A Comparative Study of Suboptimal Filters for Parameter Estimation

Olsson, Gustaf; Holst, Jan

LUND UNIVERSITY

PO Box 117
221 00 Lund
+46 46-222 00 00

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GUSTAF OLSSON
JAN HOLST

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ABSTRACT

Approximative filters, used to track unknown parameters in linear multivariable systems, are examined mainly from computational aspects in this report. Three different filters are compared and tested, an Extended Kalman predictor, an Extended Kalman filter and a Single State Iteration Filter.

Two aspects are emphasized in the report, filter consistency and filter convergence.

Consistency is examined in terms of parameter bias and variance. It is not a trivial task to judge the size of the bias, and it is even more difficult to find relevant correction schemes. The sizes of bias and variance depend critically on the assumed artificial statistics of the unknown parameters. The asymptotic filter properties are quite similar in this respect.

Convergence properties, however, are quite different for the three filters, and it is demonstrated that the iterations are important. It is, however, critical to choose an adequate iteration accuracy.

Several numerical difficulties arise when the parameter errors are large. Those difficulties can be derived partly to the integration scheme, partly to accuracy problems in the Riccati equation. This means that filter divergence sometimes also have been a result of numerical inaccuracies.

For a special first order case, filter convergence can be analytically established and numerically confirmed.

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1. INTRODUCTION AND PROBLEM STATEMENT

1.1. General Background

In this report the problem of simultaneous state and parameter estimation will be treated.

It is well known, that there is no general method for identification of non-linear and multivariable systems with time-variable parameters. When the structure of the system is determined, however, the unknown parameters in the proposed structure might be adjusted to measurement data. This can be done either off-line by calculating on the whole set of data simultaneously, or on-line with some recursive technique.

In this work three different recursive schemes, all basicly based on the extended Kalman filter are tested and compared.

The problem of recursive parameter estimation has gained much attention for some years, first in the space program and then in industrial applications.

Earth satellite orbit determination and reentry trajectory estimation were two early application fields. A survey is found in [18]. Other contributions are [29],[30],[37],[48],[49].

Examples of industrial applications are chemical reactors [35],[45] nuclear reactors [17],[33], oxygen furnace [46], and paper machine head box control [39].

The recursive parameter problem can be considered a filtering problem. The unknown parameter vector is estimated as a part of an extended state variable.

This approach always leads to the solution of a non-linear estimation problem. Generally truly non-linear filters are always infinite dimensional, and a partial differential equation of the conditional probability for the state, given the measurements

$$p(x|Y_t) \tag{1.1}$$

has to be solved [7],[9],[15],[24].

The optimal filtering algorithm cannot be implemented on a computer, except in special cases. Practical algorithms for non-linear filtering must be finite dimensional and consequently suboptimal.

Attempts have been made to solve the real non-linear filtering problem.

In [8] [10] the conditional probability (1.1) have been approximated by discrete points. Special numerical techniques have been applied to make the solution reasonable. Another approach is shown in [1]. The probability functions have been evaluated as a finite sum of Gaussian distributions. The resulting filter consists of a number of parallel Kalman filters.

The most common way to solve the problems is by assuming the above mentioned conditional probability Gaussian.

The non-linearities are expanded into Taylor series around a nominal trajectory or they are approximated by polynomial expressions. Linear approximations result in the so called extended Kalman filter. Also higher order filter, achieved by truncating the Taylor expansions, can be derived. A lot of different schemes have been suggested in the literature for different types of system, see e.g. [3],[5],[6], [11],[18],[22],[25],[26],[36],[38],[41],[42],[43],[44].

It has been shown [25], that a higher order filter can behave worse than a linearized one for special non-linear systems.

In other cases [5] it has been shown, that the added complexity of a higher order filter does not always pay off in terms of better convergence. All the filters, based on Taylor expansions, are suboptimal, and therefore a consistent solution cannot be guaranteed.

The linearized filter, the extended Kalman filter, has been widely discussed in many papers and used for many applications.

Another filter, based on the extended Kalman filter involves a local iteration algorithm. The filter is the so called single stage iteration filter [20] [48]. The purpose of the iterations is to improve the reference trajectory and thus the estimate, in the presence of the non-linearities. The algorithm seems to have certain advantages, and there are few problems to implement it on a computer. Another approach to an iterative filter is found in [27].
It has been mentioned, that the suboptimal filters do not give consistent estimates. The linearized filters often give biased errors [5], [18] and correction terms are sometimes included, e.g. second order terms of the Taylor expansions. One way to check the estimate consistency is by making statistical tests of the residuals. The filter characteristics and noise covariance are changed according to these residual tests. Several different schemes are suggested in the literature e.g. [26][28][34].

It is very difficult to get a general comparison between different suboptimal filters. The filter performance depends very much on the actual system dynamics and noise characteristics. Simulation is the most common tool for filter comparisons. This is of course time consuming, and it is often difficult to understand the filter behaviour conceptually.

Specific comparisons of suboptimal filters applied to special systems have been performed by simulations in [29][30][41][48] and [49].

1.2. Problem Statement

As shown in previous section it is very difficult to understand many suboptimal filters conceptually. It is difficult to evaluate their performance in general terms. Therefore, a filter compasion can only be valid for a special system structure.

In the present work the system structure has been chosen linear with unknown parameters. The report is devoted to some principal problems that arise in the filtering process. Three different filters, two forms of the extended Kalman filter and the single stage iteration filter, have been compared by numerical experiments.

The purpose of the report is twofold. As convergence cannot be established generally, this problem has been examined for some test systems. The consistency of the estimates is another problem. The filters are tested for some simple test systems and bias and parameter variances are examined and compared to the assumed statistics.

There exist other methods for single input- single output systems [3], but the actual filters here can be used also for non-linear systems and time variable parameters. The simple test systems are used in order to get an easier insight into the essential properties of the filters.

The system equations are considered continuous, while the measurements are made in discrete time.

In chapter 2 the system structure and the major approximations are displayed. The filter equations are formulated in chapter 3. Three different filters are discussed, namely two forms of the Extended Kalman filter (EK1 and EK2) and the Single Stage Iteration filter (SSI).

The consistency of the filter estimates are discussed briefly.

Because the state of the system has been extended with the unknown parameter vector some artificial noise has to be constructed. The influence of this noise is discussed in chapter 4. It will affect both the bias and the estimation accuracy. Chapter 4 ends with a discussion of the importance of the input signal amplitude.

Some numerical comparisons of estimate convergence are made in chapter 5. The initial error of the parameter is small and convergence rates are compared for the three filters on some test systems. Both deterministic and stochastic systems are considered.

The filters are also examined for large parameter errors, and boundaries for filter convergence are studied and compared in chapter 6. Data is generated from deterministic test systems.

It is shown, that the SSI filter is superior, because the iterations compensate for the non-linearities.

It is also shown, that e.g. sampling time, position of the unknown parameter in the system matrix and noise characteristics are crucial for the parameter convergence.

Several numerical problems arise in the filter algorithms, especially for extremely large errors of the initial values of the parameters. Such numerical inaccuracies may cause filter divergence. Therefore, it is mandatory to decide, whether filter divergence occurs due to numerical problems or due to the suboptimal character of the filter. The numerical error sources are discussed in chapter 7 and 8.

Finally, in chapter 9, the results are summarized.

2. STRUCTURE OF THE PROCESS

The report is devoted to the examination of the filter behaviour for parameter and state estimation in linear systems. The suboptimal filters can, however, be applied also to general non-linear systems. As the state vector is extended with the unknown parameter vector, also a linear system with unknown parameters results in a non-linear filtering problem. Therefore the derivations and equations are shown for the general case of a non-linear system with additive noise. The unknown parameter vector may also be time variable.

The basic system equations and the noise assumptions are presented in 2.1. The main assumtions of the estimation procedure are described in 2.2. In section 2.3 the linearizations of the system equations are performed. Finally, in 2.4, the special system structure, used in the present report, is motivated and presented.

2.1. Basic System Equations

The problem consists of estimation of the state z and the unknown parameter vector α of the system.

$$z(t+1) = F_1(z(t),\alpha(t), u(t)) + w_1(t) \tag{2.1}$$

where z is the state vector (dimension n_1)
u is the control vector (dimension n_u)
α is the unknown parameter vector (dimension n_2)

The noise term w_1 is a sequence of n_1-dimensional, zero mean, independent, Gaussian random variables with covariance matrix.

$$E(w_1(t)w_1^T(t)) = R_1'(t) \tag{2.2}$$

The function F_1 is the solution at time t+1 of the differential equation

$$\frac{dz}{dt} = f_1(z(t),\alpha(t),u(t)) \tag{2.3}$$

with the state z given at time t. The control is considered constant over one sampling interval.

The estimation is based on noisy measurements at discrete times t

$$y(t) = G_1(z(t),\alpha(t),u(t)) + e(t) \tag{2.4}$$

The additive noise is a sequence of n_y-dimensional zero-mean, independent, Gaussian random variables. The covariance function is

$$E(e(t)e^T(t)) = R_2(t) \tag{2.5}$$

The measurement noise is also assumed independent of the process noise w_1.

The initial state z(0), is assumed to be a Gaussian random variable,

$$E(z(0)) = m_o \tag{2.6}$$

$$E(z(0)\ z^T(0)) = R_o$$

The noise terms w_1 and e are independent of z(0)

The unknown parameter is assumed to be governed by a difference equation

$$\alpha(t+1) = \alpha(t) + w_2(t) \tag{2.7}$$

The term w_2 is an artificial noise. It is assumed to be a n_2-dimensional sequence of zero-mean, independent, Gaussian random variables with covariance function

$$E(w_2(t)w_2^T(t)) = R_1''(t) \tag{2.8}$$

The parameter may well be time variable. It has been shown [47] earlier by numerical experiments, that the artificial zero-mean noise w_2 takes care of time variations of the unknown parameter.

The initial value of α is unknown. As the filter does not converge if the initial parameter error is too large, the choice of $\alpha(0)$ is crucial. This problem is discussed more in chapter 6.

The covariance of the initial value of α is of course also unknown. This value is, however, not as crucial as the mean value. It can be guessed in a standard manner, as shown in chapter 4 (and 5).

The noise level of w_2 (R_1'') is a measure of the assumed variability of the parameter. It is emphasized, that the choice of artificial noise characteristics is difficult and is a crucial step of the filtering procedure (cf chapter 4 and 5).

In order to formulate the state and parameter estimation problem to a state estimation problem the state vector z is extended by the parameter vector α. The extended vector x is defined

$$x = \begin{pmatrix} z \\ \alpha \end{pmatrix} \tag{2.9}$$

The extended system matrices can now be expressed by the enlarged vector x,

$$x(t+1) = F(x(t), u(t)) + w(t) \tag{2.10}$$

$$y(t) = G(x(t), u(t)) + e(t) \tag{2.11}$$

The matrix functions F and G are extended accordingly from (2.1), (2.4) and (2.7). The measurement noise e is identical with the noise term in (2.4).

The function F is considered the solution at time t+1 of the differential equation

$$\frac{dx}{dt} = f(x,u,t) = \begin{pmatrix} f_1(z,\alpha,u) \\ \hline 0 \end{pmatrix} \tag{2.12}$$

given the state at time t.

The vector function G (2.11) is a transformation of G_1 (2.4) with the same dimension but different arguments.

The extended noise

$$w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

is considered a sequence of zero-mean, independent, Gaussian random variables with covariance function

$$E(w(t)w^T(t)) = E\begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} (w_1^T(t) \quad w_2^T(t)) =$$

$$= \begin{pmatrix} Ew_1(t)w_1^T(t) & Ew_1(t)w_2^T(t) \\ E(w_2(t)w_1^T(t) & Ew_2(t)w_2^T(t) \end{pmatrix} = \begin{pmatrix} R_1^{1} & R_1^{12} \\ R_1^{21} & R_1^{\prime\prime} \end{pmatrix} = R_1(t) \qquad (2.13)$$

The matrix R_1 is nxn-dimensional, where $n = n_1 + n_2$. For generality it is not assumed block diagonal. The matrices R_1^{12} and R_1^{21} are thus considered non-zero.

2.2. The Estimation Problem

As already mentioned in the introduction, the problem of estimating the state in a non-linear system like (2.10) - (2.11) is a difficult task. In order to get a finite-dimensional description some approximations must be made.

Three different state estimation problems are considered. Let $\hat{x}(j|Y_k)$ (sometimes written $\hat{x}(j|k)$) denote the estimate of $x(j)$ based on measurements $y(o), y(1), \ldots, y(k)$. Then the following estimates are computed

$\hat{x}(t|Y_t)$

$\hat{x}(t|Y_{t-1})$

$\hat{x}(t-1|Y_t)$

If the estimation criterion is to minimize the mean square error, then the solution is given by the conditional mean [20]

$$E(x(j)|Y_k) = \int x(j)p(x(j)|Y_k)dx(j) \tag{2.14}$$

Now two crucial assumptions are made concerning the noise. The first is, that the conditional probability

$$p(x(j)|Y_k)$$

is Gaussian with a known mean value

$$\hat{x}(j|k) = E(x(j)|Y_k) \tag{2.15}$$

and known covariance P.

This is true only if the system is linear with linear measurements [4][20].

The second approximation is, that $x(k)$, $x(k-1)$ and $y(k)$, $y(k-1)$ are jointly Gaussian. This is generally not true for non-linear systems.

In the next section the linearization procedure of the system dynamics will be shown. The two assumptions of the noise and the linearization procedure makes the problem fit into the assumptions of the Kalman-Bucy filter [23].

2.3. Linearization of the System Equations

The extended Kalman filters and the Single Stage Iteration filter are derived by linearizing the non-linear equation around a nominal trajectory x^*. This type of linearization procedure is shown elsewhere, e.g. [20] [48].

For clarity the notation T in the following is used for the sampling interval, when the equations are not written in difference equation form.

The linearized system equations can now be stated. The function $f(\cdot,u)$ is linearized over every sampling interval around a nominal trajectory determined by an initial condition vector $x^*(t)$. Linearizing (2.12) we have

$$\frac{dx(\tau)}{d\tau} \approx f(x^*(\tau),u) + \left[\frac{\partial f(x(\tau),u(\tau))}{\partial x}\right]_{x^*(\tau)} (x(\tau) - x^*(\tau)) \qquad (2.16)$$

$t \leq \tau < t + T$

The function $x^*(\tau)$ is the nominal trajectory and satisfies

$$\frac{dx^*(\tau)}{d\tau} = f(x^*(\tau),u(\tau)) \qquad (2.17)$$

with the initial condition
$$x^*(t) = \begin{cases} \hat{x}(t|Y_t) & \text{for EK1 filter} \\ \hat{x}(t|Y_{t-1}) & \text{" EK2 filter} \end{cases} \qquad (2.18)$$

Further is assumed, that $u(\tau) = u(t), \tau \in [t, t+T]$ i.e. that the control vector is constant during the sampling interval. The Jacobian $(\frac{\partial f}{\partial x})$ is defined as the matrix

$$\left\{\frac{\partial f}{\partial x}\right\}_{i,j} = \left(\frac{\partial f_i}{\partial x_j}\right) \qquad (2.19)$$

The conditional mean is simply achieved from (2.16) by calculating the conditional expectation.

$$E\left[\frac{dx(\tau)}{d\tau} \middle| Y_t\right] =$$

$$= \frac{d\hat{x}(\tau)}{d\tau} \approx f(x^*(\tau),u) \quad + \left[\frac{\partial f(x(\tau),u(\tau))}{\partial x}\right]_{x^*(\tau)} \quad (\hat{x}(\tau) - x^*(\tau)) \qquad (2.20)$$

Y_t can also be replaced by Y_{t-1}, see (2.18).

The initial condition of (2.19) is

$$\hat{x}(t) = \begin{cases} \hat{x}(t|t-1) & \text{for EK1 filter} \\ \hat{x}(t|t) & \text{for EK2 filter} \end{cases} \qquad (2.21)$$

Now (2.20) can be solved at time T^-. The notation T^- is used to mark, that the last measurement (at t+T or at t) has not yet been registered.

$$E(x(t+T^-)|Y_t) = E[F(x(t),u(t)) + w(t)|Y_t] \qquad (2.22)$$

according to (2.10) Y_{t-1} may replace Y_t as in (2.20)

The fact that

$$E(w(t)|Y_t) = E(w(t)|Y_{t-1}) = 0 \qquad (2.23)$$

makes

$x(t+T^-)|Y_t) = E(F(x(t),u(t))|Y_t)$

Thus

$\hat{x}(t+T^-) = E(x(t+T^-)|Y_t)$

is considered an approximative solution to (2.20)

Let us define

$$\hat{x}(t+T^-) = \tilde{x}(t+T) \tag{2.24}$$

In the calculations $x(t+T^-)$ therefore, is determined as the solution at time t+T of

$$\frac{dx}{d\tau} = f(x(\tau),\ u(\tau))$$

with the initial condition $x(t) = \hat{x}(t)$

It is possible to write a solution of the linearized equation (2.16) in the form

$$x(t+T) - x^*(t+T) = \phi(t+T,t)[x(t) - x^*(t)] \qquad (2.25)$$

where ϕ is the fundamental matrix, satisfying

$$\frac{d}{d\tau} \phi(\tau,t) = \left[\frac{\partial f(x(\tau),u(\tau))}{\partial x}\right]_{x^*(\tau)} \phi(\tau,t) \qquad (2.26)$$

where $t \leq \tau < t+T$
and $x^*(\tau)$ defined by (2.17) - (2.18).

The initial condition for ϕ at time t is

$$\phi(t,t) = I$$

Now, it can be shown [48], that the propagation of the error covariance $P(t)$, which is part of the filter equations, contains this ϕ-matrix. This is discussed in 3.1 and 3.2.

The measurement function $G(\bullet)$ (2.11) can be linearized around a nominal state $x^*(t)$

$$y(t) \approx G(x^*(t),u(t)) +$$

$$+ \left[\frac{\partial G(x(t),u(t))}{\partial x}\right]_{x^*(t)} \cdot (x(t) - x^*(t)) + e(t) \qquad (2.27)$$

where the Jacobian $(\frac{\partial G}{\partial x})$ is defined

$$\left\{\frac{\partial G}{\partial x}\right\}_{i,j} = \frac{\partial G_i}{\partial x_j} \triangleq \{\theta\}_{i,j} \tag{2.28}$$

The θ-matrix has the dimension $n_y \times (n_1 + n_2)$

2.4. System Structure used in the Examination

In the present report linear systems with unknown, constant parameters are considered. This simplifies eg. (2.3) to

$$\frac{dz}{dt} = A(\alpha)z(t) + B(\alpha)u(t) \tag{2.29}$$

where the input is considered known.

