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The Servo Problem

Wittenmark, Björn

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DESIGN OF DIGITAL CONTROLLERS - THE SERVO PROBLEM

BJÖRN WITTENMARK

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LUND INSTITUTE OF TECHNOLOGY
DEPARTMENT OF AUTOMATIC CONTROL
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Björn Wittenmark
Department of Automatic Control
Lund Institute of Technology
Lund, Sweden

ABSTRACT

In this paper the regulator and servo problems are discussed for discrete time systems. The servo problem is defined as to follow reference signals. This includes how to introduce models and integrators into the closed loop system. The purpose is to consider the case when the system is known to be able to derive structures for controllers that later can be used for instance in adaptive controllers to introduce integrators and reference values.

DESIGN OF DIGITAL CONTROLLERS - THE SERVO PROBLEM

1. INTRODUCTION

Linear quadratic (LQ) control and pole placement through state feedback are primarily used to solve the regulator problem. I.e. how to eliminate disturbances due to initial values. Additional control signals are then introduced to make it possible for the system to follow reference values. I.e. to solve the servo problem. This includes how to introduce reference models and integrators into the system. These problems are only to some extent discussed in text books on linear quadratic control, see for instance Anderson and Moore (1971) and Kwakernaak and Sivan (1972). Model matching for multivariable continuous time systems is discussed in Hikita (1981). The servo problem for multivariable systems is also discussed in the PhD thesis by Bengtsson (1973). In Anderson and Moore (1971 p 247) the following definitions are given:

> "If the plants outputs are to follow a class of desired trajectories, e.g. all polynomials up to a certain order, the problem is referred to as a servo (servomechanism) problem; if the desired trajectory is a particular prescribed function of time, the problem is called a tracking problem. When the outputs of the plant are to follow the response of another plant (or model) to either a specific command input or class of command inputs, the problem is referred to as the model-following problem."

These problems are here discussed for single-input-single-output discrete time systems. We will not make any hard distinctions between the different cases defined above, but will simple refer to them as the servo problem. The purpose is to give suitable structures for the controller which makes it possible to solve the servo problem when the system is known. The goals of the paper are to investigate the problems for known systems and to discuss how the solutions can handle input and output disturbances and errors in the model of the process. The problems are formulated in state space form. The input-output relations are, however, of primary interest since one goal is to derive structures that can be used in adaptive controllers to solve the servo problem. In the literature several ways to introduce integrators in self-tuning regulators are discussed, see for instance Åström (1980), Modén (1981), Allidina and Hughes (1982) and Wittenmark and Åström (1983).

The paper is organized in the following way. In Section 2 the problem is defined and different specifications are given. The regulator problem is solved in Section 3. The model-following and tracking problems are solved in Sections 4 and 5. Introduction of integrators is discussed in Section 6. Section 7 gives some examples. Conclusions are given in Section 8 and references in Section 9.

2. PROBLEM FORMULATION

The regulator and servo problems are discussed for discrete time systems. The same approaches can also be used for continuous time systems.

The process

Let the process be described by

$$\begin{aligned} x(k+1) &= \Phi x(k) + \Gamma(u(k) + v(k)) \\ y(k) &= C\,x(k) + e(k) \end{aligned} \tag{2.1}$$

where the matrices Φ, Γ and C are known. The signals $v(k)$ and $e(k)$ are input and output disturbances that are not measurable. The system is of n:th order. It is assumed that the system is reachable and observable. Further only single-input-single-output systems are treated. The input-output description of the system is defined by the pulse transfer operator

$$\begin{aligned} y(k) &= C(qI - \Phi)^{-1}\Gamma(u(k) + v(k)) + e(k) \\ &= \frac{B(q)}{A(q)}\,(u(k) + v(k)) + e(k) \\ &= H(q)\,\,(u(k) + v(k)) + e(k) \end{aligned} \tag{2.2}$$

where q is the forward shift operator and $\deg A = n$ and $d = \deg A - \deg B$. The polynomials A and B are defined as

$$A(q) = q^n + a_1 q^{n-1} + \dots + a_n \tag{2.3}$$

$$B(q) = b_0 q^{n-d} + b_1 q^{n-d-1} + \dots + b_{n-d} \tag{2.4}$$

The pole excess, d, is the same as the time delay of the process.

Specifications

The specifications for the regulator case are:

Regulator case: Influence of initial values should be eliminated following the dynamics specified by the characteristic polynomial $A_r(q)$.

For the servo case we will give two different types of specifications:

Servo case 1: The response of the system from the reference signal, u_c, to the output, y, should be given by

$$y_m(k) = H_m(q)u_c(k) = \frac{B_m(q)}{A_m(q)} u_c(k) \tag{2.5}$$

where the known polynomials B_m and A_m don't have any common factors. It is assumed that

$$\deg A_m = n_m$$

$$d \le \deg A_m - \deg B_m = d_m \tag{2.6}$$

Further A_m is a stable polynomial, i.e. all its zeros are inside the unit circle.

Servo case 2: In this case it is required that the output of the process should be 'close' to the output, y_m, of the model (2.5). The reference signal, u_c, is restricted to be a step.

The problems are specified by the polynomials A_r, B_m, and A_m. Since the system is assumed to be reachable and observable then there are no restrictions on A_r. The assumption (2.6) implies that the model has a time delay that is at least as long as that of the process. Different restrictions on A_m and B_m are discussed in connection with the solutions.

In the terminology given in Anderson and Moore (1971) Servo case 1 corresponds to model-following while Servo case 2 corresponds to the tracking problem.

In some situations it is useful to have the model (2.5) described in terms of a minimal order state space model

$$
\begin{aligned}
x_m(k+1) &= \Phi_m x_m(k) + \Gamma_m u_c(k) \\
y_m(k) &= C_m x_m(k)
\end{aligned}
\tag{2.7}
$$

The purpose of the control is to follow y_m. Depending on the knowledge of the properties of y_m and the process different solutions will be obtained. The solutions will also depend on how the disturbances are taken care of in the problem formulation.

3. THE REGULATOR PROBLEM

The system (2.1) is reachable and observable. If all states are measurable then the regulator problem is solved by the state feedback regulator

$$u(k) = - L\ x(k) + \tilde{u}_c(k) \tag{3.1}$$

In the regulator case $\tilde{u}_c = 0$, but will be used later to solve the servo problem. The feedback vector L is chosen such that

$$\det\ (zI - (\Phi - \Gamma L)) = A_r(z) \tag{3.2}$$

This gives the closed loop system

$$y(k) = C[qI - (\Phi - \Gamma L)]^{-1}\Gamma \tilde{u}_c(k) = \frac{B(q)}{A_r(q)}\ \tilde{u}_c(k) \tag{3.3}$$

The reachability condition implies that the poles of the closed loop system can be placed arbitrarily. The zeros are, however, not changed.

If all the states are not measurable then they can be estimated using an observer since the system (2.1) is observable. Let the dynamics of the observer be A_o. The characteristic polynomial of the closed loop system with an observer is then $A_r A_o$. The observer dynamics is not controllable from $\tilde{u}_c$ which implies that the input-output relation still is given by (3.3).

The regulator problem can also be solved using linear quadratic control. The feedback vector L is then chosen in order to minimize a loss function. The loss function is here restricted to be of the form

$$J(\varrho) = \sum_{k=1}^{\infty} [y(k)^2 + \varrho\ u(k)^2] \tag{3.4}$$

The vector L that minimizes (3.4) is

$$L = (\varrho + \Gamma^T S \Gamma)^{-1} \Gamma^T S \Phi$$

The characteristic polynomial of closed loop system, $A_r(z)$, is given by

$$r A_r(z) A_r(z^{-1}) = \varrho A(z) A(z^{-1}) + B(z) B(z^{-1}) \tag{3.5}$$

where

$$r = \Gamma^T S\ \Gamma + \varrho$$

and S is the solution to the steady state Riccati equation

$$S = \Phi^T S\ \Phi + Q_1 - \Phi^T S \Gamma (\varrho + \Gamma^T S \Gamma)^{-1} \Gamma^T S \Phi \tag{3.6}$$

where

$$Q_1 = C^T C$$

See for instance Aström and Wittenmark (1984), or any standard text book on linear quadratic control.

4. THE MODEL-FOLLOWING PROBLEM

The solution to the regulator problem results in a state feedback controller that shifts the poles of the closed loop system into desired locations. The servo problem is solved by determining a feedforward system from the reference signal u_c to the signal $\tilde{u}_c$ in (3.1). Depending on the specifications there are several solutions to the servo problem. Servo case 1 or the model-following problem is first considered. It is now desired that the closed loop system from u_c to y is given by the model H_m, i.e. perfect model-following is desired. To achieve this it is necessary to have, compare (3.3),

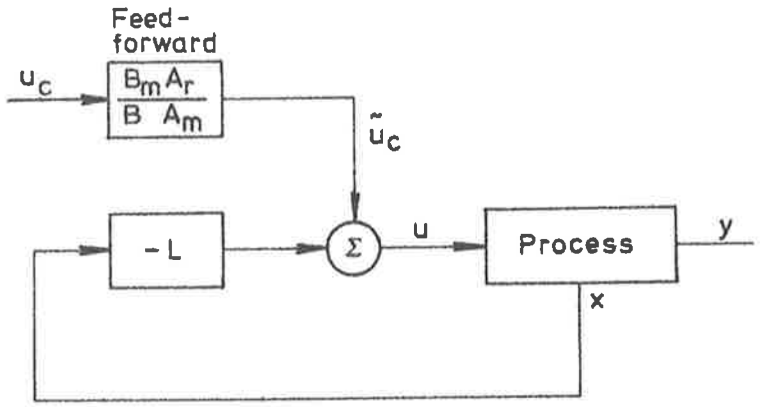

Fig 4.1 Control of the process using the controller (3.1) and (4.1).

$$\tilde{u}_c(k) = \frac{B_m(q)A_r(q)}{B(q)A_m(q)} u_c(k) \tag{4.1}$$

see Fig 4.1. The order of the feedforward system is

$$n + n_m - d$$

if there are no common factors between BA_m and B_mA_r. Notice that the orders of the polynomials A_r and A_m are not necessarily the same. Also notice that (4.1) contains the inverse of the process when it is controlled by (3.1), i.e. A_r/B. There are three properties of the controller (4.1) that are required.

