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Multipliers for Stability Analysis of Systems with Nonlinearities

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Abstract

Stability analysis of systems with nonlinearities is considered. Multipliers that describe the nonlinearities will be used for the analysis. It will in particular be shown how Popov multipliers can be combined with multipliers for slope restricted nonlinearities. The stability analysis can be approximated by a feasibility test for linear matrix inequalities. This requires that we choose a finite dimensional subspace for the set of multipliers. The choice of suitable subspace is discussed and an example is given where a duality argument gives a bound for the possible performance of the multiplier for slope restricted nonlinearities.

Key words
Stability, Nonlinearity, Multipliers, Popov criterion

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1. Introduction

This paper considers stability analysis based on Integral Quadratic Constraints (IQC), see [9] and [11], for systems with nonlinearities. The systems are assumed to consist of a nominal plant G in a positive feedback interconnection with a causal and bounded perturbation Δ, see Figure 1. The perturbation can contain unmodeled dynamics, parametric uncertainty and nonlinearities.

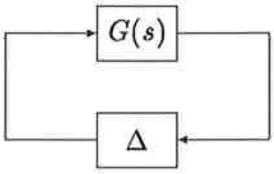

Figure 1 Feedback interconnection of a nominal linear and time invariant system G with a perturbation Δ.

The idea behind the IQC approach for stability analysis is to find a description of Δ in terms of a bounded and Hermitian valued matrix function Π that define a valid IQC in the sense that

$$\int_{-\infty}^{\infty} \begin{bmatrix} \widehat{y}(j\omega) \\ \widehat{\Delta(y)}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \widehat{y}(j\omega) \\ \widehat{\Delta(y)}(j\omega) \end{bmatrix} d\omega \geq 0, \tag{1}$$

for all square integrable inputs y to Δ. The matrix function Π is called multiplier.

The IQC methodology gives a unified approach for multiplier based stability analysis that has several advantages compared to the classical framework in for example [2]. In particular it allows easy combination of different multipliers that define valid IQC descriptions of Δ to improve the accuracy of the description.

An important stability criterion for systems with nonlinearities is the Popov criterion. The Popov multiplier is nonproper and can therefore not be used directly in (1). We will assume that the nominal plant is strictly proper. We can then use a reformulation of the main stability result in [9] and [11], which allows the use of Popov multipliers in a straightforward way, see [5]. This will allow us to combine Popov multipliers with the multipliers corresponding to slope restricted nonlinearities that were developed in [15].

The problem of finding suitable multipliers in stability analysis was noticed by Zames in his influential work [14]. The search for multipliers in IQC based stability analysis is generally an infinite dimensional problem. However, if we restrict this problem to a finite dimensional subspace then it is possible to formulate it as an optimization problem in terms of Linear Matrix Inequalities (LMI). Recent progress on numerical software for solving LMIs, see for example [4] and [3], has made this approach for stability analysis very attractive.

The Popov multiplier is useful in stability analysis of large systems since it is defined by a small number of parameters, in general only one. It is therefore computationally inexpensive to determine this multiplier.

The choice of a suitable finite dimensional subspace for the multipliers that describe slope restricted nonlinearities is particularly hard. This problem, which has been discussed before in [1], will be addressed briefly. We will also give an example where a nontrivial upper bound for the achievable performance with these multipliers can be obtained by use of a simple duality argument.

Notation and Preliminaries

M^*	Hermitian conjugation of a matrix.
$\lvert\cdot\rvert$	The Euclidean norm $\lvert x\rvert = \sqrt{x^T x}$.
$\mathbf{RL}_\infty^{n\times n}$	The space consisting of proper real rational matrix functions with no poles on the imaginary axis. If $H \in \mathbf{RL}_\infty^{m\times m}$ then $H^*(s) = H(-s)^T$.
$\mathbf{RH}_\infty^{m\times m}$	The subspace of $\mathbf{RL}_\infty^{m\times m}$ consisting of functions with no poles in the closed right half plane.
P_T	The projection operator defined by $P_T u(t) = u(t)$ when $t \le T$ and $P_T u(t) = 0$ when $t > T$.
$\mathbf{L}_2^m[0,\infty)$	The Lebesgue space of $\mathbf{R}^m$ valued signals with norm defined by $$\|u\|^2 = \int_0^\infty \lvert u(t)\rvert^2 dt.$$ $\mathbf{L}_2^m(-\infty,\infty)]$ is defined similarly as $\mathbf{L}_2^m[0,\infty)$.
$\mathbf{L}_{2e}^m[0,\infty)$	The vector space of functions f satisfying the condition $P_T f \in \mathbf{L}_2^m[0,\infty)$ for all $T > 0$.