The measurements are made in discrete time according to

$$y(t) = Cz(t) + e(t) \tag{2.30}$$

where C is a $n_y \times n_1$ matrix.

This system structure is transformed to the form (2.10) - (2.11).

Equation (2.12) corresponds to the differential equation of order $n = n_1 + n_2$

$$\frac{dx}{dt} = \frac{d}{dt}\begin{pmatrix} z \\ \alpha \end{pmatrix} = \left[\begin{array}{c|c} A(\alpha) & 0 \\ \hline \underbrace{0}_{n_1} & \underbrace{0}_{n_2} \end{array}\right] \begin{pmatrix} z \\ \alpha \end{pmatrix} \tag{2.31}$$

The noise characteristics are the same as for the non-linear case. The non-linearities in the extended differential equation (2.12) are quadratic in the state variables, i.e. of the form $\alpha_i z_j$.

In most test examples the B matrix is assumed to be known. If only B contains the unknown parameter, then the estimation problem gets simple. This is the case, because u(t) is known. Therefore no non-linearities appear, and thus the filter is linear.

In the present paper also the C-matrix is assumed known. It is no principal problem to consider C unknown. In this examination, however, it has been found, that the most interesting problems occur, when the unknown parameters in the A-matrix are calculated. Moreover, for many systems one can find the C parameters much easier a priori compared to the A parameters.

The fundamental matrix (2.26) is achieved from the equation

$$\frac{d}{d\tau}\,\phi(\tau,t) = \begin{array}{c} \left[\begin{array}{c|c} A(\alpha^*(t)) & D(z^*(\tau)) \\ \hline 0 & 0 \end{array}\right] \\ \leftarrow n_1 \rightarrow \leftarrow n_2 \rightarrow \end{array} \phi(\tau,t) \tag{2.32}$$

where z^* and α^* are the two subvectors of the nominal trajectory at time τ, $t \leq \tau < t+T$. Observe, that $\alpha(\tau) = \alpha(t)$.

The $n_1 \times n_2$-dimensional matrix $D(z^*)$ is a matrix with elements 0 or z_j.

The fundamental matrix can be expressed in the form

$$\phi = \begin{array}{cc} \left[\begin{array}{c|c} e^{A\cdot T} & H \\ \hline 0 & I \end{array}\right] & \begin{array}{c} \uparrow \\ n_1 \\ \downarrow \\ \uparrow \\ n_2 \\ \downarrow \end{array} \\ \leftarrow n_1 \rightarrow \leftarrow n_2 \rightarrow & \end{array} \tag{2.33}$$

where T is the sampling time and H is a $n_1 \times n_2$-matrix. Generally H cannot be simply expressed in z and α.

The measurement matrix θ (2.28) is simply achieved from the C-matrix by adding n_2 columns of zeroes.

The problem of identifiability naturally occurs in connection with parameter estimation problems [3]. In the present paper this problem is not considered. If a parameter would not be identifiable, the filter will not converge or it will diverge. In the third order diagonal system (section 6.3.1.) this fact is demonstrated.

3. THE FILTER EQUATIONS

The assumptions of the noise in 2.2 and the linearizations in 2.3 make it possible to directly apply the Kalman-Bucy filter for the estimation problem.

The equations of the extended Kalman filters and the SSI filter are derived elsewhere [20], [48]. For reference purposes, however, the filter equations are written down here. In 3.1 the two different extended Kalman filters EK1 and EK2 are given. In the filter EK1 the conditional mean $\hat{x}(t|t-1)$ is calculated, while in the filter EK2 $\hat{x}(t|t)$ is computed.

In the following section 3.2 the SSI filter is presented. In section 3.3 the problem of consistency is discussed.

3.1. The Extended Kalman Filters

The system is given by (2.10) - (2.11) and the process and measurement noise terms are characterized by (2.13) and (2.5) respectively.

3.1.1. Extended Kalman Filter 1 (EK1)

The estimated state at time t+1 is a conditional mean of the state x at time t+1, conditioned on measurements until time t. The filter equations are

$$\hat{x}(t+1|t) = \bar{x}(t+1) + K(t)[y(t) - \theta(t)\hat{x}(t|t-1)] \tag{3.1}$$

where

$$K(t) = \phi P(t)\theta^T [\theta P(t)\theta^T + R_2]^{-1} \tag{3.2}$$

$$P(t+1) = [\phi - K(t)\theta]P(t)\phi^T + R_1 \tag{3.3}$$

$$P(t_o) = \mathrm{cov}[x(t_o)] = P_o = \left[\begin{array}{c|c} P_o^{1} & P_o^{12} \\ \hline P_o^{21} & P_o^{11} \end{array}\right] \tag{3.4}$$

The matrices ϕ and θ are defined in (2.26) and (2.28), while the covariance matrices R_1 and R_2 are given by (2.13) and (2.5) respectively.

The matrices in (3.2) - (3.3) are time variable in the following sense

$$\phi = \phi(t+1,t) \; ; \quad \theta = \theta(t) \; ; \quad R_1 = R_1(t) \; ; \quad R_2 = R_2(t)$$

$\bar{x}(t+1)$ is defined in (2.24) as the solution of (2.20) with the initial condition $\hat{x}(t|t-1)$.

The P matrix is the error covariance matrix

$$P(t) = E\{[x(t) - \hat{x}(t|t-1)][x(t) - \hat{x}(t|t-1)]^T\} \tag{3.5}$$

under the condition that $E(x(t) - \hat{x}(t|t-1)) = E\,\tilde{x}(t|t-1) = 0$

It should be noted, that the gain matrix K(t) and the error covariance P(t) cannot be calculated in advance, contrary to the linear Kalman filter. The reason is, that ϕ is not known à priori because it is achieved by linearization around the last estimate.

3.1.2. Extended Kalman Filter 2 (EK2)

This filter is based on the same system equations as the EK1 filter. The estimate is

$$\hat{x}(t+1|t+1) = \bar{x}(t+1) + K(t+1)[y(t+1) - \theta(t+1)\bar{x}(t+1)] \tag{3.6}$$

where

$$K(t+1) = S(t+1)\theta^T(t+1)[\theta S(t+1)\theta^T + R_2]^{-1} \tag{3.7}$$

$$S(t+1) = \phi \cdot P(t) \cdot \phi^T + R_1 \tag{3.8}$$

$$P(t) = S(t) - K(t)\theta(t)S(t) \tag{3.9}$$

$$P(t_o) = \mathrm{cov}(x(t_o)) = P_o \tag{3.10}$$

where

$$R_2 = R_2(t+1), \; R_1 = R_1(t+1), \; \phi = \varphi(t+1,t), \; \theta = \theta(t+1).$$

The variables have the same definition as for the EK1 filter. One difference, however, is that $\bar{x}(t+1)$ (see (2.24)) has the initial condition $\hat{x}(t|t)$.

The matrix P(t) is defined

$$P(t) = E\{[x(t) - \hat{x}(t|t)][x(t) - \hat{x}(t|t)]^T\}$$

see [3].

3.2. The Single Stage Iteration Filter (SSI)

The meaning of the iteration is, by using the à priori probability density of the state, to make a less poor approximation of the non-linear filtering problem. The filter is already presented in [48], why only the final equations are stated here.

In (3.6) an estimate $\hat{x}(t+1|t+1)$ was achieved by using the initial condition $\hat{x}(t|t)$. Now, including one more measurement $y(t+1)$ in the filter, more information is available, and $\hat{x}(t|t+1)$ can be estimated, i.e. a smoothed value, which ought to be a more accurate estimate of x at the time t. This new estimate is used as a new initial value of a new estimate $\hat{x}^{(2)}(t|t+1)$. A new smoothed estimate of x is calculated based on $\hat{x}^{(2)}(t|t+1)$. At iteration number i the estimate is thus $\hat{x}^{(i)}(t|t+1)$. The calculations continue iteratively for this single sampling interval until the error

$$\varepsilon = ||\hat{x}^{(i+1)}(t|t+1) - \hat{x}^{(i)}(t|t+1)|| \qquad (3.11)$$

is small enough. We could also have considered the criterion

$$\varepsilon = ||\hat{x}^{(i+1)}(t+1|t+1) - \hat{x}^{(i)}(t+1|t+1)|| \qquad (3.13)$$

but from the parameter identification point of view the two criteria are equivalent because of the parameter difference equation (2.7). Then the estimation continues to the next sampling interval.

The smoothing formulas can be derived quite analogously to the estimations, e.g. by using projection technique for normal stochastic variables (see [4] chapter 7 or [20] chapter 7). At iteration stage i the interpolation is

$$\hat{x}^{(i+1)}(t|t+1) = \hat{x}^{(i)}(t|t+1) + P(t)\phi^T \theta^T [\theta S^{(i)}(t+1)\theta^T + R_2(t+1)]^{-1} *$$

$$*[y(t+1) - \theta\bar{x}^{(i)}(t+1)] \tag{3.12}$$

The meaning of $\bar{x}^{(i)}(t+1)$, $S^{(i)}$, are the same as in section 3.1 and

$$\phi = \phi^{(i)}(t+1,t)$$
$$\theta = \theta^{(i)}(t+1)$$

To summarize, the SSI filter equations are

Smoothing part

$$\hat{x}^{(i+1)}(t|t+1) = \hat{x}^{(i)}(t|t+1) + P(t)\phi^{T(i)}(t+1,t)UK^{(i)}(t+1)\tilde{y}^{(i)}(t+1)$$

$$UK^{(i)}(t+1) = \theta^{T(i)}[\theta^{(i)}S(t+1)^{(i)}\theta^{T(i)} + R_2(t+1)]^{-1}$$

$$\tilde{y}^{(i)}(t+1) = [y(t+1) - \theta^{(i)}\bar{x}^{(i)}(t+1)]$$

$$S^{(i)}(t+1) = \phi^{(i)}(t,t+1)P(t)\phi^{T(i)}(t,t+1) + R_1(t+1)$$

(3.14)

Filtering part ((f) denotes final)

$$\hat{x}(t+1|t+1) = \bar{x}^{(f)}(t+1) + K(t+1)\tilde{y}^{(f)}(t+1)$$

$$K(t+1) = S^{(f)}(t+1)UK^{(f)}(t+1)$$

$$P(t+1) = S^{(f)}(t+1) - K(t+1)\theta^{(f)}(t+1)S^{(f)}(t+1)$$

$$P(t_o) = \mathrm{cov}[x(t_o)] = P_o$$

Thus the converged values of $\hat{x}^{(i)}(t|t+1)$ and $S^{(i)}$ are used for the new filter estimates $\hat{x}(t+1|t+1)$ and $P(t+1)$. Note that, when no iterations are performed the SSI filter is identical to the EK2 filter. Observe, that there is no general proof of convergence. Practical applications, however, have shown the validity of the filters in different systems.

It is found by numerical experiments [20], [48] that the filter behaves quite favourably in the presence of non-linearities. It is found, that the parameter estimate converges much faster due to the iterations. The SSI filter also tolerates larger initial parameter errors than the EK filters. Questions of this nature will be discussed in chapter 5 and 6.

3.3. Consistency of the estimate

The major question concerning suboptimal filters is that of consistency. The filters treated in this report are all based on a Taylor series expansion of a nonlinear function as well as some crucial statistical assumptions. The assumptions are valid in a sufficiently small neighbourhood of the true parameter values.

As the filtering problem is inherently nonlinear it is generally impossible to find a sufficient statistics [23] of the conditional density by using a finite number of moment equations.

Some important assumptions are necessary to make concerning the statistical properties of the filters. The residuals r are defined as

$$r(t|t-1) = y(t) - E(y(t)|Y_{t-1}) \tag{3.15}$$

for the EK1 filter as well as the EK2 filter

The residuals are assumed to be a sequence of independent, normal stochastic variables with zero mean and covariance

$$E(r(t|t-1)r^T(t|t-1)) = \theta(t)P(t)\theta^T(t) + R_2(t) \tag{3.16}$$

for the EK1 filter and

$$E(r(t|t-1)\ r^T(t|t-1)) = \theta(t)S(t)\theta^T(t) + R_2(t) \tag{3.17}$$

for the EK2 filter respectively.
The two expressions (3.16), (3.17) can be given exactly the same interpretation.

The residuals satisfy these equations only in the case, that the filter is truly optimal in the mean square sense. The residuals therefore can be used as a detection of the goodness of the approximation. If the filter parameters are corrected according to the real data e.g. by using the residuals the filter is called adaptive. This is, however, not the only way to define the idea of adaptive filters, see [28]. Here some basic ideas of adaptive filters are pointed out. In this report, however, they are not examined further. For more detailed treatments of adaptive filters, see also [20], [26], [28].

3.3.1. Covariance matching technique

The covariance - matching technique is used by some authors [20], [28]. Consider, e.g., the sequence r(t) (3.15) which has the theoretical covariance (3.16 - 3.17). If it is found, that the empirical covariance of r(t) is significantly larger than the value (3.16 - 3.17) given by the filter, then the covariance R_1 should be increased, i.e. in our case the artificial parameter noise covariance should primarily be changed. This change also affects the value of P and brings the theoretical residual covariance closer to the empirical one (3.16 - 3.17). The empirical covariance usually is approcimated by

$$\frac{1}{m}\sum_{1}^{m} r(t)r^T(t) \tag{3.18}$$

where m is chosen by experience.

Let us discuss the EK1 filter case. The EK2 filter follows trivially. An equation for R_1 is obtained by using (3.16)

$$\theta(t)P(t)\theta^T(t) + R_2(t) = E(rr^T) \tag{3.19}$$

or

$$\theta(t)[(\phi(t,t-1)-K(t-1)\theta(t-1))P(t-1)\phi^T(t,t-1) + R_1]\theta^T(t) + R_2 = E(rr^T) \tag{3.20}$$

or

$$\theta(t)R_1\theta^T(t) = E(r\,r^T) - R_2 - \theta(t)(\phi(t,t-1) - K(t-1)\theta(t-1))P(t-1)\phi^T(t,t-1)\theta^T(t) \tag{3.21}$$

Eq. (3.21) does not give a unique solution of R_1 unless the rank of $\theta(t)$ is n. In our case the rank of $\theta(t)$ is always less than n, because of the special structure (2.13), (2.32) of the system. The product

$$\theta R_1\theta^T = (\theta_1 \mid 0)\begin{pmatrix} R_1' & R_1^{12} \\ R_1^{21} & R_1'' \end{pmatrix}\begin{pmatrix} \theta_1^T \\ 0 \end{pmatrix} \tag{3.22}$$

is independent of R_1^{12}, R_1^{21}, and R_1''. Therefore eq. (3.20) has to be developed further, so that P(t-1) is expressed in P(t-2)

$$\theta(t)[(\phi(t,t-1) - K(t-1)\theta(t-1))\,P(t-1)\phi^T(t,t-1) + R_1]\theta^T(t) + R_2 =$$

$$= \theta(t)\{(\phi(t,t-1) - K(t-1)\theta(t-1))\,[(\phi(t-1,t-2) - K(t-2)\theta(t-2))\,P(t-2)\phi^T(t-1,t-2) +$$

$$+ R_1]\phi^T(t,t-1) + R_1\}\theta^T(t) + R_2 = E(rr^T)$$

Then

$$\theta(t)\,[R_1 + (\phi(t,t-1) - K(t-1)\theta(t-1))\,R_1\,\phi(t,t-1)^T\,]\theta^T =$$

$$= -\theta(t)\,\{(\phi(t,t-1) - K(t-1)\theta(t-1))\,(\phi(t-1,t-2) - K(t-2)\theta(t-2))\,P(t-2)\phi^T(t-1,t-2)\cdot$$

$$\cdot\,\phi^T(t,t-1)\quad\theta^T(t) - R_2 + E(rr^T) \tag{3.23}$$

Then also R_1^{12}, R_1^{21} and $R_1^{''}$ can affect the residuals.

If the number of unknowns in R_1 is restricted, a unique solution can be obtained. Notice, however, that the obtained result is only approximate since P(t-1) and P(t-2) do not represent the actual error covariances when the true values of R_1 (and perhaps R_2) are unknown. For this reason, the addition of the covariance matching technique does not necessarily guarantee any convergence of the filter.

3.3.2 The ε-technique

The problem of adaptivity can also be treated from another point of view, model error sensitivity. A system model is always an approximation of the physical world, and the model error can be sometimes large, sometimes negligible. This type of structure errors exist even if all parameters are considered known. In the parameter estimation model there is still another model error, because of the unknown parameters. This occurs also for simulated data. Therefore the filter model is not exact and it will degrade the filter performance. The sensitivity for model errors is high, when the noise levels are low, as the filter then can learn the state very accurately. This sensitivity might cause filter divergence. One simple attempt to handle this problem is to increase the covariance matrix.

An idea of this type is the so called ε-technique [40]. The covariance P is increased with the diagonal matrix $\varepsilon \cdot I$ which is chosen empirically according to the model inaccuracy, thus assuring that the filter gain does not go to zero too quickly. This technique, however, is not very successful in this application. It is shown e.g. by the gain equation (3.2). If the equation is partitioned into two parts, we get

$$K = \begin{pmatrix} K_1 \\ \hline K_2 \end{pmatrix} = \left(\begin{array}{c|c} \phi_{11} & \phi_{12} \\ \hline 0 & I \end{array}\right) \left[\left(\begin{array}{c|c} P_{11} & P_{12} \\ \hline P_{21} & P_{22} \end{array}\right) + \varepsilon I \right] \begin{pmatrix} \theta_1 \\ \hline 0 \end{pmatrix} \times$$

$$\times \left(\qquad \right)^{-1} = \begin{pmatrix} K_1^0 \\ \hline K_2^0 \end{pmatrix} + \begin{pmatrix} K_1^\varepsilon \\ \hline K_2^\varepsilon \end{pmatrix}$$

The terms K_1^ε and K_2^ε are the additional terms, caused by ε. By simple multiplication it is found that $K_2^\varepsilon = 0$. The value of K_2 is not affected until next time step.

The covariance matching technique is a more advanced version to correct P or the residuals than the ε-technique. However, as the parameter noise is artificial, still no general method exists to learn the value of R_1''.

Other methods, which have been used extensively, use overweight in some way of recent measurements. Exponential weighting and moving window techniques are described elsewhere [43]

3.3.3. Bias correction

Generally the estimate $\hat{x}(0)$ is biased because of the unknown parameters. If the mean value of the estimate error $x-\hat{x}$ is non-zero, the mean value of the residuals will also be non-zero. Thus mean value check of the residuals is a detection of parameter bias. The filter equations are derived under the assumption, that

$$E(x-\hat{x}) = 0$$

and thus P represents the error covariance. If the mean value is non-zero, then P instead is the second order moment. More details are discussed in 4.1..

