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On the use of robust controllers in adaptive control

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Abstract

Using concepts from non–adaptive robust control, the stability and robustness of a general indirect adaptive control algorithm is analyzed. It is shown that the use of robust controllers, particularly those obtained from $\mathcal{H}^\infty$ optimization techniques, can improve local robustness as long as the controller parameters are sufficiently smooth functions of the estimated parameters.

1. Introduction

Ever since the work of Rohrs and co–workers [19], much of the effort of adaptive control research has been centered on the design of robust adaptive controllers. A quick glance at the results available in this field shows that almost all of the algorithms presented achieve their robustness through modifications of the parameter identification scheme, see [11], [14], [17] for examples. During the same period, considerable progress has been made in the analysis and design of non–adaptive, robust controllers. In particular, robust controllers designed on the basis of $\mathcal{H}^\infty$ and l_1 optimization techniques have been developed, eg. [4], [9]. It then seems reasonable to study the possible improvements to the robustness of adaptive controllers that could be obtained through the use of one of these robust controllers in adaptive control.

In this paper we examine this question for a discrete–time, indirect adaptive controller. In robust control papers, the plant is assumed to consist of a nominal plant $G_o(z)$, as well as a separate transfer function $\Delta_G(z)$ used to represent plant uncertainty. Two common approaches are to model these uncertainties as additive or multiplicative perturbations on $G_o(z)$; see [6] for a discussion. An alternative expression for the plant uncertainty; where Δ_G represents additive perturbations to the stable coprime factors of the nominal plant, will be used. This type of expression for model uncertainty has been advocated by Vidyasagar, see [20], where it is shown to have some advantages over other approaches. For example, it allows the number of unstable poles to vary as the plant is perturbed. Glover and McFarlane show in [9], [10], that the design of $\mathcal{H}^\infty$–optimal controllers for this problem is surprisingly explicit, suggesting possible applications to adaptive control.

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The robustness to stable factor perturbations of a general time–varying system is considered in [15]. This analysis, however, does not include the identification scheme, and so does not truly represent an adaptive controller. A model reference adaptive controller subject to stable factor perturbations of known magnitude is considered by Krause *et al.* [13]; where the robustness of the scheme is achieved by using the known magnitude to introduce a dead zone in the identification scheme.

The results in this paper can be considered as an extension of the work of [15], in that the analysis of the identification algorithm will be included. This will be done using linearization and total stability theory as in Anderson *et al.* [1]. A similar analysis for a pole placement scheme is considered in Phillips *et al.* [16]. It will be shown that provided the mapping from estimated parameters to controller transfer functions is a sufficiently smooth function, then the use of this design techniques will improve the robustness of adaptive controllers.

Section 2 gives some preliminary facts and definitions that will be used throughout the rest of the paper. Section 3 describes the type of systems and of controllers considered. In Section 4 the local stability analysis of the adaptive controller is carried out, and in Section 5 the implications of using robust controllers adaptively, in particular, the robust controller of [10] are discussed. The results are finally summarized in Section 6.

2. Preliminaries

In the sequel, q will represent the unit delay operator $qx(k) = x(k+1)$. A causal, linear, time–invariant system will be represented by its transfer function $G(z)$, where the argument z represents the usual $\mathcal{Z}$–transform. The transfer function $G(z)$ is stable if all its poles are in $|z| < 1$. Consequently, every stable transfer function G can be expressed as $G(z) = \sum_{k=0}^{\infty} g_k z^{-k}$. Moreover, if G is strictly causal, then $g_0 = 0$.

We now define some norms which will be used throughout, see [5] for a further discussion. Let $x \in \mathcal{R}^n$; then $\|x\|_2 := \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$, and $\|x\|_\infty := \sup_{1\leq i\leq n} |x_i|$. If $A \in \mathcal{R}^{m\times n}$, the induced matrix norms will be denoted by $\|A\|_{i2} = \max_{1\leq i\leq n} \sigma_i(A)$ and $\|A\|_{i\infty} = \max_{1\leq i\leq m} \sum_{j=1}^{n} |a_{ij}|$ respectively.

A sequence of vectors $\{x(k) : k \geq 0\}$, is said to belong to l^2, and l^∞ if they satisfy: $\sum_{k=0}^{\infty} \|x(k)\|_2^2 < \infty$ and $\sup_{k\geq 0} \|x(k)\|_\infty < \infty$ respectively. The norms on l^2 and l^∞ will also be denoted by $\|\cdot\|_2$ and $\|\cdot\|_\infty$. There should no ambiguity with the vector norms, since any vector x can be considered as a sequence $\{x(k) : x(0) = x,\ x(k) = 0, \forall k > 0\}$.

A system with transfer function G is said to be l^2–stable if it maps l^2 signals into l^2. If G is time–invariant, this is equivalent to requiring

$$\|G\|_\infty := \sup_{0\leq\omega\leq 2\pi} \|G(e^{i\omega})\|_{i2} < \infty.$$

l^∞–stability is defined in a similar manner, and is equivalent to requiring that

$$\|G\|_1 := \sum_{k=1}^{\infty} \|g_k\|_{i\infty} < \infty.$$

We note that $\|G\|_\infty \leq \|G\|_1$. Moreover, for time–invariant, finite–dimensional systems, the following relation between the two norms exists.

