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Rapport 3094

Dimensional Analysis:

A Magic Art in Fire Research?

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Lund, 1997

Preface

This discussion of dimensional analysis was stimulated by my giving lectures at Lund University. As is widely recognised, dimensional analysis can be regarded as a formal, powerful tool, but there are aspects seen by some as a magic art. Monographs and text books which discuss dimensional analysis tend to deal only with the formal aspects, but here these receive less attention than some of those aspects which, while still having a formal aspect, are too often seen at best as art, at worst as guesswork.

The examples taken for discussion and comment are all from the fire literature. There are no new results but there are perhaps new ways of deriving of some topics that are discussed in the literature but deserving further comments.

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1 Introduction

The relevance and application of dimensional analysis to fire problems has been discussed in general terms by Hottel [1] and by Williams [2]. They identified many dimensionless ratios but the main problem is deciding which ones to omit: in any particular application they usually make up a majority! Kanury [3] following Spalding [4] discussed a number of particular problems, and recently Quintiere [5] has extended the discussion of the application of dimensional analysis to fire problems.

The purpose of this paper is to take this exploration further, to discuss certain examples critically and to comment on the question of the analysis of experimental data in the context of dimensional analysis. However, we begin with a general discussion, emphasising some less well publicised points.

2 Theory of dimensional analysis

Classical treatments of dimensional analysis include "The Method of Dimensions" [6] a 1933 monograph by Professor A W Porter, who described it as "the first treatise (small though it is) upon this subject to be published in Gt. Britain", "Dimensional Analysis and Theory of Models" [7] by Professor H L Langhaar; and "Dimensional Analysis and Scale Factors" [8] by RC Pankhurst who gives over 30 references. We do not deal here with the many formalities discussed by Langhaar, with Buckingham's Theorem nor with any examples other than those taken from studies of fire behaviour. This is a topic which draws heavily on heat transfer and fluid dynamics, the historic developments of which owe much to dimensional analysis.

2.1 Units, scaling and dimensional analysis

One must first distinguish between dimensionless numbers derived from physics and others expressing some arithmetic relationship, e.g. a fraction or a percentage. Although dimensionless, these are not to be confused with ratios such as the Reynolds number although they may appear together - as may an index in a power law formula expressing, e.g. the relationship between heat transfer coefficient and velocity in turbulent flow. Consider

$$N_{Nu} = A \cdot N_{Re}^n \quad (1)$$

Here

N_{Nu} is the Nusselt number a dimensionless heat transfer coefficient " h "

N_{Re} is the Reynolds number, which can be regarded as a dimensionless velocity or as a dimensionless length or scale (see below)

A a dimensionless coefficient

and n an index also dimensionless.

N_{Nu} and N_{Re} have a physical basis, A and n an arithmetic one and they depend on the particular problem being discussed, e.g. a flat plate or a pipe. N_{Nu} is defined by $\frac{hD}{K}$. D is a characteristic dimension (usually the same as in the definition of N_{Re} which is UD/ν).

2.1.1 The combination of dimensionless variables

It is possible to combine dimensionless numbers to maximize the convenience to the user or the analyst.

Here, we could use the quotient $\frac{N_{Nu}}{N_{Re}}$ as the dependent variable. This removes D and

produces a new and useful dimensionless number

$$\frac{N_{Nu}}{N_{Re}} = \frac{h\nu}{KU}$$

or more usually the quotient of this and the ratio of the terms controlling molecular processes, thermal diffusivity i.e. " α " divided by the kinematic viscosity " ν " (Prandtl No) i.e.

$$\frac{N_{Nu}}{N_{Re}} \frac{k}{\nu} = \frac{h}{U\rho c_p} \text{ (a Stanton Number)}$$

Consider the classic conduction problem of a semi-infinite solid at uniform temperature with thermal conductivity K , density ρ and specific heat c_p (all constant) heated on the surface in such a way that the surface is instantaneously raised by a steady value θ_o . We seek the temperature rise θ at a depth x at time t ; a familiar text book problem. The first question to ask is whether reference is made to **all** the necessary physical quantities. This requires a physical (or chemical) judgement. This is no less a judgement than is any analysis requiring confirmation by experiment. We assume that we have concluded that

$$\theta(x,t) = \text{Function}(\theta_o, x, t, K, \rho, c_p)$$

We now turn to a dimensional analysis of a familiar problem which can be described by a differential equation, initial and boundary conditions.

2.1.2 Differential equations, heat and energy

The conventional procedure consequent on writing equation (2) is to put indices on each term, and to equate the powers of the basic independent dimension, e.g. mass M , length L , time T and in this example temperature and heat. Leaving aside philosophical questions, we must in

an equation have the same units and dimensions on each side. Heat is energy, and some authors advocate not using the term "heat", but to expose its true nature by referring to it as energy - or perhaps thermal energy. This, in practice, is not consistent with the fact that often in dimensional analysis in fire matters we must treat heat as a separate dimension " H " not as energy which in M, L, T terms is ML^2/T^2 .

Why?

Briefly, because if - but only if - there is no exchange between heat and mechanical energy one treats heat as ML^2/T^2 so that K has dimensions

$$\frac{ML}{T^3\theta} \quad (2)$$

one has from equation

$$\theta = a \theta_o^\alpha x^b t^\gamma K^\Delta \rho^\epsilon c_p^f \quad (3)$$

for the power of temperature

$$1 = \alpha - f \quad (4)$$

the power of mass

$$0 = -\alpha + \epsilon \quad (5)$$

the power of length

$$0 = \beta + \Delta - 3\epsilon + 2f \quad (6)$$

and of time

$$0 = \gamma - 3\Delta - 2f \quad (7)$$

Hence

$$\frac{\theta}{\theta_o} \propto \left(\frac{K \theta_o t^3}{\rho x^4} \right)^{\alpha-1} \left(\frac{K t}{\rho c_p x^2} \right)^{-f} \quad (8)$$

α and f are "unknown" but another pair could be chosen and the two sets of terms exchanged for any other pair formed by combination of the two in equation (8). The latter group of

quantities is the familiar Fourier number but $\frac{K \theta_o t^3}{\rho x^4}$ or any term containing θ_o formed from

this and the Fourier number e.g. $c_p \theta_o t^2 / x^2$ is included only if ML^2/T^2 is used for heat.

Putting heat as ML^2/T^2 in this context causes confusion unless the exchange between mechanical and thermal energy is a significant contribution to the process of heat transfer. If we start from the supposedly exact differential equation, we have

$$\rho c_p \frac{\delta \theta}{\delta t} = K \frac{\delta^2 \theta}{\delta x^2} \quad (9)$$

with the boundary conditions

$$\theta = 0 \begin{cases} x > 0 \\ t = 0 \end{cases}$$

$$\theta = \theta_o \begin{cases} x = 0 \\ t > 0 \end{cases}$$

These can be rearranged as

$$\frac{\delta(\theta/\theta_o)}{\delta\left(\frac{Kt}{\rho c_p}\right)} = \frac{\delta^2(\theta/\theta_o)}{\delta x^2} \quad (10)$$

$$\theta/\theta_o = 0 \text{ at } t = 0 \quad x > 0$$

$$\theta/\theta_o = 1 \text{ at } t > 0 \quad x = 0$$

It formally follows that

$$\theta/\theta_o = \text{function} \left\{ \frac{Kt}{\rho c_p x^2} \right\} \quad (11)$$

This can be confirmed by inserting equation (11) into equation (10), from which one obtains an equation involving F (the function) and its first and second differentials in terms only of $Z = Kt/\rho c_p x^2$.

This function is not a power law but it can be expressed as a series of powers of $\frac{Kt}{\rho c_p x^2}$.

In general, one can obtain dimensionless groups from the complete set of equations (not necessarily differential) and, where appropriate, the initial and boundary correlations. One part of a chapter on dimensional analysis in a text by a well known specialist does not refer to the initial and boundary conditions and is not as helpful as the author intended!

x defines a position and

$\sqrt{\frac{Kt}{\rho c_p}}$ is a distance that is the "scale" of the thermal penetration.

