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Bengtsson, Gunnar

1976

Document Version:
Publisher's PDF, also known as Version of record

Link to publication

Citation for published version (APA):
Bengtsson, G. (1976). *Feedback Realizations in Linear Multivariable Systems*. (Research Reports TFRT-3128). Department of Automatic Control, Lund Institute of Technology (LTH).

Total number of authors:
1

LUND UNIVERSITY

PO Box 117
221 00 Lund
+46 46-222 00 00

FEEDBACK REALIZATIONS IN LINEAR MULTIVARIABLE SYSTEMS

G. BENGTSSON

Report 7608(C) February 1976
Department of Automatic Control
Lund Institute of Technology

FEEDBACK REALIZATIONS IN LINEAR MULTIVARIABLE SYSTEMS *

by

Gunnar Bengtsson † +

ABSTRACT.

A feedback system consists of two objects a fixed parent system and a controller connected to it. The relationship between external and internal descriptions of such systems is described. It is shown that redundancy in a feedback system can be expressed as a partial reduction with respect to controllability and observability. Applying this reduction yields a controller of lowest possible order. It is also shown that the requirement of internal stability can be expressed as a requirement on the external description of the feedback system only. This leads to a nice formalism for internally stable control synthesis using rational matrices. This is demonstrated in two important control problems, the model matching problem and the algebraic regulator problem, and by examples.

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† Lund Institute of Technology, Div. of Automatic Control, P.O.Box 725, S-22007 Lund, SWEDEN

+ This work was done while the author held a visiting position at Univ. of Toronto, Dept. of Electrical Engineering, Toronto , CANADA

* This work was supported by the Research Council of Canada under Grant Nos. A-7399 and A-4165

1. INTRODUCTION.

A linear dynamic system can be described either as an external object (a transfer matrix) or an internal object (a state equation). The relationship between them is explained in realization theory, which is basically an outgrowth of the work on controllability and observability Kalman [3], Kalman et al [4], Gilbert [1]. These concepts describe the redundancy i.e. show how a given transfer matrix can be realized by a dynamic minimal order state equation. By now there exists a well established body of literature on this topic, see e.g. [2, 5, 12, 14] .

Realization theory in its conventional form deals essentially with uncontrolled systems (open loop). In a feedback system, schematically described in fig.1, there are two systems, a plant Σ and a controller Σ_f.

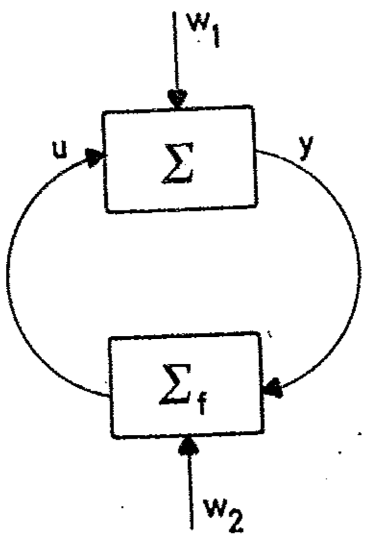


Fig.1. A feedback system

In the feedback system, the dynamics Σ describing the plant is fixed. Only the controller part Σ_f can be chosen. As the external description of such a system we take the input(w_1,w_2)/output(u,y) map. As the internal descriotion we take as usual a state equation. The realization problem is then to find a controller system Σ_f which produces a given input/ output map. The realization problem is unconventional since in the state equation there appear matrices which must be fixed, describing the fixed plant Σ.

The first one who seems to have made a more extensive use of the external description of a feedback system is Wolowich [15]. He shows how to convert a feedforward compensator into a feedback compensator where certain dynamical blocks are minimized in order and stability is produced. The results are presented in the form of an algorithm,[15]ch.7 . This paper makes a considerable generalization and is different in method and approach. Concerning the feedback realization question, the results of [15] are here contained as a special case of the sufficiency part of Theorem 4.

As in conventional realization theory, we investigate the questions of redundancy and minimality. The redundancy in a feedback system can be expressed in terms of partial reduction with respect to uncontrollability and unobservability (Theorem 1 and 2). This means that the notion of minimality will be different form the conventional open loop case (Theorem 3). Of special interest is the question of internal stability. Unlike open loop realizations, different minimal feedback realization may have different stability properties. Necessary and sufficient conditions for existence of an internally stable feedback realization are given (Theorem 4 and 6). It is also demonstrated that the requirement of internal stability can be posed as a property of the external(transfer matrix) description only (Theorem 5).

One consequence of our results is a simple algebra for internal stable control synthesis in a transfer matrix setting.

This will then be an alternative to vector space algebra, Wonham [16], and polynomial algebra, Wolowich [15] and Rosenbrock [13]. The basic idea is in the same spirit as [15] , i.e. to separate between the control synthesis question and the implementation (realization), but is considerably more general. For instance, internal stability is identified in terms of transfer matrices, regulator problems are included and minimality of the feedback realization is guaranteed provided state feedback is allowed. The application to control synthesis is demonstrated by formalizing two well known problems, the model matching problem and the algebraic regulator problem as defined in [16, 17], and by some simple examples.

The formalization of the feedback realization problem is done in Section 2. Redundancy and minimality is discussed in Section 3. Stable feedback realizations are the topic of Section 4. Some computational aspects are discussed in Section 5. Finally, internally stable control synthesis using transfer matrices is discussed in Section 6.

Notations an Preliminaries.

$\mathbb{C}$, $\mathbb{R}$ and $\mathbb{R}(s)$ denote the fields of complex numbers, real numbers and rational functions in s with coefficients in R. Script letters $\mathcal{X}$, $\mathcal{Y}$,...

denote linear vectorspaces over $\mathbb{R}$ of finite dimension , and capital Roman letters A, B, ... denote linear maps between real vector spaces. For addition of subspaces we use the symbol (+) and for the direct sum the symbol ($\oplus$). The symbol ($\approx$) denotes isomorphism between vector spaces. Im A and Ker A denote the image and the kernel of A respectively. The image of A is sometimes also written A. The subspace AV is the image of V, and $A|V$ denotes the restriction of A to V. $A_1 \approx A_2$ means that A_1 is similar to A_2.

Let $A:X \to X$ be a linear map with minimal polynomial $\alpha(s)$ and let $\mathbb{C} = \mathbb{C}^+ \cup \mathbb{C}^-$ be a fixed disjoint partition of the complex plane, where $\mathbb{C}^-$ is symmetric w.r.t. the real axis. Then $\alpha(s)$ factors uniquely as $\alpha(s) = \alpha^+(s)\alpha^-(s)$, where all the roots of $\alpha^+(s)$ $(\alpha^-(s))$ are within $\mathbb{C}^+$ ($\mathbb{C}^-$).

We define

$$X^{\pm}(A) = \text{Ker } \alpha^{\pm}(A)$$

Let $A:X \to X$, $B:U \to X$ and $C:X \to Y$ be a triple of linear maps and let $n = \dim(X)$. The <u>controllable subspace</u> R for the pair (A,B) is the subspace $R = B + AB + \ldots + A^{n-1}B$. The <u>unobservable</u> subspace N for the pair (A,C) is the subspace $N = \bigcap_{i=1}^{n} \text{Ker } CA^{i-1}$. The pair (A,B) is controllable if $R = X$, and the pair (A,C) is observable if $N = 0$.