Lemma 1 (*Boyd and Doyle* [3])
If $G(z)$ has a state-space representation of order n, then

$$\|G\|_1 \leq (2n+1)\|G\|_\infty.$$

If in addition, $G(z)$ is strictly causal, then

$$\|G\|_1 \leq 2n\|G\|_\infty.$$

□

Finally, if G is time-varying, and its input-output relation can be written as $y(k) = \sum_{j=0}^{k} g(k,j)u(j)$, then the system is l^∞ stable iff

$$\sup_{k\geq 0}\{\sum_{j=0}^{k} \|g(k,j)\|_{i\infty}\} =: \|G\|_{S1} < \infty.$$

It is easy to see that if G is time-invariant, then $\|G\|_{S1} = \|G\|_1$.

3. Plant and Controller Structure

This section presents a description of both the system and controller that are to be considered.

3.1 Nominal Plant

In order to control the true system, a model of the plant must be available to the designer. For this reason, the input-output behavior of the plant is *approximated* by the model

$$y(k) = G_0(q)u(k). \tag{1}$$

It is assumed that the transfer function $G_0(z)$ can be described by the strictly causal relation

$$G_0(z) = \frac{b_1 z^{n-1} + \ldots + b_n}{z^n + a_1 z^{n-1} + \ldots + a_n} =: \frac{z^n B(z)}{z^n A(z)} \tag{2}$$

where $B(z)$ and $A(z)$ are coprime polynomials in z^{-1}. This notation is used so that $A(z)$ and $B(z)$ are both stable transfer functions. The input-output behavior (1) can be expressed as a linear combination of the plant parameters and of filtered inputs and outputs in the usual manner:

$$y(k) = \phi(k)^T\theta_0$$

where

$$\phi(k) := [\, u(k-1) \quad \ldots \quad u(k-n) \quad -y(k-1) \quad \ldots - y(k-n) \,],$$

$$\theta_0 := [\, b_1 \quad \ldots \quad b_n \quad a_1 \quad \ldots \quad a_n \,].$$

In practice the value of θ_0 is not known; the vector θ will represent an estimate of the plant. Associated with each θ is the transfer function

$$G(z,\theta) := \frac{B(z,\theta)}{A(z,\theta)},$$

formed by replacing θ_0 with θ in the obvious manner.

It will be useful to consider coprime factorizations of G over the set of stable transfer functions. Two stable functions $A(z)$ and $B(z)$ are coprime if there exists stable functions $X(z)$ and $Y(z)$ satisfying the Bezout identity

$$A(z)X(z) + B(z)Y(z) = 1.$$

Definition 1

Let $G(z)$ be a transfer function. The pair $[N(z), M(z)]$ is a coprime factorization *of $G(z)$ if the following three conditions hold:*

(1) $M(z) \not\equiv 0$.

(2) $N(z)$, $M(z)$ *are stable, coprime* transfer functions.

(3) $G(z) = N(z)/M(z)$.

□

For any given transfer function G, the choice of N and M is not unique. For example, since the polynomials in (2) are coprime, then $[B(z), A(z)]$ is a valid coprime factorization. So is any pair $[B(z)/\xi(z), A(z)/\xi(z)]$ where $\xi(z)$ is any polynomial in z^{-1} with no zeros in $|z| \geq 1$. In the sequel it is assumed that such a $\xi(z)$ has been chosen so that N_θ and M_θ are unique functions of θ. Note that ξ may also depend on θ. For extensions of Definition 1 the reader is referred to [20].

3.2 True System

Even if θ_0 were known, it can not be assumed that the transfer function $G_0(z)$ models the true system exactly, since G_0 is only a low order approximation of the system. In the sequel it will be assumed that the true system is linear, causal, time–invariant, and that it can be described by the transfer function

$$G(z) = \frac{N_0(z) + \Delta_N(z)}{M_0(z) + \Delta_M(z)},$$

where the pair $[N_0 + \Delta_N, M_0 + \Delta_M]$ is a coprime factorization of G. The perturbation functions Δ_N and Δ_M are not known, and can be of arbitrarily high degree. Nevertheless, by property (2) of Definition 1, and the fact that the set of stable transfer functions forms a ring, then both Δ_N and Δ_M are stable. This does not imply, however, that G and G_0 share the same number of unstable poles.

3.3 Controller

In this section the type of controllers to be used are described. With each parameter vector θ, associate a controller with transfer function $K(z, \theta)$, such that $K(z, \theta)$ stabilizes $G(z, \theta)$. *Stabilizes* means that the closed–loop system is internally stable: ie. the transfer functions $S(z, \theta)$, $K(z, \theta)S(z, \theta)$, and $G(z, \theta)S(z, \theta)$ are all stable, where $S(z, \theta) := (1 - G(z, \theta)K(z, \theta))^{-1}$.

Associated with each $K(z, \theta)$, is the coprime factorization $[U(z, \theta), V(z, \theta)]$. There is an unlimited number of possible factorizations. In the sequel, $[U(z, \theta), V(z, \theta)]$, will be required to satisfy the Bezout identity

$$M(z, \theta)V(z, \theta) - N(z, \theta)U(z, \theta) = 1 \tag{3}$$

for all z. The control input $u(k)$ is given by

$$V(q)u(k) = U(q)y(k) + W(q)r'(k).$$

In this paper, the design and robustness of the feedforward transfer function W will not be discussed, so $r := Wr'$ will be regarded as the external reference input. The solution to equation (3) is not unique. Nevertheless the choice of U, and V can be specified uniquely. The following two assumptions will be made.

