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Sufficient Conditions for Dynamical Output Feedback Stabilization via the Circle Criterion

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Abstract— **This paper suggests sufficient conditions for asymptotically stable dynamical output feedback controller design based on the *circle criterion*. It is shown that a dynamic output feedback stabilization problem with impending problems of finite escape time, previously attacked by observer-based design, can be successfully solved using circle criterion design. Stability of the closed-loop system is global and robust to parameter uncertainty.**

I. Introduction

Stabilizing a nonlinear system by output feedback is often a difficult problem. Equipped with a range of available asymptotic (robust) stability criteria, one could try to design a dynamical output feedback controller so that the closed loop system satisfies one of these stability criteria. This simple idea sometimes leads to a problem that could be solved, but quite often it results in a problem that is intractable. This paper is devoted to the discussion of dynamic output feedback and the *circle criterion*, as a stability test for the closed loop system [6], [9]. One of standard initial ideas for controlling dynamical system by output feedback is based on the *separation principle*, that is, one needs to find a stabilizing *full-state* feedback controller and to determine an observer with asymptotically stable error dynamics; an output controller is then chosen to coincide with the derived *full-state* feedback controller where instead of the unmeasured true states of the dynamical system, the system states are substituted by observer states. This approach includes three steps—design of a *full-state* controller, observer design, analysis of the closed loop system—where well-known stability criteria, like the *circle criterion*, could be applied to conclude stability. Indeed, one could use this test for checking stability of the system with a *full-state* feedback controller; or use it for checking that a particular structure of an observer results in stable error dynamics; or use this test at the final point to verify that the closed-loop system derived via *certainty equivalence principle* is stable. These arguments have been used for checking stability via the *circle criterion* of error dynamics for an observer [1], [2].

The main contribution of this paper comes from the observation that for the large class of systems treated in [1], [2], there is no need to introduce an explicit observer and assumptions and arguments relevant for the *certainty equivalence principle* can be relaxed. Instead, a fixed structure of an output feedback controller can be imposed, search for its parameters and asymptotic stability can be approached by means of the *circle criterion* for the closed-loop system. It happens that for the large class of the systems considered in [1], [2], this argument works. Thus, time-varying systems and systems with structural uncertainties can be approached whereas both extensions seem infeasible using the certainty equivalence principle [1], [2].

The paper is organized as follows: Section II suggests an illustrative example considered in details. The problem statement, assumptions and main result are given in Sec. III, a brief discussion of results being added in Sec. IV.

II. Motivating Example

Consider the following dynamical system [1]

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\left(u - x_2^5\right) \quad (1)$$

$$y = x_1 \quad (2)$$

The relation between the relevant results of [1] and the current development is discussed later. The problem is to *design a controller that renders the origin of the system* (1) asymptotically stable. Let us consider a dynamical controller of the form

$$\begin{aligned} \frac{d}{dt}z &= \lambda_3 x_1 + \lambda_4 z \\ u &= \lambda_1 x_1 + \lambda_2 z + (c_1 x_1 + c_3 z)^5 \end{aligned} \quad (3)$$

where λ_i, c_j are real constants to be defined. With such a controller, the dynamics of the closed loop system are

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ z \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 \\ \lambda_1 & 1 & \lambda_2 \\ \lambda_3 & 0 & \lambda_4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} w \\ w &= (c_1 x_1 + c_3 z)^5 - x_2^5 \end{aligned} \quad (4)$$

One can easily check that the nonlinearity w of Eq. (4) and the linear virtual output of the closed loop system (4)

$$v = c_1 x_1 - x_2 + c_3 z$$

satisfies a *passivity relationship* for any x_1, x_2, z

$$v \cdot w = (c_1 x_1 - x_2 + c_3 z)[(c_1 x_1 + c_3 z)^5 - x_2^5] \geq 0 \quad (5)$$

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Introducing the matrices and the state space vector

$$A_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \ A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ X = \begin{bmatrix} x_1 \\ x_2 \\ z \end{bmatrix},$$

one can rewrite the closed loop system (4) as follows

$$\begin{aligned} \frac{d}{dt}X &= (A_0 + \sum_{i=1}^{4} \lambda_i A_i)X + Bw \\ v &= CX = c_1 x_1 - x_2 + c_3 z \end{aligned} \qquad (6)$$

