Stability Analysis of an On-Line Algorithm for Torque Limited Path Following

Dahl, Ola; Nielsen, Lars

1990

Document Version:
Publisher's PDF, also known as Version of record

Link to publication

Citation for published version (APA):
Dahl, O., & Nielsen, L. (1990). *Stability Analysis of an On-Line Algorithm for Torque Limited Path Following*. (Technical Reports TFRT-7449). Department of Automatic Control, Lund Institute of Technology (LTH).

Total number of authors:
2

CODEN: LUTFD2/(TFRT-7449)/1-9/(1990)

Stability Analysis of an On-line Algorithm for Torque Limited Path Following

Ola Dahl and Lars Nielsen

Department of Automatic Control
Lund Institute of Technology
March 1990

Department of Automatic Control
Lund Institute of Technology
P.O. Box 118
S-221 00 Lund Sweden

Document name
Report

Date of issue
January 1990

Document Number
CODEN: LUTFD2/(TFRT-7449)/1-9/(1990)

Author(s)
Ola Dahl and Lars Nielsen

Supervisor

Sponsoring organisation
Swedish National Board for Technical Development

Title and subtitle
Stability Analysis of an On-line Algorithm for Torque Limited Path Following

Abstract

Paper at the 1990 IEEE International Conference on Robotics and Automation

Key words

Classification system and/or index terms (if any)

Supplementary bibliographical information

ISSN and key title

ISBN

Language
English

Number of pages
9

Security classification

Recipient's notes

The report may be ordered from the Department of Automatic Control or borrowed through the University Library 2, Box 1010, S-221 03 Lund, Sweden, Telex: 33248 lubbis lund.

Stability Analysis of an On-line Algorithm for Torque Limited Path Following

Ola Dahl and Lars Nielsen
Dept. of Automatic Control, Lund Inst. of Technology
Box 118, S-221 00 Lund, Sweden

Abstract A secondary controller for on-line time scaling of nominal high performance trajectories has been proposed to handle path following with torques close to the limits. Such nominal reference trajectories are typically available from an off-line optimization. The problem studied in this paper is the stability properties of the closed loop system, including the robot, the primary controller, and the new secondary controller.

1. Introduction

A feedback scheme for torque limited path following by on-line trajectory time scaling is proposed in (Dahl and Nielsen, 1989), where a secondary controller is used for modification of the robot speed along a geometric path. The scheme has been evaluated by simulations and experimental results, and a stability analysis provides further motivation of the feasibility of the proposed secondary controller.

Fast motion along a geometric path is important in several applications where the robot performance is limiting the production speed e.g. gluing, arc welding of small pieces, and laser or high pressure water cutting. An off-line procedure (Bobrow, Dubowsky, and Gibson, 1983, 1985; Shin and McKay, 1985; Pfeiffer and Johanni, 1986) is used to obtain a nominal trajectory to be used as a reference trajectory for the robot's control system. These trajectories typically lead to torques at the limit, hence leaving no control authority to take care of disturbances or modeling discrepancies. One way to handle this is to reduce the assumed torque bounds (Asada and Slotine, 1986, Section 6.6), or have bounds on model errors (Shin and McKay, 1987). Another idea is to do the best possible in nominal time, but hence leave the path (Slotine and Spong, 1985).

It is obvious from the optimization problem but also from physical reasons that if the robot is behind the nominal trajectory then it is impossible to catch up if the controller already is at the torque limit. A constructive way to handle the problem is to slow down the reference trajectory if path tracking is at priority. Further, it is untractable to be conservative already in the trajectory planning stage because of productivity reasons. These considerations are the background to our research on schemes for on-line modification of the time scaling of the reference trajectory, where the goal is to keep following the path at the expense of an increase in the path traversal time. Ideally, the traversal time should only increase if needed and then as little as possible.

This paper contains a stability analysis of such an algorithm proposed in (Dahl and Nielsen, 1989). Section 2 is a review of earlier results. The key ideas can be summarized as follows. The primary controller is left unchanged, only the reference trajectory is changed. This keeps the dynamic properties of the closed loop system, e.g. robustness is preserved. A secondary control loop is used for modification of only a scalar variable, which simplifies the design and analysis of the secondary controller. A crucial step is the observation that the primary controller can be parameterized in the path parameter instead of in time. Section 3 summarizes the assumptions used in the stability analysis, and the results are given in Section 4. The analysis shows that if the nominal velocity profile is within certain limits, the result is a bounded actual velocity profile. Stability of the closed loop system including the robot, the primary and the secondary controller is then obtained by requiring a specified tracking performance for the primary controller.

2. Torque Limited Path Following

This section is a review of our scheme for torque limited path following by trajectory time scaling, proposed in (Dahl and Nielsen, 1989).