Assumption A1

There exists a non empty, open region $\Theta \ni \theta_0$, and a mapping $\mathcal{K} : \theta \rightarrow [U(z,\theta), V(z,\theta)]$ such that $\forall \epsilon > 0, \exists \delta > 0$, so that if $\theta_1, \theta_2 \in \Theta$,

$$\|\theta_1 - \theta_2\|_2 < \delta \Longrightarrow \|[U(z,\theta_1) - U(z,\theta_2), V(z,\theta_1) - V(z,\theta_2)]\|_\infty < \epsilon.$$

□

Assumption A2

$K(z,\theta_0) = U(z,\theta_0)V(z,\theta_0)^{-1}$ *stabilizes the true system* $G(z)$. □

Assumption A1 says that the controller is continuous in the *graph metric*, see [20]. Assumption A2 guarantees that the problem is well posed. It implies that the nominal plant models the true system sufficiently well so that the latter can be controlled. This assumption is implicit in any control system, whether adaptive or not. The following lemma presents a sufficient condition for A2 to be satisfied. For notational simplicity, let $K_0 := K(z,\theta_0)$ and $S_0 := S(z,\theta_0)$.

Lemma 2 (*Vidyasagar* [20])

Assumption A2 holds provided that

$$\left\| \begin{bmatrix} S_o M_o^{-1} \\ K_o S_o M_o^{-1} \end{bmatrix} [\Delta_M \ \Delta_N] \right\|_\infty < 1. \tag{4}$$

□

This lemma motivates the design of an $\mathcal{H}^\infty$ optimal robust controller: a controller which, while requiring that K_0 stabilize G_0, minimizes the expression

$$\left\| \begin{bmatrix} S_o M_o^{-1} \\ K_o S_o M_o^{-1} \end{bmatrix} \right\|_\infty .$$

The solution of this problem for the continuous-time case is derived by Glover and McFarlane in [9], and [10] where the coprime-factorization is specified to be *normalized*; that is:

$$N_0(z)N_0(z^{-1}) + M_0(z)M_0(z^{-1}) = 1.$$

This normalization allows one to bypass the expensive iterative techniques that are normally associated with $\mathcal{H}^\infty$ controllers; see [7] and [8]. A state-space description of the controller can be given explicitly in terms of the system's state-space matrices, and the solution to two uncoupled Riccati equations. More of this will be said in Section 5, where it will be shown that this controller has some other properties beneficial to adaptive controllers. Note that requiring that $[N_0, M_0]$ be normalized is the same as requiring ξ to be the spectral factor: $A(z)A(z^{-1}) + B(z)B(z^{-1}) = \xi(z)\xi(z^{-1})$.

In an adaptive control setting, θ_0 is not known, so K_0 can not be implemented directly. Instead, the *certainty equivalence* controller is used. That is, at each time k, if the plant estimate is $\theta(k)$, then $K(\theta(k))$ is applied to the system. In order to consider the robustness of the adaptive control algorithm in a unified fashion, assume that the

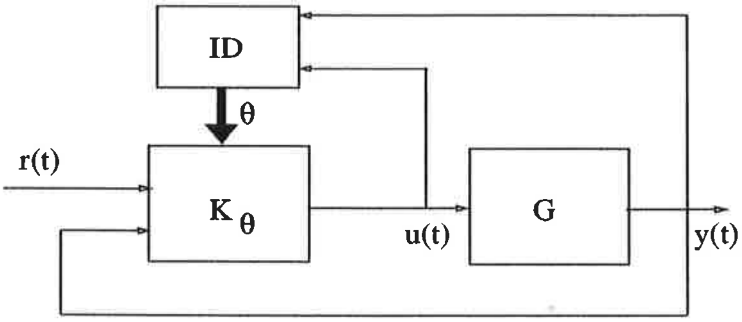

Figure 1. General adaptive controller.

difference between the nominal controller, and the frozen parameter controller $K(z,\theta)$; the controller obtained by freezing the parameter estimates at θ, can be represented as

$$K(z,\theta) = \frac{U_0(z) + \Delta_U(z,\theta)}{V_0(z) + \Delta_V(z,\theta)},$$

where $[U_0 + \Delta_U(z,\theta), V_0 + \Delta_V(z,\theta)]$ is a coprime factorization of $K(z,\theta)$. Although the perturbation functions $\Delta_U(z,\theta)$ and $\Delta_V(z,\theta)$ are stable for each time k, the time-varying operators need not be stable. Nevertheless, if the parameter estimates vary sufficiently slowly, closed loop stability will be retained, see eg. [5], page 147.

4. Local Analysis

The adaptive control problem can be depicted as in Figure 1. The plant is time-invariant. The only time variations in the closed loop system are due to the changing controller, which varies according to the estimated plant parameters. To keep the discussion as simple as possible, assume that the plant estimates change according to a projection identification scheme. That is,

$$\theta(k+1) = \theta(k) + \gamma \frac{\phi(k)e(k,\theta)}{1 + \gamma\phi(k)^T\phi(k)}, \tag{5}$$

where $e(k,\theta)$ is the prediction error, defined by

$$e(k,\theta) := y(k) - \phi(k)^T\theta(k).$$

The constant $\gamma > 0$ is the ***step-size*** or ***gain*** of the identification algorithm. Since the system is operating in closed loop, $y(k)$ and $u(k)$ depend implicitly on the estimated parameter $\theta(k)$. In the sequel this dependence will be made explicit.