An operator $H : \mathbf{L}_{2e}^m[0,\infty) \to \mathbf{L}_{2e}^m[0,\infty)$ is said to be causal if $P_T H P_T = P_T H$ for all $T \ge 0$. This means that the output value at a certain time instant does not depend on future values of the input.

A causal operator H on $\mathbf{L}_{2e}^m[0,\infty)$ is bounded if $H(0) = 0$ and if the gain defined as

$$\|H\| = \sup_{\mathbf{L}_2^m[0,\infty)\ni u\neq 0} \frac{\|Hu\|}{\|u\|} \tag{2}$$

is finite. Note that the gain is defined in terms of functions in $\mathbf{L}_2^m[0,\infty)$ and not the corresponding extended space. However, the definition in (2) implies boundedness on $\mathbf{L}_{2e}^m[0,\infty)$, since $\|P_T H u\| \le \|H\|\cdot\|P_T u\|$ for all $u \in \mathbf{L}_{2e}^m[0,\infty)$ and all $T \ge 0$. It can be shown that $\|H\|$ is the smallest such bound, see [13].

Linear time invariant convolution operators with transfer function $H \in \mathbf{RH}_\infty^{m\times m}$ are causal and bounded on $\mathbf{L}_{2e}^m[0,\infty)$. A bounded convolution operator with transfer function $H \in \mathbf{RL}_\infty^{m\times m}$ defines a (possibly non-causal) map of $\mathbf{L}_2[0,\infty)$ into $\mathbf{L}_2(-\infty,\infty)$.

Let $H = H^* \in \mathbf{RL}_\infty^{m\times m}$ and $u \in \mathbf{L}_2^m[0,\infty)$, then the quadratic form $\langle\cdot,\cdot\rangle$ is defined by

$$\langle u, Hu\rangle = \int_0^\infty u(t)^T (Hu)(t)dt = \int_{-\infty}^\infty \widehat{u}(j\omega)^* H(j\omega)\widehat{u}(j\omega)d\omega,$$

where Hu denotes the convolution $h * u$ between u and the impulse response h corresponding to H, and $\widehat{u}$ denotes the Fourier transform of u, which we define as

$$\widehat{u}(j\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty u(t)e^{-j\omega t}dt.$$

We define the $\mathbf{L}_1$-norm of the impulse response h corresponding to a strictly proper transfer function $H \in \mathbf{RL}_\infty^{1\times 1}$ as $\|h\|_1 = \int_{-\infty}^\infty \lvert h(t)\rvert dt$.

We will finally make some remarks on absolute continuity, see [12]. Absolute continuity of a function $x : \mathbf{R}^+ \to \mathbf{R}^m$ implies that the time derivative $\dot{x} := \frac{d}{dt}x$ exists and is finite almost everywhere. Furthermore, an absolutely continuous function x is the indefinite integral of its derivative, i.e. the relation $x(t) = x_0 + \int_0^t \dot{x}(\tau)d\tau$ holds for all $t \geq 0$. From now on it is assumed that x is absolutely continuous when make assumptions on the derivative $\dot{x}$.

We will use that the condition that if $x, \dot{x} \in \mathbf{L}_2^m[0,\infty)$, then it follows x is bounded and $x(t) \to 0$ as $t \to \infty$, see [2].

2. Stability Analysis Based on IQCs

The idea behind the IQC approach for stability analysis is to obtain descriptions of the nonlinear, uncertain and time-varying components of a system in terms of quadratic integrals. Let Δ be a causal and bounded operator on $\mathbf{L}_{2e}^m[0,\infty)$. Then Δ can be described by combinations of bounded and Hermitian valued multipliers with Popov multipliers.

DEFINITION 1
Let $\Pi_1 = \Pi_1^* \in \mathbf{RL}_\infty^{2m\times 2m}$ and let Π_2 be the Popov multiplier

$$\Pi_2(j\omega) = \begin{bmatrix} 0 & -j\omega\Lambda \\ j\omega\Lambda & 0 \end{bmatrix},$$

where $\Lambda = \Lambda^T \in \mathbf{R}^{m\times m}$. We say that Δ satisfies the IQC defined by $\Pi = \Pi_1 + \Pi_2$ if there exists a positive constant γ such that

