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PARAMETRIC MODELS OF LINEAR MULTIVARIABLE SYSTEMS FOR ADAPTIVE CONTROLLERS

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PARAMETRIC MODELS OF LINEAR MULTIVARIABLE SYSTEMS FOR ADAPTIVE CONTROLLERS

bstract

Algebraic design theory as proposed by Pernebo(1978) is used as a tool for design of adaptive controllers. Parametric models that are suitable for both continuous time and discrete time systems are derived under the weak condition that only the internal structure matrix has to be known in order to find a parametric model that is suitable for identification of a control that guarantees a stable closed loop system.

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1 INTRODUCTION

1.1 Introduction

The aim of this study is to give design rules for adaptive controllers suitable for a large class of linear, multivariable systems.

Chapter 2 contains a review of important algebraic properties of generalized polynomials as presented by Pernebo.

Chapter 3 contains a review of linear pole-placement controllers and model matching. Possibilities to use linear pole-placement design and model matching as a basis for adaptive control are discussed in this chapter. A new pole placement design scheme suitable for adaptive controllers is presented.

A new parametric model for linear adaptive systems is proposed in chapter 4. The model structure is chosen such that the analysis becomes simple. The relation between the pole placement methods of the previous chapter and the new parametric models is investigated.

Chapter 5 contains examples of the algorithm.

1.2 Acknowledgements

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2 ALGEBRAIC THEORETICAL BACKGROUND

2.1 Introduction

This chapter contains a review of some algebraic properties of generalized polynomial matrices and related subjects as discussed by Pernebo. ([Per 1]:II:2))

2.2 Generalized Polynomial Matrices

Let R and C denote the fields of real and complex numbers respectively. Let polynomials with coefficients in R be denoted R[z]. The rational functions will be denoted R(z).

Let the class of rational function matrices of dimensions nxm be denoted $R^{nxm}(z)$.

The concepts of 'poles' and 'zeros' are now introduced as known from complex analysis.

Consider the sets Z_- and Z_+ where

$$Z_- \cup Z_+ = C \cup \infty$$

Let the set of rational functions in the variable z with no poles in a the subset Z_- of the complex numbers C be called Z-generalized polynomials and let this set be denoted by

$$R_Z[z]$$

The set of <u>Z-generalized polynomial matrices</u>

$R_Z^{nxm}[z]$

consist of matrices, in which the elements are rational functions without poles in the region $Z_- \in C$.

The set of matrices of dimensions nxm of rational functions with poles only at infinity, i.e. the class of polynomial matrices, will be denoted $R^{nxm}[z]$.

The Z-generalized polynomials, $R_Z[z]$, with multiplication and addition constitute a euclidean ring.

A rational function is said to be <u>Z-stable</u>, if there are no poles in Z_-. A matrix of rational functions is Z-stable, if the elements have no poles in Z_-.

A matrix $M \in R_Z^{nxn}[z]$ is <u>Z-unimodular</u>, if there is a matrix B $\in R_Z^{nxn}[z]$ such that AB = I.

<u>Remark 2.1</u>

Pernebo uses the prefix notation 'Λ-' to indicate the properties obtained when avoiding poles in a prohibited region Λ. The prefix notation in this presentation should, according to this notation standard, be 'Z_--'. For convenience the simpler notation of 'Z-' will however be used here.

2.3 System description

Consider a linear, time-invariant, causal, finite dimensional, dynamical system. Let the system to be controlled be called S_0 and let it be characterized by the transfer operator G_0. The transfer operators will be called transfer functions throughout this presentation. The input-output relation is

$$y_0(t) = G_0(z)u_0(t) \tag{2.1}$$

where $G_0(z)$ is assumed to be an nxm matrix of rational functions in the variable z. For a discrete time system z denotes the forward shift operator q. For continuous time systems z denotes the differential operator p.

Let the system input vector u_0 of dimension mx1 be decomposed as

$$u_0 = \begin{bmatrix} u \\ d \end{bmatrix} \quad \text{and} \quad d = \begin{bmatrix} v \\ w \end{bmatrix} \tag{2.2}$$

where u is an m_u x 1 vector of control variables and d is an m_d x 1 vector of disturbance inputs to the system. As indicated above the disturbance vector d is further decomposed into v and w where v denotes measured disturbances, which may be used for feedforward compensation. The disturbance vector w represents the non-measurable inputs to the system. Furthermore, let y_0 be decomposed as

$$y_0 = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{2.3}$$

where y_1 denotes the n_1 vector of outputs to be controlled while y_2 denotes the n_2 vector of additional outputs.

The transfer function $G_0(z)$ may be partitioned as

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = G_0(z)\ u_0(t) = \begin{bmatrix} G_{1u}(z) & G_{1d}(z) \\ G_{2u}(z) & G_{2d}(z) \end{bmatrix} \begin{bmatrix} u(t) \\ d(t) \end{bmatrix}$$

or

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} G_{1u} & G_{1v} & G_{1w} \\ G_{2u} & G_{2v} & G_{2w} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{2.4}$$

2.4 Fractional Representations

Any transfer function matrix $G(z)$ of dimensions $n \times m$ may be decomposed into any of the fractional representations below. The matrices A_y, B_u, A_ξ and B_ξ are Z-generalized polynomial matrices. It is required that the matrices $A_y \in R_Z^{n\times n}[z]$ and $A_\xi \in R_Z^{m\times m}[z]$ are full rank polynomial matrices.

The system is then described by

$$A_y(z)\ y(t) = B_u(z)\ u(t) \quad \Rightarrow \quad G(z) = A_y^{-1}(z)\ B(z) \tag{2.5}$$

This factorization is called a left matrix fractional description (left m.f.d.).

Another description is given by

$$\begin{cases} A_\xi(z)\ \xi(t) = u(t) \\ y(t) = B_\xi(z)\ \xi(t) \end{cases} \Rightarrow \quad G(z) = B_\xi(z)\ A_\xi^{-1}(z) \tag{2.6}$$

This factorization is called a <u>right matrix fractional description</u> (right m.f.d.).

Consider a m.f.d (A,B) of the type (2.5) or (2.6) and such that $A(z) \in R^{nxn}$ and $B(z) \in R^{nxm}$ (or $A(z) \in R^{mxm}$ and $B(z) \in R^{nxm}$ resp.) loose rank simultaneously for a $z_0 \in Z_-$. Then z_0 will be called a <u>decoupled Z-unstable factor</u>. It is however possible to find another representation which does not contain any common factor such that A and B loose rank simultaneously in Z. A left m.f.d. with this property is a <u>left Z-coprime factorization</u> (l.Z.c.f) and is unique up to multiplication from the left by a Z-unimodular matrix.

Similarly, a <u>right Z-coprime factorization</u> (r.Z.c.f) is unique up to multiplication from the right by a Z-unimodular matrix.

The A-matrices in the above representations can be made triangular or block triangular by using the non uniqueness of the factorization.

2.5 Stability and Causality

A discrete time transfer function G(q) is stable, if it has no poles outside the unit disc. Analogously a continuous time transfer function G(p) is stable if it has no poles in the right half plane. The unstable regions are

$$Z_- = \{z: z\in C \text{ and } \operatorname{Re} z \geq 0\} \setminus \{0\} \quad ; \text{ (Continuous Time Case)}$$

(2.7)

$$Z_- = \{z: z\in C \text{ and } |z| \geq 1\} \setminus \{1\} \quad ; \text{ (Discrete Time Case)}$$

If stronger stability concepts are required, this will be called <u>practical stability</u>. In general an unstable region Z_- is defined, in which it is undesirable to have poles of a transfer function G(z). The stable region will be denoted by Z_+. Thus we have

$$Z_- \cup Z_+ = C \cup \{\infty\}$$

The transfer functions considered below will be required to be proper i.e.

$$\lim_{z\to\infty} \|G(z)\| < \infty \tag{2.8}$$

In the discrete time case this means that the system is causal, i.e. the output does not depend on future values of the inputs.

The same requirement for a continuous time system is the condition to avoid pure differentiators.

Properness may thus be guaranteed by avoiding poles at infinity. Since we will only treat proper systems in the sequel, we will impose the following conditions on the instability region Z_-. Then it is possible by to treat properness and stability by the same formalism.

