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ON THE UNIQUENESS OF MAXIMUM LIKELIHOOD IDENTIFICATION FOR DIFFERENT STRUCTURES

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T. Söderström

ABSTRACT.

Maximum likelihood identification of a linear dynamic system is performed as a minimization of a loss function. The concept of uniqueness of the parameter estimates is closely related to the number of local minimum points of this loss function. The number of local minimum points is examined for some different models. Asymptotic expressions for the loss function are used. Conditions are given which imply a unique local minimum point.

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I. INTRODUCTION.

The maximum likelihood (ML) method is a useful tool for estimation of parameters in system equations. The ML estimate $\hat{\theta}_{ML}$ is the global maximum point of the likelihood function $L(\hat{\theta})$, i.e.

$$L(\hat{\theta}_{ML}) \geq L(\hat{\theta}) \text{ all } \hat{\theta}$$

In most cases there is no analytical expression for the maximum point $\hat{\theta}_{ML}$. The maximization of $L(\hat{\theta})$ has to be done computationally using some search routine. Such a search routine may converge to a local maximum point θ^* of $L(\hat{\theta})$, i.e.

$$L(\theta^*) \geq L(\hat{\theta}) \quad \text{all } \hat{\theta} \text{ close to } \theta^*$$

It is then valuable to know if the likelihood function has a unique local maximum point or not.

This issue is closely related to the concept of identifiability, see Bellman-Åström (1970). The purpose of this report is to analyze the local maximum points of the likelihood function for some different structures. Bohlin (1971) has given some tests, which can be used for detecting if a local maximum or generally an arbitrary point is a global maximum point or not.

The report is organized as follows: In this chapter some basic assumptions are given. In the next chapter the mathematical tools of the analysis are penetrated. Chapter III contains an examination of the global maximum points for the different structures. It is desirable that the true value θ is a global maximum point and preferably a unique one.

Moreover, this examination simplifies the analysis of the local maximum points, since it describes all "desirable" points. The last three chapters deal with the examination of the local maximum points for some specific likelihood functions.

Consider a system given by

$$y(t) = G(\theta;q^{-1})u(t) + H(\theta;q^{-1})e(t)$$

where

$$G(\theta;q^{-1}) = \sum_{0}^{\infty} g_i(\theta)q^{-i}$$

$$H(\theta;q^{-1}) = \sum_{0}^{\infty} h_i(\theta)q^{-i}$$

$u(t)$ is the input, $y(t)$ the output and $e(t)$ gaussian white noise with zero mean and standard deviation λ. q^{-1} is the backward shift operator. It is assumed that $h_0(\theta) \equiv 1$. The system can be illustrated by the figure below.

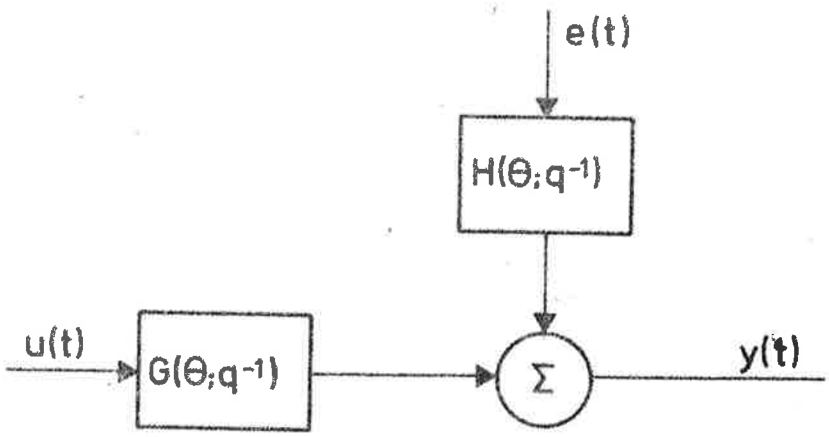

Figure 1 - Block diagram of the system.

The purpose of an identification is to estimate the value of the vector θ based on an input-output record. The true value will be denoted by θ.

In this report some different transfer functions G and H will be considered. It is assumed that G and H are rational functions in q^{-1}. The coefficients are functions of θ.

Under these assumptions the maximization of the likelihood function is equivalent to the minimization of the loss function, see Åström-Bohlin (1966).

$$V(\hat{\theta},\theta) = \frac{1}{2N} \sum_{t=1}^{N} \varepsilon^2(t) \tag{1.1}$$

where the residuals $\varepsilon(t)$ are defined by

$$y(t) = G(\hat{\theta};q^{-1})u(t) + H(\hat{\theta};q^{-1})\varepsilon(t) \tag{1.2}$$

while the output is given from

$$y(t) = G(\theta;q^{-1})u(t) + H(\theta;q^{-1})e(t) \tag{1.3}$$

The ML estimate $\hat{\theta}_{ML}$ of θ is thus given by

$$V(\hat{\theta}_{ML},\theta) = \min_{\hat{\theta}} V(\hat{\theta},\theta)$$

assuming that a global minimum exists.

The residuals can be written as

$$\varepsilon(t) = \frac{G(\theta;q^{-1}) - G(\hat{\theta};q^{-1})}{H(\hat{\theta};q^{-1})} u(t) + \frac{H(\theta;q^{-1})}{H(\hat{\theta};q^{-1})} e(t) \tag{1.4}$$

In the analysis of the loss function (1.1) ergodic theory will be used.

The generalized least squares method has been treated elsewhere by the author in Söderström (1972), where it is shown that the loss function in this case has a unique local minimum point when the signal to noise ratio is high enough. For small values of this ratio there may exist several local minimum points.

For the other cases treated here it is shown (under suitable assumptions) that all local minimum points are global minimum points. There will be a unique global (and local) minimum point if the correct order of the transfer functions is used.

II. MATHEMATICAL PRELIMINARIES.

In this chapter the basic mathematical tools for the analysis of the loss functions are given.

First some conventions used in the report are presented. Then some polynomial equations are studied. A lemma giving sufficient conditions for the existence of

$$\lim_{N\to\infty} V(\hat{\theta},\theta)$$

is considered. Finally the concept of persistently exciting signals is treated and some applications are made. Some of the lemmas are given in Söderström (1972). They are stated here too in order to clarify their use in the analysis.

In order to simplify the notations the following conventions will be used throughout the report.

<u>Convention 2.1.</u> Polynomial operators will be denoted by capital letters, e.g. $A(q^{-1})$. The number of coefficients will be denoted by n or $\hat{n}$ with a corresponding lower case letter as a subscript.

<u>Examples</u>:

$$A(q^{-1}) = 1 + \sum_{1}^{n_a} a_i q^{-i} \qquad \hat{A}(q^{-1}) = 1 + \sum_{1}^{\hat{n}_a} \hat{a}_i q^{-i}$$

$$B(q^{-1}) = \sum_{1}^{n_b} b_i q^{-i} \qquad \hat{B}(q^{-1}) = \sum_{1}^{\hat{n}_b} \hat{b}_i q^{-i}$$

The expression

$$\sum_{1}^{n_a} a_i q^{-i}$$

is interpreted as zero if $n_a = 0$.

Convention 2.2. Given two polynomials

$$A(z) = \sum_{i=0}^{n_a} a_i z^i \qquad B(z) = \sum_{i=0}^{n_b} b_i z^i$$

the notation $A(z) \equiv B(z)$ means

$$a_i = b_i \qquad 0 \leq i \leq \min(n_a, n_b)$$

and

$$\text{if } n_a > n_b \qquad a_i = 0 \qquad n_b < i \leq n_a$$
$$\text{if } n_b > n_a \qquad b_i = 0 \qquad n_a < i \leq n_a$$

Convention 2.3. Given the polynomials A(z) and B(z) (and C(z)). They are said to be relatively prime if there is no common factor to all the polynomials. The physical interpretation is that the system

$$A(q^{-1})y(t) = B(q^{-1})u(t)$$

$$(A(q^{-1})y(t) = B(q^{-1})u(t) + C(q^{-1})e(t))$$

is controllable and observable.

Convention 2.4. $\mathcal{E}x(t)$ denotes

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} x(t)$$

If x(t) is an ergodic stochastic process $\mathcal{E}x(t) = Ex(t)$. All stochastic processes in this report are ergodic. The notation is used for deterministic signals as well.

The following elementary two lemmas from the theory of equations will be useful. The proofs are not very difficult and they are given here.

The first lemma deals with an equation, which will occur several times in the forthcoming analysis.

Lemma 2.1. Given the polynomials

$$A(z) = 1 + \sum_{i=1}^{n_a} a_i z^i$$

and

$$B(z) = \sum_{i=1}^{n_b} b_i z^i$$

Consider the following equation in the unknowns $(\hat{a}_1, \ldots, \hat{a}_{\hat{n}_a}, \hat{b}_1, \ldots, \hat{b}_{\hat{n}_b})$ with $n_\ell = \min(\hat{n}_a - n_a, \hat{n}_b - n_b) \geq 0$

$$\hat{A}(z)B(z) - A(z)\hat{B}(z) \equiv 0 \tag{2.1}$$

Assume that A(z) and B(z) are relatively prime.

i) If $n_\ell = 0$ the only solution is given by

$$\hat{A}(z) \equiv A(z)$$
$$\hat{B}(z) \equiv B(z) \qquad (2.2)$$

ii) If $n_\ell > 0$ all solutions are given by

$$\hat{A}(z) \equiv A(z)L(z)$$
$$\hat{B}(z) \equiv B(z)L(z) \qquad (2.3)$$

where

$$L(z) = 1 + \sum_{i=1}^{n_\ell} \ell_i z^i$$

The coefficients $(\ell_i)_1^{n_\ell}$ are arbitrary.

<u>Proof</u>. Since $A(z) \not\equiv 0$, $\hat{A}(z) \not\equiv 0$ the equation can be written

$$\frac{B(z)}{A(z)} = \frac{\hat{B}(z)}{\hat{A}(z)} \quad \text{all } z$$

Noting that the right hand side must have the same zeros and poles as the left hand side the assertions are obvious.

Q.E.D.

<u>Corr</u>. If B(z) is of the form

$$B(z) = 1 + \sum_{1}^{n_b} b_i z^i$$

and $\hat{B}(z)$ of the form

$$\hat{B}(z) = 1 + \sum_{1}^{\hat{n}_b} \hat{b}_i z^i$$

the lemma remains true without changes.

<u>Lemma 2.2</u>. Consider the following matrix of order $\max(\hat{n}_a+n_b,\ n_a+\hat{n}_b) \times (\hat{n}_a+\hat{n}_b)$

$$P = \left[\begin{array}{cccc|cccc}
0 & & & & 1 & & & \\
b_1 & & 0 & & \cdot & \ddots & 0 & \\
\vdots & \ddots & & & \vdots & & \ddots & \\
 & & & & \cdot & & & 1 \\
b_{n_b} & & \ddots & & a_{n_a} & & & \cdot \\
 & \ddots & & b_1 & & \ddots & & \vdots \\
 & 0 & \ddots & \vdots & & 0 & \ddots & \vdots \\
 & & & b_{n_b} & & & & a_{n_a}
\end{array}\right] \qquad (2.4)$$

$$\underbrace{\hphantom{xxxxxxxx}}_{\hat{n}_a \text{ columns}} \qquad \underbrace{\hphantom{xxxxxxxx}}_{\hat{n}_b \text{ columns}}$$

(At least one of the figures b_{n_b}, a_{n_a} is on the last row.)

Let $A(z)$ and $B(z)$ have m common zeros. Assume that $\hat{n}_a \geq n_a$, $\hat{n}_b \geq n_b$.

Then rank $P = \max(\hat{n}_a+n_b,\ n_a+\hat{n}_b) - m$.