$$\left\langle \begin{bmatrix} y \\ v \end{bmatrix}, \Pi_1 \begin{bmatrix} y \\ v \end{bmatrix} \right\rangle + \left\langle \begin{bmatrix} \dot{y} \\ v \end{bmatrix}, \begin{bmatrix} 0 & \Lambda \\ \Lambda & 0 \end{bmatrix} \begin{bmatrix} \dot{y} \\ v \end{bmatrix} \right\rangle =$$
$$\int_{-\infty}^{\infty} \begin{bmatrix} \widehat{y}(j\omega) \\ \widehat{v}(j\omega) \end{bmatrix}^* \Pi_1(j\omega) \begin{bmatrix} \widehat{y}(j\omega) \\ \widehat{v}(j\omega) \end{bmatrix} + \int_0^{\infty} 2v^T\Lambda\dot{y}dt \geq -\gamma|y_0|^2,$$

for all $y, \dot{y}, v \in \mathbf{L}_2^m[0,\infty)$ such that $v = \Delta(y)$. Here $\widehat{y}$ and $\widehat{v}$ denotes the Fourier transforms of y and v respectively. We used the notation $y_0 = Cx_0$. □

REMARK 1
Differentiability is only necessary for the components of the vector y that corresponds to nonzero rows of Λ. The condition $\dot{y} \in \mathbf{L}_2^m[0,\infty)$ is thus not necessary if $\Pi_2 = 0$.

The next example, which is adopted from [2] and [14], illustrates the use of Definition 1 for describing nonlinearities with Popov multipliers.

EXAMPLE 1
Let $\varphi : \mathbf{R} \to \mathbf{R}$ be a measurable function satisfying $\varphi(0) = 0$ and a sector condition $\alpha x^2 \leq \varphi(x)x \leq \beta x^2$, where $-\infty < \alpha < \beta < \infty$. Then φ satisfies the IQC defined by the Popov multiplier

$$\Pi(j\omega) = \begin{bmatrix} 0 & -j\omega\lambda \\ j\omega\lambda & 0 \end{bmatrix},$$

where $\lambda \in \mathbf{R}$. This follows since

$$2\lambda \lim_{T\to\infty} \int_0^T \varphi(y)\dot{y}dt = 2\lambda \lim_{T\to\infty} \int_{y_0}^{y(T)} \varphi(\sigma)d\sigma = -2\lambda \int_0^{y_0} \varphi(\sigma)d\sigma \geq -\gamma|y_0|^2,$$

for all $y, \dot{y} \in \mathbf{L}_2^m[0,\infty)$, where we use $\gamma = |\lambda|\max(|\alpha|,|\beta|)$. The third equality follows since $y(T) \to 0$ as $T \to \infty$. □

We will consider stability of the system

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad x(0) = x_0, \\ y &= Cx, \\ u &= \Delta(y) + g, \end{aligned} \tag{3}$$

where it is assumed that $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$, $C \in \mathbf{R}^{m\times n}$ and that A is Hurwitz. It is further assumed that $g \in \mathbf{L}_{2e}^m[0,\infty)$ and that Δ is a bounded and causal operator on $\mathbf{L}_{2e}^m[0,\infty)$.

We define well-posedness and stability for the system in (3) as follows.

Definition 2

The system in (3) is *well-posed* if for any initial condition x_0 and for any input $g \in \mathbf{L}_{2e}^m[0,\infty)$ there exists a unique solution $(x,v) \in \mathbf{L}_{2e}^n[0,\infty) \times \mathbf{L}_{2e}^m[0,\infty)$, where x is an absolutely continuous function. Furthermore, the map from g to (x,v) should be causal. □

Definition 3

The system in (3) is *stable* if it is well-posed and if there exist constants $c > 0$ and $\rho > 0$ such that

$$\int_0^T (|y|^2 + |u|^2)dt \leq \rho|x_0|^2 + c\int_0^T |g|^2dt,$$

for all $T > 0$ and for arbitrary $x_0 \in \mathbf{R}^n$ and $g \in \mathbf{L}_{2e}^m[0,\infty)$. □

We can now use the following stability result.

Theorem 1

Assume that there exists a continuous parametrization Δ_τ, $\tau \in [0,1]$ of the operator Δ such that

1. $\Delta = \Delta_1$ and Δ_τ is a causal and bounded operator on $\mathbf{L}_{2e}^m[0,\infty)$ for any $\tau \in [0,1]$.
2. For some $\kappa > 0$ we have that

$$||\Delta_{\tau_2}(y) - \Delta_{\tau_1}(y)|| \leq \kappa|\tau_2 - \tau_1| \cdot ||y||,$$

   for all $y \in \mathbf{L}_2[0,\infty)$.
3. For any $\tau \in [0,1]$, Δ_τ satisfies the IQC defined by the multiplier $\Pi = \Pi_1 + \Pi_2$, where $\Pi_1 = \Pi_1^* \in \mathbf{RL}_\infty^{2m\times 2m}$ and Π_2 is a Popov multiplier.