Whereas the explicit form of the nonlinearity w—in this case given as (4)—is not important, it is important that the passivity relation (5) be valid. By the *circle criterion*, the system (6) is asymptotically stable provided that

1) the *frequency condition*

$$\mathrm{Re}\,\{C(j\omega I_3 - (A_0 + \sum_{i=1}^{4} \lambda_i A_i))^{-1}B\} < 0 \qquad (7)$$

holds for any $\omega \in \mathbb{R}_+$, the *negative real* notation convention being used in the inequality (7);

2) the matrix $(A_0 + \sum_{i=1}^{4} \lambda_i A_i)$ is strictly Hurwitz.

As known, these conditions are equivalent to the fact that there exists the 3×3 matrix $P = P^T > 0$, so that

$$\begin{aligned} (A_0 + \sum_{i=1}^{4} \lambda_i A_i)^T P + P(A_0 + \sum_{i=1}^{4} \lambda_i A_i) < 0 \\ PB = -C^T \end{aligned} \qquad (8)$$

Thus, development of a dynamical stabilizing controller (3) based on the *circle criterion* and the choice of quadratic constraints (5) require determination of parameters

$$\lambda_1, \lambda_2, \lambda_3, \lambda_4, c_1, c_3 \qquad (9)$$

so that all points 1)–2) are valid, or equivalently, the Bilinear Matrix Inequality (BMI) of (8) is solvable. The routine computations made for this example result in equivalent statements written in terms of parameters λ_i, c_j:

1) Introduce the quantities

$$\begin{aligned} \alpha &= 1 - c_1 \\ \beta &= c_2\lambda_3 - \lambda_2\lambda_3 + c_2\lambda_3\lambda_4 - c_1\lambda_4^2 - c_1\lambda_1 + \lambda_4^2 \\ \gamma &= (\lambda_1\lambda_4 - \lambda_2\lambda_3)(c_2\lambda_3 - c_1\lambda_4) \end{aligned}$$

The validity of the *frequency condition* leads to the two cases

a) if the parameters (9) are so that

$$\beta < 0, \qquad (10)$$

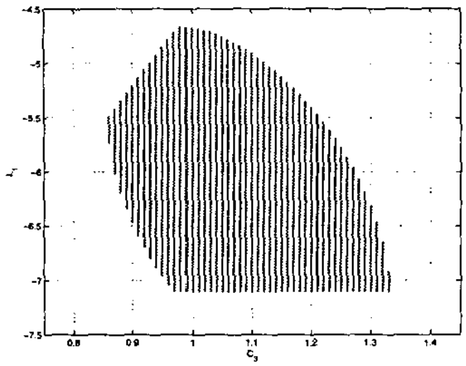

Fig. 1. The red area corresponds to λ_1 and c_3 that satisfy the constraints (10)–(13) while the other parameters (9) have the nominal value (14).

then (7) is equivalent to the inequalities

$$\gamma < 0, \quad \alpha < 0 \qquad (11)$$

b) Otherwise, (7) is equivalent to the inequalities

$$4\alpha\gamma - \beta^2 > 0, \quad \alpha < 0 \qquad (12)$$

2) The condition for the matrix $(A_0 + \sum_{i=1}^{4} \lambda_i A_i)$ to be strictly Hurwitz is equivalent to the inequalities

$$\lambda_4 < -1, \quad \lambda_1 < \lambda_4, \quad \lambda_1\lambda_4 > \lambda_2\lambda_3 \qquad (13)$$

The set of parameters (9) satisfying the constraints (10)–(13) is not empty. It can be checked that the vector

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4, c_1, c_3) = (-5, -2, -7, -3, 2, 1) \qquad (14)$$

belongs to this set. This means that for this choice of the controller parameters (9), the corresponding BMI (8) becomes a Linear Matrix Inequality (LMI), and it has a solution. In fact, these values (14) have been found in [1] via an appropriate observer design and checking the validity of *certainty equivalence principle*. Below it will be shown, what, in addition, to the asymptotic stability of the closed loop system found in [1], could be gained from the fact that the BMI (8) is solvable.