<u>Proof</u>. Consider the equation

$$\hat{A}(z)B(z) - A(z)\hat{B}(z) \equiv 0$$

From lemma 2.1 it is known that the general solution is of the form

$$\hat{A}(z) \equiv \bar{A}(z)L(z)$$

$$\hat{B}(z) \equiv \bar{B}(z)L(z)$$

where

$\bar{A}(z)$ and $\bar{B}(z)$ are relatively prime

$$L(z) = 1 + \sum_{1}^{n_\ell} \ell_i z^i$$

$$n_\ell = \min(\hat{n}_a - n_a, \hat{n}_b - n_b) + m$$

Introduce new variables $c_1 \ldots c_{\hat{n}_a}$, $d_1 \ldots d_{\hat{n}_b}$ by

$$C(z) = \sum_{1}^{\hat{n}_a} c_i z^i \equiv \hat{A}(z) - \bar{A}(z)$$

$$D(z) = \sum_{1}^{\hat{n}_b} d_i z^i \equiv \hat{B}(z) - \bar{B}(z)$$

The equation is then

$$C(z)B(z) - A(z)D(z) \equiv 0$$

with the general solution

$$C(z) \equiv \bar{A}(z)\big(L(z) - 1\big)$$

$$D(z) = \bar{B}(z)\big(L(z) - 1\big)$$

However, this equation can be written as

$$P \cdot \begin{bmatrix} c_1 \\ \vdots \\ c_{\hat{n}_a} \\ -d_1 \\ \vdots \\ -d_{\hat{n}_b} \end{bmatrix} = 0$$

The expression of the general solution implies that
$\dim N(P) = n_\ell$.
Thus rank P is given by

$$= \dim R(P^T) = \hat{n}_a + \hat{n}_b - \dim N(P)$$

$$= \hat{n}_a + \hat{n}_b - \min(\hat{n}_a - n_a,\ \hat{n}_b - n_b) - m$$

$$= \max(\hat{n}_a + n_b,\ n_a + \hat{n}_b) - m$$

Q.E.D.

Remark. In the case $\hat{n}_a = n_a$, $\hat{n}_b = n_b$ (P is square) P is nonsingular if and only if $m = 0$. This fact is already shown by e.g. Dickson (1922). In this report, however, the general case will be needed.

In the analysis of the loss functions ergodic expressions will be used. The loss functions are all of the form

$$\frac{1}{2N} \sum_{t=1}^{N} \varepsilon^2(t)$$

with $\varepsilon(t)$ given by (1.4). The following lemma gives sufficient conditions for convergence of such expressions.

Lemma 2.3. Consider the system

$$y(t) = G(q^{-1})u(t) + H(q^{-1})e(t)$$

where $G(q^{-1})$ and $H(q^{-1})$ are asymptotically stable filters of finite orders, and $e(t)$ is white noise with finite fourth moment, independent of $u(t)$.

The input $u(t)$ is the sum of two terms, $u_1(t)$ and $u_2(t)$, of which one may vanish. The term $u_1(t)$ is deterministic such that to every $\varepsilon > 0$ there is a periodic function $u_1'(t)$ fulfilling

$$|u_1(t) - u_1'(t)| < \varepsilon \quad \text{all } t$$

The second term is given by

$$u_2(t) = F(q^{-1})v(t)$$

where $F(q^{-1})$ is an asymptotically stable filter of finite order and $v(t)$ white noise with finite fourth moment.

Let $D_1(q^{-1})$ and $D_2(q^{-1})$ be two arbitrary asymptotically stable filters of finite order. Then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} [D_1(q^{-1})y(t) + D_2(q^{-1})u(t)] \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} \tag{2.5}$$

exists with probability one and in mean square.

If $u(t)$ and $y(t)$ are stochastic processes the limit is

$$E[D_1(q^{-1})y(t) + D_2(q^{-1})u(t)] \begin{bmatrix} y(t) \\ u(t) \end{bmatrix}$$

Proof. See Söderström (1972).

The notion of persistent excitation introduced in Åström--Bohlin (1966) is very useful in the analysis of the loss function.

Definition 2.1. u(t) is said to be persistently exciting of order n if

i) $$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} u(t) = \bar{u} \quad \text{and}$$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} [u(t) - \bar{u}][u(t+\tau) - \bar{u}] = r_u(\tau)$$

exist and

ii) the n by n symmetric matrix

$$R_u = \begin{bmatrix} r_u(0) & r_u(1) & \cdots & r_u(n-1) \\ & \ddots & & \vdots \\ & & & \\ & & & r_u(0) \end{bmatrix}$$

is positive definite.

Some simple properties of persistently exciting signals and a characterization of this concept in the frequency domain is given in Ljung (1971). In this report the following properties will be used [proved in Ljung (1971)].

Lemma 2.4. u(t) is persistently exciting of order n if and only if the spectral density corresponding to the sample covariance function is non zero (in distributive sense) in at least n different points.

If $u(t)$ is periodic, the spectral density will be discrete and consist of a number of δ-functions. The distribution $\delta(x)$ is here considered as non zero in $x = 0$.

Corr. Let $y(t) = H(q^{-1})u(t)$. If $u(t)$ is persistently exciting of order n and $H(q^{-1})$ is stable and has no zeros on the unit circle, then $y(t)$ is persistently exciting of order n.

A simple application is made in

Lemma 2.5. Let

$$y(t) = H(q^{-1})u(t)$$

$$H(q^{-1}) = \sum_{i=0}^{n-1} h_i q^{-i}$$

i) If $y(t) \equiv 0$ with probability one and $u(t)$ is persistently exciting of order n, then $h_i = 0$, $i = 0,\ldots,n-1$.

ii) If $u(t)$ is not persistently exciting of order n, then there exists $H(q^{-1}) \not\equiv 0$ such that $y(t) \equiv 0$ with probability one.

Proof. See Söderström (1972).

A combination of Lemma 2.1 and Lemma 2.5 gives a further result.

Lemma 2.6. Given $A(q^{-1})$, $B(q^{-1})$ and $u(t)$. Assume that $A(q^{-1})$ and $B(q^{-1})$ are relatively prime and that $n_\ell = \min(\hat{n}_a - n_a, \hat{n}_b - n_b) \geq 0$.
Consider the equation:

$$[\hat{A}(q^{-1})B(q^{-1}) - A(q^{-1})\hat{B}(q^{-1})]u(t) = 0 \quad \text{a.s.} \tag{2.6}$$

Let $m = \max(\hat{n}_a + n_b,\ n_a + \hat{n}_b)$.

i) If $u(t)$ is persistently exciting of order m the general solution is given by

$$\begin{cases} \hat{A}(q^{-1}) \equiv A(q^{-1})L(q^{-1}) \\ \hat{B}(q^{-1}) \equiv B(q^{-1})L(q^{-1}) \end{cases} \tag{2.7}$$

where

$$L(q^{-1}) = 1 + \sum_{1}^{n_\ell} \ell_i q^{-1} \qquad \text{if } n_\ell \geq 1$$

$$1 \qquad \text{if } n_\ell = 0$$

The numbers ℓ_i are arbitrary.

ii) If $u(t)$ is not persistently exciting of order m there is at least one more solution of (2.6) than (2.7).

Proof. If $u(t)$ is persistently exciting of order m it follows from Lemma 2.5 that

$$\hat{A}(q^{-1})B(q^{-1}) - A(q^{-1})\hat{B}(q^{-1}) \equiv 0$$

The general solution is then obtained from Lemma 2.1.

If u(t) is not persistently exciting of order m, Lemma 2.5 implies the existence of

$$H(q^{-1}) = \sum_{1}^{m} h_i q^{-i} \not\equiv 0$$

such that

$$H(q^{-1})u(t) = 0$$

Writing the equation

$$\hat{A}(q^{-1})B(q^{-1}) - A(q^{-1})\hat{B}(q^{-1}) \equiv H(q^{-1})$$

and invoking Lemma 2.2 the assertion ii) follows.

Q.E.D.

The concept of persistent excitation is now applied to a matrix consisting of covariances of the input and the output.

Definition 2.1. Let $y(t) = G(q^{-1})u(t)$. The following matrix of order $(m_a+m_b) \times (m_a+m_b)$ will be called the system covariance matrix of type (m_a, m_b).

$$R = \left[\begin{array}{ccc|ccc} r_y(0) & \cdots & r_y(m_a-1) & -r_{yu}(0) & \cdots & -r_{yu}(m_b-1) \\ \vdots & & \vdots & \vdots & & \vdots \\ r_y(m_a-1) & \cdots & r_y(0) & -r_{yu}(1-m_a) & \cdots & -r_{yu}(m_b-m_a) \\ \hline -r_{yu}(0) & \cdots & -r_{yu}(1-m_a) & r_u(0) & \cdots & r_u(m_b-1) \\ \vdots & & \vdots & \vdots & & \vdots \\ -r_{yu}(m_b-1) & \cdots & -r_{yu}(m_b-m_a) & r_u(m_b-1) & \cdots & r_u(0) \end{array}\right]$$

Lemma 2.7. Let

$$y(t) = \frac{B(q^{-1})}{A(q^{-1})} u(t)$$

where $A(q^{-1})$ and $B(q^{-1})$ are relatively prime. Consider the system covariance matrix R of type (m_a, m_b). Assume that u(t) is persistently exciting of order $\max(m_a+n_b, n_a+m_b)$ and let $n_\ell = \min(m_a-n_a, m_b-n_b)$.

i) Then R is positive definite if and only if $n_\ell \leq 0$.

ii) If $n_\ell > 0$ the null space of R has dimension n_ℓ and is spanned by vectors of the following form:

$[c_1 \ldots c_{m_a}, d_1 \ldots d_{m_b}]^T$ with

$$C(q^{-1}) = \sum_1^{m_a} c_i q^{-i} \equiv A(q^{-1})L(q^{-1})$$

$$D(q^{-1}) = \sum_1^{m_b} d_i q^{-i} \equiv B(q^{-1})L(q^{-1})$$

$$L(q^{-1}) = \sum_1^{n_\ell} \ell_i q^{-i}$$

The numbers ℓ_i are arbitrary.

Proof. In order to investigate the null space of R consider the equation

$$x^T R x = 0 \tag{2.8}$$

Let $x^T = [c_1 \; \ldots \; c_{m_a} \; d_1 \; \ldots \; d_{m_b}]$ and introduce the corresponding operators

$$C(q^{-1}) = \sum_1^{m_a} c_i q^{-i}, \quad D(q^{-1}) = \sum_1^{m_b} d_i q^{-i}$$

Then

$$x^T R x = E\Big[[-y(t-1) \; \ldots \; -y(t-m_a) u(t-1) \; \ldots \; u(t-m_b)]x\Big]^2$$

$$= E\big[-C(q^{-1})y(t) + D(q^{-1})u(t)\big]^2$$

The equation (2.8) is thus equivalent to

$$\frac{C(q^{-1})B(q^{-1}) - D(q^{-1})A(q^{-1})}{A(q^{-1})} u(t) = 0 \text{ a.s.}$$

From Lemma 2.4 Corr and Lemma 2.6 it follows that this equation can be replaced by

$$C(q^{-1})B(q^{-1}) - D(q^{-1})A(q^{-1}) \equiv 0$$

Using new variables given by

$$\hat{A}(q^{-1}) = 1 + \sum_1^{\hat{n}_a} \hat{a}_i q^{-i} = A(q^{-1}) + C(q^{-1}); \; \hat{n}_a = \max(m_a, n_a)$$

$$\hat{B}(q^{-1}) = \sum_1^{\hat{n}_b} \hat{b}_i q^{-i} = B(q^{-1}) + D(q^{-1}); \; \hat{n}_b = \max(m_b, n_b)$$

the equation is written as

$$\hat{A}(q^{-1})B(q^{-1}) - A(q^{-1})\hat{B}(q^{-1}) \equiv 0$$

From Lemma 2.1 itthus follows that:

i) if $n_\ell \leq 0$ $x = 0$ is the only solution,

ii) if $n_\ell > 0$ the general solution is given by

$$C(q^{-1}) \equiv A(q^{-1})L(q^{-1})$$

$$D(q^{-1}) \equiv B(q^{-1})L(q^{-1})$$

$$L(q^{-1}) = \sum_{1}^{n_\ell} \ell_i q^{-i}$$

where the numbers ℓ_i are arbitrary.
As a consequence N(R) has dimension n_ℓ.

Q.E.D.

The following two lemmas were originally used in the author's previous work, Söderström (1972), where also proofs can be found.

Lemma 2.8. Consider the equation

$$F(x) \equiv f(x) + \varepsilon g(x) = 0 \tag{2.9}$$

where dim f = dim g = dim x. Let Ω denote a set with the following properties:

f and g are twice differentiable,
$f(x) = 0$ implies $x = x_0$,
$f'(x_0)$ is non singular.