In order to analyze how the model and controller uncertainties influence y and u, begin by reconfiguring the system as in Figure 2, according to the definitions of Section 2. Using the linearity, and the time invariance of G_0 and K_0, the input–output behavior can be written as

$$x(k,\theta) = \mathcal{P}_G r(k) + \mathcal{T}_G\Delta_G x(k,\theta) + \mathcal{T}_K v(k,\theta(k)) \tag{6}$$

where the following abbreviations have been introduced:

$$x(k,\theta) := \begin{bmatrix} y(k,\theta) \\ u(k,\theta) \end{bmatrix}, \qquad \mathcal{P}_G := \begin{bmatrix} G_0S_0V_0^{-1} \\ S_0V_0^{-1} \end{bmatrix}$$

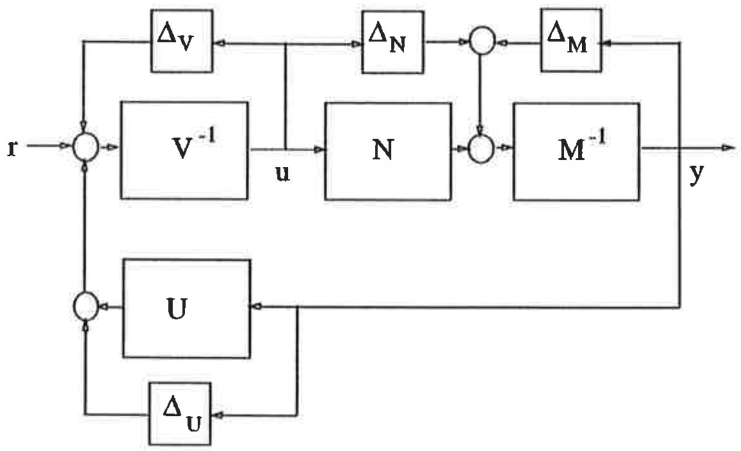

Figure 2. Adaptive control system with nominal plant and controller stable factors perturbations.

$$\mathcal{T}_G := \begin{bmatrix} S_o M_o^{-1} \\ K_o S_o M_o^{-1} \end{bmatrix} = \begin{bmatrix} V_0 \\ U_0 \end{bmatrix}, \tag{7}$$

$$\mathcal{T}_K := \begin{bmatrix} G_o S_o V_o^{-1} \\ S_o V_o^{-1} \end{bmatrix} = \begin{bmatrix} N_0 \\ M_0 \end{bmatrix}, \tag{8}$$

$$\Delta_G := [\Delta_M \ \Delta_N],$$

and

$$\begin{aligned} v(k,\theta(k)) &:= \left([\Delta_{V(\theta)} \ \Delta_{U(\theta)}] \begin{bmatrix} y \\ u \end{bmatrix} \right)(k) \\ &=: \sum_{j=0}^{k} \delta(k,j) x(j,\theta). \end{aligned}$$

Equalities (7) and (8) have been obtained from the Bezout identity (3). Collecting the x terms on the left side, and using equation (4) which guarantees that $I - \mathcal{T}_G \Delta_G$ is invertible, we can divide both sides by $I - \mathcal{T}_G \Delta_G$. Using the matrix identity $(I - X)^{-1} = I + X(I - X)^{-1}$, equation (6) becomes

$$\begin{aligned} x(k,\theta) &= \mathcal{P}_G r(k) + \mathcal{T}_G \Delta_G (I - \mathcal{T}_G \Delta_G)^{-1} \mathcal{P}_G r(k) \\ &\quad + (I - \mathcal{T}_G \Delta_G)^{-1} \mathcal{T}_K v(k,\theta(k)) \\ &=: x_0(k) + \tilde{x}(k) + \bar{x}(k,\theta). \end{aligned} \tag{9}$$

Since x_0 depends only on the nominal plant and controller, it can be regarded as the ideal system input and output. In contrast $\tilde{x}(k)$ corresponds to input and output perturbations that are introduced by the existence of unmodelled dynamics. These perturbations are guaranteed to remain bounded by (4). Finally, the vector $\bar{x}(k,\theta)$ represents perturbations that are due to the time-varying, incorrectly identified plant. These are not guaranteed to remain stable from previous assumptions. Note also that any bound on the size of $\bar{x}(k,\theta)$ will be magnified by the presence of unmodelled dynamics. This can be seen, from the appearance of the $(I - \mathcal{T}_G \Delta_G)^{-1}$ term in front of $\mathcal{T}_K v(k,\theta(k))$ in (9). The following lemma gives a sufficient condition under which $\bar{x}(k,\theta)$ will remain stable.

