), Albert and Gardner (1967), Wazan (1969) and Aizerman et al (1970). Various criteria have been suggested.

It is possible to classify the conditions into three main classes.

Eq. (2.2) can be written

$$x_n = x_{n-1} + \gamma_n [f(x_{n-1}) + (Q(x_{n-1}, e_n) - f(x_{n-1}))]$$

The term $Q(x_{n-1}, e_n) - f(x_{n-1})$ can heuristically be regarded as noise added to the deterministic algorithm

$$x_n = x_{n-1} + \gamma_n f(x_{n-1}) \tag{2.3}$$

Since the noise term is not likely to improve the convergence of (2.2), it is reasonable to require first of all that the algorithm (2.3) should converge to x^*. Convergence of (2.3) which is an ordinary difference equation, depends on the "step size" γ_n, as well as on the function f(x). A

necessary property of f(x) is that (2.3) shall converge for sufficiently small γ_n. The corresponding maximal magnitude of γ_n may depend on the initial value x_0. Conditions that assure this property will be referred to as stability conditions.

The variable γ_n tends to zero as n tends to infinity. Now, in the beginning of the recursion (2.3) γ_n can still be too large and $|x_n|$ may increase rapidly. If then also $|f(x_n)|$ increases, the sequence $\{\gamma_n\}$ may not decrease sufficiently fast to yield small corrections $\gamma_n f(x_n)$ and convergence of (2.3). Therefore conditions that assure boundedness of x_n must be introduced. These will be called boundedness conditions. Boundedness and stability conditions together imply convergence of (2.3) to the desired point x*, if

$$\gamma_n \to 0 \quad \text{as} \quad n \to \infty \quad \text{and} \quad \sum_1^\infty \gamma_n = \infty \tag{2.4}$$

The second condition is necessary, since otherwise x_n could move only a given distance from the initial value.

Furthermore, certain conditions on the noise $Q(x,e_n) - f(x)$, and on the sequence γ_n must be introduced to assure that the influence of the randomness in algorithm (2.2) is sufficiently small. Such conditions will be referred to as noise conditions.

2.2. Stability Conditions.

In the original paper by Robbins and Monro (1951), only mean square convergence of (2.2) was considered. Convergence w.p.1 was first shown by Blum (1954a). He considered the scalar case and introduced the following conditions on f(x):

$$\begin{aligned} f(x) &> 0 \quad \text{for} \quad x < x^* \\ f(x) &< 0 \quad \text{for} \quad x > x^* \end{aligned} \tag{2.5}$$

where x^* is the solution of $f(x) = 0$. Blum also assumed that

$$\inf_{\delta_1 < |x-x^*| < \delta_2} |f(x)| > 0 \quad \text{for all } \delta_2 > \delta_1 > 0$$

In this simple case the stability of the iterative methods to find the solution of $f(x) = 0$ is, of course, determined by the way the curve f(x) intersects the x-axis.

Albert and Gardner (1967) use as criterion that $\frac{d}{dx} f(x)$ be negative for all x.

Dvoretzky's (1956) criterion is valid not only for the RM case, but also for the more general algorithm (1.15). The criterion is based on a contraction mapping property and implies (2.5) for algorithm (2.2) in the case of scalar valued x. For vector valued x and Q Dvoretzky suggests the following criterion:

$$|x_n - x^* - \gamma_n f(x_n)| < F_n |x_n - x^*|$$

where

$$\prod_{n=1}^{\infty} F_n = 0$$

Blum (1954b) has suggested to use functions with Lyapunov properties. The existence of a function V(x) with properties

$$V(x) \geq 0$$

$$\inf_{\varepsilon < |x-x^*|} |V(x) - V(x^*)| > 0 \qquad \forall\ \varepsilon > 0$$

$$\sup_{\varepsilon < |x-x^*|} V'(x)\, f(x) < 0 \qquad \forall\ \varepsilon > 0$$ [1)]

is assumed. These properties obviously guarantee that the ordinary differential equation

$$\frac{d}{dt} x = f(x)$$

is stable.

Braverman and Rozonoer (1969) and Aizerman et al (1970) have given similar and slightly more general stability criteria.

In e.g. Braverman and Rosenoer (1969) and Krasulina (1972) special attention is paid to the case when the equation

$$f(x) = 0$$

has several roots.

Gladysjev (1965) has introduced a less general criterion,

---

1) The derivative V' is regarded as a row vector.

which is more easily checked than the one above. It has the form

$$\inf_{\varepsilon<|x-x^*|<1/\varepsilon} (x-x^*)^T f(x) > 0 \qquad \forall\ \varepsilon > 0 \qquad (2.6)$$

Obviously it is a special case of Blum's approach, with $V(x) = \frac{1}{2}|x - x^*|^2$. The criterion (2.6) is also used by Tsypkin (1971).

When the RM scheme is applied to minimize a function as in (1.13) the function $P(x) = E_v J(x,v)$ can be used as a Lyapunov function for the problem. This approach has been persued by Litvakov (1968) and Devyaterikov et al (1969).

2.3. Boundedness Conditions.

Consider first an example that shows how divergence in the algorithm (2.3) can occur, even if the stability conditions are satisfied.

Example 2.1. Let $f(x) = - x^3$. Then (2.3) gives

$$x_n = x_{n-1} - \gamma_n x_{n-1}^3 \qquad (2.7)$$

Clearly f(x) satisfies any of the cited stability conditions. However, with $\gamma_n = 1/n$ and $x_0 = 2$ the sequence of estimates is

$$x_1 = -6 \qquad x_2 = 102 \qquad x_3 = -353634 \quad \ldots$$

and $|x_n|$ tends to infinity as n tends to infinity. For this initial value, convergence would be obtained if

$\gamma_n \leq 1/2$, $n = 1,\ldots$ In general γ_n must be less than $2/x_0^2$ to assure convergence. There consequently exists no sequence $\{\gamma_n\}$ that yields convergence for (2.7) irrespectively of the initial value. □

Some restrictions to rule out cases like Example 2.1 must be introduced. For (2.2) there is a non zero probability that x_n may belong to a given area arbitrarily far away from x^*. Therefore such restrictions cannot be obtained by conditions on the sequence $\{\gamma_n\}$ related to the initial value x_0, but conditions on $f(x)$ must be introduced.

Blum (1954a) uses

$$|f(x)| \leq a + b|x|$$

for the scalar case. Albert and Gardner (1967) consider truncated algorithms of the type

$$\begin{aligned} x_n' &= x_{n-1} + \gamma_n Q(x_{n-1}, e_n) \\ x_n &= x_n' && \text{if } B < x_n' < A \\ x_n &= B && \text{if } x_n' \leq B \\ x_n &= A && \text{if } x_n' \geq A \end{aligned} \qquad (2.8)$$

For the Lyapunov function approach, additional assumptions on the Lyapunov function $V(x)$ must be introduced. Blum (1954b) assumes that

$$V_a(x) = E_e W_a(x,\theta,e) \leq K \quad \text{for all } a,x$$

where

$$W_a(x,\theta,e) = Q(x,e)^T V''\big(x+\theta a Q(x,e)\big)Q(x,e)$$

The variable θ is a number between 0 and 1, that may depend on e. $V''(x)$ is the matrix of second order derivatives (the Jacobian) of $V(x)$, the function introduced in Section 2.2.

In Aizerman et al (1970) a more general condition is assumed:

$$E_e\left[\max_{0\leq\theta\leq 1} W_{\gamma_n}(x,\theta,e)\right] \leq C_1 V(x) - C_2 V'(x) f(x) + C_3 \qquad (2.9)$$

Devyaterikov et al (1969) use a similar condition.

Gladysjev (1965), Litvakov (1968) and Tsypkin (1971) use a criterion that is similar to the one due to Blum in the scalar valued case:

$$E_e[Q(x,e)^T Q(x,e)] \leq C_1(1+x^T x)$$

This is a special case of (2.9), with $V(x) = |x - x^*|^2$.

All these boundedness criteria state, in various forms, that $Q(x,e)$ must not increase faster than $|x|$ as $|x|$ tends to infinity.

2.4. Noise Conditions.

In statistical literature, like in Blum (1954ab), Burkholder (1956), the problem (2.1) is often formulated as follows. Consider a family of stochastic variables $Y(x)$ having distribution $H(\cdot|x)$ and conditional expectations $f(x) = EY(x)$. The function $f(x)$ is called the regression function with respect to the family $Y(x)$. The equation $f(x) = 0$ is then solved recursively

$$x_{n+1} = x_n - \gamma_n y_n$$

where y_n is a random variable whose conditional distribution has the property

$$H(\cdot|x_n,x_{n-1},\ldots,x_1;y_{n-1},\ldots,y_1) = H(\cdot|x_n)$$

For the formulation (2.2) this means that the distribution of $Q(x_n,e_{n+1})$ given $Q(x_k,e_{k+1}),k = n-1,\ldots,0$ and $x_k,k = n,\ldots,0$ may depend only on x_n. In particular, the distribution of e_{n+1} must not depend on $e_n,\ldots,e_0$. Consequently, the sequence $\{e_i\}$ must consist of independent random variables. When the Lyapunov function approach is used, as in e.g. Blum (1954b), Aizerman et al (1970), Litvakov (1968), this independence assumption is critical to evaluate

$$E_e\left\{V'(x_n)Q(x_n,e_{n+1})|V(x_1),\ldots,V(x_n)\right\}$$

Comer (1964) points out that the assumption on independence is not very realistic in process control applications. He considers the case

$$Q(x_n,e_{n+1}) = f(x_n) + e_{n+1}$$

where the variables $\{e_n\}$ fulfil a condition weaker than independence. However, he does not show convergence with probability one for such a process. Wasan (1969) uses Comer's result for a convergence theorem for dependent noise. This theorem, however, does not seem to be correct.

A more general stochastic approximation algorithm is considered by Albert and Gardner (1967). They allow time varying regression functions $f_n(x)$ and

$$\gamma_n = \gamma_n(x_1,\ldots,x_n)$$

They give a sufficient condition for convergence of such an algorithm also for dependent noise. This result is not applicable in the present case, since it requires that

$$\sum_{n=1}^{\infty} \inf_x f_n'(x) \quad \inf_x \gamma_n = \infty$$

$$\sum_{1}^{\infty} \sup_x \gamma_n < \infty$$

which obviously cannot be fulfilled for time invariant regression functions.

The conclusion is that the assumption on independent observations has been critical to prove convergence w.p.1 for (2.2). In this report results valid also for dependent observations are presented.

Together with this assumption, it is also usually assumed that

$$E_e[Q(x,e) - f(x)]^T[Q(x,e) - f(x)] \leq \sigma^2(x) \quad \text{for all } x \qquad (2.10)$$

and

$$\sum_{1}^{\infty} \gamma_n^2 < \infty$$

In some papers a condition (2.10) is included in conditions of type (2.9).

Krasulina (1969) has shown convergence for the Kiefer-Wolfowitz procedure in case

$$E_v \mid J(x,v) - P(x) \mid^p \leq C \qquad 1 < p < 2$$

and

$$\sum_{n=1}^{\infty} (\gamma_n / a_n)^p < \infty$$

where a_n is the search length as in Example 1.2. It is thus not assumed that the variance of the noise exists. Krasulina (1972) has also shown convergence in case

$$E_e \mid Q(x,e) - f(x) \mid^2 \geq \sigma^2 > 0 \quad \text{for all } x$$

$$E_e \mid Q(x,e) - f(x) \mid^{4+\delta} \leq C < \infty \quad \delta > 0$$

$$\gamma_n = n^{-1/2}$$

where the condition

$$\Sigma \gamma_n^2 < \infty$$

is not satisfied.

In this report convergence with probability one for a set of algorithms that includes (2.2) is considered. Some new conditions of the discussed types are derived. The noise conditions in the convergence theorem of this report are more general than those discussed above. Also, less restrictions will in general be imposed on the sequence $\{\gamma_n\}$.

3. A SEPARATION THEOREM.

Consider the basic algorithm (1.8)

$$x_n = x_{n-1} + \gamma_n Q_n(x_{n-1}, e_n) \tag{3.1}$$

where $\{\gamma_n\}$ is a sequence of scalar valued variables that may be random:

$$\gamma_n = \gamma_n(e_n, \ldots, e_0)$$

The RM scheme (2.2) is a special case of (3.1). In the previous chapter different criteria to assure convergence of the RM scheme were discussed. It was found to be convenient to classify the criteria into three classes: A) Noise Conditions, B) Boundedness Conditions, and C) Stability Conditions.

In this chapter conditions for the convergence w.p.1 of (3.1) are given. They are of general nature and can usually be applied in practice only after further investigations. Such analysis is given in the following chapters. The main idea of the theorem is that the question of convergence is separated into three conditions that can be studied as problems of their own.

In Section 3.1 a heuristic interpretation of the theorem is given. The separation theorem is stated in Section 3.2 and in Section 3.3 some examples are given.

3.1. A Heuristic Interpretation.

Suppose that we shall solve the equation (1.9)

$$E_e Q(x,e) = f(x) = 0$$

where measurements $Q(x,e_n)$ are available for any chosen x. An intuitive and simple-minded approach to this problem is as follows:

1) Fix an x^i.

2) Obtain a large number of samples $Q(x^i,e_k)$, $k=1,\ldots,n$.

3) Form an estimate of $f(x^i)$ as a weighted sum of these samples:

$$\hat{f}(x^i) = \sum_{k=1}^{n} \beta_k^n Q(x^i,e_k) = z_n(x^i)$$

If β_k^n can be expressed as

$$\beta_k^n = \gamma_k \prod_{i=k+1}^{n} (1-\gamma_i)$$

the sum can also be defined recursively as

$$z_k(x^i) = z_{k-1}(x^i) + \gamma_k[Q(x^i,e_k) - z_{k-1}(x^i)] \; ; z_0=0 \qquad (3.2)$$

4) Based on this estimate, determine a new x-value

$$x^{i+1} = x^i + \tilde{\gamma}_{i+1} z_i(x^i) \qquad (3.3)$$

5) Take this x^{i+1} as the new x and repeat from 2).

This scheme has two phases: an estimation phase (3.2) and a decision phase (3.3). Now, let the number of samples in each estimation phase tend to infinity. The resulting, hypothetic, algorithm would then converge if a) the estimation phases give consistent estimates: $z_n(x^i) \to f(x^i)$ w.p.1 as $n \to \infty$, and b) the decision phase, which is a deterministic difference equation with $z_n(x^i)$ replaced by $f(x^i)$, converges to the solution of $f(x) = 0$.

Now, the Robbins-Monro scheme (3.1) can in fact be seen as an ingenious mixing of the two phases. A decision is taken in each step, but as n tends to infinity, more effort is paid to the estimation, since γ_n tends to zero.

The separation theorem states that, in spite of the mixing of the phases, convergence of (3.1) still follows from consistency in the estimation phase and convergence in the decision phase. More precisely, the conditions

a) $z_n(x^0) \to f(x^0)$ w.p.1 as $n \to \infty$ for all x^0 where

$$z_k = z_{k-1}(x^0) + \gamma_k\left(Q(x^0,e_k) - z_{k-1}(x^0)\right)$$

b) $x_n \to x^*$ as $n \to \infty$ where $f(x^*) = 0$ and

$$x_k = x_{k-1} + \gamma_k f(x_{k-1})$$

are the main conditions for convergence of (3.1) to the desired value x^*.

In Theorem 3.1 the condition b) above is split up into a boundedness condition and a stability condition.

3.2. Separation.

Theorem 3.1. Consider the algorithm (3.1)

$$x_n = x_{n-1} + \gamma_n Q_n(x_{n-1}, e_n) \qquad x \in R^m$$

and let D be a compact subset of R^m. Let $Q_n(x,e)$ be Lipschitz continuous in an open region $D^0 \supset D$ for fixed e, with Lipschitz constant $K_n(e)$. Assume that the sequence of positive scalars (random variables) $\{\gamma_n\}$ satisfies

$$\gamma_n \to 0 \text{ as } n \to \infty \quad \text{and} \quad \sum \gamma_n = \infty \qquad \text{(w.p.1)}$$

Let $z_n(x^0)$ and r_n be recursively defined by

$$z_n(x^0) = z_{n-1}(x^0) + \gamma_n[Q_n(x^0, e_n) - z_{n-1}(x^0)] \qquad z_0 = 0 \qquad (3.4)$$

$$r_n = r_{n-1} + \gamma_n[K_n(e_n) - r_{n-1}] \qquad r_0 = 0 \qquad (3.5)$$

where x^0 is a fixed element in D.

Assume that

a) $z_n(x^0)$ converges w.p.1 for any $x^0 \in D$ and define the function

$$f(x) = \lim_{n\to\infty} z_n(x)$$

r_n converges w.p.1.

b) $x_n \in D$ infinitely often (i.o.) w.p.1. (This means that w.p.1 there exists a subsequence $\{x_{n_k}\}$ that belongs to the region.)

c) The ordinary differential equation

$$\frac{d}{dt} x = f(x) \qquad (3.6)$$ [1)]

has a stationary point x^* which is an asymptotically stable solution with domain of attraction $D_1 \supset D$. (That is, all solutions with initial values in D_1 tend to x^* as t tends to infinity.)

Then $x_n \to x^*$ w.p.1 as $n \to \infty$. □

The proof is given in Appendix A.

Notice that the only condition in Theorem 3.1 for the separation to hold is that $Q_n(x,e)$ is Lipschitz continuous for fixed e. This is quite a weak condition. In particular, the separation is obtained without any conditions on the noise e.

Condition a) in Theorem 3.1 will be called the noise condition. It concerns the convergence w.p.1 of the two algorithms (3.4) and (3.5). These have the same structure as the simple example (1.12) of the RM scheme where the mean value of a stochastic variable is estimated. The convergence of these algorithms is investigated in Chapter 4. There $Q_n(x^0,e_n)$ is considered as a random variable for which $E_e Q_n(x^0,e_n)$ exists. If $z_n(x^0) \to f(x^0)$ w.p.1, it then follows under weak assumptions, that

$$f(x^0) = \lim_{n\to\infty} E_e Q_n(x^0,e)$$

This connects the two different definitions of f(x), the

---

1) Existence and uniqueness of solutions to (3.6) follows from the Lipschitz continuity of Q and from condition a).

one in (2.1) and the one in Theorem 3.1.

Condition b) will be called the boundedness condition. In its given form it is clearly necessary for convergence, but may be difficult to apply directly. Conditions that imply the boundedness condition are given in Chapter 5. Notice that if it does not hold, and D can be taken as any compact region, this means that x_n tends to infinity with non zero probability.

Condition c) clearly is the stability condition. It can be checked using Lyapunov stability theory, see e.g. Krasovskij (1963).

The techniques to find Lyapunov functions are not discussed here. In practical situations, sufficient insight into the stability properties of (3.6) might be obtained by numerical solution. The importance of the ODE (3.6) is, however, not restricted to the question of convergence of (3.1). In Chapter 6 it is shown that the trajectories of (3.6) are related to the asymptotic behaviour of (3.1). Therefore, when investigating the properties of (3.1), numerical solution of (3.6) can be a valuable complement to simulation of (3.1).

Remark. Basically, the theorem does not deal with convergence in a stochastic setting. A fixed realization for which a) holds on a dense subset of D and for which b) and the conditions on $\{\gamma_n\}$ hold is considered throughout the proof. Convergence of (3.2) is shown under these conditions. The theorem thus also can be applied for each realization, and the stochastic convergence concept "w.p.1" can be omitted. In particular, this means that the limit function $f(x)$, as well as the convergence point x^* might be random variables: $f(x) = f(x,\omega)$. Then in condition c) the ODE $\dot{x} = f(x,\omega)$ should be asymptotically stable with

stationary point $x^*(\omega)$ for almost every ω, i.e. c) must hold w.p.1.

Several extensions of the theorem are possible. It holds also for the algorithm

$$x_n = x_{n-1} + \gamma_n Q_n(x_{n-1},\dots,x_0;e_n)$$

if the dependence of old x_i on Q_n decreases sufficiently fast. In Ljung-Wittenmark (1974) such an extension is proved for a special class of such algorithms.

In some applications, like real time least squares (Example 1.3), Q_n is multiplied by a matrix that depends on old values of the noise:

$$x_n = x_{n-1} + \gamma_n S_n(e_1,\dots,e_n)Q_n(x_{n-1},e_n) \qquad (3.7)$$

This case is considered in the following corollary.

Corollary 1. Consider algorithm (3.7). Suppose the conditions of Theorem 3.1 hold with Q_n replaced by S_nQ_n. (The limit function f(x) may be a random variable.) Then $x_n \to x^*$ w.p.1 as $n \to \infty$.

In case there are several stationary points of (3.6) it may be easier to use the following variant of the stability condition.

Corollary 2. Consider algorithm (3.1). Suppose that conditions a) and b) of Theorem 3.1 hold. Assume that there exists a twice differentiable function $V(x)$, $x \in D_1$, where D_1 is an open set that contains D, such that

$V'(x)f(x) \leq 0$, $\forall\, x \in D_1$, and $V'(x)f(x) = 0 \Leftrightarrow x \in D_c$. Assume further that $V(\bar{x}) = \text{const}$ and $V'(\bar{x})f(\bar{x}) < 0$ for $\bar{x}$ belonging to the boundary of D. (This assumption can be omitted if D_1 can be taken as R^m.) Then $x_n \to D_c$ w.p.1 as $n \to \infty$ [1]. Furthermore, if D_c consists of isolated points, then $x_n \to x^*(\omega)$ w.p.1 as $n \to \infty$, where $x^*(\omega) \in D_c$.