* Causality
* Stability
* Robustness against disturbances and modelling errors

The controller (4.1) is causal if and only if condition (2.6) is fulfilled. This implies that the time delay in the model is at least as long as that of the process.

The feedforward controller is stable if $A_m(q)$ and $B(q)$ have their zeros inside the unit circle. If B has zeros outside the unit circle it is necessary that those also are zeros of B_m. This implies that only stable process zeros are allowed to be cancelled. A thorough discussion of this is given in Åström and Wittenmark (1984). Also there may be common factors in the stable polynomials A_r and A_m, which should be cancelled before (4.1) is implemented.

The feedforward controller (4.1) contains both exact information (the model) and information that may be uncertain (the process). It is desired that the total system is robust against small errors in the process model. Also it should be possible to eliminate the different types of disturbances acting on the system, see (2.1) or (2.2). This will be analyzed in the end of this section.

State-space formulation

The controller (4.1) can be implemented in different ways depending on the process and the model. A couple of different cases will be considered. Assume that a realization of the feedforward system is

$$\begin{aligned} x_f(k+1) &= \Phi_f x_f(k) + \Gamma_f u_c(k) \\ \tilde{u}_c(k) &= -L_f x_f(k) + \ell_c u_c(k) \end{aligned} \qquad (4.2)$$

A general linear controller that uses the states of the process and the feedforward system is

$$u(k) = -L\, x(k) - L_f x_f(k) + \ell_c u_c(k) \qquad (4.3)$$

We will now analyze the properties of the closed loop system. The disturbances are disregarded for the moment. Using (4.3) to control (2.1) gives

$$\begin{bmatrix} x(k+1) \\ x_f(k+1) \end{bmatrix} = \begin{bmatrix} \Phi-\Gamma L & -\Gamma L_f \\ 0 & \Phi_f \end{bmatrix} \begin{bmatrix} x(k) \\ x_f(k) \end{bmatrix} + \begin{bmatrix} \Gamma \ell_c \\ \Gamma_f \end{bmatrix} u_c(k)$$

The input-output relation is now

$$y(k) = [C \quad 0] \begin{bmatrix} qI-(\Phi-\Gamma L) & \Gamma L_f \\ 0 & qI-\Phi_f \end{bmatrix}^{-1} \begin{bmatrix} \Gamma \ell_c \\ \Gamma_f \end{bmatrix} u_c(k)$$

$$= [C \quad 0] \begin{bmatrix} [qI-(\Phi-\Gamma L)]^{-1} & -[qI-(\Phi-\Gamma L)]^{-1}\Gamma L_f[qI-\Phi_f]^{-1} \\ 0 & [qI-\Phi_f]^{-1} \end{bmatrix} \begin{bmatrix} \Gamma \ell_c \\ \Gamma_f \end{bmatrix} u_c(k)$$

We thus get the pulse transfer function

$$y(k) = C[qI-(\Phi-\Gamma L)]^{-1}\Gamma \cdot [\ell_c - L_f(qI-\Phi_f)^{-1}\Gamma_f] \; u_c(k)$$

or

$$y(k) = \frac{B(q)}{A_r(q)} \cdot \frac{B_f(q)}{A_f(q)} u_c(k) \qquad (4.4)$$

The polynomial A_f is assumed to be monic. The first part of the pulse transfer function is the pole shifting due to the solution of the regulator problem. The denominator of the second part is the characteristic polynomial of Φ_f in (4.2). The numerator of the second part has the same order as the denominator. The controller has the structure shown in Fig 4.2.

The polynomials A_r, A_f, and B_f can be arbitrarily chosen through L, L_f, and ℓ_c. Perfect model-following is obtained if

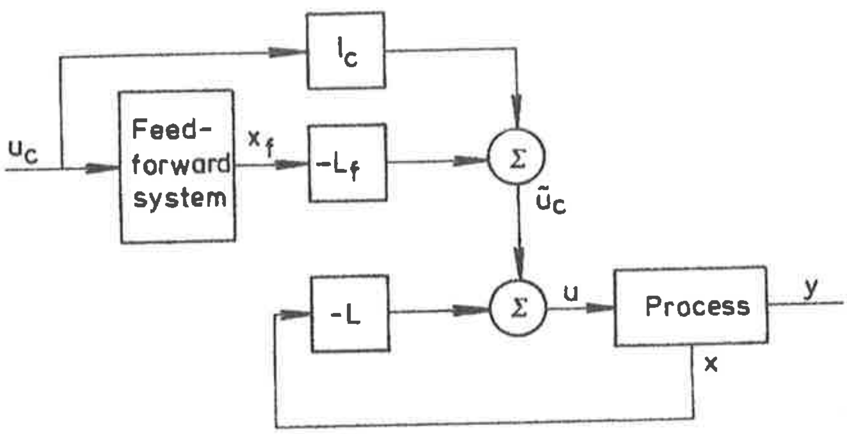

Fig 4.2 The control scheme obtained when the controller (4.3) is used.

$$B_f(q) = A_r(q)B_m(q)/b_0$$
$$A_f(q) = A_m(q)B(q)/b_0$$

The assumption (2.6) implies that B_f/A_f is causal. The order of the controller is then

$$n + n_m - d$$

which is the same as that of (4.1).

Notice that it is possible to rewrite (4.1) as

$$\tilde{u}_c(k) = \frac{B_m(q)A_r(q)}{B(q)A_m(q)} u_c(k)$$

$$= \frac{B_m(q)}{A_m(q)} u_c(k) + \frac{B_m(q)[A_r(q) - B(q)]}{A_m(q)B(q)} u_c(k) \qquad (4.5)$$

The first term is y_m and contains only known parts while the second term also contains parts that are dependent of the possibly uncertain process. If A_m and B don't have any common factors it is easy to show that the feedforward system (4.2) can be implemented as a block diagonal system of order $n + n_m - d$ with two output signals that corresponds to the two terms in (4.5), see Appendix. The importance of this separation will be apparent in Section 6 when integrators are introduced.

In the next case it is assumed that the process (2.1) and the model (2.7) are 'compatible'. Without giving a formal definition this means that:

* The orders of the system and the model are the same and that the states have the same physical interpretation.
* The model dynamics can be obtained from the process through feedback, i.e.

$$\Phi_m = \Phi - \Gamma L$$

It is then meaningful to look at the difference between the states of the process and the model. Further assume that

$$A_r(q) = A_m(q)$$

The system can now be implemented as shown in Fig 4.3. Notice that the

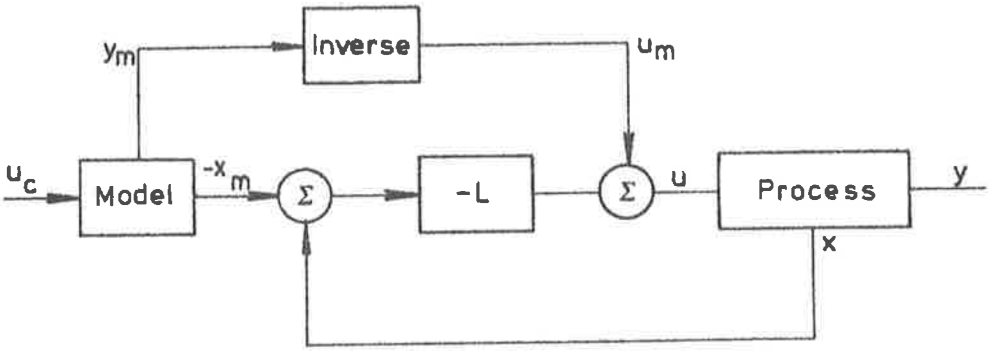

Fig 4.3 The control structure for solving the regulator and servo problems when the states of the process and the model are compatible.

system contains an inverse of the process which may be noncausal. The model together with the inverse can, however, be implemented as a causal system, see above. The advantage with the implementation is that some nonlinearities of the process can be taken care of.

Bengtsson (1973) has proposed a method for model-following for multivariable systems. Consider Fig 4.4. The controller contains four parts

* A model of the form (2.7). The order of the model is arbitrary.
* An inverse of the process
* A process model
* A state feedback regulator

The desired output, y_m, is obtained by applying the signal u_m to the process. The signal y_m is generated by the model and u_m by the inverse of the process. Let the process model be described by

$$x_{pm}(k+1) = (\Phi - KC)\, x_{pm}(k) + \Gamma\, u_m(k) + K\, y_m(k) \qquad (4.6)$$

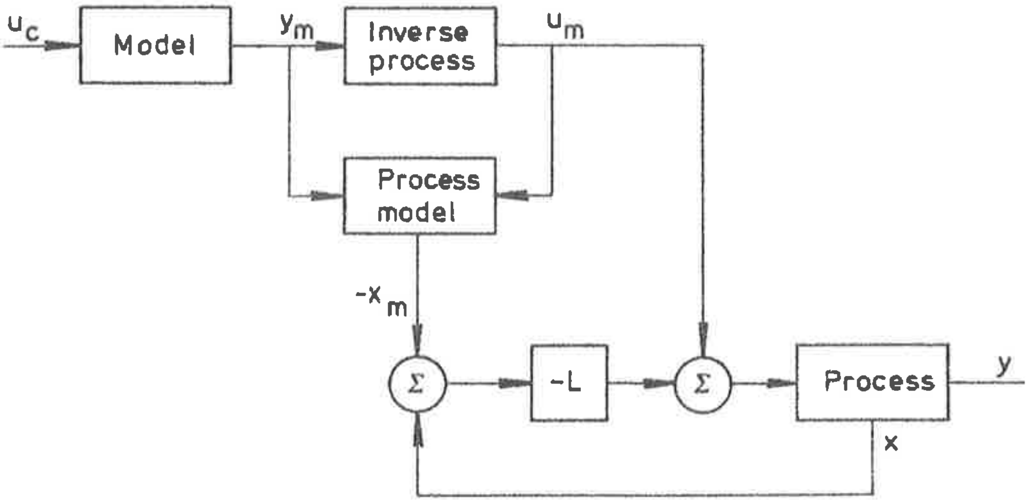

Fig 4.4 Bengtsson's method to implement a model-following controller.