Firstly, the inequalities (10)–(13) have quite a rich set of solutions. To show that the inequalities (10)–(13) suggest the controllers that cannot be obtained by the *certainty equivalence principle* elaborated in [1], let us check possible values for the parameter c_3 that is postulated in [1] to be equal to 1. Figure 1 shows approximation for a set of parameters λ_1 and c_3 that correspond to stabilizing controller provided that the rest of (9) have the nominal value (14). To illustrate an advantage of the solution based on solvability of BMI (8) *vs.* the design via *certainty equivalence principle*, let us include parametric uncertainty in the system. Consider the system (1) with uncertainty factor ε in front of the

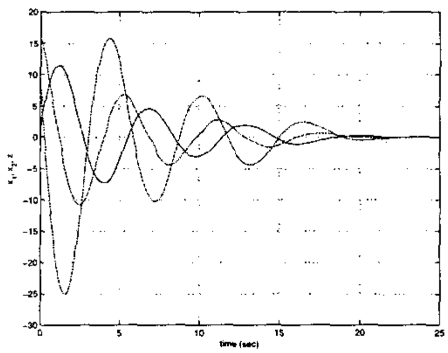

Fig. 2. The solution of the closed loop system (15), (3) with $\varepsilon = 0.243$ for the initial conditions $x_1 = 1$, $x_2 = -3$, $z = 10$.

nonlinearity in (1), *i.e.*

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}(u - \varepsilon x_2^5) \quad (15)$$

$$y = x_1 \quad (16)$$

Suppose that the nominal value ε_0 for ε is chosen as $\varepsilon_0 = 1$ and the dynamical stabilizing controller (3) with the parameters (14) has been designed as discussed above. Then, the closed-loop system is

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -5 & 1 & -2 \\ -7 & 0 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} w$$

$$w = (2x_1 + z)^5 - \varepsilon \cdot x_2^5 \quad (17)$$

The key *passivity relation* between the nonlinearity w in (17) and the linear output $v = 2x_1 - x_2 + z$ does not hold unless ε has its nominal value. But for this case one can introduce a new linear output of the system (17) such as

$$v_{new} = 2x_1 - \sqrt[5]{\varepsilon} \cdot x_2 + z.$$

Then, the *passivity relation* between w and v_{new} holds

$$v_{new} \cdot w \geq 0, \quad \forall x_1, x_2, z$$

Checking the *circle criterion* for this quadratic constraint reveals some allowed bounds for ε.

Statement 1: Consider the nonlinear system (15) with the dynamical controller (3) with the parameters λ_i, c_j as in (14), that designed to stabilize the system (15) with nominal value $\varepsilon = 1$. If the constant parameter ε is within the interval

$$\varepsilon \in [0.243, 7.26], \quad (18)$$

then the closed loop system (15), (3) remains globally asymptotically stable. In other words, the controller (3) with the parameters λ_i, c_j as in (14) robustly stabilizes (15). ■

Figures 2 and 3 show the response of the system (15), (3) for $\varepsilon = 0.243$ and $\varepsilon = 1$ with the same initial conditions. It is important to realize that this uncertainty cannot be treated in the certainty equivalence design arguments elaborated in [1]: for each new value of parameter ε, one need to change the observer! Therefore, the value of ε should be known precisely. Furthermore, the interval of an allowed uncertainty for the constant parameter ε derived in Statement 1, could be approximated for any stabilizing controllers determined by relations (10)–(13), where one could be interested in enlarging the allowed uncertainty range (Fig. 4). Another advantage of solution based on solvability of BMI (8) *vs.* the design via *certainty equivalence principle* comes from the observation that the analysis based on *separation principle* does not allow to tackle system with time-dependent right-hand side. Indeed, it is partly based on the analysis of ω-limit sets and the Barbashin-Krasovski (LaSalle) stability theorem. At the same time, the design based on solvability of the BMI (8) allows time dependence. To clarify this point, consider the modified system (1), (2)