There are several methods to correct for bias. The Taylor expansion (2.16) can be extended to second order terms, as in [5]. This procedure will decrease the bias, but not necessarily to zero, and it complicates the filter equations. It ensures that

$$E(y(t) - \hat{y}(t|t-1)) = 0$$

up to second order.

Filter convergence is still not established, as remarked in chapter 1.

In the present report no bias correction is introduced. Instead the size of the bias for some test systems and different filters will be examined (See section 4.2). Some obvious necessary conditions to detect and avoid biased residuals can be stated. From (3.1) and (3.6) it is easy to realize, that a necessary condition for parameter convergence is that the gain K is non-zero, in the sense that the product

$$K(y - \theta\hat{x})$$

is small in mean only if the residuals are small in mean. The covariance matrix R_1 (2.13) should therefore assure K non-zero. However, if R_1'' (2.13) is chosen too large other problems occur (see chapters 5 and 6), which make convergence difficult.

4. CHOICE OF FILTER CHARACTERISTICS

In the calculations it has been assumed that the characteristics of the real process noise and the measurement noise are known. The filter calculations are performed to examine the consequences of the choice of the artificial noise parameters R_1'' (2.8), $P(0)''$ (3.4), (3.10), and initial mean value $E\alpha(0)$. The covariance terms will influence both convergence rate, accuracy of the parameter estimate, and parameter bias. In 4.1 the principal questions concerning the artificial noise are discussed. The size of R_1'' and the covariance matrix P are compared to the true parameter estimate variance. Some numerical results are given in 4.2. The parameter estimate properties under different conditions will be discussed, especially the bias problems. The initial parameter errors are assumed to be zero. Convergence problems will be discussed in chapters 5 and 6.

If the input signal amplitude is decreased for unchanged disturbance levels the parameter accuracy will be decreased. Some numerical results are presented in 4.3.

4.1 Principal Problems

In this section the influence of $R_1''(t)$, $P(0)''$ and $E\,\alpha\,(0)$ will be discussed qualitatively. It has already been remarked, that R_1'' illustrates the variability of the unknown parameters. If a rapidly changing parameter has to be tracked, corresponding elements of the submatrix R_1'' have to be large. Observe, however, that one has to consider a compromize between the trackability and the fluctuations of the estimate [47].

In a system with constant but unknown parameters the submatrix R_1'' determines the parameter convergence rate from the initial value to the final one. A large value of R_1''

generally causes a faster convergence than a small value. On the other hand the estimate trajectory is smoother for a small R_1''. This is shown in 4.2. If R_1'' is too large the filter may even diverge for a certain initial parameter value, see chapter 5.

The choice of $P(0)''$ is not as crucial for the convergence as the choice of R_1''. Generally speaking $P(0)''$ shall be large for large initial parameter errors.

A discussion based on linear theory can be made to illustrate the choice of $P(0)''$. $P(t)$ is defined by (3.5). The mean value of the error vector

$$E\tilde{x}(t) = E(x(t) - \hat{x}(t)) = m(t) = \begin{pmatrix} m_z(t) \\ \hline m_\alpha(t) \end{pmatrix}$$

is mostly non-zero already from the beginning in our problem. Generally the filter estimates get biased and therefore $P(t)$ illustrates the second moment instead of the covariance. Thus the covariance P^* is

$$P^*(t) = P(t) - mm^T$$

The matrix P^* satisfies the same discrete Riccati equation as P, but the initial conditions are different,

$$P^*(0) = P(0) - m(0)\,m(0)^T \tag{4.1}$$

Now, if P converges to zero, both P^* and m will converge to zero. In the linear case the initial parameter error is assumed

$$m_\alpha(0) = E\tilde{\alpha}(0) = E[\alpha(0) - \hat{\alpha}(0)]$$

If the error of the mean of the physical state is zero, then

$$m(0) = \begin{pmatrix} 0 \\ \hline m_\alpha(0) \end{pmatrix}$$

The initial second moment is then

$$P(0) = P^*(0) + \begin{pmatrix} 0 & | & 0 \\ \hline 0 & | & m_\alpha m_\alpha^T \end{pmatrix} \begin{matrix} \}n_1 \\ \}n_2 \end{matrix} \tag{4.2}$$

As the vector m_α is unknown, the only thing one can do is to increase $P^*(0)$ with a sufficiently large submatrix, consisting of the dyad $m_\alpha \cdot m_\alpha^T$.

In the general case the value of $m_\alpha(0)$ is unknown. However, in the test cases, in order to examine the influence of $P(0)$" the right value of $m_\alpha(0)$ has sometimes been inserted into $P(0)$". Sometimes $P(0)$" was chosen standardwize $1.0 \cdot I_{n_2}$ or $10.0 \cdot I_{n_2}$. The parameter convergence properties are approximately the same. This will be discussed more in chapters 5 and 6.

A major question in this examination is how the parameter initial value influences on the filter behaviour. It is clear that a large initial error (or a large parameter change) will cause convergence problems, because of the Taylor expansion of the nonlinearities. Particularly for large errors the choices of R_1" and $P(0)$" are important.

4.2 Influence of the artificial noise covariance on estimate bias and accuracy

A number of noisy systems with constant but unknown parameters have been simulated. Refer to app. 1 for a description of the systems. The purpose has been to find the size of the bias and the influence of R_1'' on the parameter estimate. The sample mean and covariance of the estimate based on 400 samples have been computed. Thus the sample covariance of the estimate ($\hat{\sigma}^2$) may be compared with both $P(0)''$ and R_1. The initial value of the parameter error was chosen equal to zero.

As the system is forced by stochastic processes it should be necessary to consider a large number of estimate realizations in order to judge the filter performance. In this case especially the parameter bias will be considered. In order to overcome the problem of a large number of realizations, the variance of the parameter estimate mean value has been calculated according to formulas given in Appendix 3.

The EK1 and EK2 filters have been compared in the respects discussed above. Also the SSI filter was tried out in some cases in order to verify the fact, that the iterations cannot improve the filter performance, as soon as the parameter values are close to the true values.

Table 4.1 A-E describes the estimates of the system parameter of the first order system with different noise terms added. Generally speaking, the accuracy in terms of parameter variance is practically the same for the EK1 and EK2 filters. The measurement information is approximately the same. The last measurement, which comes into the EK2 filter, does not affect the sample variance significantly. The mean values of the estimates however, are quite different for the two filters, as shown by the tables.

The results verify the statement, that a small value of R_1'' decreases the variance (but causes a slower convergence, as will be shown in chapter 5). However, the parameter sample variance $\hat{\sigma}^2$ decreases slower than R_1''. In the case when process noise and no measurement noise is present (table 4.1 A,B) the value of $\hat{\sigma}^2$ decreases a factor of 3 - 4 while R_1'' decreases a factor of 10. In the case of only measurement noise (table 4.1 C,D,E) the corresponding decrease of $\hat{\sigma}^2$ is only 2 - 3.

The value of $\hat{\sigma}^2$ can be both less than and greater than the assumed value R_1''. According to the discussion in 4.1 (valid for linear systems) the value of P'' should be thought of as the error variance or the second moment of the parameter.

It is, however, demonstrated by the computations that the value of P'' has little to do with the real variance or second moment. This statement just illustrates the fact that R_1'' is chosen arbitrarily. In the test cases the true value of the unknown parameter is constant. Moreover, the initial value is chosen equal to the true value. Therefore a relevant value of R_1'' should be zero, to describe the variability. Then, however, the filter cannot track the parameter at all. In a realistic situation therefore, the value of R_1'' is overestimated. This also causes too a large value of P. The mean value error also increases the P value.

The initial value of P may have some influence on the parameter accuracy. The term $P(0)''$ might be small in this case, because the parameter initial value is good.

TABLE 4.1 A - E

Comparison between the EK1 and EK2 filters in stochastic systems with different artificial noise. The filters estimate the parameter $\alpha(=-0.5)$ of the continuous system

$$\frac{dz}{dt} = \alpha z + u$$

$$y = z$$

(system 1.2, 1.3, 1.4, 1.5 and 1.6 in App. 1). The system has been sampled with $\Delta T = 1.0$ and discrete noise has been added.

Common data

u = PRBS sequence of unit amplitude

Number of samples = 400

$E\ \hat{z}(0) = 0$

$\hat{\alpha}(0) = -0.5$ (=true value)

$$R_1 = \begin{pmatrix} r_{11} & 0 \\ 0 & r_{22} \end{pmatrix}$$

$$P(0) = \begin{pmatrix} 10^{-5} & 0 \\ 0 & 1.0 \end{pmatrix}$$

Integration step length = $\Delta T/4$

Floating numbers are written in the format .xxx-x. Thus .123-4 means $0.123 \cdot 10^{-4}$.

The values of var $(\tilde{\alpha})$ are calculated according to Appendix 3, either with real numerical values or for an exponential standard case.

TABLE 4.1 A System 1.2

r_{11} = 0.00632 (corresponding to continuous noise with covariance 0.01 dt)

R_2 = 10^{-6} (no measurement noise)

		1.0	0.1	0.01	0.001
r_{22}		1.0	0.1	0.01	0.001
$\bar{\alpha}$	EK 1	-	-.477	-.499	-.508
	EK 2	-.467	-.494	-.507	-.512
$\hat{\sigma}^2=\mathrm{var}(\hat{\alpha})$	EK 1	-	.259-1	.753-2	.218-2
	EK 2	.604-1	.243-1	.812-2	.221-2
$\hat{\sigma}$	EK 1	-	.161	.868-1	.467-1
	EK 2	.246	.156	.901-1	.470-1
$\eta=\sqrt{\mathrm{var}(\bar{a})}$	EK 1*	-	.12-1	.91-2	.59-2
	EK 2**	.14-1	.12-1	.82-2	.59-2
P"	EK 1	-	.220	.315-1	.686-2
	EK 2	1.085	.111	.450-1	.777-2
Bias 1 $(\alpha-\bar{\alpha})$	EK 1		$-0.14\hat{\sigma}$	$-0.01\hat{\sigma}$	$0.17\hat{\sigma}$
	EK 2	$-0.14\hat{\sigma}$	$-0.037\hat{\sigma}$	$+0.079\hat{\sigma}$	$+0.24\hat{\sigma}$
Bias 2 $(\alpha-\bar{\alpha})$	EK 1	-	-1.9η	-0.1η	1.4η
	EK 2	-2.4η	-0.5η	$+0.9\eta$	2.0η

*) (standard approximation)

**) calculate value

TABLE 4.1 B System 1.3

r_{11} = 0.632 (corresponding to continuous noise with covariance 1.0 dt)

$R_2 = 10^{-6}$ (no measurement noise)

r_{22}		1.0	0.1	0.01	0.001
$\bar{\alpha}$	EK 1	-	-.329	-.453	-.477
	EK 2	-.721	-.603	-.520	-.488
$\hat{\sigma}^2=\mathrm{var}(\hat{\alpha})$	EK 1	-	.175	.437-1	.127-1
	EK 2	.957	.172	.431-1	.114-1
$\hat{\sigma}$	EK 1	-	.418	.209	.113
	EK 2	.978	.415	.208	.107
$\eta=\sqrt{\mathrm{var}(\bar{\alpha})}$*		.54-1	.31-1	.22-1	.14-1
P"	EK 1	-	.507	.141	.383-1
	EK 2	1.030	.298	.106	.286-1
Bias 1 $(\alpha-\bar{\alpha})$	EK 1	-	$-.41\hat{\sigma}$	$-0.23\hat{\sigma}$	$-0.20\hat{\sigma}$
	EK 2	$0.23\hat{\sigma}$	$0.25\hat{\sigma}$	$0.096\hat{\sigma}$	$-0.12\hat{\sigma}$
Bias 2 $(\alpha-\bar{\alpha})$	EK 1	-	-5.5η	-2.1η	-1.6η
	EK 2	4.09η	3.3η	0.9η	-0.9η

*) (standar approximation)

TABLE 4.1 C System 1.4

$r_{11} = 10^{-5}$ (no process noise)

$R_2 = 10^{-2}$

r_{22}		1.0	0.1	0.01	0.001	
$\tilde{\alpha}$	EK 1	-	-.491	-.505	-.506	
	EK 2	-.502	-.510	-.509	-.507	
$\hat{\sigma}^2 = \mathrm{var}(\hat{\alpha})$	EK 1	-	.229-1	.510-2	.133-2	
	EK 2	.733-1	.231-1	.508-2	.135-2	
$\hat{\sigma}$	EK 1	-	.151	.714-1	.365-1	
	EK 2	.274	.152	.713-1	.367-1	
$\eta = \sqrt{\mathrm{var}(\tilde{\alpha})}$*		.14-1	.11-1	.74-2	.47-2	*) (standar approxima-tion)
P''	EK 1	-	.245	.339-1	.665-2	
	EK 2	1.057	.119	.518-1	.681-2	
Bias 1 $(\alpha - \tilde{\alpha})$	EK 1	-	$-0.06\hat{\sigma}$	$0.07\hat{\sigma}$	$0.12\hat{\sigma}$	
	EK 2	$+0.007\hat{\sigma}$	$0.07\hat{\sigma}$	$0.13\hat{\sigma}$	$0.19\hat{\sigma}$	
Bias 2 $(\alpha - \bar{\alpha})$	EK 1	-	-0.8η	0.7η	1.3η	
	EK 2	0.14η	0.9η	1.2η	1.5η	

TABLE 4.1 D System 1.5

$r_{11} = 10^{-5}$ (no process noise)

$R_2 = 0.25$

r_{22}		1.0	0.1	0.01	0.001	
$\bar{\alpha}$	EK 1	-	-.520*)	-.512	-.501	*) 100 samples
	EK 2	-.629	-.575	-.533	-.509	
$\hat{\sigma}^2=\text{var}(\hat{\alpha})$	EK 1	-	.949-1*)	.247-1	.885-2	
	EK 2	.602	.109	.250-1	.735-2	
$\hat{\sigma}$	EK 1	-	.308*)	.157	.941-1	
	EK 2	.776	.330	.158	.857-1	
$\eta=\sqrt{\text{var}(\bar{\alpha})}$	EK 1**)	-	.23-1	.16-1	.37-2	**) (standard approximation)
	EK 2 +)	.69-1	.36-1	.22-1	.15-1	
P''	EK 1	-	.577*)	.117	.230-1	+) calculated value
	EK 2	1.356	.522	.121	.211-1	
Bias 1 $(\alpha-\bar{\alpha})$	EK 1	-	$0.09\hat{\sigma}$*)	$0.08\hat{\sigma}$	$0.01\hat{\sigma}$	
	EK 2	$0.17\hat{\sigma}$	$0.23\hat{\sigma}$	$0.20\hat{\sigma}$	$0.099\hat{\sigma}$	
Bias 2 $(\alpha-\bar{\alpha})$	EK 1	-	0.9η	0.8η	0.3η	
	EK 2	1.86η	2.1η	1.5η	0.6η	

TABLE 4.1 E System 1.6

$r_{11} = 10^{-5}$ (no process noise)

$R_2 = 1.0$

r_{22}		1.0	0.1	0.01	0.001
$\bar{\alpha}$	EK 1	-	-.585	-.600	-.576
	EK 2	-.961	-.720	-.630	-.583
$\hat{\sigma}^2 = \mathrm{var}(\hat{\alpha})$	EK 1	-	.170	.407-1	.110-1
	EK 2	1.143	.171	.448-1	.113-1
$\hat{\sigma}$	EK 1	-	.412	.202	.105
	EK 2	1.069	.414	.212	.106
$\eta = \sqrt{\mathrm{var}(\bar{\alpha})}$*		.58-1	.31-1	.21-1	.13-1
p"	EK 1	-	.410	.112	.326-1
	EK 2	8.49	.876	.132	.325-1
Bias 1 $(\alpha - \bar{\alpha})$	EK 1	-	$0.21\hat{\sigma}$	$0.50\hat{\sigma}$	$0.72\hat{\sigma}$
	EK 2	$0.43\hat{\sigma}$	$0.53\hat{\sigma}$	$0.61\hat{\sigma}$	$0.78\hat{\sigma}$
Bias 2 $(\alpha - \bar{\alpha})$	EK 1	-	2.7η	4.8η	5.8η
	EK 2	7.9η	7.1η	6.2η	6.4η

*) (standard approximation)

The bias problem has been considered. It has been illustrated from two aspects. The variance of the parameter mean value ($var(\bar{\alpha})$) has been calculated according to app. 3. The bias (called bias 2) is significant, also for small values of R_1''.

From a practical point of view the value of "bias 1" is interesting. If the variance of the estimate is large, then a bias of the estimate is less important. From the tables one can conclude that the bias 1 value is acceptable in most cases.

Finally, when R_1'' decreases, the absolute value of the bias decreases.

The filters have also been applied to a third order system (system 3, appendix 1) with one, two or three unknown parameters in the A-matrix (table 4.2). The same conclusions as for the first order system can be made. However, some additional remarks also can be stated. Not only the artificial noise but also the position of the parameter in the A matrix is important.

TABLE 4.2 A - C

Comparison between the EK1 and EK2 filters in a third order stochastic system. The filters estimate one, two or three parameters α_i in the continuous system

$$\frac{dz}{dt} = \begin{pmatrix} \alpha_1 & 1 & 0 \\ \alpha_2 & 0 & 1 \\ \alpha_3 & 0 & 0 \end{pmatrix} z + \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x$$

(system 3.2 in App. 1). The system has been sampled with $\Delta T = 0.3$ and measurement noise has been added.

Common data

u = PRBS sequence of unit amplitude
Number of samples = 400

$E\ \hat{z}(0) = (0, 0, 0)^T$
$\hat{\alpha}(0)$ = true value

Integration step length = $\Delta T/4$

The concepts of "lower" and "upper" limits for $\sqrt{\text{var}(\bar{\alpha})}$ are discussed in appendix 3.