The process model looks like an observer. The reason for the structure is the following: The process model and the inverse process are used to generate reference trajectories for the states of the process. The signal u_m can be regarded as an input to the process which should give the output y_m. The signals u_m and y_m are thus regarded as the desired input and output of the process and are used to generate the states of the process model. These states are then used as reference states for the process. The process model (4.6) can be interpreted as an observer which generates the reference trajectory for the states of the process using the signals u_m and y_m. The vector K can be chosen such that the observer gets arbitrary eigenvalues if the process (2.1) is observable.

Seen from y_m to y we have a unit pulse transfer operator when using the controller in Fig 4.4. The advantage with the method is that some types of nonlinearities can be taken care of. Bengtsson (1973) has also shown that it is possible to generate the reference values of the states directly from the inverse of the process. The method require that B(z) has all zeros inside the unit circle. If this is not the case the inverse has to be approximated. The order of the feedforward controller with the inverse and the process model is

$$2n + n_m - d$$

Input-output formulation

One way to solve the servo problem when the process is given in input-output form is shown in Fig 4.5. The controller is now

$$u(k) = \frac{B_m}{A_m}\left[\frac{A}{B} + \frac{S}{R}\right] u_c(k) - \frac{S}{R}\, y(k) \qquad (4.7)$$

This gives the closed loop system

$$y(k) = \frac{B}{A}\, u(k) = \frac{BAB_m}{BAA_m}\, u_c(k) - \frac{SB}{RA}\left[y(k) - \frac{B_m}{A_m}\, u_c(k)\right]$$

or

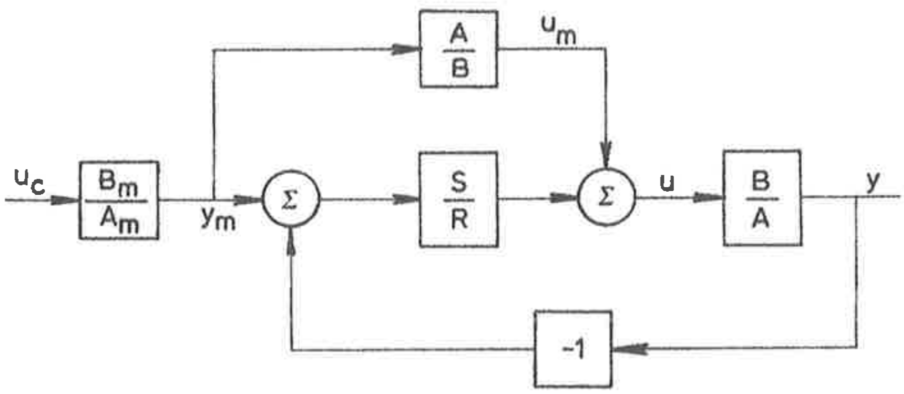

Fig 4.5 Input-output form to achieve model-following.

$$y(k) = \frac{B_m}{A_m} \cdot \frac{B(AR + BS)}{B(AR + BS)} u_c(k) = \frac{B_m}{A_m} u_c(k)$$

The cancelled factors contain the observer dynamics, the regulator dynamics, and the cancelled process zeros, i.e.

$$AR + BS = A_o A_r$$

This polynomial equation has a solution if A and B don't have any common factors and if

$$\deg A_o \geq n - 1$$

$$\deg S < n$$

$$\deg R = \deg A_o$$

Using (4.7) it is possible to make a separation between the regulator and the servo problems. This solution corresponds to the structure given in Fig 4.2, but with an observer included in the loop. The total controller has the order

$$2n + n_m - d - 1$$

The order is n - 1 higher than that of (4.1). This corresponds to the degree of the observer.

If we restrict the problem to the case $A_m = A_r$ then (4.7) can be simplified to

$$R(q)u(k) = -S(q)y(k) + T(q)u_c(k) \tag{4.8}$$

where

$$T = B_m A_o$$

$$AR + BS = BA_m A_o$$

The polynomial equation has a solution if A and B don't have any common factors and if

$$\deg A_o \geq n - n_m + d - 1$$

$$\deg S < n$$

$$\deg R = \deg A_o + n_m - d$$

This solution corresponds to the structure in Fig 4.3, but with an observer included.

The pole placement method base on polynomials is thoroughly discussed in Aström and Wittenmark (1984).

Disturbance rejection

We will now analyze how the input and output disturbances in (2.1) will influence the closed loop system.

The output disturbance will not be eliminated in the state space formulation since y is not used in the controller. A step disturbance on any of the measured states can, however, be eliminated if there is an integrator in the system. Consider the control structure in Fig 4.5. A step in the output disturbance can be eliminated only if there is an integrator in the process or in the controller, i.e. if

$$A(1)R(1) = 0$$

The signal $\tilde{u}_c$ is a feedforward signal. As in all cases where feedforward is used it is important to have exact knowledge of the process or to combine the feedforward controller with a feedback controller. In the state space case, Fig 4.1, modelling errors will cause an error in $\tilde{u}_c$. This will have the same effect as an input disturbance. This implies that both input disturbances and modelling errors will give errors in the closed loop system, mainly due to the lack of integrators in the controller.

In the input output formulation, Fig 4.5, there is a combination of feedforward and feedback. An integrator in the regulator will make it possible to eliminate input step disturbances. This integrator will also make it possible to tolerate errors in the inverse of the process model. The control system will then integrate the error between the desired output, y_m, and the true output. Notice that y_m is generated without any knowledge about the process, compare (4.5). The structure of the controller used in Fig 4.5 is thus robust against both disturbances and modelling errors.

5. THE TRACKING PROBLEM

When Servo case 2 or the tracking problem is considered it is not required to have perfect model-following. It is instead desired that the outputs of the process and the model are close. Let the closeness of y and y_m be measured by the performance index

$$J(\varrho) = \sum_{k=1}^{\infty} [(y(k) - y_m(k))^2 + \varrho\ u(k)^2] \tag{5.1}$$

where y_m is the output of (2.7) when u_c is a step. Introduce a new state which generates the step. The model can now be described by

$$x'_m(k+1) = \begin{bmatrix} \Phi_m & \Gamma_m \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_m \\ u_c \end{bmatrix} = \Phi'_m\ x'_m(k)$$

$$y_m(k) = [C_m \quad 0]\ x'_m(k) = C'_m\ x'_m(k)$$

The optimal controller can now be determined by converting the problem to a standard linear quadratic problem. Introduce

$$z = \begin{bmatrix} x \\ x'_m \end{bmatrix}$$

This gives the new system

$$z(k+1) = \begin{bmatrix} \Phi & 0 \\ 0 & \Phi'_m \end{bmatrix} z(k) + \begin{bmatrix} \Gamma \\ 0 \end{bmatrix} u(k)$$

$$= \tilde{\Phi}\ z(k) + \tilde{\Gamma}\ u(k) \tag{5.2}$$

and the performance index is transformed to

$$J = \sum_{k=1}^{\infty} [z(k)^T \tilde{Q}_1 z(k) + \varrho\ u(k)^2]$$

where

$$\tilde{Q}_1 = \begin{bmatrix} C^T C & -C^T C'_m \\ -C'^T_m C & C'^T_m C'_m \end{bmatrix}$$

The steady state Riccati equation is now

$$\tilde{S} = \tilde{\Phi}^T\tilde{S}\tilde{\Phi} + \tilde{Q}_1 - \tilde{\Phi}^T\tilde{S}\tilde{\Gamma}(\varrho + \tilde{\Gamma}^T\tilde{S}\tilde{\Gamma})^{-1}\tilde{\Gamma}^T\tilde{S}\tilde{\Phi}$$

The loss function will be finite only if the unstable eigenvalues of Φ'_m also are eigenvalues of Φ, see Anderson and Moore (1971). For instance if u_c is a step then Φ'_m has one eigenvalue in 1. The process must then contain an integrator if the loss function should be finite. This is natural since u otherwise will not be zero in steady state. To get a finite loss function the penalty on the control signal could be changed to $\varrho(u - u_m)^2$, where u_m is the input level in steady state which gives y_m.

To analyze the result partition $\tilde{S}$ as

$$\tilde{S} = \begin{bmatrix} S & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix}$$

then we get

$$S = \Phi^T S\Phi + C^T C - \Phi^T S\Gamma(\varrho + \Gamma^T S\Gamma)^{-1}\Gamma^T S\Phi$$

$$S_{12} = \Phi^T S_{12}\Phi'_m - C^T C'_m - L^T\Gamma^T S_{12}\Phi'_m$$

$$S_{22} = \Phi'^T_m S_{22}\Phi'_m + C'^T_m C'_m - \Phi'^T_m S_{12}^T\Gamma(\varrho + \Gamma^T S\Gamma)^{-1}\Gamma^T S_{12}\Phi'_m$$

where

$$L = (\varrho + \Gamma^T S\Gamma)^{-1}\Gamma^T S\Phi$$

The optimal feedback vector is now

$$\tilde{L} = (\varrho + \Gamma S\Gamma)^{-1}[\Gamma^T S\Phi \quad \Gamma^T S_{12}\Phi'_m]$$

and the control law is

$$u(k) = -\tilde{L}\ z(k) = -L\ x(k) - L'_m x'_m(k) \tag{5.3}$$

where

$$L'_m = (\varrho + \Gamma^T S\Gamma)^{-1}\Gamma^T S_{12}\Phi'_m$$

Notice that the same L is obtained as when only the regulator problem is solved, see Section 3. Also notice that the control law is independent of S_{22} and that S_{12} is obtained from

$$(\Phi - \Gamma L)^T S_{12} \Phi'_m = C^T C'_m + S_{12}$$

This linear matrix equation has a unique solution if and only if

$$\lambda_i(\Phi - \Gamma L)\lambda_j(\Phi'_m) \neq 1 \qquad \forall\ i,\ j$$

where $\lambda_i(\Phi)$ denotes the eigenvalues of the matrix Φ.

The Liapunov equation for S_{22} has no unique solution if Φ_m' has any eigenvalue equal to one.