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}(u - \sin^2(t) \cdot x_2^5)$$

$$y = x_1$$

where the nonlinearity now contains the time-varying factor $\sin^2(t)$. The arguments used for (1), (2) with the controller (3), (9) could be directly applied for this modified system and the modified dynamical controller

$$\frac{d}{dt}z = \lambda_3 x_1 + \lambda_4 z$$

$$u = \lambda_1 x_1 + \lambda_2 z + \sin^2(t) \cdot (c_1 x_1 + c_3 z)^5$$

Indeed, this property is due to the fact that the key *passivity relation* (5) between the input

$$w = \sin^2(t) \cdot \left[(c_1 x_1 + c_3 z)^5 - x_2^5\right]$$

and the linear output $v = c_1 x_1 - x_2 + c_3 z$ remains valid.

Recently, Arcak *et al.* showed that controller design for the system (1), (2) including full-state observer feedback might lead to finite-time escape [1]. To guarantee stability, the parameters λ_1 and c_1 of the controller of Eq. (3) should be non-zero. To get an additional insight into this observation, let us consider a dynamical output controller of the form

$$\frac{d}{dt}z = \Lambda_3 x_1 + \Lambda_4 z$$

$$u = \lambda_1 x_1 + \Lambda_2 z + (c_1 x_1 + C_3 z)^5 \quad (19)$$

where $z \in \mathbb{R}^m$ is a vector of internal states of the controller; λ_1 and c_1 are constants; and Λ_2–Λ_4, C_3 are matrices of appropriate dimensions. The controller (19) differs from (3) by number of internal states, and they coincide when $m = 1$. Again, the closed-loop system (1), (2), (19) could be analyzed via the *circle criterion* with the quadratic constraint (5) where

$$w = (c_1 x_1 + C_3 z)^5 - x_2^5, \quad v = c_1 x_1 + C_3 z - x_2$$

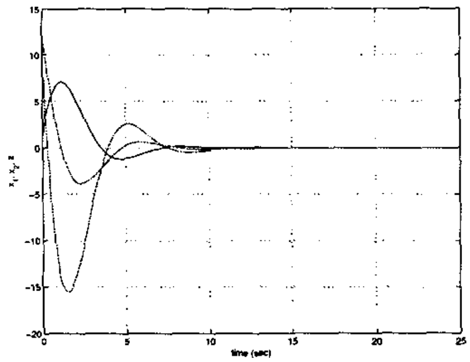

Fig. 3. The solution of the closed loop system (15), (3) with the nominal value $\epsilon = 1$ for the initial conditions $x_1 = 1$, $x_2 = -3$, $z = 10$.

Necessary conditions for BMI solvability (8) suggest introduction of a *reduced order observer* [1].

Statement 2: Consider the nonlinear system (1), (2), (19). The *frequency condition* (7) takes the form

$$\text{Re}\,\{\tilde{v}^* \tilde{w}\} = \frac{q(\lambda)}{p(\lambda)} |\tilde{w}|^2 < 0$$

where $\tilde{v} = G(j\omega)\tilde{w}$, $\lambda = \omega^2$,

$$q(\lambda) = (1 - c_1)\lambda^{(m+1)} + q_m \lambda^m + \cdots + q_1\lambda + q_0$$

$$p(\lambda) = \lambda^{(m+2)} + p_{m+1}\lambda^{(m+1)} + \cdots + p_1\lambda + p_0$$

and $G(s)$ is the transfer function of linear part of the closed loop system (1), (2), (19). Then, the *frequency condition* holds at $\omega \to +\infty$ only if the inequality $1 - c_1 < 0$ holds. ■

III. Main Results

A. Problem Formulation and Preliminary Comments

Consider a nonlinear control system of the form

$$\frac{d}{dt}x = Ax + B_1 u + B_2 \Delta(d, t) \quad (20)$$

$$y = N_1 x, \qquad d = N_2 x \quad (21)$$

where $x \in \mathbb{R}^n$ is the state vector; $y \in \mathbb{R}^m$ is the measurable output; $d \in \mathbb{R}^k$ is the vector of variables that serves as input to the *scalar* nonlinear block Δ; A, B_1, B_2, N_1, N_2 are matrices of appropriate dimensions. As in [2], [1], the nonlinear block Δ could be seen as a nonlinear operator satisfying particular properties.