The proof of this corollary is indicated in Appendix A.

3.3. Examples of Application of the Theorem.

Theorem 3.1 is applicable to a variety of algorithms. Some examples of cases that can be treated are given below. In general no convergence results can be obtained at this stage, since the noise and the boundedness conditions remain to be analysed. Hence it is demonstrated only how the separation can be achieved and how the conditions can be reformulated. Some of the examples are continued in Chapter 7.

Example 3.1 - Minimization of a function when derivatives are available.

This common case can be formulated: Minimize $P(x) = E_v J(x,v)$, $x \in R^m$, with respect to x. The derivative $\frac{\partial}{\partial x} J(x,v)$ is available, and the Robbins-Monro scheme can be applied to solve

$$f(x) = -E_v \frac{\partial}{\partial x} J(x,v) = 0$$

---

1) By this is meant that $\inf_{\tilde{x} \in D_c} |x_n - \tilde{x}| \to 0$ as $n \to \infty$ w.p.1.

for x. Examples include various estimation problems as shown in Chapter 1. In this case the boundedness and stability conditions can be interpreted in a special way.

Suppose that P(x) is continuously differentiable. Under weak conditions it follows that $f^T(x) = - P'(x)$. It is now possible to choose P(x) as the function V(x) in Corollary 2. Then $V'(x)f(x) = P'(x)f(x) = - f^T(x)f(x) \leq 0$ $\forall\ x \in R^m$. Consequently, if x_n belongs to some bounded region i.o. it will tend to a solution of $f(x) = 0$. If it is not bounded i.o. it tends to infinity. To summarize: If the noise condition is satisfied, x_n either tends to infinity or to a stationary point of P(x) w.p.1 as $n \to \infty$.

Example 3.2 - The Kiefer-Wolfowitz procedure.

If the minimum of P(x) in the previous example is to be found and no derivatives are available, the Kiefer-Wolfowitz procedure can be applied, as described in Example 1.2. The sequence $\{a_n\}$ tends to zero as n tends to infinity. Consider (1.14). To apply the theorem, take

$$Q_n(x_{n-1},e_n) = \left[\left(J(x_{n-1}-a_nu_1,v_{j_1}) - J(x_{n-1},v_i)\right)/a_n,\ldots \right]^T$$

Suppose that P(x) is twice continuously differentiable with respect to x. Then

$$Q_n(x_{n-1},e_n) = -P'(x_{n-1}) + a_nP''(\xi_n) + q_n(x_{n-1},e_n)/a_n$$

where

$$q_n(x_{n-1},e_n) = \Big[J(x_{n-1}-a_nu_1,v_{j_1}) - E_{v_{j_1}}J(x_{n-1}-a_nu_1,v_{j_1}) -$$

$$- J(x_{n-1},v_i) - E_{v_i}J(x_{n-1},v_i),\ldots\Big]^T$$

The condition on Lipschitz continuity of Q_n gives a condition

$$|q_n(x_1,e) - q_n(x_2,e)| \leq K_n(e)|x_1 - x_2| \tag{3.8}$$

The noise condition takes the form

$$z_n(x^0) = z_{n-1}(x^0) + \gamma_n\left(Q_n(x^0,e_n) - z_{n-1}(x^0)\right) \Rightarrow \tag{3.9}$$

$$\Rightarrow z_n(x^0) \to -P'(x^0) \text{ w.p.1 as } n \to \infty.$$

$$r_n = r_{n-1} + \gamma_n\left(K_n(e_n) - r_{n-1}\right) \Rightarrow r_n \to r_\infty \text{ w.p.1 as } n \to \infty \tag{3.10}$$

Now, $Q_n(x^0,e)$ contains a deterministic component $-P'(x^0) + a_nP''(\xi_n)$ where $\xi_n \to x^0$ as $n \to \infty$. This implies that $P''(\xi_n)$ is bounded and so $a_nP''(\xi_n) \to 0$ as $n \to \infty$. It is easy to show that (3.9) is equivalent to

$$\tilde{z}_n(x^0) = \tilde{z}_{n-1}(x^0) + \gamma_n\left(q_n(x^0,e_n)/a_n - \tilde{z}_{n-1}(x^0)\right)$$

$$\Rightarrow \tilde{z}_n(x^0) \to 0 \text{ w.p.1 as } n \to \infty \tag{3.11}$$

Consequently, if (3.8), (3.10) and (3.11) are satisfied, then x_n tends w.p.1 either to infinity or to a stationary point of $P(x)$.

Example 3.3 - Minimization of a function using noise corrupted measurements.

A common special case of Example 3.3 is that

$$J(x,e_n) = P(x) + e_n$$

where the distribution of e_n does not depend on x. If the KW procedure is applied to this case, the noise conditions are simpler than in Example 3.2. Obviously, since q_n does not depend on x,(3.8) and (3.10) are trivially satisfied. Hence x_n tends w.p.1 either to infinity or to a stationary point of P(x) if $z_n \to 0$ w.p.1 as $n \to \infty$, where

$$z_n = z_{n-1} + \gamma_n(e_n/a_n - z_{n-1}); \quad z_0 = 0$$

Example 3.4 - Real time least squares.

In Example 1.3 it was found that the solution of (1.23) can be written as

$$x_{n+1} = x_n + \frac{1}{n+1} R_n^{-1}\left[\xi_n y(n+1) - \xi_n \xi_n^T x_n\right] \tag{3.12}$$

where

$$R_n = \frac{1}{n+1} \sum_{k=0}^{n} \xi_k \xi_k^T \tag{3.13}$$

Assume that $\{\xi_k\}$ is a stationary stochastic process. Then

$$R_n \to R = E\xi_k \xi_k^T \quad \text{as } n \to \infty \text{ w.p.1}$$

It is assumed that R is nonsingular. In this case

$$Q_n(x,e) = R_n^{-1}\xi_n y(n+1) - \xi_n \xi_n^T x_n$$

This function clearly is Lipschitz continuous, since

$$|Q_n(x_1,e_n) - Q_n(x_2,e_n)| \leq |R_n^{-1}\xi_n\xi_n^T||x_1 - x_2| \leq$$

$$\leq \left[|R^{-1}\xi_n\xi_n^T| + |R_k^{-1} - R^{-1}||\xi_n\xi_n^T|\right]|x_1 - x_2| =$$

$$= K_n(e_n)|x_1 - x_2|$$

Therefore Corollary 1 of the separation theorem can be applied to (3.12). The noise condition is that

$$z_n(x^0) = z_{n-1}(x^0) + \frac{1}{n}\left[R_n^{-1}\left(\xi_n\xi_n^T x^0 + \xi_n y(n+1)\right) - z_{n-1}(x^0)\right] \quad (3.14)$$

converges w.p.1 to some limit $f(x^0)$.

The solution of (3.17) is

$$z_n(x^0) = \frac{1}{n}\sum_{k=1}^{n} R_k^{-1}\left[\xi_k\xi_k^T x^* - \xi_k\xi_k^T x^0 + \xi_k e(k+1)\right] =$$

$$= R^{-1}\left[\left\{\frac{1}{n}\sum_{1}^{n}\xi_k\xi_k^T\right\}\left\{x^* - x^0\right\} + \frac{1}{n}\sum_{1}^{n}\xi_k e(k+1)\right] +$$

$$+ \frac{1}{n}\sum_{1}^{n}(R_k^{-1}-R^{-1})\left[\xi_k\xi_k^T(x^*-x^0) + \xi_k e(k+1)\right]$$

It is not difficult to show that

$$z_n(x^0) \to R^{-1}E\xi_k\xi_k^T(x^*-x^0) = x^* - x^0 = f(x^0) \quad \text{w.p.1 as } n \to \infty$$

The second noise condition is that r_n converges w.p.1 where r_n is defined by (3.4). $K_n(e_n)$ is equal to $|R^{-1}\xi_n\xi_n^T| + |R_n^{-1} - R^{-1}||\xi_n\xi_n^T|$ and

$$r_n = \frac{1}{n} \sum_1^n |R^{-1}\xi_k\xi_k^T| + \frac{1}{n} \sum_1^n |R_k^{-1} - R^{-1}||\xi_k\xi_k^T|$$

which converges w.p.1 since $|R^{-1}\xi_k\xi_k^T|$ is a stationary process and since $|R^{-1} - R_k^{-1}||\xi_k\xi_k^T| \to 0$ w.p.1 as $n \to \infty$.

The ODE in the stability condition becomes

$$\frac{d}{dt} x = x^* - x$$

which clearly is globally asymptotically stable.

If the boundedness condition does not hold, x_n would tend to infinity with non zero probability. This is easily contradicted. Hence all conditions in Theorem 3.1 are satisfied and

$$x_n \to x^* \text{ w.p.1 as } n \to \infty$$

follows.

4. THE NOISE CONDITION.

The noise condition in Theorem 3.1 is that the two algorithms

$$z_n(x^0) = z_{n-1}(x^0) + \gamma_n[Q_n(x^0,e_n) - z_{n-1}(x^0)] \qquad z_0 = 0 \quad (4.1)$$

$$r_n = r_{n-1} + \gamma_n[K_n(e_n) - r_{n-1}] \qquad r_0 = 0 \quad (4.2)$$

converge w.p.1 (for fixed x^0).

In Theorem 3.1 no assumption about the statistics of the variables $Q_n(x^0,e_n)$ was made. In this chapter $Q_n(x^0,e_n)$ is considered as a vector valued random variable for which $E_{e_n}Q_n(x^0,e_n)$ exists for every n. Furthermore, the sequence $\{\gamma_n\}$ is here supposed to consist of deterministic scalars as in (1.5). If the original sequence $\{\gamma_n\}$ in (3.1) is stochastic, γ_n and Q_n in (3.1) can be redefined as

$$\gamma_n Q_n(x,e_n) = \bar{\gamma}_n\left(1 + \frac{\tilde{\gamma}_n}{\bar{\gamma}_n}\right)Q_n(x,e_n) = \bar{\gamma}_n Q_n^*(x,e_n^*)$$

where $\bar{\gamma}_n = E\gamma_n$ and $\tilde{\gamma}_n = \gamma_n - \bar{\gamma}_n$ which gives a deterministic sequence $\{\bar{\gamma}_n\}$.

For a specific application, it is possible to study the convergence of (4.1) using the special structure of the problem as in Example 3.4. In this chapter general conditions to assure convergence of (4.1) w.p.1 are discussed. The objective is to give results that cover practical situations, rather than to elaborate on the sharpness of the theorems.

In Section 4.1 it is shown that the convergence problem

for (4.1) can equivalently be formulated for a simpler algorithm. In Section 4.2 the common case when the sequence $\{\gamma_n\}$ asymptotically decreases as $1/n$ is considered.

Expressions for the absolute moments of z_n are derived in Section 4.3 and these are used in Section 4.4 to obtain more general convergence results.

4.1. An Equivalent Problem Formulation.

In this section it is shown that it is sufficient to analyze convergence of the algorithm

$$y_n = y_{n-1} + \gamma_n[f_n - y_{n-1}] \qquad y_0 = 0 \tag{4.3}$$

where $\{y_n\}$ are scalars, $\{\gamma_n\}$ a sequence of deterministic positive scalars and $\{f_n\}$ a sequence of scalar valued random variables with $Ef_n = 0$ all n.

Algorithm (4.1) is more complex than (4.3). It involves vector valued random variables with time varying mean values.

Lemma 4.1. Let $z_n(x^0)$ and y_n be defined by (4.1) and (4.3) respectively. Suppose that

$$E_e Q_n(x^0,e) \to f(x^0) \text{ as } n \to \infty \tag{4.4}$$

and that

$$0 \leq \gamma_n \leq 1 \qquad \sum_1^\infty \gamma_n = \infty$$

Then

$$z_n(x^0) \to f(x^0) \quad \text{w.p.1 as } n \to \infty$$

if and only if

$$y_n \to 0 \quad \text{w.p.1 as } n \to \infty \quad \text{for} \quad f_n = Q_n^{(i)}(x^0, e_n) - E_e Q_n^{(i)}(x^0, e_n)$$

$i = 1,\ldots,m$, where $Q^{(i)}$ denotes the i:th row of the column vector Q.

Proof. Since (4.1) is linear in z_n

$$z_n^{(i)}(x_0) = y_n + v_n$$

where v_n is defined by

$$v_n = v_{n-1} + \gamma_n[d_n - v_{n-1}] \qquad v_0 = 0$$

where

$$d_n = E_e Q_n^{(i)}(x^0, e_n)$$

It clearly is sufficient to show that (4.4) implies that $v_n \to f^{(i)}(x^0)$ as n tends to infinity. This is done as follows.

Eq. (4.4) means that $|d_n - f^{(i)}(x^0)| < \varepsilon$ for $n > N_0(\varepsilon)$. Then

$$v_{N_0+m} = \prod_{j=N_0+1}^{N_0+m} (1-\gamma_j) v_{N_0} + \sum_{j=N_0+1}^{N_0+m} \beta_j^{N_0+m} d_j$$

where

$$\beta_j^N = \gamma_j \prod_{i=j+1}^{N} (1-\gamma_i) \quad \text{if } j < N \quad \text{and} \quad \beta_N^N = \gamma_N \tag{4.5}$$

Now $v_{N_0} = 1$ and $d_j = 1$ gives $v_j = 1$; $j \geq N_0$, which means that

$$\prod_{j=N_0+1}^{N_0+m} (1-\gamma_j) + \sum_{j=N_0+1}^{N_0+m} \beta_j^{N_0+m} = 1$$

and consequently

$$\left|v_{N_0+m} - f^{(i)}(x^0)\right| \leq \left|\prod_{j=N_0+1}^{N_0+m} (1-\gamma_j)\right| \left|d_{N_0} - f^{(i)}(x^0)\right| +$$

$$+ \left|\sum_{j=N_0+1}^{N_0+m} \beta_j^{N_0+m}\right| \varepsilon$$

Now

$$\prod_{j=N_0+1}^{N_0+m} (1-\gamma_j) \to 0 \quad \text{as } m \to \infty \quad \text{since} \quad \sum_{1}^{\infty} \gamma_j = \infty$$

Hence

$$|v_n - f^{(i)}(x^0)| < 2\varepsilon$$

for sufficiently large n and so

$$\lim_{n\to\infty} v_n = f^{(i)}(x^0)$$

□

Remark. Assumption (4.4) can be replaced by the weaker condition $v_n \to f^i(x^0)$ as $n \to \infty$, where v_n is defined as in the proof of the lemma.

As a consequence of the lemma it is sufficient to study convergence of the simple algorithm (4.3).

4.2. The Case $\gamma_n \sim A/n$ For Large n.

It has already been remarked that Eqs. (4.1) and (4.3) correspond to estimation of mean values. As shown in Example 1.1 a suitable choice of $\{\gamma_n\}$ then is $\gamma_n = 1/n$. Also in the case (3.1) $\gamma_n = B/n$ for large n (but not for small n, cf. Chapter 6) seems to be a good choice. The convergence properties for the case $\gamma_n = b_n/n$ where $b_n \to B$ as $n \to \infty$, therefore deserve special interest.

In algorithm (4.3) $\gamma_n Q_n(x_n, e_{n+1}) = b_n/n \; Q_n(x_n, e_{n+1})$ can be redefined as

$$1/n \; Q_n^*(x_n, e_{n+1}) \text{ where } Q_n^* = b_n Q_n$$

Apply the separation theorem and (4.1) becomes

$$z_{n+1}(x^0) = z_n(x^0) + \frac{1}{n}\left(Q_n^*(x^0, e_{n+1}) - z_n(x^0)\right)$$

Consequently the convergence analysis for (4.3) with $\gamma_n = 1/n$ covers all sequences that asymptotically behave like B/n.

With $\gamma_n = 1/n$ in (4.3)

$$y_n = \frac{1}{n} \sum_{k=1}^{n} f_k \tag{4.6}$$

and ergodic theory can be applied to obtain convergence of y_n.

According to Cramer-Leadbetter (1967) (4.6) converges to zero w.p.1 if

$$Ef_k f_s \leq \frac{k^p + s^p}{1 + |k - s|^q} \qquad 0 \leq 2p < q < 1 \tag{4.7}$$

The condition (4.7) imposes a restriction on the dependence of the sequence $\{f_k\}$, that is quite weak.

4.3. Asymptotic Moments of y_n.

We will now consider a general sequence $\{\gamma_n\}$ and general distributions of f_n. As a convenient regularity condition on f_n will be chosen that the absolute moments up to a certain order p exist. The corresponding moments of y_n then also exist. In this section upper bounds for these moments are calculated.

To facilitate the calculation, certain conditions on the sequence $\{\gamma_n\}$ are introduced. They are chosen as

$$\begin{aligned} &\text{a)} \quad 0 \leq \gamma_n \leq 1 \\ &\text{b)} \quad \sum_{n=1}^{\infty} \gamma_n = \infty \\ &\text{c)} \quad \gamma_{n+1} \geq \gamma_n(1-\gamma_{n+1}) \\ &\text{d)} \quad \{\gamma_n\} \text{ is decreasing as a function of } n \end{aligned} \tag{4.8}$$

Notice that it is always possible to redefine γ_n and Q_n so that the sequence γ_n can be scaled arbitrarily. Condition (4.8a) therefore is not restricting. Condition (4.8c) states that in the sum y_n each observation f_k has a weight no less than the previous one. As will be discussed in Chapter 6, this is the interesting case. The conditions (4.8) are satisfied for the common choice

$$\gamma_n = A/n^{\alpha} \quad 0 < \alpha < 1 \quad \text{all } A; \quad \alpha = 1 \quad A \geq 1 \tag{4.9}$$

Some conditions on the dependence between the variables f_k also must be imposed. Conditions involving only second moments as (4.7) are not sufficient in the general case. Since most stochastic processes occuring in control theory have been generated as white noise through some linear (time varying) filter, we adopt the following condition:

Let f_n be obtained from white noise as

$$f_n = \sum_{k=0}^{\infty} h_{k,n} e_{n-k} \quad \text{where} \quad |h_{k,n}| < \alpha_n \lambda^k \quad \lambda < 1 \tag{4.10}$$

and $(e_k, k = 0, \pm 1, \ldots)$ is a sequence of independent random variables with zero mean values.

Remark. If $\{f_n\}$ is a stationary, regular stochastic process, it can always be represented as filtered white noise as in (4.10), Doob (1953). The conditions on $h_{k,n}$, however, do not follow automatically from stationarity only.

It is now possible to prove the following lemma.

Lemma 4.2. Consider the algorithm (4.3)

$$y_n = y_{n-1} + \gamma_n(f_n - y_{n-1}) \qquad y_0 = 0$$

Assume that the sequence $\{\gamma_n\}$ satisfies (4.8). Assume further that f_n satisfies (4.10) and that $\{\alpha_n\}$ is a non decreasing sequence of numbers and

$$E|e_k|^p < C$$

which implies

$$E|f_n|^p < C' \cdot \alpha_n^p$$

where p is an even integer. Then

$$E|y_n|^r \leq K_r(\alpha_n)^r(\gamma_n)^{r/2} \qquad 1 < r \leq p \tag{4.11}$$

The proof is given in Appendix B. □

The lemma extends the results given by Chung (1954). There, (4.11) is obtained in the special case

$$\alpha_n = 1 \qquad \gamma_n = n^{-\alpha} \qquad 1/2 < \alpha \leq 1 \qquad \text{and } f_n \text{ indep. variables.}$$

However, Chung considers a more general regression function.

The estimates on the moments can be used to obtain convergence criteria. This is treated in the next section.

4.4. Convergence With Probability One.

Using the estimates of the absolute moments of y_n it is easy to establish convergence of y_n to zero w.p.1:

Theorem 4.1. Consider the algorithm (4.3) with the same assumptions as in Lemma 4.2. Suppose

$$\sum_{n=1}^{\infty} \gamma_n^{p/2} \alpha_n^p < \infty \quad \text{where } p \text{ is defined in Lemma 4.2.}$$

Then $y_n \to 0$ as $n \to \infty$ w.p.1.

Proof. From Chebysjev's inequality and Lemma 4.2

$$P(|y_n| > \varepsilon) \leq \frac{E|y_n|^p}{\varepsilon^p} \leq \frac{k_p \gamma_n^{p/2} \alpha_n^p}{\varepsilon^p}$$

and

$$\sum_{n=1}^{\infty} P(|y_n| > \varepsilon) \leq \frac{k_p}{\varepsilon^p} \sum_{n=1}^{\infty} \gamma_n^{p/2} \alpha_n^p < \infty$$

The Borel Cantelli lemma now assures

$y_n \to 0$ as $n \to \infty$ w.p.1. □

Theorem 4.1 shows that it is possible to trade off conditions on the sequence $(\gamma_n, n = 1,\ldots)$ against conditions on the moments of f_n.

Thus the usually given criterion

$$\sum_{n=1}^{\infty} \gamma_n^2 < \infty \quad \text{for the Robbins-Monro case} \tag{4.12}$$

is in fact not necessary to achieve convergence w.p.1. It can be violated if more regularity of f_n is required. For example, if all moments of f_n exist and are uniformly bounded, it is sufficient that

$$\sum_{n=1}^{\infty} \gamma_n^p < \infty \quad \text{for some sufficiently large } p$$

which is satisfied e.g. for $\gamma_n = n^{-\alpha}$, $0 < \alpha \leq 1$.

However, (4.12) can be violated only if higher moments of f_n exist. This is seen from the following example.