The closed loop system

We will now analyze the closed loop system when (5.3) is used on the system (5.2). Partition L_m' as

$$L'_m = [L_m \ \ -\ell_c]$$

The structure of the controller is thus the same as (4.3) and the closed loop system is

$$z(k+1) = \begin{bmatrix} \Phi & 0 & 0 \\ 0 & \Phi_m & \Gamma_m \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x(k) \\ x_m(k) \\ u_c(k) \end{bmatrix} + \begin{bmatrix} \Gamma \\ 0 \\ 0 \end{bmatrix} (-Lx(k) - L_m x_m(k) + \ell_c u_c(k))$$

$$y(k) = [\ C \quad 0 \quad 0]\ z(k)$$

Computations analogous to those leading to (4.4) gives now the input output relation

$$y(k) = \frac{B(q)}{A_r(q)} \cdot \frac{\tilde{A}(q)}{A_m(q)} u_c(k)$$

The poles of the closed loop system are thus given by the roots of A_r and A_m, i.e. the closed loop system is of order

$n + n_m$

What can be said about $\tilde{A}(q)$? Calculation of a couple of examples gives the following indications

* If B(q) has all zeros inside the unit circle and if $\varrho = 0$ then $A_r = q^d B(q)$ and $\tilde{A}(q)$ will have d_m roots in the origin and the rest at the roots of B_m. This is true even if B_m has zeros outside the unit circle.
* If $\varrho \to \infty$ then $\tilde{A}(q) \to A_m$.
* There does not seem to be any simple relation that characterizes $\tilde{A}$, see Example 7.3 in Section 7.

The controller has in this case the same structure as the state space controllers in Section 4. The closed loop system will then also have the same properties with respect to disturbances and modelling errors. The only way to take care of the disturbances is to give a model for them and to include that into the total system. The resulting optimal controller will then contain a feedback from the disturbance. If the disturbance is not measurable it has to be estimated. This will be discussed in the next section.

6. INTRODUCTION OF INTEGRATORS

In the previous section it was indicated that the loss function (5.1) with $\varrho \neq 0$ is finite only if the process (2.1) contains an integrator. If the process does not have an integrator one must be introduced and it is necessary to change the loss function to get a finite loss. Also from classical control design we know that integrators will make it possible to eliminate steady state errors. The integrator will also make the closed loop system less sensitive to modelling errors. There are several ways to introduce integrators. We will only discuss how to solve the model-following problem. The tracking problem is solved in an analogous way.

Differentiating the input

One way to get an integrator into the system is to introduce a new control signal by taking the difference of the control signal, i.e. to let

$$\nabla u(k) = u(k) - u(k-1)$$

be the control signal. The old control signal u(k-1) will then become a new state of the system. The augmented control system is

$$z(k+1) = \begin{bmatrix} \Phi & \Gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \Phi_f \end{bmatrix} \begin{bmatrix} x(k) \\ u(k-1) \\ x_f(k) \end{bmatrix} + \begin{bmatrix} \Gamma \\ 1 \\ 0 \end{bmatrix} \nabla u(k)$$

A general linear controller has the form

$$\nabla u(k) = -Lx(k) - \ell_u u(k-1) - L_f x_f(k) + \ell_c u_c(k) \qquad (6.1)$$

The structure of the controller is given in Fig 6.1.

If $\ell_u \neq 0$ then the integrator will be eliminated and there may still be steady state errors if there are unmodelled disturbances acting on the system. If $\ell_u = 0$ then is the choice of the poles of the closed loop system limited. This implies that this way of introducing an integrator is inflexible.

Integrating the output

Another method to get an integrator is to integrate the difference between the desired output and the output of the process i.e. to introduce the new state

$$x_{n+1}(k+1) = x_{n+1}(k) + y_m(k) - y(k)$$

From (4.5) and Appendix it is seen that y_m can be generated on the form

$$y_m(k) = C_f x_f(k)$$

for a suitable C_f. The augmented system now becomes

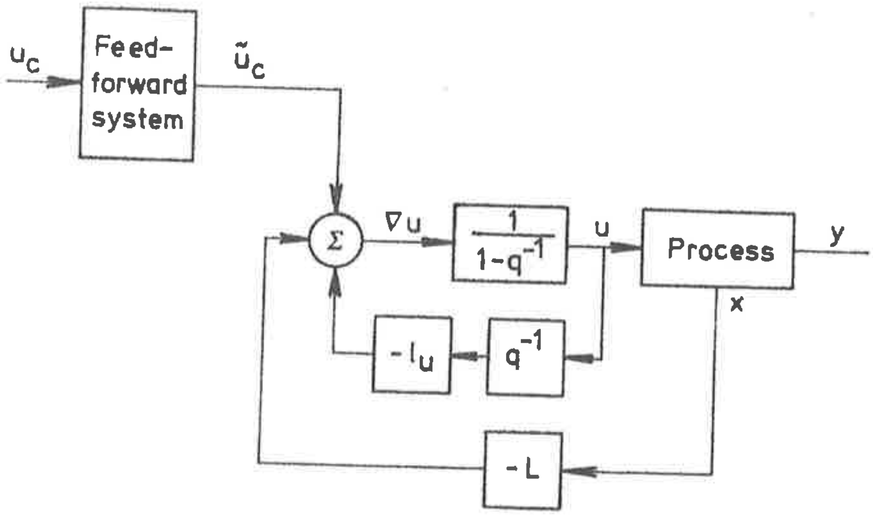

Fig 6.1 The controller structure (6.1) when an integrator is introduced by differentiating the input signal.

$$z(k+1) = \begin{bmatrix} \Phi & 0 & 0 \\ -C & 1 & C_f \\ 0 & 0 & \Phi_f \end{bmatrix} \begin{bmatrix} x(k) \\ x_{n+1}(k) \\ x_f(k) \end{bmatrix} + \begin{bmatrix} \Gamma \\ 0 \\ 0 \end{bmatrix} u(k)$$

This leads to a controller of the form, see Fig 6.2,

$$u(k) = -Lx(k) - \ell_{n+1}x_{n+1}(k) - L_f x_f(k) + \ell_c u_c(k) \qquad (6.2)$$

Assume that

$$\tilde{u}_c = -L_f x_f + \ell_c u_c$$

is given by (4.1). Using (6.2) gives the closed loop system

$$y(k) = \frac{B_m[(q-1)A_r - \ell_{n+1}B]}{A_m[(q-1)A_r - \ell_{n+1}B]} \cdot \frac{BA_m}{BA_m} u_c(k) = \frac{B_m}{A_m} u_c(k)$$

where

$$A_r(q) = \det[qI - (\Phi - \Gamma L)]$$

The closed loop system will behave as the model. The cancelled factor $(q-1)A_r - \ell_{n+1}B$ can be given arbitrary roots. In Section 4 it was shown that y_m

can be generated from B_m/A_m only. This implies that the closed loop system will be robust against modelling errors. Further can step disturbances in both the input and the output be eliminated since the error $y_m - y$ is integrated. This way to introduce an integrator can thus be recommended.

Estimating the disturbance

A third way to eliminate the influence of disturbances is to estimate the disturbance and compensate for it. Consider the case when e = 0 and when v is a unknown level in (2.1). Further assume that all the states can be measured. The system can now be augmented with a new state

$$v(k+1) = v(k)$$

which should be estimated. Let the augmented system be

$$z(k+1) = \begin{bmatrix} \Phi & \Gamma \\ 0 & 1 \end{bmatrix} z(k) + \begin{bmatrix} \Gamma \\ 0 \end{bmatrix} u(k) = \Phi' z(k) + \Gamma' u(k)$$

$$y_z(k) = [I \quad 0]\, z(k) = C' z(k)$$

where

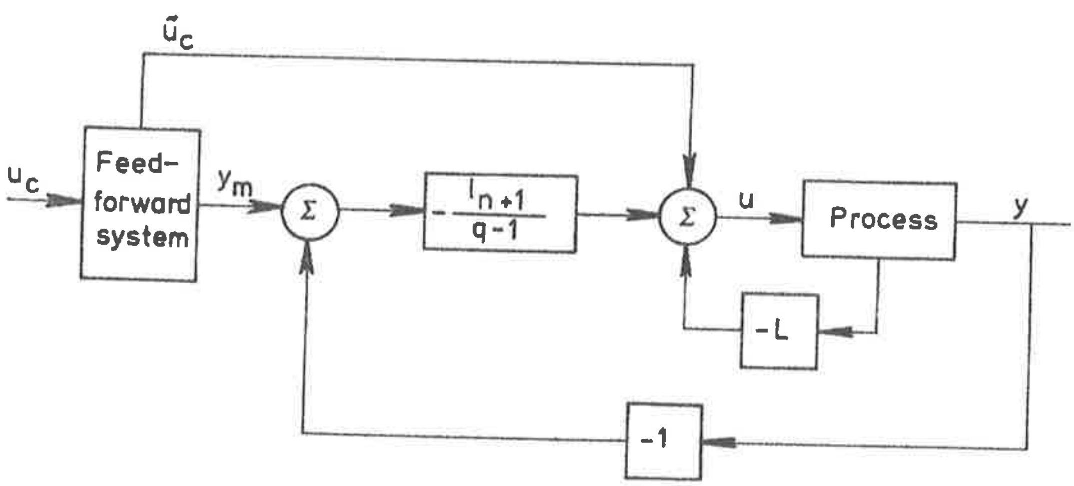

Fig 6.2 The controller structure when the integrator is introduced by integrating the error of the output of the process, (6.2).

$$z(k) = \begin{bmatrix} x(k) \\ v(k) \end{bmatrix}$$

An observer that gives the estimates of z(k) given data up to and including time k is

$$\begin{aligned} \hat{z}(k) &= \Phi'\hat{z}(k-1) + \Gamma'u(k-1) \\ &\quad + K[y_z(k) - C'(\Phi'\hat{z}(k-1) + \Gamma'u(k-1))] \\ &= [I - KC'][\Phi'\hat{z}(k-1) + \Gamma'u(k-1)] + Ky_z(k) \end{aligned}$$

Since all the states of the process (2.1) are measured it is possible to use a first order Luenberger observer to estimate the only unknown state, v. This is obtained by choosing K such that

$$C'K = I$$

or

$$K = \begin{bmatrix} I \\ K_v \end{bmatrix}$$

The observer now becomes

$$\begin{aligned} \hat{x}(k) &= x(k) \\ \hat{v}(k) &= (1 - K_v\Gamma)\hat{v}(k-1) + K_v[x(k) - \Phi x(k-1) - \Gamma u(k-1)] \end{aligned} \qquad (6.3)$$

The error $\tilde{v} = v - \hat{v}$ is described by

$$\tilde{v}(k) = (1 - K_v\Gamma)\tilde{v}(k-1)$$

The measured states are estimated exactly. The gain K_v is a row vector of order n. By choosing K_v it is possible to decide how fast the estimator will be. For instance

$$K_v\Gamma = 1$$

gives a dead beat observer.