Assumption 1: The *scalar* nonlinearity $\Delta(d, t)$ is such that there exists a $k \times 1$ matrix Π_Δ so that for any functions $d_1(t)$, $d_2(t) \in L^k_{2e}$: $\exists \{t_n\}_{n=1}^{+\infty}$, $t_n \to +\infty$ as $n \to +\infty$, such that $\forall n$

$$\int_0^{t_n} (d_1(\tau) - d_2(\tau))^T \Pi_\Delta \left(\Delta(d_1(\tau), \tau) - \Delta(d_2(\tau), \tau)\right) d\tau \geq 0 \quad (22)$$

For asymptotic stabilization of the origin of the system (20) by output feedback, try a dynamical controller of the form

$$u = R_z z + R_y y + R_\Delta \Delta(C_y y + C_z z, t) \quad (23)$$

$$\frac{d}{dt} z = \Lambda_z z + \Lambda_y y + \Lambda_u u + \Lambda_\Delta \Delta(C_y y + C_z z, t) \quad (24)$$

where $z \in \mathbb{R}^k$ is the internal state of the controller; Λ_z, Λ_y, Λ_u, Λ_Δ, R_z, R_y, R_ϕ, R_Δ are constant matrices of appropriate dimensions. The closed-loop system is then

$$\frac{d}{dt}\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} A + R_y N_1 & B_1 R_z \\ (\Lambda_y + \Lambda_u R_y) N_1 & (\Lambda_z + \Lambda_u R_z) \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \quad (25)$$

$$+ \begin{bmatrix} B_1 R_\Delta \\ \Lambda_u R_\Delta + \Lambda_\Delta \end{bmatrix} \Delta(C_y y + C_z z, t) + \begin{bmatrix} B_2 \\ 0 \end{bmatrix} \Delta(d, t)$$

Assumption 2: There exists a linear transformation

$$T = \begin{bmatrix} T_x & T_{xz} \\ 0 & T_z \end{bmatrix}, \quad \begin{bmatrix} x_n \\ z_n \end{bmatrix} = T \begin{bmatrix} x \\ z \end{bmatrix} \quad (26)$$

with $\det T \neq 0$, and there exist matrices R_Δ, Λ_u, Λ_Δ of such that $T_x B_1 R_\Delta + T_{zx}(\Lambda_u R_\Delta + \Lambda_\Delta) = -T_x B_2$. ■

Assumption 3: The *scalar* nonlinearity $\Delta(d, t)$ is such that there exist a 2×2 matrix

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12} & 0 \end{bmatrix} \quad (27)$$

and there exists $M \in \mathbb{R}^{1 \times k}$ so that for any function $d(t) \in L^k_{2e}$: $\exists \{t_n\}_{n=1}^{+\infty}$, $t_n \to +\infty$ as $n \to +\infty$, such that $\forall n$

$$\int_0^{t_n} \begin{bmatrix} Md(\tau) \\ \Delta(d(\tau), \tau) \end{bmatrix}^T \Pi \begin{bmatrix} Md(\tau) \\ \Delta(d(\tau), \tau) \end{bmatrix} d\tau \geq 0 \quad (28)$$

Assumptions 1, 2 and 3 enable us to rewrite the closed loop system (20), (23), (24) in the input-output form

$$\frac{d}{dt}\begin{bmatrix} x_n \\ z_n \end{bmatrix} = \mathcal{A} \begin{bmatrix} x_n \\ z_n \end{bmatrix} + \mathcal{B}_1 w_1 + \mathcal{B}_2 w_2 \quad (29)$$

with

$$v_1 = Md = MN_2 T_x^{-1}\left(x_n - T_{xz} T_z^{-1} z_n\right) \quad (30)$$

$$v_2 = \Pi_\Delta^T (C_y y + C_z z - d) = \Pi_\Delta^T C_z T_z^{-1} z_n + \quad (31)$$