Lemma 3

The system from r to $[y,u]$ will be l^∞ stable provided that the time–varying perturbations

$\Delta_{K(\theta)} := [\Delta_{V(\theta)}\ \Delta_{U(\theta)}]$ *are stable, and satisfy*

$$\frac{\|\Delta_{K(\theta)}\|_{S1}\|\mathcal{T}_K\|_1}{1-\|\mathcal{T}_G\Delta_G\|_1} \le \mu < 1. \tag{10}$$

Proof: From (9):

$$\begin{aligned}\|x(k,\theta)\|_\infty &\le \|x_0+\tilde{x}\|_\infty + \|(I-\mathcal{T}_G\Delta_G)^{-1}\mathcal{T}_K v\|_\infty\\ &\le \|x_0+\tilde{x}\|_\infty + \|(I-\mathcal{T}_G\Delta_G)^{-1}\mathcal{T}_K\|_1\|v\|_\infty\\ &\le \|x_0+\tilde{x}\|_\infty + (1-\|\mathcal{T}_G\Delta_G\|_1)^{-1}\|\mathcal{T}_K\|_1\|\Delta_{K(\theta)}\|_{S1}\|x(k,\theta)\|_\infty.\end{aligned}$$

Collecting the $\|x(k,\theta)\|_\infty$ terms on the left, gives

$$\left(1-\frac{\|\Delta_{K(\theta)}\|_{S1}\|\mathcal{T}_K\|_1}{1-\|\mathcal{T}_G\Delta_G\|_1}\right)\|x(k,\theta)\|_\infty \le \|x_0+\tilde{x}\|_\infty$$

which, if (10) holds, and since x_0 and $\tilde{x}$ are bounded implies $\|x(\cdot,\theta)\|_\infty \le (1-\mu)^{-1}\times \|x_0+\tilde{x}\|_\infty < \infty$ as required.

□

Lemma 3 provides a simple, albeit conservative condition for the stability of the overall time–varying system. Consider the identification algorithm (5). Note that the regression vector $\phi(k,\theta)$ is obtained by a linear filtering operation on $x(k,\theta)$, say $\phi = \Gamma x$. Then $\phi(k,\theta)$ can be divided into the sum of three components ϕ_0, $\tilde{\phi}$ and $\bar{\phi}(k,\theta)$ according to (9). Similarly, the prediction error $e(k,\theta)$ can be separated as follows:

$$\begin{aligned}e(k,\theta) &= y(k,\theta) - \phi(k,\theta)^T\theta(k)\\ &= y_o(k) - \phi(k,\theta)^T\theta(k) + \tilde{y}(k) + \bar{y}(k,\theta)\\ &= -\phi(k,\theta)^T(\theta(k)-\theta_0) + \tilde{y}(k) - \tilde{\phi}(k)^T\theta_0 + \bar{y}(k,\theta) - \bar{\phi}(k,\theta)^T\theta_0\\ &=: -\phi(k,\theta)^T\tilde{\theta}(k) + \tilde{e}(k) + \bar{e}(k,\theta)\end{aligned} \tag{11}$$

where the third equality was obtained using the fact that $y_o(k) = \phi_o(k)^T\theta_0$. The expressions for $\bar{e}(k,\theta)$ and $\tilde{e}(k)$ can be evaluated explicitly in terms of the unmodelled dynamics Δ_G and controller perturbations $\Delta_{K(\theta)}$. Using the fact that $[M_0\ \ -N_0]\mathcal{T}_G = 1$ and $[M_0\ \ -N_0]\mathcal{T}_K = 0$ gives the following expressions:

$$\begin{aligned}\tilde{e}(k) &= \tilde{y}(k) - \theta_0^T\tilde{\phi}(k) = A_0\tilde{y}(k) - B_0\tilde{u}(k)\\ &= \xi\,[M_0\ \ -N_0]\tilde{x}(k)\\ &= \xi\,[M_0\ \ -N_0]\begin{bmatrix} S_oM_o^{-1}\\ K_oS_oM_o^{-1}\end{bmatrix}\Delta_G(I-\mathcal{T}_G\Delta_G)^{-1}x_0(k)\\ &= \xi\, M_0[I\ \ -G_0]\begin{bmatrix} I\\ K_0\end{bmatrix} S_0M_0^{-1}\Delta_G(I-\mathcal{T}_G\Delta_G)^{-1}x_0(k)\\ &= \xi\,\Delta_G(I-\Delta_G\mathcal{T}_G)^{-1}x_0(k)\end{aligned} \tag{12}$$

and similarly,

$$\begin{aligned}\bar{e}(k,\theta) &= \bar{y}(k,\theta) - \theta_0^T\bar{\phi}(k,\theta) = A_0\bar{y}(k,\theta) - B_0\bar{u}(k,\theta)\\ &= \xi\,[M_0\ \ -N_0](I-\mathcal{T}_G\Delta_G)^{-1}\mathcal{T}_K(\Delta_{K(\theta)}x)(k)\\ &= \xi\,[M_0\ \ -N_0]\{I+\mathcal{T}_G\Delta_G(I-\mathcal{T}_G\Delta_G)^{-1}\}\\ &\qquad\times\mathcal{T}_K(\Delta_{K(\theta)}x)(k)\\ &= \xi\,\Delta_G(I-\mathcal{T}_G\Delta_G)^{-1}\mathcal{T}_K(\Delta_{K(\theta)}x)(k).\end{aligned} \tag{13}$$

Replacing $e(k,\theta)$ in (5), with equation (11) gives

$$\tilde{\theta}(k+1) = \left(I - \gamma\frac{\phi(k,\theta)\phi(k,\theta)^T}{1+\gamma\phi(k,\theta)^T\phi(k,\theta)}\right)\tilde{\theta}(k) + \frac{\gamma\phi(k,\theta)(\tilde{e}(k)+\bar{e}(k,\theta))}{1+\gamma\phi(k,\theta)^T\phi(k,\theta)}. \tag{14}$$