Robot Dynamics and Torque Constraints

The robot is described by the rigid body dynamics

$$H(q)\ddot{q} + v(q,\dot{q}) + d(q)\dot{q} + g(q) = \tau \quad (1)$$

where $q \in R^n$ is the vector of joint variables, $\tau \in R^n$ is the vector of input torques, $H(q)$ is the inertia matrix, $v(q,\dot{q})$ is the vector of coriolis and centrifugal forces, $d(q)$ is the viscous friction matrix, and $g(q)$ is the vector of gravitational forces. The torque constraints are in general given by a region $E(q,\dot{q})$ where the admissible torques satisfy $\tau \in E$, but typically each component of the torque vector is upper and lower limited by constants leading to a hyper rectangle as torque constraint region E.

Primary Controller

A primary controller for reference trajectory tracking is assumed available. The primary controller is designed for good performance, disturbance rejection etc., and is kept unchanged. It is however written in a special form, suitable for the design of the secondary controller.

Path Following by Trajectory Time Scaling

The path is given in parametrized form $f(s) \in R^n$ where s is the scalar path parameter. A nominal reference trajectory $f(s_n(t))$ where $s_n(t)$ is the nominal path parameter, is available from an off-line procedure. The time scaling is done by on-line modification of the nominal path parameter $s_n(t)$, resulting in an actual path parameter $s(t)$. The modified reference trajectory $f(s(t))$ is then used as input to the primary controller. The goal is to have an actual path parameter $s(t)$ resulting in fast motion without violating the torque constraints.

Nominal Trajectory

The nominal path parameter $s_n(t)$ is represented by the nominal velocity and acceleration profiles $v_n(s)$ and $a_n(s)$. These functions are computed from the nominal path parameter as

$$\begin{aligned} \dot{s}_n(t) &= v_n(s_n(t)) \\ \ddot{s}_n(t) &= a_n(s_n(t)) = \frac{d}{ds_n}(\frac{1}{2}v_n(s_n(t))^2) \end{aligned} \quad (2)$$

Note that the functions v_n and a_n depend only on the nominal trajectory $s_n(t)$, $\dot{s}_n(t)$, and $\ddot{s}_n(t)$, and can thus be precomputed and stored in advance.

Secondary Controller

The secondary controller uses measurements of the robot's speed and position to control the path parameter s, and is given by the dynamical system

$$\begin{aligned} a_r &= \beta a_n(s) + \frac{1}{2}\alpha(v_n(s)^2 - \dot{s}^2) \\ \ddot{s} &= sat(a_r, s, \dot{s}, e, \dot{e}) \\ \dot{s}(0) &= \dot{s}_n(0) \\ s(0) &= s_n(0) \end{aligned} \quad (3)$$

The secondary controller (3) computes the path acceleration $\ddot{s}$ as the limitation of the auxiliary variable a_r. The path acceleration is limited by the function *sat*, by computing bounds on the path acceleration from the torque constraints, using information about the actual position and speed of the robot. The function *sat* will be defined more precisely in the following subsections. The result of using the secondary controller (3) is that the path acceleration is limited so that the torque constraints are not violated. The term $\beta a_n(s)$ is a feedforward term from the nominal path acceleration, and the term $\frac{1}{2}\alpha(v_n(s)^2 - \dot{s}^2)$ is a feedback from the nominal and actual path velocities. The parameter β is chosen as $\beta = 1$ or $\beta = 0$, where $\beta = 0$ corresponds to the case where the nominal path acceleration $a_n(s)$ is not specified. The limitation of $\ddot{s}$ occurs only if needed, i.e. if the unlimited path acceleration a_r leads to violation of the torque constraints. Otherwise, the feedback term results in $\dot{s}$ approaching the nominal velocity profile $v_n(s)$.

Path Parametrization of the Primary Controller

The function *sat* will now be more precisely defined. The bounds on path acceleration used by the function *sat* are computed from a parametrization of the primary controller. The parametrization has the form

$$\tau = b_1(s,\dot{s},q,\dot{q})\ddot{s} + b_2(s,\dot{s},q,\dot{q}) \quad (4)$$

As an example of the controller parametrization, a computed torque controller, given by

$$\begin{aligned} \tau = &\hat{H}(q)(\ddot{q}_r + K_p e + K_v \dot{e}) + \hat{v}(q,\dot{q}) \\ &+ \hat{d}(q)\dot{q} + \hat{g}(q) \end{aligned}$$

is parametrized as

$$\begin{aligned} b_1(s,q) &= \hat{H}(q)f'(s) \\ b_2(s,\dot{s},q,\dot{q}) &= \hat{H}(q)(f''(s)\dot{s}^2 + K_p e + K_v \dot{e}) \\ &\quad + \hat{v}(q,\dot{q}) + \hat{d}(q)\dot{q} + \hat{g}(q) \end{aligned}$$

where the reference trajectory is denoted q_r, the tracking error e is defined as $e = q_r - q$, K_p and K_v are feedback matrices, and the variables $\hat{H}$, $\hat{v}$, $\hat{d}$, and $\hat{g}$ represent an available model of the robot.