- Z_- contains the regions of asymptotic instability
- Z_- is symmetric with respect to the real axis
- Z_- may not contain all points on the real axis
- Z_- contains the 'infinity point'

(2.9)

The fractional representations with polynomial entries of z corresponding to p or q are thus not allowed, since all polynomials have poles at infinity. This will be avoided by a transformation of the variable z such that Z_- including infinity is mapped into a bounded region Z_-^*.

Consider the transformation

$$z^* = \frac{b}{z - a} \tag{2.10}$$

where $a \in Z_+ \cap R$ and $b > 0$. Infinity will then be mapped into the origin, and the unstable region Z_-^* will be a bounded subset of C. The positive constant b may be chosen arbitrarily. In this presentation b is however always chosen such that $z^*=1$ corresponds to $z=1$ for discrete time and $z=0$ for continuous time systems. This means that $z^*=1$ corresponds to the static properties both of discrete time and continuous time systems.

For discrete time controllers we will choose $a=0$. This means that z^* corresponds to the backward shift operator q^{-1}.

For continuous time systems the parameter a is chosen as a negative real number.

A matrix M^* will denote a Z^*-generalized polynomial matrix expressed in the variable z^*. In particular, all polynomial matrices in z^* are Z^*-generalized polynomial matrices.

According to standard multivariable theory it can now be stated that there exist polynomial, fractional representations (A_y^*, B_u^*) and (A_ξ^*, B_ξ^*), which are l.Z^*.c.f and r.Z^*.c.f. respectively.

2.6 The Structure Matrix

Consider a Z^*-generalized polynomial matrix $M^* \in R_{Z^*}^{n \times m}[z^*]$ of rank r. The matrix M^* may be factorized as

$$M^*(z^*) = L^*(z^*)\ S^*(z^*)\ R^*(z^*) \tag{2.11}$$

where $L^* \in R_{Z^*}^{n \times n}[z^*]$, $S^* \in R^{n \times m}[z^*]$ and $R^* \in R_{Z^*}^{m \times m}[z^*]$. Here L^* is left Z^*-invertible, R^* is right Z^*-invertible and the polynomial matrix S^* contains all the Z^*-zeros of M^* and has no zeros outside Z_-^*. The matrix S^* is called the Z^*-Smith form of M^* (cf. [Ros]:2:3 and [Per 1]:II:2).

$$S^* = \begin{bmatrix} D^* & 0_{n,m-n} \end{bmatrix}, \qquad n < m$$

$$S^* = D^* , \qquad n = m$$

$$S^* = \begin{bmatrix} D^* \\ 0_{n-m,m} \end{bmatrix}, \qquad n > m \tag{2.12}$$

where D^* is $\mathrm{diag}(s_1^*, \ldots, s_r^*, 0, \ldots, 0)$.

The matrix D^* is a diagonal, polynomial matrix in the variable z^* and contains the invariant polynomials s_i^* on the principal diagonal. Each non-zero polynomial s_i^* is monic and divides s_{i+1}^* for i=1,2,....,r-1. The polynomials s_i^* have all their zeros in Z_-^*. The Z^*-Smith form is unique.

Another decomposition with less restrictions will be used in this presentation. The Smith-form is a particular decomposition of the form

$$M^* = M_L^* \; M_S^* \; M_R^* \tag{2.13}$$

with the same dimensions as L^*, S^*, R^* and D^*. Let the divisibility conditions of the invariant polynomials s_i^* be relaxed so that any ordering of the invariant polynomials on the principal diagonal is sufficient.

Let furthermore the condition of the monic, invariant polynomials be replaced by the requirement that

$$s_i^*(1) = 1 \qquad \text{for} \quad 1\leq i \leq r \tag{2.14}$$

for the class of decompositions given by (2.13). This requirement is always possible to satisfy in the ordinary stability case where $\{1\}\in Z_+^*$. A necessary condition for this decomposition to hold in the case of a practical stability requirement with $\{1\}\in Z_-^*$ is that $M^*(1)\neq 0$.

This decomposition will be used in lemma 3.1 and onwards.

Introduce the following concepts.

Definition 2.1: M_S^* is an internal Z^*-structure matrix of M^*.

The concept 'internal structure matrix' is introduced since it will be shown to - loosely speaking - represent the information transmission properties of the system. This concept is new and not found in Pernebo's presentation.

Definition 2.2: $M_{LS}^* = M_L^* M_S^*$ is the left Z^*-structure matrix of M^*.

Pernebo has shown that the left structure matrix of a system determines the servo properties of the system ([Per 1]:II:5).

Definition 2.3: $M_{SR}^* = M_S^* M_R^*$ is the right Z^*-structure matrix of M^*.

The right structure matrix of a system determines the properties of an input reconstructor acting on the output ([Per 1]:II.7).

Diagonalization of M_{LS}^* by multiplication from the right by a Z^*-generalized matrix N^* may be achieved. The matrix N^* is however not in general Z^*-unimodular, which implies that $\det(M^*N^*)$ in general will contain additional Z^*-zeros compared to $\det(M^*)$. Resulting, diagonal, Z^*-generalized polynomial matrices will be denoted by subscript 'D' e.g. M_D^*.

3 MULTIVARIABLE POLE PLACEMENT DESIGN

3.1 Introduction

The pole-placement controllers for systems with known parameters will be reviewed in this chapter.

In the first part the general concepts of pole placement and its limitations will be reviewed. Thereafter in §3.13 model matching and pole-placement algorithms compatible with linear parametric models and suitable for recursive estimation will be proposed.

3.2 Control Objective

The closed loop system is required to:

- be stable
- have no uncontrollable or unobservable Z^*-unstable modes
- be able to asymptotically follow a reference signal generated by a specified model
- be able to reject disturbances (3.1)

3.3 General Design Constraints

The control object will be required to be

- Time invariant
- Finite dimensional
- Causal
- Without feedthrough from u to y

- Controllable from u
- Observable from y (3.2)

Consider a control law of the type

$$R^{*}(z^{*})u(t) = - S^{*}(z^{*})y(t) + T^{*}(z^{*})u_{c}(t) \qquad (3.3)$$

An <u>admissible</u> pole placement controller is

- Linear
- Causal
- Able to stabilize the closed loop system

It is also required that

- R^{*} and S^{*} are relatively left Z^{*}-coprime (3.4)

All pole placement problems do not have causal solutions. The conditions for existence can conveniently be expressed by the structure matrix. ([Per 1]:II:7).

The structure matrix is important to know for all pole placement design in general and for adaptive pole placement design in particular. Explicit conditions will be described below.

When designing polynomial identities to obtain certain closed loop poles, it is necessary to consider that any attempt to cancel the Z^{*}-zeros of the control object will introduce decoupled unstable modes or - when exact cancellation does not occur - new unstable poles.

The usual pole-placement algorithms do not avoid any cancellations of unstable zeros in a control object unless all zeros that should remain in the transfer function are explicitly specified.

The required explicit knowledge of the Z^*-zeros of the control object $G^*=B^*A^{*-1}$ may be represented in several ways. The quantities of least complexity containing all the Z^*-zeros are

$$\det G^*(z^*)$$

or

$$\det B^*(z^*) \tag{3.5}$$

It is in general not sufficient to know only (3.5) in order to solve the model matching problem. As will be indicated in §3.4 it is necessary to know a left structure matrix

$$B^*_{LS} \tag{3.6}$$

in order to guarantee a solution to the model matching problem. Although the proposed solution of the adaptive problem is based on a model matching problem, it will be shown to be sufficient to know an internal structure matrix

$$B^*_S \tag{3.7}$$

which is an entity of less complexity then B^*_{LS}.

3.4 Servo Design Constraints

Consider a r.Z^*.c.f. representation of a control object.

$$\begin{cases} A^*_\xi \xi = u \\ y = B^*_\xi \xi \end{cases} \tag{3.8}$$

Choose a linear controller in a l.Z^*.c.f. form

$$R^*_u u = - S^*_y y + T^* u_c \tag{3.9}$$

The closed loop system is given by

$$y = B^*_\xi \left[R^*_u A^*_\xi + S^*_y B^*_\xi \right]^{-1} T^* u_c \tag{3.10}$$

The left structure matrix B^*_{LS} of the B^*-matrix represents the Z^*-non-invertible part of the system.