A rational function $t(s)=\frac{q(s)}{p(s)}$, where q(s) and p(s) are relatively prime, is <u>proper</u> if $\deg(p(s)) \geq \deg(q(s))$. It is <u>strictly proper</u> if the inequality is strict. It is <u>stable w.r.t. $\mathbb{C}^-$</u> if all the roots of p(s) are within $\mathbb{C}^-$. A matrix T(s) of rational functions is proper, strictly proper, stable if all its elements are proper, strictly proper , stable respectively

The coordinate free representation of linear systems used in this paper is mainly due to Morse and Wonham[10,11] ; see also Wonham [16].

2. FORMALIZATION OF THE FEEDBACK REALIZATION PROBLEM.

Consider a linear time invariant system Σ:

$$\dot{x} = Ax + Bu + Ew \qquad (2.1)$$
$$y = Cx \quad ; \quad z = Hx$$

where $x \in X$ ($\approx R^n$) is the vector of states, $u \in U$ ($\approx \mathbb{R}^r$) is the vector of control inputs, $w \in W$ ($\approx \mathbb{R}^m$) the vector of exogenous inputs, $y \in Y$ ($\approx \mathbb{R}^q$) is the vector of accessible outputs and $z \in Z$ ($\approx \mathbb{R}^p$) the vector of controlled outputs. Here, A, B, C, E and H are linear maps (matrices) between the appropriate vector spaces. It is assumed that (A, B, C) is controllable and observable and that B is of full column rank. The system (2.1) represents the plant and is called the parent system.

A controller for Σ is a second dynamic system Σ_f driven by y and w and with u as output:

$$\dot{x}_f = A_f x_f + K_f y + G_f w$$
$$u = Fy + F_f x_f + Gw \qquad (2.2)$$
$$x_f \in X_f$$

The system (2.2) should be thought of as connected to (2.1). The vector w in (2.1) and (2.2) represents all external stimuli on the systems such as disturbances, reference inputs etc.. If w is simply a reference input, it is connected to the feedback system only through (2.2), i.e. E = 0 in (2.1). If w represents both disturbances and reference inputs, the columns of E corresponding to the reference inputs are taken to be zero. The systems (2.1) and (2,2) taken together comprise the feedback system.

Now, let

$$x_e \triangleq (x, x_f) \quad ; \quad y_e \triangleq (y, x_f) \quad ; \quad u_e \triangleq (u, u_f)$$

where $x_f \in X_f$, $u_f \in U_f$ and $X_f \in U_f$. The feedback system can then also be written as

$$\dot{x}_e = A_e x_e + B_e u_e + E_e w$$

$$y_e = C_e x_e \tag{2.3a}$$

$$u_e = F_e y_e + G_e w$$

with

$$A_e \triangleq \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} ; \quad B_e \triangleq \begin{pmatrix} B & 0 \\ 0 & B_f \end{pmatrix} ; \quad C_e \triangleq \begin{pmatrix} C & 0 \\ 0 & I_f \end{pmatrix} \tag{2.3b}$$

$$E_e \triangleq \begin{pmatrix} E \\ 0 \end{pmatrix} ; \quad F_e \triangleq \begin{pmatrix} F & F_f \\ B_f^{-1}K_f & B_f^{-1}A_f \end{pmatrix} ; \quad G_e \triangleq \begin{pmatrix} G_f \\ B_f^{-1}G_f \end{pmatrix}$$

where $B_f : U_f \approx X_f$ is an isomorphism and I_f is the identity on X_f. Note that F_e, G_e and n_f contain all the system data for the controller Σ_f. We may thus also regard a controller as the triple

$$(F_e, G_e, n_f) \tag{2.4}$$

where $n_f = \dim X_f$ is the <u>order</u> of the controller . Here we have applied the standard dynamic extension technique used in the geometric state space theory [16]. This representation of the controller will be used from now on.

Like a system of the usual type, a controller has also an external description. By a formal computation in (2.3), we obtain the input(w)/output(u) map as

$$u(s) = T_f(s)w(s) \tag{2.5}$$

$$= (QF_eC_e(s - A_e - B_eF_eC_e)^{-1}(B_eG_e + E_e) + QG_e)w(s)$$

where $Q: \mathcal{U} \oplus \mathcal{U}_f \to \mathcal{U}$ is the projection such that $u + u_f \mapsto u$ for all $u \in \mathcal{U}$ and all $u_f \in \mathcal{U}_f$. Here, $T_f(s)$ is a mapping $\mathbb{R}^m(s) \to \mathbb{R}^r(s)$ and is called the input/output map for the controller Σ_f. The signal flows for an external and an internal description of a feedback system is shown in Fig. 1.

Having identified an external description of a controller we immediately recognize the converse problem, i.e. given $(\Sigma, T_f(s))$, find, if possible, a control system (F_e, G_e, n_f) which yields $T_f(s)$ as input/output map of the feedback system.

DEFINITION 1.

The triple (F_e, G_e, n_f) is an (output) feedback realization of $(\Sigma, T_f(s))$ if it satisfies (2.5).

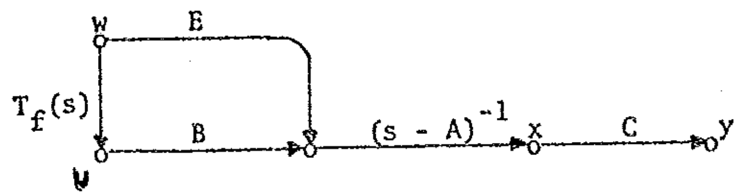
We may thus characterize the class of feedback realizations as the class of implementations in the form of a dynamic feedback control which give the same control signal u and the same output signal y for all exogenous signals w.

Unlike open loop realizations, different minimal feedback realizations may have different stability properties. Let

$$\mathbb{C} = \mathbb{C}^+ \cup \mathbb{C}^- \tag{2.6}$$

be a disjoint partition of the complex plane, where $\mathbb{C}^-$ represents the "good" part and $\mathbb{C}^+$ the "bad" part. To avoid compatibility problems (a polynomial with real coefficients must have complex conjugate roots), we assume that $\mathbb{C}^-$ is symmetric w.r.t. the real axis and contains at least one real point. Note specifically that $\mathbb{C}^-$ is quite arbitrary and not just the open left halfplane, say.

(a)



(b)

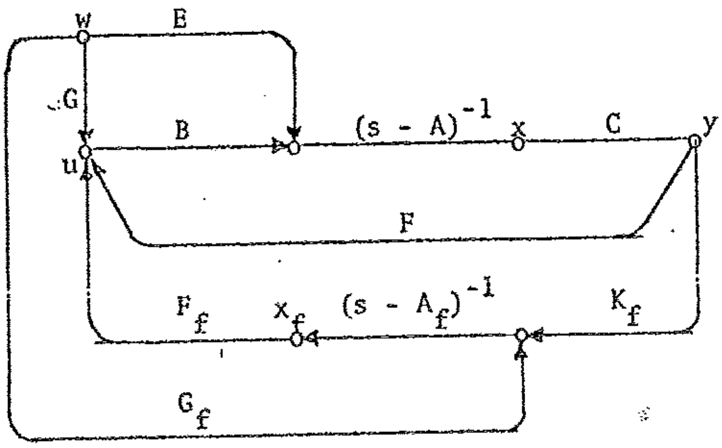


Fig.1. The structures of an external (a) and an internal (b) representatio of a feedback system.