On-line Constraints on Path Acceleration

On-line constraints on the path acceleration $\ddot{s}$ can be computed from equation (4). First, note that since $e = q_r - q$ and $q_r = f(s)$, we can equivalently regard b_1 and b_2 as functions of s, $\dot{s}$, e, and $\dot{e}$. The controller parametrization is then rewritten as

$$\tau = b_1(s,\dot{s},e,\dot{e})\ddot{s} + b_2(s,\dot{s},e,\dot{e}) \tag{5}$$

On-line bounds on $\ddot{s}$ are now computed as the maximum and minimum values of $\ddot{s}$ that results in $\tau \in E$, when τ is given by (5). Note that for certain values of s, $\dot{s}$, e, and $\dot{e}$, these bounds do not exist, since there may be no $\ddot{s}$ resulting in $\tau \in E$. This reasoning is similar to the analysis used in off-line trajectory planning (Bobrow, Dubowsky, and Gibson, 1983, 1985; Shin and McKay, 1985; Pfeiffer and Johanni, 1986). There, it is assumed that the robot is moving along the path, resulting in an equation for the robot dynamics of the form

$$a_1(s)\ddot{s} + a_2(s)\dot{s}^2 + a_3(s)\dot{s} + a_4(s) = \tau \tag{6}$$

This equation is used to compute off-line constraints on $\dot{s}$ and $\ddot{s}$, which then can be used in the trajectory planning. Here, we get on-line constraints from equation (5), resulting in constraints on $\dot{s}$ and $\ddot{s}$ depending on the tracking errors e and $\dot{e}$.

The saturation function *sat*

The function *sat* is used for limitation of $\ddot{s}$ by the on-line bounds on path acceleration computed from the controller parametrization (5). Given s, e, and $\dot{e}$, it depends on $\dot{s}$ whether these bounds exist or not. There are thus requirements on the path speed $\dot{s}$. The limitation function *sat* is meaningful only if the requirements on $\dot{s}$ are satisfied. We therefore define *sat* as follows.

DEFINITION 1
The function $sat(x, s, \dot{s}, e, \dot{e})$ is defined as the limitation of x by the on-line bounds on $\ddot{s}$, computed from the controller parametrization (5), if $\dot{s}$ is admissible, and 0 otherwise. The path speed $\dot{s}$ is admissible if $\dot{s} \in S(s,e,\dot{e})$, where $S(s,e,\dot{e})$ is a set of $\dot{s}$-values such that $\dot{s} \in S$ implies that there exists on-line bounds on the path acceleration $\ddot{s}$.

Remark 1. The set S is typically an interval $0 \leq \dot{s} \leq \dot{s}_{max}$, but may also be more than one interval, corresponding to islands in the phase plane in off-line trajectory planning (Shin and McKay, 1985). The set S used in the stability analysis in Section 4 is defined below in Definition 2.

Remark 2. The value 0 is chosen as a safety value. If the constraints on $\dot{s}$ are violated, the path speed $\dot{s}$ is kept constant. This is convenient for the analysis but other methods could of course be considered. □

Time Scale Transformation

We have introduced the nominal velocity profile v_n and the nominal acceleration profile a_n in (2). We introduce in the same way for the actual trajectory $s(t)$, the actual velocity and acceleration profiles, v and a, as

$$\dot{s}(t) = v(s(t)), \quad \ddot{s}(t) = a(s(t))$$

Introducing y and y_r by

$$y(s) = \frac{1}{2}v(s)^2, \quad y_r(s) = \frac{1}{2}v_n(s)^2 \tag{7}$$

the nominal and actual path accelerations are given by

$$a_n(s) = \frac{dy_r(s)}{ds}, \quad \ddot{s} = \frac{dv(s)}{ds}v(s) = \frac{dy(s)}{ds} \tag{8}$$

The algorithm (3) can now be written in the transformed time scale as

$$\begin{aligned} a_r &= \beta\frac{dy_r}{ds} + \alpha(y_r - y) \\ \frac{dy}{ds} &= sat(a_r, s, \dot{s}, e, \dot{e}) \\ y(s_0) &= \frac{1}{2}\dot{s}_{n_0}^2 \end{aligned} \tag{9}$$

where $\dot{s}_{n_0} = \dot{s}_n(0)$. We see that by using the transformed time scale, the secondary controller, given by the algorithm (3) and rewritten in (9), can be regarded as a first order system, where the derivative of the

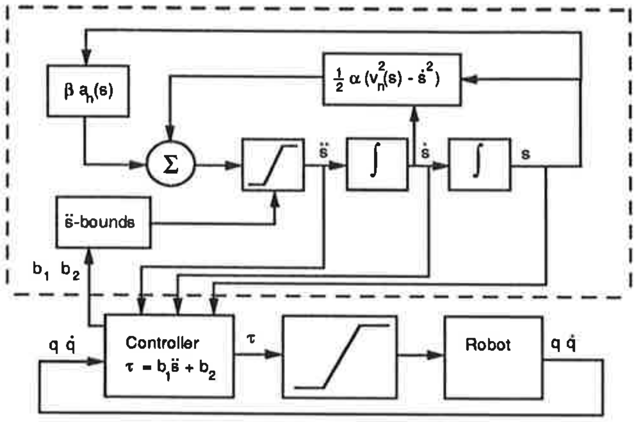

Figure 1. A block diagram of the robot with primary and secondary control loop. The primary controller is parametrized in s. Observe that b_1 and b_2 depend on measured quantities.