Then there is a $\varepsilon_1 > 0$ such that $0 < \varepsilon \leq \varepsilon_1$ implies that (2.9) has a unique solution $\bar{x}$ in Ω. $\bar{x}$ fulfils

$$\bar{x} - x_0 = O(\varepsilon), \qquad \varepsilon \to 0$$

<u>Lemma 2.9</u>. Consider the function

$$V(x,y,\varepsilon) = \frac{1}{2}\, x^T P(y) x + \varepsilon h(x,y)$$

where (x,y) belongs to a set Ω, for which $P(y)$ is a positive definite matrix for all y, twice differentiable with respect to y and $h(x,y)$ a twice differentiable function. ε is considered as a fix parameter.

The following necessary and sufficient conditions for local minimum points in Ω are true.

There is a constant $\varepsilon_0 > 0$ such that if $0 < \varepsilon \leq \varepsilon_0$ the following is true.

i) Every stationary point of $V(x,y,\varepsilon)$ in Ω fulfils

$$(x,y) = (0,y_0) + \big(0(\varepsilon), o(1)\big), \quad \varepsilon \to 0 \tag{2.10}$$

where y_0 is a solution of

$$h'_y(0,y) = 0 \tag{2.11}$$

If (x,y) is a local minimum point it is necessary that $h''_{yy}(0,y_0)$ is positive definite or positive semidefinite.

ii) If y_0 is a solution of (2.11) and $h''_{yy}(0,y_0)$ is positive definite then there <u>exists</u> a <u>unique</u> local minimum of the form (2.10), and the point will in fact satisfy

$$(x,y) = (0,y_0) + \big(0(\varepsilon), 0(\varepsilon)\big), \quad \varepsilon \to 0$$

The matrix of second order derivatives is positive definite in the minimum point.

III. GLOBAL MINIMUM POINTS FOR DIFFERENT STRUCTURES.

In this chapter the global minimum points of loss functions of the type

$$\begin{cases} V(\hat{\theta},\theta) = \frac{1}{2N} \sum_{t=1}^{N} \varepsilon^2(t) \\ \\ \varepsilon(t) = \frac{G(\theta;q^{-1}) - G(\hat{\theta};q^{-1})}{H(\hat{\theta};q^{-1})} u(t) + \frac{H(\theta;q^{-1})}{H(\hat{\theta};q^{-1})} e(t) \end{cases} \quad (3.1)$$

are analyzed.

For finite N the analysis has to be done in a probabilistic setting. In order to do the analysis reasonable ergodic theory will be used.

The following assumptions are made:

- Let Ω = {θ; such that the poles of $G(\theta;z)$, the poles of $H(\theta;z)$ and the zeros of $H(\theta;z)$ are outside the circle $|z| = 1 + \varepsilon$, where $\varepsilon > 0$ is some small number}. It is assumed that $\theta \in \Omega$ and only points $\hat{\theta}$ in the set Ω are considered. This limitation is motivated from the representation theorem, Åström (1970), and the demand of a finite variance of the output.

- The input is assumed to be a periodic signal or filtered white noise (or a sum of these two types).

- The input signal and the noise $e(t)$ are independent.

Under these assumptions it follows from Lemma 2.3 that $V(\hat{\theta},\theta)$ has a limit $W(\hat{\theta},\theta)$ (with probability one and in mean square) as N tends to infinity. The function $W(\hat{\theta},\theta)$ is given by

$$W(\hat{\theta},\theta) = \frac{1}{2}\,\mathcal{E}\left[\frac{G(\theta;q^{-1}) - G(\hat{\theta};q^{-1})}{H(\hat{\theta};q^{-1})}\,u(t)\right]^2 +$$

$$+\ \frac{1}{2}\ E\left[\frac{H(\theta;q^{-1})}{H(\hat{\theta};q^{-1})}\,e(t)\right]^2 \tag{3.2}$$

Let

$$\tilde{H}(q^{-1}) = \frac{H(\theta;q^{-1})}{H(\hat{\theta};q^{-1})} = 1 + \sum_{i=1}^{\infty} \tilde{h}_i q^{-i}$$

Then

$$W(\hat{\theta};\theta) \geq \frac{1}{2}\ E\left[e(t) + \sum_{i=1}^{\infty} \tilde{h}_i e(t-i)\right]^2 =$$

$$= \frac{1}{2}\,\lambda^2\left[1 + \Sigma\tilde{h}_i^2\right] \geq \frac{\lambda^2}{2} \tag{3.3}$$

But $W(\theta;\theta) = \frac{1}{2}\lambda^2$ which implies that $\hat{\theta} = \theta$ always is a global minimum point of $W(\hat{\theta},\theta)$. However, $\hat{\theta} = \theta$ is not necessarily a unique solution of

$$W(\hat{\theta},\theta) = \inf_{\theta^*} W(\theta^*,\theta) \tag{3.4}$$

This equation can in view of (3.3) be written as

$$\begin{cases} \dfrac{G(\theta;q^{-1}) - G(\hat{\theta};q^{-1})}{H(\hat{\theta};q^{-1})}\,u(t) = 0 \\ \\ H(\theta;q^{-1}) = H(\hat{\theta};q^{-1}) \end{cases} \tag{3.5}$$

The equations (3.5) will now be discussed for different structures of the system. The input signal will be assumed to be persistently exciting of a sufficiently high order. The first part of (3.5) will then in fact be replaced by

$$G(\hat{\theta};q^{-1}) = G(\theta;q^{-1})$$

Most of the material is well-known and parts of it have been treated by the author before in Söderström (1972), Åström-Söderström (1973). These parts are included here to get a more complete survey.

As a general result it can be said that the loss functions for the different cases have a unique global minimum if a model of correct order is applied. If the model order is too high there is in most cases no unique global minimum point.

To simplify the notations the second argument in W will be dropped in the rest of the report.

Structure 1: The Least Squares (LS) Method.

The system is in this case given by

$$A(q^{-1})y(t) = B(q^{-1})u(t) + e(t) \tag{3.6}$$

so

$$G(\theta;q^{-1}) = \frac{B(q^{-1})}{A(q^{-1})} \qquad H(\theta;q^{-1}) = \frac{1}{A(q^{-1})}$$

The equations (3.5) become

$$\begin{cases} \dfrac{\hat{A}(q^{-1})B(q^{-1}) - A(q^{-1})\hat{B}(q^{-1})}{A(q^{-1})} u(t) = 0 \\ \hat{A}(q^{-1}) \equiv A(q^{-1}) \end{cases}$$

or simplified

$$[\hat{B}(q^{-1}) - B(q^{-1})]u(t) = 0$$
$$\hat{A}(q^{-1}) \equiv A(q^{-1}) \qquad (3.7)$$

The consistency properties of this method are well-known, Åström (1968).

Lemma 3.1. Assume that $n_\ell = \min(\hat{n}_a - n_a, \hat{n}_b - n_b) \geq 0$ and that u(t) is persistently exciting of order $n_{\hat{b}}$. Then there is a unique global minimum point given by $\hat{A}(q^{-1}) \equiv A(q^{-1})$, $\hat{B}(q^{-1}) \equiv B(q^{-1})$. There are no other local minimum points.

Proof. The first statement follows immediately from Lemma 2.5 and (3.7). The second statement is true since V is convex.

Q.E.D.

Structure 2: The General Least Squares (GLS) Model.

The structure is given by Clarke (1967), Söderström (1972)

$$A(q^{-1})y(t) = B(q^{-1})u(t) + \frac{1}{C(q^{-1})} e(t) \qquad (3.8)$$

Thus

$$G(\theta;q^{-1}) = \frac{B(q^{-1})}{A(q^{-1})}, \qquad H(\theta;q^{-1}) = \frac{1}{A(q^{-1})C(q^{-1})}$$

The equations (3.5) become

$$\begin{cases} \dfrac{\hat{C}(q^{-1})}{A(q^{-1})} [\hat{A}(q^{-1})B(q^{-1}) - A(q^{-1})\hat{B}(q^{-1})]u(t) = 0 \\ \\ \hat{A}(q^{-1})\hat{C}(q^{-1}) \equiv A(q^{-1})C(q^{-1}) \end{cases} \tag{3.9}$$

The solution of these equations is treated in Söderström (1972).

Lemma 3.2. Assume that $n_\ell = \min(\hat{n}_a - n_a, \hat{n}_b - n_b) \geq 0$, $(\hat{n}_c - n_c) \geq 0$, $u(t)$ is persistently exciting of order $\max(\hat{n}_a + n_b, n_a + \hat{n}_b)$, and that $A(q^{-1})$ and $B(q^{-1})$ are relatively prime. Then the solutions of (3.9) fulfil

$$\begin{aligned} \hat{A}(q^{-1}) &\equiv A(q^{-1})L(q^{-1}) \\ \hat{B}(q^{-1}) &\equiv B(q^{-1})L(q^{-1}) \\ L(q^{-1})\hat{C}(q^{-1}) &\equiv C(q^{-1}) \end{aligned} \tag{3.10}$$

where

$$L(q^{-1}) = 1 + \sum_{1}^{n_\ell} \ell_i q^{-i} \quad \text{if } n_\ell \geq 1$$

$$= 1 \qquad\qquad \text{if } n_\ell = 0$$

Proof. The assumptions of the theorem imply that the first equation in (3.9) can be replaced by

$$\hat{A}(q^{-1})B(q^{-1}) - A(q^{-1})\hat{B}(q^{-1}) \equiv 0$$

Lemma 2.1 gives the rest of the proof.

Q.E.D.

Remark 1. If $n_\ell = 0$ $\hat{\theta} = \theta$ is the unique solution.

Remark 2. Note that when $n_\ell \geq 1$ there are only a finite number of solutions of (3.9). This is particular for the GLS case. The reason for this property is the special structure of the system equation.

Remark 3. If u(t) is not persistently exciting of order $\max(\hat{n}_a+n_b, n_a+\hat{n}_b)$ there may exist global minimum points which do not fulfil (3.10). An example is given in Söderström (1972).

It is well-known, Söderström (1972), that the number of local minimum points depends on the signal to noise ratio.

Structure 3: Time Series.

In this case stochastic processes of the form

$$A(q^{-1})y(t) = C(q^{-1})e(t) \tag{3.11}$$

are considered. Then

$$G(\theta;q^{-1}) = 0 \qquad H(\theta;q^{-1}) = \frac{C(q^{-1})}{A(q^{-1})}$$

The equations for the global minimum point (3.5) are

$$A(q^{-1})\hat{C}(q^{-1}) - \hat{A}(q^{-1})C(q^{-1}) \equiv 0 \tag{3.12}$$

Lemma 3.3. Assume that

i) $A(q^{-1})$ and $C(q^{-1})$ are relatively prime

ii) $n_\ell = \min(\hat{n}_a - n_a, \hat{n}_c - n_c) \geq 0$

The solutions of (3.12) are

$$\hat{A}(q^{-1}) \equiv A(q^{-1})L(q^{-1}) \qquad (3.13)$$
$$\hat{C}(q^{-1}) = C(q^{-1})L(q^{-1})$$

where

$$L(q^{-1}) = 1 + \sum_{1}^{n_\ell} \ell_i q^{-i} \quad \text{if } n_\ell \geq 1$$

$$= 1 \quad \text{if } n_\ell = 0$$

The parameters ℓ_i are arbitrary. These points are the only stationary points as well.

Proof. See Åström-Söderström (1973).

Remark. If $n_\ell = 0$ $\hat{\theta} = \theta$ is the only global as well as local minimum point.

Structure 4.