The stability of the controlled system is determined by the spectrum of map

$$A_e^* = A_e + B_e F_e C_e$$

DEFINITION 2.

<u>A feedback realization</u> (F_e, G_e, n_f) <u>is internally stable w.r.t.</u> $\mathbb{C}^-$ <u>if the spectrum of</u> A_e^* <u>is within</u> $\mathbb{C}^-$.

The main problem considered in this paper is the following

REALIZATION PROBLEM.

<u>Given</u> $(\Sigma\ , T_f(s))$ <u>and the region</u> $\mathbb{C}^-$, <u>find</u>, <u>if possible, a feedback realization</u> (F_e, G_e, n_f) <u>of lowest possible order</u> n_f which is stable <u>w.r.t.</u> $\mathbb{C}^-$.

Note the presence of the fixed matrices A, B, C and E in the expression for $T_f(s)$. In conventional realization theory, the matrices describing the state equation are completely free to choose. The case with a partially known state equation has not been solved, but is here solved for the special (but important) feedback structure case. Note that the problem can not be solved by a straightforward application of conventional realization theory e.g. by taking a minimal realization of the controller part in fig. 1. The realization thus obtained is not minimal in general.

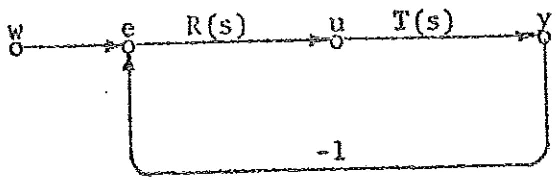
The feedback structure with input dynamics used in [15] is less general than feedback structure (2.2). Moreover, $E = 0$ in [15].

The case $E \neq 0$ is important since it means that important disturbance problems such as e.g. the algebraic regulator problem is included in our formalization, cf. Section 6.

The following example gives an intuitive illustration of the feedback realization problem

Example 1.

Assume that the feedback system is represented by the following signal flow graph



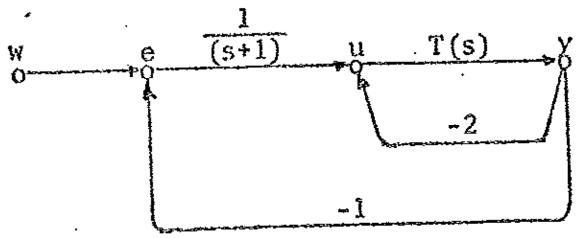
where R(s) and T(s) are the transfer matrices of the regulator and the plant respectively. Let

$$R(s) = \frac{(s-1)}{(s+1)^2} \quad ; \quad T(s) = \frac{1}{(s-1)}$$

Note that there is a right halfplane cancellation in this case. An obvious feedback realization is obtained by taking a minimal realization of R(s) and connecting it in the way the signal flow graph indicates. It is easily verfied that

$$\mathrm{spec}(A_e + B_e F_e) = \{ +1 \; ; \; -1 \pm j \}$$

i.e. the controlled system is unstable. Consider instead the following signal-flow graph, describing the same control



It is easily verified that the input(w)/output(e,u,y) maps are the same as in the signal-flow graph above. However, in this case, if we take a minimal realization of $\frac{1}{s+1}$ and connect as the signal flow graph indicates, we have

$$\text{spec}\ (A_e + B_e F_e) = \{-1 \pm j\}$$

i.e. a lower order and stable controlled system. It will be shown later that if the "regulator" and the " plant" in this example are interchanged there exists no stable feedback realization.

2. REDUNDANCY AND MINIMALITY IN FEEDBACK SYSTEMS.

The redundancy in an open loop system with respect to its input/output map is expressed in terms of the basic notions of controllability and observability. The purpose of this section is to develop rules for order reduction in a feedback system and to show that these reductions in fact leads to a minimal order feedback system.

In this section it is assumed that state feedback is allowed, i.e. $\mathcal{Y} = \mathcal{X}$ $(C_e = I_e)$.

Consider the controlled system (2.3), and introduce the input(w)/output(x) map P(s) defined by

$$\begin{aligned} x(s) &\triangleq P(s)w(s) \\ &\triangleq (s - A)^{-1}(BT_f(s) + E)w(s) \qquad (3.1) \\ &= P_x(s - A_e - B_eF_eC_e)^{-1}(B_eG_e + E_e)w(s) \end{aligned}$$

where $P_x : \mathcal{X} \oplus \mathcal{X}_a \to \mathcal{X}$ is the projection such that $P(x + x_a) = x$ for all $x \in \mathcal{X}$ and all $x_a \in \mathcal{X}_a$. The first expression is obtained using (2.1) and (2.5) and second (2.3).

To proceed, we need two lemmas

LEMMA 1.

Consider two controllers (F_e, G_e, n_f) and $(\hat{F}_e, \hat{G}_e, \hat{n}_f)$. They have the same input/output map $T_f(s)$ iff they give rise to the same P(s), i.e. iff the following equality holds

$$P_x(s - A_e - B_eF_eC_e)^{-1}(B_eG_e + E_e) = \hat{P}_x(s - \hat{A}_e - \hat{B}_e\hat{F}_e\hat{C}_e)^{-1}(\hat{B}_e\hat{G}_e + \hat{E}_e) \quad (3.2)$$

where $(\hat{\cdot})$ denotes the corresponding objects for $(\hat{F}_e, \hat{G}_e, \hat{n}_f)$.

PROOF.

By (3.1), P(s) is the same for all feedback realizations of $(\Sigma\ , T_f(s))$. Conversely, if $P(s) = \hat{P}(s)$, there follows $(s - A)^{-1}(BT_f(s) + E) = (s - A)^{-1}(B\hat{T}_f(s) + E)$ by (3.1). Multiplying with $\hat{B}(s - A)$ from the left, where $\hat{B}B = I_r$, gives the result. □

LEMMA 2.

Let $(A_e, B_e, E_e, P_x, X_f, U_f, n_f)$ and $(\hat{A}_e, \hat{B}_e, \hat{E}_e, \hat{P}_x, \hat{X}_f, \hat{U}_f, \hat{n}_f)$ denote two dynamic extensions of Σ according to above. Assume that $n_f \geq \hat{n}_f$, and let $P_f: X_f \to \hat{X}_f$ be a projection. If

$$P = I \oplus P_f \triangleq \begin{pmatrix} I_x & 0 \\ 0 & P_f \end{pmatrix} \tag{3.3}$$

then

$$PA_e = \hat{A}_e P \;;\quad PE_e = \hat{E}_e \;;\quad P_x = \hat{P}_x P \;;\quad PB_e = \hat{B}_e S \tag{3.4}$$

for some projection $S: U \oplus U_f \to U \oplus \hat{U}_f$.

PROOF.

This is just a coordinate free interpretation of the block structure in (2.3). □

The reduction of order in a feedback system can now be expressed as a partial reduction w.r.t. controllability and observability in the the system (2.3). Regard

$$x = P_x x_e \tag{3.5}$$

as output. The input/output map of this system is defined by (3.1)

THEOREM 1.