output y is limited by the function sat. The effect of using the algorithm is that the saturation function limits the slope of the actual velocity profile $v(s) = \dot{s}$. If the bounds on path acceleration are not activated, we get $\ddot{s} = \frac{dy}{ds} = a_r$, resulting in a linear system with an s-time constant $\frac{1}{\alpha}$. The algorithm (9) is further described in (Dahl and Nielsen, 1989; Dahl, 1989) where it is also used as a part of an algorithm for the minimum time case, where it is not sufficient to consider only the path acceleration constraints. An additional scaling of the nominal velocity profile is introduced and the scaling factor is updated on-line. The algorithm (3) is illustrated in Figure 1, where a block diagram of the system with the secondary control loop is shown.

3. The Closed Loop System

The stability analysis of the nonlinear closed loop system, shown in Figure 1, is based on the following assumptions.

Robot Model The robot dynamics are given by

$$H(q)\ddot{q} + v(q,\dot{q}) + d(q)\dot{q} + g(q) = \tau \qquad (1)$$

Path The path is given in parametrized form $f(s)$ and the derivatives $f'(s)$ and $f''(s)$ exist and are bounded.

Primary Controller The reference trajectory used by the primary controller is $q_r(t) = f(s(t))$. Typically, reference values for speed and acceleration are available, resulting in a primary controller having the inputs

$$q_r = f(s), \; \dot{q}_r = f'(s)\dot{s}, \; \ddot{q}_r = f''(s)\dot{s}^2 + f'(s)\ddot{s}$$

The primary controller can be parametrized as

$$\tau = b_1(s,\dot{s},e,\dot{e})\ddot{s} + b_2(s,\dot{s},e,\dot{e}) \qquad (5)$$

and the output τ of the primary controller is constrained by $\tau \in E$, where E typically is given by constant bounds, resulting in $\tau \in E$ if $\tau_i^{min} \leq \tau_i \leq \tau_i^{max}$ for all i, $1 \leq i \leq n$, the number of joint variables.

Nominal Trajectory The nominal reference trajectory is represented by the nominal velocity and acceleration profiles, $v_n(s)$ and $a_n(s)$, equation (2).

Secondary Controller The secondary controller is given by (3), and written in the transformed time scale, using (7) and (8), as

$$\begin{aligned} a_r &= \beta\frac{dy_r}{ds} + \alpha(y_r - y) \\ \frac{dy}{ds} &= sat(a_r, s, \dot{s}, e, \dot{e}) \\ y(s_0) &= \frac{1}{2}\dot{s}_{n_0}^2 \end{aligned} \qquad (9)$$

The saturation function sat is given by Definition 1.

Restrictions on Path Speed The set S in Definition 1 is defined as

DEFINITION 2
Given $b_1(s,\dot{s},e,\dot{e})$ and $b_2(s,\dot{s},e,\dot{e})$ in the controller parametrization (5). The set $S(s,e,\dot{e})$ is then defined as $\dot{s} \in S$ if there exist finite $\ddot{s}$-bounds $\ddot{s}_{min}(s,\dot{s},e,\dot{e})$ and $\ddot{s}_{max}(s,\dot{s},e,\dot{e})$ such that $\ddot{s}_{max} \geq 0$, and $\ddot{s}_{min} \leq 0$. □

Note that this definition of S can be interpreted as a restriction on the path speed $\dot{s}$, since there may exist $\dot{s}$ such that the bounds exist but have the same sign. Note also that with S according to Definition 2, $sat(x,s,\dot{s},e,\dot{e})$ is finite and that $| sat(x,s,\dot{s},e,\dot{e}) | \leq | x |$ for all x, s, $\dot{s}$, e, and $\dot{e}$.

4. Stability Analysis

The purpose of the stability analysis is now to show that given a nominal velocity profile, the actual velocity profile results in $\tau \in E$ and tracking errors e and $\dot{e}$ smaller than a given tolerance.

Outline of Proof

The idea in the stability proof is to first show that given a bounded nominal velocity profile, the result of using the secondary controller (9) is that the actual velocity profile is also bounded, and the bounds are independent of the tracking errors e and $\dot{e}$. This is the result of Theorem 1 below. The stability of the closed loop system is then shown by the following reasoning, which is formalized in Theorem 2.

1. Given a nominal velocity profile, bounds on the actual velocity profile are given by Theorem 1. Furthermore, the path speed $\dot{s}$ should be admissible when $\dot{s}$ is below the bound given by Theorem 1, *and* the tracking errors are smaller than a given tolerance, e.g. given as $| e(t) | \leq e_{max}$ and $| \dot{e}(t) | \leq \dot{e}_{max}$ for all t. This means that the resulting torques will also be admissible as long as the tracking errors are smaller than the given tolerance.
2. The primary controller should give tracking errors below the specified tolerance, when the reference trajectory is satisfying the bounds computed by Theorem 1.
3. Since the bounds computed in 1, gives tracking errors below the tolerances, $\dot{s}$ will be admissible during the complete motion, and therefore the torques will also be admissible.