The temperature distribution is continuous and a very simple model of the thermal process shows $\sqrt{\frac{Kt}{\rho c_p}}$ is a measure of the penetration. Thus if δ is the effective penetration then

$K\theta_o/\delta$ is a measure of the rate at which heat enters the solid. The heat content is thus of order

$K(\theta_o/\delta)t$ and this must approximately be $\frac{\rho c_p \theta_o \delta}{2}$. Hence $\delta = \sqrt{\frac{Kt}{\rho c_p}}$, neglecting a

coefficient of order 1. We must expect

$$\frac{\theta}{\theta_o} \rightarrow 0 \text{ for } x \gg \sqrt{\frac{Kt}{\rho c_p}}.$$

What happens if the solid is not semi-infinite but is a slab of thickness L ? Provisionally we take a simple condition at $x=L$, e.g. $\theta=0$. Clearly one can define the position by $\frac{x}{L}$ to produce another variable.

$$\text{i.e.} \quad \frac{\theta}{\theta_o} = \text{function} \left[\frac{Kt}{\rho c_p x^2}, \frac{x}{L} \right] \quad (12)$$

$$\text{or} \quad \frac{\theta}{\theta_o} = \text{function} \left[\frac{Kt}{\rho c_p L^2}, \frac{x}{L} \right] \quad (13)$$

More formally one can multiply both sides of equation (10) by L^2 and thereby define $\frac{Kt}{\rho c_p L^2}$

and $(x/L)^2$ as dimensionless variables: this leads directly to equation (13).

2.2 *Physical meaning and implications*

We have already illustrated by a simple model the physical and geometric significance of the

ratios $\frac{Kt}{\rho c_p x^2}$ and $\frac{Kt}{\rho c_p L^2}$ in a thermal conditions problems.

Clearly if $\frac{x}{L} \ll 1$ we cannot expect the thermal wave to have reached the rear surface. Hence, L is not then relevant and must be removed.

i. e. $\frac{Kt}{\rho c_p L^2} \div \frac{x^2}{L^2}$ i. e. $\frac{Kt}{\rho c_p x^2}$ is the relevant variable.

After a long heating time the "steady state" solution does not depend on " t " if there is cooling but the existence of a "steady state" depends on the condition at the second surface as well as on x/L . Order of magnitude discussions of terms in differential equations so as to simplify them was commonplace before CFD permitted numerical solutions of complex equations with or without sensitivity analyses!

In this connection one recognises that the physical significance of a Reynolds number is that of a ratio of a characteristic inertial force ρU^2 to a characteristic viscous force $\mu U/\Delta$ where U/Δ is representative of the gradient. This ratio is $\frac{U\Delta}{\mu/\rho} \left(= \frac{U\Delta}{\nu} \right)$ and we ought to expect that if the terms are representative of the forces the importance of this Reynolds number depends on whether it is $\ll 1$ or $\gg 1$.

The reason for critical Reynolds numbers which defines the transition from laminar to turbulent flow being of order $10^3 \rightarrow 10^4$ is that the length usually employed in Reynolds numbers is a physical length, e. g. a distance from a leading edge or diameter D , whilst Δ is physically a boundary layer often being orders of magnitude smaller than D .

In its various forms, the Froude number N_{Fr} is the basis of much modelling in fire studies. It is defined as the ratio of inertial to buoyancy forces and a common definition is

$$\frac{\rho U^2}{g \Delta \rho H_B}$$

where $\Delta \rho$ is the difference in density producing the buoyancy force, ρU^2 expresses the inertial force and H_B is, in principle a height over which there is a buoyancy effect. Often $\Delta \rho/\rho$ appears as θ/T_o which expresses the buoyancy caused by thermal expansion. Occasionally one sees $U/\sqrt{g \theta H_B/T_o}$ the square root of this expression for Froude number.

It is important **not** to treat this as simply the square root of the former because it is not only the square root of the ratio of forces but, as a ratio of velocities, it can be related to a ratio of volumetric quantities, which in turn is related to air/fuel ratios and entrainment. This arithmetic translation is physically ambiguous.

Anticipating a discussion yet to come below, one notes that physically g is associated with a vertical height; its appearance with a horizontal distance must require an answer to the question it raises. In brief, the answer anticipated is akin to that involved in the Reynolds number where the physically important **dependent** quantity is replaced by an **independent** geometric parameter.

A third point: if a temperature rise, say, is being correlated with a Froude number, the temperature rise in the buoyancy may be omitted when correlating data with a formula based only on dimensionless variable: if

$$y = F_1 (x/y)$$

then $y = F_2 (x)$

3. Other examples

3.1 Self-heating and thermal explosion

The classical Semenov [9] self-heating theory equates the heat loss from the surface ‘S’ of a uniformly heated object, i.e.

$$h S(T - T_o)$$

to the rate of generation of heat $V \dot{q}'''$ where \dot{q}''' refers to the heat generation per unit volume throughout the volume V by a zero order reaction obeying the Arrhenius Law

$$\dot{q}''' = Q f \rho e^{-E/RT} \quad (14)$$

where Q is the heat release for unit mass.

$$\text{Hence} \quad hS(T - T_o) = Q f \rho V e^{-E/RT} \quad (15)$$

Obviously RT/E is a dimensionless temperature but this is not always the most useful one. We discuss the behaviour of the quantity $(T - T_o) e^{+E/RT}$ in the above equation.

Equation (15) can be rewritten as

$$(T - T_o) e^{E/RT_o} e^{\frac{-E(T-T_o)}{TT_o}} = \frac{QfV}{hS}$$

$$\text{i.e.} \quad \theta e^{-\frac{\theta}{1 + \frac{RT_o}{E}\theta}} = \frac{QfV}{hS} e^{-E/RT_o} \cdot \frac{E}{RT_o^2}$$

$$\text{where} \quad \theta = \frac{E}{RT_o^2} (T - T_o)$$

This dimensionless temperature **difference** becomes important for the low values of RT_o/E

typical in many problems. $\theta e^{-\theta \left[1 + \frac{RT_o}{E} \theta \right]}$ has a critical value

$$\theta = \frac{E}{RT_o^2} (T - T_o) = 1 \text{ for } \frac{RT_o}{E} \ll 1$$

$$\frac{hSRT_o^2}{E} \theta e^{-\theta} = Q\rho fV e^{-E/RT_o}$$

The "error" is a term of order $e^{-\theta^2 RT_o/E}$, a few per cent.

Because $\theta e^{-\theta}$ has a maximum value at $\theta = 1$ the Semenov criterion for the existence of a stable solution is

$$\frac{EQ\rho fV}{RT_o^2 Sh} e^{-E/RT_o} < \frac{1}{e}$$

Frank-Kamenetskii [10] initially allowed for gradients within the material (assumed rigid) but assumed an infinite value for h . This results in h being replaced by $\alpha \frac{KS}{V}$ where α is $O(1)$.

$$\text{i.e. } \frac{EQ\rho fV^2}{RT_o^2 KS^2} e^{-E/RT_o} < \delta \quad (16)$$

where δ is $O(1)$.

δ depends on the geometry and shape of the material, and there is an extensive mathematical literature on this topic. It follows from dimensional analysis that for any $\frac{hV}{KS}$ uniform boundary conditions the equilibrium conditions are given by

$$\frac{EQ\rho f}{RT_o^2} \left(\frac{V}{S} \right)^2 e^{-E/RT_o} = \text{function} \left(\frac{hV}{KS} \right) \quad (17)$$

and, from the preceding arguments, that for

$$x \rightarrow 0, \text{function}(x) \propto x$$

and for $x \rightarrow \infty, \text{function}(x) \rightarrow \text{constant}$,

which depends on the shape of the body.

$Q \rho f V e^{-E/RT_o}$ is characteristic of the heat release whilst $\frac{hS}{K} \frac{RT_o^2}{E}$ is characteristic of the heat loss. $\frac{hV}{KS}$ is proportional to the ratio of the external to the internal conductance. If there is no solution to the equation because

$$\frac{E}{RT_o^2} \frac{Q \rho f}{K} \left(\frac{V}{S} \right)^2 e^{-E/RT_o} > \text{maximum of function} \left(\frac{hV}{KS} \right)$$

there can be no equilibrium: the material heats up indefinitely unless some limiting process is introduced, e.g. reactant consumption or diffusion. Consider again the approximation that led to the introduction of the alternative dimensionless temperature -

$$e^{-E/RT} \approx e^{-E/RT_o} e^{\frac{E(T-T_o)}{RT_o^2}}$$

This approximation can be used even when there are **large** temperature differences PROVIDED T_o is chosen not as an ambient temperature but as some datum near to the maximum temperature, e.g. as T_I when a slab is heated on one side from T_o to T_I and the temperature inside the solid is able to rise only slightly above T_I . The fractional error in the role of chemical heating on the cool side of the slab is enormous but the actual and the approximate expressions for the heat generation in the cool region are such that **both** may be negligible in the heat balance.

The choice of dimensionless temperature or temperature rise is helped by a physical analysis.