$$+ \Pi_\Delta^T \{C_y N_1 - N_2\} T_x^{-1}\left(x_n - T_{xz} T_z^{-1} z_n\right)$$

$$\mathcal{A} = T \begin{bmatrix} A + R_y N_1 & B_1 R_z \\ (\Lambda_y + \Lambda_u R_y) N_1 & (\Lambda_z + \Lambda_u R_z) \end{bmatrix} T^{-1} \quad (32)$$

$$\mathcal{B}_1 = \begin{bmatrix} 0 \\ T_z(\Lambda_u R_\Delta + \Lambda_\Delta) \end{bmatrix} \quad \mathcal{B}_2 = \begin{bmatrix} T_x B_2 \\ 0 \end{bmatrix} \quad (33)$$

while w_1, w_2 are the *scalar* nonlinearities written in the original coordinates x, z

$$w_1 = \Delta(C_y y + C_z z, t), \; w_2 = \Delta(d, t) - \Delta(C_y y + C_z z, t)$$

and v_1, v_2 are *scalar passive* outputs (in integral sense) of the closed loop system (29), *i.e.*, $\exists \{t_{n_i}\}$ with $t_{n_i} \to +\infty$ as $n_i \to +\infty$, $i = 1, 2$, so that

$$\int_0^{t_{n_1}} \begin{bmatrix} v_1(\tau) \\ w_1(\tau) \end{bmatrix}^T \Pi \begin{bmatrix} v_1(\tau) \\ w_1(\tau) \end{bmatrix} d\tau \geq 0, \quad \int_0^{t_{n_2}} v_2(\tau) w_2(\tau) d\tau \geq 0$$

Denote

$$\mathcal{C}_1 = MN_2T_x^{-1}\left[I_n,\ -T_{xz}T_z^{-1}\right]$$
$$\mathcal{C}_2 = \Pi_\triangle^T\left[\{C_yN_1 - N_2\}T_x^{-1},\ \Pi_\triangle^T C_z T_z^{-1} - \{C_yN_1 - N_2\}T_x^{-1}T_{xz}T_z^{-1}\right]$$

By the *circle criterion*, the system (29) is asymptotically stable provided that

1) there exist $\tau_1 \geq 0$, $\tau_2 \geq 0$, $\tau_1 + \tau_2 > 0$ such that $\forall \xi_1$, $\forall \xi_2 \in \mathbb{C}^1$ the *frequency condition*

$$\tau_2 \mathrm{Re}\left\{\xi_2^* \mathcal{C}_2 \mathcal{A}_{j\omega}^{-1}(\mathcal{B}_1\xi_1 + \mathcal{B}_2\xi_2)\right\} + \tag{34}$$
$$+\tau_1 \begin{bmatrix} \mathcal{C}_1\mathcal{A}_{j\omega}^{-1}(\mathcal{B}_1\xi_1 + \mathcal{B}_2\xi_2) \\ \xi_1 \end{bmatrix}^* \Pi \begin{bmatrix} \mathcal{C}_1\mathcal{A}_{j\omega}^{-1}(\mathcal{B}_1\xi_1 + \mathcal{B}_2\xi_2) \\ \xi_1 \end{bmatrix} < 0$$

holds $\forall \omega \in \mathbb{R}_+$[1]. Here $\mathcal{A}_{j\omega}^{-1} = (j\omega I_{n+l} - \mathcal{A})^{-1}$;

2) the matrix $\mathcal{A}$ is strictly Hurwitz.

As known, these conditions are equivalent to the fact that there exists the $(n+l) \times (n+l)$ matrix $P = P^T > 0$ so that

$$\mathcal{A}^T P + P\mathcal{A} + \tau_1 \mathcal{C}_1^T \Pi_{11} \mathcal{C}_1 < 0,$$
$$P\mathcal{B}_1 = -\tau_1 \Pi_{12} \mathcal{C}_1^T, \quad P\mathcal{B}_2 = -\frac{\tau_2}{2}\mathcal{C}_2^T \tag{35}$$

Thus, a circle-criterion development of a dynamical stabilizing controller requires determination of matrices

$$\Lambda_z,\ \Lambda_y,\ \Lambda_u,\ \Lambda_\triangle,\ R_z,\ R_y,\ R_\triangle,\ C_z,\ C_y \tag{36}$$

so that all points 1)–3) are valid, or equivalently, that the BMI (35) have a solution.