Note that in 2, the torque limits do not have to be taken into account. The primary controller gives bounds on the tracking errors, given the bounds on the reference trajectory. The bounds on the reference trajectory are independent of the tracking errors, and given by Theorem 1. This means that the tracking error bounds will be satisfied during the motion. Since the path speed $\dot{s}$ is admissible for all tracking errors below the tracking error bounds, the resulting torques will also be admissible.

Stability of the Secondary Controller

A stability result for the algorithm (9) is given below. The result is based on the observation that the algorithm can be interpreted as a first order system with input y_r and output y, and where the derivative of the output, $\frac{dy}{ds}$, is limited. The limitation is done by the function *sat*, and occurs only if the bounds used in the limitation have different signs, i.e. if $\dot{s}$ is admissible, see Definitions 1 and 2. If this is not the case, the value of *sat* is 0, resulting in $\frac{dy}{ds} = 0$.

Theorem 1—Stability of the Secondary Controller
If the secondary controller is given by the algorithm (9), and

1. $y_r(s) = \frac{1}{2}v_n(s)^2$ satisfies $\Delta + \epsilon < y_r(s) < y_{max} - \Delta$, $\Delta, \epsilon > 0$ for all $s \geq s_0$, and $| \beta \frac{dy_r}{ds} |$ is bounded for all $s \geq s_0$.
2. The feedback gain α satisfies $\alpha \geq | \frac{\beta}{\Delta} \frac{dy_r}{ds}(s) |$ for all $s \geq s_0$.
3. $y(s) = \frac{1}{2}v(s)^2$ satisfies the initial conditions $\epsilon \leq y(s_0) \leq y_{max}$, $v(s_0) > 0$.

then

* $\epsilon \leq y(s) \leq y_{max}$ for all $s \geq s_0$, and $v(s) > 0$ for all $s \geq s_0$. The resulting path acceleration $\ddot{s} = \frac{dy}{ds}$ is bounded by $| \frac{dy}{ds} | \leq | \beta \frac{dy_r}{ds} | + | \alpha y_{max} |$.

Proof: The upper bound $y(s) \leq y_{max}$ is first shown. Suppose that for some s, $s = s_2 > s_0$, $y(s_2) > y_{max}$. Then, since $y(s_0) \leq y_{max}$, there are one or more values of $s, s_0 \leq s < s_2$ where $y(s) = y_{max}$. Let s_1 be the largest of these values. This gives $y(s_1) = y_{max}$, $y(s) > y_{max}$ for $s_1 < s < s_2$. Then, by the Mean Value Theorem, there is a point $\sigma_1, s_1 < \sigma_1 < s_2$, such that

$$\frac{dy}{ds}(\sigma_1) = \frac{y(s_2) - y(s_1)}{s_2 - s_1} > 0$$

Using equation (9), this gives

$$\frac{dy}{ds}(\sigma_1) = sat(a_r(\sigma_1), \sigma_1, v(\sigma_1), e(\sigma_1), \dot{e}(\sigma_1)) > 0$$

Since the lower bound in the limitation by *sat* is never positive, this implies $a_r(\sigma_1) > 0$. This gives

$$a_r(\sigma_1) = \beta \frac{dy_r}{ds}(\sigma_1) + \alpha(y_r(\sigma_1) - y(\sigma_1)) > 0$$

But $y(\sigma_1) > y_{max}$ and $y_r(\sigma_1) < y_{max} - \Delta$ implies $y_r(\sigma_1) - y(\sigma_1) < -\Delta < 0$. This results in $a_r(\sigma_1) \leq 0$ if $\alpha \geq | \frac{\beta}{\Delta} \frac{dy_r}{ds}(\sigma_1) |$ which gives a contradiction. Hence, $y(s) \leq y_{max}$ for all $s \geq s_0$.

The lower bound $y(s) \geq \epsilon$ is now shown by the same method. Suppose that for some s, $s = s_4 > s_0$, $y(s_4) < \epsilon$. Then, since $y(s_0) \geq \epsilon$, there are one or more values of $s, s_0 \leq s < s_4$ where $y(s) = \epsilon$. Let s_3 be the largest of these values. This gives $y(s_3) = \epsilon$, $y(s) < \epsilon$ for $s_3 < s < s_4$. Then there is a point $\sigma_2, s_3 < \sigma_2 < s_4$, such that

$$\frac{dy}{ds}(\sigma_2) = \frac{y(s_4) - y(s_3)}{s_4 - s_3} < 0$$

Using equation (9), this gives

$$\frac{dy}{ds}(\sigma_2) = sat(a_r(\sigma_2), \sigma_2, v(\sigma_2), e(\sigma_2), \dot{e}(\sigma_2)) < 0$$