Furthermore, assume that the parametrized system

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad x(0) = x_0, \\ y &= Cx, \\ u &= \Delta_\tau(y) + g, \end{aligned} \tag{4}$$

satisfies the following properties

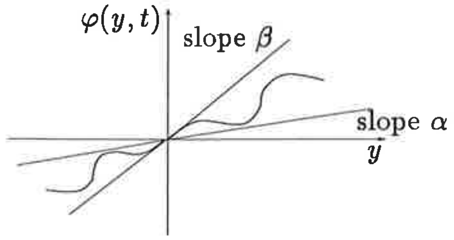

Figure 2 Nonlinearity in sector$[\alpha, \beta]$.

1. It is stable when $\tau = 0$.
2. It is well-posed for any $\tau \in [0,1]$.

Then the system in (4) is stable for all $\tau \in [0,1]$ if there exists $\varepsilon > 0$ such that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* (\Pi_1(j\omega) + \Pi_2(j\omega)) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \leq -\varepsilon I, \quad \forall \omega \geq 0,$$

where $G(s) = C(sI - A)^{-1}B$.

Proof: A full proof is given in [5] □

REMARK 2
The parametrization of Δ can often be taken as $\Delta_\tau = \tau\Delta$.

REMARK 3
It follows from the proof that we could allow the operator G to have a direct term, i.e. $G(s) = C(sI - A)^{-1}B + D \in \mathbf{RH}_\infty^{m\times m}$, under the condition that $\Lambda D = 0$, where Λ is the parameter in the Popov multiplier Π_2.

REMARK 4
It is sometimes useful to verify exponential convergence to zero of the state vector x. If for example $g = 0$, then the conditions in Theorem 1 imply exponential convergence if the operator Δ is memoryless and bounded. This follows from a result in [9].

3. IQC Descriptions for Nonlinearities

We will in this section consider multiplier descriptions for nonlinearities. It is assumed that the nonlinearities are regular enough to define operators on $\mathbf{L}_{2e}[0,\infty)$.

3.1 Sector Bounded Nonlinearities

Sector bounded nonlinearities are defined as follows, see Figure 2.

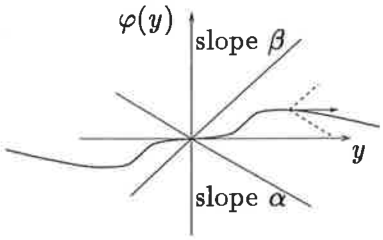

Figure 3 Nonlinearity with slope restricted in the interval $[\alpha, \beta]$.

DEFINITION 4
We define sector$[\alpha, \beta]$, to be the class of nonlinearities $\varphi : \mathbf{R}^2 \to \mathbf{R}$, which satisfy the following conditions

(i) $\varphi(0, t) = 0, \quad \forall t \in \mathbf{R}^+$.
(ii) $\alpha y^2 \le \varphi(y, t) y \le \beta y^2, \quad \forall y \in \mathbf{R},\ t \in \mathbf{R}^+$.

where α and β are constants satisfying $-\infty < \alpha < \beta < \infty$. □

It is easy to verify that a sector bounded nonlinearity satisfies the IQC used in the circle criterion, i.e. any $\varphi \in$ sector$[\alpha, \beta]$ satisfies the IQC defined by

$$\Pi(j\omega) = \begin{bmatrix} \frac{\beta}{\beta-\alpha} & -\frac{1}{\beta-\alpha} \\ -\alpha & 1 \end{bmatrix}^T \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\beta}{\beta-\alpha} & -\frac{1}{\beta-\alpha} \\ -\alpha & 1 \end{bmatrix}.$$

This follows since

$$\begin{bmatrix} y \\ \varphi(y,t) \end{bmatrix}^T \Pi \begin{bmatrix} y \\ \varphi(y,t) \end{bmatrix} = \frac{2}{\beta - \alpha}(\beta y - \varphi(y,t))(\varphi(y,t) - \alpha y) \ge 0,$$

where the last inequality follows from the sector bound.

3.2 Static Nonlinearities

Static (time-invariant) nonlinearities satisfying the conditions of Example 1 can be described by the Popov multipliers

$$\Pi(j\omega) = \begin{bmatrix} 0 & -j\omega\lambda \\ j\omega\lambda & 0 \end{bmatrix},$$

where λ generally can be taken as an arbitrary real number.