Example 4.1. Let f_n, $n = 1,\ldots,$ be a sequence of independent random variables where f_n has the distribution

$$f_n = \begin{cases} 1/\gamma_n & \text{with probability } (\gamma_n)^r \\ 0 & \text{with probability } 1 - (\gamma_n)^r \end{cases}$$

Then $P(|\gamma_n f_n| \geq 1) = (\gamma_n)^r$.

The moments $E|f_n|^s$ are uniformly bounded only for $s \leq r$.

Assume that

$$\sum_{n=1}^{\infty} \gamma_n^r = \infty \qquad \text{and} \qquad \sum_{n=1}^{\infty} \gamma_n^{r+\varepsilon} < \infty \qquad \text{for some } \varepsilon > 0$$

Then

$$\sum_{n=1}^{\infty} P(|\gamma_n f_n| \geq 1) = \sum_{n=1}^{\infty} \gamma_n^r = \infty$$

and since the variables (f_n) are independent

$$|\gamma_n f_n| \geq 1 \qquad \text{i.o. w.p.1}$$

from the Borel-Cantelli lemma. With algorithm (4.3)

$$y_n = (1-\gamma_n)y_{n-1} + \gamma_n f_n$$

y_n will consequently w.p.1 not converge to any limit. To be able to apply the theorem $E|f_n|^{2(r+\varepsilon)}$ would have to be uniformly bounded. Thus the moment conditions on f_n cannot be dispensed with.

□

A common case is when the variables f_n are normally distributed. Then also y_n has normal distribution. The probability that $|\gamma_n| > \varepsilon$ can now be determined directly which gives the following theorem:

Theorem 4.2. Consider algorithm (4.3)

$$y_n = y_{n-1} + \gamma_n(f_n - y_{n-1})$$

Let the variables f_n have normal distribution with zero mean value. Assume that

$$|Ef_n f_{n+r}| < \lambda^r \alpha_n \alpha_{n+r} \quad \text{all } n \text{ and some } \lambda < 1,$$

where $\{\alpha_n\}$ is a non decreasing sequence.

Let the sequence $\{\gamma_n\}$ satisfy (4.8) and suppose that

$$\sum_{n=1}^{\infty} e^{-\varepsilon/\gamma_n \alpha_n^2} < \infty \quad \text{for all } \varepsilon > 0$$

Then $y_n \to 0$ as $n \to \infty$ w.p.1.

Proof. As in Lemma 4.2 it follows that

$$Ey_n^2 < K_2 \gamma_n \alpha_n^2$$

Now

$$P(|y_n| > \varepsilon) = \frac{1}{\sqrt{2\pi Ey_n^2}} \int_{|x|>\varepsilon} e^{-x^2/2Ey_n^2} dx \leq \frac{1}{\varepsilon} Ce^{-\varepsilon^2/K_2\gamma_n\alpha_n^2}$$

Application of the Borel-Cantelli lemma completes the proof.

□

So far in this section the cases with dependent and independent random variables have been dealt with simultaneously. If we confine ourselves to the case of independent variables a refinement of Theorem 4.1 can be obtained:

Theorem 4.3. Consider the algorithm (4.3)

$$y_n = y_{n-1} + \gamma_n(f_n - y_{n-1})$$

where $\{f_n\}$ is a sequence of independent random variables. Suppose $\{\gamma_n\}$ satisfies (4.8). Let $E|f_n|^p \leq \alpha_n^p$ for some real $p > 1$, where $\{\alpha_n\}$ is a non decreasing sequence, and suppose that

$$\sum_{n=1}^{\infty} \gamma_n^{p'} \alpha_n^p < \infty \quad \text{where } p' = \min(p, 1+p/2)$$

Then

$y_n \to 0$ as $n \to \infty$ w.p.1.

The proof is given in Appendix C. □

Remark. The condition (4.8cd) is not used in case $1 < p \leq 2$.

Krasulina's (1969) result corresponds to the case $1 < p < 2$. The usual condition

$$Ef_n^2 < C \quad \text{and} \quad \Sigma\gamma_n^2 < \infty$$

is obtained as a special case of the theorem (p=2).

Summing up, Theorems 4.1 - 4.3 give weaker and more general conditions for convergence of (4.1) than usually reported. Dependent random variables can also be treated. These results can now be applied to the general algorithm (3.1) via Theorem 3.1. In Chapter 7 some applications of this kind are given.

5. THE BOUNDEDNESS CONDITION.

In the separation theorem in Chapter 3 it is assumed known that the estimates $\{x_n\}$, obtained from

$$x_n = x_{n-1} + \gamma_n Q_n(x_{n-1}, e_n) \tag{5.1}$$

with probability one, infinitely often are inside a certain bounded region D. Criteria which assure such behaviour will be discussed in this chapter. Here $\{\gamma_n\}$ is assumed to be a sequence of deterministic scalars.

In Section 2.3 some suggested criteria were reviewed. Condition (2.9) restricts the choice of Lyapunov functions for the problem. A similar condition is discussed in Section 5.1. It is shown that under weak conditions on the noise and on the sequence $\{\gamma_n\}$ convergence in probability to the desired value x^* can be established. Now, convergence in probability implies convergence w.p.1 along a subsequence. Consequently, x_n will w.p.1 belong to any open region containing x^* i.o. This gives the desired boundedness property for any region D.

From a practical point of view the question of boundedness of the estimates may seem uninteresting. In many cases the desired convergence point x^* of (5.1) is a priori known to belong to some bounded region. It is therefore natural to construct the estimates x_n such that they belong to this area. A straightforward way is to project the right hand side of (5.1) into the area in question. Such algorithms are discussed in Section 5.2.

5.1. Lyapunov Function Approach.

In this section Q is supposed to be time invariant. With slight modifications the results hold also for general Q_n. Denote

$$E_e Q(x,e) = f(x)$$

A Lyapunov function for the ODE

$$\frac{d}{dt} x = f(x); \quad f(x) = 0 \Rightarrow x = x^* \tag{5.2}$$

will now be introduced:

Let V(x) be a twice continuously differentiable function satisfying

$$\begin{aligned} &a) \quad V(x) \geq 0; \quad V(x) = 0 \Leftrightarrow x = x^* \\ &b) \quad V'(x)f(x) = W(x) \leq -CV(x), \; C > 0 \\ &c) \quad E_e[Q(x,e)^T V''(\xi_n(e))Q(x,e)] \leq -AW(x) + B; \quad x \in R^m, \; A,B > 0 \end{aligned} \tag{5.3}$$

where $\xi_n(e) = x + \theta\gamma_n Q_n(x,e) \quad 0 \leq \theta \leq 1 \quad n \geq N_0$

Condition (5.3c) restricts the choice of functions V(x). All Lyapunov functions to (5.2) do not satisfy (5.3c). Also, as will be shown below, there exists functions f(x) for which (5.3abc) cannot be satisfied for any V(x), but for which (5.3ab) can be satisfied.

Notice that in (5.3c) $\xi_n(e)$ depends on e. This condition thus also requires certain regularity of the noise.

In (5.3) like everywhere before, the expectation of a function $h(x,e)$

$$E_e h(x,e)$$

is taken with respect to e, while x is regarded as a fixed parameter. Let now x_n be generated by the algorithm (5.1). Then x_n depends on the noise terms $e_1,\ldots,e_n$. If $\{e_i\}$ is a sequence of independent variable e_n is independent of x_{n-1}. Hence

$$E_{e_n} h(x_{n-1},e_n) = E[h(x_{n-1},e_n)|x_{n-1}]$$

where the RHS denotes conditional expectation given x_{n-1} and

$$Eh(x_{n-1},e_n) = E_{x_{n-1}} E_{e_n} h(x_{n-1},e_n) \qquad (5.4)$$

If $\{e_i\}$ are dependent, also x_{n-1} and e_n are dependent and (5.4) is no longer true. However, x_{n-1} contains information about recent e_k only to a decreasing extent. Therefore, under quite mild conditions on the noise $\{e_i\}$, x_{n-1} and e_n become less dependent as n increases. Then the following relation holds:

$$Eh(x_{n-1},e_n) - E_{x_{n-1}} E_{e_n} h(x_{n-1},e_n) \to 0 \quad \text{as} \quad n \to \infty \qquad (5.5)$$

for any function h for which Eh exists.

Based on the function V in (5.3) a theorem that guarantees convergence in probability can be shown.

Theorem 5.1. Consider the algorithm

$$x_n = x_{n-1} + \gamma_n Q(x_{n-1}, e_n) \tag{5.6}$$

where

$$\sum_1^\infty \gamma_n = \infty \quad \text{and} \quad \gamma_n \to 0 \quad \text{as} \quad n \to \infty$$

Let the function $V(x)$ satisfy (5.3). Assume that the stochastic process $\{e_i\}$ is such that condition (5.5) is satisfied and that $EV(x_n)$ is finite for all n (but not necessarily uniformly bounded).

Then $x_n \to x^*$ in probability as $n \to \infty$ and consequently $x_n \in D$ i.o. w.p.1, where D is any open region containing x^*.

Proof. By expansion into Taylor series:

$$V(x_n) = V[x_{n-1} + \gamma_n Q(x_{n-1}, e_n)] = V(x_{n-1}) + \gamma_n h(x_{n-1}, e_n)$$

where

$$h(x_{n-1}, e_n) = V'(x_{n-1}) Q(x_{n-1}, e_n) + \gamma_n Q(x_{n-1}, e_n)^T V''(\xi_n) \cdot$$

$$\cdot Q(x_{n-1}, e_n)$$

$$E_{e_n} h(x_{n-1}, e_n) = V'(x_{n-1}) f(x_{n-1}) + \gamma_n E_{e_n} Q(x_{n-1}, e_n)^T V''(\xi_n) \cdot$$

$$\cdot Q(x_{n-1}, e_n) \leq W(x_{n-1}) - \gamma_n A W(x_{n-1}) + B\gamma_n \leq$$

$$\leq - (1 - \gamma_n A) C V(x_{n-1}) + B\gamma_n$$

Now

$$EV(x_n) = EV(x_{n-1}) + \gamma_n Eh(x_{n-1},e_n) =$$

$$= E[V(x_{n-1}) + \gamma_n E_{e_n} h(x_{n-1},e_n)] + \gamma_n g_n \leq$$

$$\leq E[(1-\gamma_n C+CA\gamma_n^2)V(x_{n-1})+B\gamma_n^2] + \gamma_n g_n$$

where $g_n \to 0$ as $n \to \infty$ according to (5.5)

For sufficiently large n, $\gamma_n < 1/2A$ and then we have

$$EV(x_n) \leq EV(x_{n-1}) + \gamma_n \frac{C}{2}\left[(g_n+B\gamma_n)\frac{2}{C} - EV(x_{n-1})\right]$$

It now follows from Lemma 4.1 that since $g_n+B\gamma_n \to 0$, we have $EV(x_n) \to 0$ as $n \to \infty$. This implies, according to (5.3a) that $x_n \to x^*$ in probability as $n \to \infty$. □

Remark. Notice that the function V(x) assures the boundedness condition as well as the stability condition in Theorem 3.1. If also the noise condition is satisfied, the conclusion of the theorem can be strenghtened to yield convergence w.p.1.

Example 5.1. Consider the simple case

$$Q(x,e) = e - x, \quad Ee = 0; \quad Ee^2 = 1, \quad \{e_n\} \text{ indep. variables}$$

which gives

$$x_n = x_{n-1} + \gamma_n(e_n-x_{n-1}); \quad \gamma_n \to 0 \quad \sum_{n=1}^{\infty} \gamma_n = \infty$$

Choose as function $V(x) = x^2$. Then (5.3a) is trivially satisfied. Since

$$W(x) = V'(x)E_e(e-x) = -2x^2$$

also condition (5.3b) is satisfied with $C = 2$. Now

$$E_e[(e-x)2(e-x)] = 2 + 2x^2$$

and so condition (5.3c) holds with $B = 2$ and $A = 1$. Theorem 5.1 now states that $x_n \to 0$ in probability.

If, on the other hand,

$$Q(x,e) = e - x^3$$

we can still try $V(x) = x^2$. Again (5.3ab) are satisfied, but

$$E_e[(e-x^3)2(e-x^3)] = 2 + 2x^6$$

and (5.3c) cannot be satisfied for any B and A. Indeed, x_n may very well tend to infinity as shown in Example 2.1.

□

5.2. Projection Algorithms.

In most applications algorithm (5.1) will in fact be

$$x_n = [x_{n-1} + \gamma_n Q_n(x_{n-1}, e_n)]_D \tag{5.7}$$

where

$$[f]_D = \begin{cases} f \text{ if } f \in D \\ \text{some interior or boundary point of } D \text{ if } f \notin D \end{cases}$$

where D is some closed bounded region.

The sequence $\{x_n\}$ will thus by definition belong to a bounded area. This means that condition b) of Theorem 3.1 is automatically satisfied. However, the theorem cannot be straightforwardly applied, since the behaviour of $\{x_n\}$ close to the boundary of D is not governed by (5.1). To use the separation theorem, it must be shown that

$$x_n \in D^0 \quad \text{i.o. w.p.1}$$

where D^0 is a subset of D such that $\partial D^0 \cap \partial D = \phi$ (∂D = boundary of D). Within the set D^0, the projection algorithm (5.7) coincides with (5.1) for large n.

It is assumed that D is described by

$$D = \{x | U(x) \leq A\}$$

where U is a twice continuously differentiable non negative function. The region D cannot be chosen arbitrarily. Loosely, the algorithm (5.1) shall not have a tendency to move out of the region. It is thus assumed that the trajectories of (5.2) do not intersect ∂D "outwards", i.e.

$$\sup_{x \in \partial D} U'(x)f(x) < 0 \qquad (5.8)$$

where

$$f(x) = \lim_{n \to \infty} E_e Q_n(x,e)$$

(Assume that the convergence is uniform in $x \in D$).

Assume further that

$$E_e\left[Q_n(x,e)^T U''(\xi_n(e)) Q_n(x,e)\right] \leq B \qquad x \in D \qquad (5.9)$$

where

$$\xi_n(e) = x + \theta \gamma_n Q_n(x,e) \quad \text{some } \theta \quad 0 \leq \theta \leq 1$$

This condition much resembles (5.3c). However, (5.3c) is basically a, rather restrictive, condition on $Q(x,e)$ as a function of x as shown in Example 5.1. Condition (5.9) is a quite weak condition on the noise e.

Theorem 5.2. Consider algorithm (5.7) where D is defined as above. Assume that (5.8) and (5.9) hold, and that $f(x)$ is continuous in a neighbourhood of ∂D. Let $\{\gamma_n\}$ and $\{e_n\}$ satisfy the conditions of Theorem 5.1. Then $x_n \in D^0$ i.o. w.p.1 where D^0 is a subset of D, such that $\partial D^0 \cap \partial D = \phi$.

Proof. Since $W(x) = U'(x)f(x)$ is continuous, (5.8) implies that

$$\sup_{x \in \Delta D_\varepsilon} W(x) = \hat{\delta} < 0$$

where ΔD_ε is some neighbourhood of ∂D. Define D^0 as the difference between D and ΔD_ε:

$$D^0 = D \cap [D \cap \Delta D_\varepsilon]^* \qquad * = \text{complement}$$

Suppose that

$$x_n(\omega) \notin D^0 \quad \text{all } n > N(\omega) \text{ for } \omega \in \Omega' \text{ where } P(\Omega') = \delta > 0.$$

Define the random variable

$$N(\omega) \begin{cases} \text{as above for } \omega \in \Omega' \\ 0 \text{ for } \omega \notin \Omega' \end{cases}$$

and consider the following modification of algorithm (5.7), yielding the estimates $\{\bar{x}_n\}$.

Let $\tilde{x}_n$ be defined as

$$\tilde{x}_n = \bar{x}_{n-1} + \gamma_n Q_n(\bar{x}_{n-1}, e_n)$$

and let

$$\bar{x}_n = \bar{x}_{n-1} \qquad \text{if } n \geq N(\omega) \text{ and } \bar{x}_{n-1} \in D^0$$

$$\bar{x}_n = [\tilde{x}_n]_D \qquad \text{otherwise}$$

Then $\bar{x}_n(\omega) = x_n(\omega)$ for $\omega \in \Omega'$ and

$$P\left(x_n(\omega) \in \Delta D_\varepsilon\right) > \delta/2 \quad \text{for } n > N_0$$

Introduce

$$\tilde{U}(x) = \begin{cases} U(x) & x \in \Delta D_\varepsilon \\ 0 & x \in D^0 \end{cases}$$

Then

$$E\tilde{U}(\bar{x}_{n+1}) - E\tilde{U}(\bar{x}_n) \leq E\tilde{U}(\tilde{x}_{n+1}) - E\tilde{U}(\bar{x}_n) \leq$$

$$\leq \gamma_n \Big\{ -\hat{\delta}\,\delta/2\; P(N<n) + AP(N\geq n) + EU'(x_n) \cdot$$

$$\cdot \left[E_e Q_n(x_n,e) - f(x_n)\right] + \gamma_n B \Big\}$$

But $P(N\geq n) \to 0$ and $E_e Q_n(x_n,e) - f(x_n) \to 0$ as $n \to \infty$. Since $\Sigma\gamma_n = \infty$ this implies that $E\tilde{U}(\bar{x}_n) \to -\infty$ which is impossible.

Hence $P(\Omega') = 0$, i.e. $x_n \in D^0$ i.o. w.p.1.

□

Remark. By modifying the proof, condition (5.5) can be replaced by the (essentially stronger) condition a) of Theorem 3.1.

Example 5.2. Consider again the case $Q(x,e) = e - x^3$. Choose $D = [-\sqrt{A},\sqrt{A}]$ and $U(x) = x^2$. It is easy to see that $U(x)$ satisfies (5.6) and (5.7). Theorem 5.2 now guarantees that x_n is strictly interior to D i.o. if algorithm (5.6) is used. From Theorem 3.1 then follows that $x_n \to 0$ w.p.1 as $n \to \infty$ under weak conditions on the noise e_n. The projection into a bounded area is thus not only a formal trick to achieve theorems on convergence. It also makes divergent schemes converge. Intuitively, the estimate "rests" at the boundary until γ_n is so small that the adjustments $\gamma_n Q(x_{n-1},e_n)$ force x_n into the interior of the area D.

6. CONVERGENCE RATE.

In this chapter the importance of the ordinary differential equation (3.6) associated with the algorithm (1.8) is discussed. It is shown that the ODE is intimately connected with the selection of the sequence $\{\gamma_n\}$ to obtain fast convergence of $\{x_n\}$. The usefulness of slow convergence of $\{\gamma_n\}$ is illustrated and explained. The importance of small γ_n initially is also discussed.

In Section 6.1 some numerical examples are given, where slow convergence of $\{\gamma_n\}$ is favourable. A heuristic analysis of the connection between convergence rate and choice of $\{\gamma_n\}$ is given in Section 6.2. In Section 6.3 it is shown that the sequence $\{\gamma_n\}$ in many cases must be bounded from above to obtain acceptable stability properties. A theorem that connects the trajectories of (3.6) with the sequence $\{x_n\}$ defined by (1.8) is proved in Section 6.4. There also the implications of this result on the choice of $\{\gamma_n\}$ are illustrated.

6.1. Choice of $\{\gamma_n\}$.

Consider as in the Robbins-Monro case the problem to solve

$$E_e Q(x,e) = f(x) = 0 \tag{6.1}$$

for x. As remarked in Chapter 1, a suitable estimate of x at time n is obtained as the solution of

$$\frac{1}{n} \sum_{k=1}^{n} Q(x,e_k) = 0 \tag{6.2}$$

In the simple case (1.12) when $Q(x,e) = e - x$ the equation (6.2) is linear in x and the solution can be obtained recursively as

$$x_n = x_{n-1} + \frac{1}{n}(e_n - x_{n-1}) \tag{6.3}$$

which is the RM scheme with $\gamma_n = 1/n$. The same analysis can also be done in the more general case when x is a vector, see (1.24). The resulting algorithm is of the form (1.7) with $\gamma_n = 1/n$ and S_n tending to a constant matrix.

However, in general when $Q(x,e)$ does not depend linearly on x, a more complex situation arises. This is the case for the adaptive algorithms in Examples 1.5 and 1.6, for the general recursive estimation algorithms of Example 1.4, and a variety of other cases. It is recognized by most people who have applied such algorithms that a considerable increase in convergence rate is obtained if γ_n is chosen to decrease more slowly than 1/n. Some specific examples are given below.

Example 6.1 - Self-tuning regulator.

An example with a self-tuning regulator (see Example 1.5) is shown in Fig. 6.1 (from Wittenmark (1973)). It is readily seen that the parameters tend to the desired values more rapidly for a constant $\gamma_n = \gamma_0$ than for $\gamma_n = 1/n$.

Example 6.2 - Recursive maximum likelihood.

In Fig. 6.2 (from Söderström (1973)) the result of recursive approximate maximum likelihood estimation is shown. Cf. Example 1.4. Again the curve that corresponds to a slower decrease in γ_n shows faster convergence.