Equation (14) is a non–linear difference equation. Notice that $\bar{e}(k,\theta) \neq 0$ implies that $\theta = \theta_0$ will no longer be a fixed point of (14). To analyze the stability properties of (14) proceed as follows. Freeze the time variations at some parameter $\bar{\theta}$. This means that the time–varying operator $\Delta_{K(\theta)}$ becomes the time–invariant transfer function $\Delta_{K(\bar{\theta})}$. Replacing this equation in (14) and linearizing about the nominal model θ_0, yields the following equation

$$\tilde{\bar{\theta}}(k+1) = \Lambda(k)\tilde{\bar{\theta}}(k) + f(k)\tilde{\bar{\theta}}(k) + g(k) + O(\|\tilde{\bar{\theta}}\|_\infty^2) \tag{15}$$

where

$$\begin{aligned}
\Lambda(k) &:= I - \gamma\phi(k,\theta_0)\phi(k,\theta_0)^T/d(k),\\
f(k) &:= \gamma\left(\phi(k,\bar{\theta}_0)\frac{\partial\bar{e}(k,\bar{\theta})}{\partial\bar{\theta}}|_{\bar{\theta}=\theta_0} + \tilde{e}(k)\frac{\partial\bar{\phi}(k,\theta)}{\partial\bar{\theta}}|_{\theta=\theta_0}\right)/d(k)\\
&\quad - \gamma^2\left(\tilde{e}(k)\phi(k,\theta_0)\phi(k,\theta_0)^T\frac{\partial\bar{\phi}(k,\theta)}{\partial\bar{\theta}}|_{\bar{\theta}=\theta_0}\right)/d(k)^2,\\
g(k) &:= \gamma\phi(k,\theta_0)\tilde{e}(k)/d(k),\\
d(k) &:= 1 + \gamma\phi(k,\theta_o)^T\phi(k,\theta_o).
\end{aligned}$$

The first analysis of this type was carried out for Rohrs' counterexamples, see [2]. Since then considerable work has been done rigorizing the validity of the steps above see [1], and [18]. The value of the analysis in this paper is that the bounds of the magnitudes of f and g can be expressed in terms of the transfer function norms of of $\mathcal{T}_G$, $\mathcal{T}_K$, etc.. For notational simplicity, all $\|\cdot\|$ will refer to the induced l^∞ norm, and the time dependence of all functions will be omitted. Let $\|\Delta_G\mathcal{T}_G\| \leq \|\Delta_G\|\|\mathcal{T}_G\| \leq (\epsilon)\ (1/\epsilon_0)$, and $\|\mathcal{T}_K\| \leq 1/\epsilon_{K_0}$. Then

$$\begin{aligned}
\|f\| &\leq \gamma^{\frac{1}{2}}\left\|\frac{\gamma^{\frac{1}{2}}\phi(\theta_0)}{1+\gamma\phi(\theta_o)^T\phi(\theta_o)}\right\|\left\|\frac{\partial\bar{e}(\bar{\theta})}{\partial\bar{\theta}}|_{\bar{\theta}=\theta_0}\right\| + 2\gamma\|\tilde{e}(k)\|\left\|\frac{\partial\bar{\phi}(\bar{\theta})}{\partial\bar{\theta}}|_{\bar{\theta}=\theta_0}\right\|\\
&\leq \frac{1}{2}\gamma^{\frac{1}{2}}\frac{\epsilon/\epsilon_{K_0}}{(1-\epsilon/\epsilon_o)^2}\|\xi\|\left\|\frac{\partial\Delta_{K(\bar{\theta})}}{\partial\bar{\theta}}|_{\bar{\theta}=\theta_0}\right\|\|x_o\|\\
&\quad + 2\gamma\frac{\epsilon/\epsilon_{K_0}}{(1-\epsilon/\epsilon_o)^3}\|\xi\|\ \|\Gamma\|\left\|\frac{\partial\Delta_{K(\bar{\theta})}}{\partial\bar{\theta}}|_{\bar{\theta}=\theta_0}\right\|\|x_o\|^2\\
&= \frac{1}{2}\gamma^{\frac{1}{2}}\frac{\epsilon/\epsilon_{K_0}{}^2}{(1-\epsilon/\epsilon_o)^2}\|\xi\|\left\|\frac{\partial\Delta_{K(\bar{\theta})}}{\partial\bar{\theta}}|_{\bar{\theta}=\theta_0}\right\|\|r\|\left\{1+\frac{4\gamma^{\frac{1}{2}}}{\epsilon_{K_0}(1-\epsilon/\epsilon_o)}\|\Gamma\|\ \|r\|\right\}
\end{aligned}$$

and

$$\|g\| \leq \frac{1}{2}\gamma^{\frac{1}{2}}\frac{\epsilon/\epsilon_{K_0}}{1-\epsilon/\epsilon_o}\|\xi\|\ \|r\|.$$

These inequalities yield the following local stability theorem for the difference equation (15).