3.2 Ignition (external heating)

Ignition studies after WW2 concentrated on ignition by radiation and it was recognised that cellulosic materials ignited because they produced flammable decomposition products which could be ignited in air by an auxiliary ignition source e.g. a spark or a small flame. Such fuel could only be produced by thermal heating causing decomposition.

The incident heat flux in effect raises the surface temperature by conventional heat transfer processes until the temperature is high enough for any chemical heating to cause thermal

instability; subsequent chemical decomposition produces flammable gaseous products rapidly enough.

Dimensional analysis of the problem arose naturally because the main component of the induction period, i.e. the delay time, was the thermal capacity of the material. Sauer [11], in the USA, developed the procedure which was exploited by Martin [12] and by Simms [13].*

Martin and his colleagues omitted surface cooling as a secondary factor. This was a simplification and was perhaps appropriate so long as one is concerned with ignition times under severe exposure. If one is concerned with whether ignition occurs - a matter of importance in civil defence, then (and now too, to judge by the attention paid to evaluating a critical condition in opposed flow spread of flame) - cooling is important. Simms included cooling but treated it as a Newtonian linear simplification (see Appendix 1).

In general the thermal conduction solution is

$$\frac{h\theta(x)}{q''} = function \left[\frac{x}{L}, \frac{Kt}{\rho c_p L^2}, \frac{h^2 \tau}{K \rho c_p}, \frac{K \tau}{\rho c_p L^2} \right] \quad (18)$$

where L is the slab thickness

q'' a characteristic incident flux

and τ a time controlling the variation of the imposed heat flux with time (assuming the initial values of heat flux and temperatures are zero)

For $\theta = \theta_{ig}$ - the effective ignition temperature rise at $x = 0$,

$$\frac{t_{ig}}{\tau} = function \left(\frac{h^2 \tau}{K \rho c_p}, \frac{q''}{h \theta_{ig}}, \frac{k \tau}{L^2} \right)$$

where $k = K/\rho C_p$

For a thick solid $L \gg \sqrt{\frac{Kt}{\rho c_p}}$ and L is removable from equation 18.

i.e.
$$\frac{h \theta_{ig}}{q''} = function \left(\frac{t}{\tau}, \frac{h^2 \tau}{K \rho c_p} \right)$$

* There is much contemporary discussion of ignition and critical temperature which could benefit from a reading of early literature.

If there is no τ as when $q'' = q''_s = \text{constant}$

then
$$\frac{t_{ig} K \rho c_p}{h^2} = \text{function} \left(\frac{q''_s}{h \theta_{ig}}, \frac{hL}{K} \right) \quad (19)$$

and if additionally cooling is negligible (i.e. h is omitted) and the solid is thick (L is omitted).

$$\frac{t_{ig} K \rho c_p \theta_{ig}^2}{q''^2} = \text{constant}$$

viz. the conventional square law.

Sauer & Simms also introduced a chemical component using the dimensionless temperature

$$\frac{RT}{E} \text{ and a nominal chemical heat release } Qf\rho \text{ kW/m}^3.$$

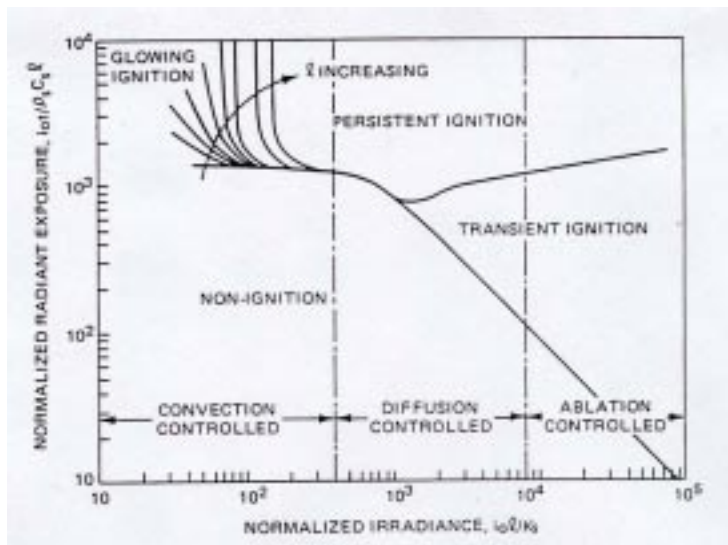
This leads to the inclusion of dimensionless terms, one of which is similar to the δ in self heating theory (see above). This approach seems to depend too much on chemical terms of uncertain value and form to be practical and of value.

Kanury [14] describes how Martin used equation [19] in his ignition studies

Fig. 1

$$y' = \frac{q'' t}{\rho c_p L}$$

$$x' = \frac{q'' L}{K}$$

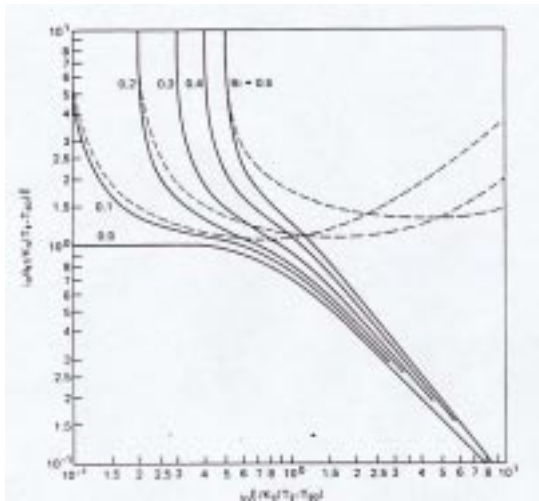


y' and x' are not dimensionless. They would be if divided by the supposedly constant ignition temperature rise. This procedure was used to find θ_{ig} from experimental data from which Fig. 1 was derived.

Fig. 2 Theoretical correlation

$$y = y'/\theta_{ig}$$

$$x = x'/\theta_{ig}$$



Neglecting cooling one has for **thin** materials in Fig. 2

$$\frac{q''L}{K} \rightarrow 0 \text{ i.e. } y \rightarrow 1$$

so that for $\frac{q''L}{K} \rightarrow 0$ and B_i , the Biot Number in Fig. 2, equal to $\frac{hL}{K} = 0$

one has
$$\frac{q''t_{ig}}{\rho c_p L \theta_{ig}} = 1$$

For a thick solid, theory shows as in Fig. 2

$$\frac{q''t_{ig}}{\rho c_p L \theta_{ig}} = A \left(\frac{q''L}{K \theta_{ig}} \right)^{-1} \text{ where } A = \pi/4$$

From the data underlying Fig. 1

$$\theta_{ig}^2 = 1.2 \cdot 10^6 \text{ i.e. } \theta_{ig} = 1000^\circ K$$

The reason why θ_{ig} appears to be over 1000 °(C or F?) is unclear. 600K above ambient was what Martin actually reported. Simms gave a lower figure 500°C-550°C claiming he had made various corrections which were essential to analysing experimental data. Leaving aside the arithmetic we have demonstrated how the variables can be presented in various ways and used to estimate a parameter not measured directly.

3.3 *Vertical plumes*

Axi-symmetric, and to a lesser extent two-dimensional plumes, have received considerable attention in the fluid mechanics and fire literature.

Early studies were based on total similarity at **all** horizontal sections above the source. Early works, e.g. that by Sir Geoffrey Taylor [15] who developed the work of Schmidt [16], Rouse, Yih and Hyphreys [17] and Yokoi [18] whose study of plumes emerging from openings is a classic, assumed an eddy diffusivity to determine the bell-shaped cross sectional distribution.

After the introduction of the alternative presumption - local similarity - with a constant entrainment coefficient - most analyses were still based on assumed horizontal distributions of velocity and temperature, Gaussian or 'Top Hat'.

However, the only independent variables considered in any of these developments were -

- (a) initial mass and momentum flux (or one of these and orifice size),
- (b) thermal or buoyancy constancy.*

The dependent variables which were chosen were the plume width, velocity and temperature rise. Turbulent eddy diffusivities were described in term of these variables. Entrainment coefficients are usually taken as constant, but are otherwise (19) dependent on temperature rise, itself a dependent variable.

* The following arguments if applied to plumes with radiation loss, become sufficiently complicated to warrant resorting to CFD calculations.

Dimensional analysis does not have to make assumptions about Gaussian or Top Hat distributions, nor about the constancy of the entrainment coefficient E . These are internal dependent variables.