TABLE 4.2 A System 3.2, one unknown

$R_1 = \text{diag}\ (10^{-5}, 10^{-5}, 10^{-5}, r_{22})$

$P(0) = \text{diag}\ (10^{-5}, 10^{-5}, 10^{-5}, 1.0)$

	α_1 unknown (=-6)			α_2 unknown (=-11)			α_3 unknown (=-6)		
r_{22}	0.1	0.01	0.001	0.1	0.01	0.001	0.1	0.01	0.001
$\tilde{\alpha}$ EK 1	-6.410	-6.354	-6.231	-	-	-	-5.805	-5.847	-5.908
$\tilde{\alpha}$ EK 2	-6.330	-6.365	-6.239	-11.273	-11.246	-11.353	-5.867	-5.848	-5.908
$\hat{\sigma}^2 = var(\hat{\alpha})$ EK 1	.532	.162	.870-1	-	-	-	.354	.798-1	.179-1
$\hat{\sigma}^2 = var(\hat{\alpha})$ EK 2	.610	.165	.874-1	.764	.230	.721-1	.407	.797-1	.180-1
$\hat{\sigma}$ EK 1	.729	.403	.295	-	-	-	.595	.282	.134
$\hat{\sigma}$ EK 2	.781	.406	.296	.874	.480	.269	.638	.282	.134
$\sqrt{var(\bar{\alpha})}$ lower limit	.043	.022	.016	.048	.026	.015	.035	.016	.007
$\sqrt{var(\bar{\alpha})}$ upper limit	.12	.060	.043	.013	.071	.040	.094	.042	.020
P'' EK 1	2.280	.519	.147	-	-	-	1.952	.519	.126
P'' EK 2	2.160	.506	.147	2.50	.601	.175	1.847	.508	.125

Table 4.2B - System 3.2, 2 unknowns.

$R_1 = \mathrm{diag}(10^{-5}, 10^{-5}, 10^{-5}, r_{22}, r_{22})$; $P(0) = \mathrm{diag}(10^{-5}, 10^{-5}, 10^{-5}, 1.0, 1.0)$

		α_1, α_2 unknown (-6, -11)			α_1, α_3 unknown (-6, -6)		
r_{22}		0.1	0.01	0.001	0.1	0.01	0.001
$\bar{\alpha}_i$	EK1		-6.336 -11.238			-6.374 -6.020	
	EK2	-6.349 -11.317	-6.348 -11.245	-6.230 -11.355	-6.491 -6.266	-6.376 -6.019	-6.244 -5.988
$\hat{\sigma}_i^2 = \mathrm{var}(\hat{\alpha}_i)$	EK1		.163 .215			.169 .852-1	
	EK2	.616 .707	.163 .215	.871-1 .700-1	.948 .538	.166 .844-1	.772-1 .139-1
$\hat{\sigma}_i$	EK1		.403 .464			.411 .292	
	EK2	.785 .841	.404 .464	.295 .274	.974 .733	.407 .291	.278 .118
$\sqrt{\mathrm{var}(\bar{\alpha}_i)}$	lower limits	.043 .043	.022 .025	.016 .015	.054 .040	.023 .016	.015 .007
	upper limits	.120 .120	.060 .068	.043 .040	.14 .11	.060 .043	.041 .017
$(P'')_{1,2}$	EK1						
	EK2	2.084 2.538	.478 .628	.149 .179	2.295 1.971	.575 .589	.165 .138

Table 4.20 - System 3.2, 3 unknowns.

$R_1 = \mathrm{diag}(10^{-5}, 10^{-5}, 10^{-5}, r_{22}, r_{22}, r_{22})$; $P(0) = \mathrm{diag}(10^{-5}, 10^{-5}, 10^{-5}, 1.0, 1.0, 1.0)$

		$\alpha_1, \alpha_2, \alpha_3$ unknown (-6, -11, -6)			
r_{22}		0.1	0.01	0.001	10^{-5}
$\bar{\alpha}_i$	EK1	-6.493 -11.274 -6.100	-6.354 -11.236 -6.011	-6.225 -11.347 -5.983	-6.174 -11.410 -6.005
	EK2	-6.405 -11.383 -6.156	-6.357 -11.242 -6.012	-6.223 -11.353 -5.985	
$\hat{\sigma}_i^2 = \mathrm{var}(\hat{\alpha}_i)$	EK1	.654 .517 .396	.170 .219 .851-1	.787-1 .754-1 .130-1	.581-1 .399-1 .320-2
	EK2	.846 .704 .437	.164 .209 .830-1	.762-1 .758-1 .129-1	
$\hat{\sigma}_i$	EK1	.809 .719 .629	.412 .458 .292	.281 .275 .114	.241 .200 .566-1
	EK2	.920 .839 .661	.405 .457 .288	.276 .275 .114	
$\sqrt{\mathrm{var}(\bar{\alpha}_i)}$	lower limits	.045 .040 .035	.023 .025 .016	.015 .016 .006	.013 .011 .003
	upper limits	.135 .120 .097	.060 .067 .042	.042 .041 .017	.035 .029 .008

Table 4.2C - Contd.

		$\alpha_1, \alpha_2, \alpha_3$ unknown (-6, -11, -6)			
r_{22}		0.1	0.01	0.001	10^{-5}
p''	EK1	2.34 2.83 2.01	.563 .660 .567	.168 .181 .139	.644-1 .659-1 .241-1
	EK2	2.22 2.69 1.92	.546 .645 .554	.167 .180 .138	

Consider Table 4.2A, which shows the estimation of one parameter at a time in the system. The accuracy of α_3 is generally much better than that of α_1, if the variance is considered. Also the bias, measured in $\hat{\sigma}$ or by $\sqrt{var(\bar{\alpha})}$, is smaller for α_3. The same comparison can be made between α_2 and α_3, and the α_3 estimate is better in the senses mentioned.

In Table 4.2B the results of estimation of two parameters simultaneously is displayed. Compare the variances for $r_{22} = 10^{-3}$. These values give an indication how accurate the parameter can be estimated. Then α_3 is again the most accurate one, while α_1 and α_2 are approximately the same and not so good as α_3.

A similar conclusion can be made from Table 4.2C, where three unknowns are estimated.

The parameter variances depend on r_{22}, as mentioned before. It is worth noting, however, that if r_{22} decreases e.g. from 0.1 to 0.001, the corresponding changes of $\hat{\sigma}^2(\alpha_i)$ are quite different. When r_{22} is changed a factor of 100, the parameter variances are changed a factor as follows:

$\hat{\sigma}^2(\alpha_1) \sim 6 - 12$

$\hat{\sigma}^2(\alpha_2) \sim 7 - 11$

$\hat{\sigma}^2(\alpha_3) \sim 20 - 40$

Thus, α_3 is more sensitive to changes in r_{22} than the other parameters. This can also be illustrated by Fig. 4.1.

The accuracy of the estimated parameters seems to be almost the same if one parameter at a time is estimated or if two or three are estimated simultaneously. Consider the case $r_{22} = 10^{-3}$ in the Tables 4.2A-C.

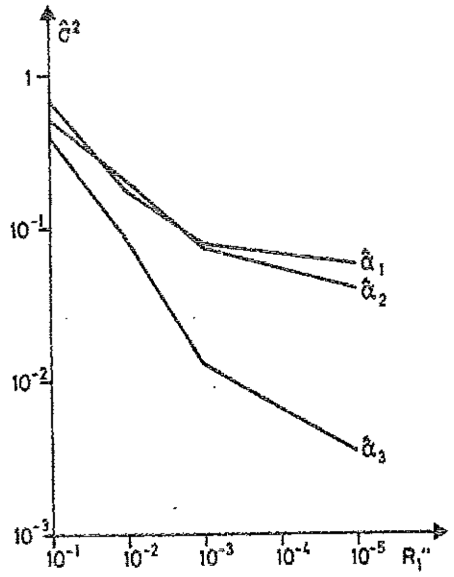


Fig. 4.1 - The sample covariance as function of the assumed values of $r_{2\dot{2}}$. The values are taken from Table 4.2C and are shown for the EK1 filter.

The empirical accuracy can be compared with the Fisher information matrix, calculated from off-line identifications [1G

There the loss function is defined

$$V(\theta) = \int_0^T [y_m - y(\theta)]^2 dt$$

where θ = unknown parameter vector
y_m = measured output
$y(\theta)$ = model output

The matrix $V_{\theta\theta}$ is calculated for the deterministic system 3 for the case three unknown parameters.

$$V_{\theta\theta} = \begin{pmatrix} 0.446 & -0.299\cdot 10^{-4} & -0.334 \\ -0.299\cdot 10^{-4} & 0.335 & 0.228\cdot 10^{-2} \\ -0.334 & 0.228\cdot 10^{-2} & 1.173 \end{pmatrix}$$

The elements of this matrix should not change for different number of parameters. The parameter accuracy is proportional to the diagonal elements of the inverse, and these elements may change considerably for different orders.

In this case most of the off-diagonal elements are quite small. The inverse of the (1,1) element is 2.24. Corresponding inverse of the upper left 2×2 submatrix of $V_{\theta\theta}$ is

$$\begin{pmatrix} 2.24 & 2.00\cdot 10^{-4} \\ 2.00\cdot 10^{-4} & 2.99 \end{pmatrix}$$

and the accuracy of α_1 is the same as before. In the three parameter case the inverse is

$$V_{\theta\theta}^{-1} = \begin{vmatrix} 2.85 & -.530-2 & .814 \\ -.530-2 & 2.99 & -.733-2 \\ .814 & -.733-2 & 1.084 \end{vmatrix}$$

The diagonal element corresponding to α_1 is thus slightly changed to 2.85.

The variances of the parameters are estimated as in [2]

$$\sigma^2(\alpha_1) = k \cdot 2.85$$
$$\sigma^2(\alpha_2) = k \cdot 2.99$$
$$\sigma^2(\alpha_3) = k \cdot 1.08$$

This calculation shows at least qualitatively, that α_3 is more accurate than the other two parameters.

Consider the bias e.g. for $r_{22} = 10^{-3}$ and compare the Tables 4.2A, B, C. The bias for α_i is about the same if α_i is estimated alone or together with other parameters. The estimate of α_3 is better than the other α_i also in terms of bias, measured in absolute value or normalized with $\hat{\sigma}$ or $\sqrt{var(\bar{\alpha})}$

4.3. Influence of Signal-to-Noise Ratio on Parameter Accuracy.

With some numerical experiments it is illustrated, how the emphirical value of the parameter variance is influenced by the input amplitude.

The result is shown in Table 4.3. Consider the values of $\hat{\sigma}^2$. For a small value of r_{22} (which is the most relevant choice) the value of $\hat{\sigma}^2$ increases up to a finite value for $u = 0$. On the other hand, when $r_{22} = 1.0$, some other effects dominate. The reason is, that r_{22} is assumed so large, that corresponding $\hat{\sigma}^2$ is very large. Therefore its value does not change very much as a function of $|u|$.

Notice, that P" is also depending on $|u|$. The reason is, that P cannot be calculated in advance. Rather it is a function of $\hat{x}$, so the parameter estimates also are reflected into P.

Table 4.3 - Influence of input amplitude $|u|$ on parameter estimate for the EK2 filter.
The filter estimates the parameter α (= -0.5) in the continuous system

$$\frac{dz}{dt} = \alpha z + u$$

$$y = z$$

(system 1.2 in App. 1).

Number of samples = 400.

$$R_1 = \begin{pmatrix} 0.00632 & 0 \\ 0 & r_{22} \end{pmatrix}$$

$$R_2 = 10^{-6}$$

$$P(0) = \begin{pmatrix} 10^{-5} & 0 \\ 0 & 1.0 \end{pmatrix}$$

Integration step length = $\Delta T/4$.

(The values for $|u|$ = 1 coincides with those of Table 4.1A)
The α estimates are illustrated with their $\hat{\sigma}$ limits in Fig. 4.2.

r_{22}	$\lvert u \rvert$	1.0	0.1	0.01	0.001
$\bar{\alpha}$	1	-.467	-.494	-.507	-.512
	10^{-1}	-.670	-.659	-.597	-.551
	10^{-2}	-.844	-.643	-.547	-.474
	10^{-3}	-.898	-.719	-.601	-.526
	0	-.813	-.672	-.570	-.477
$\hat{\sigma}^2 = \mathrm{var}(\hat{\alpha})$	1	.604-1	.243-1	.812-2	.221-2
	10^{-1}	.824	.158	.333-1	.769-2
	10^{-2}	1.002	.159	.386-1	.117-1
	10^{-3}	1.147	.159	.392-1	.132-1
	0	.941	.172	.581-1	.226-1
$\hat{\sigma}$	1	.246	.156	.901-1	.470-1
	10^{-1}	.908	.397	.182	.877-1
	10^{-2}	1.001	.399	.196	.108
	10^{-3}	1.071	.399	.198	.115
	0	.970	.415	.241	.150
P"	1	1.085	.111	.450-1	.777-2
	10^{-1}	2.313	.361	.832-1	.293-1
	10^{-2}	1.140	.317	.938-1	.352-1
	10^{-3}	4.086	.538	.121	.414-1
	0	1.781	.576	.182	.483-1
Bias 1 $\alpha - E\hat{\alpha}$ $(*\hat{\sigma})$	1	-0.14	-0.037	0.079	0.24
	10^{-1}	0.19	0.40	0.53	0.58
	10^{-2}	0.34	0.37	0.24	-0.24
	10^{-3}	0.37	0.55	0.51	0.23
	0	0.32	0.41	0.29	-0.16

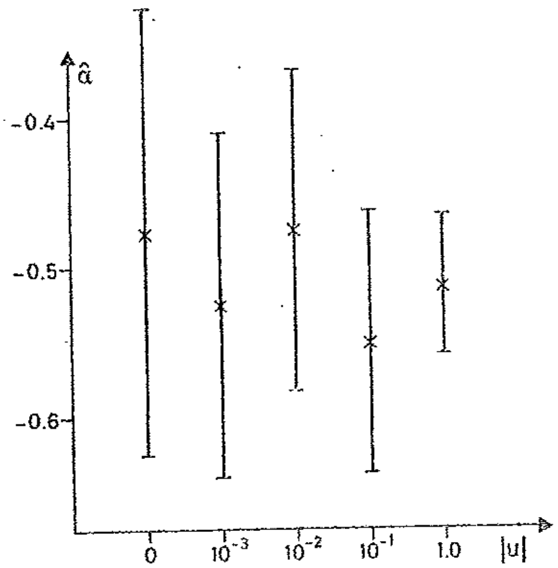


Fig. 4.2 - Estimation of α $(\pm\hat{\sigma})$ by the EK2 filter as function of input amplitude.
In all cases $r_{22} = 10^{-3}$.

5. NUMERICAL STUDIES OF PARAMETER CONVERGENCE DUE TO FILTER COEFFICIENTS

The derivation of the filters are basedon local Taylor expansions of the non-linear functions around the nominal estimate. If the parameter error is small enough, the higher order terms of the Taylor expansion can be neglected, and the filter does not diverge. Numerical problems are in general quite reasonable in this case (see further chap.7 and 8).

It will be illustrated numerically that the choice of artificial parameter noise also affects the divergence properties of the filter. The convergence is discussed out of three aspects. Convergence for small parameter errors was discussed in chapter 4.1. Typically, it does not pay off very much to perform iterations in the SSI filter.

Convergence for large parameter errors is discussed in this chapter. For such errors it still pays off to do iterations.

In chapter 6 filter divergence due to parameter initial values is discussed.

The choice of the filter parameters was discussed qualitatively in 4.1. In this chapter some quantitative results will be shown concerning the convergence rate.

A major result out of this examination may be the fact that the SSI filter is powerful for large parameter errors. With the SSI filter it is possible to combine a high convergence speed of the estimation with a small variance of the final estimate. This result corresponds in a sense to that of adaptive filters. There the covariance matrix is large in order to get a fast convergence and small to achieve a small parameter variance.

Deterministic systems are considered in 5.1, while stochastic systems are discussed in 5.2 and 5.3 for EK filters and for SSI filters respectively.

5.1. Convergence Rate in Deterministic Systems.

The section is divided into two parts. In 5.1.1 a test system of first order and with one unknown parameter is discussed. In 5.1.2 a third order system with one, two or three unknown parameters is considered.

5.1.1. First order test system.

Some numerical experiments have been performed on the first order system 1.8 (see App. 1).

The discrete-time noise is assumed to be

$$R_1 = \begin{pmatrix} 10^{-5} & 0 \\ 0 & r \end{pmatrix} \tag{5.1}$$

$$R_2 = 10^{-6} \tag{5.2}$$

$$P(0) = \begin{pmatrix} 10^{-5} & 0 \\ 0 & p_0 \end{pmatrix}. \tag{5.3}$$

where r and p_0 determine the artificial noise.

The influence of r and p_0 can be illustrated by a number of figures. All of them show the parameter estimate as function of time until the relative parameter error is less than 10^{-4}.

Fig. 5.1A,B show the result of the EK1 filter. It is clear from several runs, that a small r gives a smoother parameter trajectory than a large r. On the other hand, the time for convergence is longer for a small r. If r is too large, as in Fig. 5.1A, the filter can diverge.

The Figs. 5.1 also illustrate, that the size of p_0 dominates over r in the beginning, as the difference between the trajectories is not clear until $t = 3$. The influence of p_0 is illustrated by comparing Figs. 5.1 and 5.2. In the former case p_0 is chosen

$$p_0 = m_\alpha m_\alpha^T$$

and in the latter case $p_0 = 1.0$.

Also observe, that no correction takes place at $t = 1$. This depends on the matrix partitions, and the value of the parameter Kalman gain is always zero at $t = 1$. This can easily be verified by Eq. (3.2).

As p_0 directly affects the Kalman gain a too large value of p_0 can give such a large correction, that divergence follows.

The sampling time is important, a fact, that will be discussed further in chapter 6. Fig. 5.3 illustrates the convergence for sampling time $T = 1.0$ for the same system as previously (system 1.1). The influence of r is, however, negligible in this case. An initial error of 10.0 will cause filter divergence for this sampling time.

A similar examination can also be done for the EK2 and SSI filters. For these filters the convergence rate is much higher for deterministic systems when the parameter error is large, because $\hat{x}(t|t)$ is used instead of $\hat{x}(t|t-1)$. This will be illustrated further in chapter 6.

Figs. 5.4A,B show the convergence from ±10 for the SSI filter. The curves should be compared to Fig. 5.1 A and B. The iterations make the convergence very fast.

In the figures there is no obvious difference between different choices of r in the first part of the estimation trajectory. There the iterations have more influence on the convergence than the choice of r (convergence in the large). When the estimate has approached the true value, the number of iterations is small (convergence in the small). Then the convergence rate is determined by r only. Also here it is true, that convergence rate is small for small values of r.

If no iterations are made in the SSI filter the EK2 filter is achieved. In Figs. 5.5A,B the parameter convergence for this filter and the same test system as before is shown. In 5.5B there is an overshoot in the estimate depending on the value of P(0). The SSI filter is more insensitive to the choice of P(0), e.g. illustrated by 5.4B.