The system is assumed to be governed by

$$A(q^{-1})y(t) = B(q^{-1})u(t) + A(q^{-1})e(t) \qquad (3.14)$$

so

$$G(\theta;q^{-1}) = \frac{B(q^{-1})}{A(q^{-1})} \qquad H(\theta;q^{-1}) = 1$$

The equations (3.5) are thus replaced by

$$\frac{[\hat{A}(q^{-1})B(q^{-1}) - A(q^{-1})\hat{B}(q^{-1})]}{A(q^{-1})\hat{A}(q^{-1})} u(t) = 0 \text{ a.s.} \tag{3.15}$$

Lemma 3.4. Assume that $n_\ell = \min(\hat{n}_a - n_a, \hat{n}_b - n_b) \geq 0$, $A(q^{-1})$ and $B(q^{-1})$ are relatively prime, and that $u(t)$ is persistently exciting of order $\max(\hat{n}_a + n_b, n_a + \hat{n}_b)$. Then the solutions of (3.15) are

$$\hat{A}(q^{-1}) \equiv A(q^{-1})L(q^{-1})$$
$$\hat{B}(q^{-1}) \equiv B(q^{-1})L(q^{-1}) \tag{3.16}$$

where

$$L(q^{-1}) = 1 + \sum_{1}^{n_\ell} \ell_i q^{-i} \qquad n_\ell \geq 1$$

$$= 1 \qquad n_\ell = 0$$

The numbers ℓ_i are arbitrary.

Proof. Lemma 2.4 and Lemma 2.1 give the result.

Q.E.D.

Remark. If $n_\ell = 0$ $\hat{\theta} = \theta$ is the only global minimum point.

Structure 5.

This structure is discussed e.g. in Åström-Bohlin (1966). It is given by

$$A(q^{-1})y(t) = B(q^{-1})u(t) + C(q^{-1})e(t) \tag{3.17}$$

This means that

$$G(\theta;q^{-1}) = \frac{B(q^{-1})}{A(q^{-1})} \qquad H(\theta;q^{-1}) = \frac{C(q^{-1})}{A(q^{-1})}$$

and (3.5) can be replaced by

$$\begin{cases} \dfrac{\hat{A}(q^{-1})B(q^{-1}) - A(q^{-1})\hat{B}(q^{-1})}{A(q^{-1})\hat{C}(q^{-1})}\, u(t) = 0 \\ \\ \hat{A}(q^{-1})C(q^{-1}) - A(q^{-1})\hat{C}(q^{-1}) \equiv 0 \end{cases} \tag{3.18}$$

Lemma 3.5. Assume that $n_\ell = \min(\hat{n}_a - n_a,\ \hat{n}_b - n_b,\ \hat{n}_c - n_c) \geq 0$, $u(t)$ is persistently exciting of order $\max(\hat{n}_a + n_b,\ n_a + \hat{n}_b)$, and that $A(q^{-1})$, $B(q^{-1})$ and $C(q^{-1})$ are relatively prime. Then the general solution of (3.18) is given by

$$\begin{aligned} \hat{A}(q^{-1}) &\equiv A(q^{-1})L(q^{-1}) \\ \hat{B}(q^{-1}) &\equiv B(q^{-1})L(q^{-1}) \\ \hat{C}(q^{-1}) &\equiv C(q^{-1})L(q^{-1}) \end{aligned} \tag{3.19}$$

where

$$\begin{aligned} L(q^{-1}) &= 1 + \sum_{1}^{n_\ell} \ell_i q^{-i} \quad \text{if } n_\ell \geq 1 \\ &= 1 \qquad\qquad\quad\ \text{if } n_\ell = 0 \end{aligned}$$

The coefficients ℓ_i are arbitrary.

Proof. Define $\tilde{A}(q^{-1})$, $\tilde{B}(q^{-1})$ and $D(q^{-1})$ from

$$A(q^{-1}) = \tilde{A}(q^{-1})D(q^{-1})$$
$$B(q^{-1}) = \tilde{B}(q^{-1})D(q^{-1})$$

$\tilde{A}(q^{-1})$, $\tilde{B}(q^{-1})$ are relatively prime

$$D(q^{-1}) = 1 + \sum_{1}^{n_d} d_i q^{-i} \qquad (n_d \geq 0)$$

The first equation of (3.18) can be replaced by (Lemma 2.4)

$$\hat{A}(q^{-1})\tilde{B}(q^{-1}) - \tilde{A}(q^{-1})\hat{B}(q^{-1}) \equiv 0$$

The solution is (Lemma 2.1)

$$\hat{A}(q^{-1}) \equiv \tilde{A}(q^{-1})M(q^{-1})$$
$$\hat{B}(q^{-1}) \equiv \tilde{B}(q^{-1})M(q^{-1}) \qquad (3.20)$$

$$M(q^{-1}) = 1 + \sum_{1}^{n_m} m_i q^{-i}$$

$$n_m = n_d + n_\ell$$

$(m_i)_{i=1}^{n_m}$ are determined only by the second equation in (3.18)

The last equation of (3.18) gives

$$M(q^{-1})C(q^{-1}) - D(q^{-1})\hat{C}(q^{-1}) \equiv 0 \qquad (3.21)$$

According to the assumptions of the lemma $C(q^{-1})$ and $D(q^{-1})$ have no common factors.

The solution of (3.21) w.r.t. $M(q^{-1})$ and $\hat{C}(q^{-1})$ is

$$\begin{aligned} \hat{C}(q^{-1}) &\equiv C(q^{-1})L(q^{-1}) \\ M(q^{-1}) &\equiv D(q^{-1})L(q^{-1}) \\ L(q^{-1}) &= 1 + \sum_{1}^{n_\ell} \ell_i q^{-i} \end{aligned} \tag{3.22}$$

$(\ell_i)_1^{n_\ell}$ arbitrary

The combination of (3.20) and (3.22) gives the desired solution (3.19).

Q.E.D.

Remark. If $n_\ell = 0$ $\hat{\theta} = \theta$ is the only global minimum point.

Structure 6.

In this section the structure used by Bohlin (1970) is considered

$$y(t) = \frac{B(q^{-1})}{A(q^{-1})} u(t) + \frac{C(q^{-1})}{D(q^{-1})} e(t) \tag{3.23}$$

The equations (3.5) turn out to be

$$\begin{cases} \dfrac{\hat{D}(q^{-1})}{\hat{C}(q^{-1})} \dfrac{\hat{A}(q^{-1})B(q^{-1}) - A(q^{-1})\hat{B}(q^{-1})}{A(q^{-1})\hat{A}(q^{-1})} u(t) = 0 \text{ a.s.} \\ \\ \hat{C}(q^{-1})D(q^{-1}) \equiv C(q^{-1})\hat{D}(q^{-1}) \end{cases} \tag{3.24}$$

<u>Lemma 3.6</u>. Assume that

$n_\ell = \min(\hat{n}_a - n_a,\ \hat{n}_b - n_b) \geq 0$

$n_m = \min(\hat{n}_c - n_c,\ \hat{n}_d - n_d) \geq 0$

$A(q^{-1})$ and $B(q^{-1})$ are relatively prime

$C(q^{-1})$ and $D(q^{-1})$ are relatively prime

$u(t)$ persistently exciting of order $\max(\hat{n}_a + n_b,\ n_a + \hat{n}_b)$

Then the general solution of (3.24) is

$$
\begin{aligned}
\hat{A}(q^{-1}) &\equiv A(q^{-1})L(q^{-1}) \\
\hat{B}(q^{-1}) &\equiv B(q^{-1})L(q^{-1}) \\
\hat{C}(q^{-1}) &\equiv C(q^{-1})M(q^{-1}) \\
\hat{D}(q^{-1}) &\equiv D(q^{-1})M(q^{-1})
\end{aligned}
\tag{3.25}
$$

where

$$L(q^{-1}) = 1 + \sum_1^{n_\ell} \ell_i q^{-i}$$

$$M(q^{-1}) = 1 + \sum_1^{n_m} m_i q^{-i}$$

$(\ell_i)_1^{n_\ell}$, $(m_i)_1^{n_m}$ arbitrary

<u>Proof</u>. The result follows from Lemma 2.1 and Lemma 2.4.

Q.E.D.

<u>Remark</u>. If $n_\ell = n_m = 0$, $\hat{\theta} = \theta$ is the only global minimum point.

IV. LOCAL MINIMUM POINTS FOR STRUCTURE 4.

In this chapter the local minimum points for the case with white measurements noise are treated. It will be shown that $n_a = \hat{n}_a = 1$, n_b arbitrary will imply a unique local minimum point. The loss function can in certain cases have "singular" saddle points corresponding to $\hat{B}(q^{-1}) \equiv 0$. The analysis can unfortunately not be extended to the case $n_a > 1$.

For the structure with white measurement noise the system is described by

$$A(q^{-1})y(t) = B(q^{-1})u(t) + A(q^{-1})e(t)$$

The loss function for this structure is given by

$$2W(\hat{\theta}) = \mathcal{E}\left[\frac{\hat{A}(q^{-1})B(q^{-1}) - A(q^{-1})\hat{B}(q^{-1})}{A(q^{-1})\hat{A}(q^{-1})}\, u(t)\right]^2 + \lambda^2 \qquad (4.1)$$

Assume that $\hat{n}_a \geq n_a$, $\hat{n}_b \geq n_b$, $B(q^{-1}) \not\equiv 0$ and that $u(t)$ is persistently exciting of order $\max(\hat{n}_a+n_b,\ n_a+\hat{n}_b)$.

The stationary points of the function are the solutions of $W_{\hat{\theta}}(\hat{\theta}) = 0$, which is written as

$$\mathcal{E}\left[\frac{\hat{A}(q^{-1})B(q^{-1}) - A(q^{-1})\hat{B}(q^{-1})}{A(q^{-1})\hat{A}(q^{-1})}\, u(t)\right]\left[\frac{\hat{B}(q^{-1})}{\hat{A}(q^{-1})^2}\, q^{-i}u(t)\right] = 0 \qquad 1 \leq i \leq \hat{n}_a$$

$$(4.2)$$

$$\mathcal{E}\left[\frac{\hat{A}(q^{-1})B(q^{-1}) - A(q^{-1})\hat{B}(q^{-1})}{A(q^{-1})\hat{A}(q^{-1})}\, u(t)\right]\left[\frac{-1}{\hat{A}(q^{-1})}\, q^{-i}u(t)\right] = 0 \qquad 1 \leq i \leq \hat{n}_b$$

It is not possible to find all solutions of (4.2) in an easy way. The following attempt of analysis will be made. Let

$$H(q^{-1}) = \sum_{1}^{m} h_i q^{-i} = \hat{A}(q^{-1})B(q^{-1}) - A(q^{-1})\hat{B}(q^{-1})$$

with $m = \max(\hat{n}_a + n_b,\ n_a + \hat{n}_b)$. The equations (4.2) will be rewritten as

$$Q(\hat{\theta}) \begin{bmatrix} h_1 \\ \vdots \\ h_m \end{bmatrix} = 0$$

where $Q(\hat{\theta})$ is a matrix of order $(\hat{n}_a + \hat{n}_b) \times m$. If rank $Q(\hat{\theta})$ is m for all $\hat{\theta}$ it can be concluded that $h_i = 0$ for $i = 1, \ldots, m$. This gives the equation for the global minimum points, Lemma 3.4.

Put

$$v(t) = \frac{1}{A(q^{-1})\hat{A}(q^{-1})^2}\, u(t)$$

Then (4.2) is equivalent to

$$\mathcal{E} \begin{bmatrix} -\hat{B}(q^{-1})A(q^{-1})v(t-1) \\ \vdots \\ -\hat{B}(q^{-1})A(q^{-1})v(t-\hat{n}_a) \\ \hat{A}(q^{-1})A(q^{-1})v(t-1) \\ \vdots \\ \hat{A}(q^{-1})A(q^{-1})v(t-\hat{n}_b) \end{bmatrix} \begin{bmatrix} \hat{A}(q^{-1})v(t-1) & \ldots & \hat{A}(q^{-1})v(t-m) \end{bmatrix} \begin{bmatrix} h_1 \\ \cdot \\ \cdot \\ h_m \end{bmatrix} = 0$$

which can be written as

$$\begin{bmatrix} 0 & -\hat{b}_1 & \cdot & \cdot & -\hat{b}_{\hat{n}_b} & 0 & & \\ & \cdot & & & & \cdot & & \\ 0 & & \cdot & & & & \cdot & \\ & & 0 & -\hat{b}_1 & \cdot & \cdot & -\hat{b}_{\hat{n}_b} & \\ \hline 1 & \cdot & \cdot & \cdot & \hat{a}_{\hat{n}_a} & 0 & & \\ & \cdot & & & & \cdot & & \\ 0 & & \cdot & & & & \cdot & \\ & & 1 & \cdot & \cdot & \cdot & \hat{a}_{\hat{n}_a} & \end{bmatrix} \cdot P_0 \cdot \begin{bmatrix} h_1 \\ \cdot \\ \cdot \\ \cdot \\ h_m \end{bmatrix} = 0 \qquad (4.3)$$

where P_0 is the following matrix of order $(\hat{n}_a+\hat{n}_b) \times m$.

$$P_0 = \mathcal{E} \begin{bmatrix} A(q^{-1})v(t-1) \\ \vdots \\ A(q^{-1})v(t-\hat{n}_a-\hat{n}_b) \end{bmatrix} \begin{bmatrix} \hat{A}(q^{-1})v(t-1) & \dots & \hat{A}(q^{-1})v(t-m) \end{bmatrix} \qquad (4.4)$$

To continue the analysis it is necessary to examine the rank of the two matrices in (4.3). It will be necessary to separate three different cases.