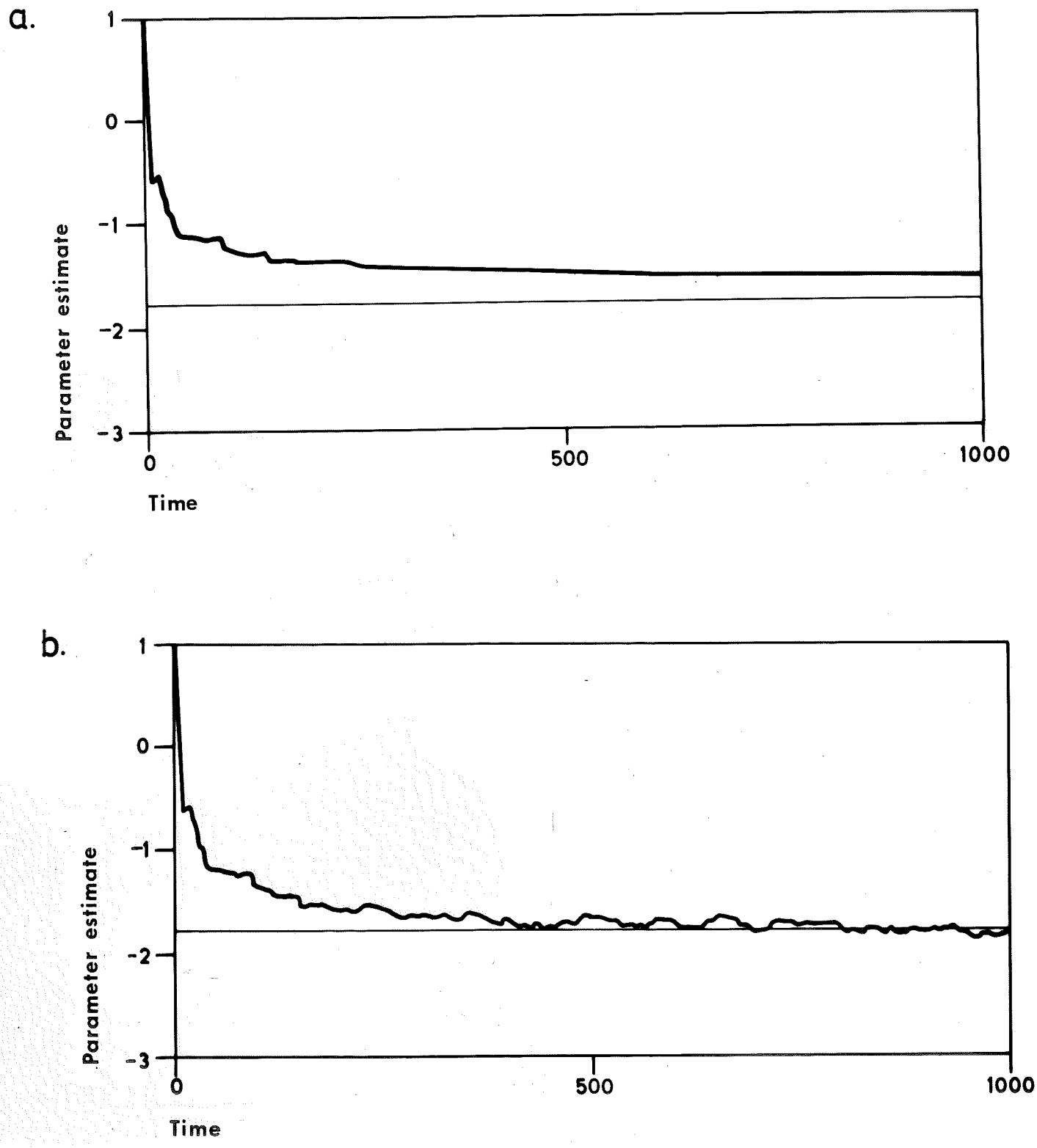


Fig. 6.1 (Wittenmark (1973)) - Convergence for a self-tuning regulator, with one parameter. The algorithm is of type (1.24), (1.30) with a) $\gamma_n = 1/n$ and b) γ_n = constant = 0.01.

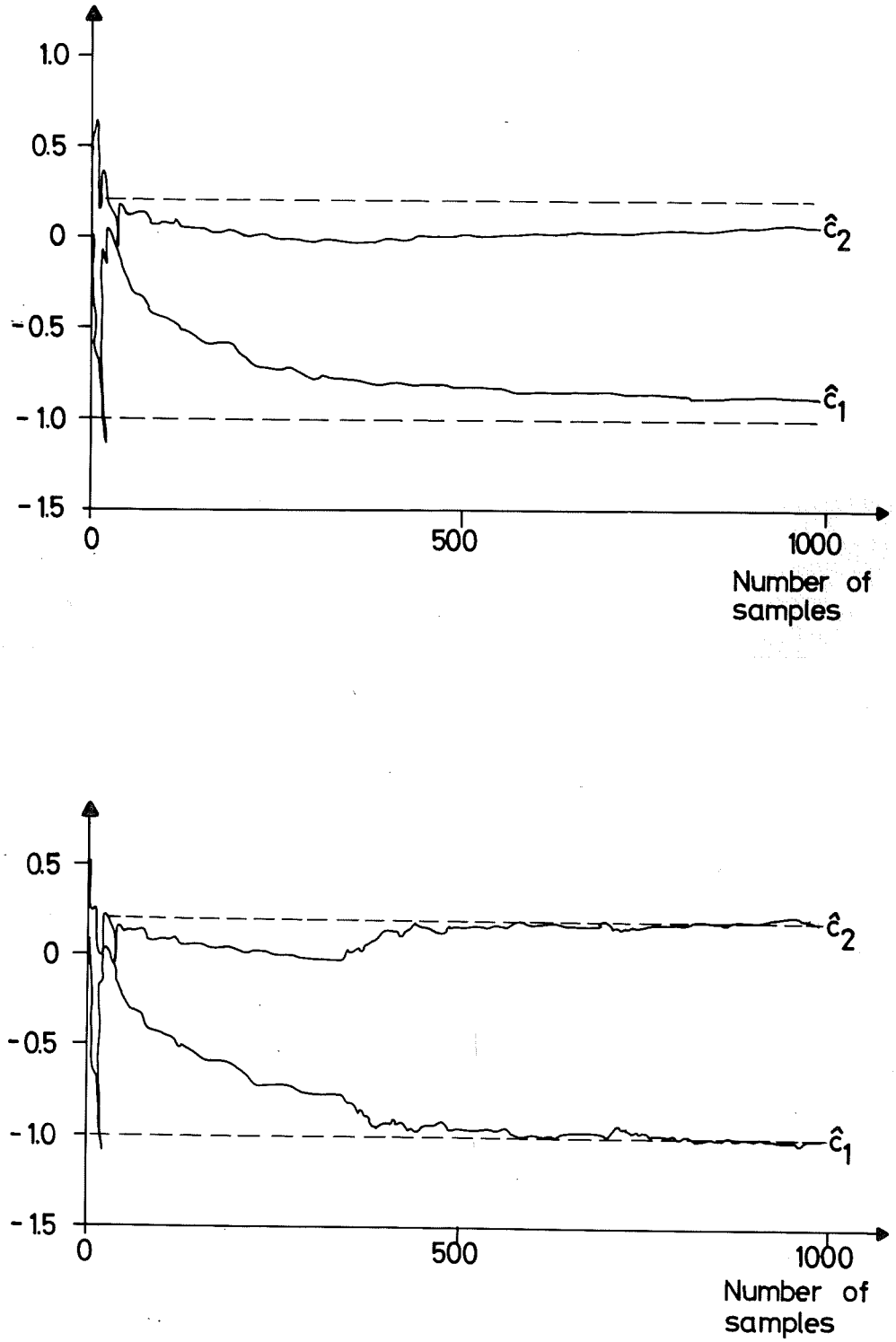


Fig. 6.2 (Söderström (1973)) - Convergence for C parameter estimates of a linear system (1.25). An approximate maximum likelihood method (1.29) has been used. The upper curves correspond to $\gamma_n = 1/n$ in (1.29). The lower curve is obtained if restarts are used. These make γ_n decrease more slowly than $1/n$.

Also in the RM scheme with improved convergence rate, suggested by Kesten (1958) $\{\gamma_n\}$ is reduced slowly.

6.2. Heuristic Analysis.

Consider again the problem to solve (6.1) from measurements $Q(x,e_k)$, $k = 1,\ldots$ It is convenient to introduce

$$\tilde{Q}(x,e) = Q(x,e) + x$$

Then (6.1) can be written

$$E_e\tilde{Q}(x,e) = x \tag{6.4}$$

The Robbins-Monro scheme (1.10) is one possibility to obtain estimates x_n recursively:

$$x_n = x_{n-1} + \gamma_n Q(x_{n-1},e_n); \quad x_0 = 0 \tag{6.5}$$

It is straightforward to show that x_n defined by (6.5) can be written

$$x_n = \sum_{k=1}^{n} \beta_k^n \tilde{Q}(x_{k-1},e_k) \tag{6.6}$$

where

$$\beta_k^n = \gamma_k \prod_{j=k+1}^{n} (1-\gamma_j) \qquad k < n; \quad \beta_n^n = \gamma_n$$

In this section (6.6) is heuristically interpreted as an approximate solution of (6.2), which can be written as

$$x_n = \frac{1}{n} \sum_{k=1}^{n} \tilde{Q}(x_n, e_k) \triangleq \tilde{f}_n(x_n) \tag{6.7}$$

Eq. (6.7) is to be solved for x_n. This cannot be done straightforwardly. One difficulty is that $\tilde{f}_n(x)$ is not available as a function of x. Even if it were, (6.7) is a non linear equation that must be solved. This can e.g. be done in the iterative manner

$$x_n^{(i)} = \tilde{f}_n\left(x_n^{(i-1)}\right) \tag{6.8}$$

If just one iteration is made and the previous estimate x_{n-1} is used as a starting value we have

$$x_n = \tilde{f}_n(x_{n-1}) = \frac{1}{n} \sum_{k=1}^{n} \tilde{Q}(x_{n-1}, e_k) \tag{6.9}$$

Now, $\tilde{Q}(x_{n-1}, e_k)$ are not known, and $\tilde{f}_n(x_{n-1})$ cannot be calculated. One possibility is to approximate $\tilde{Q}(x_{n-1}, e_k)$ with $\tilde{Q}(x_{k-1}, e_k)$ and so the sum in (6.9) is replaced by

$$\frac{1}{n} \sum_{k=1}^{n} \tilde{Q}(x_{k-1}, e_k) \tag{6.10}$$

Now, the last terms in (6.10) are likely to be better approximations than the first ones. Therefore it is reasonable to assume that a weighted sum

$$\sum_{k=1}^{n} \beta_k^n \tilde{Q}(x_{k-1}, e_k) \tag{6.11}$$

is a better approximation of the RHS of (6.9) if β_k^n increases with k, than if $\beta_k^n = 1/n$ as in (6.10). Combining (6.11) with (6.9) the following approximate solution to

(6.7) is obtained

$$x_n = \sum_{k=1}^{n} \beta_k^n \tilde{Q}(x_{k-1}, e_k)$$

where β_k^n are suitable weighting coefficients:

$$\sum_{k=1}^{n} \beta_k^n = 1 \qquad \beta_k^n < \beta_{k+1}^n \qquad \text{all } k < n \tag{6.12}$$

The estimate given by (6.12) is exactly of the form that the RM scheme gives, i.e. (6.6).

It is interesting to see what the property (6.12) means in terms of γ_n. Some calculation shows that

$$\beta_k^n < \beta_{k+1}^n \Leftrightarrow \frac{1}{\gamma_{k+1}} - \frac{1}{\gamma_k} < 1 \tag{6.13}$$

This means that $1/\gamma_k$ increases more slowly than k, or that γ_k decreases more slowly than 1/k. The fact that such choices give faster convergence rates in practice can therefore be explained with the more suitable weighting of old observations in (6.6).

Remark. The analysis is just a heuristic one. The function $\tilde{Q}(x,e)$ was chosen as $Q(x,e) + x$. A similar analysis could be performed for

$$\tilde{Q}_\lambda(x,e) = Q(x,e) + \lambda x$$

The estimate x_n defined by (6.5) then is

$$x_n = \sum_{k=1}^{n} \beta_k^n(\lambda) Q(x_{k-1}, e_k) \quad \text{where} \quad \beta_k^n(\lambda) = \gamma_k \prod_{k+1}^{n-1} (1-\lambda\gamma_i)$$

Also the iterative scheme (6.8) has another convergence dynamics for this $\tilde{Q}$. The goodness of the correction obtained in one iteration, as in (6.9) depends on λ. There is consequently some interaction between the convergence dynamics of (6.8) and the weighting coefficients in (6.6). This is not stringently accounted for in the above analysis.

6.3. Bounds on the Sequence $\{\gamma_n\}$.

The choice of $\{\gamma_n\}$ affects not only the convergence rate. In many cases too large values of γ_n may cause instability effects in the algorithm (6.5).

Example 6.3. Consider the following scheme:

$$x_n = x_{n-1} + \gamma_n(e_n - Ax_{n-1}) \tag{6.14}$$

with

$$A = \begin{pmatrix} \delta & \omega \\ -\omega & \delta \end{pmatrix} ; \quad \delta > 0$$

Introduce $P_n = Ex_n x_n^T$. Suppose that $\{e_n\}$ is a sequence of independent variables with normal distribution and zero mean value. Then

$$P_{n+1} = P_n - \gamma_n(AP_n + P_nA^T) + \gamma_n^2[AP_nA^T + \Sigma]$$

where $\Sigma = Ee_n e_n^T$. Suppose $\Sigma = I$. It is then straightforward to show that tr P_{n+1} > tr P_n unless

$$\gamma_n < \frac{2\delta \text{ tr } P_n}{2 + (\delta^2+\omega^2)\text{tr } P_n}$$

A necessary condition for this relation to hold obviously is

$$\gamma_n < \frac{2\delta}{\delta^2 + \omega^2} \tag{6.15}$$

However, applying Theorems 5.1, 3.1 and 4.3 to this algorithm, it can be shown that $x_n \to 0$ w.p.1 for any sequence $\{\gamma_n\}$ such that

$$\Sigma\gamma_n = \infty \quad \text{and} \quad \Sigma\gamma_n^p < \infty \text{ some real } p > 0 \tag{6.16}$$

Consequently, bounds on $\{\gamma_n\}$ like (6.15) are not necessary to achieve convergence. Such bounds, however, are of great importance to obtain convergence in practice. Algorithm (6.14) with $\omega = 5$ and $\delta = 0.5$ has been simulated for some choices of $\{\gamma_n\}$ that all satisfy (6.16) and theoretically give convergence. The results are shown in Table 6.1. □

Another example of bounded sequences $\{\gamma_n\}$ is the estimation algorithm (1.21):

$$x_{n+1} = x_n + \gamma_{n+1}\left\{\xi_n y(n+1) - \xi_n\xi_n^T x_n\right\}$$

To avoid that x_n assumes too large values $\{\gamma_n\}$ must be normalized. The choices

$$\gamma_n = \frac{1}{n}\left(\xi_n^T \xi_n\right)^{-1}$$

or

$$\gamma_n = \left(\sum_{k=1}^{n} \xi_k^T \xi_k\right)^{-1} .$$

have the effect that the sequence $\{\gamma_n\}$ is appropriately bounded. Notice that the real time least squares algorithm corresponds to

$$\gamma_n S_n = \left(\sum_{1}^{n} \xi_k \xi_k^T\right)^{-1}$$

Table 6.1 - Simulation of (6.14) with $\delta = 0.5$, $\omega = 5$ and $\Sigma = I$ for some sequences $\{\gamma_n\}$. The numbers shown are $|x_n|$.

n \ γ_n	1/n	$1/n^{0.1}$	0.04/n	5/(500+n)
0	1.42	1.42	1.42	1.42
1	6.99	6.99	1.39	1.41
2	17.55	31.56	1.41	1.41
5	66.99	2691.2	1.43	1.42
10	129.21	$3 \cdot 10^6$	1.42	1.42
100	121.28	$\sim 10^{55}$	1.36	1.31
25000	8.70	$\sim 10^{75}$	1.22	0.02

6.4. Trajectories.

So far, it has been observed that $\{\gamma_n\}$ must be chosen to be sufficiently small to avoid unstable behaviour of (6.5). We have also indicated that (6.13) should be satisfied in order to improve the convergence rate. However, we have not been able to give any rules or quantitative estimates how to choose $\{\gamma_n\}$. It may be argued that the more $Q(x,e)$ changes with x, the less weight should the first terms in the sum (6.11) have. Clearly, if Q is independent of x, all terms should have the same weight. We will try and formalize such an argument.

The separation theorem 3.1 states that the ODE

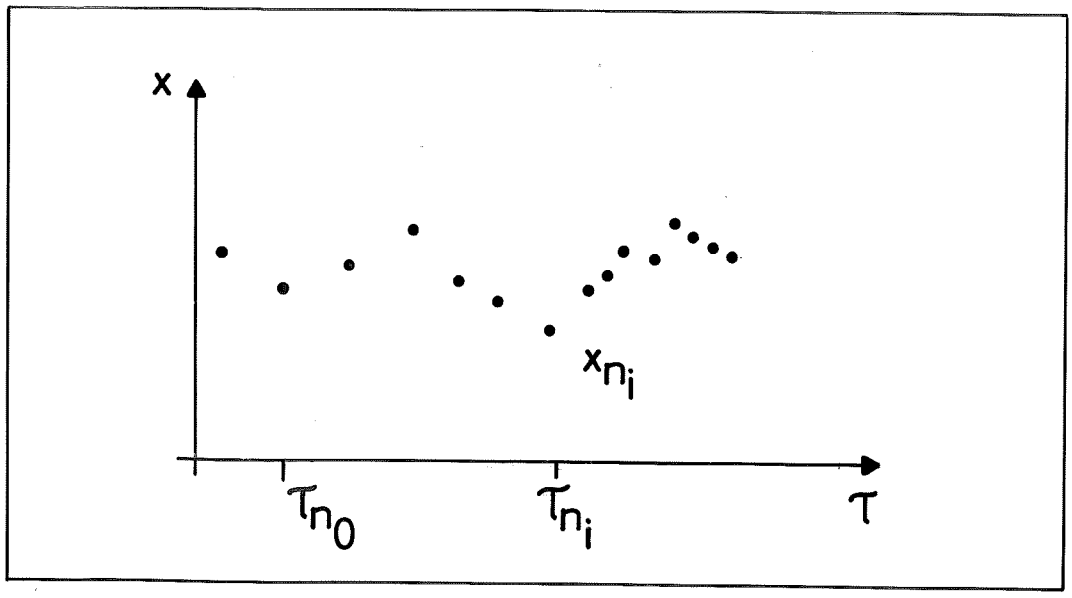
$$\frac{d}{d\tau} x = f(x) = E_e Q(x,e) \tag{6.17}$$

is important to decide convergence of (6.5). It can, in fact, be shown that the trajectories of (6.17) also govern the behaviour of the estimates x_n, obtained from (6.6). Loosely, the trajectories are the "expected paths" of $\{x_n\}$.

The result is formulated as follows. Let x_i, $i = n_0, \ldots,$ be generated by (6.5). The values can be plotted with the sample numbers i as the abscissa. It is also possible to introduce a fictitious time τ by

$$\tau_n = \sum_{i=1}^{n} \gamma_i$$

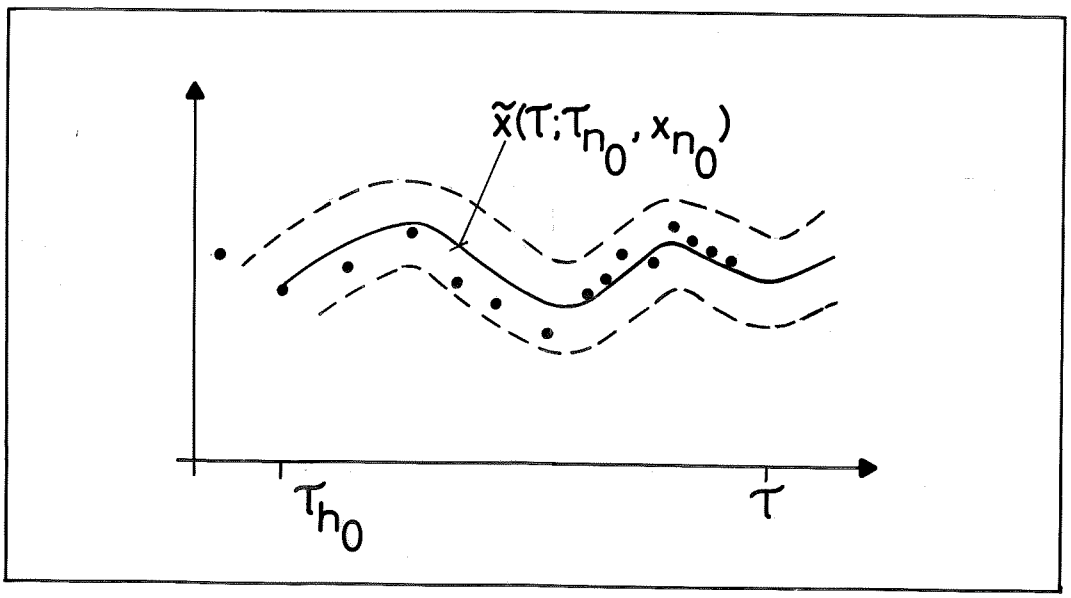
Suppose that the estimates x_i are plotted against this time τ:



Let $\tilde{x}(\tau;\tau_{n_0},x_{n_0})$ be the solution of (6.17) with initial value x_{n_0} at time τ_{n_0}:

$$\tilde{x}(\tau_{n_0};\tau_{n_0},x_{n_0}) = x_{n_0}$$

Plot also this solution in the same diagram:



Let I be a set of integers. The probability that all points x_i, $i \in I$, simultaneously are within a certain distance ε from the trajectory is estimated in the following theorem.