B. Sufficient Conditions for BMI Solvability (35)

To formulate the main result, we need to postulate some additional properties of the system (20).

Assumption 4: A feedback controller

$$u = K_1 y + K_2 d + K_\triangle \Delta(d,t) \tag{37}$$

with some matrices K_1, K_2, and $K_\triangle$ renders the closed-loop system (20), (37) asymptotically stable. Furthermore this asymptotic stability can be verified from the *circle criterion* applied to the quadratic constraint in Assumption 3 for matrices K_1, K_2, $K_\triangle$ such that

1) The *frequency condition* holds $\forall \omega \in \mathbb{R}_+$, *i.e.*,

$$\begin{bmatrix} MN_2A_{j\omega}^{-1}(B_1K_\triangle + B_2) \\ 1 \end{bmatrix}^* \Pi \begin{bmatrix} MN_2A_{j\omega}^{-1}(B_1K_\triangle + B_2) \\ 1 \end{bmatrix} < 0$$
$$A_{j\omega}^{-1} = (j\omega I_n - \{A + B_1(K_1N_1 + K_2N_2)\})^{-1}$$

[1]When the inequality degenerates at $\omega = +\infty$, the strict inequality should hold in a limit if the matrix is multiplied by a factor ω^2.

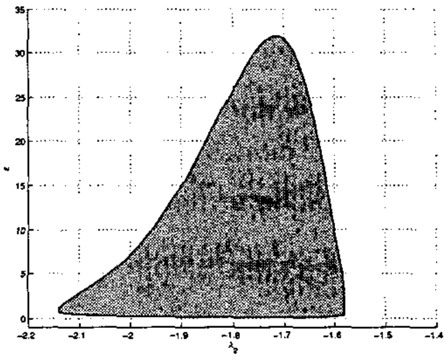

Fig. 4. The red area corresponds to values of ε as a function of the parameter λ_2 for which the asymptotic stability of the closed system preserved. The other parameters (9) have the nominal value (14). The largest uncertainty interval for ε is attained for $\lambda_2 = -1.71$, and looks as $\varepsilon \in [0.132, 31.97]$.

2) the matrix $(A + B_1(K_1N_1 + K_2N_2))$ is strictly Hurwitz. ■

For the further development it will be convenient to make linear transformation of the system (20), choosing a matrix N_0 such that the $n \times n$ matrix

$$N = \begin{bmatrix} N_0 \\ N_1 \\ N_2 \end{bmatrix}$$

has full rank, *i.e.*, $\det N \neq 0$, where the matrices N_1 and N_2 are from (21) and (21). In new coordinates

$$\begin{bmatrix} r \\ y \\ d \end{bmatrix} = Nx = \begin{bmatrix} N_0 \\ N_1 \\ N_2 \end{bmatrix} x \tag{38}$$

the system (20) takes an equivalent form

$$\dot r = A_{11}r + A_{12}y + A_{13}d + B_{11}u + B_{12}\Delta(d,t) \tag{39}$$
$$\dot y = A_{21}r + A_{22}y + A_{23}d + B_{21}u + B_{22}\Delta(d,t) \tag{40}$$
$$\dot d = A_{31}r + A_{32}y + A_{33}d + B_{31}u + B_{32}\Delta(d,t) \tag{41}$$

Assumption 5: B_{12}, B_{22} are zero matrices. ■

Assumption 6: There exists a matrix Φ such that $A_{31} = \Phi A_{21}$ and such that the system

$$\frac{d}{dt}e = (A_{33} - \Phi A_{23})e + B_{32}(\Delta(d,t) - \Delta(d-e,t)) \tag{42}$$

is asymptotically stable for any function $d \in L_{2e}$, its stability being verified by the *circle criterion* applied to the constraint in Assumption 1. Namely