Since the upper bound in the limitation by *sat* is always nonnegative, this implies $a_r(\sigma_2) < 0$. This gives

$$a_r(\sigma_2) = \beta \frac{dy_r}{ds}(\sigma_2) + \alpha(y_r(\sigma_2) - y(\sigma_2)) < 0$$

But $y(\sigma_2) < \epsilon$ and $y_r(\sigma_2) > \epsilon + \Delta$ implies $y_r(\sigma_2) - y(\sigma_2) > \Delta > 0$. This results in $a_r(\sigma_2) \geq 0$ if $\alpha \geq \mid \frac{\beta}{\Delta}\frac{dy_r}{ds}(\sigma_2) \mid$ which gives a contradiction. Hence, $y(s) \geq \epsilon$ for all $s \geq s_0$.

We have shown $\epsilon \leq y(s) \leq y_{max}$ for all $s \geq s_0$. Since $v(s_0) > 0$, and $y(s) = \frac{1}{2}v(s)^2 > 0$ for all $s \geq s_0$, $\dot{s} = v(s)$ will continue to be positive, i.e. $v(s) > 0$ for all $s \geq s_0$. Furthermore, the function *sat* has the property $\mid sat(x, s, v, e, \dot{e}) \mid \leq \mid x \mid$ for all x, s, v, e, and $\dot{e}$, which gives the bound on $\ddot{s} = \frac{dy}{ds}$ as

$$\mid \frac{dy}{ds} \mid = \mid sat(\beta\frac{dy_r}{ds} + \alpha(y_r - y), s, v, e, \dot{e}) \mid \leq$$
$$\mid \beta\frac{dy_r}{ds} \mid + \mid \alpha(y_r - y) \mid \leq \mid \beta\frac{dy_r}{ds} \mid + \mid \alpha y_{max} \mid$$

□

Theorem 1 shows that if the gain α is chosen sufficiently large, and $y_r(s)$ and $\beta\frac{dy_r}{ds}$ are bounded, then $y(s)$ and $\frac{dy}{ds}$ are also bounded, and the bounds are independent of the tracking errors e and $\dot{e}$.

Stability of the Closed Loop System

We can now state the following result.

THEOREM 2—Stability of the Closed Loop System
If

1. The nominal velocity profile $v_n(s)$ satisfies $\Delta + \epsilon < \frac{1}{2}v_n^2(s) < y_{max} - \Delta$, $\Delta, \epsilon > 0$, for all $s \geq s_0$, and $\mid \beta\frac{dy_r}{ds} \mid$ is bounded for all $s \geq s_0$, and the feedback gain α satisfies $\alpha \geq \mid \frac{\beta}{\Delta}\frac{dy_r}{ds}(s) \mid$ for all $s \geq s_0$.
2. The actual velocity profile $\dot{s} = v(s)$ satisfies the initial conditions $\epsilon \leq \frac{1}{2}v(s_0)^2 \leq y_{max}$, $v(s_0) > 0$.
3. The motion starts at $s = s_0$ with zero tracking error, i.e. $s(0) = s_0$, $e(0) = \dot{e}(0) = 0$.
4. The primary controller has the property that $\mid e \mid \leq e_{max}$ and $\mid \dot{e} \mid \leq \dot{e}_{max}$ for all $s \geq s_0$, if $\mid \ddot{s} \mid \leq \mid \beta\frac{dy_r}{ds} \mid + \mid \alpha y_{max} \mid$ and $\mid \dot{s} \mid \leq \sqrt{2y_{max}}$ for all $s \geq s_0$, and $e(0) = \dot{e}(0) = 0$.
5. The path speed $\dot{s}$ is admissible if $\dot{s} \leq \sqrt{2y_{max}}$, $\mid e \mid \leq e_{max}$, and $\mid \dot{e} \mid \leq \dot{e}_{max}$.

then

* The closed loop system is stable in the sense that $\mid e \mid \leq e_{max}$ and $\mid \dot{e} \mid \leq \dot{e}_{max}$ for all $s \geq s_0$, and τ is admissible for all $s \geq s_0$.

Proof: Assumptions 1 and 2, together with Theorem 1, give $0 < \dot{s} \leq \sqrt{2y_{max}}$ for all $s \geq s_0$, and $\mid \ddot{s} \mid \leq \mid \beta\frac{dy_r}{ds} \mid + \mid \alpha y_{max} \mid$ for all $s \geq s_0$. Assumptions 3 and 4 then give $\mid e \mid \leq e_{max}$ and $\mid \dot{e} \mid \leq \dot{e}_{max}$ for all $s \geq s_0$. Assumption 5 together with the bound on $\dot{s}$ then implies that $\dot{s}$ will be admissible during the motion, i.e. for $s \geq s_0$, which in turn implies that τ will be admissible for all $s \geq s_0$.