Theorem 6.1. Consider algorithm (6.5). Let $Q(x,e)$ be Lipschitz continuous for fixed e with Lipschitz constant $K(e)$. Assume that $E|Q(x^0,e_n)|^{2p} < C$ and $E|K(e_n)|^{2p} < C$ and suppose that $Q(x^0,e_n)$, $K(e_n)$ and $\{\gamma_n\}$ satisfy the conditions of Lemma 4.2. Assume that $f(x)$ is continuously differentiable. Denote

$$\sum_{i=1}^{n} \gamma_i = \tau_n$$

and denote the solution of (6.17) with initial condition $x(\tau_{n_0}) = x^0$ by $\tilde{x}(\tau;\tau_{n_0},x^0)$. Consider the ODE (6.17) linearized around this solution:

$$\frac{d}{d\tau}\Delta x = f'\left(x(\tau;\tau_{n_0},x_{n_0})\right)\Delta x$$

Assume that there exists a quadratic Lyapunov function for this linear, time varying ODE (see e.g. Brockett (1970)). Let I be a set of integers such that $\inf|\tau_i-\tau_j| = D > 0$ where $i \neq j$ and $i,j \in I$. Then there exists a K and an ε_0 such that for $\varepsilon < \varepsilon_0$

$$P\left\{\sup_{n\in I}|x_n - \tilde{x}(\tau_n;\tau_{n_0},x_{n_0})| > \varepsilon\right\} \leq \frac{K^r}{\varepsilon^{4r}} \sum_{j=n_0}^{N} (\gamma_j)^r \qquad r\leq p \qquad (6.18)$$

where $N = \sup_{i\in I} i$, which may be ∞.

The proof is given in Appendix D.

□

If the sum $\Sigma\gamma_n^r$ is convergent, the right hand side of (6.18) can, for fixed ε, be chosen arbitrarily small by taking n_0 sufficiently large. Thus the theorem states that the trajectories of the ODE (6.17) arbitrarily well de-

scribe the behaviour of the algorithm (6.5) for sufficiently large time points.

We have in Section 6.3 observed that in some applications $\{\gamma_n\}$ has to be bounded by quite a small constant. For example, in the estimation algorithm (1.21) with $\gamma_n = (\Sigma \xi_k^T \xi_k)^{-1}$ a large observation $|\xi_k|$ causes all γ_n to be small. In these cases

$$\sum_{i=k}^{\infty} \gamma_i^r$$

is small and so the probability that the points x_n are outside a certain region around the trajectory also is small.

Although the proof of the theorem provides an estimate of K from given constants, we do not intend to use (6.18) to obtain numerical bounds for the probability. The point of the theorem is that a connection between the ODE (6.17) and the algorithm (6.5) is established.

Example 6.4. Consider again Example 6.3, with $\delta = 0.5$, $\omega = 5$ and $\Sigma = I$. Then, according to (6.5), γ_n has to be smaller than 0.04 to assure a stable behaviour. The sequence $\{\gamma_n\}$ has been chosen as

a) $\gamma_n = 0.04/n$

b) $\gamma_n = 1/(100+n)$

c) $\gamma_n = 0.01$

In Fig. 6.3 the results from simulations are shown. The

sequence $\{x_n\}$ is there plotted against n and against the fictitious time

$$\tau_n = \sum_1^n \gamma_k$$

Corresponding phase planes are also shown.

The different choices of $\{\gamma_n\}$ can be seen as different scaling of the time. The slower $\{\gamma_n\}$ decreases, the faster runs the corresponding time.

Due to Theorem 6.1 the estimates have to follow the solution of the associated ODE. This is shown in Fig. 6.4. The way to make the estimates approach the desired point $x^* = 0$ fast, is to speed up the time, i.e. to make

$$\sum_1^n \gamma_k$$

as large as possible, while keeping the bound $\gamma_k < 0.04$. This clearly implies that $\{\gamma_n\}$ shall decrease slowly. However, the effect of the noise must be taken into consideration when the estimates are close to the origin. A comparison between Fig. 6.3c and Fig. 6.4 shows that the effect is not negligable.

The agreement between the simulations and the trajectories depends critically on the bound on γ_n that has to be chosen to avoid instability effects. If the trajectories are "straight", γ_n can be chosen larger and the noise has greater relative influence. In Fig. 6.5 simulations of (6.14) with $\delta = 5$, $\omega = 5$ and $\Sigma = I$ are shown. In this case large $\{\gamma_n\}$ can be chosen initially, and x_n quickly gets close to the origin. Then γ_n must decrease faster than in

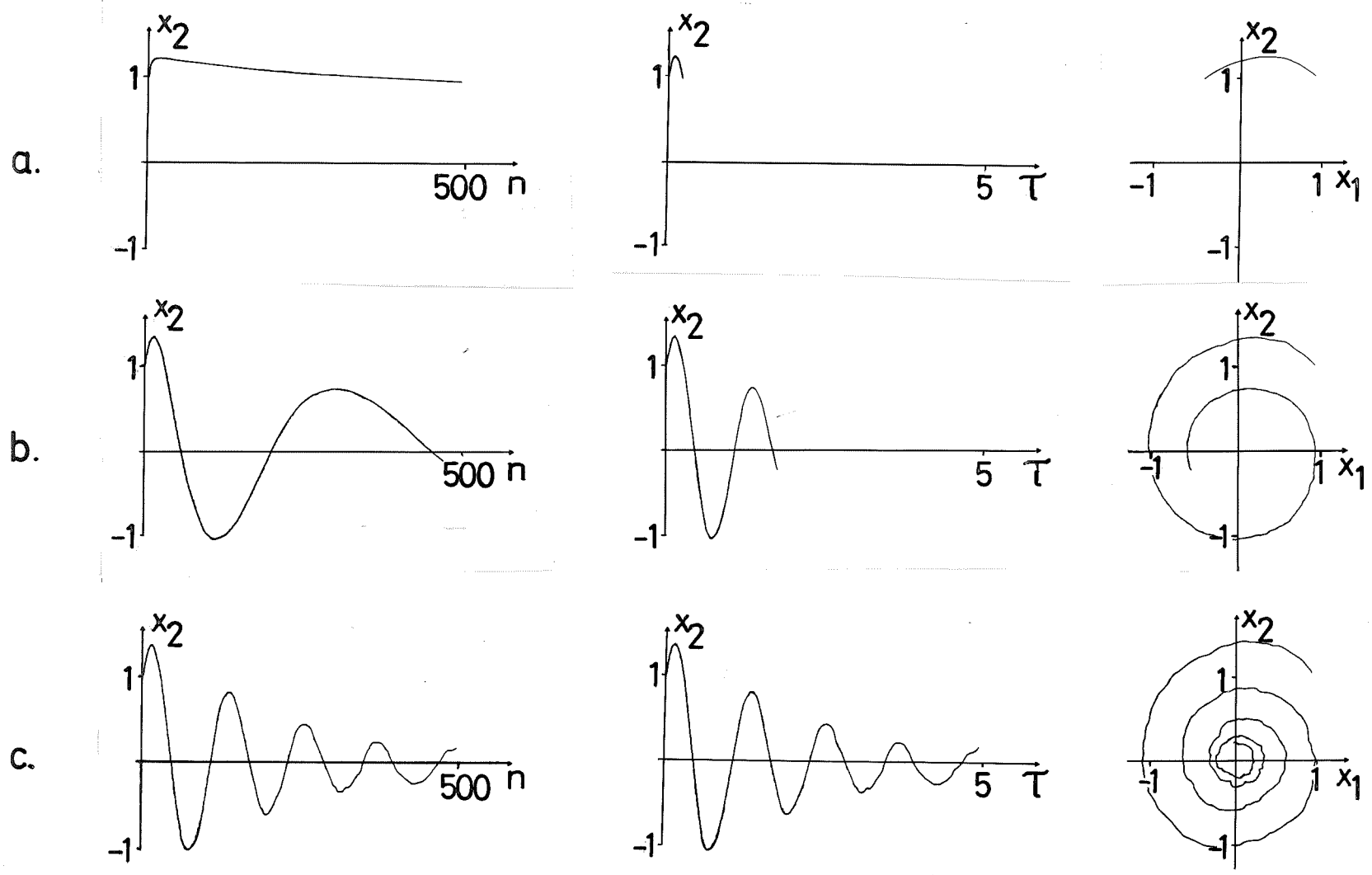


Fig. 6.3 - Simulation of (6.14) with $\delta = 0.5$, $\omega = 5$ and $\Sigma = I$. The second component of x_n is plotted against n and τ_n and against the first component of x_n for different choices of $\{\gamma_n\}$ a) $\gamma_n = 0.04/n$ b) $\gamma_n = 1/(100+n)$ c) $\gamma_n = 0.01$.

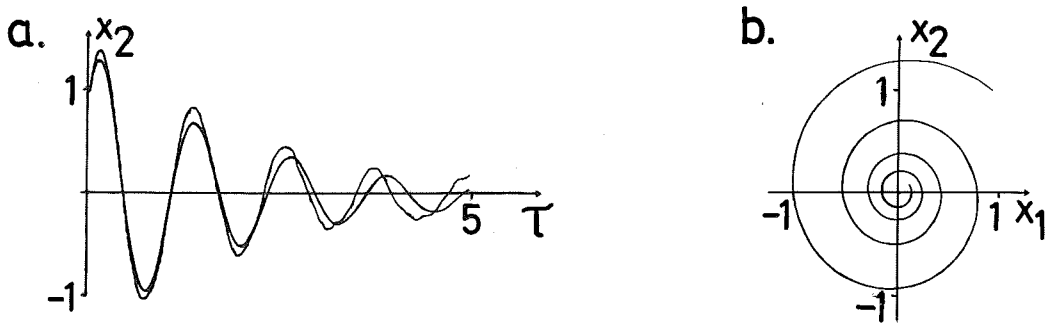


Fig. 6.4 - Solution of the ODE associated to the algorithm (6.14) with $\delta = 0.5$, $\omega = 5$. In Fig. a, also the curve of Fig. 6.3c is shown.

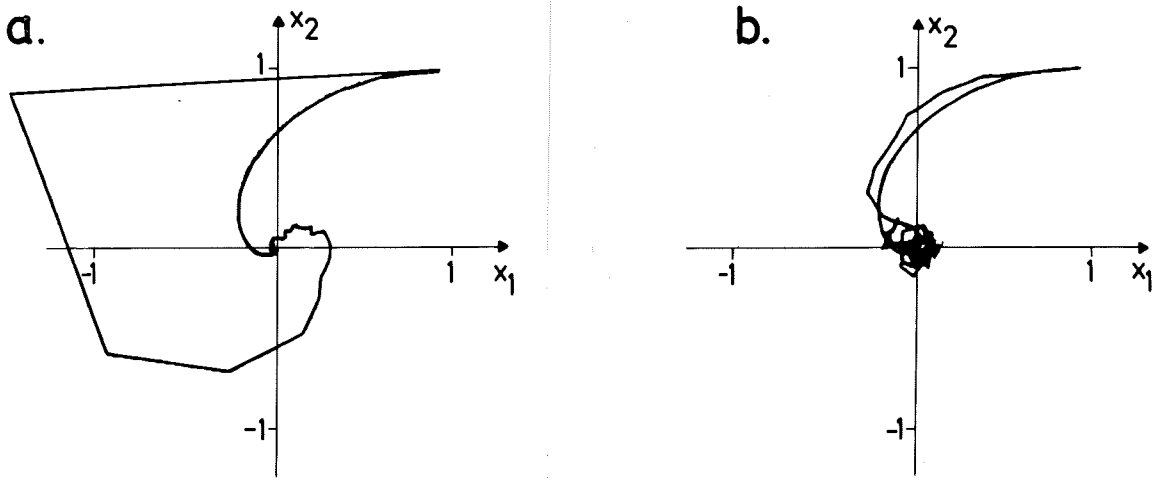


Fig. 6.5 - Simulation of (6.14) with $\delta = 5$, $\omega = 5$ and $\Sigma = I$ for a) $\gamma_n = 0.25/n$ and b) $\gamma_n = 0.05$, $n = 1,\ldots,100$. The solution of the associated ODE is also shown.

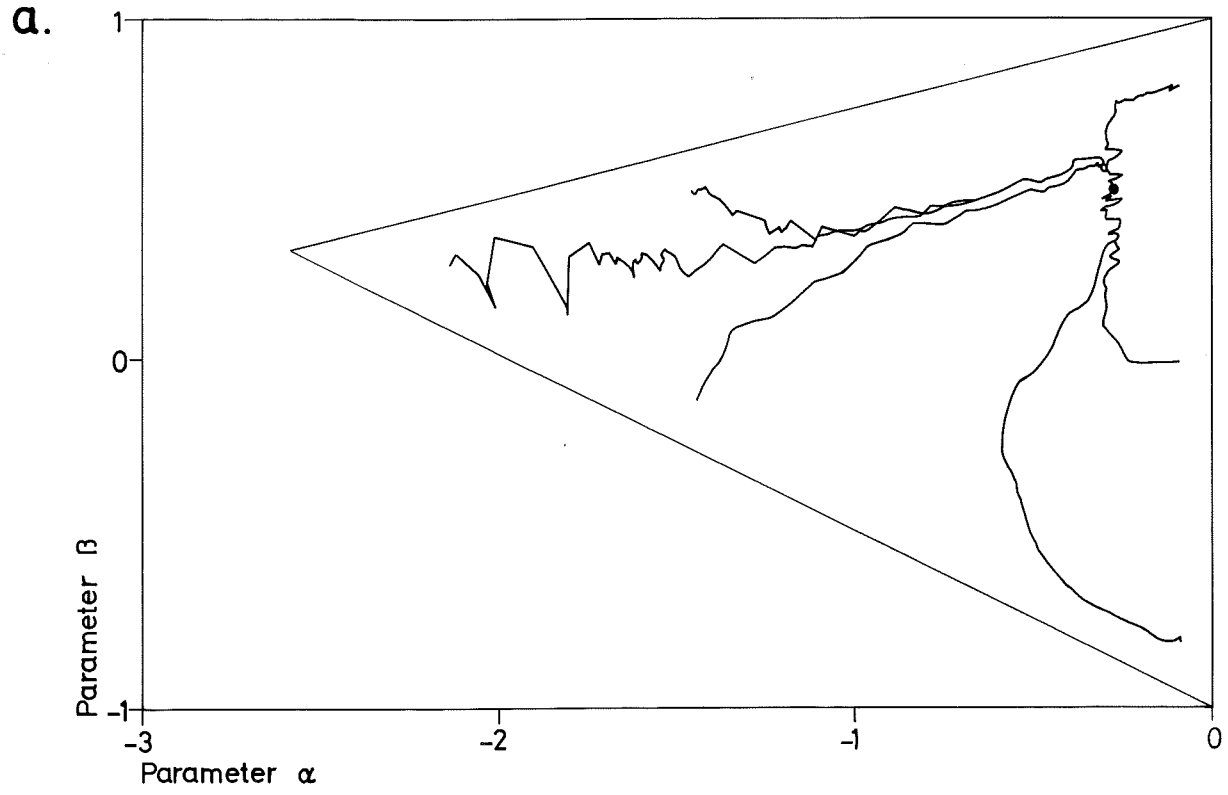
the previous case, in order to reduce the predominating influence of the noise.

In Fig. 6.6 simulations of the self-tuning regulator of Example 1.5 are shown. The regulator parameter estimates α and β for the system

$$y(t+1) - 0.99y(t) = u(t) + 0.5u(t-1) + e(t) - 0.7e(t-1)$$

where $\{e(t)\}$ is white noise, are plotted in a phase plane. These curves are compared with the trajectories of the corresponding ODE. It is seen that the trajectories well describe the behaviour of the algorithm.

To summarize, Theorem 6.1 and these examples show that numerical solution of the ODE (6.17) can be a valuable complement to simulation of the algorithm (6.5). The effect of various choices of the sequence $\{\gamma_n\}$ can also be understood in terms of this ODE. The advantage with sequences that decrease slowly (in the beginning of the procedure) can be clearly seen as in Example 6.4. This conclusion is the same as that of the heuristic analysis in Section 6.2.



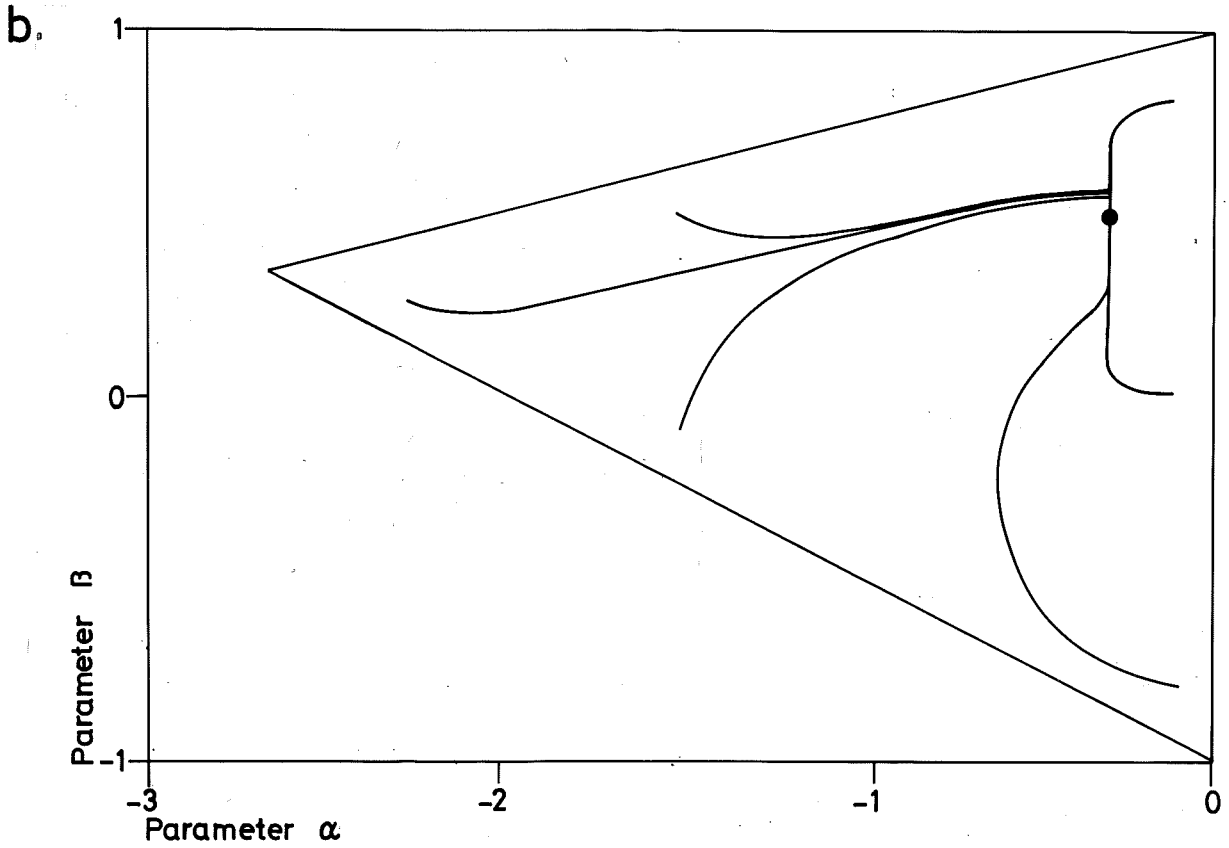


Fig. 6.6 - a) Simulation of the self-tuning regulator (1.21) with (1.30). The sequence γ_n is chosen as $0.002n^{-0.1}$.

b) Trajectories of the corresponding ODE.

7. SUMMARY AND DISCUSSION.

The separation theorem together with the results of Chapters 4 and 5 can be combined into several theorems on convergence. For a specific application it is, of course, possible also to tailor noise and boundedness conditions. In Section 7.1 two convergence theorems are given that seem to cover a large variety of applications. Some examples are given in Section 7.2. In Section 7.3 the significance of the results of the report are discussed.

7.1. Convergence Theorems.

Suppose in this section that $\{\gamma_n\}$ is deterministic. The sequence $\{\gamma_n\}$ in the algorithm

$$x_n = x_{n-1} + \gamma_n Q_n(x_{n-1}, e_n) \tag{7.1}$$

is often chosen such that

$$\gamma_n n \to A \text{ as } n \to \infty \tag{7.2}$$

For this choice Theorem 3.1 can be combined with (4.7) and Theorem 5.2 to yield the following result.

Theorem 7.1. Consider algorithm (7.1) with $\{\gamma_n\}$ satisfying (7.2). Assume that the estimates are projected into an area $D = \{x \mid U(x) \leq A\}$ where U satisfies conditions (5.8) and (5.9). Let $Q_n(x, e_n)$ be Lipschitz continuous in D for fixed e_n with Lipschitz constant $K_n(e_n)$. Introduce

$$f(x) = \lim_{n \to \infty} E_e Q_n(x, e)$$

and assume that the convergence is uniform in $x \in D$.