Let (F_e, G_e, n_f) be a controller with input/output map $T_f(s)$ and let R_e be the controllable subspace for the pair $(A_e + B_e F_e \;,\; B_e G_e + E_e)$ Then there exists another controller $(\hat{F}_e, \hat{G}_e, \hat{n}_f)$ with the same input/output map and of order

$$\hat{n}_f = \dim(R_e \cap X_f) \leq n_f \tag{3.6}$$

where equality holds only if

$$R_e \supset X_f \tag{3.7}$$

i.e. only if all vectors of the form $(0\ ,\ x_f)$ are reachable by w in the controlled system (2.3).

PROOF.

Let $X_f = X_c \oplus R_e \cap X_f$ for some complement X_c and let $\hat{X}_f \triangleq R_e \cap X_f$ be the new extension space. Moreover, let $P_f : X_f \to \hat{X}_f$ be any projection such that Ker $P_f = X_c$. By Lemma 2, (3.4) holds, with P defined as in (3.3). Let us construct a pair $\hat{F}_e$ and $\hat{G}_e$ which satisfies (3.2), i.e. is a feedback realization. Since Ker P = X_c and $R_e \cap X_c = 0$, there follows that $\dim(PR_e) = \dim(R_e)$, and therefore there is an $\tilde{F}_e$ such that

$$\tilde{F}_e P | R_e = F_e | R_e \quad . \tag{3.8}$$

Let S be as in (3.4) and take $(\hat{F}_e, \hat{G}_e, \hat{n}_f)$ where

$$\hat{F}_e \triangleq S\tilde{F}_e \quad ; \quad \hat{G}_e \triangleq SG_e \quad ; \quad \hat{n}_f \triangleq \dim(R_e \cap X_e) \quad . \tag{3.9}$$

We must show that this is a feedback realization of $T_f(s)$. Using (3.4) and (3.9), it follows directly that

$$P(B_e G_e + E_e) = \hat{B}_e \hat{G}_e + \hat{E}_e$$

and by induction

$$\begin{aligned} P(A_e+B_eF_e)^{k+1}(B_eG_e+E_e) &= P(A_e+B_eF_e)(A_e+B_eF_e)^k(B_eG_e+E_e) \\ &= P(A_e+B_e\tilde{F}_eP)(A_e+B_eF_e)^k(B_eG_e+E_e) \\ &= (\hat{A}_e+\hat{B}_e\hat{F}_e)P(A_e+B_eF_e)^k(B_eG_e+E_e) \\ &= (\hat{A}_e+\hat{B}_e\hat{F}_e)^{k+1}(\hat{B}_e\hat{G}_e+\hat{E}_e) \end{aligned} \tag{3.10}$$

where the second equality follows by (3.8), the third by (3.4) and (3.9) and the fourth by induction. By Lemma 1, $(\hat{F}_e,\hat{G}_e,\hat{n}_f)$ realizes $(\Sigma\ ,\ T_f(s))$ iff (3.2) holds i.e. iff

$$P_x(A_e+B_eF_e)^k(B_eG_e+E_e) = \hat{P}_x(\hat{A}_e+\hat{B}_e\hat{F}_e)^k(\hat{B}_e\hat{G}_e+\hat{E}_e)$$

for all integers $k \geq 0$. This follows, however, immediately from (3.10) and the fact that $P_x = \hat{P}_xP$. □

A corresponding reduction can also be done w.r.t. to unobservability in the controlled system (2.3) with (3.5) as output:

THEROEM 2.

Let (F_e,G_e,n_f) be a controller with input/output map $T_f(s)$ and let N_e be the unobservable subspace for the pair $(A_e+B_eF_e,P_x)$. Then there exists another controller $(\hat{F}_e,\hat{G}_e,\hat{n}_f)$ with the same input/output map and of order

$$\hat{n}_f = n_f - \dim(N_e) \leq n_f \tag{3.11}$$

where equality holds only if

$$N_e = 0 \tag{3.12}$$

i.e. only if no vector of the form $x_e=(\ 0\ ,\ x_f)$ is unobservable in the controlled system (2.3) with (3.5) as output.

PROOF.

First, $N_e \subset \operatorname{Ker} P_x = X_f$, and therefore we can find a complement X_c such that $X_f = X_c \oplus N_e$. Take $\hat{X}_f \triangleq X_c$ as the new extension space and let $P_f: X_f \to \hat{X}_f$ be a projection such that $\operatorname{Ker} P_f = N_e$. By Lemma 2, (3.4) is satisfied with P as in (3.3).

Since $A_e X_f = 0$, there also follows $A_e N_e = 0$. Since N_e is $(A_e + B_e F_e)$-invariant, we now have

$$N_e \supset (A_e+B_eF_e)N_e = B_eF_eN_e$$

Taking images under P . . and using (3.4), we have $0 = \hat{B}_e SF_e N_e$, and since $\hat{B}_e$ is monic, $SF_eN_e = 0$. Since Ker $P = N_e$, there is an $\hat{F}_e$ such that

$$SF_e = \hat{F}_e P \tag{3.13}$$

Also take

$$\hat{G}_e \triangleq SG_e \quad ; \hat{n}_f \triangleq \dim(X_c) = n_f - \dim(N_e) \tag{3.14}$$

Let us show that $(\hat{F}_e,\hat{G}_e,\hat{n}_f)$ is a feedback realization of $(\Sigma\ ,\ T_f(s))$. Using (3.4) and (3.13), we have

$$P(A_e+B_eF_e) = (\hat{A}_e+\hat{B}_e\hat{F}_e)P$$

and therefore

$$\begin{aligned} P_x(A_e+B_eF_e)^k(B_eG_e+E_e) &= \hat{P}_x P(A_e+B_eF_e)^k(B_eG_e+E_e) \\ &= \hat{P}_x(\hat{A}_e+\hat{B}_e\hat{F}_e)^k(\hat{B}_e\hat{G}_e+\hat{E}_e) \end{aligned}$$

By Lemma 2 we conclude that $(\hat{F}_e,\hat{G}_e,\hat{n}_f)$ and (F_e,G_e,n_f) have the same $T_f(s)$ ◻

It now remains to verify that the reductions in order expressed by Theorem 1 and Theorem 2 in fact represents all the the reductions that are possible within a feedback system. Let us call a controller system (F_e,G_e,n_f) minimal if there is now other controller system with the same input/output map but of lower order.

We then have

THEOREM 3.

Assume that state feedback is allowed. A controller (F_e, G_e, n_f) is minimal if and only if (3.7) and (3.12) hold i.e. iff in the controlled system (2.3)

(i) all vectors of the form $(0, x_f)$ are reachable by w

(ii) no vector of the form $(0, x_f)$ is unobservable through x

Any controller can be reduced to minimal form using the reductions in Thm. 1 and Thm. 2.

PROOF.