Theorem 1

Consider the difference equation

$$\tilde{\bar{\theta}}(k+1) = \Lambda(k)\tilde{\bar{\theta}}(k) + f(k)\tilde{\bar{\theta}}(k) + g(k) \tag{16}$$

Suppose that there exists a ball $B(\theta_0, r) \subset \Theta$ such that: Assumption A1; equation (10); and $\left\| \frac{\partial \Delta_{K(\theta)}}{\partial \theta} \right\| < c$ hold for all $\bar{\theta} \in B(\theta_0, r)$. If $\exists T$, α, such that

$$0 < \alpha I \leq \sum_{i=0}^{T-1} \phi(k+i, \theta_0)\phi(k+i, \theta_0)^T < \infty \tag{17}$$

for all $k \geq 0$, then $\exists \epsilon^*$, C, such that $\forall \epsilon$, $0 < \epsilon < \epsilon^*$ and $\|\tilde{\bar{\theta}}(0)\| < \frac{r}{C}$, the solution to (16) will remain in $B(\theta_0, r)$, $\forall k \geq 0$.

Proof: Equation (17) guarantees that the linear time–varying system

$$\tilde{\bar{\theta}}(k+1) = \Lambda(k)\tilde{\bar{\theta}}(k)$$

is exponentially stable. Thus, there exists constants $0 \leq a < 1$ and $C > 0$ such the state transition matrix $F(k, 0)$ satisfies $\|F(k, 0)\| \leq Ca^k$. Let $\|\tilde{\bar{\theta}}(0)\| < r/C$ and choose ϵ^* small enough so that constants β_1 and β_2 satisfy $C(\beta_1 + \beta_2) + a < 1$ where $\|f\| \leq \beta_1$ and $\|g\| < \beta_2 r$. This choice is possible since $\|f\|$ and $\|g\|$ are both bounded by continuous, increasing functions of ϵ in $B(\theta_0, r)$, and equal to 0 for $\epsilon = 0$. It is then possible to use the the discrete–time total stability theorem of [1], page 26, to show that $\|\tilde{\bar{\theta}}(k)\| \leq r$, $\forall k \geq 0$. □

As long as assumption A1 is satisfied , the results of Theorem 1 will be valid, independent of the particular regulator design law that is implemented in the adaptive controller. The fact that a persistently exciting regression vector will give an identification scheme with a good measure of robustness has been well documented in the literature, see [1] and the references therein. Nevertheless Theorem 1 attempts to quantify the robustness attained by different regulators in terms of transfer function norms, and so gives guidelines for the design of "optimally robust" adaptive controllers.

In an adaptive control setting, the goal of a robust regulator must be to increase the region of attraction $B(\theta_0, r)$ as much as possible, while still maintaining the desired nominal system performance x_0.

A necessary condition arising for the local analysis of the theorem is that all $\tilde{\bar{\theta}} \in B(\theta_0, r)$ must provide a stabilizing frozen regulator parameter regulator $K_{\bar{\theta}}$. A sufficient condition for this is given in Lemma 3. Note that a less conservative condition can be obtained from

$$\left\| [\mathcal{T}_G \ \mathcal{T}_K] \begin{bmatrix} \Delta_G \\ \Delta_{K(\theta)} \end{bmatrix} \right\|_\infty < 1$$

This equation, as with Lemma 2 motivates the design of an $\mathcal{H}^\infty$ controller minimizing $\|[\mathcal{T}_G \ \mathcal{T}_K]\|_\infty$. This controller will make $\|\bar{e}(k)\|_2$ and $\|e(k, \bar{\theta})\|_2$ as small as possible for given ξ, x_0, Δ_G and $\Delta_{K(\theta)}$. The design of this regulator is considered in the next section.

5. Adaptive Robust Controllers

As explained in the previous section, a controller that minimizes

$$\|[\mathcal{T}_G \ \mathcal{T}_K]\|_\infty = \left\| \begin{bmatrix} S_0 M_0^{-1} & G_0 S_0 V_0^{-1} \\ K_0 S_0 M_0^{-1} & S_0 V_0^{-1} \end{bmatrix} \right\|_\infty$$

will improve the robustness of the adaptive control algorithm. The solution to this problem is simple if the coprime factorization is chosen to be normalized as in [9]. To ease notation the subscript "0" will be dropped in the following theorem.

Theorem 2

Let $G = NM^{-1}$, where $[N, M]$ is a normalized coprime factorization of G and $[U, V]$ be the set of all controller coprime factorizations given by identity (3). Then K stabilizes G and minimizes

$$\left\| \begin{bmatrix} SM^{-1} \\ KSM^{-1} \end{bmatrix} \right\|_\infty \tag{18}$$

iff K stabilizes G and minimizes

$$\left\| \begin{bmatrix} SM^{-1} & GSV^{-1} \\ KSM^{-1} & SV^{-1} \end{bmatrix} \right\|_\infty . \tag{19}$$

Moreover, the optimal values are $(1+\alpha^2)^{1/2}$, and $\{1+\alpha^2/2+\alpha\sqrt{1+\alpha^2/4}\}^{1/2}$, respectively, where $\alpha \geq 0$. The value of α is obtained in [9].