An admissible controller will result in a closed loop system of the type

$$y = B^*_{LS} F^*_u u_c \tag{3.11}$$

where F^*_u is any Z^*-generalized polynomial matrix. The properties of the left structure matrix in servo design are investigated in ([Per 1]:II:5 and [Per 2]).

3.5 Disturbance Rejection

In regulator design we are mainly concerned with a class of controllers that give good properties of the transfer function between disturbance inputs and the controlled outputs. The expression

$$G^*_{dy} = \left[G^*_{vy} \quad G^*_{wy} \right] \tag{3.12}$$

obtained below in §3.10 for a right m.f.d. of the control object has however no transfer function that is easy to analyse. In order to give some hints about disturbance rejection the alternative fractional representation will be used to indicate some necessary properties.

The feedforward characteristics for closed loop systems derived from right m.f.d. of the control object will be reconsidered in §4.4 below, where some matrix identity results are available to facilitate the analysis.

Let the system be described by the $1.Z^*$.c.f. representation

$$A^*_y(z^*)\ y(t) = B^*_u(z^*)\ u(t) + C^*_d(z^*)\ d(t) \tag{3.13}$$

and the controller by the $r.Z^*$.c.f. expression

$$\begin{cases} R^*_\xi(z^*)\ \xi(t) = y(t) \\ u(t) = -\ S^*_\xi(z^*)\ \xi(t) \end{cases} \tag{3.14}$$

The closed loop system is given by

$$y(t) = R^*_\xi(z^*)\left[A^*_y(z^*)R^*_\xi(z^*) + B^*_u(z^*)S^*_\xi(z^*)\right]^{-1}C^*_d(z^*)d(t)$$

(3.15)

The zeros of

$$U^{*}(z^{*}) = \left[A_{y}^{*}(z^{*})R_{\xi}^{*}(z^{*}) + B_{u}^{*}(z^{*})S_{\xi}^{*}(z^{*})\right] \tag{3.16}$$

become poles of the closed loop system. Any admissible controller, which stabilizes the system, requires $U^{*}(z^{*})$ to be Z^{*}-unimodular. Therefore no zeros of C_{d}^{*} may be cancelled.

The matrix R_{ξ}^{*} is essentially determined by the choice of U^{*}. The invariant part of the transfer function is thus determined by the R_{ξ}^{*}-matrix of the controller and the right Z^{*}-structure matrix of C_{d}^{*} of the disturbance inputs to the system.

If the right structure matrix C_{SR}^{*} of C_{d}^{*} is factorized to the right, it is seen that any admissible controller will result in a closed loop system of the type

$$y(t) = F_{d}^{*}(z^{*})\ C_{SR}^{*}(z^{*})\ d(t) \tag{3.18}$$

where F_{d}^{*} is some stable Z^{*}-generalized polynomial matrix essentially determined by R_{ξ}^{*}

The conclusion to make is that although (3.18) has a formal similarity to the corresponding servo expression (3.11) there is less freedom of choice in the disturbance rejection case. There are strong conditions imposed on F_{d}^{*} while F_{u}^{*} is essentially free to choose. The restrictions on R_{ξ}^{*} are determined by the requirements of (3.16), where R_{ξ}^{*} has to be chosen such that it 'compensates' the Z^{*}-zeros of B_{u}^{*}.

The properties of disturbance rejection in a more general setting are investigated by Pernebo([Per 1]:II:ch.7).

3.6 The Control Object

After these two comments on the role of the structure matrix we will return to the full problem introduced in §2.3.

From now on the control object will be presented by a r.Z^*.c.f. pair (A^*, B^*) and the controller by a l.Z^*.c.f. triple (R^*, S^*, T^*) unless otherwise specified. All indices introduced above regarding the type of factorization will be omitted when not explicitly needed.

A general fractional representation for a linear, multivariable system S_O is given by a block triangular r.Z^*.c.f.

$$\begin{bmatrix} A^*_{uu}(z^*) & A^*_{uv}(z^*) & A^*_{uw}(z^*) \\ 0 & A^*_{vv}(z^*) & A^*_{vw}(z^*) \\ 0 & 0 & A^*_{ww}(z^*) \end{bmatrix} \begin{bmatrix} \xi_u(t) \\ \xi_v(t) \\ \xi_w(t) \end{bmatrix} = \begin{bmatrix} u(t) \\ v(t) \\ w(t) \end{bmatrix}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} B^*_{1u}(z^*) & B^*_{1v}(z^*) & B^*_{1w}(z^*) \\ B^*_{2u}(z^*) & B^*_{2v}(z^*) & B^*_{2w}(z^*) \end{bmatrix} \begin{bmatrix} \xi_u(t) \\ \xi_v(t) \\ \xi_w(t) \end{bmatrix}$$

(3.23)

where

y_1 – Measured outputs to be controlled

y_2 – Other measured outputs

u – Control input

v – Measured disturbance inputs

w – Non-measurable disturbance inputs

It is assumed that A^*_{uu}, A^*_{vv} and A^*_{ww} are quadratic polynomial matrices of full rank which means no restriction. It is also assumed that A^*_{vv} and A^*_{ww} are Z^*-unimodular matrices. This requirement is needed to guarantee the existence of a feedback stabilizing linear controller which may also be seen from (3.36) below. The system is not stabilizable, if A^*_{vv} or A^*_{ww} are not Z^*-unimodular. Attempts to compensate for a v with unstable poles originating from zeros in Z^*_- of A^*_{vv} would give a design scheme with cancellations of common factors with Z^*-zeros. The case of an A^*_{vv} with zeros in Z^*_- will therefore not be considered further.

Assumption 3.1

A particular case, where the number of independent control inputs is equal to the number of full rank controlled outputs, will be emphasized below. The rank condition does obviously never allow the number of full rank controlled outputs to surpass the number of linearly independent control inputs.

This specialization only imposes the mild restriction that any remaining control inputs will be formally referred to as a part of the disturbance input vector v for which no controller parametrizations will be derived.

A particular right matrix factorization of the transfer function between u and y_1 will now be presented in order to facilitate the analysis below. This factorization is merely a particular choice among all possible choices of factorizations of the control object and does not impose any restrictions as will be seen in the following lemma.

<u>Lemma 3.1</u>

A quadratic (nxn) transfer function G^* of full rank corresponding to a strictly proper transfer function G may be decomposed into a relatively r.Z^*.c.f. (A^*, B^*). The factorization is such that A^* contains all the Z^*-poles of G^* and B^* contains all Z^*-zeros of G^* and such that

$$G^* = B^* A^{*-1} = B_L^* B_S^* B_R^* A^{*-1} \qquad (3.24)$$

where

A^* is a quadratic polynomial matrix

B_L^* and B_R^* are polynomial Z^*-unimodular matrices

B_S^* is a diagonal polynomial matrix (3.25)

satisfying

$A^*(0) = I$

$B_S^*(1) = I$

$B_R^*(0) = \begin{bmatrix} b_{ij} \end{bmatrix}_R$ is upper right triangular and invertible

$(b_{ii})_R = 1$ for $1 \le i \le n$

where B_S^* contains all zeros of G^* in Z_-^* but has no zeros in Z_+^*. The stability region considered is assumed to be according to (2.7).

Remark 3.1

It is necessary to demand $\|G^*(1)\| \neq 0$ for the lemma to hold if a stronger practical stability requirement with $\{1\} \in Z_-^*$ is desired(cf.(2.14)). The critical point is that the monic polynomials of the Z^*-Smith form may not rescaled such that $s_i^*(1)=1$. This is essentially a static, full rank condition which is necessary to impose in order to satisfy static gain requirements for y_1.

A proof is given in appendix 3:1 in the form of a construction of one such decomposition. This is based on the Z^*-Smith form.

A decomposition of the transfer function G_{11}^*, according to lemma 3.1, is

$$G_{11}^* = B_{1u}^* A_{uu}^{*-1} \qquad (3.26)$$

where A^*_{uu} and B^*_{1u} are polynomial matrices in z^*

$$A^*_{uu}(0) = I$$

and

$$B^*_{1u} = B^*_L B^*_S B^*_R \qquad (3.27)$$

with the properties given in the lemma.