3.3 Slope Restricted Nonlinearities

We will here consider memoryless nonlinearities with restricted slope.

DEFINITION 5
We define slope$[\alpha, \beta]$ to be the class of memoryless nonlinearities $\varphi : \mathbf{R} \to \mathbf{R}$ which have the following properties:

(i) $\varphi(0) = 0$.
(ii) $\alpha(y_1 - y_2)^2 \le (\varphi(y_1) - \varphi(y_2))(y_1 - y_2) \le \beta(y_1 - y_2)^2, \quad \forall y_1, y_2 \in \mathbf{R}$.

where α, β are constants with $-\infty < \alpha < \beta < \infty$. □

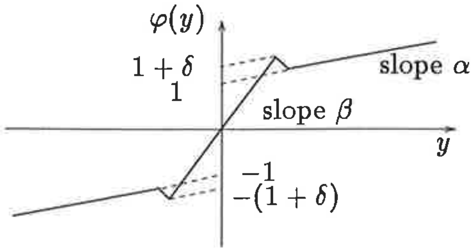

Figure 4 Stiction Nonlinearity.

The above definition is illustrated in Figure 3. At every point, the slope of $\varphi(y)$ lies in the sector $[\alpha, \beta]$.
The slope restricted nonlinearities satisfies the IQCs defined by, see [15],

$$\Pi = \begin{bmatrix} \dfrac{\beta}{\beta-\alpha} & -\dfrac{1}{\beta-\alpha} \\ -\alpha & 1 \end{bmatrix}^T \begin{bmatrix} 0 & 1+H(j\omega) \\ 1+H(j\omega)^* & 0 \end{bmatrix} \begin{bmatrix} \dfrac{\beta}{\beta-\alpha} & -\dfrac{1}{\beta-\alpha} \\ -\alpha & 1 \end{bmatrix}, \tag{5}$$

where H is a strictly proper rational transfer function with corresponding impulse response h satisfying the following constraints

1. The $\mathbf{L}_1$-norm constraint $||h||_1 \leq 1$.
2. $h(t) \geq 0$ for all $t \in \mathbf{R}$. If φ is an odd function then this constraint is no longer needed.

We note that the multipliers in (5) contain the multipliers for sector bounded nonlinearities. This follows if we use $H = 0$. The multiplier description in (5) also holds for multivariable slope restricted nonlinearities, see [1].

3.4 A Stiction Nonlinearity

We define *stiction nonlinearities* to be nonlinear functions that may change slope from β to α in a small region where the slope is arbitrary negative, see figure 4. The formal definition is as follows

DEFINITION 6
We define stiction$[\alpha, \beta]$, to be the class of nonlinearities $\varphi : \mathbf{R} \to \mathbf{R}$ satisfying the following properties.

(i) φ is odd.

(ii) For positive values of y, the function φ is defined as

$$\varphi(y) = \begin{cases} \beta y, & y \leq (1+\delta)/\beta. \\ \in [1+\alpha y, 1+\delta+\alpha y], & y > (1+\delta)/\beta. \end{cases}$$

where α, β and δ are constants with $0 < \alpha < \beta < \infty$, $\beta > 0$, and $\delta > 0$.
□

The nonlinearities in stiction$[\alpha, \beta]$ can be described by the same multipliers as in (5) with the exception that the $\mathbf{L}_1$-norm of the impulse response corresponding to H satisfies $||h||_1 \leq 1/(\delta+1)$. This follows from the same argument

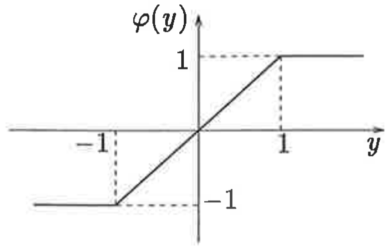

Figure 5 Saturation Nonlinearity.

as in [10], where the case with $\beta = \infty$ is treated. We have

$$\begin{aligned}(\beta y - \varphi(y))(\varphi(y) - \alpha y + h * (\varphi(y) - \alpha y))(t) \\ \geq (\beta y - \varphi(y))(\varphi(y) - \alpha y - \operatorname{sign}(y) \sup_y |\varphi(y) - \alpha y| \cdot \|h\|_1)(t) \\ \geq (\beta y - \varphi(y))(\varphi(y) - \alpha y - \operatorname{sign}(y))(t) \geq 0, \quad \forall t \geq 0,\end{aligned}$$

where the second inequality follows since $|\varphi(y) - \alpha y| \leq 1+\delta$, and $\|h\|_1 \leq 1/(1+\delta)$. The last inequality follows since for positive y, we have $\varphi(y) - \alpha y - 1 \geq 0$, when $\beta y - \varphi(y) > 0$. The reverse inequalities hold for negative y. Integration with respect to time from zero to infinity gives the desired IQC.