It is interesting to compare the number of calculations of $\hat{x}(t|t)$ in the EK2 filter with the total number of calculations of $\hat{x}(t|t+1)$ and $\hat{x}(t+1|t+1)$ in the SSI filter. If the tables in Figures 5.4 and 5.5 are compared, it is shown that the SSI filter needs more calculation time, due to the extra iterations in the first time steps.

The number of iterations is intimately connected to the choice of ε (3.11). It determines when to accept the difference between two successive iterative estimates as zero.

The larger the residuals are the more iterations are made. This means that iterations are performed, in systems with low level noise, only in the beginning of the estimation, compare Fig. 5.4. By making ε larger (eq.(3.11)) the number of iterations will be smaller in each time step. On the other hand, the number of time steps might increase. This is illustrated for the first order system 1.7 in Table 5.1. There is no process or output noise.

ε	Number of time steps	Total number of calculations of $\hat{x}(t\|t+1)$	Max number of calculations of $\hat{x}(t\|t+1)$ per time step
1.0	7	10	3
10^{-1}	6	13	5
10^{-2}	6	16	7
10^{-3}	6	21	9
10^{-4}	6	27	11
10^{-5}	6	30	13
10^{-6}	6	37	16

Table 5.1 - Number of steps to convergence and number of calculations of $\hat{x}(t|t+1)$ as a function of ε for system 1.7. The parameter has converged when the relative error is less than 10^{-5}. Initial value of the parameter +10.0. P(0)" = 1.0.

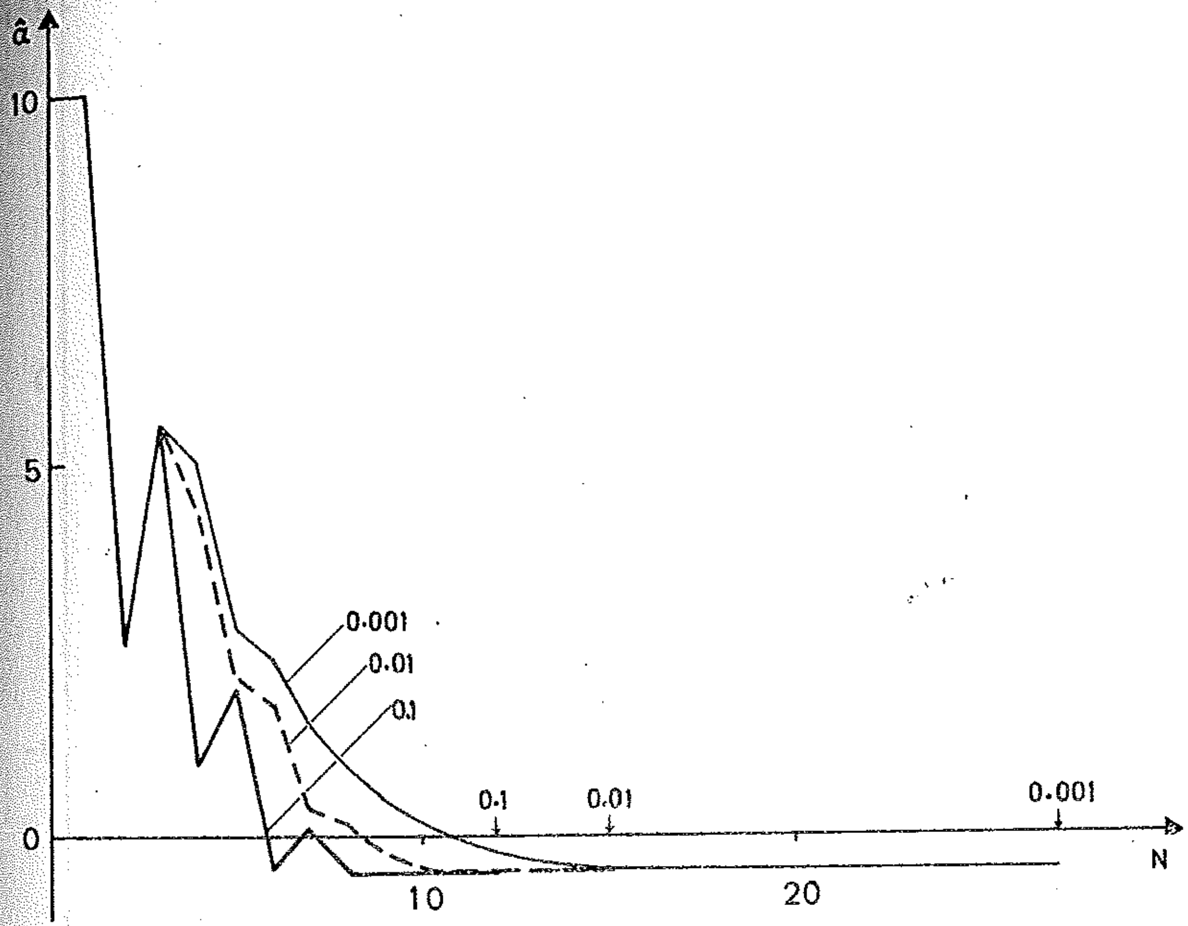


Fig. 5.1 - Parameter estimates $\hat{\alpha}$ of the EK1 filter for the first order system 1.8.

5.1A

$$P(0) = \begin{pmatrix} 10^{-5} & 0 \\ 0 & 10.5^2 \end{pmatrix} \quad R_1 = \begin{pmatrix} 10^{-5} & 0 \\ 0 & r \end{pmatrix} \quad r = 1.0,\ 0.1,\ 0.01,\ 0.001$$

The arrows along the time axis indicate when the parameter estimate has converged within the relative error 10^{-4}. $r = 1.0$ gives a divergent filter.

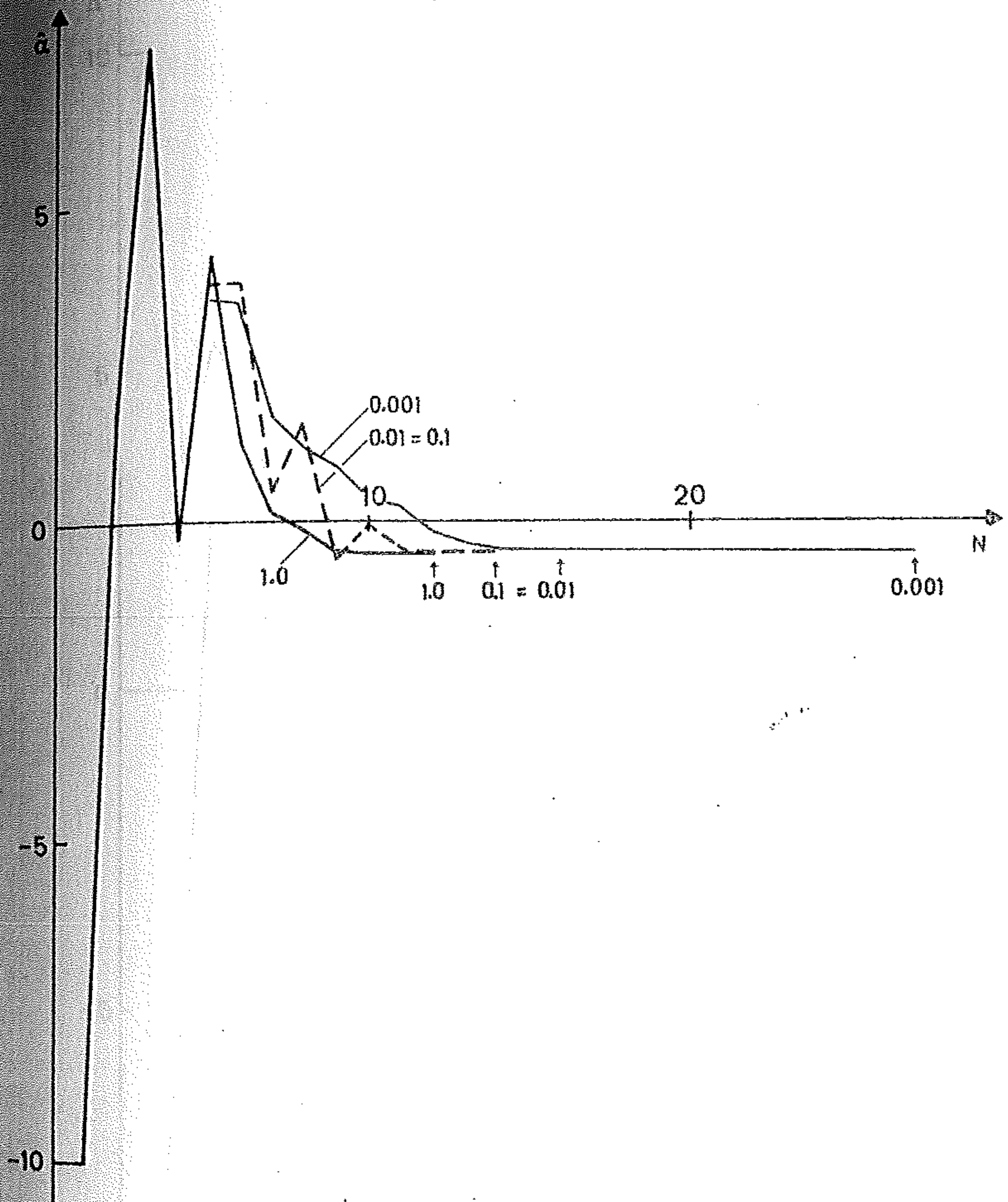


Fig. 5.1B $P(0) = \begin{pmatrix} 10^{-5} & 0 \\ 0 & 9.5^2 \end{pmatrix} \quad R_1 = \begin{pmatrix} 10^{-5} & 0 \\ 0 & r \end{pmatrix}$

$r = 1.0,\ 0.1,\ 0.01,\ 0.001$

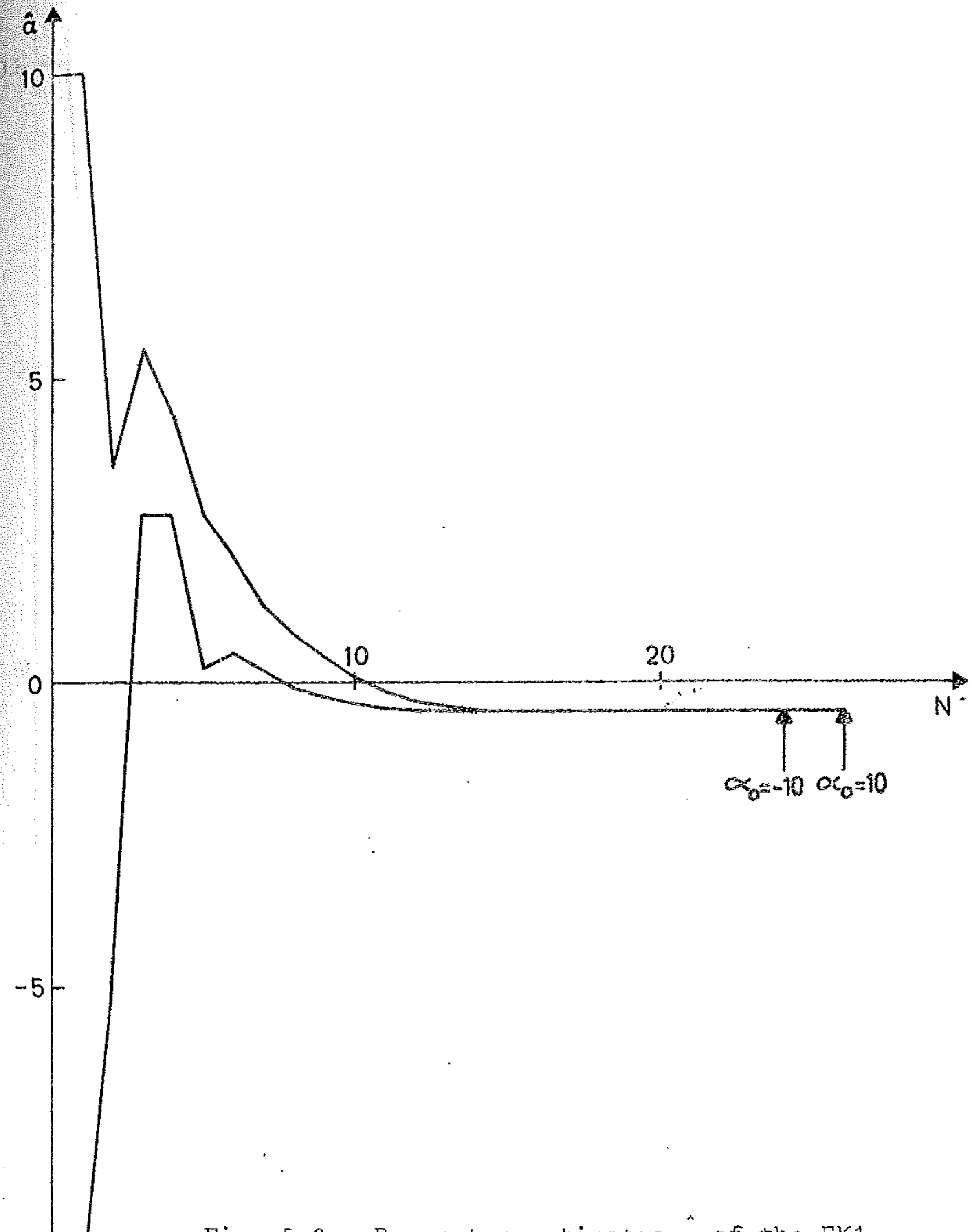


Fig. 5.2 - Parameter estimates $\hat{\alpha}$ of the EK1 filter for the first order system 1.8.

$$P(0) = \begin{pmatrix} 10^{-5} & 0 \\ 0 & 1 \end{pmatrix} \quad R_1 = \begin{pmatrix} 10^{-5} & 0 \\ 0 & 10^{-3} \end{pmatrix}$$

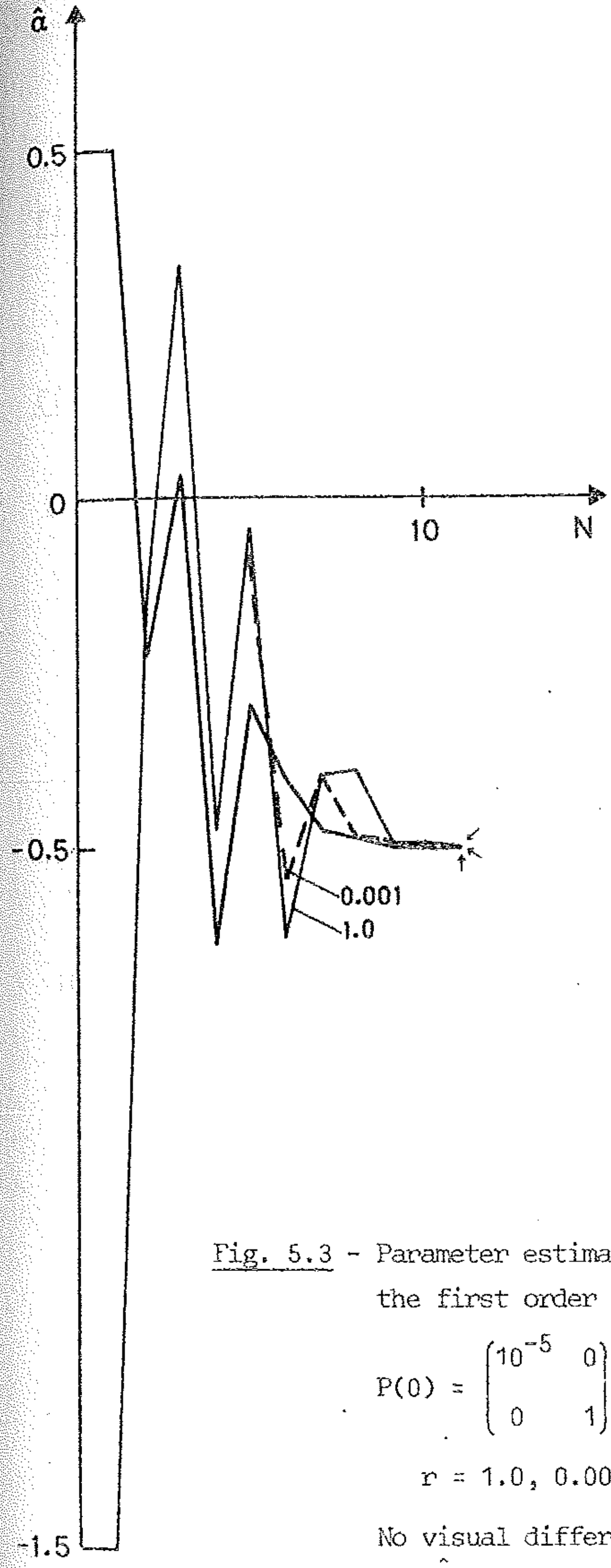


Fig. 5.3 - Parameter estimates $\hat{\alpha}$ of the EK1 filter for the first order system 1.1.

$$P(0) = \begin{pmatrix} 10^{-5} & 0 \\ 0 & 1 \end{pmatrix} \quad R_1 = \begin{pmatrix} 10^{-5} & 0 \\ 0 & r \end{pmatrix}$$

$r = 1.0,\ 0.001$

No visual difference appears for different r and $\hat{\alpha}(0) = 0.5$.

A small difference for different r can be seen for $\hat{\alpha}(0) = -1.5$.

All estimates have converged within 10^{-4} at $N = 11$.

Fig. 5.4 - Parameter estimates $\hat{\alpha}$ of the SSI filter for the first order system 1.8.