The estimated level $\hat{v}$ can now be used to compensate for v. The state feedback controller is now

$$z(k) = \begin{bmatrix} x(k) \\ v(k) \end{bmatrix}$$

An observer that gives the estimates of z(k) given data up to and including time k is

$$\begin{aligned} \hat{z}(k) &= \Phi'\hat{z}(k-1) + \Gamma'u(k-1) \\ &\quad + K[y_z(k) - C'(\Phi'\hat{z}(k-1) + \Gamma'u(k-1))] \\ &= [I - KC'][\Phi'\hat{z}(k-1) + \Gamma'u(k-1)] + Ky_z(k) \end{aligned}$$

Since all the states of the process (2.1) are measured it is possible to use a first order Luenberger observer to estimate the only unknown state, v. This is obtained by choosing K such that

$$C'K = I$$

or

$$K = \begin{bmatrix} I \\ K_v \end{bmatrix}$$

The observer now becomes

$$\hat{x}(k) = x(k)$$

$$\hat{v}(k) = (1 - K_v\Gamma)\hat{v}(k-1) + K_v[x(k) - \Phi x(k-1) - \Gamma u(k-1)] \tag{6.3}$$

The error $\tilde{v} = v - \hat{v}$ is described by

$$\tilde{v}(k) = (1 - K_v\Gamma)\tilde{v}(k-1)$$

The measured states are estimated exactly. The gain K_v is a row vector of order n. By choosing K_v it is possible to decide how fast the estimator will be. For instance

$$K_v\Gamma = 1$$

gives a dead beat observer.

The estimated level $\hat{v}$ can now be used to compensate for v. The state feedback controller is now

$$u(k) = -Lx(k) - \hat{v}(k) - L_f x_f(k) + \ell_c u_c(k)$$

If we instead of an input disturbance have an output disturbance, e, it is obvious that this disturbance can be estimated as

$$\hat{e}(k) = y(k) - Cx(k)$$

if all the states still can be measured.

7. EXAMPLES

The methods described in the previous sections will now be demonstrated in some examples.

Example 7.1 - State space formulation

Consider the second order system

$$x(k+1) = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.8 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} [u(k) + v(k)]$$

$$y(k) = [1 \quad 1] x(k)$$

The pulse transfer operator is

$$H(q) = \frac{2q-1}{(q-0.2)(q-0.8)}$$

Let the desired model be

$$H_m(q) = \frac{0.35q}{q^2 - q + 0.35}$$

and let

$$A_r(q) = q^2$$

The regulator problem is solved by choosing

$$L = [-1 \quad 16]/15$$

The feedforward system, which solves the model-following problem is

$$\tilde{u}_c(k) = \frac{0.175\, q^3}{(q-0.5)(q^2-q+0.35)}$$

One state space representation of the feedforward system is

$$x_f(k+1) = \begin{bmatrix} 1 & -0.5 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} x_f(k) + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u_c(k)$$

$$\tilde{u}_c(k) = -L_f x_f(k) + \ell_c u_c(k)$$

$$= [0.25 \quad 0.525 \quad 1.25] x_f(k) + 0.175 u_c(k) \qquad (7.1)$$

Fig 7.1 shows the output and the control signal when the controller (7.1) is used. The initial value of the first state of the process is 1 and the reference value is changed from 0 to 1 at t = 30 and a disturbance of magnitude -0.25 is added to the input at t = 60. The controller will not give the correct steady state value if the steady state gain of the process is changed. Also a step disturbance at the input as well as at the output will give an error, since

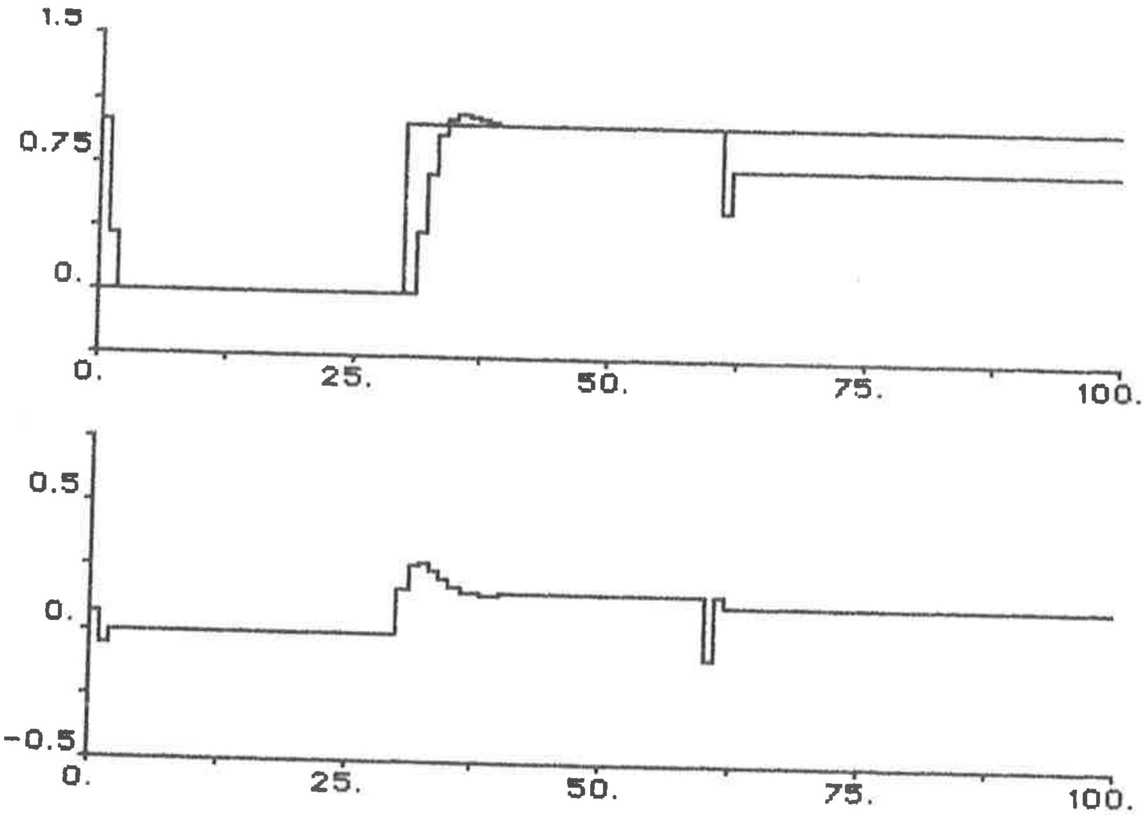

Fig 7.1 Output, reference value, and control signal when (7.1) is used to control the process in Example 7.1.

neither the process nor the controller contain an integrator.

□

Example 7.2 - Motor, input-output formulation

A model of a continuous time motor is in this example controlled by a digital controller. The continuous time transfer function is

$$G(s) = \frac{1}{s(s + 1)}$$

Sampling the system with the sampling interval $h = 0.5$ gives the pulse transfer operator

$$H(q) = \frac{0.11(q + 0.85)}{(q - 1)(q - 0.61)}$$

The sampled open loop system has a zero on the negative real axis. The control signal will introduce a ringing in the system if this zero is cancelled. To avoid this it is assumed that the model has the same zero. It is assumed that the model has poles that corresponds to damping $\zeta = 0.7$ and a natural frequency $\omega = 1$ rad/s. The model is then

$$H_m(q) = \frac{B_m(q)}{A_m(q)} = K \frac{q + 0.85}{q^2 + p_1 q + p_2}$$

where

$$p_1 = -2e^{-\zeta\omega h}\cos(\sqrt{1 - \zeta^2}\ \omega h) = -1.32$$

$$p_2 = e^{-2\zeta\omega h} = 0.50$$

$$K = (1 + p_1 + p_2)/(1 + 0.85) = 0.10$$

Further we assume that the desired regulator response has a damping $\zeta = 1$ and a natural frequency $\omega = 2$. This gives

$$A_r(q) = q^2 - 0.736q + 0.135$$

Using the regulator structure in Fig 4.5 gives the regulator

$$u(k) = \frac{B_m(1)A(q)}{B(1)A_m(q)} u_c(k) + \frac{S(q)}{R(q)} [y_m(k) - y(k)] \qquad (7.2)$$

Assume that the observer polynomial is

$$A_o(q) = q$$

and determine R and S from

$$AR + BS = A_o A_r$$

where deg R = deg S = 1. This gives

$$R(q) = q + 0.381$$

$$S(q) = 4.595q - 2.564$$

Fig 7.2 shows the output and the control signal when (7.2) is used. The initial values of the output and of the output velocity are both 1. The reference value is zero until t = 10 when it is changed to 1. An input disturbance, -0.5, is applied at t = 20. The figure shows that the servo and regulator designs can be separated. The unmodelled input disturbance will not be eliminated since there is no integrator in the regulator. An integrator can be introduced by solving the polynomial equation above, but with