Case 1.

Consider points such that $\hat{A}(q^{-1})$ and $\hat{B}(q^{-1})$ are relatively prime. Then the first matrix of (4.3) is non singular (Lemma 2.2). Define a square matrix P of order $m \times m$.

$$P(A,\hat{A},v) = \mathcal{E} \begin{bmatrix} A(q^{-1})v(t-1) \\ \vdots \\ A(q^{-1})v(t-m) \end{bmatrix} \begin{bmatrix} \hat{A}(q^{-1})v(t-1) & \dots & \hat{A}(q^{-1})v(t-m) \end{bmatrix} \qquad (4.5)$$

which consists of the upper square part of P_0.

If P is non singular for all possible $(\hat{a}_i)_{i=1}^{\hat{n}_a}$ then it is possible to conclude that $h_i = 0$ is the only solution of (4.3).

The properties of $P(A,\hat{A},v)$ are described in

Lemma 4.1.

i) Assume that $n_a = 1$, $\hat{n}_a = 1$. Then $P(A,\hat{A},v)$ is non singular for all A, all $\hat{A}$ and all v(t), such as v(t) is persistently exciting of order m.

ii) There are A, $\hat{A}$ and v(t) such that $n_a = 1$, $\hat{n}_a = 2$, $m = 3$, v(t) persistently exciting of order m and $P(A,\hat{A},v)$ singular.

Proof.

i) Let $x = [x_1 \; \ldots \; x_m]^T$ be an arbitrary vector and define

$$X(q^{-1}) = \sum_1^m x_i q^{-i}$$

Then

$$x^T P x = \mathcal{E}\,[A(q^{-1})X(q^{-1})v(t)][\hat{A}(q^{-1})X(q^{-1})v(t)] =$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{Re}[A(e^{i\omega})\hat{A}(e^{-i\omega})]\,|X(e^{i\omega})|^2 \phi_v(\omega)d\omega$$

The function $\phi_v(\omega)$ is the spectral density associated with the asymptotic sample covariance function.

But

$$\text{Re}[A(e^{i\omega})\hat{A}(e^{-i\omega})] =$$

$$= 1 + a\hat{a} + (a+\hat{a})\cos\omega \geq 1 + a\hat{a} - |a+\hat{a}| \geq$$

$$\geq (1-|a|)(1-|\hat{a}|) > 0$$

Thus $x^T Px \geq 0$ and equality implies $|X(e^{i\omega})|^2 \phi_v(\omega) \equiv$ $\equiv 0$. From Lemma 2.4 it is concluded that this implies $X(e^{i\omega}) \equiv 0$ or $x = 0$.

ii) Let v(t) be white noise with unit variance. Take $A(q^{-1}) = 1 + aq^{-1}$ and $\hat{A}(q^{-1}) = (1-aq^{-1})^2$. Then

$$P(A,\hat{A},v) = \begin{bmatrix} 1-2a^2 & a & 0 \\ -2a+a^3 & 1-2a^2 & a \\ a^2 & -2a+a^3 & 1-2a^2 \end{bmatrix}$$

det P = $1 - 2a^2 + 3a^4 - 4a^6$ is a continuous function of a. Det[P(a=0)] = 1 and det[P(a=1)] = -2 imply that there is $|a| < 1$ such that det P = 0.

Q.E.D.

<u>Case 2</u>.

Consider points such that $\hat{A}(q^{-1})$ and $\hat{B}(q^{-1})$ are not relatively prime, but $\hat{B}(q^{-1}) \not\equiv 0$.

Define

$$\bar{A}(q^{-1}) = 1 + \sum_{1}^{\bar{n}_a} \bar{a}_i q^{-i}; \quad \bar{B} = \sum_{1}^{\bar{n}_b} \bar{b}_i q^{-i}; \quad \bar{L}(q^{-1}) = 1 + \sum_{1}^{\bar{n}_\ell} \bar{\ell}_i q^{-i}$$

by

$\hat{A}(q^{-1}) = \bar{A}(q^{-1})\bar{L}(q^{-1})$

$\hat{B}(q^{-1}) = \bar{B}(q^{-1})\bar{L}(q^{-1})$

$\bar{A}(q^{-1})$ and $\bar{B}(q^{-1})$ are relatively prime

$\bar{n}_a = \hat{n}_a - \bar{n}_\ell ; \; \bar{n}_b = \hat{n}_b - \bar{n}_\ell$

Change the definition of $H(q^{-1})$, m and v(t) to

$$H(q^{-1}) = \sum_1^m h_i q^{-i} = \bar{A}(q^{-1})B(q^{-1}) - A(q^{-1})\bar{B}(q^{-1})$$

$$m = \max(\bar{n}_a + n_b, \; n_a + \bar{n}_b)$$

$$v(t) = \frac{1}{A(q^{-1})\bar{A}(q^{-1})\hat{A}(q^{-1})} u(t)$$

Then the equation $V_{\hat{\theta}}(\hat{\theta}) = 0$ can be written as

$$\begin{array}{l} \hat{n}_a \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. \\ \hat{n}_b \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. \end{array} \begin{bmatrix} 0 & -\bar{b}_1 & \cdots & -\bar{b}_{\bar{n}_b} & & 0 \\ & \ddots & & & \ddots & \\ 0 & & 0 & -\bar{b}_1 & \cdots & -\bar{b}_{\bar{n}_b} \\ \hline 1 & \bar{a}_1 & \cdots & \bar{a}_{\bar{n}_a} & & 0 \\ & \ddots & & & \ddots & \\ 0 & & 1 & \bar{a}_1 & \cdots & \bar{a}_{\bar{n}_a} \end{bmatrix} \cdot Q_0 \cdot \begin{bmatrix} h_1 \\ \vdots \\ h_m \end{bmatrix} = 0 \quad (4.6)$$

$\hat{n}_a + \hat{n}_b - \bar{n}_\ell$ columns

where Q_0 is the following matrix of order $(\hat{n}_a + \hat{n}_b - \bar{n}_\ell) \times m$

$$Q_0 = \mathcal{E}\begin{bmatrix} A(q^{-1})v(t-1) \\ \vdots \\ A(q^{-1})v(t-\hat{n}_a-\hat{n}_b+\bar{n}_\ell) \end{bmatrix}\begin{bmatrix} \hat{A}(q^{-1})v(t-1) & \ldots & \hat{A}(q^{-1})v(t-m) \end{bmatrix} \tag{4.7}$$

According to Lemma 2.2 the first matrix of (4.6) has rank $\hat{n}_a + \hat{n}_b - \bar{n}_\ell$. Thus $h_i = 0$ is the only solution if rank $Q_0 =$ $= m$. This condition, however, is already analyzed in the previous case.

Case 3.

Consider points such that $\hat{B}(q^{-1}) \equiv 0$. Such singular points may look uninteresting from a theoretical point of view. For two reasons they are studied here, besides the purpose to give general information of the loss function W. The first reason is that in a practical case it is not trivial to determine if $\hat{b}_i = 0$. The other reason is that the result of this chapter will be used later on in Chapter 6. For this case the equations (4.2) turn out to be

$$\mathcal{E}\left[\frac{B(q^{-1})}{A(q^{-1})}\, u(t)\right]\left[\frac{q^{-i}}{\hat{A}(q^{-1})}\, u(t)\right] = 0 \qquad 1 \leq i \leq \hat{n}_b \tag{4.8}$$

If $\hat{n}_a > \hat{n}_b$ this system is overdetermined and may have an infinite number of solutions such that $\hat{A}(z)$ has zeros outside the unit circle. In Appendix A this case is further considered. It is also shown that the stationary points satisfying $\hat{B}(q^{-1}) \equiv 0$ always are saddle points.

The equation (4.8) implies

$$\mathcal{E}\left[\frac{B(q^{-1})}{A(q^{-1})}\, u(t-1)\right]\left[\frac{B(q^{-1})}{\hat{A}(q^{-1})}\, u(t-1)\right] = 0 \tag{4.9}$$

Put

$$v(t) = \frac{B(q^{-1})}{A(q^{-1})\hat{A}(q^{-1})} u(t)$$

If $n_a = \hat{n}_a = 1$ it follows from Lemma 4.1 that (4.9) cannot be satisfied.

In Appendix B the special case of u(t) as white noise is treated. It is shown that the mild condition $\hat{n}_a = \hat{n}_b \geq \max(n_a, n_b)$ implies that the global minimum points are the only stationary points.

Summing up, the analysis has given the following information of the loss function $W(\hat{\theta})$. Assume that u(t) is persistently exciting of order $\max(\hat{n}_a + n_b,\ n_a + \hat{n}_b)$

1. If $\hat{n}_a = n_a = 1$, $\hat{n}_b \geq n_b$, n_b arbitrary then $W(\hat{\theta})$ has a unique stationary point, namely the local (and global) minimum point $\hat{\theta} = \theta$.

2. If $\hat{n}_a > n_a$ and $\hat{n}_b > n_b$ there is no unique <u>global</u> minimum point.

3. The analysis gives no information of the number of local minimum points if $\hat{n}_a \geq 2$.

4. There are systems such that $\hat{n}_a = n_a = 2$, $\hat{n}_b = n_b = 1$ and with a set of saddle points satisfying $\hat{B}(q^{-1}) \equiv 0$. An immediate implication is that it is not sufficient to consider only $W'(\hat{\theta}) = 0$ in the general analysis of the number of local minimum points.

5. If u(t) is white noise and $\hat{n}_a = \hat{n}_b \geq \max(n_a, n_b)$ then the global minimum points are the only stationary points.

V. LOCAL MINIMUM POINT FOR STRUCTURE 5.

In this part structure 5 is considered. Partial results on the number of local minimum points will be given. Only cases with very high or very low signal to noise ratios are treated. The mathematical tools are Lemma 2.8 and Lemma 2.9. These two lemmas deal with the effects of a disturbance term $\varepsilon g(x)$ resp. $\varepsilon h(x,y)$. The application of them will be made on the loss function. ε will be inverse proportional or proportional to the signal to noise ratio.

<u>Theorem 5.1</u>. Consider the system

$$A(q^{-1})y(t) = B(q^{-1})u(t) + C(q^{-1})e(t) \qquad (3.17) = (5.1)$$

and the loss function

$$2W(\hat{\theta}) = E\left[\frac{\hat{A}(q^{-1})B(q^{-1}) - A(q^{-1})\hat{B}(q^{-1})}{A(q^{-1})\hat{C}(q^{-1})}u(t)\right]^2 +$$

$$+ E\left[\frac{\hat{A}(q^{-1})C(q^{-1})}{A(q^{-1})\hat{C}(q^{-1})}e(t)\right]^2 \qquad (5.2)$$

Assume that

i) $n_\ell = \min(\hat{n}_a - n_a,\ \hat{n}_b - n_b,\ \hat{n}_c - n_c) = 0$

ii) $u(t)$ is persistently exciting of order $\max(\hat{n}_a + n_b,\ n_a + \hat{n}_b)$

iii) $A(q^{-1})$, $B(q^{-1})$ and $C(q^{-1})$ are relatively prime.