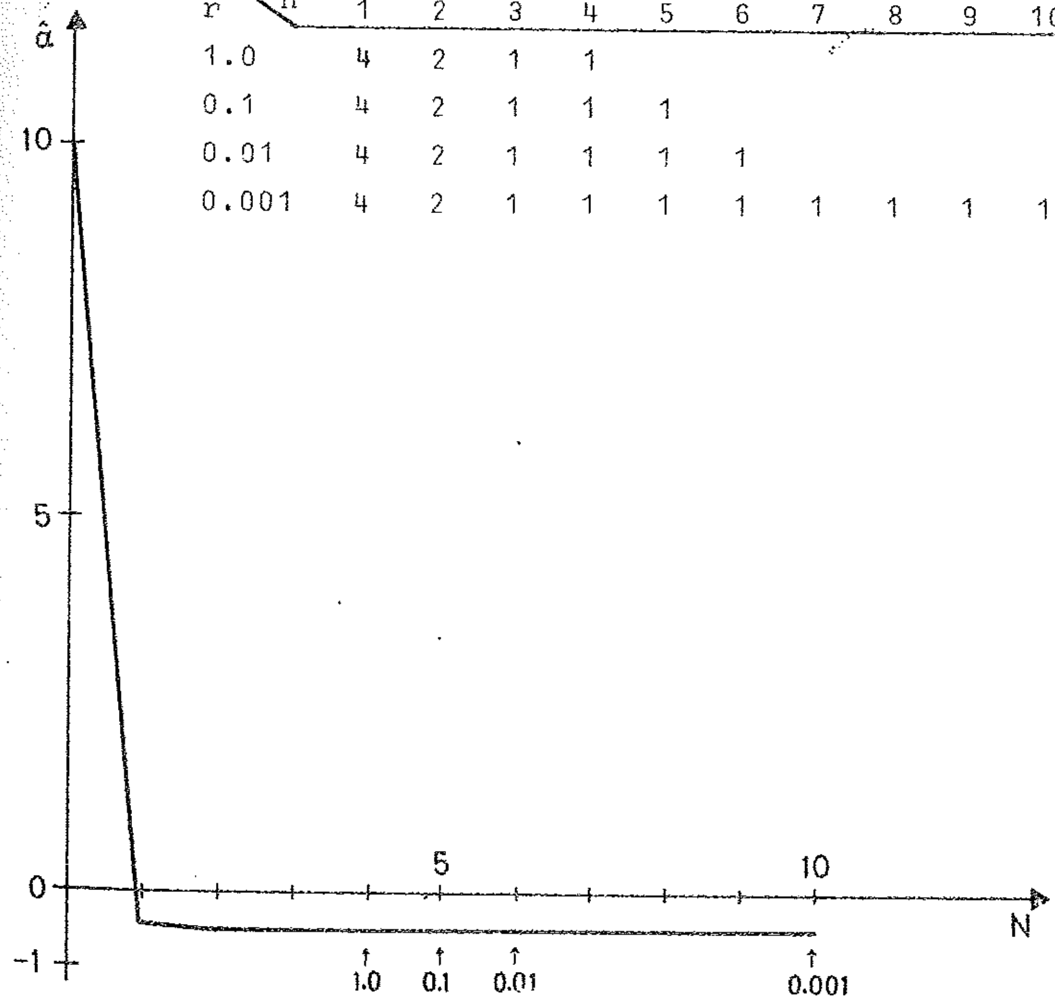
Fig. 5.4A $P(0) = \begin{pmatrix} 10^{-5} & 0 \\ 0 & 10.5^2 \end{pmatrix}$ $R_1 = \begin{pmatrix} 10^{-5} & 0 \\ 0 & r \end{pmatrix}$

$r = 1.0,\ 0.1,\ 0.01,\ 0.001$

Filter parameter $\varepsilon = 10^{-2}$ (Eq. (3.11)). The arrows along the time axis indicate when the parameter has converged within 10^{-4} from the right value.

Number of estimate calculations at each step.

r \ n	1	2	3	4	5	6	7	8	9	10
1.0	4	2	1	1						
0.1	4	2	1	1	1					
0.01	4	2	1	1	1	1				
0.001	4	2	1	1	1	1	1	1	1	1



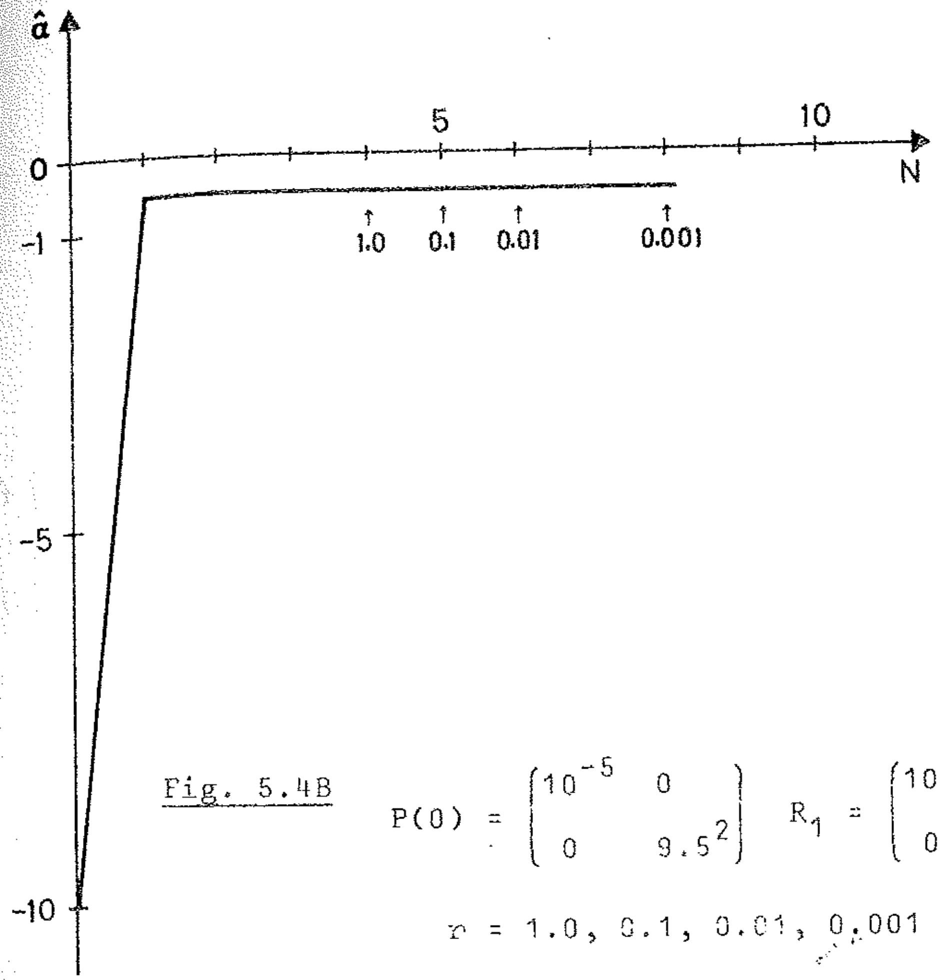


Fig. 5.4B

$$P(0) = \begin{pmatrix} 10^{-5} & 0 \\ 0 & 9.5^2 \end{pmatrix} \quad R_1 = \begin{pmatrix} 10^{-5} & 0 \\ 0 & r \end{pmatrix}$$

$r = 1.0,\ 0.1,\ 0.01,\ 0.001$

Number of estimate calculations at each step.

r \ n	1	2	3	4	5	6	7	8
1.0	4	2	1	1				
0.1	4	2	1	1	1			
0.01	4	2	1	1	1	1		
0.001	4	2	1	1	1	1	1	1

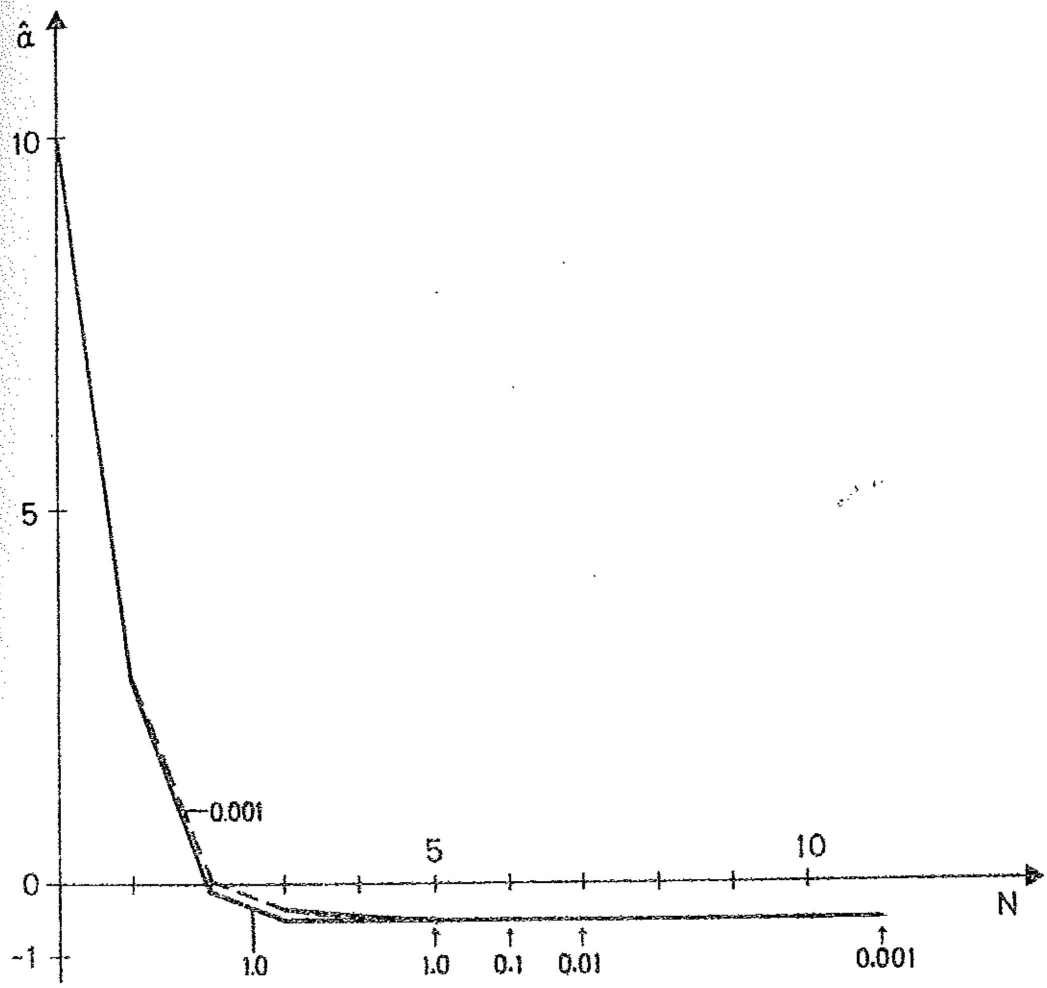


Fig. 5.5A - Parameter estimates $\hat{\alpha}$ of the EK2 filter for the system from Fig. 5.1. R_1 and P(0) are the same as in Fig. 5.4A.

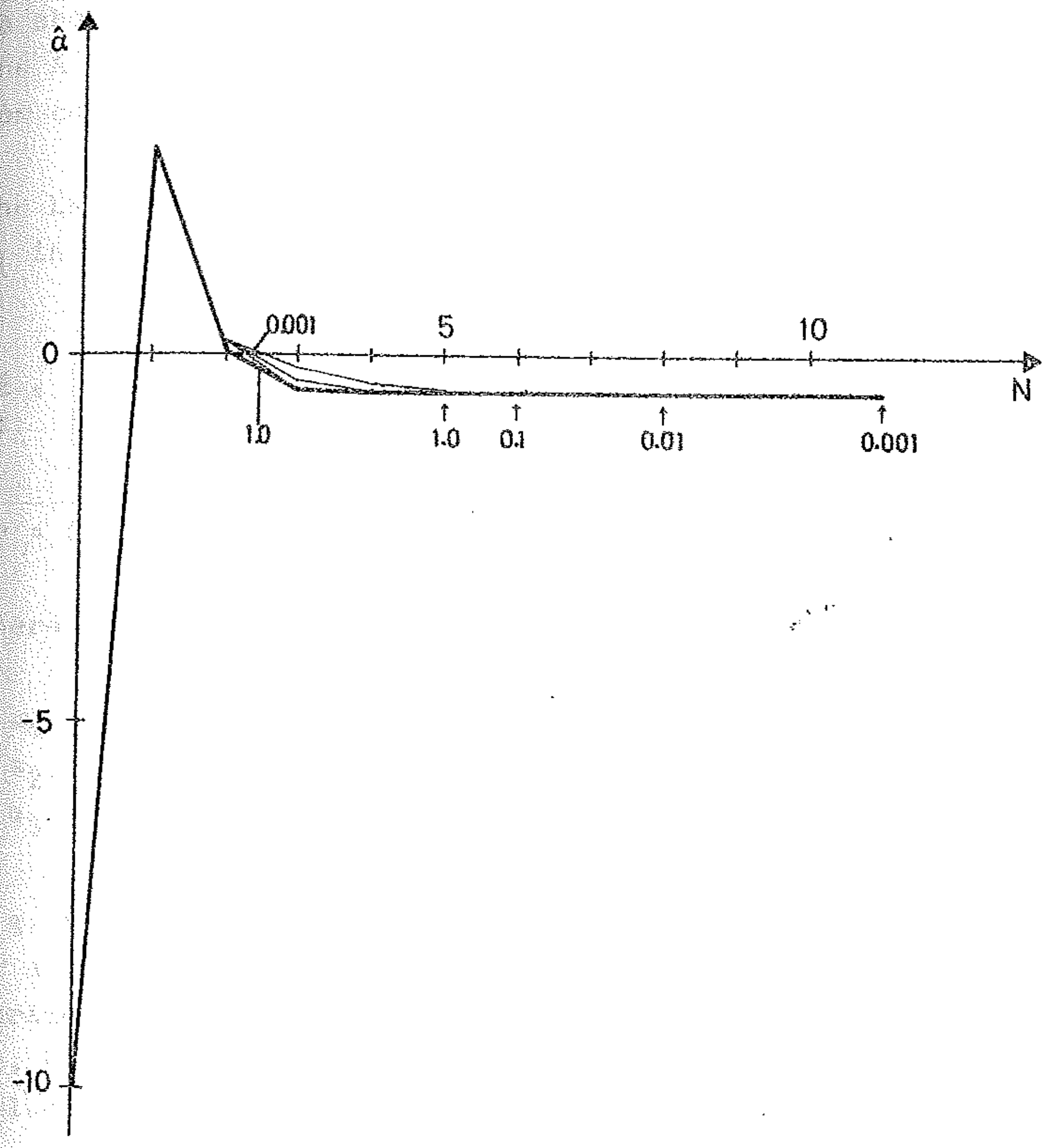


Fig. 5.5B - R_1 and $P(0)$ are the same as in Fig. 5.4B.

5.1.2. Third order test system.

By the third order test system 3.1 (see App. 1) it will be shown, that the position of the unknown parameter in the A matrix is important not only for accuracy but also for convergence rate.

The EK1 filter has been used to estimate the parameters α_i in the continuous deterministic system 3.1.

$$\frac{dz}{dt} = \begin{pmatrix} \alpha_1 & 1 & 0 \\ \alpha_2 & 0 & 1 \\ \alpha_3 & 0 & 0 \end{pmatrix} z + \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix} u \tag{5.4}$$

$$y = (1 \quad 0 \quad 0) z$$

where the true values are

$$\begin{cases} \alpha_1 = -6 \\ \alpha_2 = -11 \\ \alpha_3 = -6 \end{cases} \tag{5.5}$$

Fig. 5.6 illustrates the convergence, when one parameter at a time is unknown. As z_1 is measured directly, the α_1 parameter is influenced earlier than the α_3 parameter. It is, however, interesting to note, that α_1 gets its final value after fewer steps than α_2 and α_3.

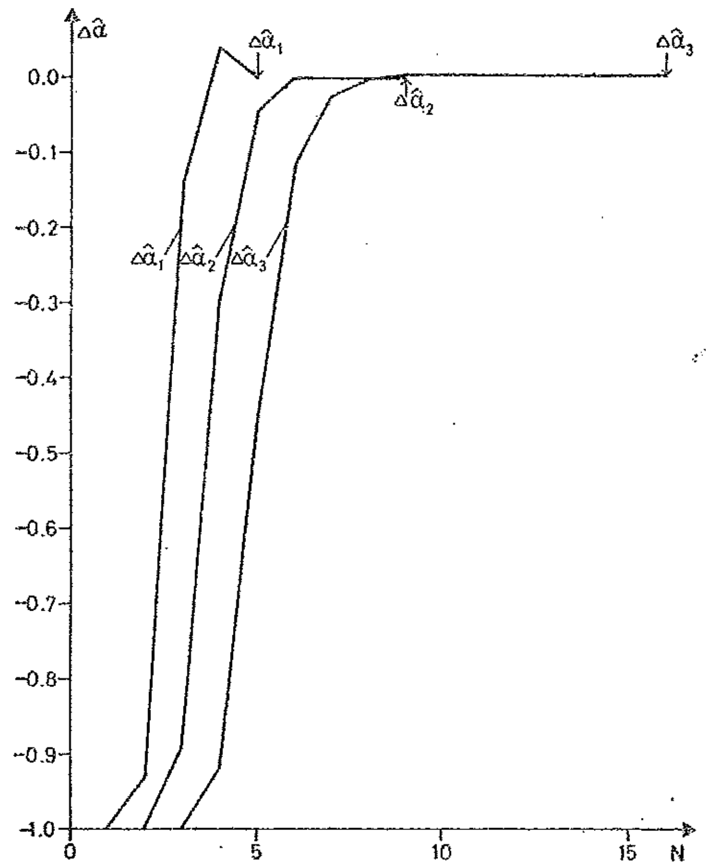


Fig. 5.6 - Parameter estimates $\hat{\alpha}$ of the EK1 filter in the system 3.1 (5.3). One parameter at a time is assumed unknown with $\hat{\alpha}(0) - \alpha = \Delta\hat{\alpha} = -1.0$

5.2. EK Filter Convergence in Stochastic Systems.

The two different questions on accuracy and convergence will now be discussed together. In 4.3 it was demonstrated, that a small R_1'' improves the accuracy and in 5.1 it has been shown that a small R_1'' causes slow convergence, at least in deterministic systems.

The general conclusions are analogous to those of chapter 4 but different types of problems occur, when iterations are involved. Therefore the discussion is devoted to the EK1 and EK2 filters in 5.2 and to the SSI filter in 5.3.

The first order system has been used as a test system, either with process noise (system 1.2) or output noise (system 1.5).

As the purpose is to find out the influence of R_1'' on the convergence rate the initial value of P has been chosen small, in order not to interfere too much

$$P(0) = \mathrm{diag}(10^{-6}, 10^{-4})$$

The first test system is system 1.5 with additive output noise

$$R_2 = 0.25 \tag{5.6}$$

The process noise is

$$R_1 = \mathrm{diag}(10^{-6}, r)$$

where r has been given different values.

Fig. 5.7A-C show the EK1 filter performance for different choices of r, while the lower figure is an expanded version of the three upper figures.

For a small r the convergence is slow but the final estimate is relatively good.

In Figs. 5.8 corresponding curves for the EK2 filter are shown. It is clearly demonstrated that after the "transient" period the realizations of the estimates are very similar in the EK1 and EK2 filters for the same r value. This fact was also noted in 4.3. The only major difference is, that the realization in the EK1 filter $[\hat{x}(t|t-1)]$ is delayed one sampling interval compared to the EK2 filter $[\hat{x}(t|t)]$.