It is interesting to note that if we, for the case when β is infinity, let φ be multi-valued at zero with $\varphi(0) \in [-1-\delta, 1+\delta]$, then we obtain a model for friction with stiction, see [10].

3.5 Combining Multipliers

Common nonlinearities, such as for example saturations and dead-zones, belong to all classes above. They can therefore be described by the multiplier $\Pi = \sum \alpha_i \Pi_i$, where $\alpha_i \geq 0$ and where Π_i is a representative from any of the classes above.

EXAMPLE 2

The saturation nonlinearity in Figure 5 satisfies the IQCs defined by the multipliers

$$\Pi = \begin{bmatrix} 0 & \lambda - j\omega q + H(j\omega) \\ \lambda + j\omega q + H(j\omega)^* & -2(\lambda + \operatorname{Re} H(j\omega)) \end{bmatrix},$$

where $\lambda \geq 0$, $q \in \mathbf{R}$, and where $H \in \mathbf{RL}_\infty$ is strictly proper with impulse response h, satisfying $\|h\|_1 \leq \lambda$.

In general we find an IQC description of an operator Δ in terms of a convex cone Π_Δ of multipliers, which then can be used for the stability analysis as described in the next section.

4. LMI Computations

Assume that we have a description of Δ in terms of a convex cone Π_Δ of multipliers. The stability analysis can then be formulated as the following convex feasibility test.

Stability Test: Find $\Pi \in \Pi_\Delta$ such that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0, \quad \forall \omega \in [0, \infty]. \tag{6}$$

The convex cone Π_Δ is in general infinite dimensional. We need to restrict the search to a finite dimensional subspace in order to have a tractable problem. It is possible to parametrize a finite dimensional convex subset of Π_Δ such that the corresponding restricted stability test can be reformulated as a feasibility problem involving LMIs. We can, for example, use the parametrization

$$\Pi(j\omega) = \Psi(j\omega)^* M(\lambda) \Psi(j\omega),$$

where Ψ in general is a nonproper rational function in order to include the Popov multiplier, and where $M(\lambda)$ is a symmetric matrix that depends linearly on the parameter vector λ. The range of λ needs to be restricted in general. We assume that this can be done with the LMI constraints $\Phi_k^T M(\lambda) \Phi_k \leq 0,\ k = 1, \ldots, K$.

Next introduce realizations $\Psi(s) = C_\psi(sI - A_\psi)^{-1} B_\psi + D_\psi + E_\psi s$ and $G(s) = C(sI - A)^{-1} B + D$ where E_ψ is associated with the Popov part of the multiplier and where it is assumed that A_ψ does not have any eigenvalues on the imaginary axis. If $E_\psi D = 0$, then standard transfer function multiplication gives a realization $\Phi_0(s) = C_\phi(sI - A_\phi)^{-1} B_\phi + D_\phi$ of

$$\Phi_0 = \Psi \begin{bmatrix} G \\ I \end{bmatrix},$$

and the stability test above can, by the positive real lemma, be formulated as the following LMI test

LMI Test: Find λ and $P = P^T$ such that

$$\begin{bmatrix} A_\phi & B_\phi \\ I & 0 \\ C_\phi & D_\phi \end{bmatrix}^T \begin{bmatrix} 0 & P & 0 \\ P & 0 & 0 \\ 0 & 0 & M(\lambda) \end{bmatrix} \begin{bmatrix} A_\phi & B_\phi \\ I & 0 \\ C_\phi & D_\phi \end{bmatrix} < 0,$$
$$\Phi_k^T M(\lambda) \Phi_k \leq 0, \ k = 1, \ldots, K.$$

In our format, multiplier descriptions for diagonally structured operators can easily be obtained by addition and augmentation of multiplier descriptions for the elements in the structure, see [6].