(If) Consider a feedback realization (F_e, G_e, n_f) satisfying (3.7) and (3.12) and assume there is another feedback realization $(\hat{F}_e, \hat{G}_e, \hat{n}_f)$ such that $\hat{n}_f < n_f$. We may then assume that the second realization also satisfies (3.7) and (3.12), since otherwise $\hat{n}_f$ can be reduced still further using Theorem 1 and 2. Since Ker $P_x = X_f$, there now follows

$$\begin{aligned} \dim(P_x R_e) &= \dim R_e - \dim R_e \cap X_f \\ &= \dim R_e - \dim X_f \end{aligned} \tag{3.15}$$

where the second equality follows from (3.7). In the same way, we have

$$\dim(\hat{P}_x \hat{R}_e) = \dim \hat{R}_e - \dim \hat{X}_f \tag{3.16}$$

Now, $(A_e+B_eF_e, B_eG_e+E_e, P_x)$ and $(\hat{A}_e+\hat{B}_e\hat{F}_e, \hat{B}_e\hat{G}_e+\hat{E}_e, \hat{P}_x)$ are both realizations (of conventional type) of P(s), cf. Lemma 1. Since they are both observable the dimensions of their controllable subspaces must be the same, i.e. $\dim R_e = \dim \hat{R}_e$. There also follows that $\dim P_x R_e = \dim \hat{P}_x \hat{R}_e$ since $P_x (A_e + B_e F_e)^k (B_e G_e + E_e) = \hat{P}_x (\hat{A}_e + \hat{B}_e \hat{F}_e)^k (\hat{B}_e \hat{G}_e + \hat{E}_e)$ for all nonnegative integers k. From (3.15) and (3.16) we have directly that $\dim X_f = \dim \hat{X}_f$, which contradicts our initial assumption $n_f > \hat{n}_f$. Hence, the feedback realization is minimal

(only if) Follows directly from Theorem 1 and Theorem 2.

Remark.

Since complete reduction with respect to controllability and observability can not be made in general, it may happen that the order of a feedback system with a minimal feedback realization is greater than the order of a minimal (open loop) realization of the input(w)/output(u,z) map

$$\begin{pmatrix} H(s - A)^{-1}(BT_f(s) + E) \\ T_f(s) \end{pmatrix}$$

even if the parent system is controllable and observable. The reason for this is the presence of the fixed matrices A, B, C and E.

Remark.

These results are completely new. The rules for order reduction are not identified in [15].

4. INTERNALLY STABLE FEEDBACK REALIZATIONS.

Let us now turn to internally stable output feedback realizations, i.e . we require in addition

$$\mathrm{spec}(A_e + B_e F_e C_e) \subset \mathbb{C}^- \tag{4.1}$$

where $\mathbb{C}^-$ is a quite arbitrary region of the complex plane, cf.(2.6). In the realization of uncontrolled systems, stability is not an issue since all minimal realizations are isomorphic and therefore have the same characteristic polynomial, see e.g. [5]. For feedback realizations this is no longer true. In fact, there may exist two minimal feedback realizations of the same input/output map, one being stable and the other unstable.

Our construction of an internally stable (output) feedback realization will not necessarily result in a minimal one. However, if in addition state feedback is allowed, the feedback realization is in fact both minimal and stable. We will treat two cases separately:

$$\begin{array}{ll} \text{(a)} & (F_e, G_e, n_f) \\ \text{(b)} & (F_e, 0, n_f) \end{array} \tag{4.2}$$

In the first case, all the external signals w are assumed to be accessible for measurement, while in the second case this assumption is dropped.

External Signal Available.

The admissible feedback realizations are of the form (4.2a)

First, we must guarantee that feedback realizations exist

PROPOSITION 1.

Assume that $(\Sigma\ ,\ T_f(s))$ are given where $T_f(s)$ is proper. There always exists a feedback realization (F_e, G_e, n_f) for any choice of C in (2.1)

PROOF.

This is almost immediate from conventional realization theory. Just take a minimal realization (A_f, G_f, F_f, G) of $T_f(s)$ and let $F = 0$ and $K_f = 0$ in (2.2). We have then just represented $u(s) = T_f(s)w(s)$ by a system Σ_f in cascade with Σ □

The feedback realization constructed in Proposition 1 is generally unsatisfactory since the corresponding signal flow is open loop. However, existence is assured and the signal flow will become closed loop if we insist on stability w.r.t. the region $\mathbb{C}^-$, i.e. that (4.1) is satisfied.

The existence of stable (output) feedback realizations can now be stated as

THEOREM 4.

There exists an internally stable (output) feedback realization (F_e, G_e, n_f) of $(\Sigma\ ,\ T_f(s))$ iff

(i) $T_f(s)$ is proper

(ii) $P(s)$ is stable w.r.t. $\mathbb{C}^-$

where $P(s)$ is defined by (3.1). Moreover, if there exists any (state) feedback realization of order n_f, and (i)-(ii) hold , then there exists an internally stable (output) feedback realization of order $\hat{n}_f \leq n_f + n - q$

To prove this we need some preliminary results

LEMMA 3.

Let $T(s) = C(s-A)^{-1}B$ and let R be the controllable subspace for the pair (A,B). Then, $T(s)$ is stable w.r.t. $\mathbb{C}^-$ iff

$$R^+ \subset \mathrm{Ker}\, C$$

where

$$R^+ \triangleq X^+(A) \cap R .$$

PROOF.

Follows from Kalman's structure theorem [6]. □

LEMMA 4.

Let (F_e, G_e, n_f) be an arbitrary feedback realization of $(\sum, T_f(s))$ and let R_e be the controllable subspace for the pair $(A_e+B_eF_eC_e, B_eG_e+E_e)$. Then $P(s)$ is stable w.r.t. $\mathbb{C}^-$ iff

$$R_e^+ \subset X_f$$

where

$$R_e^+ \triangleq R_e \cap X_e^+(A_e+B_eF_eC_e) .$$

PROOF.

Follows from Lemma 3 and (3.1) since $\mathrm{Ker}\, P_x = X_f$ □

PROOF OF THEOREM 4.

(If) According to Proposition 1 there always exists a feedback realization. Using Theorems 1 and 2 we can then find a (state) feedback realization (F_e, G_e, n_f) so that $R_e \cap X_f = 0$ and $N_e = 0$, where R_e and N_e are as in the theorems. Since P(s) is stable w.r.t. $\mathbb{C}^-$, there follows from Lemma 4 that $R_e^+ \subset X_f = \mathrm{Ker}\, \dot{P}_x$. In addition, R_e^+ is $(A_e+B_eF_e)$-invariant, and therefore $R_e^+ \subset N_e = 0$. We have thus shown that the map

$$(A_e + B_eF_e)|R_e$$

has its spectrum within $\mathbb{C}^-$. Now, write

$$u_e = F_ex_e + G_ew + \Delta u_e \tag{4.3}$$

where Δu_e is for the moment undetermined. Describe the controlled system (2.3) with (4.3) as control in a basis adapted to $X_e = X_e \oplus R_e$, where X_e is any complement of R_e. Since R_e is $(A_e+B_eF_e)$ - invariant, and $R_e \supset \mathrm{Im}(B_eG_e+E_e)$, the controlled system becomes in this basis

$$\dot{x}_e = \begin{pmatrix} A_{e1} & 0 \\ A_{e2} & A_{e3} \end{pmatrix} x_e + \begin{pmatrix} 0 \\ D_e \end{pmatrix} w + \begin{pmatrix} B_{e1} \\ B_{e2} \end{pmatrix} \Delta u_e \tag{4.4}$$

$$u_e = [\,F_{e1} \quad F_{e2}\,]x_e + G_ew + \Delta u_e$$

First, $A_{e3} \cong (A_e+B_eF_e)|R_e$ and therefore A_{e3} has its spectrum within $\mathbb{C}^-$. Since (A_e, B_e) is a controllable pair, there follows that (A_{e1}, B_{e1}) is also a controllable pair. Take $\Delta u_e = [\,\Delta F_e \quad 0]$ so that $A_{e1}+B_{e1}\Delta F_e$ is stable. Taking

$$\hat{F}_e = [\,F_{e1}+\Delta F_e \quad F_{e2}] \tag{4.5}$$

there follows that $(\hat{F}_e, G_e, n_f)$ is an internally stable (state) feedback realization. Since (A,C) is an observable pair, this realization can be implemented by use of an observer, yielding an output feedback realization

$(\tilde{F}_e, \tilde{G}_e, \tilde{n}_f)$, where $\tilde{n}_f = n_f + n - q$, provided a standard reduced order observer is used, Luenberger [7].