Denote the signal to noise ratio by S. There is a number S_0 (which may depend on Ω), such that if $S_0 \leq S < \infty$ then

$W(\hat{\theta})$ has a unique local minimum in Ω, namely $\hat{\theta} = \theta$.

Proof. This proof is a modification of Appendix E in Söderström (1972). Perform a change of variables by

$$x = \begin{bmatrix} \hat{a}_1 - a_1 \\ \vdots \\ \hat{a}_{n_a} - a_{n_a} \\ \vdots \\ \hat{a}_{\hat{n}_a} \\ \hat{b}_1 - b_1 \\ \vdots \\ \hat{b}_{n_b} - b_{n_b} \\ \vdots \\ \hat{b}_{\hat{n}_b} \end{bmatrix} \qquad y = \begin{bmatrix} \hat{c}_1 \\ \cdot \\ \cdot \\ \cdot \\ \hat{c}_{\hat{n}_c} \end{bmatrix} \tag{5.3}$$

Assume that $A(q^{-1}) \equiv \tilde{A}(q^{-1})D(q^{-1})$, $B(q^{-1}) \equiv \tilde{B}(q^{-1})D(q^{-1})$ where $\tilde{A}(q^{-1})$ and $\tilde{B}(q^{-1})$ are relatively prime and

$$D(q^{-1}) = 1 + \sum_{1}^{n_d} d_i q^{-i} \qquad (n_d \geq 0)$$

The loss function can be written

$$W(x,y) = \frac{1}{2} x^T P(y) x + \varepsilon \tilde{h}(x,y) \tag{5.4}$$

with $P(y)$ as the system covariance matrix of

$$A(q^{-1})y^F(t) = B(q^{-1})u^F(t), \qquad u^F(t) = \frac{1}{\hat{C}(q^{-1})} u(t)$$

where $u^F(t)$ is the input and $y^F(t)$ the output.

P(y) may be singular, but the null space of P(y) is independent of y. This is obvious, since from Lemma 2.7 the null space is spanned by vectors of the form

$$\begin{bmatrix} f_1 \\ \vdots \\ f_{\hat{n}_a} \\ g_1 \\ \vdots \\ g_{\hat{n}_b} \end{bmatrix}$$

with

$$F(q^{-1}) = \sum_1^{\hat{n}_a} f_i q^{-i} \equiv \tilde{A}(q^{-1})L'(q^{-1})$$

$$G(q^{-1}) = \sum_1^{\hat{n}_b} g_i q^{-i} = \tilde{B}(q^{-1})L'(q^{-1})$$

$$L'(q^{-1}) = \sum_1^{k} \ell'_i q^{-i} \quad \text{is arbitrary}$$

The simplest case k = 0 is not treated explicitly in the following. In this case P(y) is non singular. It is easy to see how the proof can be simplified for this case.

Introduce now the new variables

$$x' = \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix}$$

where x_1' is of dimension k and x_2' of dimension $(\hat{n}_a+\hat{n}_b-k)$. The vector x' is defined by

$$x = Qx' = [Q_1 \mid Q_2] \begin{bmatrix} x_1' \\ -- \\ x_2' \end{bmatrix}$$

where

$$Q_1 = \left[\begin{array}{ccc}
1 & & 0 \\
\tilde{a}_1 & & \\
\vdots & \ddots & \\
\tilde{a}_{\tilde{n}_a} & & 1 \\
 & \ddots & \vdots \\
0 & & \tilde{a}_{\tilde{n}_a} \\
 & & 0 \\
\hline
0 & & 0 \\
\tilde{b}_1 & & \\
\vdots & \ddots & \\
\tilde{b}_{\tilde{n}_b} & & \tilde{b}_1 \\
 & \ddots & \vdots \\
0 & & \tilde{b}_{\tilde{n}_b} \\
 & & 0
\end{array}\right]$$

Q_1 is a $(\hat{n}_a+\hat{n}_b) \times k$ matrix and Q_2 an arbitrary $(\hat{n}_a+\hat{n}_b) \times$ $\times$ $(\hat{n}_a+\hat{n}_b-k)$ matrix with the properties $Q_1^T Q_2 = 0$ and Q non singular. Q_2 can for instance be constructed by Gram Schmidt orthogonalization.

From the discussion it follows that

Q_1x_1' is a typical element in the null space $N(P(y))$

Q_2x_2' is a typical element in the space $N(P(y))^{\perp}$

From these facts it is concluded that

$$P(y)Q_1 = 0$$

and that the matrix

$$R(y) = Q_2^T P(y) Q_2$$

of order $(\hat{n}_a+\hat{n}_b-k) \times (\hat{n}_a+\hat{n}_b-k)$ is non singular for all y.

The loss function is now written as

$$W(x_2',z) = \frac{1}{2}\, x_2'^T R(z) x_2' + \varepsilon h(x_2',z) \tag{5.5}$$

where z denotes the vector

$$\begin{bmatrix} x_1' \\ y \end{bmatrix}$$

Write the vector x_1' as

$$x_1' = \begin{bmatrix} \ell_1 \\ \vdots \\ \ell_k \end{bmatrix}$$

Then $x = Q_1x_1'$ is equivalently expressed as

$$\hat{A}(q^{-1}) \equiv A(q^{-1})\hat{L}(q^{-1}), \quad \hat{B}(q^{-1}) \equiv B(q^{-1})\hat{L}(q^{-1})$$

with

$$\hat{L}(q^{-1}) = 1 + \hat{\ell}_1 q^{-1} + \dots + \hat{\ell}_k q^{-k}$$

The function h(0,z) is written with operators as

$$h(0,z) = E\left[\frac{\hat{L}(q^{-1})C(q^{-1})}{D(q^{-1})\hat{C}(q^{-1})}\, e(t)\right] \qquad (5.6)$$

From assumption iii) and the discussion above it follows that $k \geq n_d$, $\min(k-n_d, \hat{n}_c-n_c) = 0$ and that $C(q^{-1})$ and $D(q^{-1})$ are relatively prime. From Lemma 3.3 it follows that h(0,z) has a unique local minimum point given by

$$\hat{L}(q^{-1}) \equiv D(q^{-1})$$

$$\hat{C}(q^{-1}) \equiv C(q^{-1})$$

The matrix of second order derivatives of h in this point turns out to be the system covariance matrix of the system

$$y'(t) = \frac{C(q^{-1})}{D(q^{-1})}\, u'(t); \qquad u'(t) = \frac{1}{C(q^{-1})}\, e(t)$$

It is positive definite according to Lemma 2.7.

From Lemma 2.9 it follows that V has a unique local minimum point in Ω. It fulfils

$$\hat{\theta} = \theta + \theta(1/S) \qquad S_0 \leq S \to \infty \qquad (5.7)$$

Since $\hat{\theta} = \theta$ is a minimum point it is concluded that it is the only local minimum point in Ω.

Q.E.D.

The other theorem of this chapter deals with the case of a low signal to noise ratio and utilizes Lemma 2.8.

Theorem 5.2. Consider the system (5.1) and the loss function (5.2).

Assume that

i) $\min(\hat{n}_a - n_a,\ \hat{n}_c - n_c) = 0$

ii) $u(t)$ is persistently of order $\hat{n}_b$

iii) $A(q^{-1})$ and $C(q^{-1})$ are relatively prime.

Denote the signal to noise ratio by S.

There is a number S_1 such that $0 < S \leq S_1$ implies that $W(\hat{\theta})$ has a unique local minimum in Ω, namely $\hat{\theta} = \theta$.

Proof. From the equation

$$\frac{\partial W}{\partial \hat{b}_i} = 0 \qquad 1 \leq i \leq \hat{n}_b$$

$\{\hat{b}_i\}$ can be solved as functions of $\{\hat{a}_i, \hat{c}_j\}$ according to assumption ii). Cf. the representation (5.4) of the loss function. These $\hat{b}_i$ functions are put into the remaining equations. The remaining equations can be written (after division by λ^2)

$$f(x) + \varepsilon g(x) = 0 \tag{5.8}$$

x is the vector $[\hat{a}_1 \ \ldots \ \hat{a}_{\hat{n}_a},\ \hat{c}_1 \ \ldots \ \hat{c}_{\hat{n}_c}]^T$. f(x) is the gradient of

$$E\left[\frac{\hat{A}(q^{-1})C(q^{-1})}{A(q^{-1})\hat{C}(q^{-1})}\, e(t)\right]^2$$

and g(x) is the gradient of

$$\mathcal{E}\left[\frac{\hat{A}(q^{-1})B(q^{-1}) - A(q^{-1})\hat{B}(q^{-1})}{A(q^{-1})\hat{C}(q^{-1})}\, u(t)\right]^2$$

where the expressions for $\hat{b}_i$ are used.

The quantity $\varepsilon = 1/\lambda^2$ is proportional to S.

Since {according to assumptions i) and iii)} $f(x) = 0$ has a unique solution given by

$$\hat{A}(q^{-1}) \equiv A(q^{-1}) \qquad \hat{C}(q^{-1}) \equiv C(q^{-1})$$

and f' is non singular in this point it follows from Lemma 2.8 that (5.8) has a unique solution in Ω. Since $\hat{\theta} = \theta$ is a local minimum point it follows that it is the only local minimum point of V in Ω.

Q.E.D.

<u>Discussion of Assumptions and Results.</u>

The assumptions i) - iii) of Theorem 5.1 are sufficient (and almost necessary) conditions for a unique global minimum, Lemma 3.5.

The assumptions i) - iii) of Theorem 5.2 are slightly stronger than the conditions used in Lemma 3.5.

If assumption i) in Theorem 5.1 is changed to $n_\ell > 0$ the mathematical machinery of Söderström (1972) will give that every local minimum points are close to some global minimum point. It is harder to examine if there are local minimum points which are not global minimum points. Since the case $n_\ell > 0$ is rather degenerated it is the author's point of view that a careful analysis is of little interest.

The very strong assumptions in the theorems are the restrictions of the signal to noise ratio. It is shown that a sufficiently high and a sufficiently small signal to noise ratio will imply existence of a unique local minimum point. However, it is unfortunately not practically possible to give any estimates of the bounds S_0 and S_1.

VI. LOCAL MINIMUM POINTS FOR STRUCTURE 6.

The structure is given by

$$y(t) = \frac{B(q^{-1})}{A(q^{-1})} u(t) + \frac{C(q^{-1})}{D(q^{-1})} e(t)$$

and the loss function for this structure can be written

$$2W(\hat{\theta}) = W_1(\hat{\theta}) + W_2(\hat{\theta})$$

$$W_1(\hat{\theta}) = \mathcal{E}\left[\left\{\frac{\hat{B}(q^{-1})}{\hat{A}(q^{-1})} - \frac{B(q^{-1})}{A(q^{-1})}\right\} \frac{\hat{D}(q^{-1})}{\hat{C}(q^{-1})} u(t)\right]^2 \qquad (6.1)$$

$$W_2(\hat{\theta}) = E\left[\frac{\hat{D}(q^{-1})C(q^{-1})}{\hat{C}(q^{-1})D(q^{-1})} e(t)\right]^2$$

If the operator $\hat{D}(q^{-1})/\hat{C}(q^{-1})$ has no influence on the number of stationary points of $W_1(\hat{\theta})$ the properties of this function is already known from Chapter 4. The function $W_2(\hat{\theta})$ is exactly the loss function for structure 3. In order to utilize these facts the following condition is introduced.

Definition 6.1. The function

$$\tilde{W}(\hat{\theta}) = \mathcal{E}\left[\left(\frac{\hat{B}(q^{-1})}{\hat{A}(q^{-1})} - \frac{B(q^{-1})}{A(q^{-1})}\right) u(t)\right]^2 \qquad (6.2)$$

is said to fulfil the uniqueness condition (abbreviated UC) if for u(t) persistently exciting of order $\max(\hat{n}_a+n_b, n_a+\hat{n}_b)$ it follows that all local minimum points satisfy

$$\hat{A}(q^{-1})B(q^{-1}) - A(q^{-1})\hat{B}(q^{-1}) \equiv 0 \tag{6.3}$$

From Chapter 4 it is known that UC holds at least in the case $\hat{n}_a = n_a = 1$.