3.7 Structure Matrix Considerations

The following definitions are introduced (cf. §2.6). with the factorizations of G^*_{11} given by (3.26)-(3.27).

B^*_S (The Internal Structure Matrix)

$B^*_{LS} = B^*_L B^*_S$ (The Left Structure Matrix)

The left structure matrix B^*_{LS} expresses the noninvertible invariant part of the control object under the constraints that only admissible controllers are used. Recall that admissibility requires the controller to be causal and to result in a stable closed loop system. The term servo structure matrix may also be used.

The left structure matrix of the system in §3.6 is strictly the left structure matrix of $B^*_{\cdot u}$ (3.38). However, since y_1 are the controlled outputs, it is desirable to emphasize the closed loop conditions for y_1. In order to satisfy this, B^*_{LS}, which is the square left structure matrix of B^*_{1u} and decides the transmission properties from input to output, will be referred to as the left structure matrix. This holds for all block triangular r.Z^*.c.f:s of the type represented by (3.23).

A trade-off between the restrictions imposed above and the specifications of the control objective gives rise to the general design scheme, which is to be stated below.

3.8 The Reference Model

Let the model to be followed be restricted to models of the form

$$\begin{cases} A^*_M(z^*)\ y^m_1(t) = B^*_M(z^*)\ u_c(t) \\ y^M_1(t) = B^*_{LS}(z^*)\ y^m_1(t) \end{cases} \qquad (3.28)$$

It was made plausible in §3.4 that a left structure matrix $B^*_{LS}(z^*)$ must be incorporated in any reference model to be followed perfectly by a system with the characteristics of the control object above (3.23).

The matrices A^*_M and B^*_M should be relatively left Z^*-coprime. The matrix $A^*_M(z^*)$ is required to be Z^*-unimodular since the reference model has to be stable. The factorization between A^*_M and B^*_M will be chosen such that

$$A^*_M(0) = I \qquad (3.29)$$

which does not impose any restrictions, since the transfer functions are required to be of full rank and proper. Triangularization of A^*_M by elementary row operations on A^*_M and B^*_M may also be achieved.

Two different control objectives satisfying (3.1)-(3.4) and described in terms of reference models will now be studied. The control objectives will be referred to as the regulator case and the servo case respectively.

1) The regulator case should have a reference model of the type (3.28). The closed-loop system will be required to be stable, to have certain closed-loop poles, to have a correct static gain. There are no specifications on transient cross-coupling properties of the transfer function from the command signal u_c to the output y^M.

2) The servo case should satisfy not only the requirements for the regulator case but should also have completely specified transient properties. A straightforward way to assure this is to require a diagonal transfer function from y^m_1 to y^M_1. Then $B^*_{LS}(z^*)$ in (3.28) has to be replaced by $B^*_D(z^*)$ which is the left structure matrix $B^*_{LS}(z^*)$ diagonalized from the right by a polynomial matrix T^*_3 (cf.§3.16).

The reference model for the diagonal case becomes

$$\begin{cases} A^*_M(z^*)y^m_1(t) = B^*_M(z^*)u_c(t) \\ y^M_1(t) = B^*_D(z^*)y^m_1(t) \end{cases} \tag{3.30}$$

The difference between these two schemes in the case of a known control object will be considered in §3.11 and §3.12.

The requirement that B^*_{LS} should be incorporated in the model is no particular restriction for the investigation of cross coupling and static gain properties when B^*_{LS} is known. It is however an awkward problem in the adaptive context since B^*_{LS} is unknown and essentially impossible to manipulate by the controller.

Although B^*_{LS} must be incorporated in the transfer function it is not necessary to know B^*_{LS} apriori in order to solve the adaptive problem as will be shown below in §4.5 and §4.6. If regulator transfer function properties with correct static servo properties are sufficient, it will be shown to be sufficient to know a B^*_S.

Similarly, it will be shown to be sufficient to know a B^*_S and a feasible B^*_D in order to solve the servo problem.

3.9 The Pole Placement Controller

Let the controller be given by

$$R^*_u(z^*)u(t) = - S^*_y(z^*)\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} + T^*(z^*)\begin{bmatrix} u_c(t) \\ v(t) \end{bmatrix} \qquad (3.31)$$

The matrices R^*_u and S^*_y will always be chosen as polynomial matrices with respect to z^* but may in general be chosen as any kind of Z^*-generalized polynomial matrices. The matrix T^* is required to be a Z^*-generalized polynomial matrix.

Denote

$$u_1(t) = \begin{bmatrix} u_c(t) \\ v(t) \end{bmatrix} \tag{3.32}$$

Let the $S_y^*(z^*)$-matrix be partitioned into

$$S_y^*(z^*) = \begin{bmatrix} S_1^*(z^*) & S_2^*(z^*) \end{bmatrix} \tag{3.33}$$

where each submatrix corresponds to the observations $[y_1^T \; y_2^T]$.

Let the $T^*(z^*)$-matrix be partitioned into

$$T^*(z^*) = \begin{bmatrix} T_{uc}^*(z^*) & T_{ff}^*(z^*) \end{bmatrix} \tag{3.34}$$

where T_{uc}^* and T_{ff}^* represent the feedforward compensations from the command signal u_c and the measurable disturbance inputs v respectively.

3.10 The Closed Loop System

If the expression (3.31) is substituted into (3.23), the following is obtained

$$R_u^*(z^*)\begin{bmatrix} A_{uu}^*(z^*) & A_{uv}^*(z^*) & A_{uw}^*(z^*) \end{bmatrix}\xi_u(t) + S_y^*(z^*)B_{.u}(z^*)\xi_u(t) =$$

$$= T^*(z^*)u_1(t) = T_{uc}^*(z^*)u_c(t) + T_{ff}^*(z^*)v(t) \tag{3.35}$$

The system equations for the closed loop system become

$$\left[\begin{array}{c|c|c} R_u^* A_{uu}^* + S_y^* B_{.u}^* & R_u^* A_{uv}^* + S_y^* B_{.v}^* & R_u^* A_{uw}^* + S_y^* B_{.w}^* \\ \hline 0 & A_{vv}^* & A_{vw}^* \\ \hline 0 & 0 & A_{ww}^* \end{array}\right] \left[\begin{array}{c} \xi_u \\ \hline \xi_v \\ \hline \xi_w \end{array}\right] =$$

$$= \left[\begin{array}{c|c|c} T_{uc}^* & T_{ff}^* & 0 \\ \hline 0 & I & 0 \\ \hline 0 & 0 & I \end{array}\right] \left[\begin{array}{c} u_c \\ \hline v \\ \hline w \end{array}\right]$$

$$\left[\begin{array}{c} y_1 \\ \hline y_2 \end{array}\right] = \left[\begin{array}{c|c|c} B_{1u}^* & B_{1v}^* & B_{1w}^* \\ \hline B_{2u}^* & B_{2v}^* & B_{2w}^* \end{array}\right] \left[\begin{array}{c} \xi_u \\ \hline \xi_v \\ \hline \xi_w \end{array}\right] \qquad (3.36)$$

The transfer functions of the closed loop system are then obtained as

$$G^*_{u_c y_0} = B^*_{.u}\left[R^*_u A^*_{uu} + S^*_y B^*_{.u}\right]^{-1} T^*_{uc}$$

$$G^*_{vy_0} = B^*_{.u}\left[R^*_u A^*_{uu} + S^*_y B^*_{.u}\right]^{-1}\left[T^*_{ff} + \left[R^*_u A^*_{uv} + S^*_y B^*_{.v}\right] A^{*-1}_{vv}\right] + B^*_{.v} A^{*-1}_{vv}$$

$$G^*_{wy_0} = B^*_{.u}\left[R^*_u A^*_{uu} + S^*_y B^*_{.u}\right]^{-1}\left[\left[R^*_u A^*_{uv} + S^*_y B^*_{.v}\right] A^{*-1}_{vv} A^*_{vw} - \left[R^*_u A^*_{uw} + S^*_y B^*_{.w}\right]\right] A^{*-1}_{ww}$$

$$- B^*_{.v} A^{*-1}_{vv} A^*_{vw} A^{*-1}_{ww} + B^*_{.w} A^{*-1}_{ww} \qquad (3.37)$$

where $B^*_{.u}$ etc. denotes

$$B^*_{.u} = \begin{bmatrix} B^*_{1u} \\ B^*_{2u} \end{bmatrix} \qquad (3.38)$$

Since it has already been assumed that A^*_{vv} and A^*_{ww} are polynomial and Z^*-unimodular matrices, the closed loop system will be stable iff the polynomial matrix

$$\left[R^*_u(z^*)A^*_{uu}(z^*) + S^*_y(z^*)B^*_{.u}(z^*)\right] \qquad (3.39)$$

is Z^*-unimodular. The full expression

$$\det\left[R^*_u(z^*)A^*_{uu}(z^*) + S^*_y(z^*)B^*_{.u}(z^*)\right]\det A^*_{vv}(z^*) \det A^*_{ww}(z^*) \qquad (3.40)$$

gives some of the closed loop poles of the system. In addition there are other Z^*-stable poles which are not explicitly taken account of in B^*_L and T^*_{uc}. By choosing A^*_M and B^*_M it is however possible to choose which poles to have.