(Only if) Let (F_e, G_e, n_f) be an internally stable feedback realization. Then $X_e^+(A_e + B_e F_e C_e) = 0$, so clearly $P(s)$ is stable according to Lemma 4. Moreover, $T_f(s)$ is proper by (2.5). □

The feedback realization constructed in the theorem is not necessarily minimal. However, if state feedback is allowed, we have

COROLLARY 1.

Assume that state feedback is allowed. Then the feedback realization constructed in the sufficiency part of the proof of the theorem is both minimal and internally stable.

PROOF.

Follows immediately from Theorem 3, since the conditions (i) and (ii) are satisfied in this case. □

The existence of an internally stable feedback realization depends crucially on the stability of the rational matrix $P(s)$ describing the input(w)/output(x) map in (2.3). The stability of $P(s)$ depends in turn only on properties of the external description of the control $T_f(s)$, i.e. is independent of any specific feedback realization.

The stability of $P(s)$ can also be related to the closed loop transfer matrix

$$T^*(s) = C(s - A)^{-1}(BT_f(s) + E) \tag{4.6}$$

in the following way.

THEOREM 5.

Assume that (A,C) is an observable pair and let $P(s)$ be as in Theorem 4. Then $P(s)$ is stable w.r.t. $\mathbb{C}^-$ iff $T^*(s)$ and $T_f(s)$ are both stable w.r.t. $\mathbb{C}^-$.

PROOF.

(If) Since (A,C) is an observable pair, there follows that the matrices C and $s - A$ are relatively left prime, cf. [15], i.e. there are polynomial matrices $X(s)$ and $Y(s)$ such that

$$X(s)C + Y(s)(s - A) = I$$

i.e.

$$X(s)C(s - A)^{-1} = (s - A)^{-1} - Y(s)$$

Using this, we obtain from (4.6)

$$X(s)T^*(s) = (s - A)^{-1}(BT_f(s) + E) - Y(s)(BT_f(s) + E)$$

i.e.

$$P(s) = X(s)T^*(s) + Y(s)(BT_f(s) + E)$$

Since $T^*(s)$ and $T_f(s)$ are both stable w.r.t. $\mathbb{C}^-$, there thus follows that $P(s)$ is stable w.r.t. $\mathbb{C}^-$.

(Only if) Since $T^*(s) = CP(s)$, there follows that $T^*(s)$ is stable. Multiply $P(s)$ from left by $\hat{B}(s - A)$ where $\hat{B}B = I$. We obtain

$$T_f(s) = \hat{B}(s - A)P(s) - \hat{B}E$$

and $T_f(s)$ is also stable w.r.t. $\mathbb{C}^-$. □

External Signal not Available.

In the model (2.1) w represents all external stimuli on the system. If w represents a disturbance, it is not always realistic to assume that w is available. In this case we must look for feedback realizations of the form

$$(F_e, 0, n_f) \tag{4.7}$$

Assume that state feedback is permitted, i.e. $H_e = I_e$. We also assume that E in (2.1) is of full column rank, since otherwise there exists a disturbance which is not connected to the system.

First, we have

PROPOSITION 2.

There is a feedback realization $(F_e, 0, n_f)$ of $(\Sigma, T_f(s))$ iff $T_f(s)$ is strictly proper.

PROOF.

(If)We will show that $u(s) = T_f(s)w(s)$ can be rewritten as $u(s) = F(s)x(s)$,where F(s) is proper and

$$x(s) = (s - A)^{-1}(Bu(s) + Ew(s)) \tag{4.8}$$

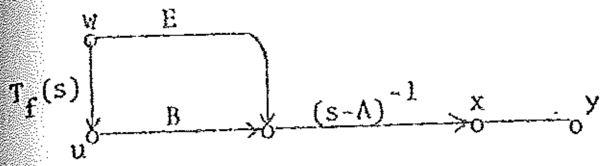
Consider

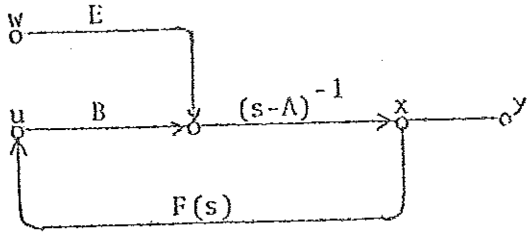
$$F(s) = (I + T_f(s)\hat{E}B)^{-1}T_f(s)\hat{E}(s - A) \tag{4.9}$$

where $\hat{E}$ is such that $\hat{E}E = I$. Since $T_f(s)$ is strictly proper, there follows directly that F(s) is proper. Furthermore, applying $u(s) = F(s)x(s)$ to (4.8) yields

$$u(s) = (I + T_f(s)\hat{E}B)^{-1}T_f(s)\hat{E}(Bu(s) + Ew(s))$$

Solving this equation for u(s) gives $u(s) = T_f(s)w(s)$. We thus have two externally equivalent ways of representing the controlled system as is indicated in the signal-flow graphs below.





Since F(s) is proper, we can implement the second signal-flow graph by taking a minimal realization of F(s) and connect as the signal-flow graph indicates. This gives a feedback realization of the desired type.

(Only if) Is obvious from (2.5). □

We can then state the following realization theorem

THEOREM 6 .

Under the assumptions above, there exists an internally stable feedback realization $(F_e,0,n_f)$ of $(\Sigma\ ,\ T_f(s))$ iff

(i) $T_f(s)$ is strictly proper

(ii) P(s) is stable w.r.t. to $\mathbb{C}^-$

where P(s) is defined as in (3.1). Furthermore, the construction in the sufficiency part of the proof of Theorem 4 yields a minimal internally stable feedback realization in this case.

PROOF.

(If) According to Proposition 2 there is a feedback realization of

of the form $(F_e, 0, n_f)$. Applying the reductions in Theorem 1 and Theorem 2 yields $(\hat{F}_e, 0, \hat{n}_f)$, with $\hat{n}_f$ minimal. Finally, if the stabilization in the sufficiency part of the proof of Theorem 2 is applied, we obtain a stable feedback realization $(\tilde{F}_e, 0, \hat{n}_f)$.
(only if) Follows from Proposition 2 and Theorem 3. □

A conventional observer for the system (2.1) requires knowledge of all inputs u and w [7] . Since w is not available in this case, it means that a feedback realization of the form $(F_e, 0, n_f)$ cannot be implemented as output feedback by use of a conventional observer if $T_f(s)$ is to be preserved. However, assuming a stochastic description of w(s) (white noise through some linear filter), it is reasonable to implement the solution via the "best possible" state estimator, i.e. a Kalman filter.

Remark.

The results of [15] are contained in the sufficiency of Theorem 4 for the special case E = 0.