For an axi-symmetric plume which remains a plume rising in a still and uniform atmosphere. One presupposes one is considering a cross section, bounded by a region or regions with no flow parallel to the plume axis and over which one can define a mass flow " M ". One obtains for the mass flow M at a height Z

$$\frac{MT_o c_p}{Q} = function \left\{ \frac{gZ^{5/3}}{\left(\frac{gQ}{\rho_o c_p T_o} \right)^{2/3}}, \frac{M_o T_o c_p}{Q}, \frac{gb_o^{5/3}}{\left(\frac{gQ}{\rho_o c_p T_o} \right)^{2/3}}, \frac{M_o}{b_o \mu}, \frac{\mu}{\rho_o k} \right\} \quad (20)$$

where Q is the constant convected heat flux and M_o is the initial mass flow. Molecular diffusion is included as it is necessarily implied when allowance is made for differing velocity and temperature rise profiles. The suffix o denotes initial and ambient conditions.

If detailed analysis is based on the balances of mass flow, momentum force and buoyancy consideration etc. then for a constant entrainment coefficient and constant and identical shapes of velocity and temperature distributions across the plume one obtains in the far field where the initial conditions are no longer an influence

$$\frac{MT_o c_p}{Q} \propto \left\{ gZ^{5/3} / \left(\frac{gQ}{\rho_o c_p T_o} \right)^{2/3} \right\}$$

i.e. $M \propto g^{1/3} Q^{1/3} Z^{5/3}$

i.e. the conventional proportionality with M proportional to $Q^{1/3}$ and $Z^{5/3}$. The same general result is equation (20) will be true for a plume with a given inclined source, the angle being dimensionless. However in the discussion of an inclined plume the velocity and temperature vary about a trajectory defined by their peak values. The same arguments - and assumptions as lead to equation (20) lead to an equation for the trajectory

$$\frac{Z}{S} = function \left[\frac{x}{S} \cdot \frac{M_o T_o c_p}{Q}, \psi_o \frac{M_o}{b_o \mu}, \frac{\mu}{\rho_o k}, \frac{b_o}{S} \right]$$

where
$$S = \left(\frac{Q}{\rho_o c_p T_o \sqrt{g}} \right)^{2/5}$$

ψ_o is the inclination of the source and x the horizontal projection of the trajectory. This discussion may seem trivial, adding nothing to the conventional and more detailed theories. This is not so. Experimental data $M(Z \cdot Q)$ is better analysed as $\frac{M}{Q}$ versus $Z^{5/3}/Q^{2/3}$ than as M versus $Z^{5/3}/Q^{1/3}$ if only because the near field data are weighted quite differently. Similar arguments apply to line plumes.

3.3.1 The Bent over Plume

Line sources with a constant entrainment velocity in still air have a constant vertical velocity in the far field

$$\omega_{ff} \propto \left(\frac{gQ'}{\rho_g c_p T_o} \right)^{1/3}$$

where Q' is the heat release per unit width of the plume.

It is presumed that the two dimensional line plume can be bent over by a side wind in excess of the entrainment velocity (15).

Dimensional analysis for such a turbulent plume in a turbulent side wind U supports a solution

$$\frac{g\theta}{T_o} \propto \frac{\omega_c^2}{Z} \text{ function} \left(\frac{gZ}{\omega_c^2}, \frac{gx}{\omega_c^2}, \frac{U}{\omega_c} \right) \quad (21i)$$

where
$$\omega_c = (gQ' / \rho_o c_p T_o)$$

For the far field of a weak plume there is a maximum $\theta_m(x)$ as Z varies from 0 to ∞ so that

$$\frac{g\theta_m}{T_o} \propto \frac{\omega_c^2}{x} \text{ function} \left(\frac{gx}{\omega_c^2}, \frac{U}{\omega_c} \right) \quad (21ii)$$

If we write this as a power law and identify the length of flame ℓ_f by the x at which $\theta_m(x)$ is a particular θ_f defining the flame length, where θ_m is a constant, we obtain

$$\frac{gl_f}{\omega_c^2} \propto (U/\omega_c)^a \quad (22)$$

where a is an unknown index. The power law cannot apply for all (U/ω_c) : for a vertical flame a is zero and the left hand side is a constant and the well known relationships obtains

$$gl_f \propto (gQ'/\rho_g c_p T_o)^{2/3}$$

For flow fully dependent on forced convection "g" must be absent ($a = -1$) so that

$$\frac{gl_f}{\omega_c^2} \propto \frac{\omega_c}{U}$$

$$\text{i.e.} \quad l_f \propto \frac{Q'}{\rho_g c_p T_o} \cdot U \quad (22i)$$

a linear relationship between l_f and Q' . In a mixed system we expect "a" to lie between 0 and -1.0 depending on the value of U/ω_c .

A set of experiments on small flames (19) from cribs were correlated by

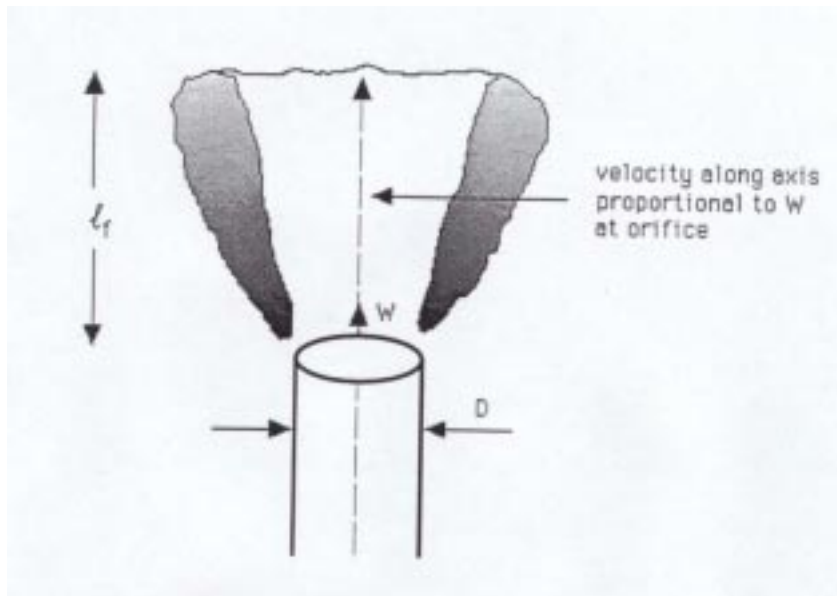
$$l_f \propto Q'^{0.74} U^{-0.21} (a = -0.21)$$

Whether a weak plume relationship can be used with a flame temperature hypothesis - as it can for vertical flames - may be questioned but the above discussion suggests that the linear approximation between l_f and Q' may be less of an approximation for forced than for natural connections.

3.4 *Flames*

Before any work was done on the size of flames from fires, there was a considerable literature on forced jet flames (see Fig. 3) because of their industrial relevance.

Fig. 3 Turbulent jet (momentum driven)



The essence of the physical theory for a given fuel, is as follows. The velocity of the flow of oxygenated air into the flame is given by -

$$\epsilon \frac{Y_{ox}}{d} \propto \frac{Y_{ox}}{l_f}$$

where ϵ is the turbulent diffusivity

Y_{ox} the oxygen concentration

and d is a mean radial distance proportional to

l_f the flame height

Only because l_f/D is generally large (see Fig. 3) does l_f determine the mean gradient for the diffusion. ϵ is dependent on a length and a velocity and there are no others in turbulent flow than respectively l_f and W .

Buoyancy forces are considered negligible or secondary. The fuel flow Q_f is proportional to WD^2 so that assuming (i) a particular degree of combustion independent of Q_f and l_f and (ii) that the flame surface over which there was diffusion is proportional to l_f^2 then with $d \propto l_f$

$$l_f^2 \left[W l_f \left(\frac{Y_{ox}}{l_f} \right) \right] \propto W D^2$$

i.e. $Y_{ox} l_f^2 \propto W D^2$

i.e. for a given Y_{ox} and a given fuel $l_f \propto D$.

Note that

$$\frac{\epsilon}{l_f} \propto W$$

in essence describes the entrainment velocity as proportional to the flow velocity.

The result $\frac{l_f}{D} = \text{constant}$ is the classic result for turbulent jets (20), detailed combustion and flow analysis determining the constant. It is interesting to compare this to laminar jets where ϵ is replaced by a viscosity μ . Then

$$\frac{\mu Y_{ox}}{l_f} \cdot l_f^2 \propto Q_{fuel}$$

I.e. $l_f \propto Q$ the classic result for laminar jets.

Introducing buoyancy instead of momentum simply replaces (21) W by \sqrt{gl} (the ratio $\Delta\rho/\rho$ the fractional density deficiency being assumed nearly constant in the flame zone).