3.11 Feasible Regulator Case Transfer Functions

The conditions on controllers of the type (3.31) will now be given for the case when the control object factorization (A^*, B^*) is known. It will be required that the model matching properties given by the reference model are satified.

Model matching means that the transfer function matrix from u_c to y_1 coincides with that of the reference model (3.28) i.e.

$$B^*_{1u}\left[R^*_u A^*_{uu} + S^*_y B^*_{\cdot u}\right]^{-1} T^*_{uc} = B^*_{LS} A^{*-1}_M B^*_M \qquad (3.41)$$

This is fulfilled, if the following polynomial matrix identities hold (cf. [Per 1]:II:2.5 and App.3:2 Step 2).

$$R^*_u A^*_{uu} + S^*_y B^*_{\cdot u} = T^*_1 A^*_M B^*_R$$

$$T^*_{uc} = T^*_1 B^*_M \qquad (3.42)$$

where T^*_1 is any Z^*-unimodular polynomial matrix that will be cancelled in the transfer function from u_c to y_1.

Remark 3.2

In the case where $B^*_{\cdot u} = B^*_{1u}$ it holds that the matrix B^*_R is a right divisor of $T^*_1 A^*_M B^*_R$ and of $S^*_y B^*_{\cdot u}$. Then B^*_R must also be a factor of $R^*_u A^*_{uu}$ for all matrices R^*_u satisfying the equation

(3.42). In the SISO-case it is standard to cancel such common factors. This is not done here in order to simplify the matrix notation.

3.12 Feasible Servo Case Transfer Functions

The polynomial identities (3.42) can in general not satisfy the equation (3.41) if B^*_{LS} is replaced by B^*_S in the reference model (3.28). Morover there does not in general exist any solution to the problem of finding a diagonal transfer function with the same number of zeros in Z^*_- as $\det(B^*_S)$. This is due to the fact that B^*_{1u} is not in general diagonalizable from the right by any Z^*-unimodular matrix.

If the reference model (3.30) is used, a solution is obtained if the identities below are satisfied

$$R^*_u A^*_{uu} + S^*_y B^*_{\cdot u} = T^*_1 A^*_M B^*_R$$

$$T^*_{uc} = T^*_1 A^*_M T^*_3 A^{*-1}_M B^*_M$$

$$B^*_{LS} T^*_3 = B^*_D \qquad (3.43a-c)$$

for a diagonal B^*_D such that T^*_3 is a polynomial matrix. In the same way as in the regulator case T^*_1 is any Z^*-unimodular polynomial matrix.

The choice of T^*_{uc} is given as a possible Z^*-generalized polynomial matrix solution. However, there are many possible choices and in the final choice for the adaptive system a slightly modified version is chosen.

It is more convenient to use polynomial matrices. Since T^*_{uc} in (3.43b) is a Z^*-generalized polynomial matrix this may be substituted by the polynomial matrix T^*_{ym} in the control law

$$T^*_{ym} = T^*_1 A^*_M T^*_3 \quad \Rightarrow \quad T^*_{uc} u_c = T^*_{ym} y^m_1 \tag{3.44}$$

Remark 3.3

'Truely' diagonal transfer functions may be achieved also by diagonalizing the open loop system after which ordinary SISO control design methods might be applied. This design scheme obviously requires the knowledge of the diagonalizing matrix and/or its inverse. Although these matrices might be found in an adaptive regime, the controller design problem remains to be solved. Since it is not in the scope of this presentation to deal with the kind of nested adaptive loops that then would occur, this case will not be elaborated.

3.13 A New Pole Placement Model For Adaptive Control

In this paragraph a pole placement basis for adaptive control is formulated. The derivations are made in a fairly general framework in order to give possibilities for adaptive implementation of a wide class of pole placement solutions that satisfy the control objectives given in §3.8.

A second objective is to find a class of adaptive, linear controllers based on the pole placement solutions indicated in (3.42) and (3.43).

A third objective is to satisfy the demands imposed by requirements of parametric models for estimation of adaptive controllers.

Consider first the case of servo design where the objective is to assign a prescribed behaviour from the command signal u_c to the controlled output y_1. This is to be treated with the methods of (3.42) and (3.43).

The regulator design with respect to measurable disturbance inputs v can be treated within the same framework, since the only conceptual difference compared with the servo case is the adaptive feedforward terms T^*_{ff}. The performance of the adaptive feedforward will not be considered here but in the chapter of parametric models (§4.9) because of the non-attractive present form of the transfer function G^*_{vy}. Some new identities (3.50a-c), which are also useful for a simplification of G^*_{vy}, will first be introduced.

The case of regulator design with respect to non measurable disturbances w will not be treated what so ever due to poor modeling possibilities of the linearly parametric properties. The transfer function properties obtained in (3.37) however guarantee stability and boundedness with respect to bounded disturbances w since the closed loop system has stable transfer functions(cf.(3.37), (3.41)). It is however not possible to neglect the importance of w in the identification algorithms. The accuracy of parameter estimates obtained from an identification algorithm is highly dependent of influence from unknown inputs to the system.

If the full left structure matrix B^*_{LS} of the system is known then it is also possible to find a suitable B^*_M to satisfy desired specifications on static gains etc. as already commented in §3.8. This is a difficulty in the case when B^*_{LS} is not known in the design schemes (3.42)-(3.43). The static gain will not be correct and there are poles of B^*_L which are not cancelled etc..

The requirement that B^*_{LS} is known is however very restrictive and to be able to make an adaptive design with only the internal structure matrix B^*_S as the apriori knowledge the following pole placement scheme is introduced to replace (3.42) and (3.43).

The diagonal B^*_S is certainly necessary to know in order to guarantee a stable closed loop system, when a regulator is designed by a pole placement algorithm of the considered types. Further comments are given at the end of this paragraph.

The servo transfer functions are repeated for convenience

$$G^*_{u_c y_1}(z^*) = B^*_{1u}(z^*)\left[R^*_u(z^*)A^*_{uu}(z^*) + S^*_y(z^*)B^*_{.u}(z^*)\right]^{-1}T^*_{uc}(z^*) \tag{3.45}$$

In particular the pole placement requirements formulated by the reference model are satisfied by the following expressions which could be regarded as a modification of

(3.42)-(3.43).

Let from now on A^*_M be a lower triangular matrix (cf. (3.29)). Let T^*_1 be a lower triangular Z^*-unimodular polynomial matrix such that

$$T^*_1(0) = I \quad \Rightarrow \quad T^*_1(0)A^*_M(0) = I \tag{3.46}$$

Consider the polynomial, internal structure matrix

$$B^*_S = \begin{bmatrix} b^*_1(z^*) & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & b^*_n(z^*) \end{bmatrix} \tag{3.47}$$

The off-diagonal elements of the lower triangular matrix

$$T^*_1 A^*_M = \begin{bmatrix} p^*_{11}(z^*) & & 0 \\ \vdots & \ddots & \\ p^*_{n1}(z^*) & \cdots & p^*_{nn}(z^*) \end{bmatrix} \tag{3.48}$$

should be chosen such that each polynomial p^*_{ij} of a certain column j in $T^*_1 A^*_M$ has the polynomial b^*_j of the B^*_S-matrix as a factor.

$$p^*_{ij}(z^*) = b^*_j(z^*)p'^*_{ij}(z^*) \quad \text{for} \quad 1<i<n \quad \text{and} \quad j<i \tag{3.49}$$

This specification is satisfied by a particular choice of T^*_1 which is lower, left triangular and Z^*-unimodular. There is no conflict with the requirement (3.46) for $z^*=0$ when direct feedthrough is not present.