Remark. Note that the result holds for any primary controller, satisfying Assumption 4. A result showing the possibility of achieving a primary controller with this property is given in (Craig, 1988, p. 30) for a computed torque controller based on an uncertain model of the robot. □

Restrictions on Path Speed

The following Lemma is given to demonstrate that the fact that $\dot{s}$ is admissible, i.e. $\dot{s} \in S$, can be interpreted as a restriction on $\dot{s}$.

LEMMA 1
Suppose that the torque constraints can be expressed as $\mid \tau \mid \leq \tau_{max}$, where $\mid \tau \mid$ denotes the vector norm of τ. A necessary and sufficient condition for $\dot{s}$ admissible is then given by $\mid b_1 \mid \neq 0$, and $\mid b_2 \mid \leq \tau_{max}$.

Proof: Suppose that $\dot{s}$ is admissible. We then have bounds on $\ddot{s}$ such that τ is admissible for all $\ddot{s}$, satisfying $\ddot{s}_{min} \leq \ddot{s} \leq \ddot{s}_{max}$, where $\ddot{s}_{min} \leq 0$, and $\ddot{s}_{max} \geq 0$. This means that τ is admissible if $\ddot{s} = 0$, i.e. $\mid b_1 0 + b_2 \mid \leq \tau_{max}$, which implies $\mid b_2 \mid \leq \tau_{max}$. Furthermore, the bounds on $\ddot{s}$ are finite, which implies $\mid b_1 \mid \neq 0$, since if $\mid b_1 \mid = 0$, then $b_1 = 0$, which results in $\mid \tau \mid = \mid b_2 \mid \leq \tau_{max}$ for all $\ddot{s}$, which contradicts the fact that the bounds on $\ddot{s}$ are finite.

Suppose now that $\mid b_2 \mid \leq \tau_{max}$, and $\mid b_1 \mid \neq 0$. We then have

$$\mid b_1\ddot{s} + b_2 \mid \leq \mid b_1 \mid \mid \ddot{s} \mid + \mid b_2 \mid$$

which gives bounds on $\ddot{s}$ as

$$\ddot{s}_{min} = -\frac{\tau_{max} - \mid b_2 \mid}{\mid b_1 \mid} \leq \ddot{s} \leq \frac{\tau_{max} - \mid b_2 \mid}{\mid b_1 \mid} = \ddot{s}_{max}$$

and we see that there exist finite bounds on $\ddot{s}$, satisfying $\ddot{s}_{max} \geq 0$, and $\ddot{s}_{min} \leq 0$, i.e. $\dot{s}$ is admissible. □

The interpretation of $\dot{s}$ admissible as a restriction on $\dot{s}$ is motivated as follows. Suppose that the controller is a computed torque controller. The controller parametrization then gives

$$\begin{aligned} b_1 &= \hat{H}(q)f'(s) \\ b_2 &= \hat{H}(q)(f''(s)\dot{s}^2 + K_v\dot{e} + K_p e) + \hat{v}(q,\dot{q}) \\ &\quad + \hat{d}(q)\dot{q} + \hat{g}(q) \end{aligned}$$

Suppose further that the tracking errors are small, so that $q \approx q_r$ and $\dot{q} \approx \dot{q}_r$. Since $q_r = f(s)$, the vector $\hat{v}$ can then be approximated as $\hat{v}(q,\dot{q}) \approx \hat{v}(q_r,\dot{q}_r) = v(f,f')\dot{s}^2$. This gives

$$\begin{aligned} b_2 &\approx \hat{H}(q_r)f''(s)\dot{s}^2 + \hat{v}(f,f')\dot{s}^2 \\ &\quad + \hat{d}(q)f'(s)\dot{s} + \hat{g}(q) \\ &= b_{21}\dot{s}^2 + b_{22}\dot{s} + b_{23} \end{aligned}$$

and we see that the equivalent formulation of $\dot{s}$ admissible as given by Lemma 1, $| b_2 | \leq \tau_{max}$, now gives restrictions on the path speed $\dot{s}$.

EXAMPLE 1

The restrictions on $\dot{s}$ will be illustrated by two numerical examples. For both examples, three curves in the s-$\dot{s}$ plane will be shown. These are the admissible region, i.e. the region where there exists bounds on the path acceleration, the region where the bounds have different signs, and the minimum time trajectory. The computations will be done under the assumptions that the tracking errors are small, i.e. the open loop system (1) is used in the computations. The purpose is to show that requiring different signs on the online bounds on path acceleration, see Definition 2, is not a significantly restrictive assumption. It will be shown that for the examples considered, the minimum time trajectory is almost inside the region where the bounds on path acceleration have different signs. The first example is a decoupled cartesian robot, described by the model

$$m_i\ddot{x}_i + d_i\dot{x}_i = \tau_i, \quad i = 1,2$$

where $m_1 = m_2 = 0.045$ and $d_1 = d_2 = 0.0048$. The torques are constrained by $| \tau_i | \leq 0.2$, and the path is $f_1(s) = 0.4(1-\cos(s))$, $f_2(s) = 0.8\sin(s)$. The result is shown in Figure 2. As can be seen from the figure, the minimum time trajectory is entirely contained in the region where the bounds have different signs. The second example is a planar two link robot with links of length one and unitary masses concentrated at the joints. The torques are constrained by $| \tau_i | \leq 10$, and