Hence

$$\begin{aligned} \epsilon &\propto L \sqrt{gl_f} \\ \epsilon &= \frac{Y_{ox}}{d} l_f^2 \propto Q_f \end{aligned}$$

and $d \propto l_f$

so that $l_f \propto Q^{2/5}$

More generally (because the above is based on a conical combustion zone and a surface of arbitrary shape) the surface area is written as $l_f^2 \text{function}(l_f/D)$ where D is the base dimension, so that

$$l_f^2 \cdot \text{function}\left(\frac{l_f}{D}\right) \propto \frac{\sqrt{g} l_f}{l_f} \propto Q_f$$

i.e.

$$\frac{l_f^2}{D^2} \cdot \text{function}\left(\frac{l_f}{D}\right) \propto \frac{Q_f^2}{D^2 \sqrt{g} l_f}$$

NB, g cannot physically be combined with anything except a vertical dimension but algebraic manipulation of the above gives

$$\frac{l_f}{D} = \text{function}\left(Q_f / D^2 \sqrt{gD}\right)$$

Note we have treated Q_f as volume per sec and the ratio $Q_f / D^2 \cdot \sqrt{gD}$ is dimensionless. To interpret Q as heat is appropriate beyond the end of the process of combustion but not within the combustion zone.

Plume and flame theories can be integrated because

$$\rho_{fuel} Q_f \propto Q_{heat} (\propto \text{mass of air})$$

and the independent variable is then

$$\frac{Q_{heat}}{\rho c_p T_o D^2 \sqrt{gD}}$$

Above we have quoted the plume theory result for axi-symmetric plumes.

$$M \propto Q_f^{1/3} Z^{5/3}$$

i.e.

$$\theta_{mean} \propto \frac{Q_f}{M} \propto Q_f^{2/3} Z^{-5/3}$$

Yokoi used this and similar plume formulae to define flame length l_f by the locus of a *particular* value of θ_{mean} i.e.

$$\theta_{flame} \propto \theta_{mean} \propto Q_f^{2/3} l_f^{-5/3}$$

$$l_f \propto Q_g^{2/5}$$

Thomas and Karlsson [22] have recently used the same argument to augment a few direct measurements of flame length by plume data for flow under the ceiling of a well ventilated compartment.

3.5 *Flame spread*

3.5.1 *Opposed flow flame spread*

There are few points at which dimensional analysis is helpful in analysing problems in flame spread, but at least one is fundamental. If a plane heat source at constant temperature θ_o moves at velocity v_a through an infinite medium of constant thermal properties at a lower uniform temperature, then the temperature rise in the medium a distance x ahead of the source is

$$\begin{aligned}\theta &= \theta_o e^{-v_a x / k_g} & x \geq 0 \\ &= \theta_o & x < 0\end{aligned}$$

There is conduction ahead of the source, its value at the source being -

$$-K_g \left(\frac{d\theta}{dx} \right)_{x=0} = \rho_g c_p v_a \theta_o$$

which is the heat energy required to raise the medium to the temperature of the source. Conduction occurs even though the right hand side does not contain K_g . If there was no conduction there could not be a gradient, only a step change in temperature. There is at best some ambiguity in the literature on this topic. It is to be noted that there is no fixed dimension in this idealisation of the problem other than k_g/v_a the characteristic scale length in a moving medium. This is also necessarily the value for a semi-infinite plane source moving through a semi-infinite medium on one side of a cold surface (see Fig. 5). The only distance in the statement of this problem is k_g/v_a . When Parker [23] dealt with flame spread on a thin material (see Fig. 4) he wrote the heat flux from the flame Q_f to the pyrolysing surface at θ_p as

$$q_f'' = \frac{K_g(\theta_{fl} - \theta_p)}{\delta}$$

whilst the heating of the thin material of thickness D requires

$$q_f'' \Delta = \rho_s c_p \theta_p V D$$

(24)

i. e.
$$\frac{K_g(\theta_{fl} - \theta_p) \Delta}{\rho_s c_p \sigma} = DV$$

(25)

Δ and δ were measured by Parker. k_g/v_a is the scale distance in all directions including that at right angles to the motion in those situations e.g. as in Fig. 5 where there is no temperature gradient on the cold boundary. This of course presumes that the cold boundary is on a body of effectively infinite conductivity or capacity. If the movement of the thin material is assumed to have no influence on the gas phase nearby then the dimensionless ratio of Δ to δ must be considered as independent of v_a . Each might depend on the combustion kinetics (not considered by simple thermal models but a factor defining both Δ and δ and perhaps their ratio).

Fig. 4 Downward spread of flame on a thin sheet moving front (heat source)

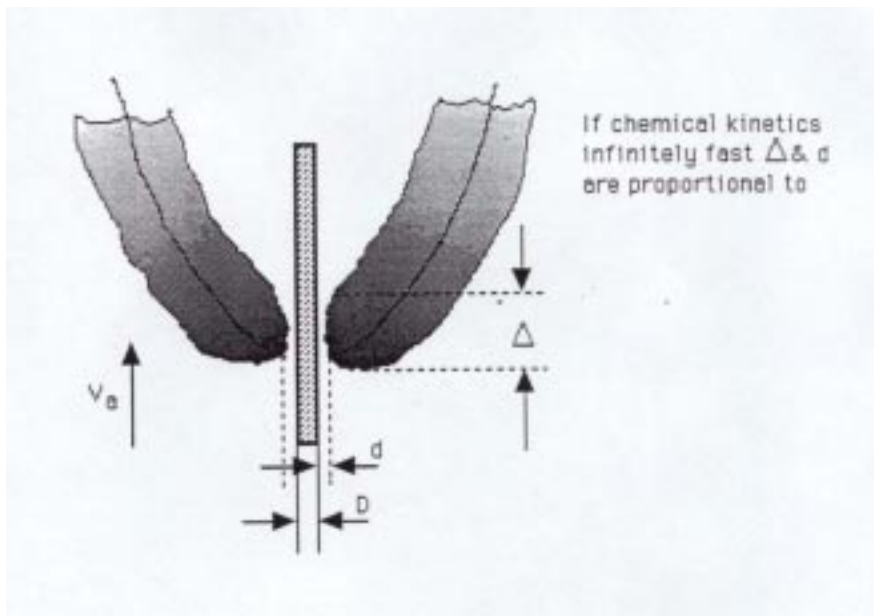
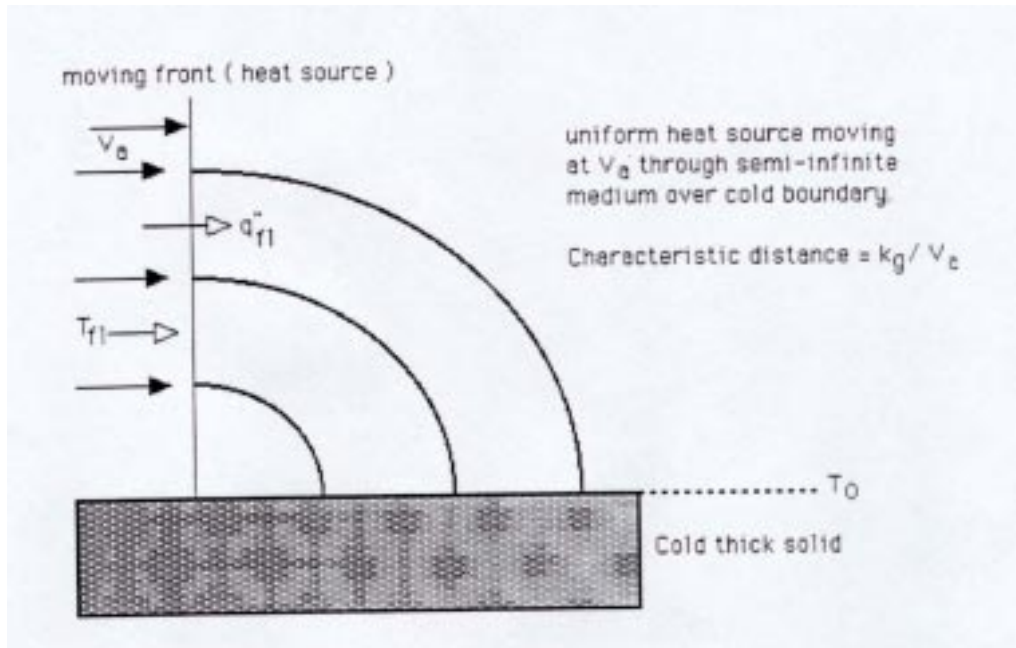


Fig. 5 A uniform heat source moving over a cold surface



The lack of dependence of V on v_a is qualified. v_a cannot be zero: heat transfer into a stationary semi-infinite gas is like that into a solid: there is no steady state!