We will next discuss a possible parametrization of the multipliers for slope restricted nonlinearities in the format above. Let $H = H_c + H_{ac} \in \mathbf{RL}_\infty$ have realizations $H_c = C_c(sI - A_c)^{-1} B_c$ and $H_{ac} = C_{ac}(sI - A_{ac})^{-1} B_{ac}$ for the causal and anti-causal part, respectively. Then we have

$$||h||_1 = \int_0^\infty |C_c e^{A_c t} B_c| dt + \int_0^\infty |C_{ac} e^{-A_{ac} t} B_{ac}| dt.$$

Introduce *basis multipliers* $H_i = H_{ic} + H_{iac},\ i = 1, \ldots, N$ on this form and use

$$H = \sum_{i=1}^N (\lambda_i^+ - \lambda_i^-) H_i,$$

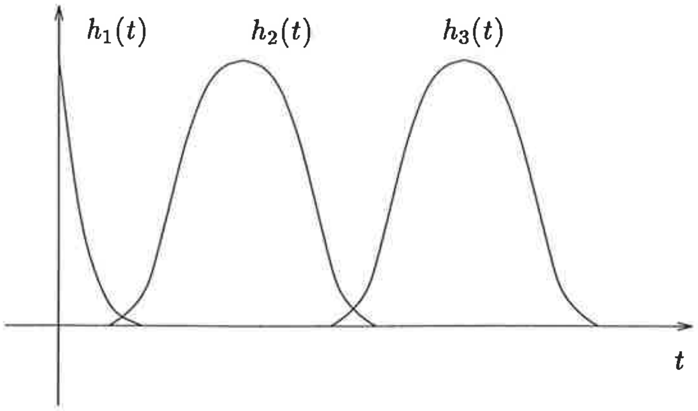

Figure 6 Impulse responses with little overlap.

where $\lambda_i^+, \lambda_i^- \geq 0$, as a finite dimensional parametrization of H in (5). Then the constraint

$$\sum(\lambda_i^+ + \lambda_i^-)||h_i|| \leq 1 \tag{7}$$

ensures that $||h||_1 \leq 1$. It is now easy to obtain a finite dimensional parametrization of the multipliers in (5) in the format suggested above, see [7].

The $\mathbf{L}_1$-norm constraint in (7) is generally conservative. The case when the slope restricted nonlinearity is not odd can be treated without any conservativity with the method developed in [1]. This is a consequence of the requirement that $h(t) \geq 0$, $\forall t \geq 0$. The multiplier for odd slope restricted nonlinearities gives extra flexibility since $h(t)$ need not be positive. In order to use this flexibility, we need to find basis multipliers H_i with impulse responses that overlap as little as possible in order to avoid unnecessary conservativity in (7), see Figure 6. This is generally hard since the impulse responses h_i are combinations of exponential functions and it would require multipliers with high order transfer functions to obtain the desired overlap condition. This affects the speed of the LMI conditions.

The above discussion indicates that it is crucial to choose suitable ***basis multipliers*** for the stability analysis. The next proposition can be used to obtain conditions for unfeasibility of the stability test in (6) for a system with an odd, slope restricted nonlinearity. A different approach to the same problem has been suggested in [8].

PROPOSITION 1—UNFEASIBILITY

Consider the problem: Find a strictly proper $H \in \mathbf{RL}_\infty^{1\times 1}$ with corresponding impulse response satisfying $||h||_1 \leq 1$, such that $\text{Re}\,(G(j\omega)(I + H(j\omega)^*)) < 0$, $\forall \omega \in [0, \infty]$. This feasibility problem has no solution if there exist frequencies $\omega_1, \ldots, \omega_{N-1} \in [0, \infty)$ and $z_1, \ldots, z_N \geq 0$ such that

$$\sum_{k=1}^{N} z_k \text{Re}\; G(j\omega_k) \geq \sup_{t\in\mathbf{R}} \left| \sum_{k=1}^{N-1} z_k \text{Re}\; (G(j\omega_k)e^{j\omega_k t}) \right|, \tag{8}$$

where $\omega_N = \infty$.

Proof: Unfeasibility means that there exist frequencies $\omega_1, \ldots, \omega_{N-1} \in [0, \infty)$ and scalars $z_1, \ldots, z_N \geq 0$ such that

$$\sum_{k=1}^{N} z_k \text{Re } (G(j\omega_k)(I + H(j\omega_k)^*)) \geq 0. \tag{9}$$

We have $H(j\infty) = 0$ and

$$H(j\omega_k) = \int_{-\infty}^{\infty} h(t) e^{-j\omega_k t} dt.$$

A simple argument shows that

$$\sum_{k=1}^{N} z_k \text{Re } G(j\omega_k) \geq \sup_{t\in\mathbf{R}} \left| \sum_{k=1}^{N-1} z_k \text{Re } (G(j\omega_k) e^{j\omega_k t}) \right| \cdot \int_{-\infty}^{\infty} |h(t)| dt,$$

is sufficient for (9) to hold (it is actually necessary as well). This proves the proposition since the last integral is less than one. □

It is clear that the choice of frequencies for the application of Proposition 1 is a very delicate task. In fact, the right hand side of (8) will be $\sum |z_k G(j\omega_k)|$ for most choices of frequencies. In this case the condition (9) of the proposition cannot hold unless the frequencies are chosen such that $G(j\omega_k) \in \mathbf{R}^+$ for all k.