1) The next *frequency condition* holds $\forall \omega \in \mathbb{R}_+$

$$\mathrm{Re}\left\{\Pi_\triangle(j\omega I_k - (A_{33} - \Phi A_{23}))^{-1}B_{32}\right\} < 0 \tag{43}$$

2) The matrix $(A_{33} - \Phi A_{23})$ is strictly Hurwitz. ■

Theorem 1: Consider the system (39)–(41). Suppose Assumptions 1–6 hold. Take an output feedback controller as

$$u = K_1 y + K_2 \hat{d} + K_\triangle \Delta(\hat{d}, t), \tag{44}$$

where matrices K_1, K_2, $K_\triangle$ satisfy Assumption 4; the variable $\hat{d}$ is defined as $\hat{d} = z + \Phi y$ with the matrix Φ from Assumption 6 and z defined as a solution to

$$\begin{aligned} \frac{dz}{dt} &= [(A_{32} - \Phi A_{22}) + (A_{33} - \Phi A_{23})\Phi] y \\ &+ (A_{33} - \Phi A_{23}) z \\ &+ (B_{31} - \Phi B_{21}) u + B_{32}\Delta(z + \Phi y, t) \end{aligned} \tag{45}$$

Then, the closed-loop system (20)–(21), (44)–(45) is globally asymptotically stable. ■

Theorem 2: Consider the system (29)–(31). Suppose Assumptions 1–6 hold, then the BMIs (35) are solvable, one solution being

$$\begin{aligned} \Lambda_z &= A_{33} - \Phi A_{23}, & C_z &= I_k, \\ \Lambda_u &= B_{31} - \Phi B_{21}, & \Lambda_\triangle &= B_{32}, \\ R_z &= K_2, & R_y &= K_1 + K_2\Phi, \\ R_\triangle &= K_\triangle, & C_y &= \Phi, \\ \Lambda_y &= (A_{32} - \Phi A_{22}) + (A_{33} - \Phi A_{23})\Phi \end{aligned} \tag{46}$$

IV. Discussion

1. In Sec. III is shown that stabilization of nonlinear systems by dynamical output feedback via the *Circle Criterion* could be reformulated as a problem of solvability of *Bilinear Matrix Inequalities* (BMI) of particular type. In general, finding even one solution for BMI could be a difficult problem, while for low dimensional system, all solutions could be found, and the result is written in the form of finite number of inequalities for the parameters of the controller.

2. This approach allows not only to find a set of all solutions (dynamical stabilizing controllers of particular structure for low dimensional systems), but also to analyze and quantify a robustness of the closed-loop system in case of uncertainty and parameter variation.

3. To compare the results with related ones in [1], one should realize that methods in [1] are devoted to stabilization of more general time-invariant dynamical systems, than the equations (20)–(21) represent. This generality reduces the ability to find a variety of stabilizing controllers, and only a few controllers was found in [1].

4. If Assumption 5 is omitted, then Theorem 1 is still true, that is, the closed loop system (20)–(21), (44)–(45) is globally asymptotically stable. In fact, this is the result of [1, Theorem 1, p.677] whereas the main result—*i.e.*, solvability of the particular BMI stated in Theorem 2—is not. The reason is that convergence of any closed loop system solution to the origin can be proven in this case by a particular choice of a Lyapunov-type function, designed only for this particular solution, while for different solutions these Lyapunov functions are different. In turn, the solvability of the BMI implies existence of only one positive definite quadratic form that provides exponential convergence of any closed-loop solution to the origin.

5. The 2×2 quadratic form Π in Assumption 3 has a particularly simple structure for use in the example. In general, one might consider any *integral quadratic constraint*, [7], [8], that leads to a design of stabilizing controller when all state variables are available. In turn, the nonlinearity $\Delta(d, t)$ should not be scalar, but it seems that such generality would obscure the main contribution of the paper.

6. The computational algorithms for solving nonconvex problems (35) are still at a developing stage [5].

V. Conclusions

In this paper, it was shown that a dynamic output feedback stabilization problem with impending problems of finite escape time, previously attacked by observer-based design, can be successfully solved using circle criterion design. Stability of the closed-loop system is global and robust to parameter uncertainty.

VI. References

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