<u>Theorem 6.1</u>. Consider the loss function

$$2W(\hat{\theta}) = \mathcal{E}\left[\left(\frac{\hat{B}(q^{-1})}{\hat{A}(q^{-1})} - \frac{B(q^{-1})}{A(q^{-1})}\right)\frac{\hat{D}(q^{-1})}{\hat{C}(q^{-1})}\, u(t)\right]^2 +$$

$$+ E\left[\frac{\hat{D}(q^{-1})C(q^{-1})}{\hat{C}(q^{-1})D(q^{-1})}\, e(t)\right]^2$$

Assume that

i) $\hat{n}_a \geq n_a,\ \hat{n}_b \geq n_b,\ \hat{n}_c \geq n_c,\ \hat{n}_d \geq n_d$

ii) $A(q^{-1})$ and $B(q^{-1})$ as well as $C(q^{-1})$ and $D(q^{-1})$ are relatively prime

iii) $u(t)$ is persistently exciting of order $\max(\hat{n}_a+n_b,\ n_a+\hat{n}_b)$

iv) The UC is fulfilled.

Then all <u>local</u> minimum points of $W(\hat{\theta})$ are <u>global</u> minimum points, i.e. they fulfil

$$\hat{A}(q^{-1})B(q^{-1}) - A(q^{-1})\hat{B}(q^{-1}) \equiv 0$$
$$\hat{C}(q^{-1})D(q^{-1}) - C(q^{-1})\hat{D}(q^{-1}) \equiv 0$$

Proof. Let θ^* be a local minimum point of $W(\hat{\theta})$. Then there is a $\delta > 0$ such that $||\theta^* - \hat{\theta}|| < \delta$ implies $W(\theta^*) \leq W(\hat{\theta})$. Let especially $\hat{\theta}$ coincide with θ^* in the $\hat{c}_i$- and $\hat{d}_i$-components. Then $W_1(\theta^*) \leq W_1(\hat{\theta})$. Thus θ^* is also a local minimum point of $W_1(\hat{\theta})$. From UC it follows that

$$\hat{A}(q^{-1})B(q^{-1}) - A(q^{-1})\hat{B}(q^{-1}) \equiv 0$$

When this expression is used in $W_{\hat{\theta}} = 0$ it follows that it is necessary that $(\hat{c}_1, \ldots, \hat{c}_{\hat{n}_c}, \hat{d}_1, \ldots \hat{d}_{\hat{n}_d})$ is a stationary point of $W_2(\hat{\theta})$, i.e.

$$\hat{C}(q^{-1})D(q^{-1}) - C(q^{-1})\hat{D}(q^{-1}) \equiv 0$$

Q.E.D.

Corr. If especially $\min(\hat{n}_a - n_a, \hat{n}_b - n_b) = 0$ and $\min(\hat{n}_c - n_c, \hat{n}_d - n_d) = 0$ the loss function has a unique local minimum point $\hat{\theta} = \theta$.

The conditions in the theorem for a unique local minimum point are partly the same conditions as used in Lemma 2.6 for a unique global minimum point, partly the uniqueness condition (UC). In contrast to the theorems for structure 5 no assumptions on the signal to noise ratio have been done.

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APPENDIX A.

This appendix deals with the degenerated solutions of (4.2). If $\hat{B}(q^{-1}) \equiv 0$ the condition $W'_{\hat{\theta}}(\hat{\theta}) = 0$ gives

$$\mathcal{E}\left[\frac{B(q^{-1})}{A(q^{-1})}\, u(t)\right]\left[\frac{q^{-i}}{\hat{A}(q^{-1})}\, u(t)\right] = 0 \qquad 1 \leq i \leq \hat{n}_b \qquad (4.8)$$

Consider for a while the following example. Let $B(q^{-1}) = bq^{-1} \not\equiv 0$, $n_a = \hat{n}_a = 2$. Further the input u(t) is assumed to fulfil $u(t) = A(q^{-1})(1+c_1q^{-1}+c_2q^{-2})w(t)$ where w(t) is white noise.

The equation (4.8) gives after a simple calculation

$$(1+a_1c_1+c_1^2+a_1c_1c_2+a_2c_2+c_2^2) + (-a_1c_2-c_1-c_1c_2)\hat{a}_1 + $$

$$+ c_2\hat{a}_1^2 - c_2\hat{a}_2 = 0 \qquad (A.1)$$

For $c_2 \not\equiv 0$ (A.1) describes a parabola in the $(\hat{a}_1,\hat{a}_2)$-plane. Let S be the subset of the $(\hat{a}_1,\hat{a}_2)$-plane such that $(\hat{a}_1,\hat{a}_2) \in S$ implies that the zeros of $1 + \hat{a}_1z + \hat{a}_2z^2 = 0$ are outside the unit circle. Depending on the values of a_1, a_2, c_1 and c_2 the parabola may intersect the set S.

In Figure A.1 the parabola and the set S are drawn for the special case $a_1 = -1.8$, $a_2 = 0.81$, $c_1 = 1.8$, $c_2 = 0.81$.

The following discussion will show that all stationary points, which satisfy (4.8), are saddle points.

Let θ^* satisfy $\hat{B}(q^{-1}) \equiv 0$ and (4.8). The matrix formed by

$$W_{\hat{b}_i\hat{b}_j} = \mathcal{E}\left[\frac{1}{\hat{A}(q^{-1})}\, u_{t-i}\right]\left[\frac{1}{\hat{A}(q^{-1})}\, u_{t-j}\right]$$

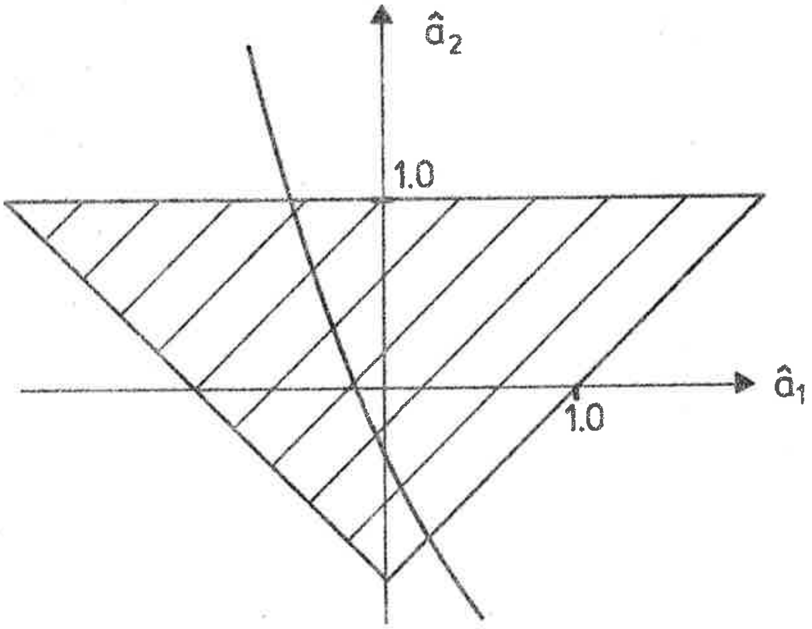

Figure A.1 - Illustration of eq. (A.1).

is positive definite for all arguments $\hat{\theta}$. If only the $\hat{b}_i$-components of θ^* are changed $W(\hat{\theta})$ will increase. If only the $\hat{a}_i$-components of θ^* are changed $W(\hat{\theta})$ will have the same value. There exists a point θ^{**} which

1) is arbitrary close to θ^*

2) differs from θ^* only in the $\hat{a}_i$-components

3) does not satisfy (4.8).

Clearly from 2) $W(\theta^{**}) = W(\theta^*)$. Given θ^{**} a new point θ^{***} is constructed. $W(\hat{\theta})$ is minimized with respect to the $\hat{b}_i$-parameters and with the $\hat{a}_i$-parameters given by θ^{**}. Since $W_{\hat{b}_i\hat{b}_j}$ is positive definite the optimization problem has a well defined solution. Call it θ^{***}. According to 3) $W(\theta^{***}) < W(\theta^{**}) = W(\theta^*)$. Finally it is observed that $||\theta^{***}-\theta^{**}||$ depends continuously on $||\theta^{**}-\theta^*||$. To summarize this means that there exists a point θ^{***} arbitrary close to θ^*, such that $W(\theta^{***}) < W(\theta^*)$. This discussion proves that θ^* must be a saddle point.

The following schematic figures are intended as an explanation of the behaviour of $W(\hat{\theta})$.

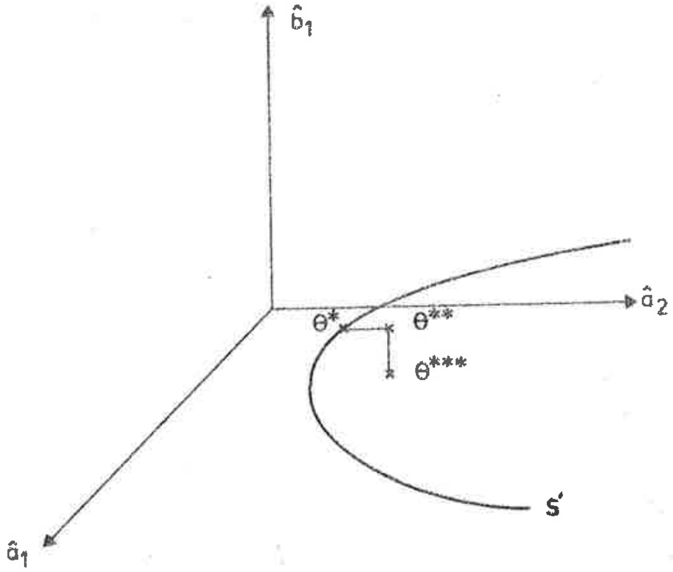

Figure A.2 - Schematic figure of θ^*, θ^{**}, θ^{***}.

The curve S' is given by (4.8) and lies in the $(\hat{a}_1, \hat{a}_2)$-plane. θ^* lies on S'. θ^{**} lies in the $(\hat{a}_1, \hat{a}_2)$-plane, close to θ^* but not on S'. θ^{***} lies below the $(\hat{a}_1, \hat{a}_2)$-plane. In Figure A.3 it is shown how $W(\hat{\theta})$ may vary in the plane spanne by θ^*, θ^{**}, θ^{***}.

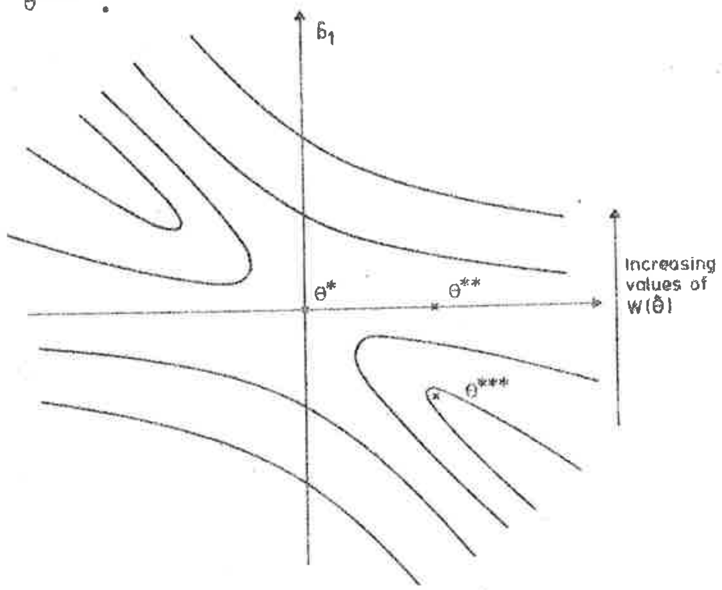

Figure A.3 - Schematic curves of $W(\hat{\theta})$ = constant.

APPENDIX B

In this appendix the structure 4 will be considered in the special case when the input signal is white noise. First the rank of the matrix Q_o defined in (4.7) will be examined. Then the equation (4.8) will be discussed.