The following lemma is shown in appendix 3:2.

Lemma 3.2

There exist polynomial matrix solutions R_u^*, S_y^*, T_2^*, T_3^* and a diagonal B_D^* satisfying the following equations for given A_M^*, A_{uu}^*, B_L^*, B_S^*, B_R^*, $B_{\cdot u}^*$ and T_1^* with the properties given by the expressions of (3.27-30) and (3.48-49). The matrix T_2^* is a polynomial Z^*-unimodular matrix.

$$T_2^*T_1^*A_M^* B_L^*B_S^* = B_S^* T_1^*A_M^*$$

$$R_u^*A_{uu}^* + S_y^*B_{\cdot u}^* = T_1^*A_M^*B_R^* \quad ; \qquad R_u^*(0) = B_R^*(0) \overset{\Delta}{=} B_0$$

$$T_2^*T_1^*A_M^*B_D^* = B_S^*T_3^* \qquad (3.50a\text{-}c)$$

The following theorem gives a model matching basis for adaptive control.

Theorem 3.1

The transfer functions of the closed loop system (3.45) and the models (§3.8) coincide if R_u^*, S_y^* and T^* are chosen as follows for polynomial matrix solutions R_u^*, S_y^*, T_2^*, T_3^* and diagonal B_S^* and B_D^*. The polynomial matrix T_2^* is Z^*-unimodular.

$$T_2^* T_1^* A_M^* B_L^* B_S^* = B_S^* T_1^* A_M^*$$

$$R_u^* A_{uu}^* + S_y^* B_{.u}^* = T_1^* A_M^* B_R^* \; ; \; R_u^*(0) = B_0$$

$$T_{uc}^* = T_2^* T_1^* B_M^* \qquad \text{(Regulator Case)}$$

$$T_2^* T_1^* A_M^* B_D^* = B_S^* T_3^*$$

$$T_{ym}^* = T_3^* \qquad \text{(Servo Case)}$$

(3.51a-e)

Proof:

Step 1

The existence of the identities (3.50a-c) are shown by lemma 3.2.

Step 2 Closed Loop Identities

The transfer function to consider is the closed loop transfer function (3.45). The identities (3.45) and (3.51a-c) give via straightforward matrix calculations the transfer functions

$$G_{u_c y_1}^*(z^*) = G_{y_m y_1}^*(z^*) G_{u_c y_m}^*(z^*) \qquad (3.52)$$

where

$$G^*_{u_c y_1}{}^m(z^*) = A_M^{*-1}(z^*)B_M^*(z^*)$$

$$G^*_{y_1 y_1}{}^m(z^*) = B^*_{1u}\left[R^*_u A^*_{uu} + S^*_y B^*_{.u}\right]^{-1} T^*_2 T^*_1 A^*_M \tag{3.53}$$

$$= B^*_L B^*_S B^*_R \,(T^*_1 A^*_M B^*_R)^{-1}\, T^*_2 T^*_1 A^*_M = \left[T^*_2 T^*_1 A^*_M\right]^{-1} B^*_S \left[T^*_2 T^*_1 A^*_M\right]$$

The regulator case which has an unknown and unspecified B^*_L but requires correct controller identities and correct static gain is satisfied for this closed loop behaviour since

$$\det\left[G^*_{u_c y_1}(z^*)\right] = \det\left[B^*_{LS}(z^*)T^*_2(z^*)A^{*-1}_M(z^*)B^*_M(z^*)\right] =$$

$$= \det\left[B^*_S(z^*)A^{*-1}_M(z^*)B^*_M(z^*)\right] \tag{3.54}$$

and for the static gain

$$G^*_{y_1 y_1}{}^m(1) =$$

$$= \left[T^*_2(1)T^*_1(1)A^*_M(1)\right]^{-1} B^*_S(1)\left[T^*_2(1)T^*_1(1)A^*_M(1)\right] = I \tag{3.55}$$

The poles of the reference model have their counterparts in the closed loop system and the static gain is that of the reference model. This case has sufficient regulator properties and is able to set point control. The transient cross coupling properties may not be investigated without knowledge of T^*_2.

The servo case is a well specified model matching problem. It makes use of the alternative T^*_{ym} (3.44) and may thus also guarantee a transient servo behaviour without cross-couplings.

The following calculations show that the model-matching problem has a satisfactory solution through the identities (3.51d-e).

$$G^{*m}_{y_1 y_1}(z^*) = B^*_L B^*_S \left[T^*_1 A^*_M\right]^{-1} T^*_3 =$$

$$= \left[\left[T^*_2 T^*_1 A^*_M\right]^{-1} B^*_S \left[T^*_1 A^*_M\right]\right] \left[T^*_1 A^*_M\right]^{-1} T^*_3 =$$

$$= \left[T^*_2 T^*_1 A^*_M\right]^{-1} B^*_S T^*_3 = \left[T^*_2 T^*_1 A^*_M\right]^{-1} \left[T^*_2 T^*_1 A^*_M\right] B^*_D = B^*_D \qquad (3.56)$$

These matrix manipulations are allowed since T^*_2, T^*_1 and A^*_M are Z^*-unimodular matrices.

<u>Remark 3.4</u>

In order to guarantee a stable closed loop it is necessary to know that the inverted matrices B^*_L and B^*_R do not contain any Z^*-zeros. This is achieved by explicit separation of all Z^*-zeros of B^*_{1u} into B^*_S. It is a general property of any explicit pole placement algorithm in order to avoid bad cancellations and not particular to the proposed adaptive design.

4 PARAMETRIC MODELS

4.1 Introduction

The first purpose of this chapter is to find a representation of the type

$$u_i = \vartheta_i^T \phi_i \tag{4.1}$$

for each control variable u_i. The ϕ-vector denotes some known data vector and ϑ denotes the controller parameters associated with a pole placement algorithm. The design properties of the pole placement controller should be fulfilled by using the control input of (4.1).

The adaptive problem is however precisely that the parameter vector ϑ_i is not known. Thus the second problem of this chapter is to find equations, which allow estimation of ϑ_i. Assume that a sequence of known data $\{[\bar{u}(t),\bar{\Phi}(t)]\}$ satisfies the same equation i.e.

$$\bar{u}_i(t) = \vartheta_i^T \bar{\Phi}_i(t) \tag{4.2}$$

Then it is possible to estimate ϑ_i by standard vector space methods.

Let this discussion be repeated as follows.

The first objective of this chapter is to find a linear expression of the type (4.1) for the control input such that the control law realizes a pole placement controller of the type given in §3.13.

The second objective of this chapter is to find a function of data, that forms a sequence $\{[\tilde{u}_i, \tilde{\Phi}_i]\}$ related to ϑ_i as in (4.2), such that the parameters ϑ_i for the control may be estimated.

There are a few problems to be solved in association with such a parameter 'translation'. It is strongly desirable to avoid anything but polynomial relations since nonlinear expressions of parameter estimates are difficult to handle. Furthermore - to avoid parameter interaction that would lead to some multi-output estimation schemes based on the parametric models, it will be shown to be desirable to keep the expression of the (diagonal) internal structure matrix B^*_S. This will be shown in §4.5, §4.6 in the derivation of expressions of the type (4.1) from the pole placement control laws. The importance of the internal structure matrix B^*_S will be shown in these paragraphs.

The parametrization problem will be solved by making a particular decomposition ((4.19), (4.22))of the open loop transfer function. This parametric model has the property that the pole placement control law is almost explicitly expressed. Another property is that the control law parameters will be shown to be possible to identify(§4.10).