Remark.

The theorems above still hold if the requirement that (A,B) is controllable is replaced by the requirement that (A,B) is stabilizable.

Example 2.

Consider the signal-flow graph in Example 1 . In this case

$$y(s) = T^{*}(s)w(s)$$
$$= \frac{1}{s^2 + 2s + 2} w(s)$$

$$u(s) = T_f(s)w(s)$$
$$= \frac{s - 1}{s^2 + 2s + 2} w(s)$$

Since $T^{*}(s)$ and $T_f(s)$ are both stable, there exists a stable feedback realization according to Theorem 4 and 5. If the regulator $R(s)$ and the plant $T(s)$ are interchanged in this example, we have

$$T^{*}(s) = \frac{1}{s^2 + 2s + 2}$$

$$T_f(s) = \frac{(s + 1)^2}{(s - 1)(s^2 + 2s + 2)}$$

Since $T_f(s)$ is unstable in this case, there exists no stable feedback realization.

5. COMPUTATIONAL ASPECTS.

The constructions in the theorems above are of the same type as order reduction with respect to uncontrollability and unobservability of a nonminimal realization in conventional realization theory. Below, the geometric constructions are converted to a computational algorithm leading to a minimal and internally stable (state) feedback realization.

We assume that $B_f = I_f$ in (2.3) (no restriction). By a basis matrix V for a subspace $\mathcal{V}$ we mean a matrix whose columns are linearly independent and span $\mathcal{V}$.

STEP 1: Find any feedback realization (F_e, G_e, n_f), e.g. the open loop realization used in Proposition 1. If w is not accessible for measurement, find any realization of the form $(F_e, 0, n_f)$, e.g. using the construction in Theorem 6.

STEP 2. Perform the order reduction of Theorem 2. More precisely, partition the observability matrix Q as

$$Q = \begin{pmatrix} P_x \\ P_x A_e^* \\ \vdots \\ P_x A_e^{*(n+n_f-1)} \end{pmatrix} = \begin{array}{cc} [T_1 & ; \; T_2] \\ \hat{n} & n_f \end{array}$$

$$P_x = \begin{pmatrix} I_n & 0 \end{pmatrix} ; \qquad A_e^* = A_e + B_e F_e \; .$$

Find a $n_f \times \hat{n}_f$ basis matrix X for the row space of T_2. If $n_f = \hat{n}_f$, no order reduction can be made and proceed to STEP 3. Otherwise, take

$$P = \begin{pmatrix} I_n & 0 \\ 0 & X^T \end{pmatrix} \qquad S = \begin{pmatrix} I_r & 0 \\ 0 & X^T \end{pmatrix}$$

and solve the linear matrix equation $SF_e = \hat{F}_e P$ for $\hat{F}_e$. Also take $\hat{G}_e = SG_e$.

The reduced feedback realization is $(\hat{F}_e, \hat{G}_e, \hat{n}_f)$

STEP 3. Perform the order reduction of Theorem 1. Compute a basis matrix R for the column space of the controllability matrix

$$[B_e^* \quad A_e^* B_e^* \quad \ldots . \quad A_e^{*(n+n_f-1)} B_e^*] \quad ; \quad A_e^* = \hat{A}_e + \hat{B}_e \hat{F}_e \quad ; \; B_e^* = \hat{B}_e \hat{G}_e + \hat{E}_e$$

Compute a $\hat{n}_f \times \tilde{n}_f$ basis matrix T for $\mathrm{Im}(R) \cap X_f$, e.g. using $\mathrm{Im}(R) \cap X_f = (\mathrm{Im}(R)^{\perp} + X_f^{\perp})^{\perp}$. If $\tilde{n}_f = \hat{n}_f$ no order reduction can be made and proceed to STEP 4. Since a basis matrix for X_f is given by $[0 \quad I_{n_f}]^T$, T must be of the form

$$T = \begin{pmatrix} 0 \\ X \end{pmatrix}$$

Find a matrix $\hat{X}$ such that $\hat{X}X = I$, e.g. $\hat{X} = (X^T X)^{-1} X^T$, and take

$$P = \begin{pmatrix} I_n & 0 \\ 0 & \hat{X} \end{pmatrix} \qquad S = \begin{pmatrix} I_r & 0 \\ 0 & \hat{X} \end{pmatrix}$$

If $\tilde{n}_f = 0$, take instead

$$P = [I_n \quad 0] \qquad S = [I_r \quad 0]$$

Solve the linear matrix equation $\bar{F}_e PR = \hat{F}_e R$ for $\bar{F}_e$ and take $\tilde{F}_e = S\bar{F}_e$ and $\tilde{G}_e = S\hat{G}_e$. The reduced feedback realization is given by $(\tilde{F}_e, \tilde{G}_e, \tilde{n}_f)$. This realization is also minimal.

STEP 4. It remains to make the feedback realization internally stable. This step involves performing a state transformation of the feedback system to obtain the structure (4.4) and thereafter select ΔF_e such that eigenvalues of $A_{e1} + B_{e1}\Delta F_e$ are all within the desired region, i.e. a standard pole assignement problem.

This algorithm should be compared with the algorithm in [15], ch.7, which solves a less general problem. Apparently, the present algorithm achieves more with considerably less computational effort.

6. APPLICATIONS TO CONTROL SYNTHESIS.

One consequence of the results of this paper is a simple algebra for internally stable control synthesis using transfer matrices. This fact is demonstrated by formalizing two important control problems, the model matching problem and the algebraic regulator problem as defined in [16,17] , and by an example. The basic approach is pure feedforward compensation. Feedback, internal stability and minimality are then properties of a feedback realization as guaranteed by the theorems above.

The following notation is used below. Consider the partition (2.6) of the complex plane. An arbitrary rational matrix T(s) can be uniquely decomposed as

$$T(s) = T(s)_+ + T(s)_- + T(s)_p \tag{6.1}$$

where $T(s)_p$ is a polynomial matrix and $T(s)_+$ and $T(s)_-$ are strictely proper rational matrices with all their poles within $\mathbb{C}^+$ and $\mathbb{C}^-$ respectively. A rational matrix being stable can then be expressed as $T(s)_+ = 0$.

The Regulator Problem.

The system is described by (2.1) where (A,B) is controllable (stabilizable) and (A,C) observable. Moreover, the matrix E has full column rank. The disturbance w is described by a model

$$\dot{x}_d = A_d x_d \quad ; \quad w = Hx_d \tag{6.2}$$

Also let $\mathbb{C}^+$ and $\mathbb{C}^-$ be the right and open left halfplane respectively. The control is allowed to be of the form

$$\dot{x}_f = A_f x_f + K_f x$$
$$u = Fx + F_f x_f \tag{6.3}$$

i.e. w is not accessible for feedback. The problem is to find a control of the form (6.3) such that (a) $y(t) \to 0$ as $t \to \infty$ for all $x_d(0)$ and (b) the feedback system disregarding disturbances is stable (w.r.t. C^-). This is the regulator problem with internal stability as formulated in [16,17,18].