Suppose that the ODE

$$\dot{x} = f(x)$$

has an asymptotically stable solution $x(t) = x^*$, with domain of attraction $\supset D$. Suppose that

$$\mathrm{Cov}\big(K_n(e_n), K_m(e_m)\big) \leq C \cdot \frac{n^p + m^p}{1 + |n - m|^q} \qquad 0 \leq 2p < q < 1$$

and

$$\mathrm{Cov}\big[Q_n(x^0, e_n), Q_m(x^0, e_m)\big] \leq C \cdot \frac{n^p + m^p}{1 + |n - m|^q} \qquad 0 \leq 2p < q < 1$$

Then $x_n \to x^*$ w.p.1 as $n \to \infty$. □

For more general sequences $\{\gamma_n\}$ the conditions on the noise terms must be somewhat strengthened:

Theorem 7.2. Consider algorithm (7.1) with the same conditions on D and f(x) as in Theorem 7.1. Suppose that $\{\gamma_n\}$ satisfies

$$\sum_{n=1}^{\infty} \gamma_n = \infty; \quad \sum_{n=1}^{\infty} \alpha_n^{2p} \gamma_n^p < \infty \text{ (p integer);}$$

$$\{\gamma_n\} \text{ decreasing; } \lim_{n\to\infty} \sup \left[\frac{1}{\gamma_{n+1}} - \frac{1}{\gamma_n}\right] < \infty$$

Assume that

$$E|Q_n(x^0,e_n)|^{2p} \leq \alpha_n^{2p} L_1(x^0) \quad \text{and} \quad E|K_n(e_n)|^{2p} \leq \alpha_n^{2p} L_2$$

where $\{\alpha_n\}$ is nondecreasing, and that the stochastic processes

$$Q_n(x^0,e_n) - E_{e_n} Q(x^0,e_n) \quad \text{and} \quad K_n(e_n) - EK_n(e_n)$$

can be considered as filtered white noise as in (4.10). Then $x_n \to x^*$ w.p.1 as $n \to \infty$.

□

These theorems are only examples of convergence results that can be synthesized from the results of Chapters 3-5. Some specific applications are given in the next section.

7.2. Applications.

1. The Robbins-Monro scheme (cf. Example 1.1)

Theorems 7.1 and 7.2 are directly applicable to the RM scheme. In terms of the classification of Chapter 2 the stability condition has another formulation, but is essentially the same as the one due to Blum (1954b). The boundedness condition (projection algorithm) is maybe more natural. The most important feature is that the noise condition is substantially weaker. Correlated observations can be handled and the conditions on the sequence $\{\gamma_n\}$ are weaker. If all moments of the noise exist, it is only required that

$$\Sigma \gamma_n^p < \infty \text{ some } p$$

This is e.g. true for $\gamma_n = n^{-\alpha}$, $0 < \alpha \leq 1$. The usual results allow just $1/2 < \alpha \leq 1$. The discussion of Chapter 6 shows that slowly decreasing sequences $\{\gamma_n\}$ are of interest as a way to obtain faster convergence.

Example 7.1. Consider the algorithm of Example 6.3. Suppose that $\{e_n\}$ is a sequence of normally distributed, zero mean valued random variables, with a covariance function $Ee_n^T e_{n+s}$ that tends to zero exponentially as s tends to infinity. In this case

$$Q(x_{n-1}, e_n) = - Ax_{n-1} + e_n$$

which clearly is Lipschitz continuous, with a Lipschitz constant that does not depend on e_n. The ODE $\dot{x} = - Ax$ is globally asymptotically stable with stationary point $x^* = 0$. There exists a quadratic Lyapunov function that satisfies (5.3). Consequently, if

$$\sum_1^\infty \gamma_n = \infty; \sum_1^\infty \gamma_n^p < \infty \text{ some } p > 0 \quad \left(\text{or even } \sum_1^\infty \exp(-\varepsilon/\gamma_n) < \infty \text{ all } \varepsilon > 0\right)$$

and

$$\lim_{n\to\infty} \sup \left[\frac{1}{\gamma_{n+1}} - \frac{1}{\gamma_n}\right] < \infty$$

then $x_n \to 0$ w.p.1 as $n \to \infty$.

2. Minimization of a function using noise corrupted measurements (the Kiefer-Wolfowitz procedure). (Cf. Examples 1.2 and 3.3)

Consider as in Example 3.3 minimization of a function $h(x)$ when only noise corrupted measurements are available:

$$J(x,e_n) = h(x) + e_n$$

where e_n is a sequence of random variables with zero mean and uniformly bounded 2p moments. Let $\{e_n\}$ be obtained from independent variables by linear, exponentially stable filtering. The minimization is performed with the Kiefer-Wolfowitz procedure with step size γ_n and increments a_n for numerical differentiation. Combining the results of Example 3.3 and Theorem 4.1 we have:

x_n tends w.p.1 either to infinity or to a stationary point of $h(x)$ if for some p

$$\sum_1^\infty \gamma_n = \infty; \quad \{\gamma_n\} \text{ decreasing}; \quad \lim_{n\to\infty} \sup\left[\frac{1}{\gamma_{n+1}} - \frac{1}{\gamma_n}\right] < \infty;$$

$$\sum_1^\infty (\gamma_n/a_n^2)^p < \infty; \quad a_n \to 0 \tag{7.3}$$

These conditions are considerably more general than the usually reported ones:

$$\Sigma\gamma_n = \infty; \quad \Sigma(\gamma_n/a_n)^2 < \infty; \quad a_n \to 0; \quad \Sigma\gamma_n a_n < \infty \tag{7.4}$$

Taking $\gamma_n = c_1 n^{-\alpha}$ and $a_n = c_2 n^{-\beta}$ conditions (7.4) imply that (α,β) must lie in the shaded region in Fig. 7.1. If

we assume that all moments of the noise exist, our conditions, (7.3), require (α,β) to lie in the triangle A, which contains the shaded region.

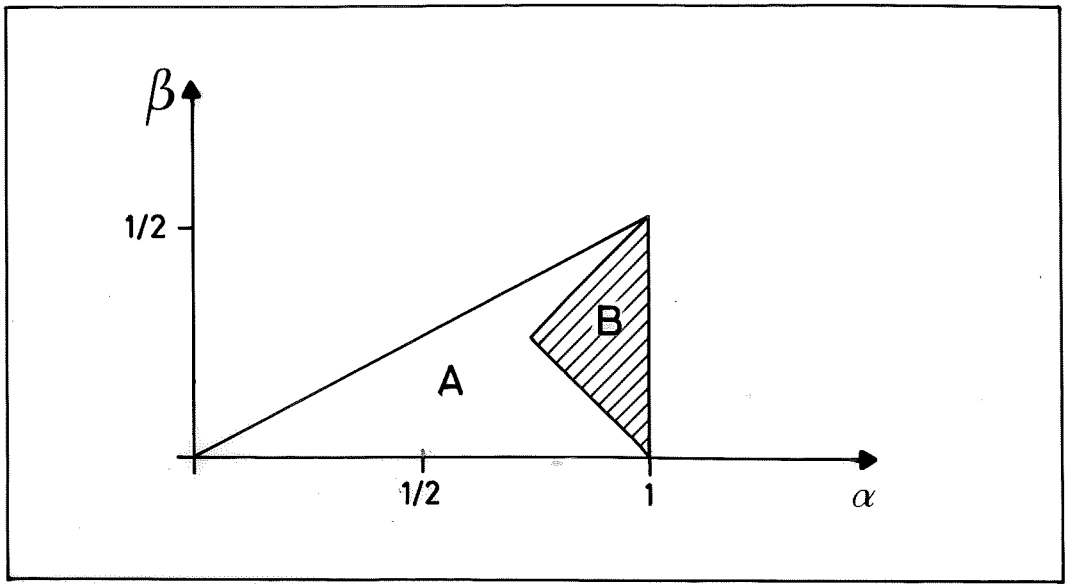


Fig. 7.1 - Values of (α,β) that give convergence in the Kiefer-Wolfowitz procedure with $\gamma_n = n^{-\alpha}$; $a_n = n^{-\beta}$.
Region A: according to (7.3). Region B: according to (7.4).

3. Adaptive systems.

Applications to real time least squares estimation has been considered in Example 3.4. The more complex problem with self-tuning regulators based on least squares estimation (Example 1.5) is treated in Ljung-Wittenmark (1974).

The self-learning classifier, suggested by Tsypkin (1968) (see also Braverman (1966)), can be analysed using the same tools:

Example 7.2. Consider the self-learning classifier defined in Example 1.6. Let e_n have the distribution shown in Fig. 7.2, consisting of two triangular distributions:

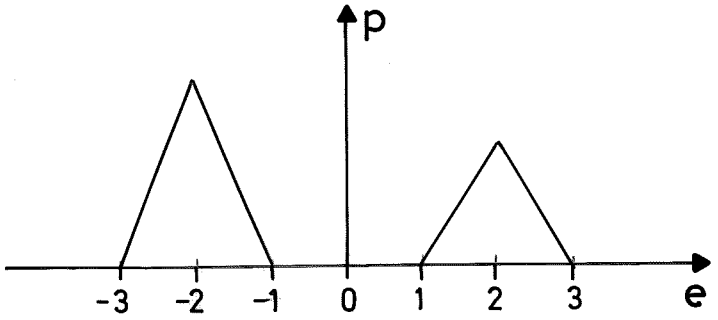


Fig. 7.2 - Probability density function of the random variable to be classified.

The probability of outcomes in the left triangle is λ. Clearly, it is desirable that the classification rule, the number c_n, should converge to some value between -1 and +1. Introduce as in Example 1.6

$$x_n = \begin{pmatrix} x_n^A \\ x_n^B \end{pmatrix}$$

Then (1.32) can be written

$$x_n = x_{n-1} + \gamma_n Q(x_{n-1}, e_n)$$

$E_e Q(x,e) = f(x)$ is readily computed as follows. For a given x the corresponding classification point is $c(x) = (x_1+x_2)/2$. $f_1(x)$ is then the mean value of the distribution left of the point c(x), minus x_1. $f_2(x)$ is found correspondingly. The algebraic expression for f(x) as a function of x and λ is lengthy and is omitted.

The trajectories of the ODE $\dot{x} = f(x)$ are shown in Fig. 7.3 for two choices of λ. In case $\lambda = 0.5$ the variable x converges to $x^* = (-2,2)$ which gives a correct classification rule $c^* = 0$. The case $\lambda = 0.99$ corresponds to a common situation, when errors that occur rather seldom (1%), "outliers", shall be detected. In this case there are two possible convergence points, $x^* = (-2,2)$ and $x^{**} = (-2.3, -1.4)$. The latter gives a classification rule $c^{**} = -1.8$ which classifies 39% of "correct values" as outliers. For any starting value x_0 there is a positive probability that the classification rule converges to c^{**}.

In this example it is straightforward to numerically solve the corresponding ODE. It is quite cumbersome to find suitable Lyapunov functions for the problem.

□

7.3. Conclusion.

In this report it has been shown that an ordinary differential equation can be defined for certain classes of recursive, stochastic algorithms. These algorithms cover a variety of control applications. The ODE has been shown to contain information about convergence of the algorithms as well as about convergence rates and behaviour of the algorithm. In the analysis it has been possible to separate the stochastic part from the rest of the problem.

Martingale theory, which is the traditional tool to show convergence, has not been used. This causes some proofs to be more technical, because instead of just referring to the martingale theorem it is in fact necessary to go through some arguments that are used in the proof of it. However, the application of martingale theory requires some extra conditions, that can be removed with our approach.

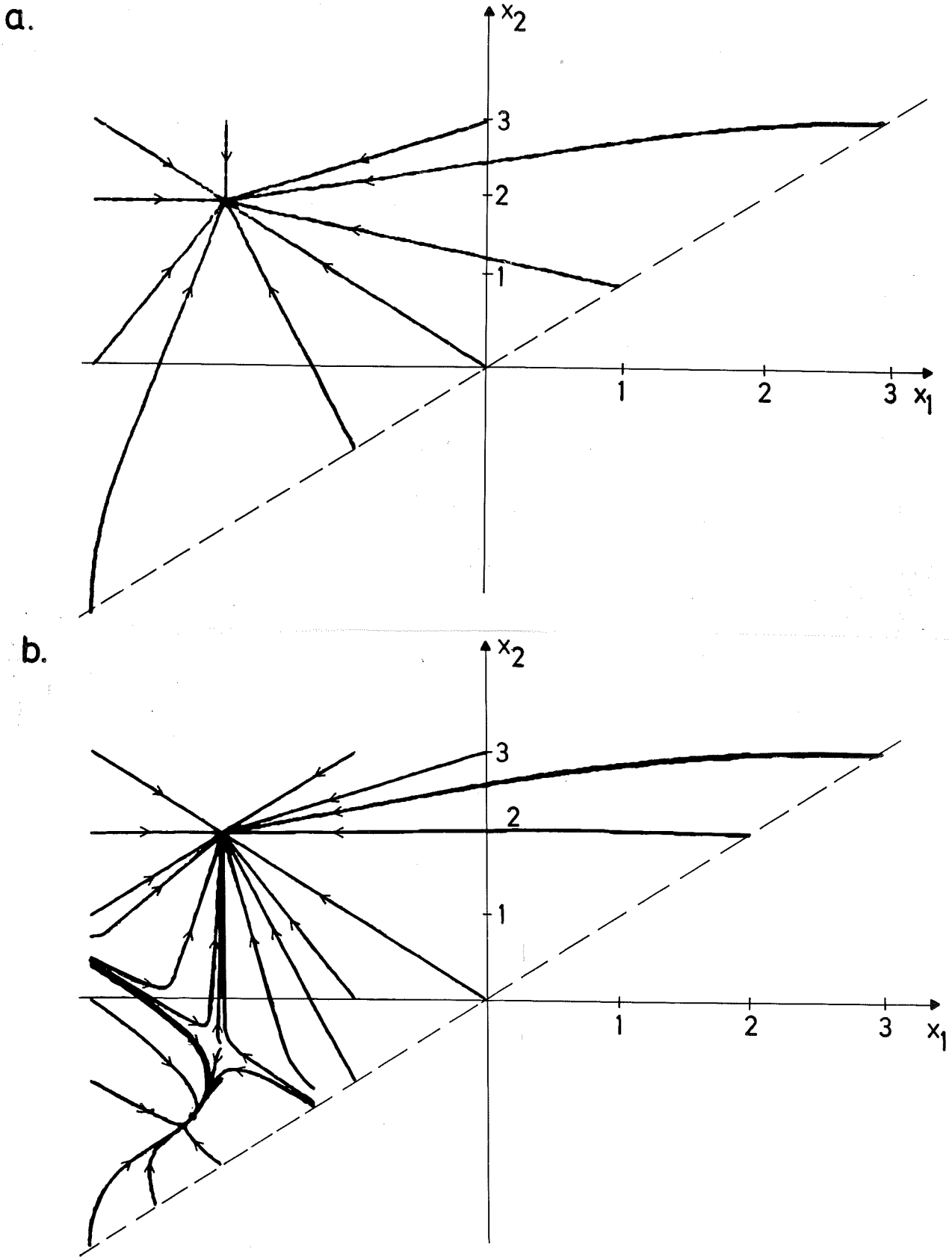


Fig. 7.3 - Trajectories for the ODE that corresponds to self-learning classification for the distributions defined in Example 7.2.

a) $\lambda = 0.5$ b) $\lambda = 0.99$

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APPENDIX A.

Proof of Theorem 3.1.

Before proceeding to the proof, let us first remark that although $z_n(x^0)$ converges w.p.1 for each fixed x^0, this does not imply convergence if x^0 is a random variable. To treat such problems in a strict way, introduce a denumerable subset of D

$$D_d = \left\{x^{(1)}, x^{(2)}, \ldots\right\} \subset D$$

which is dense in D.

Let Ω denote the sample space and denote the elements of Ω by ω.

Assumption a) implies that $z_n(x^{(i)})$ converges w.p.1, i.e. for all $\omega \in \Omega^{(i)}$, where $\Omega^{(i)}$ has measure 1. Let

$$\Omega^* = \bigcap_{i=1}^{\infty} \Omega^{(i)} \cap \Omega_A$$

where Ω_A is the set of all realizations for which r_n converges, condition b) is satisfied <u>and</u> $\gamma_n \to 0$; $\Sigma\gamma_n = \infty$. Then also Ω^* has measure 1. In the rest of the proof only such realizations ω are considered that belong to Ω^*.

The basic idea of the proof is that the sequence of estimates $\{x_n\}$ obtained from algorithm (3.1) behaves like solutions of the ODE (3.6) asymptotically and locally. This result is shown in the following lemma:

Lemma A.1. Let $\bar{x} \in D_d$ and $\omega \in \Omega^*$. Let $t \leq t_0$, where t_0 does not depend on $\bar{x}$ and ω. Define the sequence $m(n,t,\omega)$ so that

$$\sum_{k=n}^{m(n,t,\omega)} \gamma_k(\omega) \to t \text{ as } n \to \infty$$

Then, if $x_n(\omega)$ belongs to a closed subset of D^0,

$$x_{m(n,t,\omega)}(\omega) = x_n(\omega) + tf(\bar{x}) + R_1(t,n,\omega,\bar{x}) + R_2(t,n,\omega,\bar{x}) \quad (A.1)$$

where

$$|R_1(t,n,\omega,\bar{x})| < tK|x_n(\omega) - \bar{x}| + At^2$$

and

$$R_2(t,n,\omega,\bar{x}) \to 0 \text{ as } n \to \infty$$

Proof of Lemma A.1. Consider the sequence $\{z_n(\bar{x},\omega)\}$ defined by (3.4). Let $n \leq j(n,\omega) \leq m(n,t,\omega)$. Then

$$z_{j(n,\omega)}(\bar{x},\omega) = z_n(\bar{x},\omega) + \sum_{i=n+1}^{j(n,\omega)} \gamma_i(\omega)\left\{Q_i\left(\bar{x},e_i(\omega)\right) - z_{i-1}(\bar{x},\omega)\right\}$$

Let now n tend to infinity. Since

$$\lim_{n\to\infty} z_{j(n,\omega)}(\bar{x},\omega) = \lim_{n\to\infty} z_n(\bar{x},\omega) = f(\bar{x})$$

it follows that

$$\lim_{n\to\infty} \sum_{i=n+1}^{j(n,\omega)} \gamma_i(\omega)\left\{Q_i\left(\bar{x},e_i(\omega)\right) - z_{i-1}(\bar{x},\omega)\right\} = 0$$

or

$$\sum_{i=n+1}^{j(n,\omega)} \gamma_i(\omega) Q_i\left(\bar{x}, e_i(\omega)\right) - f(\bar{x}) \sum_{i=n+1}^{j(n,\omega)} \gamma_i(\omega) =$$

$$= R_3\left(j(n), n, \omega, \bar{x}\right) \tag{A.2}$$

where $R_3\left(j(n), n, \omega, \bar{x}\right) \to 0$ as $n \to \infty$.

Analogously

$$\lim_{n\to\infty} \sum_{n+1}^{m(n,\omega)} \gamma_i(\omega) K_i\left(e_i(\omega)\right) = r_\infty \lim_{n\to\infty} \sum_{n+1}^{m(n,\omega)} \gamma_i(\omega) = r_\infty t \tag{A.3}$$

where $r_\infty = \lim_{n\to\infty} r_n$

Consider now

$$x_{j(n,\omega)}(\omega) = x_n(\omega) + \sum_{n+1}^{j(n,\omega)} \gamma_i(\omega) Q_i\left(x_{i-1}(\omega), e_i(\omega)\right) =$$

$$= x_n(\omega) + \sum_{i=n+1}^{j(n,\omega)} \gamma_i(\omega) Q_i\left(\bar{x}, e_i(\omega)\right) +$$

$$+ \sum_{i=n+1}^{j(n,\omega)} \gamma_i(\omega) \left\{ Q_i\left(x_{i-1}(\omega), e_i(\omega)\right) - Q_i\left(\bar{x}, e_i(\omega)\right) \right\} \tag{A.4}$$

The first sum of the RHS of (A.4) can be approximated using (A.2). To approximate the second sum, use the Lipschitz continuity of Q_i:

$$\left| Q_i\left(x_{i-1}(\omega), e_i(\omega)\right) - Q_i\left(\bar{x}, e_i(\omega)\right) \right| \leq \left| x_{i-1}(\omega) - \bar{x} \right| K_i\left(e_i(\omega)\right)$$

(Assume for a moment that $x_i; n \leq i \leq j(n,\omega)$ belong to D^0, so that the Lipschitz continuity holds. This assumption is removed below.)

Hence

$$\left| \sum_{i=n+1}^{j(n,\omega)} \gamma_i(\omega) \left\{ Q_i\left(x_{i-1}(\omega), e_i(\omega)\right) - Q_i\left(\bar{x}, e_i(\omega)\right) \right\} \right| \leq$$

$$\leq \max_{n \leq i \leq j} |x_i(\omega) - \bar{x}| \sum_{i=n+1}^{m(n,t,\omega)} \gamma_i(\omega) K_i\left(e_i(\omega)\right) \leq$$

$$\leq \max_{n \leq i \leq m} |x_i(\omega) - \bar{x}| \{r_\infty t + R_4(t,n,\omega)\} \leq$$

$$\leq \left[\max_{n \leq i \leq m} |x_i(\omega) - x_n(\omega)|\right][r_\infty t + R_4(t,n,\omega)] \qquad (A.5)$$

where $R_4(t,n,\omega) \to 0$ as $n \to \infty$ according to (A.3).