4.2 LIP-Models

Definition 4.1

A parameter vector ϑ will be said to have a linear, identifiable form if there exist known functions of data which forms a vector $\phi_i(t)$ and a known scalar $\nu_i(t)$ which are related to ϑ such that

$$\phi_i^T(t)\ \vartheta = \nu_i(t) \tag{4.3}$$

where the coefficient of ν_i is equal to 1.

Definition 4.2

A parametric model of a system S will be said to be a linear, identifiable parametric (LIP) model, if the system output may be represented in the form (4.3) where ν_i and ϕ_i correspond to known functions of input and output data.

Example 4.1

A left m.f.d. with $A_y(0)=I$ of a system S is a LIP-model for the output y if all inputs and outputs are measurable.

Proof:

The A_y-matrix may be decomposed as

$$A_y^*(z^*) = A_y^*(0) + \left[A_y^*(z^*)-A_y^*(0)\right]$$

It is readily seen that it is possible to express

$$y(t) = -\left[A_y^*(z^*)-I\right]y(t) + B_u^*(z^*)u(t) =$$

$$= \left[-(A_y^*(z^*)-I) \;\; B_u^*(z^*)\right]\begin{bmatrix} y(t) \\ u(t) \end{bmatrix}$$

Since the polynomial matrices (A_y^*-I) and B_u^* disappear completely for $z^*=0$, the equation is explicit in y. The right hand side contains u and y operated upon by (finite) powers $\geq$ 1 of z^*. (In the discrete time case these correspond to old data and for the continuous time case this corresponds to to lowpass-filtered inputs and outputs.) Thus the right hand side is a function of known data.

Each of y_i may now be written as

$$y_i = \vartheta_i^T \phi_i \tag{4.4}$$

where ϕ_i contains u and y filtered by finite powers greater than zero of z^*.

<u>Remark 4.1</u>

A right m.f.d. is not a LIP-model since the internal variable ξ cannot be measured.

<u>Definition 4.3</u>

An adaptive controller will be said to be <u>linear in the parameter estimates</u> if there exist linear, identifiable control parameter vectors θ_i such that the output from the controllers i.e. the control variables u_i can be expressed

$$u_i = - \theta_i^T \varphi_i \qquad (4.5)$$

where the φ_i's represent finite vectors of measured inputs and outputs.

Remark 4.2

Adaptive systems are in general time varying and nonlinear systems.

This presentation is an attempt to avoid all explicitly nonlinear formulations, which involve division by parameter estimates etc.. The objective is to find bilinear expressions for the adaptive pole placement controllers that are linear in the parameter estimates as well as in measured data. i.e. applied control inputs, disturbances and outputs.

The formulation of LIP-models for each control input forms a basis for avoiding most of the explicitly nonlinear expressions.

4.3 Model Tranformations

The transfer function to the controlled outputs in (3.36) may be written (use eq.(3.51a-e) and (3.23)).

$$y_1(t) = B^*_{1u}(z^*)\xi_u(t) + B^*_{1v}(z^*)\xi_v(t) + B^*_{1w}(z^*)\xi_w(t) =$$

$$= B^*_{1u}\left[T^*_1A^*_MB^*_R\right]^{-1}\left[R^*_uA^*_{uu} + S^*_yB^*_{y.u}\right]\xi_u(t) +$$

$$+ B^*_{1v}\xi_v(t) + B^*_{1w}\xi_w(t) =$$

$$= B^*_LB^*_S\left[T^*_1A^*_M\right]^{-1}\left[R^*_uu + S^*_yy\right] + H^*_dd \qquad (4.6)$$

where

$$H^*_dd = - B^*_LB^*_S\left[T^*_1A^*_M\right]^{-1}\left[\left[R^*_uA^*_{uv} + S^*_yB^*_{y.v}\right]\xi_v + \left[R^*_uA^*_{uw} + S^*_yB^*_{y.w}\right]\xi_w\right]$$

$$+ B^*_{1v}\xi_v + B^*_{1w}\xi_w \qquad (4.7)$$

Recall that $T^*_1A^*_M$ was chosen according to (3.48)-(3.49) such that it is possible to find a Z^*-unimodular polynomial T^*_2 satisfying the relation

$$T^*_2T^*_1A^*_MB^*_LB^*_S = B^*_ST^*_1A^*_M \qquad (3.51a)$$

If the expression (4.6) is multiplied on both sides with the Z^*-unimodular polynomial matrix

$$T^*_2T^*_1A^*_M$$

it is possible to simplify the right hand side to

$$T_2^* T_1^* A_M^* y_1 = B_S^* \left[R_u^* u + S_y^* y \right] + T_2^* T_1^* A_M^* H_d^* d \qquad (4.9)$$

Consider the constant matrix

$$R_u^*(0) = B_0 \qquad (3.51b)$$

It would be benificial to invert B_0 from the left while retaining a diagonal B_S^* in order to obtain a LIP-expression for u. This is not possible since B_S^* and B_0 do not commute. However, the following observation helps in reaching the goal.

In the derivation of B_0 (3.51b) it was shown to be possible to find an expression such that $B_R^*(0) = B_0$ is upper right triangular and invertible (cf. lemma 3.1). It was also shown that the principal diagonal contains '1' in all the elements. If the constant part of R_u^* i.e. B_0 is separated as indicated by (4.11) it follows that

$$T_2^* T_1^* A_M^* y_1 = B_S^* \left[u + R_0 u + R^* u + S^* y \right] + T_2^* T_1^* A_M^* H_d^* d$$

where now (4.10)

$$R_0 = B_0 - I$$

$$R^* = R_u^* - B_0$$

$$S^* = S_y^* \qquad (4.11)$$

Assume first that d=0 and consider each row of (4.10). The parameters of interest in order to state the pole placement control law for the control input u_i consist of the parameters of row i of

$$T_2^*,\ R_0,\ R^* \text{ and } S^*$$

The parameters of each row are linearly related by an expression involving u, y and $T_1^* A_M^* y_i$ since B_S^* is diagonal. This property will be used to derive LIP-models for the control inputs.

Remark 4.3

The property that the matrix B_S^* is diagonal is crucial for the existence of expressions, which are 'pure' in the controller parameters. Any non-diagonal matrix in the place of B_S^* would result in sums etc. of the parameters.

Hence a linear, identifiable parametric model is derived for the controller.This holds for the disturbance-free case since B_S^* is a known operator and since R_0 is an upper triangular matrix with zeros on the diagonal. This imposes the mild restriction that when computing a control input u it is necessary to compute u_{i+1} before u_i. Despite of this, each u_i is still a linear expression of inputs and outputs to the system. This restriction is removed, when B_0 happens to be diagonal. Then it will be possible to compute each control input u_i without any regards to the computations of u_j for $i \neq j$ at the same time instant.

This model is satisfactory for identification when no disturbance acts on the system. In order to formulate a more general case it is necessary to investigate the error propagation in the system.

4.4 Feedforward Considerations

Consider now the disturbance term (4.7) in the proposed parametric model (4.10). Utilize the identity (3.51a)

$$T_2^* T_1^* A_M^* H_d^* d =$$

$$= \left[\left[T_2^* T_1^* A_M^* B_{1v}^* - B_S^* \left[R_u^* A_{uv}^* + S_y^* B_{.v}^*\right]\right]\right] \xi_v +$$

$$+ \left[\left[T_2^* T_1^* A_M^* B_{1w}^* - B_S^* \left[R_u^* A_{uw}^* + S_y^* B_{.v}^*\right]\right]\right] \xi_w \qquad (4.12)$$

An expression for ξ_v is obtained from (3.26). This notation is introduced in (4.12) which give

$$F_v'^* = T_2^* T_1^* A_M^* B_{1v}^* \quad \text{and} \quad F_w'^* = T_2^* T_1^* A_M^* B_{1w}^*$$

$$T_v'^* = \left[R_u^* A_{uv}^* + S_y^* B_{.v}^*\right] \quad \text{and} \quad T_w'^* = \left[R_u^* A_{uw}^* + S_y^* B_{.w}^*\right]$$

(4.13)

If this notation is introduced above

$$T_2^* T_1^* A_M^* H_d^* d = \left[F_v'^* - B_S^* T_v'^*\right] A_{vv}^{*-1} v + H_w'^* w \qquad (4.14)$$

where

$$H_w'^* = - \left[F_v'^* - B_S^* T_v'^*\right] A_{vv}^{*-1} A_{vw}^* A_{ww}^{*-1} + \left[F_w'^* - B_S^* T_w'^*\right] A_{ww}^{*-1} \qquad (4.15)$$

It is necessary to cancel the denominators introduced by the inverted A_{vv}^* in (4.14) in order to obtain a polynomial expression for the terms corresponding to the disturbance inputs v. This may be done by introduction of a new diagonal Z^*-unimodular matrix U^* chosen such that the diagonal entries consist of the l.c.d. of each row of the matrices

$$\left[F'_v A^{*-1}_{vv} \; , \; B^*_S T'^*_v A^{*-1}_{vv} \right] \tag{4.16}$$

and scaled such that

$$U^*(0) = I \tag{4.17}$$

If (4.10) is multiplied by U^* from the left, a polynomial matrix will be the result - except for the w-dependent terms. This result is presented with a somewhat abbreviated notation in (4.19).