Fig. 4 and equations (22) and (25) imply Δ is at right angles to δ and one can be puzzled by the references in the literature to δ being associated with forward conduction but this confusion arises because k_g/v_a is the characteristic dimension in all directions: there is no other on the thermally thin material. In the discussion of spread at velocity V on thick solids,

D is replaced by $\sqrt{\frac{\pi}{2}} \sqrt{\frac{k_s \Delta}{V}}$ and equation (24) gives

$$q''_{fl} \sqrt{\Delta} = \frac{\sqrt{\pi}}{2} \sqrt{K_s \rho_p c_p} \sqrt{V} (\theta_p - \theta_o)$$

i.e.
$$V = \frac{4}{\pi} \frac{q''_{fl}{}^2 \Delta}{K_s \rho_s c_p (\theta_p - \theta_o)^2}$$

$q''_{fl} \sqrt{\Delta}$ varies from one material to another but is taken as independent¹ of V . That $q''_{fl} \sqrt{\Delta}$ is a constant for a material is a hypothesis to the justified and tested by experiment!

¹ Tests of materials measure V for various initial conditions as assessed by a θ_o a determined by external radiation varying along the length of a heated specimen. This tests the hypothesis that $q''_{fl} \sqrt{\Delta}$ is a constant.

Whilst the dimension Δ is forward in the sense of being parallel to the spread the relevant conduction into the solid is at right angles to this. Removing q_{fl}'' gives

$$V = \frac{4}{\pi} \frac{K_g^2 (\theta_{fl} - \theta_p)^2 \Delta}{\delta^2 K_s \rho_s c_s (\theta_p - \theta_o)^2}$$

and de Ris's [24] result implies $\frac{\Delta}{\delta^2} \propto \frac{v_a}{k_g}$

giving the result sometimes seen

$$V = v_a \frac{K_g c_g \rho_g}{K_s c_s \rho_s} \left(\frac{\theta_{fl} - \theta_p}{\theta_p - \theta_o} \right)^2$$

Were we to include the effects of finite kinetics we could, of course, readily construct a dimensionless ratio by incorporating $\frac{k_g}{v_a}$ into the dimensionless chemical heat release terms

(see above) including

$$\frac{E}{RT_{fl}^2} \frac{Q''' \rho_g f}{K_g} e^{-E/RT_{fl}} \left(\frac{k_g}{v_a} \right)^2$$

where the reaction rate temperature datum is taken at a nominal flame temperature. One would then be able to discuss extinction.

3.5.2 Concurrent flame spread

3.5.2.1 Flame spread and the Delichatsios-Saito length scale

The essence of a quasi-steady thermal theory of concurrent flame spread is (25)

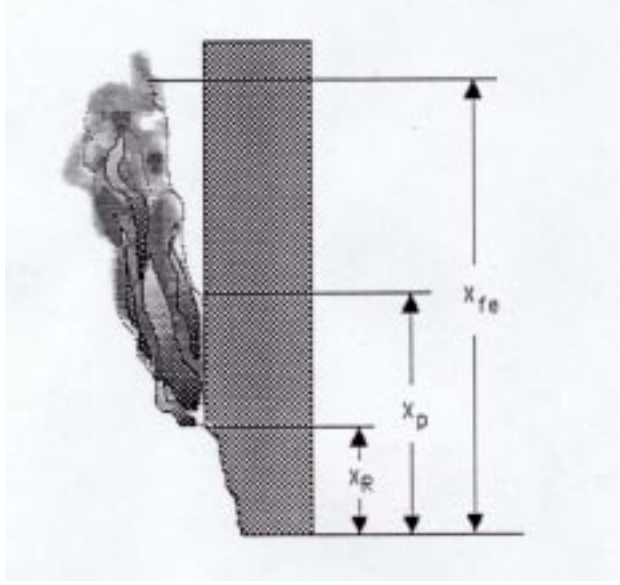
$$\frac{dX_p}{dt} = \frac{X_{fl} - X_p}{\tau_{ig}} \quad (26)$$

where X_p is the distance advanced by the pyrolysis zone (Fig. 6),
 X_{fl} is the distance advanced by the flame lip

t is time

and t_{ig} is the time for ignition under the exposure to flame

Fig. 6 Flame spread up a thick solid



One can define t_{ig} by

$$\theta_p - \theta_o = \frac{2}{\sqrt{\pi}} \frac{q_{fl}'' \sqrt{kt_{ig}}}{K_s}$$

Likewise the advance of the locus of the burnt out front (25) is given by

$$\frac{dX_R}{dt} = \frac{X_p - X_R}{t_B} \quad (27)$$

where X_R is the locus of burn out (25) and t_B is the time between ignition and burn out. We have the flame length relationship (26)

$$X_{fl} - X_R = K_n \left\{ Q_B' + \int_o^t q_{fl}''(t - t_p) \frac{dX_p}{dt_p} dt_p + q_{fl}''(t) X_{po} \right\}^n \quad (28)$$

where K_n is a constant (known from experiments)

X_{po} is X_p at time zero.

and Q_B' is the burner strength KW/m

and n an index, usually 2/3 or unity.

Complete algebraic solutions (27) are available only when $n = 1$

For the flames in which $X_{fl} - X_p \gg X_p - X_R$ we expect "n" to be 2/3 for natural convection and vertical spread. When $X_{fl} - X_p \ll X_p - X_R$ there are reasons for supposing the same relationship holds but there are few data for this condition. Section 3.5.2.1 gives some justification for the linear approximation $n = 1$.

K_n is a constant depending on the value of n . For $n = 1$ the value is roughly $0.01 \text{ m}^2/\text{Kw}$.

It is acknowledged that the approximation $n = 1$ is in one respect fundamentally different from $n \neq 1$ even if it is close to unity. Care is necessary in the interpretation of theory. If $n = 1$ indefinite spread is one possibility if $n \neq 1$ even if is close to 1 but <1 , spread is always limited.

The heat release rate per unit area produced by pyrolysis from the fuel is characterised by an initial value Q_o'' which is proportional to the mass rate per unit area of pyrolysis M_o'' i.e. $Q_o'' = qM_o''$ is a characteristic pyrolysis rate eg $q''_{net}/\Delta H_v$. q''_{net} is an effectively mean net heat transfer rate from the flame and ΔH_v is a characteristic heat of pyrolysis. For non-charring materials one has

$$Q_o'' = q''_{net} \Delta H_c / \Delta H_v$$

Dimensional analysis permits us to write

$$\frac{\ell_{fl}}{\ell} = function \left(\frac{Q_o''}{\rho_g c_p T_o \sqrt{gl}}, \frac{X_{po}}{\ell} \right) \quad (29)$$

where ℓ is a dimension characteristic of the fire.

Delichatsios and Saito [28] showed that

$$\ell = \left(\frac{q''_{net} \Delta H_c / \Delta H_v}{\rho_g c_p T_o \sqrt{g}} \right)^2 \quad (30)$$

$$\text{Hence } \ell_{fl}/\ell = M(x_{po}/\ell)^n \quad (31)$$

where M is here a dimensionless constant.

The development of theory using the $2/3$ power instead of the linear law defines a dimensionless parameter (see Appendix 2)

$$E = Q'_B \left(1 + t_{ig}/t_B\right)^3 / K_{2/3}^2 (qm_o'')^3 \quad (32)$$

which with t/t_{ig} defines the behaviour of flame on a thick solid with the $2/3$ power flame length law.

It can be shown that

$$E = Q'_B \left(1 + t_{ig}/t_B\right)^3 (\ell_B/\ell)^{3/2}$$

where $\ell_B = \left(Q'_B / x_{po} \rho_g c_p T_o \sqrt{g}\right)^2$

is analogous to ℓ but is characteristic of the burner and the initial pyrolysis length.

In summary, whichever power law applies to flame length we have, following Delichatsios and Saito,

$$\frac{X_p}{\ell} = function \left(\frac{X_{po}}{\ell}, \frac{t_{ig}}{t_B}, \frac{t}{t_{ig}}, \ell/\ell_B \right)$$

If the width of the spreading zone was finite and of width D then D is a dimension characteristic of the source and ℓ/D is an additional ratio on the right hand side.

If there is preheating ahead of the flame it is represented by an additional distance which can be expressed to a first approximation as having two components - one a constant and another proportional to the appropriate scale length. This can be either X_{po} or ℓ . Since their ratio is already included in the set of independent ratios in the functional equation we can write the heating as taking place over the distance $m+n (X_{fl} - X_p)$ instead of over $X_{fl} - X_p$, so introducing two additional terms $\frac{m}{\ell}$ and n .