During the "transient" part of the realization the EK2 filter converges faster than the EK1 one. This was also found in 5.1.1.

The next test system is system 1.2, disturbed by process noise but not by output noise,

$$R_1 = \mathrm{diag}(0.00632, r)$$

$$R_2 = 10^{-6}$$

Fig. 5.9 shows a number of realizations of the parameter estimate in the EK1 filter. Fig. 5.10 contains corresponding estimates with the EK2 filter. Also for this system the parameter "transients" are faster for the EK2 system. The later parts of the curves, however, look similar for corresponding values of r.

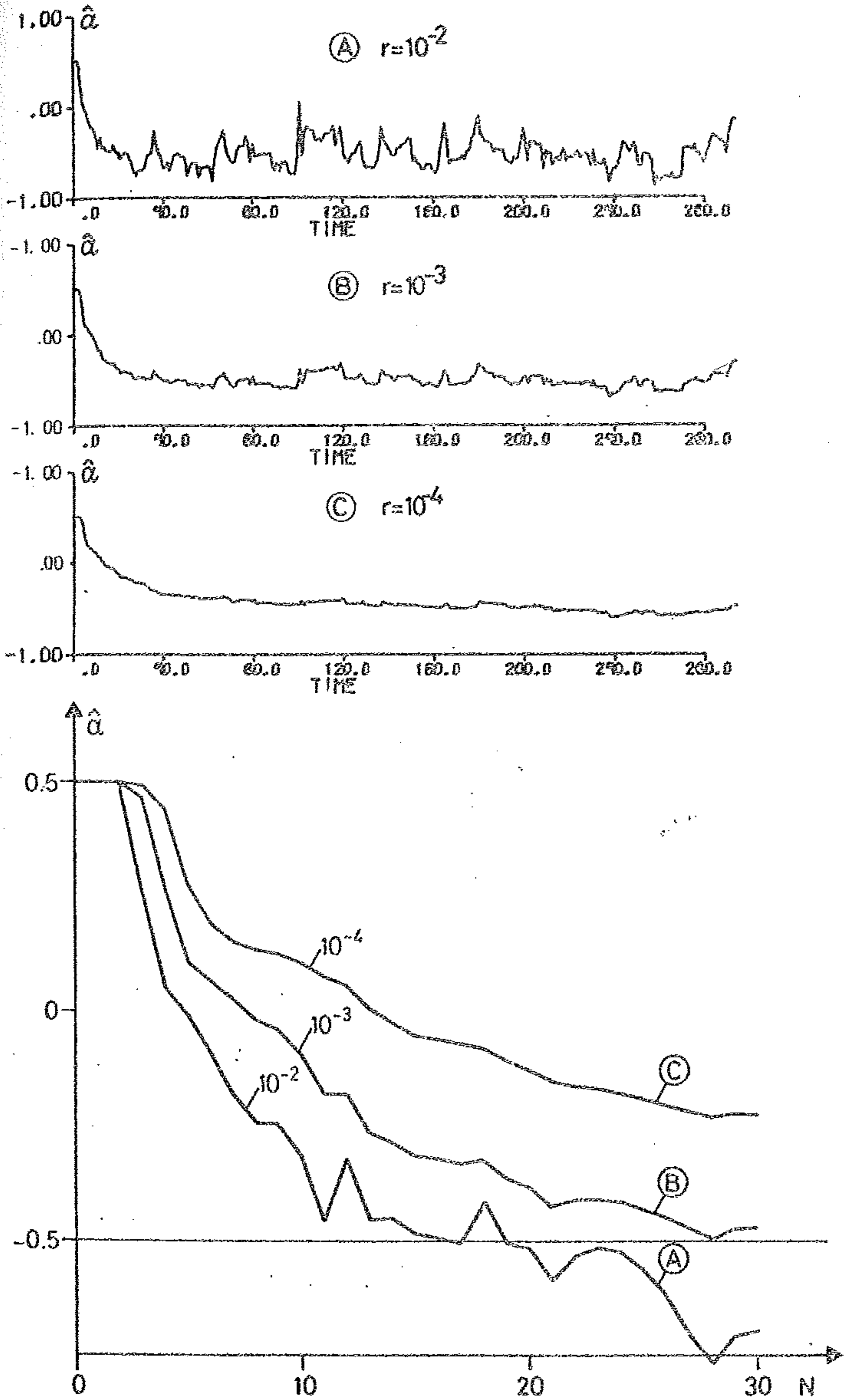


Fig. 5.7. - EK1 filter parameter estimates of the stochastic system 1.5 with output noise $R_2 = 0.25$. $R_1 = \text{diag.}\ (10^{-6}, r)$. The lower figure is an expanded version of the three upper figures.

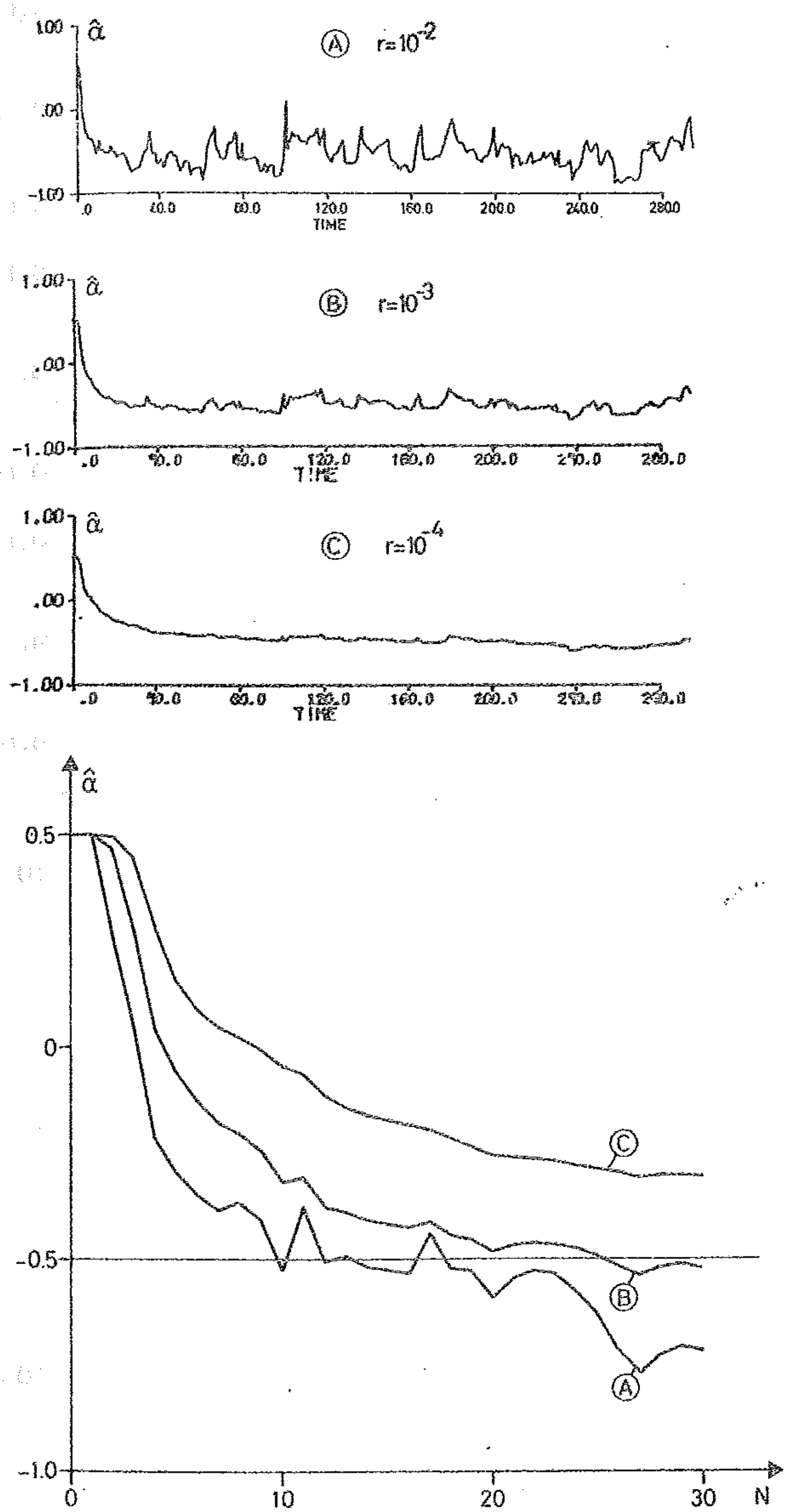


Fig. 5.8. - EK2 filter parameter estimates of the stochastic system 1.5 with output noise $R_2 = 0.25$ $R_2 = \mathrm{diag}(10^{-6}, r)$. The lower figure is an expanded version of the three upper figures.

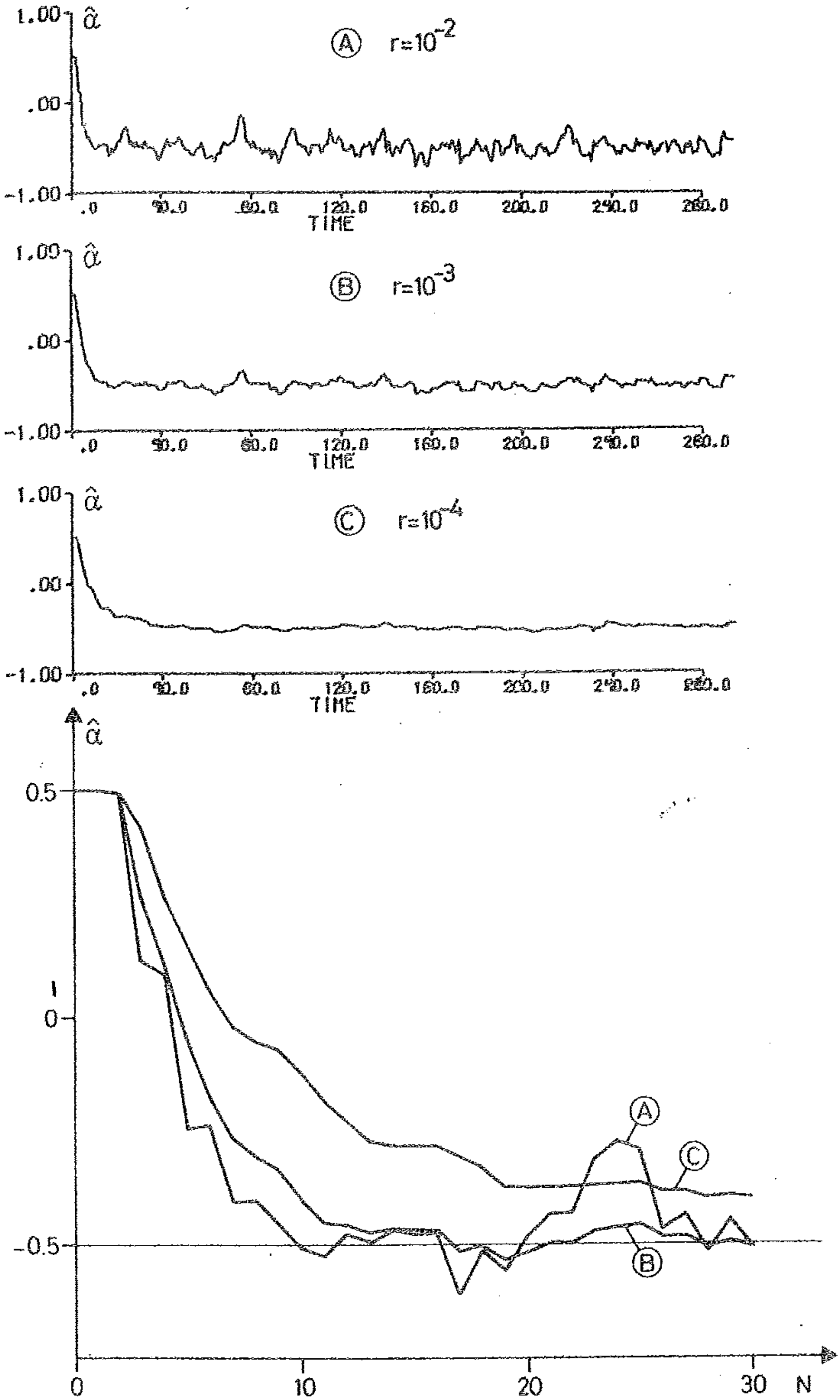


Fig. 5.9 - EK1 filter parameter estimates of the stochastic system 1.2 with process noise $R_1 = (0.00632, r)$. The lower figure is an expanded version of the the three upper figures.

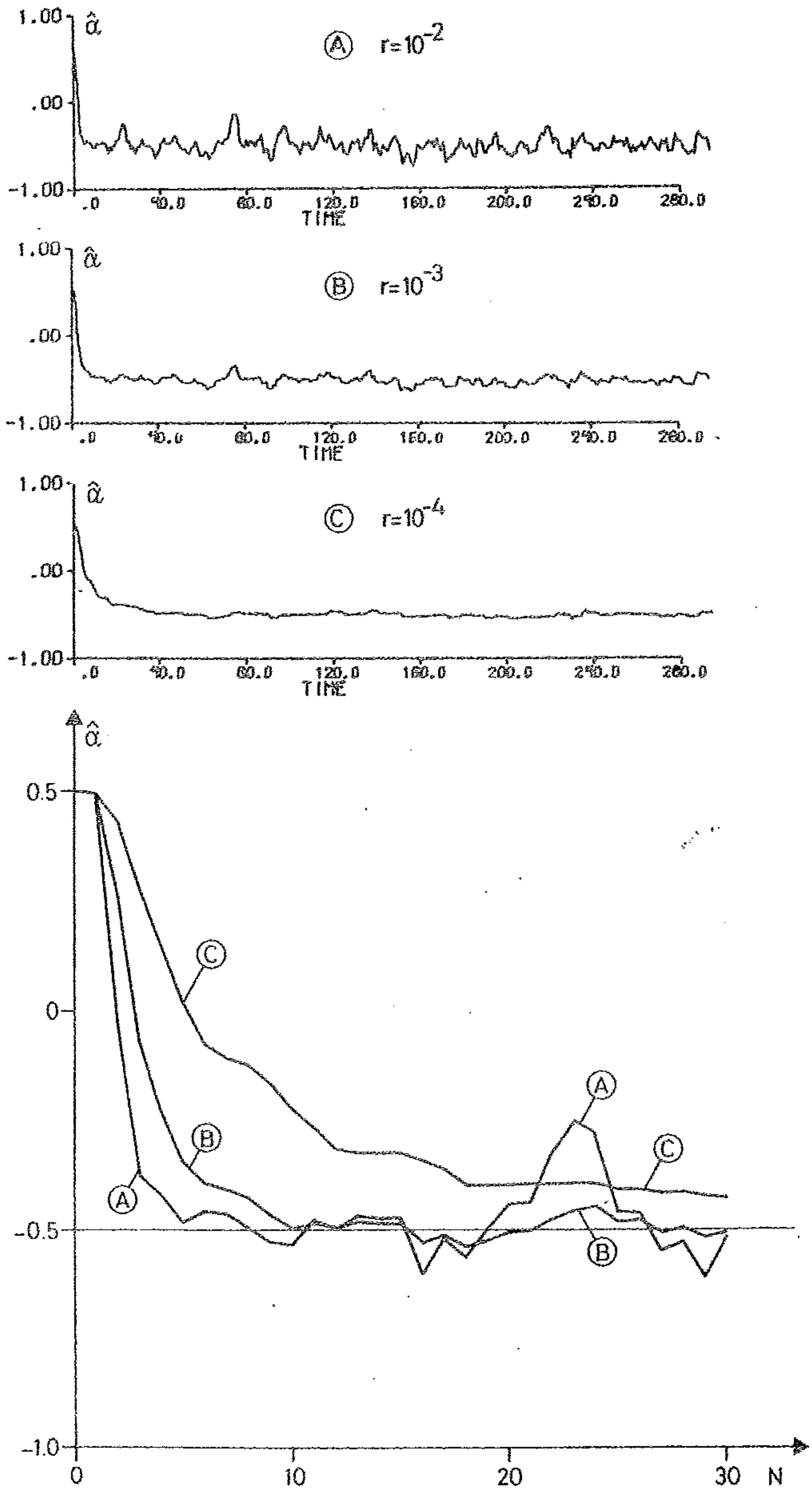


Fig. 5.10 - EK2 filter parameter estimates of the stochastic system 1.2 with R_1 = diag(0.00632, r). The lower figure is an expanded version of the three upper figures.

5.3. SSI Filter Convergence in Stochastic Systems.

In 5.1 it was shown that the iterations in the SSI filter make the convergence in the deterministic systems faster than for the EK filters. In this section it will be shown that this is true also for stochastic systems.

There are, however, some additional problems, which occur in the stochastic systems. They have to do with the choice of the iteration parameter ε (3.11).

The increasing ability of tracking the parameter that was demonstrated in 5.1 with decreasing ε may lead to a larger variance of the parameter estimation. See Fig. 5.11 where the variance is calculated on the 100 last parameter estimates. A small ε also results in many iterations per sampling interval and the iterations occur not only in the beginning of the estimation. This is illustrated in Fig. 5.12 and 5.13C for the first order system with output noise. Note that the variance of the parameter estimate does not change very much when ε is changed from 0.1 to 0.01 but the computing time is much bigger in the latter case. Some more examples illustrating the above discussion is found in [19].

Still another effect of a small ε is illustrated in Fig. 5.13 for system 1.5. ε is chosen to 10^{-2}. The other filter constants are the same as in Figs. 5.7B, C and 5.8B, C. The parameter "transient" is all the time very fast, and it is worthwhile to make some iterations in every sampling interval.