The next example illustrates the use of the proposition.

EXAMPLE 3

Let Δ in the system (3) be an odd slope restricted nonlinearity $\varphi \in \text{slope}[0, k]$, see Definition 5, and let the linear part of the system have the transfer function

$$G(s) = \frac{s^2}{(s^2 + \alpha)(s^2 + \beta) + 10^{-4}(14s^3 + 21s)},$$

where $\alpha = 0.9997$ and $\beta = 9.0039$. This is a system with two very distinct resonances at $\omega \approx 1$ and $\omega \approx 3$. The purpose of the example is to find an upper bound on k such that stability of the system is guaranteed. The multiplier $H(s) = -\frac{6.25}{(s+2.5)^2}$ can be used to prove stability for $k = 0.0048$. If we use the dual with $\omega_1 = 1$ and $\omega_2 = 3$, then the condition in (8) is satisfied if $k = 0.0061$. Hence, the duality gap is reasonable small despite the low order of the multiplier H. Figure 7 shows the Nyquist curves for $G(j\omega) - \frac{1}{k}$ when $k = 0.0061$ and $k = 0.0048$, respectively. Note that we use the convention of using a positive feedback loop. This implies that the multiplier we search for should be such that $(G(j\omega) - \frac{1}{k})(1 + H(j\omega)^*)$ is in the left half plane.

If we use the Popov multiplier combined with a multiplier for the circle criterion, then the stability criterion becomes

$$\text{Re } ((G(j\omega) - \frac{1}{k})(1 - \lambda j\omega)) < 0, \quad \forall \omega.$$

For the case with $k = 0.0048$ we obtain the following necessary condition for this stability test

$$\text{Re } ((G(j) - \frac{1}{k})(1 + j\lambda)) < 0,$$
$$\text{Re } ((G(j3) - \frac{1}{k})(1 + j3\lambda)) < 0,$$

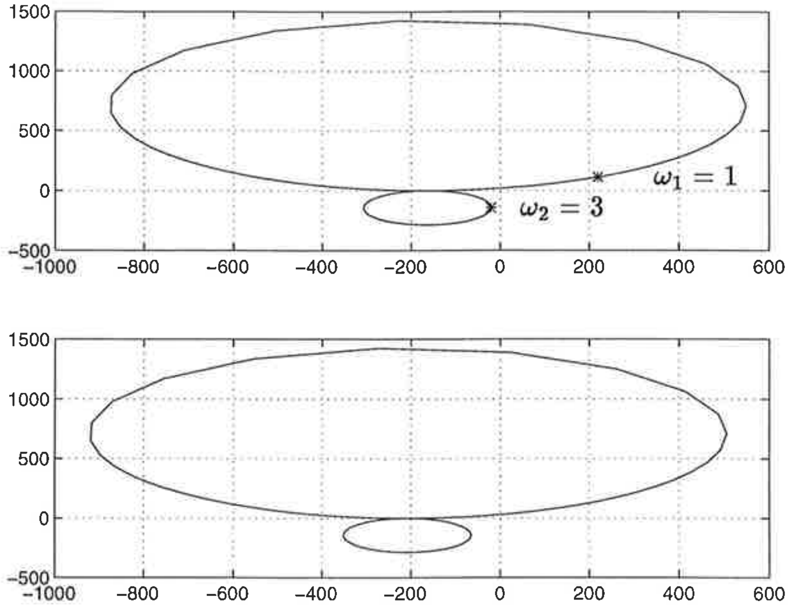

Figure 7 The upper plot shows the Nyquist diagram of $G(j\omega) - 1/k$, when $k = 0.0061$. There is no solution to the feasibility test in Proposition 1 for this value of k. The lower plot shows the Nyquist diagram of $G(j\omega) - 1/k$, when $k = 0.0048$. The multiplier $H = 6.25/(6 + 2.5)^2$ proves stability for this value of k.

which has no solution since $G(j) - \frac{1}{k} = 175.5 + j111.9$ and $G(j3) - \frac{1}{k} = -65.5 - j144.2$. □

5. Conclusions

The paper illustrates how Popov's multiplier for stability analysis of systems with nonlinearities can be combined conveniently with multipliers for slope restricted nonlinearities in an IQC framework. The stability analysis can be formulated as a feasibility test for linear matrix inequalities.

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