The form of this functional relation is, in principle, independent of geometry, e.g. a corner, unless this is characterised by a relevant dimension "D".

3.6. *The upper gas layer temperature*

The McCaffrey, Quintiere, Harkleroad (MQH) regression [29]

$$\frac{\theta}{T_o} \alpha \left[\frac{Q}{\rho_g c_p T_o A_v \sqrt{gH}} \right]^m \left(\frac{h A_T}{\rho_g c_p A_v \sqrt{gH}} \right)^n \quad (33)$$

determines a mean temperature of the upper gas layer in terms of the rate of heat release and various room parameters where the symbols are as defined by McCaffrey, Quintiere and Harkleroad and m is given as 2/3 and n as -1/3.

This was used to correlate mean temperatures in upper gas layers in rooms with length ℓ , breadth "b" and height "s" with a vertical opening of area A_v , and height H . In this situation one assumes that θ depends on T_o , Q , ρ_g , c_p , A_T , A_v , g , H , h , and one presumes on ℓ , s and b as well. We include T_o in the set of variables because physical arguments recognise the role of buoyancy for which we include $\Delta\rho/\rho$ i.e. θ/T_o .

The ratios $\frac{\ell}{s}$, $\frac{b}{s}$, $\frac{H}{s}$, $\frac{H^2}{A_v}$, $\frac{A_T}{A_v}$ are purely geometric and we only need one of the terms to discuss the formation of dimensionless groups so long as we remember to add the others in a functional description for θ . We pick A_v . (It does not matter which at this stage).

Hence we write

$$\theta = Q^a \cdot T_o^\beta (\rho_g c_p)^\gamma A_v^\Delta h^\epsilon g^\mu \text{function} \left(\frac{\ell}{s}, \frac{b}{s}, \frac{H}{s}, \text{etc} \right)$$

N.B. we are assuming a power laws. If not, we express the function as an infinite series of powers. We recognise that only ρ and c_p contain the dimension of mass so we use their product to remove mass.

Comparing indices of

$$\begin{aligned}
&\text{"heat" gives} & \alpha + \varepsilon + \gamma = 0 \\
&\text{"length" gives} & \mu - 3\gamma + 2\Delta - 2\varepsilon = 0 \\
&\text{"time" gives} & -\alpha - \varepsilon - 2\mu = 0 \\
&\text{and "temperature" gives} & \beta - \gamma - \varepsilon = 1
\end{aligned}$$

These give four of the six indices and we arbitrarily pick α and ε as given so that

$$\begin{aligned}
\beta &= 1 - \alpha \\
\gamma &= -[\alpha + \varepsilon] \\
\mu &= -(\alpha - \varepsilon)/2 = \gamma/2 \\
\Delta &= \frac{-5\alpha}{4} - \frac{\varepsilon}{4}
\end{aligned}$$

Hence
$$\frac{\theta}{T_o} \alpha \left(\frac{Q}{\rho_g c_p T_o A_v \sqrt{g A_v^{1/2}}} \right)^\alpha \left(\frac{h A_T}{\rho_g c_p A_v \sqrt{A_v^{1/2}}} \right)^\varepsilon \left(\frac{H^2}{A_v} \right)^d \left(\frac{H^2}{A_T} \right)^f \text{ etc} \quad (34)$$

The indices d, f etc. express only that $\frac{\theta}{T_o}$ may depend on any of the ratios of lengths.

These groups in equation (34) can be multiplied and divided without restriction provided the remaining groups are independent and no variable is lost.

Arguing physically we expect h to be associated with A_T , that " g " is associated with a vertical dimension and θ/T_o is the dependent variable on the left-hand side of the equation. We pick gH instead of $gA_v^{1/2}$. Hence instead of equation (34) we write without loss of generality

$$\frac{\theta}{T_o} = \text{function} \left(\frac{Q}{\rho_g c_p T_o A_v \sqrt{gH}}, \frac{h A_T}{\rho_g c_p A_v \sqrt{gH}}, \frac{A_v}{A_T}, \frac{H^2}{A_v} \text{ etc} \right)$$

We have reduced the 9 variables (plus extra geometric ones) to 2 (plus the extra ones).

There is no formal justification for omitting any of these variables. So how did MQH do so?

For convenience in this discussion we omit from the data any variations in l , s and b (other than in A_v , A_T and H) i.e our hypothesis is

$$\frac{\theta}{T_o} \propto \left(\frac{Q}{\rho_g c_p T_o A_v \sqrt{gH}} \right)^m \left(\frac{h A_T}{\rho_g c_p A_v \sqrt{gH}} \right)^n \left(\frac{A_v}{A_T} \right)^p \left(\frac{H^2}{A_v} \right)^q$$

We accommodate the experimental variation in Q , A_v , H , A_T and write $\theta \propto Q^\alpha A_v^\beta H^\gamma A_T^\Delta$.

With 4 variables we are able to identify values of m , n , p , q with α , β , γ and Δ . The hypothesis in MQH's regression is that -

$$p = q = 0$$

We have for

$$Q, \quad m = \alpha$$

$$A_v, \quad -m - n + p = \beta$$

$$H, \quad \frac{m}{2} - \frac{n}{2} + p = \gamma$$

$$A_T, \quad n = \Delta$$

Therefore

$$p = \alpha + \beta + \Delta$$

and
$$q = \frac{\alpha + \Delta}{4} + \frac{\gamma}{2}$$

From the statistical analysis of the data McCaffery, Quintiere and Harkleroad presumably found that any values of p and q were not significant. More data would be required to make more sensitive tests. A more physically based approach is possible.

A crude energy balance is

$$Q \approx ahA_T\theta + b\rho_g c_p A_v \omega \theta$$

where " a " and " b " are unknown but constant coefficients, " ω " is a mean velocity which might be characterised by either

$$\sqrt{gH\theta/T_o} \text{ or } \left(\frac{gQ'}{\rho_g c_p T_o} \right)^{1/3}$$

In principle, ω depends on the depth of the hot layer for it extends below the top of a doorway, the gas temperature etc. These are accommodated in the functional relationship. This can be represented generally as

$$\frac{\omega}{\sqrt{gH}} = \text{function} \left\{ \frac{\theta}{T_o}, \left(\frac{gQ}{\rho_g c_p T_o} \right)^{1/3} / \sqrt{gH}, \frac{A_v}{H^2} \text{ etc} \right\}$$

Hence

$$\frac{Q}{h_k A_T T_o} = \text{function} \left\{ \frac{\theta}{T_o}, \frac{hA_T}{\rho_g c_p A_o \sqrt{gH}}, \left(\frac{gQ}{\rho_g c_p T_o} \right)^{1/3} / \sqrt{gH}, \frac{A_v}{H^2} \text{ etc} \right\}$$

Rearranging, we have

$$\frac{\theta}{T_o} = \text{function} \left\{ \frac{Q}{\rho_g c_p A_v \sqrt{gH}}, \frac{hA_T}{\rho_g c_p A_v \sqrt{gH}} \left(\frac{Q}{\rho_g c_p T_o H^2 \sqrt{gH}} \right), \frac{A_v}{H^2} \text{ etc} \right\}$$

The third variable can be replaced by the first and A_v/H^2 and so is redundant.

The above discussion shows a little of the relationship between dimensional analysis and physical argument. Important for this and other examples is the recognition of the limitation of power laws: a power law as in equation (33) is nonsense for $h_k \rightarrow 0$, but one can often produce approximations over a limited range between linear and power laws especially for this problem [30].

Thus, if the second term on the right hand side of equation (33) is written a $X^{-1/3}$ it can be replaced by $\left(\frac{1+X}{2} \right)^{-2/3}$ matched to be exact at $X=1$ and 4% less at $X=2$ and $X=1/2$ an error of $\pm 2\%$ over a range of 4 to 1 in X . Without a detailed analysis of the actual data one cannot "a priori" say that a correlation based on

$$1 + \frac{hA_T}{\lambda \rho c_p \sqrt{gH}}$$

with λ a disposable constant would be superior to the power law. Additional "error" may be acceptable in view of the removal of the singularity when $h_k A_T \rightarrow 0$.