Assume that

$$\max_{n \leq i \leq m(n,t,\omega)} |x_i(\omega) - x_n(\omega)| = S(n,t,\omega)$$

is attained for $i = j^*(n,\omega)$. Then taking $j = j^*$ and inserting (A.5) and (A.2) in (A.4) for this j gives

$$S(n,t,\omega) = |x_j(\omega) - x_n(\omega)| \leq \left| f(x) \sum_{n+1}^{j} \gamma_i(\omega) + R_3\left(j^{(n)}, n, \omega, \bar{x}\right) \right| +$$

$$+ [S(n,t,\omega) + |x_n(\omega) - \bar{x}|][r_\infty t + R_4(t,n,\omega)]$$

or

$$S(n,t,\omega)[1 - r_\infty t - R_4(t,n,\omega)] \leq |f(x)|t + R_3(j^{(n)},n,\omega,\bar{x}) +$$

$$+ |\bar{x} - x_n(\omega)|[r_\infty t + R_4(t,n,\omega)]$$

For sufficiently small t, $t < t_0$, and sufficiently large n, $r_\infty t + R_4(t,n,\omega) < 1/2$ and since $x_n(\omega)$ and $\bar{x} \in D$ we have

$$|x_n(\omega) - \bar{x}| \leq C_1$$

Hence

$$S(n,t,\omega) \leq 2[|f(x)| + r_\infty C_1]t + R_3(j(n),n,\omega,\bar{x}) + C_1 R_4(t,n,\omega) =$$

$$= C_2 t + R_5(j(n),n,\omega,\bar{x})$$

where $R_5(j(n),n,\omega,\bar{x}) \to 0$ as $n \to \infty$.

Now choose $j(n,\omega) = m(n,t,\omega)$ in (A.4) which gives, using (A.2) and (A.5)

$$|x_{m(n,t,\omega)}(\omega) - x_n(\omega) - tf(\bar{x})| \leq$$

$$\leq R_3(t,n,\bar{x},\omega) + |f(x)|\left[\sum_n^m \gamma_i - t\right] +$$

$$+ \left\{C_2 t + R_5(t,n,\bar{x},\omega) + |\bar{x} - x_n(\omega)|\right\}\left\{r_\infty t + R_4(t,n,\omega)\right\} =$$

$$= C_2 r_\infty t^2 + r_\infty t|\bar{x} - x_n(\omega)| + R_2(t,n,\bar{x},\omega)$$

where $R_2(t,n,\bar{x},\omega) \to 0$ as $n \to \infty$.

It now remains only to remove the assumption $x_i \in D^0$, $n \leq i \leq m(n,t,\omega)$. If this assumption does not hold, let

$i = \bar{j}(n,\omega)$ be the first time $x_i \notin D^0$. Then apply the results above to $j(n,\omega) = \bar{j}(n,\omega)$, which gives

$$|x_{\bar{j}(n,\omega)} - x_n(\omega)| \leq C_4 t + R_6 \text{ where } R_6 \to 0 \text{ as } n \to \infty.$$

For sufficiently small t, this contradicts the definition of $\bar{j}$.

□

It follows from the converse stability theorem (see Krasovskij (1963)) that assumption c) implies the existence of a function V(x) with properties

- V(x) is infinitely differentiable
- $0 \leq V(x) < 1 \Leftrightarrow x \in D_1$ and $V(x) = 0 \Leftrightarrow x = x^*$
- $\frac{d}{dt} V(x) = V'(x)f(x)$ is negative definite in D_1

Consider from now on a fixed realization $\omega \in \Omega^*$. All variables below depend on ω, but this argument will be suppressed.

An outline of the rest of the proof is as follows:

Step 1: A convergent subsequence $\{x_{n_k}\}$ tending to $\tilde{x}$ is considered. Then $\{x_{n_k}\}$ is close to $\tilde{x}$ infinitely often, and according to Lemma A.1, $x_{m(n_k,t)}$ will approximately be $x_{n_k} + tf(\tilde{x})$. This means that $V(x_{m(n_k,t)})$ is strictly less than $V(x_{n_k})$ if $\tilde{x} \neq x^*$. A complication in this step is that $\tilde{x}$ may not belong to D_d. The formal proof is somewhat lengthy and involves several elaborate choices of constants. The result is, however, intuitively clear. The proof of step 1 follows over the next few pages and extends to eq. (A.10).

Step 2: From the above result it is quite clear that x^* must be a cluster point to $\{x_n\}$, since $V(x_n)$ has a tendency to decrease everywhere in D except for $x = x^*$. That this actually is the case is shown in Lemma A.2.

Step 3: If there is another cluster point to $\{x_n\}$ than x^*, say $\hat{x}$, the sequence must move from x^* to $\hat{x}$ infinitely many times. But then $V(x_n)$ is increasing, which contradicts the result of step 1. Hence only one cluster point exists and convergence follows. The formal proof of this claim is given in Lemma A.3.

From condition b) there exists at least one cluster point $\tilde{x}$, to the sequence $\{x_k\}$ in D. Hence there is a subsequence x_{n_k} that tends to $\tilde{x}$ as $k \to \infty$. Since D_d is dense in D, there is for arbitrary $\varepsilon > 0$ an element $\bar{x} = \bar{x}(\tilde{x},\varepsilon) \in D_d$ such that $|\bar{x} - \tilde{x}| < \varepsilon/2$. Consequently

$$|x_{n_k} - \bar{x}| < \varepsilon \qquad k > K_0(\varepsilon) \tag{A.6}$$

Consider now

$$V\left(x_{m(n_k,t)}\right) - V(x_{n_k})$$

where m is defined as in Lemma A.1. Denote $n_k = k'$ and $m(n_k,t) = k''$, and use the mean value theorem. This gives

$$V(x_{k''}) - V(x_{k'}) = V'(\xi_k)\,(x_{k''}-x_{k'}) =$$

$$= V'(\bar{x})\,(x_{k''}-x_{k'}) + (\xi_k-\bar{x})^T V''(\xi_k')(x_{k''}-x_{k'}) \tag{A.7}$$

where

$$\xi_k = x_{k'} + \theta_1(x_{k''}-x_{k'}); \quad \xi_k' = x_{k'} + \theta_2(\xi_k-x_{k'}), \quad 0 \leq \theta_i \leq 1.$$

Now apply Lemma A.1 to $x_{k''} - x_{k'}$ which gives

$$x_{k''} - x_{k'} = tf(\bar{x}) + R_1(t,n_k,\bar{x}) + R_2(t,n_k,\bar{x})$$

where

$$|R_1(t,n_k,x)| < tK|x_{n_k} - \bar{x}| + At^2 \qquad (A.8)$$

and

$$R_2(t,n_k,x) \to 0 \text{ as } k \to \infty \qquad (A.9)$$

Insert this in (A.7):

$$V(x_{k''}) - V(x_{k'}) = tV'(\bar{x})\, f(\bar{x}) + R_6(t,n_k,\bar{x})$$

where

$$R_6(t,n_k,\bar{x}) = (\xi_k - \bar{x})^T V''(\xi_k')(x_{k''} - x_{k'}) + V'(\bar{x})\,\{R_1 + R_2\}$$

Now suppose that the cluster point $\tilde{x}$ is different from the desired convergence point x^*. Then $V'(\tilde{x})f(\tilde{x}) = -\delta$, $\delta > 0$. This implies that $\exists\ \varepsilon_0$ such that

$$V'(\bar{x})f(\bar{x}) < -\delta/2 \quad \text{if} \quad |\bar{x} - \tilde{x}| < \varepsilon_0$$

Denote

$$\sup_{|\xi - \bar{x}| < \varepsilon_0} |V''(\xi)| = C_1, \quad \sup_{x \in D} |V'(x)| = C_3,$$

$$|f(\bar{x})| + At_0 + \varepsilon = C_2(\varepsilon)$$

Then

$$|(\xi_k-\bar{x})^T V''(\xi_k')(x_{k''}-x_{k'})| \leq C_1[tC_2(\varepsilon) + R_2]^2$$

First choose $\varepsilon = \min(\varepsilon_0, \delta/(4C_3K))$ and $k > K_0(\varepsilon)$. Then

$$|V'(\bar{x})R_1| < t[\delta/4 + AC_3t]$$

$$|R_6(t,n_k,\bar{x})| \leq C_1[tC_2(\varepsilon) + R_2]^2 + t(\delta/4+At) + R_2 =$$

$$= t\delta/4 + t^2(C_1C_2^2(\varepsilon) + C_3A) + R_2C_3 +$$

$$+ C_1R_2^2 + 2R_2C_1C_2(\varepsilon)$$

Now choose

$$t \leq \frac{\delta}{8C_2^2(\varepsilon) + C_3A}$$

which gives

$$|R_6(t,n_k,\bar{x})| \leq 3t\delta/8 + R_2C_3 + C_1R_2^2 + 2C_1C_2(\varepsilon)R_2$$

Finally choose $K > K_0(\varepsilon)$ so that

$$R_2C_3 + C_1R_2^2 + 2C_1C_2(\varepsilon)R_2 < t\delta/16 \quad \text{for } k > K$$

which is possible since $R_2(t,n_k,x) \to 0$ as $k \to \infty$.

Hence

$$V(x_{k''}) - V(x_{k'}) < -t\delta/2 + R_6(t,n_k,\bar{x}) < -t\delta/32$$

or

$$V\left(x_{m(n_k,t)}\right) < V\left(x_{n_k}\right) - t\delta/32 \qquad k > K$$

Since $x_{n_k} \to \tilde{x}$ as $k \to \infty$ and V is continuous this implies

$$V\left(x_{m(n_k,t)}\right) < V(\tilde{x}) - t\delta/64 \qquad k > K_1 \tag{A.10}$$

This means that if $\tilde{x}$ is a cluster point different from x^* the sequence x_n will i.o. be strictly interior to $\{x | V(x) \leq V(\tilde{x}) - t\delta/64\}$. This region is compact. Consequently another cluster point must exist that yields a smaller value of V. In Lemma A.2 it is shown that this implies that also x^* must be a cluster point, i.e. that

$$\liminf_{n\to\infty} V(x_n) = 0 \tag{A.11}$$

To conclude the proof it must also be shown that

$$\limsup_{n\to\infty} V(x_n) = 0 \tag{A.12}$$

This is done in Lemma A.3.

Lemma A.2. Suppose (A.10) holds for any subsequence $\{x_{n_k}\}$ that converges to a point different from x^*. Then (A.11) holds.

Proof. Consider inf V(x) taken over all cluster points in D. Let this value be U. Since the set of cluster points in D is compact, there exists a cluster point $\hat{x}$, such that $V(\hat{x}) = U$. If now $U > 0$, $V'(\hat{x})f(\hat{x})$ will be strictly negative ($= -\delta$) and from (A.10) $V(x_k)$ takes a value less than $U - \delta t/64$ infinitely often, which contradicts U being the infimum. Hence $U = 0$, which means that x^* is a cluster point.

□

Lemma A.3. From (A.10) and (A.11) it follows that

$$\limsup_{n\to\infty} V(x_n) = 0$$

Proof. If $x_n \in D$ the difference

$$|x_{n+1} - x_n| = |\gamma_n Q_n(x_n, e_{n+1})| \leq \gamma_n |Q_n(x^*, e_{n+1})| +$$

$$+ |x_n - x^*| \gamma_n |K_n(e_{n+1})| \leq$$

$$\leq |z_{n+1}(x^*) - z_n(x^*)| + \gamma_n |z_n(x^*)| +$$

$$+ |x_n - x^*| \left\{ |r_{n+1} - r_n| + \gamma_n r_n \right\}$$

tends to zero since $z_n(x^*)$ and r_n converge. Suppose that

$$\limsup_{n\to\infty} V(x_n) = A > 0$$

Consider the interval $I = [A/3, 2A/3]$ [1]. This interval is then crossed "upwards" and "downwards" infinitely many times. Since the step size $x_{n+1} - x_n$ tends to zero when $x_n \in D$, there will be a subsequence of $V(x_n)$ that belongs to I. Consider now such a special convergent sequence of "upcrossings". Let $\{x_{n'_k}\}$ be defined as follows:

$$V(x_{n'_k - 1}) < A/3 \qquad V(x_{n'_k}) \geq A/3 \qquad V(x_{n'_k + s_k}) > 2A/3$$

where s_k is the first s for which $V(x_{n'_k + s}) \notin I$ and $x_{n'_k} \to \tilde{x}$ as $k \to \infty$. Clearly $V(\tilde{x}) = A/3$.

---

[1] If $A > 1$ take $I = [1/3, 2/3]$.

Now, from (A.10)

$$V\left(x_{m(n_k',t)}\right) < A/3 - \delta t/64$$

This means that $V(x_{n_k'+s_k}) \notin I$ where $s_k = m(n_k',t) - n_k'$. But, if t is sufficiently small, no s, smaller than s_k can have made $V(x_{n_k'+s}) > 2A/3$, according to Lemma A.1 and the continuity of V. This contradicts the definition of the subsequence n_k'.

Hence no interval I can exist, A must be zero and the lemma follows.

□

Lemma A.3 implies that $x_n \to x^*$ for the chosen realization. The set of all ω for which this holds, Ω^*, has measure 1. This concludes the proof of the theorem.

□

Proof of Corollary 2. The proof of Corollary 2 is very much like the proof of the theorem. Eq. (A.10) is obtained straightforwardly if $\tilde{x} \notin D_c$. The only differences are caused by the weaker properties of V(x) in this case. In particular need the area $\{x|V(x) \leq V(\tilde{x})\}$ not be bounded. Therefore the closure of the set

$$\{x|V(x) \leq V(\tilde{x})\} \cap D$$

is considered instead. The conclusion of the corresponding Lemma A.2 therefore is that inf V(x) taken over the cluster points in D is assumed at a cluster point $\hat{x}$, such that either $\hat{x} \in \partial D$ or $\hat{x} \in D_c$.

If $\hat{x} \in \partial D$, eq. (A.10) implies that $\{x_n\}$ is outside D infinitely often. Since it also is inside D infinitely often,

there must be another cluster point $\hat{\hat{x}}$ on the boundary. At this point the trajectories are pointing into the region D and $V(x)$ is decreasing. This contradicts $\hat{x}$ giving the infimum of $V(\hat{x})$ since $V(\hat{x}) = V(\hat{\hat{x}})$.

Hence $\hat{x} \in D_c$ is a cluster point. Now suppose that there is another cluster point $\tilde{x} \notin D_c$. Then clearly, $V(\tilde{x}) > V(\hat{x})$ and $\{x_n\}$ moves from $\hat{x}$ to $\tilde{x}$ infinitely many times, corresponding to increasing $V(x)$. As in Lemma A.3 this is contradicted, and the corollary follows.

□

APPENDIX B

Proof of Lemma 4.2.

Lemma 4.2. Consider the algorithm (4.3)

$$y_n = y_{n-1} + \gamma_n(f_n - y_{n-1}) \qquad y_0 = 0 \tag{B.1}$$

Assume that the sequence $\{\gamma_n\}$ satisfies (4.8). Assume further that f_n satisfies (4.10) and that $\{\alpha_n\}$ is a non decreasing sequence of numbers and

$$E|e_k|^p < C$$

which implies

$$E|f_n|^p < C'\alpha_n^p$$

where p is an even integer. Then

$$E|y_n|^r \leq K_r(\alpha_n)^r(\gamma_n)^{r/2} \qquad 1 < r \leq p \tag{B.2}$$

Proof It is evidently sufficient to show (B.2) for r = p, since Lyapunov's inequality

$$\left(E|x|^r\right)^{1/r} \leq \left(E|x|^{r'}\right)^{1/r'} \qquad 1 < r \leq r' < \infty$$

gives

$$\left[E\left|\frac{y_n}{\alpha_n\sqrt{\gamma_n}}\right|^r\right]^{1/r} \leq \left[E\left|\frac{y_n}{\alpha_n\sqrt{\gamma_n}}\right|^p\right]^{1/p} \leq (K_p)^{1/p} \qquad 1 < r \leq p$$

and so

$$E(|y_n|^r) \leq (K_p)^{r/p} \alpha_n^r \gamma_n^{r/2}$$

The solution of (B.1) can be written

$$y_n = \left[\prod_{k=m+1}^{n} (1-\gamma_k)\right] y_m + \sum_{k=m+1}^{n} \beta_k^n f_k \qquad (B.3)$$

where β_k^n is defined by (4.5).

Then, according to (4.8c)

$$\beta_k^n \leq \beta_n^n = \gamma_n \qquad k \leq n$$

Choose a subsequence n_k, such that

$$\sum_{i=n_k+1}^{n_{k+1}} \gamma_i \to t \quad \text{as } k \to \infty$$

This is possible since

$$\sum_{1}^{\infty} \gamma_i$$

diverges and $\gamma_n \to 0$. From (4.8d) we have

$$\sum_{n_{k-1}}^{n_k} \gamma_i \geq (n_k - n_{k-1})\gamma_{n_k}$$

and thus

$$(n_k - n_{k-1}) \leq K_1 / \gamma_{n_k} \tag{B.4}$$

Introduce

$$T_k = \sum_{i=n_{k-1}+1}^{n_k} \beta_i^{n_k} f_i$$

The lemma will now be proved by first (Lemma B.1) estimating $E|T_k|^p$ and then extending (Lemma B.2) this estimate to y_n.

Lemma B.1. With the assumptions of the theorem

$$E|T_k|^p \leq K_p \alpha_{n_k}^p \gamma_{n_k}^{p/2}$$

Proof of Lemma B.1.

$$E\, T_k^p \leq \sum_{j_1=n_{k-1}+1}^{n_k} \cdots \sum_{j_p=n_{k-1}+1}^{n_k} \beta_{j_1}^{n_k} \cdot \ldots \cdot \beta_{j_p}^{n_k} |Ef_{j_1} \cdot \ldots \cdot f_{j_p}| \leq$$

$$\leq \gamma_{n_k}^p \sum_{j_1} \cdots \sum_{j_p} |Ef_{j_1} \cdot \ldots \cdot f_{j_p}| \tag{B.5}$$

Now consider

$$\sum_{j_1} \cdots \sum_{j_p} |Ef_{j_1} \cdot \ldots \cdot f_{j_p}| \leq$$

$$\leq \sum_{j_1} \cdots \sum_{j_p} \left[\sum_{k_1=0}^{\infty} \cdots \sum_{k_p=0}^{\infty} |h_{k_1,j_1} \cdots h_{k_p,j_p} Ee_{j_1-k_1} \cdot \ldots \cdot e_{j_p-k_p}| \right] \leq$$

$$\leq \sum_{j_1} \cdots \sum_{j_p} \left[\sum_{k_1=0}^{\infty} \cdots \sum_{k_p=0}^{\infty} \alpha_{n_k}^p \lambda^{k_1+\ldots+k_p} |Ee_{j_1-k_1} \cdot \ldots \cdot e_{j_p-k_p}| \right]$$

Consider

$$Ee_{j_1-k_1} \cdot \ldots \cdot e_{j_p-k_p}$$

If some index j_i-k_i occurs only once in the product above, the expectation value is zero, since e_n and e_m are independent with mean value zero for $n \neq m$.

Suppose that $k_1,\ldots,k_p$ are fixed. The term $Ee_{j_1-k_1} \cdot \ldots \cdot e_{j_p-k_p}$ is independent of the order of the indices. Each term can be obtained in at most $p!$ ways. In every non zero term each index occurs at least twice. Therefore

$$\sum_{j_1} \cdots \sum_{j_p} |Ef_{j_1} \cdots f_{j_p}| \leq p! \alpha_{n_k}^p \sum_{j_1}' \cdots \sum_{j_p}' \sum_{k_1} \cdots \sum_{k_p} \lambda^{k_1+\ldots+k_p} \cdot$$

$$\cdot |Ee_{j_1-k_1} \cdot \ldots \cdot e_{j_p-k_p}|$$

where the prime denotes that the summation is restricted to terms for which each index j_i-k_i, $1 \leq i \leq p/2$, equals at least one of the indices j_n-k_n, $p/2+1 \leq n \leq p$, and vice versa.

Suppose now that $k_1,\ldots,k_p,j_1,\ldots,j_{p/2}$ are fixed in the sum above. Then the number of terms that are obtained as $j_{p/2+1},\ldots,j_p$ vary over $[n_{k-1},n_k]$ is less than or equal to $(p/2)^{p/2}$, and independent of n_k-n_{k-1}.

Since $|Ee_{j_1-k_1} \cdot \ldots \cdot e_{j_p-k_p}| \leq E|e|^p \leq C$ this gives

$$\sum_{j_1} \cdots \sum_{j_p} |Ef_{j_1} \cdot \ldots \cdot f_{j_p}| \leq \alpha_{n_k}^p p! \, (p/2)^{p/2} C \sum_{j_1} \cdots \sum_{j^{p/2}} \left(\sum_{k=0}^{\infty} \lambda^k \right)^p \leq$$

$$\leq \alpha_{n_k}^p C'(p,\lambda)(n_k - n_{k-1})^{p/2}$$

and from (B.5)

$$ET_k^p \leq \gamma_{n_k}^p \alpha_{n_k}^p (n_k - n_{k-1})^{p/2} C(\lambda) \leq C \gamma_{n_k}^{p/2} \alpha_{n_k}^p$$

□

Lemma B.2. Under the assumptions of the theorem

$$E|T_k|^p < B_1 \alpha_{n_k}^p \gamma_{n_k}^{p/2}$$

implies

$$E|y_{n_k}|^p < B_2 \alpha_{n_k}^p \gamma_{n_k}^{p/2} \quad \text{where } B_2 = B_1/(1-e^{t/4})^p \tag{B.6}$$

Remark. This result holds for general $p > 0$.