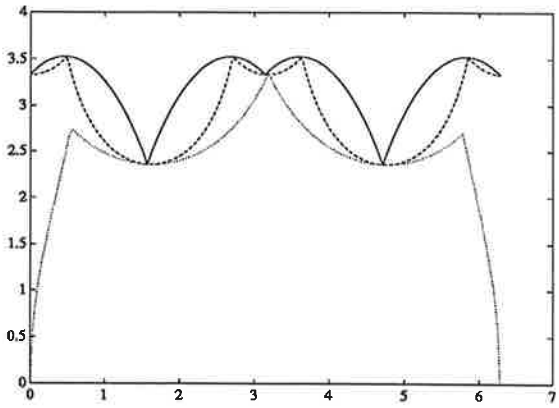

Figure 2. The different regions in the s-$\dot{s}$ plane for the cartesian robot. The solid line is the admissible region, the dashed line is the region where the bounds on path acceleration have different signs, and the dotted line is the minimum time trajectory.

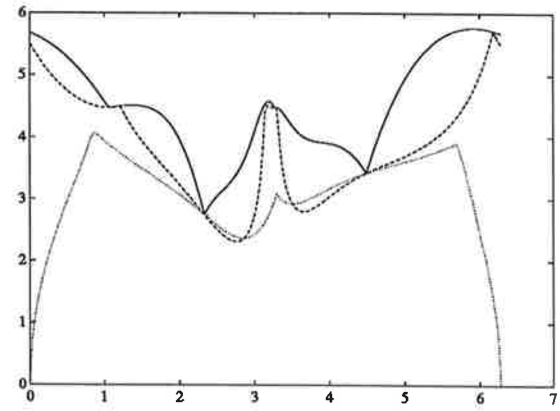

Figure 3. The different regions in the s-$\dot{s}$ plane for the two link robot. The solid line is the admissible region, the dashed line is the region where the bounds on path acceleration have different signs, and the dotted line is the minimum time trajectory.

the path is a circle in cartesian space with a radius of 0.5 and a midpoint with cartesian coordinates (1,0). The result of this example is shown in Figure 3. In this example, the minimum time trajectory is in some intervals slightly above the region where the bounds have different signs.

5. Conclusions

A stability analysis of a complete closed loop system, including the robot, the primary controller, and the time scaling secondary controller, has been presented.

The interpretation of the path parameter s as a transformed time scale is a key idea in the analysis. Using the transformed time scale, the secondary controller is written as a first order system with limitation, resulting in a simplified analysis compared to using the original time scale. The stability analysis includes a stability result for the secondary controller, showing that if the nominal velocity profile is within certain limits, the actual velocity profile will be bounded, and the bounds depend only on the nominal trajectory. The stability of the closed loop system is then shown, assuming a specified tracking performance for the primary controller. The scheme for on-line trajectory scaling has previously been shown feasible in simulations and experiments, and the stability analysis presented here provides additional insight and justification of the design of the secondary controller.

6. References

ASADA, H. and J.-J.E. SLOTINE (1986): *Robot Analysis and Control*, John Wiley and Sons, New York.

BOBROW, J.E., S. DUBOWSKY and J.S. GIBSON (1983): "On the Optimal Control of Robotic Manipulators with Actuator Constraints," *ACC-83, San Francisco.*

BOBROW, J.E., S. DUBOWSKY and J.S. GIBSON (1985): "Time Optimal Control of Robotic Manipulators Along Specified Paths," *Int.J.Robotics Research*, **4**, 3.

CRAIG, J.J. (1988): *Adaptive Control of Mechanical Manipulators*, Addison-Wesley.

DAHL, O. (1989): "Torque Limited Path Following by On-line Trajectory Time Scaling," Licentiate Thesis TFRT-3204, Department of Automatic Control, Lund Institute of Technology, Lund, Sweden.

DAHL, O. and L.NIELSEN (1989): "Torque Limited Path Following by On-line Trajectory Time Scaling," *1989 IEEE Conf. Robotics and Automation, Scottsdale, Arizona, USA.*

PFEIFFER, F. and R.JOHANNI (1986): "A Concept for Manipulator Trajectory Planning," *IEEE Conf. Robotics and Automation, San Francisco.*

SHIN, K.G. and N.D.MCKAY (1985): "Minimum-Time Control of Robotic Manipulators with Geometric Path Constraints," *IEEE Transactions On Automatic Control*, **AC-30**, No. 6, 531–541.

SHIN, K.G. and N.D.MCKAY (1987): "Robust Trajectory Planning for Robotic Manipulators Under Payload Uncertainties," *IEEE Transactions On Automatic Control*, **AC-32**, No. 12, 1044–1054.

SLOTINE, J.-J.E. and M.W. SPONG (1985): "Robust Robot Control with Bounded Inputs," *J. Robotics Systems*, **2**, 4.