4 The analysis of data

Analysing data statistically is fast becoming routine, but the principles underlying such analyses are, it seems, slowly being forgotten by many practitioners. Once such lack of understanding will be described below. Consider the two equations

$$Z = a \cdot W^{\alpha} \cdot x^{\beta} \cdot y^{\gamma} \quad (35i)$$

where Z , W , x and y are measurements.

and
$$\frac{Z}{W} = b \left(\frac{x}{W} \right)^{\Delta} \left(\frac{y}{x} \right)^{\varepsilon} \quad (35ii)$$

where $\frac{Z}{W}$, $\frac{x}{W}$ and $\frac{y}{x}$ are dimensionless is a hypothesis

The advantage of dimensionless analysis is the grouping of quantities which necessarily reduces the three indices α , β and γ to two, in other examples Δ and ε to one.

This does not alter the fact that error is attached to the measurements so that statistical analysis should be undertaken on equation (35i) not equation (35ii). It is a hypothesis that equation (35i) can be rewritten as equation (35ii), a hypothesis which, if possible, has to be justified or not contradicted by the data.

Comparing the indices

$$\beta = \Delta - \varepsilon$$

$$\gamma = \varepsilon$$

$$\alpha = 1 - \Delta$$

one requires

$$\alpha + \beta + \gamma - 1 = 0$$

This constraint is implied by the reduction of one in the number of degrees of freedom.

There are, as we shall see, more complex situations where more than one constraint can be derived. Since the original statistical analysis defines α , β and γ and their covariances and variances, one can establish the variance of $\alpha + \beta + \gamma$ to see if $\alpha + \beta + \gamma$ differs significantly from one.

Alternatively one could analyse

$$\frac{Z}{W} = b \left(\frac{x}{W} \right)^\Delta \left(\frac{y}{x} \right)^\epsilon W^\theta$$

or some extra non-dimensionless variable to see if θ exists.

These treatments of the data cannot do more than demonstrate consistency. They cannot prove anything except that either because of a shortage of data or too great a variability in a data one cannot justify a re-arrangement in dimensionless form from the data, whatever theory suggests.

One common error arises from recognising that w , x and y can be made dimensionless by incorporating say, u , etc., e.g.

$$\frac{Z}{x} \propto \left(\frac{x}{u^n} \right)^\alpha \left(\frac{y}{v^m} \right)^\beta \left(\frac{W}{\mu^p} \right)^\gamma$$

where α , β , n , m , p are coefficients. However, if there are no variations in u , v or μ , the reformulation cannot add any confidence that the original data can be part of dimensionless correlation as physically there should be.

A recently published paper correlated one dimensionless variable y against three energy flows: E_1 , E_2 and E_3 . What is suggested above requires a regression of y against E_1 , E_2 and E_3 say,

$$y \propto E_1^a, E_2^\beta, E_3^\gamma.$$

Instead the authors correlated

$$y \propto \left(\frac{E_1 E_2}{E_3^2} \right)^n$$

and so lost the possibility of checking whether $(\alpha-\beta)$ exists and whether $a+\gamma-2\beta$ exists. Perhaps they did justify their correlation, but if so they left it out of their paper.

Clearly if the checks were not satisfied it would demonstrate either that the data were not consistent with the simpler reformation or that there were insufficient data. If they were satisfied, constancy over the range of the data is demonstrated; but consistency is just that, not proof.

5 Conclusions

Various fire safety engineering questions have been discussed from the point of view of a dimensional analysis and in the course of this examples have been given of

- (i) the use of dimensional analysis (as apposed to non-dimensional numbers, fractions etc.)
- (ii) the combination of dimensionless groups as a result of physical arguments
- (iii) their use in the formulation of a solution of a differential equation with its boundary or initial conditions
- (iv) choosing one of two alternative formulations of a dimensionless variable $\frac{RT}{E}$ or $\frac{E}{RT^2}(T - T_o)$
- (v) the consequences of the difference between dependent and independent variables
- (vi) their use in evaluating a quantity (θ_{ig})
- (vii) determining the structure of a formula as a result of physical considerations (bent over plumes and flame lengths)
- (viii) the choice and significance of a characteristic length for inclusion in a dimensionless variable k_g/v_a in v_ax/k_g and ℓ in the Delichatsios-Saito length)
- (ix) the analysis of measurements of quantities claimed to be a part of a dimensionless variable.

These matters have been mentioned as they have arisen in the examples discussed in the hope that the reader feels the art is not such a mystery as perhaps once thought.

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Appendix I

The constancy of the heat transfer coefficient

If in a dimensional analysis h is to be treated as depending on the surface temperature then we first make use of the ratio of the last two terms in equation (18) which define hl/K so that equation (18) can be rewritten with h present in only one term, v.i.z.

$$\frac{K\theta_{(x)}}{q''L} = function \left(\frac{x}{L}, \frac{Kt}{\rho c_p L^2}, \frac{h^2 \tau}{K \rho c_p}, \frac{K\tau}{\rho c_p L^2} \right)$$

$$\text{Hence } \frac{K\theta_{x=o}}{q''L} = function \left(\frac{kt}{\rho c_p L^2}, \frac{h^2 \tau}{K \rho c_p}, \frac{k\tau}{\rho c_p L^2} \right)$$

If now we write as

$$h = h_o f(\theta/\theta_o)$$

by introducing h_o a constant characteristic value of h and a characteristic temperature rise θ_o equation (18) becomes

$$\frac{K\theta(x)}{q''L} = function \left(\frac{x}{L}, \frac{Kt}{\rho c_p L^2}, \frac{h_o^2 \tau}{K \rho c_p}, \frac{K\tau}{\rho c_p L^2}, \frac{K\theta_o}{q''L} \right)$$

If radiation is included in $f(\theta/\theta_o)$ then the absolute surface temperature has to be incorporated, i.e. a new dimensionless variable, e.g. $\frac{KT_o}{q''L}$ must be included.

Appendix 2

We consider equations (26) and (27)

$$\begin{aligned} M'' &= M_o'' & t < t_B \\ &= 0 & t > t_B \end{aligned}$$

and a modification of equation (28) viz

$$X_{fl} - X_{R=} = k_l \left[Q' + qM_o'' (X_p + X_R) \right]^{2/3}$$

i.e. we use the simplest form of the pyrolysis or heat release rate but employ the 2/3 power law.

With
$$\tau = \frac{t_{ig} t}{1 + t_{ig}/t_B}$$

and
$$Y = \frac{(X_p - X_R) (1 + t_{ig}/t_B)^3}{K^3 (qM'')^2}$$

we obtain

$$\frac{dY}{d\tau} = (E + Y)^{2/3} - Y$$

where
$$E = \frac{Q(1 + t_{ig}/t_B)^3}{(KqM'')^3}$$

Notation

a	a constant
A	area, A_v - ventilation area, A_τ envelope surface area, a constant
b	breadth or radius
c_p	specific heat
d	a distance, an index
D	a distance = burner diameter ($=2b_o$)
e	an index
E	energy, activation energy, a dimensionless parameter
f	an index, frequency factor
g	acceleration due to gravity
h	heat transfer coefficient
ΔH	chemical heat quantity, eg. ΔH_c - calorific value, ΔH_v - heat vaporisation
H	height of opening, H_B - buoyancy height
k	thermal diffusivity, $K/\rho c_p$
K	thermal conductivity, K_n - flame length constant
ℓ	a length, l_f - flame length
L	a distance, thickness
m	an index
M	mass flux, a dimensionless constant
n	an index
N_{Re}	Reynolds number
N_{Nu}	Nusselt number
N_{Pr}	Prandtl number
N_{Fr}	Froude number
ϕ	an index
q	heat quantity, q''' is rate of heat release per unit volume, \dot{q}'' or q'' rate of heat transfer per unit area
Q	convection heat flux, rate of heat release from chemical energy
R	universal gas constant

s	compartment height
S	surface area or $\left(Q/\rho_o c_p T_o \sqrt{g}\right)^{2/5}$
t	time
T	temperature
u	a variable
U	velocity
v	a variable
v_a	velocity of air
V	volume, rate of flame spread
ω	velocity
W	a variable
x	distance, a variable
X	distance to a moving front, a variable
y	distance, a variable
Y_{ox}	concentration of oxygen - a variable
z	height
Z	$\frac{Kt}{\rho c_p x^2}$, a variable
α	an index
β	an index
γ	an index
δ	a depth, distance, dimensionless thermal explosion or self heating parameter
Δ	an index, a distance
ε	an index, eddy diffusivity
θ	temperature difference - an index
λ	a constant
μ	viscosity, an index, a variable
ν	kinematic viscosity μ/ρ
ρ	density
$\Delta\rho$	density difference
τ	a characteristic time

Suffices and affices

o	initial, ambient
B	burner
ig	ignition
fl	flame
ff	far field
g	gas
c	characteristic value
p	pyrolysis
s	surface, solid
m	maximum
∇	per unit volume
\angle	per unit area
\notin	per unit length/width