Proof of Lemma B.2. Introduce

$$A_k = \prod_{i=n_{k-1}+1}^{n_k} (1-\gamma_i)$$

It follows that $e^{-2t} < A_k < e^{-t/2}$ for sufficiently large k. Then from (B.3)

$$y_{n_k} = T_k + A_k y_{n_{k-1}} \tag{B.7}$$

Suppose that (B.6) is valid for k-1:

$$\left[E|y_{n_{k-1}}|^p\right]^{1/p} \leq B_2^{1/p}\gamma_{n_{k-1}}^{1/2}\alpha_{n_{k-1}}$$

Minkowski's inequality, applied to (B.7) then yields

$$\left[E|y_{n_k}|^p\right]^{1/p} \leq \left[E|T_k|^p\right]^{1/p} + A_k\left[E|y_{n_{k-1}}|^p\right]^{1/p}$$

Now

$$A_k\gamma_{n_{k-1}} = \beta_{n_{k-1}}^{n_k} \leq \beta_{n_k}^{n_k} = \gamma_{n_k}$$

since γ_k satisfies (4.8). Hence

$$\left[E|y_{n_k}|^p\right]^{1/p} \leq \left[B_1^{1/p}\gamma_{n_k}^{1/2} + A_k^{1/2}B_2^{1/p}\gamma_{n_k}^{1/2}\right]\alpha_{n_k} \leq$$

$$\leq \left[B_1^{1/p} + e^{t/4}B_2^{1/p}\right]\gamma_{n_k}^{1/2}\alpha_{n_k} = B_2^{1/p}\gamma_{n_k}^{1/2}\alpha_{n_k}$$

Consequently

$$E|y_{n_k}|^p \leq B_2\gamma_{n_k}^{p/2}\alpha_{n_k}^p \qquad k = 1,2,\ldots$$

since

$$E|y_{n_1}|^p = E|T_1|^p < B_1\gamma_{n_1}^{p/2}\alpha_{n_1}^p < B_2\gamma_{n_1}^{p/2}\alpha_{n_1}^p$$

□

To complete the proof of the theorem it remains only to be shown that (B.2) is valid not only for the subsequence n_k as in (B.6) but for all values of n. But for $n_{k-1} < j \leq n_k$

$$y_j = \prod_{i=n_{k-1}+1}^{j}(1-\gamma_i)y_{n_{k-1}} + \prod_{i=j+1}^{n_k}(1-\gamma_i)^{-1}\sum_{i=n_{k-1}+1}^{j}\beta_i^{n_k}f_i =$$

$$= \prod_{i=j+1}^{n_k}(1-\gamma_i)^{-1}\left[A_k y_{n_{k-1}} + \sum_{n_{k-1}+1}^{j}\beta_i^{n_k} f_i\right]$$

As in (B.5)

$$E\left|\sum_{n_{k-1}+1}^{j}\beta_i^{n_k}f_i\right|^p \leq \gamma_{n_k}^p \sum_{j_1=n_{k-1}+1}^{j}\cdots\sum_{j_p=n_{k-1}+1}^{j}|Ef_{j_1}\ldots f_{j_p}| \leq$$

$$\leq C'\gamma_{n_k}^{p/2}\alpha_j^p$$

Application of Minkowski's inequality as in Lemma B.2 now gives

$$(E|y_j|^p)^{1/p} \leq \left[\prod_{n_{k-1}+1}^{n_k}(1-\gamma_i)^{-1}\right]B_2\gamma_{n_k}^{1/2}\alpha_j$$

and hence

$$E|y_j|^p \leq e^{2pt}B_2\gamma_j^{p/2}\alpha_j^p = C'\gamma_j^{p/2}\alpha_j^p$$

□

APPENDIX C.

Proof of Theorem 4.3.

Theorem 4.3. Consider the algorithm (4.3)

$$y_n = y_{n-1} + \gamma_n(f_n - y_{n-1})$$

where $\{f_n\}$ is a sequence of independent random variables. Suppose $\{\gamma_n\}$ satisfies (4.8). Let $E|f_n|^p \leq \alpha_n^p$ for some real $p > 1$, where $\{\alpha_n\}$ is a nondecreasing sequence and suppose that

$$\sum_{n=1}^{\infty} \gamma_n^{p'} \alpha_n^p < \infty \quad \text{where } p' = \min(p, 1+p/2)$$

Then

$$y_n \to 0 \quad \text{as} \quad n \to \infty \text{ w.p.1}$$

Proof. As in the proof of Theorem 4.1 we will introduce

$$T_k = \sum_{n_{k-1}+1}^{n_k} \beta_i^{n_k} f_i \; ; \qquad \sum_{n_{k-1}}^{n_k} \gamma_j \to t \quad \text{as } k \to \infty$$

and first obtain estimates of $E|T_k|^p$ where p is a real number > 1. Thus Lemma B.1 cannot be applied, since p is there assumed to be an even integer. Consider first $1 < p \leq 2$. In this case a theorem due to von Bahr-Esseen (1965) is applicable:

Suppose S_n is a sum of independent variables

$$S_n = \sum_1^n e_i$$

Then

$$E|S_n|^p \leq (2-1/n) \sum_1^n E|e_i|^p \qquad 1 < p \leq 2$$

In our case:

$$E|T_k|^p \leq 2 \sum_{j=n_{k-1}+1}^{n_k} \left(\beta_j^{n_k}\right)^p E|f_j|^p \leq K \sum_{j=n_{k-1}+1}^{n_k} \gamma_j^p \alpha_j^p$$

Clearly, using Chebysjev's inequality and the Borel Cantelli lemma

$$\sum_{j=1}^{\infty} (\gamma_j \alpha_j)^p < \infty \quad \text{implies that } T_k \to 0 \text{ as } k \to \infty \text{ w.p.1.}$$

As follows from the lemma below, this implies that $y_n \to 0$ as $n \to \infty$ w.p.1.

Lemma C.1 (Petrov (1972)). Let $B_n \to \infty$ as $n \to \infty$, and suppose there exists a subsequence B_{n_k} and constants $c_1 > 1$ and c_2, such that

$$c_1 \leq \frac{B_{n_k}}{B_{n_{k+1}}} \leq c_2$$

for all sufficiently large k. Let S_n be a sum of independent variables with zero mean value. Let $T_k = (S_{n_k} - S_{n_{k-1}})B_{n_k}$.

Then

$$B_n S_n \to 0 \text{ w.p.1} \quad \text{as } n \to \infty$$

if and only if

$$T_k \to 0 \text{ w.p.1} \quad \text{as } n \to \infty$$

□

Introduce

$$\delta_n = \gamma_n \prod_{i=1}^{n} (1-\gamma_i)^{-1}$$

$$S_n = \sum_{1}^{n} \delta_k f_k$$

$$B_n = \prod_{1}^{n} (1-\gamma_i)$$

Then $y_n = B_n S_n$, $\beta_k^n = \delta_k B_n$ and

$$(S_{n_k} - S_{n_{k-1}}) B_{n_k} = \sum_{n_{k-1}+1}^{n_k} \delta_j B_{n_k} f_j = \sum_{n_{k-1}+1}^{n_k} \beta_j^{n_k} f_j = T_k$$

The proof of the assertion now follows from Lemma C.1 in case $1 < p \leq 2$.

Now, we turn to the case $p > 2$. The following result, due to Dharmadhikari and Jogdeo (1969) will be used:

Let $\{e_i\}$ be a sequence of independent zero mean valued random variables

$$S_n = \sum_1^n e_i$$

Then

$$E|S_n|^p \leq C_p n^{p/2-1} \sum_1^n E|e_i|^p \qquad p \geq 2$$

Applying this result to T_k the following estimates are obtained:

$$E|T_k|^p \leq C(n_k-n_{k-1})^{(p/2)-1} \sum_{n_{k-1}+1}^{n_k} (\beta_j^{n_k})^p \alpha_j^p E|f_j|^p \leq$$

$$\leq C_p (n_k-n_{k-1})^{p/2} \gamma_{n_k}^p \alpha_{n_k}^p F_p \leq K_p' \gamma_{n_k}^{p/2} \alpha_{n_k}^p$$

The second inequality follows from property (4.8). The third inequality follows from (B.4). From Lemma B.2 now follows that also

$$E|y_n|^p \leq K_p \gamma_n^{p/2} \alpha_n^p \qquad (C.3)$$

The estimate (C.3) will now be applied to the following lemma by Zolotarev (1965).

Lemma C.2 (Zolotarev (1965)). Let $e_1,\ldots,e_n$ be independent random variables with zero mean values. Let

$$0 \leq B_n \leq B_{n-1} \leq \ldots \leq B_1$$

Let $\varphi(x)$ be a function defined for $x \geq 0$, nondecreasing,

convex and

$$\varphi(0) = 0 \qquad \varphi(xy) \leqslant \varphi(x)\varphi(y) \qquad x,y \geqslant 0$$

Let

$$S_n = \sum_1^n e_i$$

Then

$$P(\max_{m\leqslant k\leqslant n} B_k S_k \geqslant x) \leqslant \frac{1}{\varphi(x)}\Big\{\varphi(B_n)E^+\varphi(S_n) + \sum_{k=m}^{n-1}(\varphi(B_k) - \varphi(B_{k+1}))E^+\varphi(S_k)\Big\}$$

for any x.

Here

$$E^+\varphi(x) = \int_0^\infty \varphi(x)dF(x)$$

where F(x) is the distribution function for x. □

Take now in Lemma C.2.

$$B_n = \prod_1^n (1-\gamma_j); \quad \varphi(x) = |x|^p \qquad S_n = \sum_1^n \delta_k f_k$$

where

$$\delta_k = \gamma_k \prod_{i=1}^{k} (1-\gamma_i)^{-1}$$

Then $y_n = c_n S_s$ and

$$P(\max_{m\leq k\leq n} |y_k| > \varepsilon) \leq \frac{1}{|\varepsilon|^p}\left\{E|y_n|^p + \sum_{k=m}^{n-1} (B_k^p - B_{k+1}^p)E|S_k|^p\right\}$$

Now

$$B_k^p - B_{k+1}^p = B_k^p[1-(1-\gamma_{k+1})^p] \leq p\gamma_{k+1}C_k^p \text{ since } 0 \leq \gamma_{k+1} \leq 1$$

Choose a subsequence n_k such that

$$\sum_1^\infty \gamma_{n_k}^{p/2}\alpha_{n_k}^p < \infty$$

This is possible since

$$\sum_1^\infty \gamma_n^{p'}\alpha_n^p < \infty \Rightarrow \gamma_n\alpha_n \to 0$$

Then

$$\sum_{k=1}^\infty P(\max_{n_k\leq j\leq n_{k+1}} |y_j| > \varepsilon) \leq \frac{1}{|\varepsilon|^p}\left\{\sum_{k=1}^\infty E|y_{n_k}|^p + \sum_1^\infty p\gamma_k E|y_k|^p\right\} \leq$$

$$\leq C\left\{\sum_1^\infty \gamma_{n_k}^{p/2}\alpha_{n_k}^p + \sum_1^\infty \gamma_k^{p/2+1}\alpha_k^p\right\} < \infty$$

and consequently $y_n \to 0$ w.p.1 as $n \to \infty$.

This completes the proof ofethe theorem. □

APPENDIX D

Proof of Theorem 6.1.

Lemma A.1 states that the sequence $\{x_n\}$ locally and asymptotically follows the trajectories of (6.17). This result will be extended to global estimates by linking a chain of local estimates (A.1).

Order the set of indices $I = \{n_i\}$ such that $n_1 < n_2 < \ldots < n_k < n_{k+1} < \ldots$ Denote $\Delta\tau_k = \tau_{n_{k+1}} - \tau_{n_k}$. Then, taking $\bar{x} = x_n$ in (A.1) yields

$$x_{n_{k+1}} = x_{n_k} + \Delta\tau_k f(x_{n_k}) + A\Delta\tau_k^2 + R_2(\Delta\tau_k, n_k, x_{n_k}) \qquad (D.1)$$

By going through the details of the proof of Lemma A.1 it is found that

$$R_2(\Delta\tau_k, n_k, x_{n_k}) = \sum_{i=n_k+1}^{n_{k+1}} \gamma_i \Big\{ \big[2Q(x_{n_k}, e_i) - K(e_i)\big] - E\big[2Q(x_{n_k}, e_i) - K(e_i)\big] \Big\}$$

Applying Lemma 4.2 gives

$$E|R_2|^{2p} \leq L_1 \gamma_{n_k}^p$$

and so from Chebysjev's inequality

$$P\big(|R_2(\Delta\tau_k, n_k, x_{n_k})| > \varepsilon_1\big) \leq L_1 \gamma_{n_k}^p / \varepsilon_1^{2p} \qquad (D.2)$$

Also,

$$\tilde{x}(\tau_{n_k+1};\tau_{n_k},x_{n_k}) = x_{n_k} + \Delta\tau_k f(x_{n_k}) + L_2\Delta\tau_k^2 \qquad (D.3)$$

Combine (D.1) and (D.3):

$$|x_{n_{k+1}} - \tilde{x}(\tau_{n_{k+1}};\tau_{n_k},x_{n_k})| \leq (A+L_2)\Delta\tau_k^2 + R_2(\Delta\tau_k,n_k,x_k) \qquad (D.4)$$

Define $L_3 = A + L_2$.

According to the assumptions of the theorem there exists a function $V(\Delta x,\tau)$ which is quadratic in Δx and such that

$$C_1|\Delta x|^2 \leq V(\Delta x,\tau) \leq C_2|\Delta x|^2 \qquad (D.5)$$

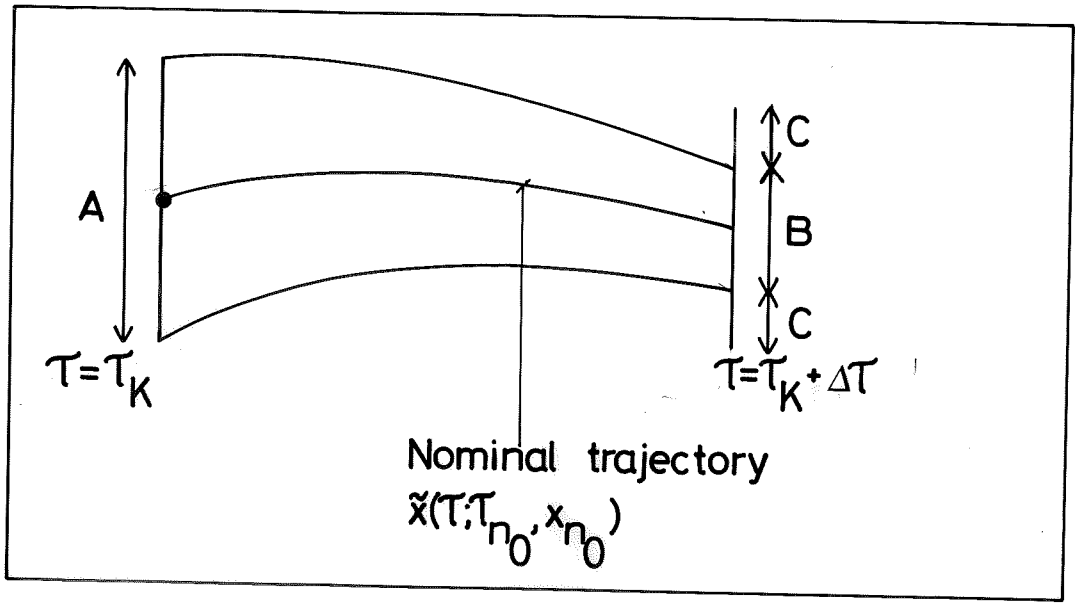
and

$$\frac{d}{d\tau} V(\Delta x,\tau) < -\lambda|\Delta x|^2 ; \quad \lambda > 0 \qquad (D.6)$$

along solutions of the variational equation.

We will now give an outline and a heuristic interpretation of the rest of the proof before we proceed to the formal treatment.

The idea of the proof can geometrically be expressed as follows:



Assume that the estimate at time τ_k is in the interval A. The trajectories that start in A belong at time $\tau_k + \Delta\tau_k$ to the interval B. The length of B is given by (D.6). If $V(\Delta x,\tau) = |\Delta x|^2$, then $B \leq (1-\lambda\Delta\tau_k)A$. Now, the estimates obtained by the algorithm differ from the trajectories with less than $L_3\Delta\tau_k^2 + R_2(\Delta\tau_k,n_k,x_{n_k})$ according to (D.4). Denote this distance by C. During the time interval $\Delta\tau_k$, the estimates have not diverged from the nominal trajectory if $A \leq B + 2C$, i.e. if

$$A \leq (1-\lambda\Delta\tau_k)A + L_3\Delta\tau_k^2 + R_2(\Delta\tau_k,n_k,x_{n_k})$$

or

$$\lambda\Delta\tau_k A \leq L_3\Delta\tau_k^2 + R_2(\Delta\tau_k,n_k,x_{n_k})$$

To achieve this, A and $\Delta\tau_k$ must be chosen with care. The interval $\Delta\tau_k$ must be large enough to let the trajectories converge sufficiently, and small enough to limit second order effects and the noise influence.

We now turn to the formal proof.

Select first ε (corresponding to A in the discussion above) such that

$$\varepsilon < \frac{8DL_3C_1C_2}{\lambda} = \varepsilon_0$$

Since $\Delta\tau_k > D$, it follows that

$$\frac{\lambda\varepsilon}{C_1C_2L_3 \cdot 8} < \Delta\tau_k \qquad \text{all } k.$$

Possibly by extending the set I, it is thus possible to obtain

$$\frac{\lambda\varepsilon}{C_1C_2L_3 \cdot 8} < \Delta\tau_k < \frac{3\lambda\varepsilon}{C_1C_2L_3 \cdot 8} \tag{D.7}$$

Now suppose that

$$|R_2(\Delta\tau_k, n_k, x_{n_k})| < \frac{\lambda^2\varepsilon^2 3}{L_3C_2^2C_1^2 \cdot 64}$$

and

$$V^{1/2}[(\tilde{x}(\tau_{n_k};\tau_{n_0},x_{n_0}) - x_{n_k}),\tau_{n_k}] \leq \varepsilon$$

Then

$$V^{1/2}\left[\left(\tilde{x}(\tau_{n_{k+1}};\tau_{n_0},x_{n_0}) - x_{n_{k+1}}\right),\tau_{n_{k+1}}\right] \leq$$

$$\leq V^{1/2}\left[\left\{\tilde{x}(\tau_{n_{k+1}};\tau_{n_0},x_{n_0}) - \tilde{x}(\tau_{n_{k+1}};\tau_{n_k},x_{n_k})\right\},\tau_{n_{k+1}}\right] +$$

$$+ V^{1/2}\left[\left[\tilde{x}(\tau_{n_{k+1}};\tau_{n_k},x_{n_k}) - x_{n_{k+1}}\right],\tau_{n_{k+1}}\right] \leq$$

$$\leq \left(1 - \frac{\lambda}{2C_1}\Delta\tau_k\right)V^{1/2}\left[\left(\tilde{x}(\tau_{n_k};\tau_{n_0},x_{n_0}) - x_{n_k}\right),\tau_{n_k}\right] +$$

$$+ C_2\left|\tilde{x}(\tau_{n_{k+1}};\tau_{n_k},x_{n_k}) - x_{n_{k+1}}\right| \leq$$

$$\leq \left(1 - \frac{\lambda}{2C_1}\Delta\tau_k\right)\varepsilon + C_2L_3\Delta\tau_k^2 + C_2\left|R_2(\Delta\tau_k,n_k,x_{n_k})\right| \leq$$

$$\leq \varepsilon + C_2L_3\left\{\left(\Delta\tau_k - \frac{\lambda\varepsilon}{8C_1C_2L_3}\right)\left(\Delta\tau_k - \frac{3\lambda\varepsilon}{8C_1C_2L_3}\right)\right\} \leq \varepsilon$$

The first inequality follows since $V^{1/2}$ is a norm, the second follows from the properties of V. The third and fourth inequalities follow from the assumptions made just above. The last inequality follows from (D.7). In other words, if

$$V^{1/2}\left[\left(\tilde{x}(\tau_{n_k};\tau_{n_0},x_{n_0}) - x_{n_k}\right),\tau_{n_k}\right] \leq \varepsilon$$

then

$$V^{1/2}\left[\left(\tilde{x}(\tau_{n_{k+1}};\tau_{n_0},x_{n_0}) - x_{n_{k+1}}\right),\tau_{n_{k+1}}\right] \leq \varepsilon$$

with probability at least

$$P\left\{|R_2(\Delta\tau_k, n_k, x_{n_k})| < \frac{3\lambda^2\varepsilon^2}{64C_1^2C_2^2L_3}\right\} >$$

$$> 1 - L_1\left(\frac{64L_3C_2^2C_1^2}{3\lambda^2}\right)^{2p} \gamma_{n_k}^p/\varepsilon^{4p}$$

Now the event

$$\Omega = \{\sup_{n\in I}|x_n - \tilde{x}(\tau_n;\tau_{n_0}, x_{n_0})| > \varepsilon\} \subset$$

$$\subset \{\sup_{n\in I} V^{1/2}[(x_n - \tilde{x}(\tau_n;\tau_{n_0}, x_{n_0})), \tau_n] > \varepsilon C_1\} \subset \bigcap_1^N \Omega_k$$

where

$$\Omega_k = \left\{V^{1/2}[(x_{n_j} - \tilde{x}(\tau_{n_j};\tau_{n_0}, x_0)), \tau_{n_j}] \leq \varepsilon C_1 \quad j \leq k;\right.$$

$$\left. V^{1/2}[(x_{n_{k+1}} - \tilde{x}(\tau_{n_{k+1}}, \tau_{n_0}, x_0)), \tau_{n_{k+1}}] > \varepsilon C_1\right\}$$

and thus

$$P(\Omega) \leq \sum_1^N P(\Omega_k) \leq \left[\frac{64C_2^2C_1L_3}{3\lambda^2}\right]^{2p} \frac{L_1}{\varepsilon^{4p}} \sum_{j=n_0}^N \gamma_j^p$$

and the theorem is proved.

□