The closed loop system can be rewritten in terms of external (transfer matrix) representations as

$$y = T_1(s)u + T_2(s)w$$
$$u = T_f(s)w \tag{6.4a}$$
$$w = D(s)x_d(0)$$

where

$$T_1(s) = C(s - A)^{-1}B \quad ; \quad T_2(s) = C(s - A)^{-1}E$$
$$D(s) = H(s - A_d)^{-1} \tag{6.4b}$$

and $T_f(s)$ is the external description of the controller, cf.(2.5). The existence of a state feedback realization with w not being accessible is equivalent to $T_f(s)$ being strictly proper and the existence of an internally stable feedback realization is equivalent to that $T_f(s)$ and $T_1(s)T_f(s) + T_2(s)$ both are stable rational matrices w.r.t. $\mathbb{C}^-$ (Theorem 5 and 6). Furthermore,

$$y = (T_1(s)T_f(s) + T_2(s))D(s)x_d(0) \tag{6.5}$$

and therefore output regulation (a) is equivalent to that the transfer matrix in (6.5) is stable.

Let us summarize this into

PROPOSITION 3

There exists a solution to the regulator problem as defined above iff there exists a strictly proper rational matrix $T_f(s)$ such that

$$(i)\ T_f(s)_+ = 0$$

$$(ii)\ (T_1(s)T_f(s) + T_2(s))_+ = 0$$

$$(iii)((T_1(s)T_f(s) + T_2(s))D(s))_+ = 0$$

An internally stable feedback control solving the problem is given as any internally stable feedback realization of $(\Sigma, T_f(s))$

Remark.

The disturbance model $D(s)$ can be taken as an arbitrary proper rational matrix, i.e. more general than (6.4b).

The Model Matching Problem.

The system is given by (2.1) with $E = 0$. The same observability and controllability assumptions as above is made. The control is allowed to be of the form (2.2). The closed loop system shall behave as a specified model

$$y = T_m(s)w$$

and shall be internally stable. Representing the control as

$$u = T_f(s)w$$

the closed loop system becomes

$$y = T(s)T_f(s)w \qquad ; \quad T(s) = C(s - A)^{-1}B$$

The existence of a stable feedback realization is equivalent to $T_f(s)$ and $T(s)T_f(s)$ both being stable rational matrices. Therefore,

PROPOSITION 4.

There exists an internally stable solution to the model matching problem iff there exists a proper rational matrix $T_f(s)$ such that

(i) $T_m(s)_+ = 0$ and $T_f(s)_+ = 0$

(ii) $T(s)T_f(s) = T_m(s)$

Remark.

This result can be compared with [15], Thm. 8.5.2 , where no stability restrictions (i) are imposed. The difference is of course our requirement of internal stability. A state space approach to model matching is treated in [8,9].

The control synthesis is simple to perform using the results above, especially if the number of inputs and outputs are few. An illustration is given in the following example.

Example. 3.

The plant is

$$y = T(s)u + H(s)v$$

$$T(s) = \begin{pmatrix} \frac{s+1}{s(s^2+1)} & \frac{s+3}{s(s^2+1)} \\ \frac{1}{s(s^2+1)} & \frac{s-1}{s(s^2+1)} \end{pmatrix} \qquad H(s) = \begin{pmatrix} \frac{s+1}{s(s^2+1)} \\ \frac{1}{s(s^2+1)} \end{pmatrix}$$

where y is the output, u the control input and w a disturbance which is not accessible for measurement. The underlying system is assumed to be a minimal realization of $[T(s) \;;\; H(s)]$. This realization is stabilizable from u. Furthermore, we assume that dynamic feedback control from the state is allowed. The latter assumption can later be dropped using arguments from observer theory.

The requirements on the closed loop system are: (a) y_1 is to follow a step y_{1r} without steady state error, (b) y_2 is to respond to y_{2r} as a model system $t_m(s) = \frac{9}{(s+3)^2}$, (c) y_{1r} is not allowed to interact with y_2, (d) there is no steady state error in y for ramp inputs v and (e) the closed loop system is internally stable.

The control is taken as

$$u = R_1(s)y_{1r} + R_2(s)y_{2r} + R_3(s)v$$

With notations as above we then have

$$T_f(s) = [\,R_1(s) \quad R_2(s) \quad R_3(s)] \quad ; \quad w^T = [\,y_{1r} \quad y_{2r} \quad v]$$

The closed loop system becomes

$$y = T(s)R_1(s)y_{1r} + T(s)R_2(s)y_{2r} + (T(s)R_3(s) + H(s))v$$

where $R_1(s)$ and $R_2(s)$ are proper and $R_3(s)$ strictly proper (v is not accessible). To ensure existence of an internally stable feedback realization, we must restrict the choice of compensators such that $R_i(s)$, i=1,2,3, and $T(s)R_i(s)$, i=1,2, and $T(s)R_3(s)+H(s)$ are all stable rational matrices (Theorems 4,5 and 6). To satisfy (c) we must take $R_1(s)$ such that

$$\begin{pmatrix} \frac{1}{s(s^2+1)} & \frac{(s-1)}{s(s^2+1)} \end{pmatrix} R_1(s) = 0$$

i.e.

$$R_1(s) = \begin{pmatrix} -(s-1) \\ 1 \end{pmatrix} q(s)$$

for some q(s). Then

$$T(s)R_1(s) = \begin{pmatrix} \frac{4+s-s^2}{s(s^2+1)} \\ 0 \end{pmatrix} q(s)$$

In order to satisfy the stability requirement, $q(s)$ must cancel the factor $s(s^2+1)$ and itself be stable and produce a proper $R_1(s)$. Take e.g.

$$q(s) = \frac{ks(s^2+1)}{(s+3)^4}$$

Since y_1 is to respond to a step input y_{1r} without steady state error, the transfer function relating y_{1r} and y_1 must have value 1 for $s=0$, i.e.

$$-\frac{(4+s-s^2)}{s(s^2+1)} \left.\frac{(s-1)ks(s^2+1)}{(s+3)^4}\right|_{s=0} = 1$$

which gives $k = 81/4$. The compensator $R_1(s)$ thus becomes

$$R_1(s) = 81/4 \begin{pmatrix} -\dfrac{(s-1)s(s^2+1)}{(s+3)^4} \\ \dfrac{s(s^2+1)}{(s+3)^4} \end{pmatrix}$$

In addition to the stability requirement, the compensator $R_2(s)$ must be chosen so that (b) is satisfied, i.e.

$$\begin{pmatrix} \dfrac{1}{s(s^2+1)} & \dfrac{s-1}{s(s^2+1)} \end{pmatrix} R_2(s) = \frac{9}{(s+3)^2}$$

An admissible compensator is

$$R_2(s) = \begin{pmatrix} \dfrac{36s(s^2+1)}{(s+3)^3} \\ \dfrac{9s(s^2+1)}{(s+3)^3} \end{pmatrix}$$

Finally, to satisfy (d) each element in $T(s)R_3(s) + H(s)$ must have a double zero at $s=0$. Some straightforward computations shows that an admissible compensator is given by

$$R_3(s) = \begin{pmatrix} \dfrac{s^3(s^2+1)}{(s+3)^5} & -1 \\ \dfrac{s^3(s^2+1)}{(s+3)^6} & \end{pmatrix}$$

Using the results above it is then possible to find an internally stable and minimal feedback realization of the synthezised feedforward compensator.

7. AKNOWLEDGEMENTS.

The author wishes to express his gratitude to Prof. W.M.Wonham for his valuable comments on the contents of this work. The author also wants to thank Prof. M.Heymann for pointing out some more general features in some earlier work done by the author.

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