Proof: Here we repeat the procedure of [9], which derives obtains the minimum of (18) to show that the optimal controllers of (18) and (19) are in fact the same. Begin by characterizing all controllers which achieve internal stability. If X and Y are stable transfer functions such that $MY - NX = 1$, then the set of all stabilizing controllers is given by

$$[U, V] = [X + MQ, Y + QN]$$

where Q is any stable transfer function, furthermore, $MV - NU = 1$. Replacing K in (19) with the set of controllers $[U, V]$ gives

$$\left\| \begin{bmatrix} Y + QN & N \\ X + QM & M \end{bmatrix} \right\|_\infty \tag{20}$$

The problem is then reduced to that of finding amongst all stable transfer functions Q, the one which minimizes (20). Since the infinity norm is invariant under multiplication of an inner matrix, premultiply the matrix in (20) by

$$\begin{bmatrix} N^* & M^* \\ M & -N \end{bmatrix}$$

where $M(z)^* := M(z^{-1})$. This gives

$$\left\| \begin{bmatrix} N^*Y + M^*X + Q & 1 \\ 1 & 0 \end{bmatrix} \right\|_\infty^2 = \frac{2+\alpha^2+\sqrt{\alpha^4+4\alpha^2}}{2} \tag{21}$$

where $\alpha = \|N^*Y + M^*X + Q\|_\infty$. Proceeding in the same fashion with equation (18), one gets that

$$\left\| \begin{bmatrix} SM^{-1} \\ KSM^{-1} \end{bmatrix} \right\|_\infty^2 = 1 + \alpha^2 \tag{22}$$

Since the equations in (21) and (22) are both monotonically increasing functions of α, the optimal values for (18) and (19) are both obtained by minimizing α. Thus the optimal Q, and hence the optimal K, is the same for both problems. The relation between the optimal values follows from (21) and (22). □

The question arises as to whether the controller obtained from Theorem 2 is more sensitive to parameter variations than other less robust regulators. Unfortunately the

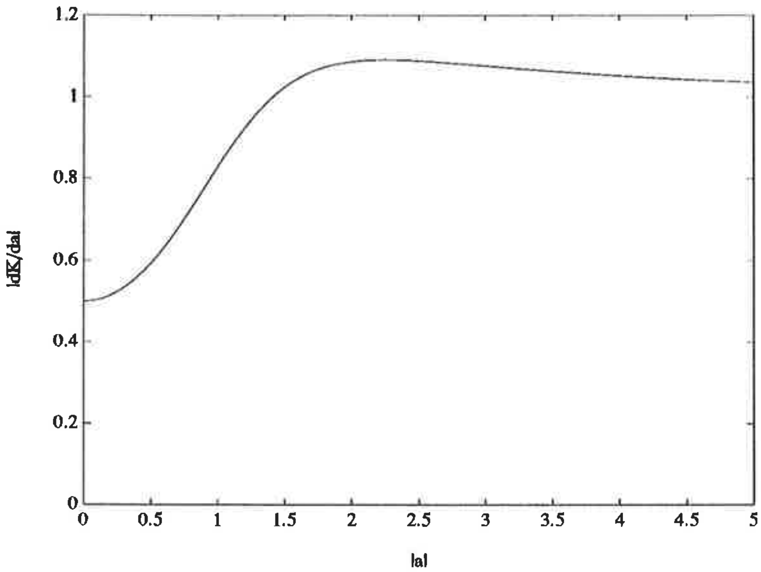

Figure 3. Plot of $\left|\frac{dk}{da}\right|$ over $|a|$ for the $\mathcal{H}^\infty$ controller of Example 1.

calculation of this is by no means trivial. The optimal controller of Theorem 2 is obatined from the solution of two Riccati equations. Obtaining values for the sensitivity of these Riccati equatins and hence the controller would give rather cumbersome formulae, see [12]. The following example should serve to illustrate this point.

Example 1

Consider a simple first order system with known gain and unknown pole: $G(z) = (z + a_1)^{-1}$. In this case $\theta = a$, and the certainty equivalence controller corresponding to the design law of Theorem 2 is a constant feedback of

$$k(a) = \begin{cases} -a - \frac{a^2+2-\sqrt{a^4+4}}{2a}, & a \neq 0 \\ 0, & a = 0. \end{cases}$$

Differentiating k with respect to a gives

$$\left|\frac{dk}{da}\right| = \begin{cases} \frac{a^4-4+(a^2+2)\sqrt{a^4+4}}{2a^2\sqrt{a^4+4}}, & a \neq 0 \\ 1/2, & a = 0 \end{cases}$$

Figure 3 shows the the graph of $\left|\frac{dk}{da}\right|$ versus a. For a pole placement algorithm, where the closed loop pole is placed at a constant location, this would be a constant 1. Thus, at small values of a, where the robustness margin is large, the $\mathcal{H}^\infty$ controller is less sensitive to parameter variations. At it's most sensitive point, the increase in sensitivy is less than 10%. Note that to be true to the analysis of Section 4, the factorization $k = u/v$ where u and v satisfy (3) should have been calculated, and then their respective derivatives computed. The equations obtained would be quite complicated and for this reason we have used the simpler formulas for k and $\frac{dk}{da}$. □

6. Conclusions

The local stability properties of an adaptive control algorithm for a discrete–time system subject to non–parametric stable factor perturbations have been analyzed. The stability analysis has been done using linearization and averaging techniques. It is shown that the use of specially designed robust regulators will improve the robustness of the overall adaptive controller. Note that the use of robust regulators will also improve the robustness of algorithms, such as those in [11], which ensure that θ approaches a point of closed loop stability and then turn the adaptation algorithm off. For the set of perturbations Δ_G, the robust controller will provide as large a region as possible where the adaptation can be disconnected. The analysis, since it is based on transfer functions can be easily carried over to continuous time algorithms, as well as to multivariable plants.

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