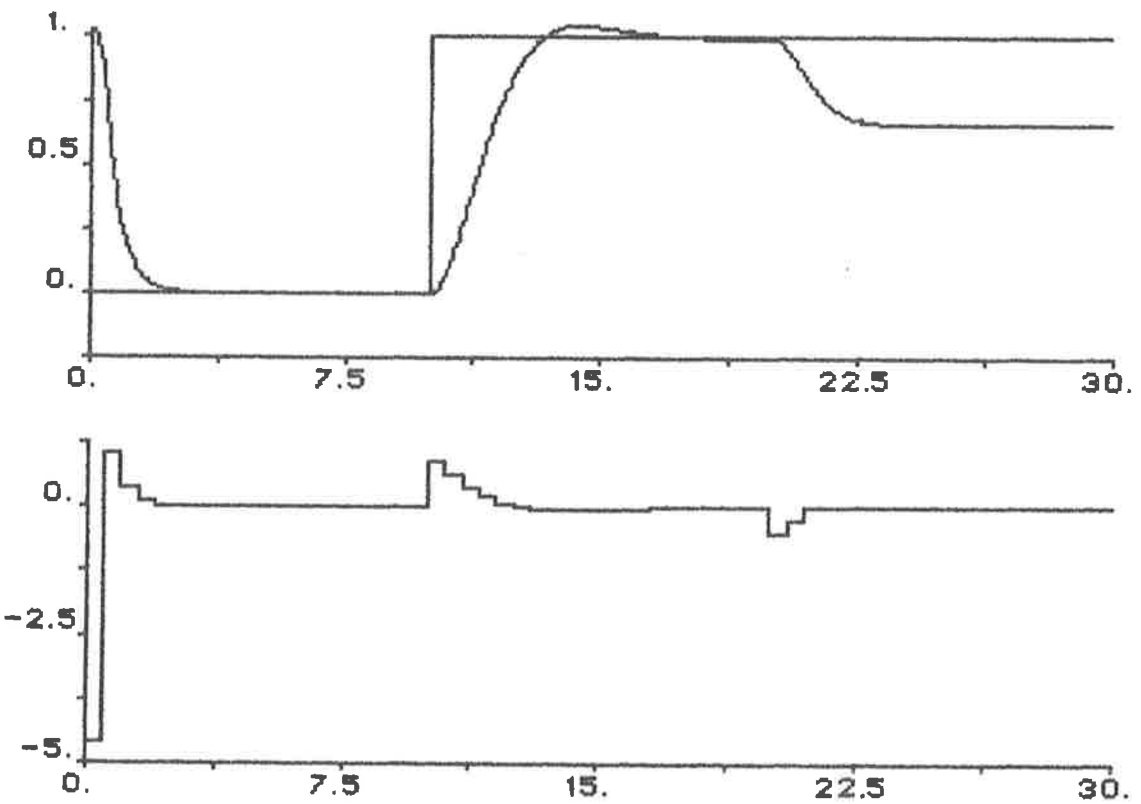

Fig 7.2 Output, reference value, and control signal when the controller (7.2) is used to control the motor process in Example 7.2.

$$A_o(q) = (q - 0.3)^2$$

$$R(q) = (q-1)(q+r)$$

and deg S = 2. The solution is

$$R(q) = (q-1)(q+0.46)$$

$$S(q) = 7.66q^2 - 9.87q + 3.20 \tag{7.3}$$

The output and the reference signal when (7.3) is used are shown in Fig 7.3. The input step disturbance is now effectively eliminated.

□

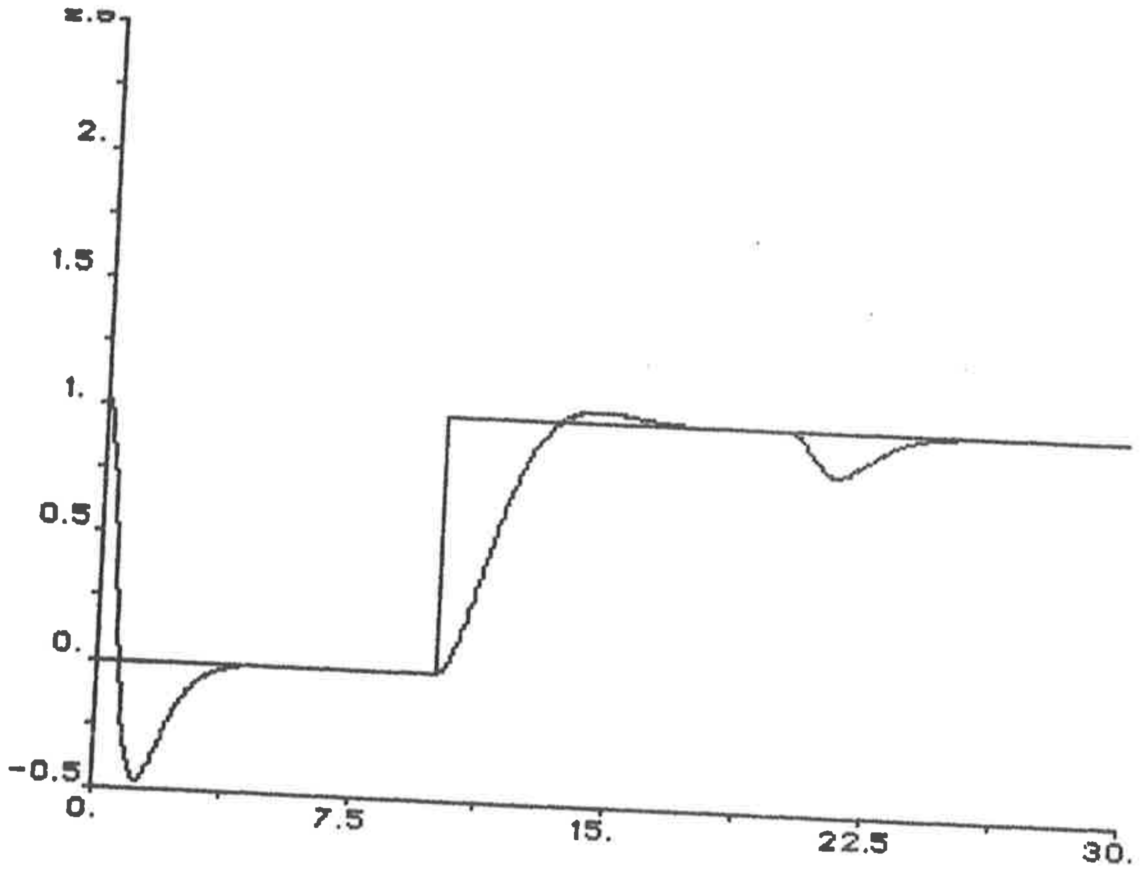

Fig 7.3 Output and reference value when the controller (7.3) is used to control the motor process in Example 7.2.

Example 7.3 - Integrator, servo case 2

Consider an integrator

$$x(k+1) = x(k) + u(k)$$
$$y(k) = x(k)$$

or

$$H(q) = \frac{1}{q - 1}$$

Let the desired model be described by

$$x_m(k+1) = 0.75x_m(k) + 0.25u_c(k)$$
$$y_m(k) = x_m(k)$$

or

$$H_m(q) = \frac{0.25}{q - 0.75}$$

Assume that the loss function is given by (5.1) and that Servo case 2 should be solved.

The stationary Riccati equation is

$$s^2 - s = \varrho$$

which has the solution

$$s = 0.5 + \sqrt{\varrho + 0.25}$$

This gives

$$\ell = \frac{s}{\varrho + s}$$

$$\varphi_c = \Phi - \Gamma L = \frac{\varrho}{\varrho + s}$$

$$A_r(q) = q - \varphi_c$$

We can now compute the matrix S_{12}

$$S_{12} = [s_1 \quad s_2]$$

where

$$s_1 = \frac{1}{\varphi_c \varphi_m - 1} \qquad \text{and} \qquad s_2 = \frac{\varphi_c \gamma_m}{1 - \varphi_c} s_1$$

This gives

$$\ell_m = \frac{\varphi_m s_1}{\varrho + s} \qquad \text{and} \qquad \ell_c = - \frac{\gamma_m s_1 + s_2}{\varrho + s}$$

The controller (5.3) is thus

$$u(k) = -\ell x(k) - \ell_m x_m(k) + \ell_c u_c(k)$$

The polynomial $\tilde{A}$ is

$$\tilde{A}(q) = \ell_c q - (\ell_c \varphi_m + \ell_m \gamma_m) = \ell_c (q - \tilde{a})$$

Fig 7.4 Shows how ℓ_c, φ_c and $\tilde{a}$ vary for different values of ϱ. For $\varrho = 1$ we get

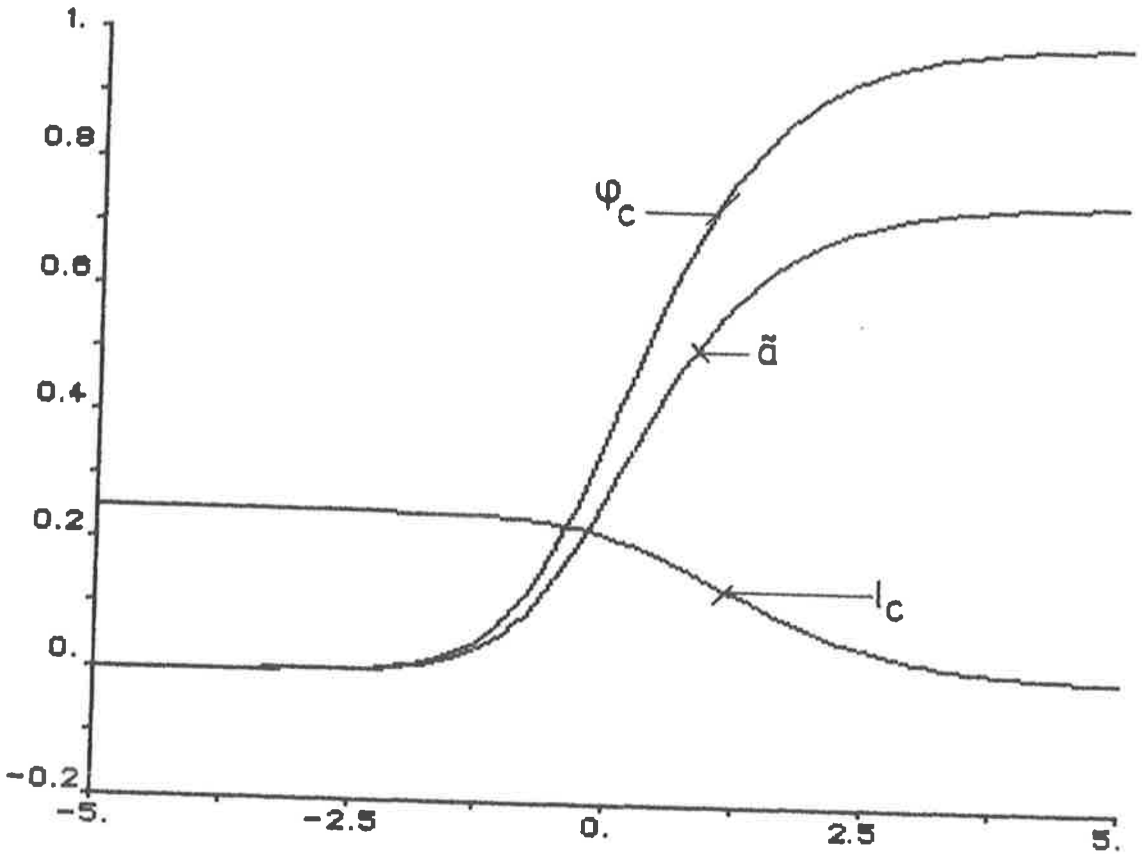

Fig 7.4 ℓ_c, φ_c and $\tilde{a}$ as a function of ϱ for the system in Example 7.3.