The filter in Fig. 5.13 is, however, very sensitive to residuals making the difference between two successive iterative estimates larger than ε. Especially when R_1'' is small this effect is striking. The change of the parameter per

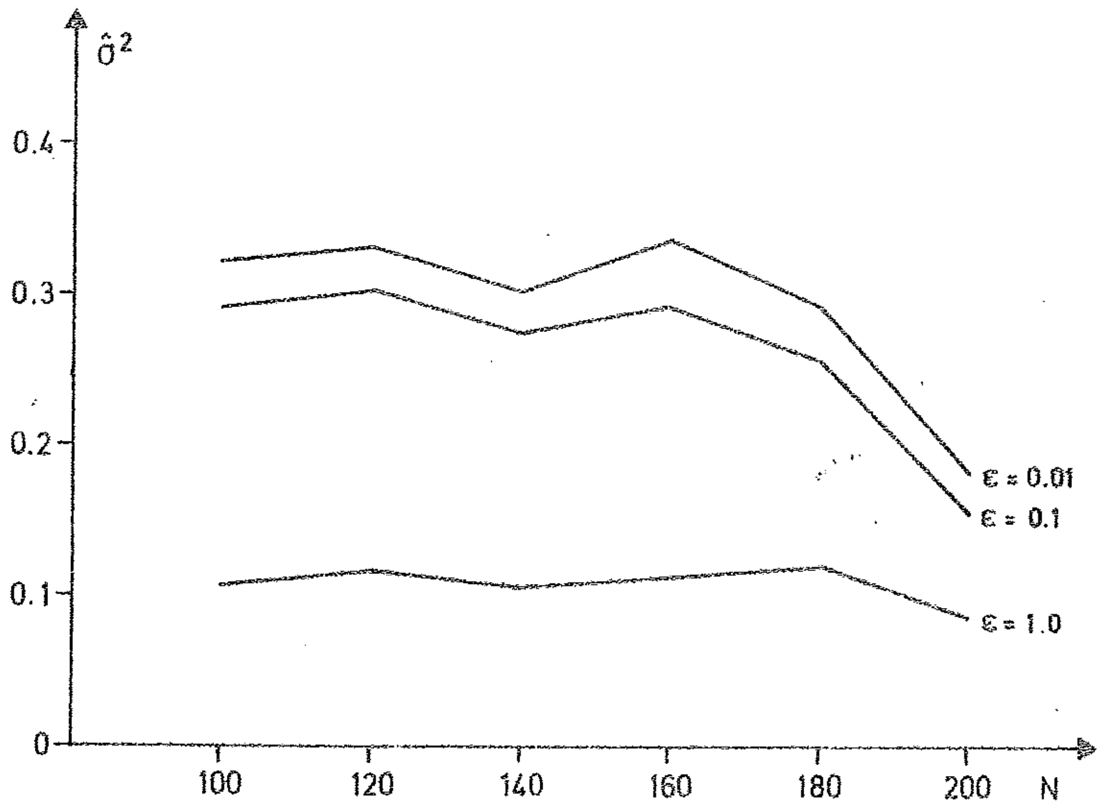


Fig. 5.11 - Empirical variance $\hat{\sigma}^2$ of the parameter estimate in system 1.4 for different values of ε at different times. $R_1'' = 1.0$
$\hat{\sigma}^2$ is calculated for the last 100 time steps.

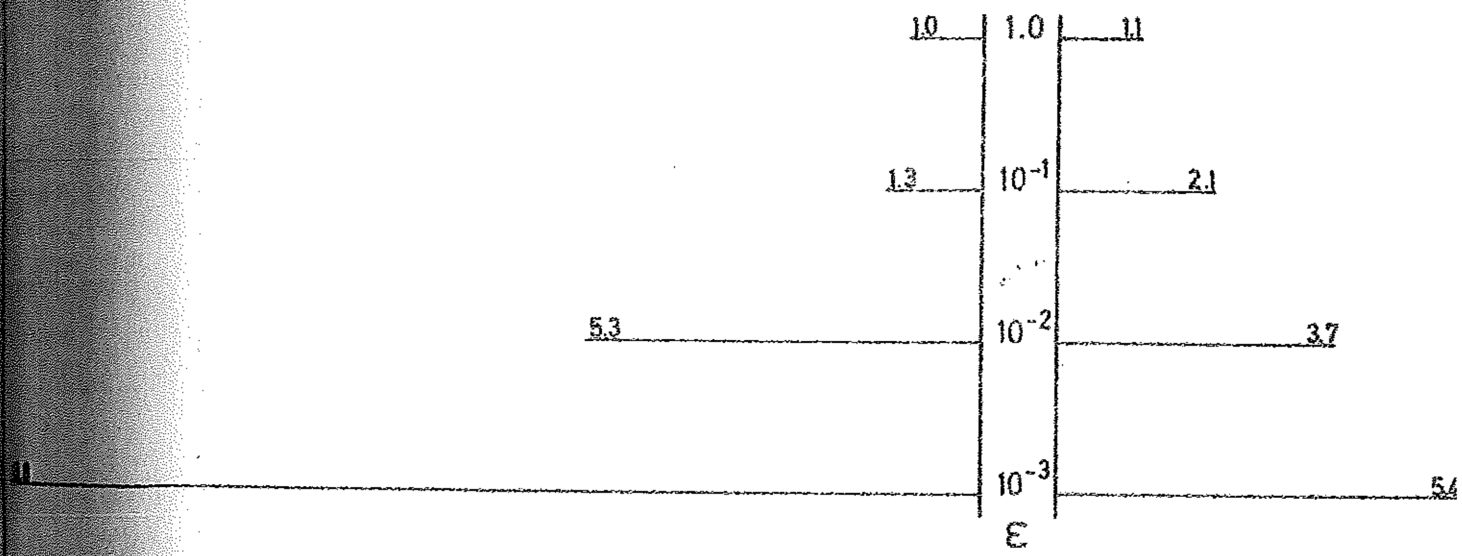


Fig. 5.12 - Mean number of iterations per sampling interval in estimation of the parameter in system 1.4 for different values of ε.

sampling interval is then small as long as no iterations are made. But here the value of ε is too small. Thus the filter is forced to make relatively many iterations.

Figure 5.13C shows the number of calculations of $\hat{x}(t|t+1)$ step by step. Whenever the curve in Fig. 5.13B has a large step it corresponds to many iterations.

The estimation can be made much better by changing ε to 10^{-1}.

Thereby the filter is less sensitive, but still some iterations are performed. For this case the parameter estimates are plotted in Fig. 5.14A, B, C. The corresponding number of iterations for the case $r = 10^{-4}$ is plotted in Figure 5.14D. The comparison shows that the last estimation (with $r = 10^{-4}$) in many respects is better than all previous estimations. However, the convergence rate is relatively low.

The results discussed can be verified by system 1.2 with process noise.

In Fig. 5.15 the SSI filter estimates are shown, and they correspond to the curves from the other filters in Figs. 5.9 and 5.10. ε has been chosen 10^{-2}. The estimates in Fig. 5.15 are not very good depending on too a small ε. If ε is made larger, however, the estimates get superior. In Fig. 5.16 ε is chosen 10^{-1}. Again, however, the convergence rate in Fig. 5.16C is relatively low.

The discussion on the choice of ε illustrates a principal difficulty with the SSI filter. Because of the iterations it is possible to choose a small R1" and still achieve a rapid "transient". The problem of choosing R1", however, is now replaced by another problem, to choose ε.

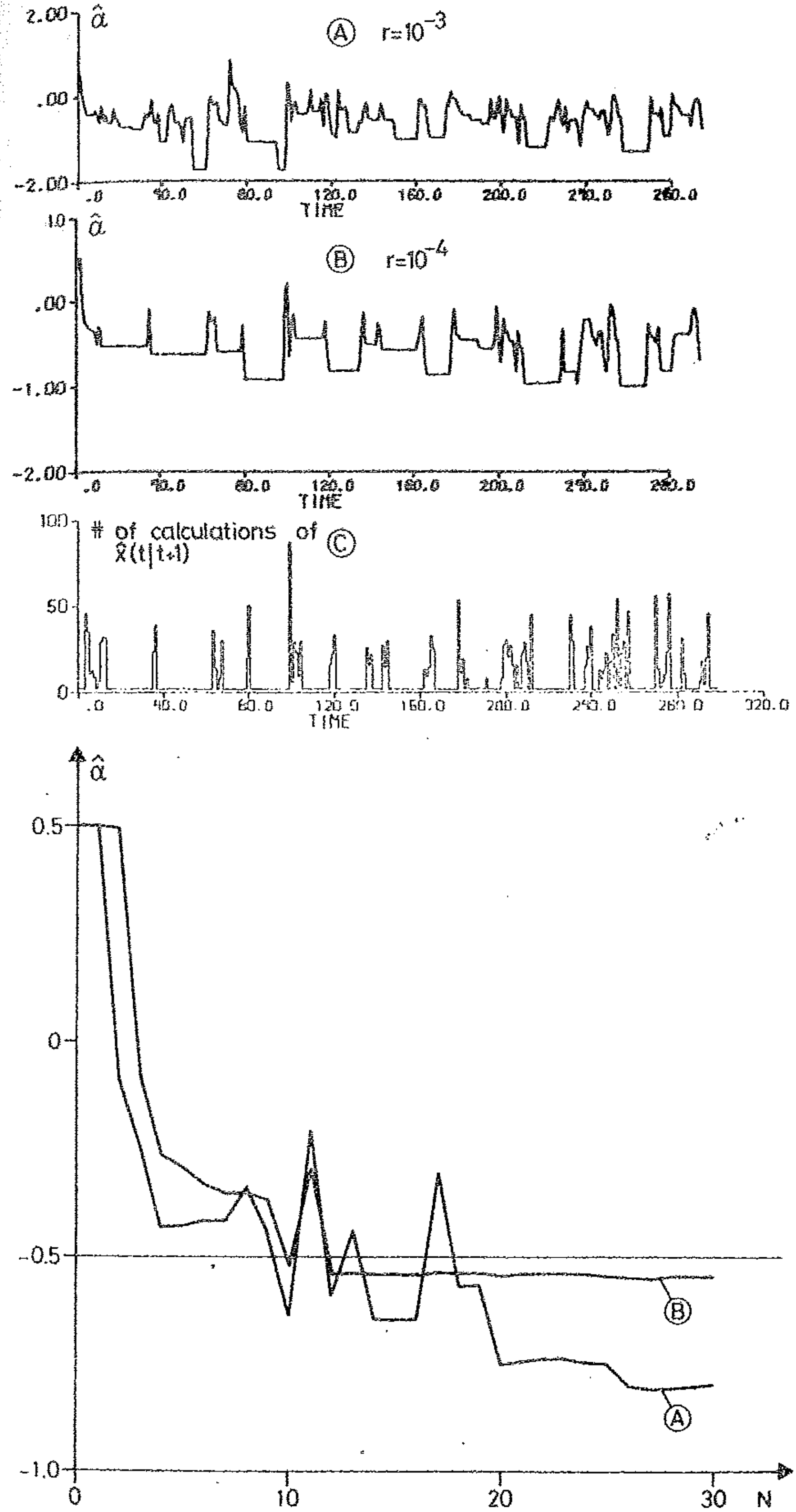


Fig. 5.13 - SSI filter parameter estimates of the system 1.5 with $R_2 = 0.25$, $R_1 = \text{diag}(10^{-6}, r)$, $\varepsilon = 10^{-2}$.
The figure C shows the number of iterations step by step when $r = 10^{-4}$.
The lower figure is an expanded version of the two upper figures.

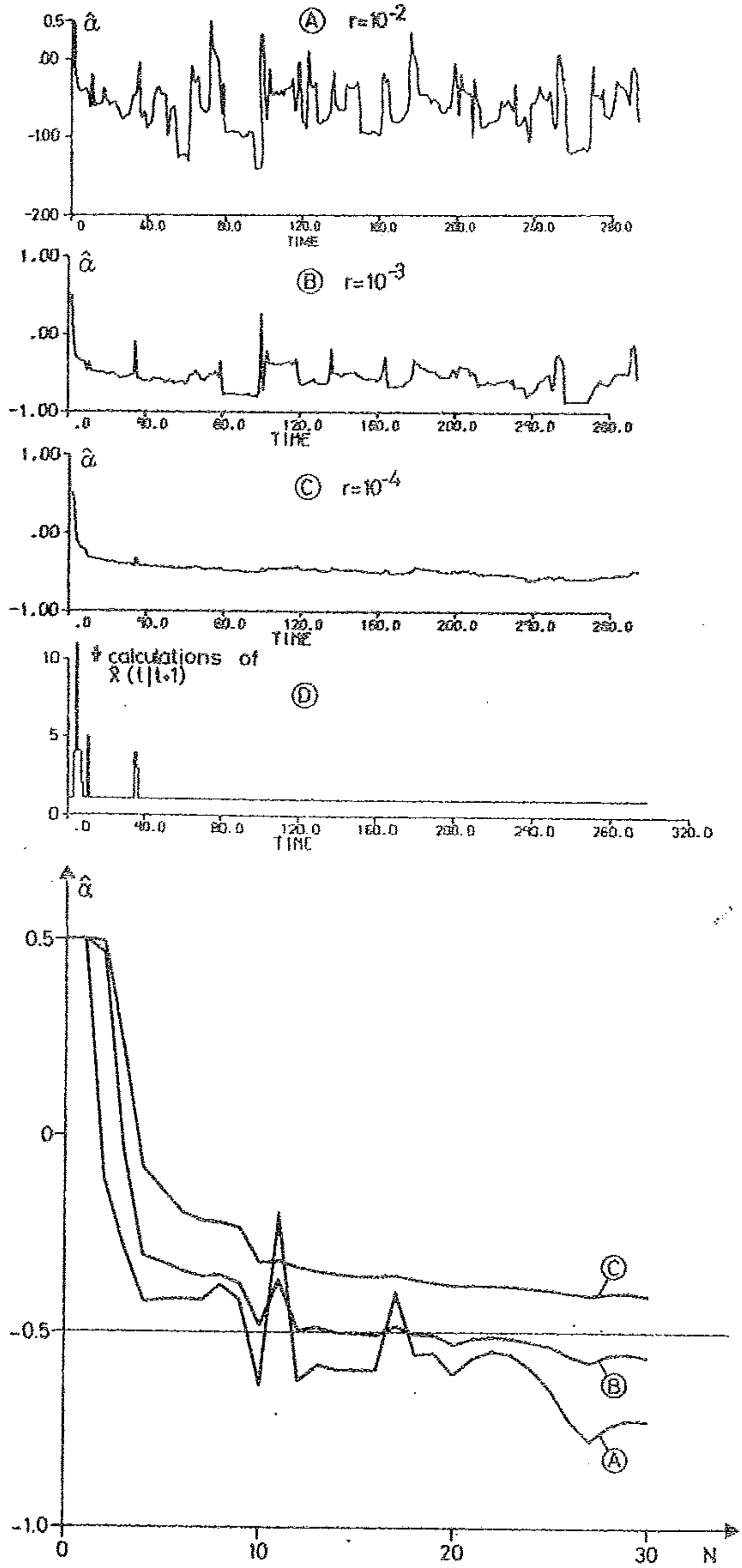
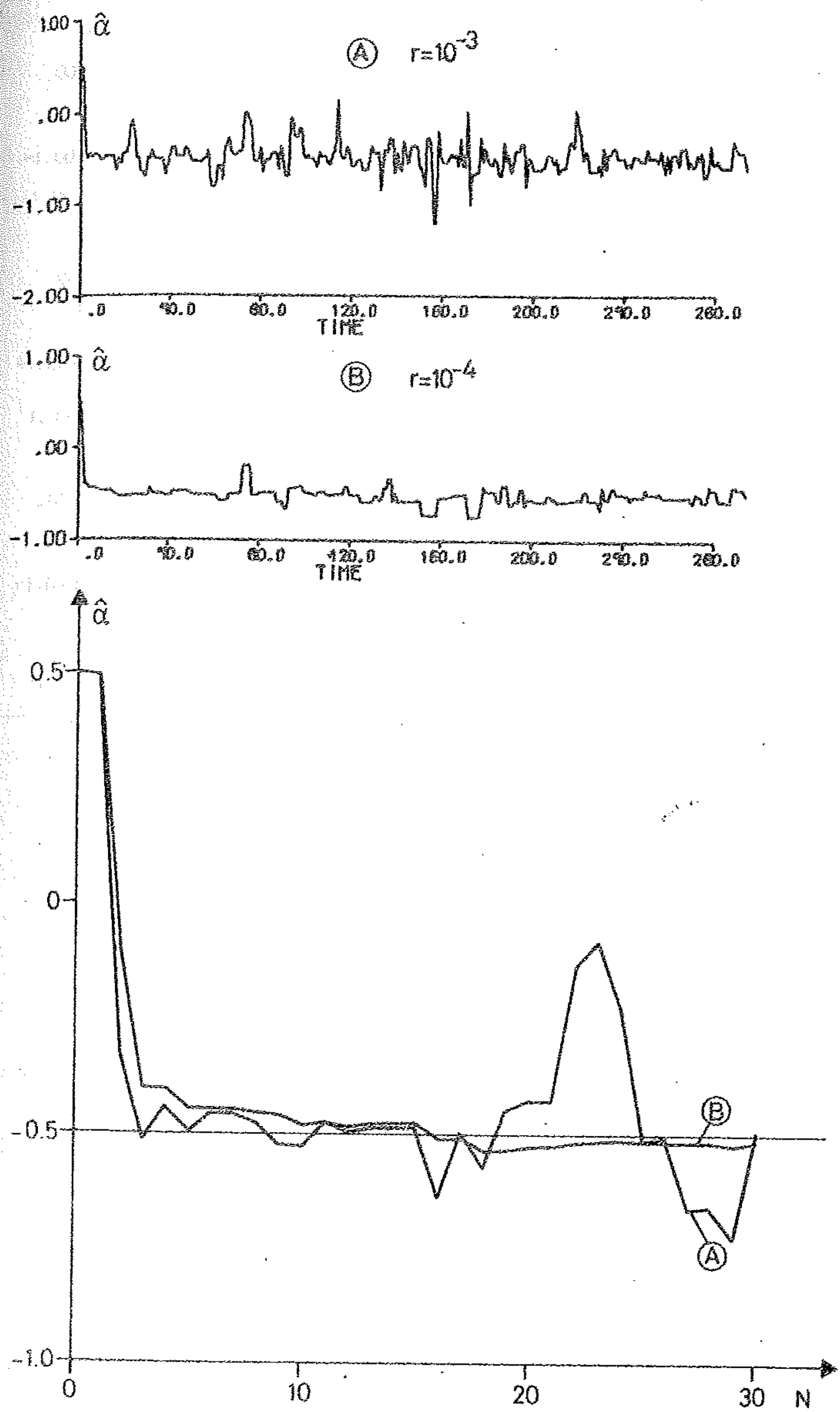


Fig. 5.14 - SSI filter parameter estimates of the system 1.5 with $R_2 = 0.25$, $R_1 = \mathrm{diag}(10^{-5}, r)$, $\varepsilon = 10^{-1}$.

The figure D shows the number of iterations at every step when $r = 10^{-4}$.

The lower figure is an expanded version of the three upper figures.



<u>Fig. 5.15</u> - SSI filter parameter estimations of the system 1.2 with R_1 = diag(0.00632, r) and $R_2 = 10^{-6}$, $\varepsilon = 10^{-2}$.
The lower figure is an expanded version of the two upper figures.