In order to simplify the analysis it is assumed that

$$\hat{n} = \hat{n}_a = \hat{n}_b, \qquad n = n_a = n_b, \qquad \hat{n} \geq n \tag{B.1}$$

This is a mild condition. Further let u(t) be white noise of unit variance. Denote $\bar{n}_a$ and $\bar{n}_b$ by $\bar{n}$, which particularly means $m = n+\bar{n}$. Define

$$A^*(z) = z^n A(z^{-1}) = z^n + \sum_{i=1}^{n} a_i z^{n-i}$$

$$\bar{A}^*(z) = z^{\bar{n}} \bar{A}(z^{-1}) = z^{\bar{n}} + \sum_{i=1}^{\bar{n}} \bar{a}_i z^{\bar{n}-i}$$

Then the ij:th element of Q_o can be written as

$$Q_{o,ij} = \frac{1}{2\pi i} \oint \frac{z^i}{\hat{A}(z)\bar{A}(z)} \frac{z^{-j} z^{n+\bar{n}}}{A^*(z)\bar{A}^*(z)} \frac{dz}{z} \qquad \begin{array}{l} i = 1,\ldots \bar{n}+\hat{n} \\ j = 1,\ldots \bar{n}+n \end{array} \tag{B.2}$$

The matrix Q_o will be factorized using ideas from Åström--Söderström (1973).

The poles inside the unit circle of (B.2) are exactly the zeros of $A^*(z)\bar{A}^*(z)$. They are relabelled by

$$A^*(z)\bar{A}^*(z) = \prod_{k=1}^{p} (z-u_k)^{t_k} \tag{B.3}$$

where $t_k \geq 1$, $u_k \neq u_\ell$ if $k \neq \ell$ and

$$\sum_{k=1}^{p} t_k = n + \bar{n} = m$$

With the use of (B.3) $Q_{o,ij}$ is evaluated as follows.

$$Q_{o,ij} = \frac{1}{2\pi i} \oint \frac{z^{m+i-j-1}}{\hat{A}(z)\bar{A}(z)} \frac{1}{\prod_{k=1}^{p} (z-u_k)^{t_k}} dz$$

$$= \sum_{\ell=1}^{p} \operatorname*{Res}_{z=u_\ell} \frac{z^{m+i-j-1}}{\hat{A}(z)\bar{A}(z)} \frac{1}{\prod_{k=1}^{p} (z-u_k)^{t_k}}$$

$$= \sum_{\ell=1}^{p} \frac{1}{(t_\ell-1)!} D^{(t_\ell-1)} [z^{m+i-j-1} F_\ell(z)]_{z=u_\ell}$$

$$= \sum_{\ell=1}^{p} \sum_{k=0}^{t_\ell-1} \frac{1}{k!(t_\ell-1-k)!} D^{(k)}[z^{i-1}]_{z=u_\ell} D^{(t_\ell-1-k)} [z^{m-j} F_\ell(z)]_{z=u_\ell} \quad \text{(B.4)}$$

where D denotes differentiation with respect to z and the functions $F_\ell(z)$ are defined by

$$F_\ell(z) = \frac{1}{\hat{A}(z)\bar{A}(z)} \frac{1}{\prod_{\substack{k=1 \\ k\neq\ell}}^{p} (z-u_k)^{t_k}} \tag{B.5}$$

Thus

$$Q_o = V\cdot\tilde{Q} = [V_1 \; V_2 \ldots \; V_p] \begin{bmatrix} \tilde{Q}_1 \\ \tilde{Q}_2 \\ \cdot \\ \cdot \\ \cdot \\ \tilde{Q}_p \end{bmatrix} \tag{B.6}$$

where $V_\ell (1\leq \ell \leq p)$ is the $(\hat{n}+\bar{n})\times t_\ell$ matrix

$$V_\ell = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ z & 1 & & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ z^{\hat{n}+\bar{n}-1} & D[z^{\hat{n}+\bar{n}-1}] & \cdots & D^{(t_\ell-1)}[z^{\hat{n}+\bar{n}-1}] \end{bmatrix}_{z=u_\ell} \tag{B.7}$$

The matrix $\tilde{Q}_\ell (1\leq\ell\leq p)$ is $t_\ell\times m$ and is given by

$$\tilde{Q}_{\ell,ij} = \frac{1}{(i-1)!(t_\ell-i)!} D^{(t_\ell-i)} [z^{m-j} F_\ell(z)]_{z=u_\ell} \tag{B.8}$$

The matrix V is a generalization of the van der Monde matrix. It follows from Kaufamn (1969) that the rank of V is m.

The matrix γ also can be factorized. In fact

$$\gamma = S \cdot X \tag{B.9}$$

where X is an m×m matrix which can be written as

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{bmatrix}$$

The matrix $X_\ell (1 \leq \ell \leq p)$ is $t_\ell \times m$ and holds

$$X_\ell = \begin{bmatrix} z^{m-1} & \cdots & 1 \\ D[z^{m-1}] & & 0 \\ \cdot & & \cdot \\ D^{(t_\ell - 1)}[z^{m-1}] & \cdots & 0 \end{bmatrix}_{z=u_\ell} \tag{B.10}$$

According to Kaufman (1969) X is nonsingular. The square matrix S can be written as

$$S = \begin{bmatrix} S_1 & & & 0 \\ & S_2 & & \\ & & \ddots & \\ 0 & & & S_p \end{bmatrix}$$

where $S_1, \ldots, S_p$ are square block matrices of the orders $t_1 \times t_1, \ldots, t_p \times t_p$. They are given by

$$S_{\ell,ik} = \begin{cases} 0 \text{ if } k > t_\ell + 1 - i \\ \dfrac{1}{(i-1)!(k-1)!(t_\ell - i+1-k)!} D^{(t_\ell - i+1-k)} [F_\ell(z)]_{z=u_\ell} \\ \qquad \text{if } k \leq t_\ell + 1 - i \end{cases}$$

This means that S_ℓ has the following structure

$$S_\ell = \begin{bmatrix} S_{\ell,11} & \cdot\ \cdot\ \cdot & S_{\ell,1t_\ell} \\ \cdot & & \cdot \\ \cdot & \cdot & \\ \cdot & & \\ S_{\ell,t_\ell 1} & & 0 \end{bmatrix}$$

The elements of the cross diagonal are given by

$$S_{\ell,i\ t_\ell+1-i} = \frac{1}{(i-1)!(t_\ell - i)!} F_\ell(u_\ell) \qquad 1 \leq i \leq t_\ell$$

and they are nonzero according to the definition (B.5) of $F_\ell(z)$. This means that S is nonsingular.

Thus it has been proven that the rank of Q_o is m.

Now the degenerated case of $\hat{B}(q^{-1}) \equiv 0$ will be treated. Consider the equation (4.8). Factorize $A^*(z)$ as

$$A^*(z) = \prod_{i=1}^{q} (z-u_j)^{s_j} \tag{B.11}$$

where $s_j \geq 1$ and

$$\sum_{j=1}^{q} s_j = n$$

Using (B.11) the equation (4.8) is written as

$$\frac{1}{2\pi i} \oint \frac{B^*(z)}{\prod_{j=1}^{q} (z-u_j)^{s_j}} \frac{z^i}{\hat{A}(z)} dz = 0 \qquad 0 \leq i \leq \hat{n} -1 \tag{B.12}$$

Straight-forward calculations analogous to (B.4) give

$$U \cdot g = [U_1 \; U_2 \; \ldots U_q] \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_q \end{bmatrix} = 0 \tag{B.13}$$

The matrix $U_\ell (1 \leq \ell \leq q)$ is $\hat{n} \times s_\ell$ and holds

$$U_\ell = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ z & 1 & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ z^{\hat{n}-1} & D[z^{\hat{n}-1}] & \ldots & D^{(s_\ell -1)}[z^{\hat{n}-1}] \end{bmatrix}_{z=u_\ell} \tag{B.14}$$

The vector g_ℓ is given by

$$g_{\ell,i} = \frac{1}{(i-1)!(s_\ell-i)!} D^{(s_\ell-i)} [f_\ell(z)]_{z=u_\ell} \qquad (B.15)$$

The functions $f_\ell(z)$ are given by

$$f_\ell(z) = \frac{B^*(z)}{\hat{A}(z) \prod_{\substack{j=1 \\ j\neq\ell}}^{q} (z-u_j)^{s_j}} \qquad (B.16)$$

Since the rank of U is n, see Kaufman (1969), it follows that

$$g_\ell = 0 \qquad 1 \leq \ell \leq q$$

Then it follows from Åström-Söderström (1973) that

$$D^{(k)}[B^*(z)]_{z=u_j} = 0 \qquad 0 \leq k \leq s_j-1, \quad 1 \leq j \leq q \qquad (B.17)$$

Thus

$$B^*(z) = \tilde{B}(z) \prod_{j=1}^{q} (z-u_j)^{s_j}$$

where $\tilde{B}(z)$ is some polynomial. $B^*(z)$ is, however, a polynomial of degree n-1, while the product $\prod_{j=1}^{q} (z-u_j)^{s_j}$ is a polynomial of degree n. This implies that $\tilde{B}(z) \equiv 0$ and the contradiction $B(z) \equiv 0$ is established.

To summarize it has been shown that if the input is white noise and the assumptions (B.1) hold, then the loss function has no other stationary points than the global minimum points.

Is it possible to extend the calculations? One extension would be to substitute (B.1) with the more general $\min(\hat{n}_a - n_a,\ \hat{n}_b - n_b) \geq 0$ and another to permit input signals which are filtered white noise. Note that it is trivial to allow the case $\hat{n} \geq \max(n_a, n_b)$. If e.g. n_a is larger than n_b the polynomial $B^*(z)$ can be multiplied by $z^{n_a - n_b}$ and the new polynomial will have n_a coefficients.

However, the extensions desired are not possible in general. The reason is that the number of poles inside the unit circle may be larger than the numbero of rows in Q_o resp. the number of equations in (B.12). This means that the matrices V and U will have a smaller number of rows than columns which causes the idea of the calculations to break down.

However, there are cases where the results can be extended further. For instance, the analysis of (4.8) can be extended in a straight-forward way to the case

$$\max(n_a, n_b) \leq \hat{n}_b, \quad \hat{n}_a \text{ arbitrary}$$

Since the extensions can not be done in general and since the assumptions (B.1) are mild, it is of minor interest to extend the calculations further.

CORRECTIONS

The abbreviation pa.b denotes page a line b.

p6.10 Read "$n_a < i \leq n_b$"

p9.6 Read "the coefficients b_{n_b}, a_{n_a}"

p11.4 Read "given by dim R(P)"

p20.4, p20.5 Read "three times differentiable"

p20.6 Read "fixed parameter"

p22.15, p24.1, p24.4, p25.3. p29.8 Read "= 0 a.s."

p24.10 Read "$\hat{A}(q^{-1}) \equiv \hat{A}(q^{-1})$"

p24.13 Read "since trivial calculations show that V is strictly convex"

p26.3, p27.13, p28.16, p31.9, p32.18, p40.14, p42.1, p46.18, p46.19, p47.11, p48.10, p52.14. The equality $\hat{\theta} = \theta$ is not consistent if the vectors are of different orders. The meaning is for p26.3
$\hat{A}(q^{-1}) \equiv A(q^{-1})$, $\hat{B}(q^{-1}) \equiv B(q^{-1})$, $\hat{C}(q^{-1}) \equiv C(q^{-1})$
For the other cases the modifications are analogous.

p27.6 Replace "=" with "≡"

p28.14, p30.8 Read "Lemma 2.4 Corr and Lemma 2.6"

p30.16 Replace the line with "which gives"

p32.16 Read "Lemma 2.1, Lemma 2.4 Corr and Lemma 2.6"

p33.3 Read "measurement"

p37.19 Read "$\overline{B}(q^{-1})=$"

p45.16 Read "$\hat{A}(q^{-1}) \equiv \tilde{A}(q^{-1})\hat{L}(q^{-1})$, $\hat{B}(q^{-1}) \equiv \tilde{B}(q^{-1})\hat{L}(q^{-1})$"

p46.8 Read "a unique stationary point"

p47.6 Add "and $\hat{n}_b \geq n_b$"

p52.5 Read "minimum point with respect to $(\hat{a}_1 \ldots \hat{a}_{\hat{n}_a}\ \hat{b}_1 \ldots \hat{b}_{\hat{n}_b})$"

p55.9 Read "Canonical"

pB.3.11 Read "Kaufman"

pB.8.13 Read "number"