$\ell_c = 0.22$

$\varphi_c = 0.38$

$\tilde{a} = 0.29$

The controller is now

$$(q-0.75)u(k) = -(0.618q-0.464)y(k)+(0.217q-0.062)u_c(k) \quad (7.4)$$

Fig 7.5 shows the output and the control signal when the controller (7.4) is used. The initial value of x is 1. The reference signal is zero until t = 30 when it is changed to 1. An input disturbance, -0.25, is added at t = 60.

□

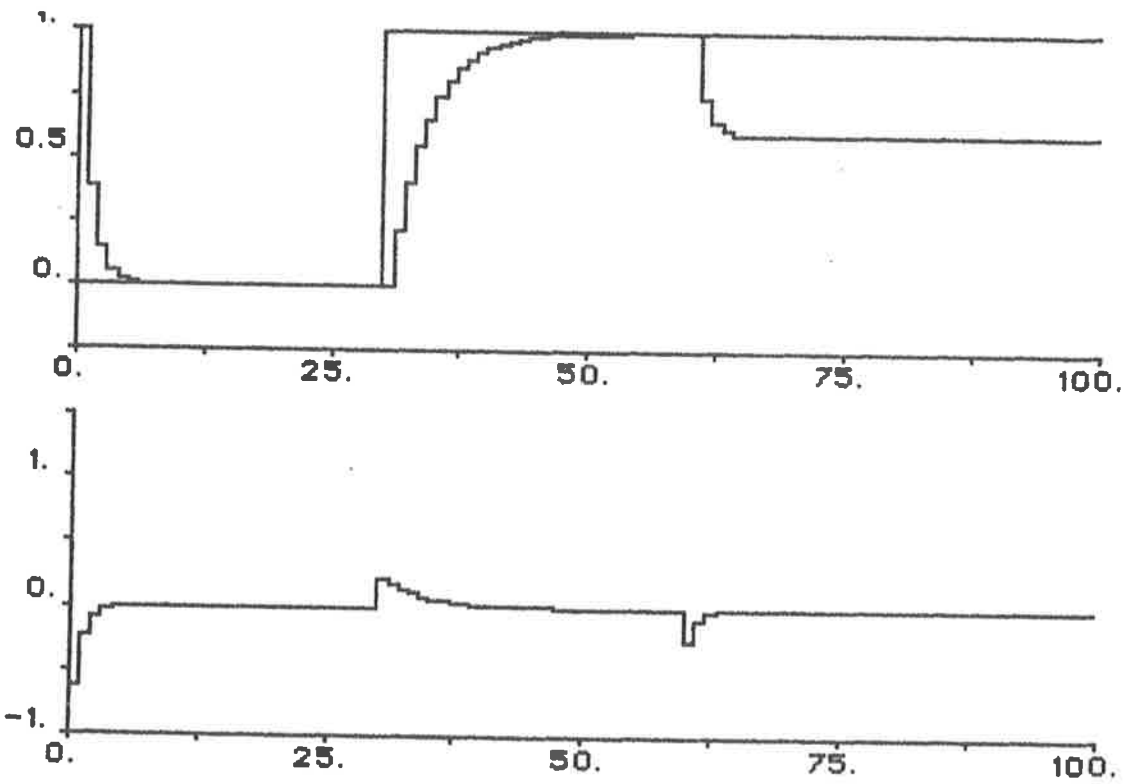

Fig 7.5 Output, reference value, and control signal when the controller (7.4) is used to control the process in Example 7.3.

Example 7.4 - Single time constant, introduction of integrator

In this example we will introduce an integrator in the controller when the process is

$$x(k+1) = \varphi\, x(k) + \gamma\, u(k)$$
$$y(k) = x(k)$$

or

$$H(q) = \frac{\gamma}{q - \varphi}$$

where $\varphi = 0.9$ and $\gamma = 1$.

The first method in Section 6 with differentiating the control signal and using (6.1) gives the closed loop system

$$y(k) = \frac{\gamma_m}{(q-\varphi_m)} \cdot \frac{(q-\varphi+\gamma\ell)[q-1+\ell_u]+\gamma\ell(1-\ell_u)}{(q-\varphi+\gamma\ell)[q-1+\ell_u]+\gamma\ell(1-\ell_u)}\, u_c(k)$$

The input output model is the desired one. The parameters ℓ and ℓ_u are used to choose the cancelled poles. Placing these in the origin gives

$$\ell = \varphi/\gamma$$

$$\ell_u = 1$$

$$\tilde{u}_c = \frac{\gamma_m q}{\gamma(q-\varphi_m)}\, u_c$$

Fig 7.6 shows the behavior of the closed loop system. The proposed method can not eliminate the input disturbance unless $\ell_u = 0$. This choice will, however, severely limit where the poles may be placed.

The second method in Section 6 gives a better result. Using (6.2) gives the closed loop system

$$y(k) = \frac{\gamma_m}{(q-\varphi_m)} \cdot \frac{(q-\varphi+\gamma\ell)(q-1)-\gamma\ell_{n+1}}{(q-\varphi+\gamma\ell)(q-1)-\gamma\ell_{n+1}}\, u_c(k)$$

Placing the cancelled poles in the origin gives

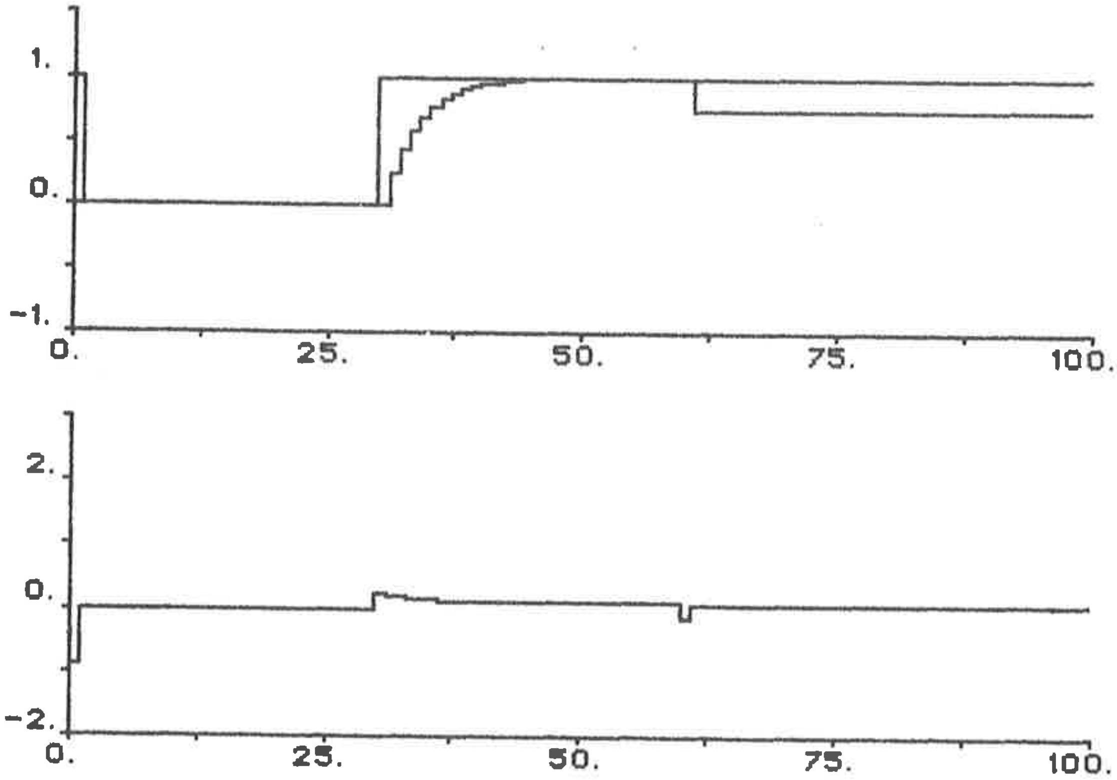

Fig 7.6 Output, reference value, and control signal when the controller (6.1) is used to control the process in Example 7.4.

$$\ell = (1+\varphi)/\gamma = 1.9$$

$$\ell_{n+1} = -1/\gamma = -1$$

$$\tilde{u}_c = \frac{0.25(q+1)}{q-0.75} u_c$$

Fig 7.7 shows the output and the control signal when (6.2) is used. The controller can now eliminate the output disturbance.

Using the third method in Section 6 gives the controller

$$u(k) = -\ell x(k) - \hat{v}(k) + \tilde{u}_c(k) \qquad (7.5)$$

where

$$\hat{v}(k) = (1-k_v\gamma)\hat{v}(k-1) + k_v[x(k) - \varphi x(k-1) - \gamma u(k-1)]$$

Dead beat response for the regulator case is obtained for

$$\ell = \varphi/\gamma$$

This gives

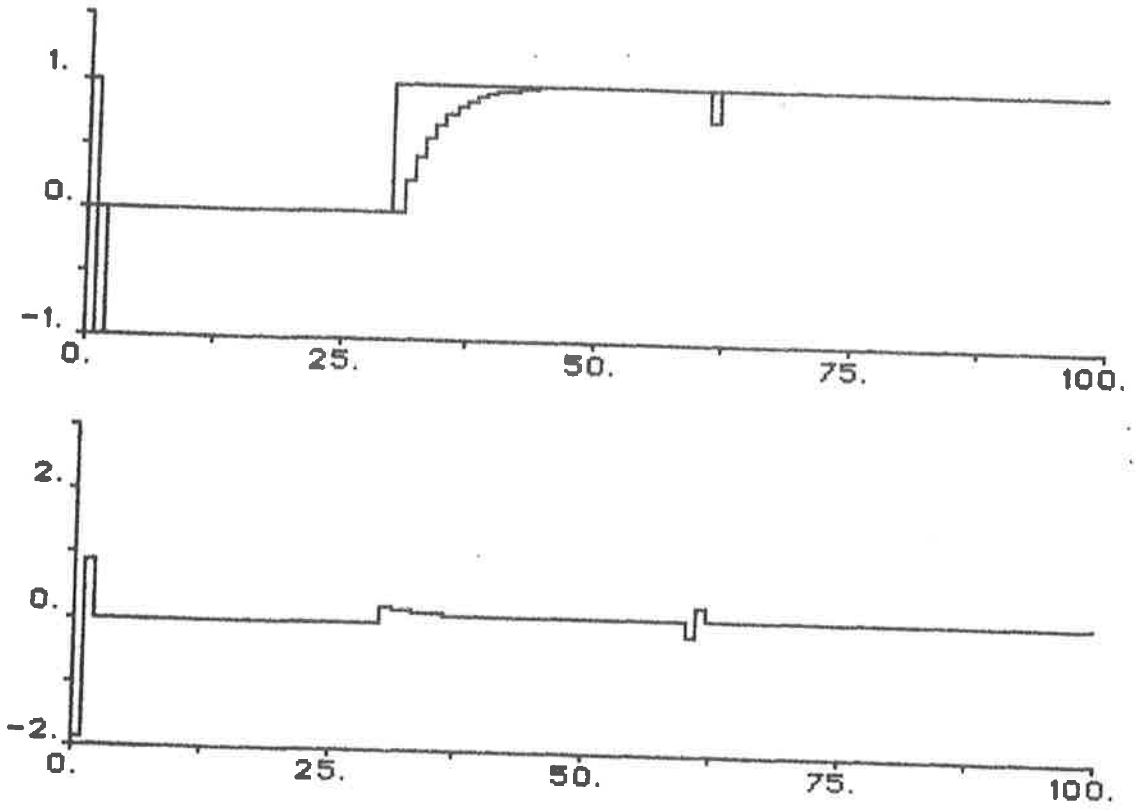

Fig 7.7 Output, reference value, and control signal when the controller (6.2) is used to control the process in Example 7.4.

$$\tilde{u}_c = \frac{\gamma_m q}{\gamma(q-\varphi_m)} u_c$$

Fig 7.8 shows the output, the reference signal and the output when (7.5) is used with $k_v = 1/\gamma$.

□

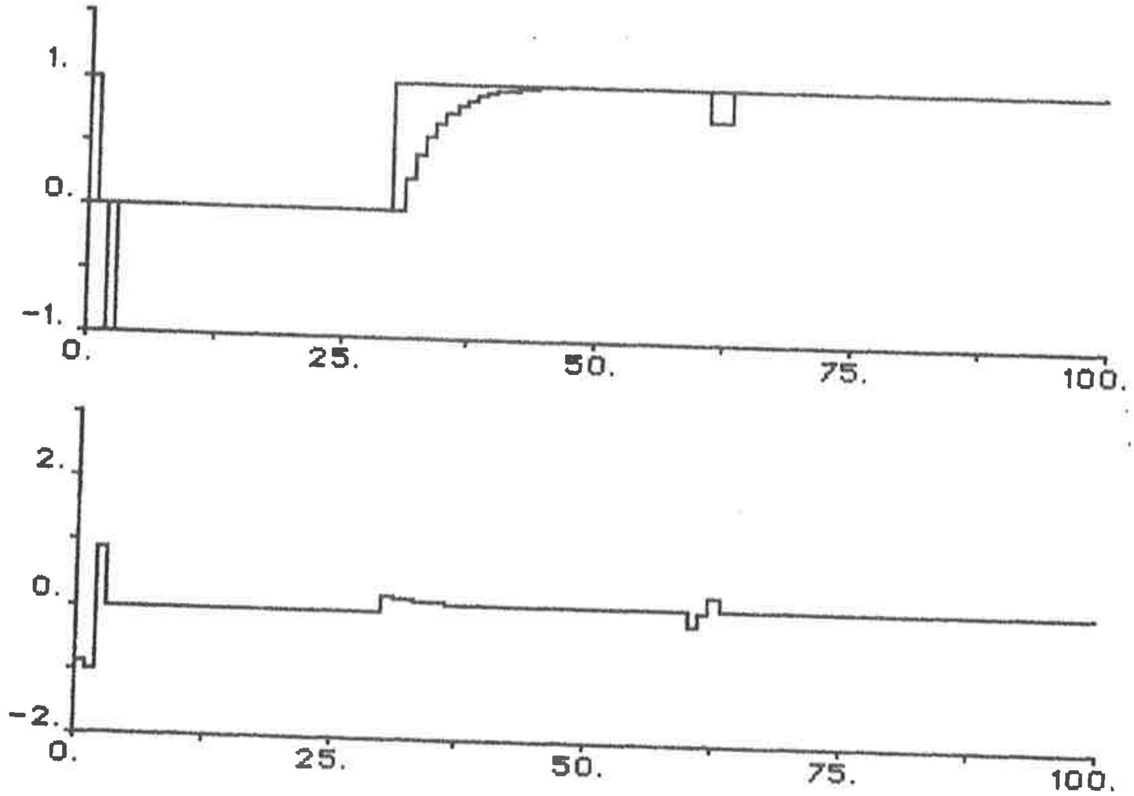

Fig 7.8 Output, reference value, and control signal when the controller (7.5) is used to control the process in Example 7.4.

8. CONCLUSIONS

These notes have discussed the regulator and servo problems for discrete time systems. The purpose has been to derive structures for input output controllers that later can be used in adaptive controllers.

It should be noticed that the servo problem in is solved using feedforward. The feedforward controller is, however, implemented such that uncertainties in the process model and unmodelled disturbances can be taken care of. This is done by introducing integrators into the system.

Three ways of introducing integrators have been discussed, but only two of them solve the problem with unmodelled disturbances and uncertain process models. The conclusion is thus that it is recommended to use the regulator structure given in Fig 6.1 or to estimate the disturbance if the problem is formulated in state space form. If only the input output form is used can the controller be implemented as in Fig 4.5. The integrator is then introduced in the controller polynomial R.

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APPENDIX

Consider the feedforward controller (4.5)

$$\tilde{u}_c(k) = \frac{B_m(q)A_r(q)}{B(q)A_m(q)} u_c(k)$$

$$= \frac{B_m(q)}{A_m(q)} u_c(k) + \frac{B_m(q)[A_r(q) - B(q)]}{A_m(q)B(q)} u_c(k)$$

$$y_m(k) + u_1(k)$$

The first term is equal to y_m. It will now be shown that the controller can be realized on the form (4.2) such that y_m can be generated based only on the exactly known model H_m.

Consider the block diagonal realization

$$x_f(k+1) = \begin{bmatrix} \Phi_m & 0 \\ 0 & \Phi_b \end{bmatrix} x_f(k) + \begin{bmatrix} \Gamma_m \\ \Gamma_b \end{bmatrix} u_c(k) \qquad (A.1)$$

where Φ_m and Γ_m are obtained from (2.7) and where

$$\det(qI - \Phi_b) = B(q)/b_0$$

Without loss of generality we may assume that Φ_m and Φ_b are on controllable canonical form. This implies that

$$\Gamma_m^T = [1 \; 0 \; \ldots \; 0]$$

and

$$\Gamma_b^T = [1 \; 0 \; \ldots \; 0]$$

Further the first rows of Φ_m and Φ_b are the coefficients with reversed signs of the characteristic polynomials of Φ_m and Φ_b respectively. This implies that the model output now can be written as

$$y_m(k) = [0 \; \ldots \; 0 \; b_{m0} \ldots \; b_{mn_m-d_m} \; 0 \; \ldots 0] x_f(k)$$

i.e. a linear combination of the states that are known exactly.

The signal $\tilde{u}_c$ can also be generated from the realization (A.1). Rewrite $\tilde{u}_c$ as

$$\tilde{u}_c = \frac{B_m A_r}{BA_m} u_c = \frac{b_{m0}}{b_0} u_c + \frac{D}{A_m B/b_0} u_c$$

where b_{m0} and b_0 are the leading coefficients in B_m and B respectively. Further

$$\deg D < \deg B + \deg A_m$$

Now if B and A_m don't have any common factors then we way write

$$\frac{D}{A_m B/b_0} = \frac{D_1}{A_m} + \frac{D_2}{B/b_0}$$

where D_1 and D_2 are given from the Diophantine equation

$$A_m D_2 + D_1 B/b_0 = D$$

This equation has a solution if and only if A_m and B don't have any common factors, see Åström and Wittenmark (1984). The signal $\tilde{u}_c$ can now be genarated as

$$\tilde{u}_c(k) = [d_{11} \dots d_{1\ n-1}\ d_{21} \dots d_{2\ n_m - d_m}] x_f(k) + \frac{b_{m0}}{b_0} u_c(k)$$

If Φ_m and Φ_b are not on controllable form then the system (A.1) can be transformed into the desired form with the transformation matrix

$$T = \begin{bmatrix} T_m & 0 \\ 0 & T_b \end{bmatrix}$$

where T_m and T_b are such that

$$T_m \Phi_m T_m^{-1}$$

and

$$T_b \Phi_b T_b^{-1}$$

are on controllable canonical from.