4.5 Parametric Models - Regulator Design

Introduce the following shorter notation

$$y_f = T^*_1 A^*_M \; y_1$$

$$e = y_1 - y^M_1$$

$$e_f = T^*_1 A^*_M \; e$$

$$T^*_f = U^* \; T^*_2 \tag{4.18a-g}$$

$$R^* = U^* \; R^*_u - B_O$$

$$S^* = U^* \; S^*_y$$

$$F_v^* = U^* F_v^{*} A_{vv}^{*-1} \quad , \; T_v^* = U^* T_v^{*} A_{vv}^{*-1} \quad \text{and} \quad H_w^* = U^* H_w^{*}$$

The full parametric model for the regulator case of (3.51a-e) is given by (4.19), where B_S^* is the known (diagonal) internal structure matrix and R_O is guaranteed to be an upper triangular constant matrix with zeros on the diagonal.

Theorem 4.1

The parametric models (4.9) and (4.19) represent the same transfer functions from (u,v) to y_1.

$$T_f^* y_f = B_S^*\left[u + R_O u + R^* u + S^* y - T_v^* v\right] + F_v^* v + H_w^* w \tag{4.19}$$

Proof

Since (4.9) and (4.19) differ only by a common Z^*-unimodular factor

$$U^*$$

multiplied from the left, this common factor may be cancelled whereby the identities coincide. The common Z^*-unimodular factors may be interpreted as decoupled stable modes, which do not appear in the resulting closed loop transfer functions.

Remark 4.4

A non-Z^*-unimodular T_2^* can always diagonalize the structure matrix from the left such that B_O becomes diagonal. However the new Z^*-zeros would introduce unstable modes in the closed loop system as is seen in equation (4.19) and is thus not allowed.

4.6 Parametric Models - Servo Design

Let the output error be defined as $e = y - y^M$ (cf.(4.18b)). If we form the output error equation of the system we obtain

$$T_f^* T_1^* A_M^* e =$$

$$= B_S^* \left[u + R_O u + R^* u + S^* y - T_v^* v \right] + F_v^* v - T_f^* T_1^* A_M^* B_D^* y^m + H_w^* w$$

$$= B_S^* \left[u + R_O u + R^* u + S^* y - T_v^* v - T_m^* y^m \right] + F_v^* v + H_w^* w \qquad (4.20)$$

where

$$T_m^* = U^* \; T_3^* \qquad (4.21)$$

where the polynomial matrix $T_2^* T_1^* A_M^* B_D^*$ has been refactorized as $B_S^* T_3^*$ (cf. 3.51d). Note that all the requirements on B_D^* are stated by the conditions of (3.51a-e). Then T_m^* may be estimated directly.

A parametric model for the servo transfer function case is

$$T_f^* e_f = B_S^* \left[u + R_O u + R^* u + S^* y - T_v^* v - T_m^* y^m \right] + F_v^* v + H_w^* w \qquad (4.22)$$

The quantities to estimate which are of interest for the controller are

R_O, R^*, S^*, T_v^* and T_m^*

while

T_f^* and F_v^*

also should be represented in the estimation algorithm in order to guarantee existence of solutions. The terms of $H_w^* w$ will however not be represented in the estimation algorithms.

4.7 Another Interpretation Of The Parametric Models

It is possible to make the following observations in the case when w=0 and where all outputs are controlled outputs, i.e. where $y_1 = y_O$.

The parametric model (4.19) may be rearranged into the form

$$\left[T_f^* T_1^* A_M^* - B_S^* S^* \right] y = B_S^* \left[I + R_O + R^* \right] u + \left[F_v^* - B_S^* T_v^* \right] v \qquad (4.23)$$

which is immediately recognized as a left m.f.d. representation of the open loop system.

Let this be expressed in the left m.f.d. notation as

$$A^*_y y = B^*_u u + C^*_v v \tag{4.24}$$

Similarly (4.22) can be interpreted as a left m.f.d. representation of the open loop system

$$\left[T^*_f T^*_1 A^*_M - B^*_S S^*_S\right] e = B^*_S\left[I + R_O + R^*\right] u$$

$$+ \left[F^*_v - B^*_S T^*_v\right] v - B^*_S\left[T^*_m - S^* B^*_D\right] y^m \tag{4.25}$$

which might be denoted in the left m.f.d fashion as

$$A^*_e e = B^*_u u + C^*_v v - B^*_{ym} y^m \tag{4.26}$$

In the case where all outputs are controlled outputs these arguments may be summed up as

Theorem 4.2

The pole placement identities (3.50a-c) perform a transformation from a right m.f.d. to a left m.f.d. of the open loop system.

Remark 4.5

The parametric model is independent of whether the system operates in closed or open loop. This is an important property useful for identification.

Proof

The parametric models have been derived only by using the polynomial matrix identities and open loop properties of the control object. The control inputs and outputs have nowhere been specified in the calculations.

Remark 4.6

This property will be used in a following presentation to show that parameter identification can be performed in both open and closed loop modes.

Remark 4.7

Notice that the resolution of a known A_y^* associated with the proposed pole placement solution i.e.

$$A_y^* = T_f^* \left[T_1^* A_M^* \right] - B_S^* S^* \qquad (4.27)$$

for known $T_1^* A_M^*$ and B_S^* is associated with the general polynomial matrix equation

$$XA + BY = C$$

where A, B and C are known matrices while X and Y are the desired solutions. Although a linear equation this is usually considered hard to solve ([Per 1]:II:2.5).

4.8 The Pole Placement Control Law

The correct control law to apply for the pole placement controller with feedforward from v and u_c is the following

$$u = - R_O u - R^* u - S^* y + T_v^* v + T_{uc}^* u_c$$

where

$$T_{uc}^* = T_f^* T_1^* B_M^* \qquad \text{(Regulator Case)} \qquad (4.28)$$

or

$$u = - R_O u - R^* u - S^* y + T_v^* v + T_{ym}^* y^m$$

$$T_{ym}^* = T_m^* \qquad \text{(Servo Case)} \qquad (4.29)$$

Remark 4.8

It is not necessary to incorporate the feedforward term $T_v^* v$ in the control law. When knowing both F_v^* and T_v^* from the on-line identification it is also possible to compute some other suitable feedforward. It would also be possible to have a fixed, chosen feedforward etc..

The terms F_v^* and T_v^* should however be incorporated in the identification algorithm, since influence of unknown inputs to the system will always corrupt identification.

The application of the control law (4.28) gives the closed loop system (cf.(3.53))

$$T_f^* T_1^* A_M^* y = B_S^* T_f^* T_1^* A_M^* y^m + F_v^* v + H_w^* w \qquad (4.30)$$

for the regulator case - and

$$T_f^* T_1^* A_M^* e = F_v^* v + H_w^* w \qquad (4.31)$$

for the servo case(cf.(3.56)).

The regulator case has the prescribed static gain properties since

$$B_S^*(1) = I \tag{3.27}$$

Remark 4.9

The pole placement control law (3.31) may be restituted from (4.27) by collecting terms and by cancellation of the common factor U^*. Alternatively it might be considered to be a control law derived with another Z^*-unimodular T_1^*.

4.9 Feedforward Properties

It is possible to ignore or modify the feedforward suggested by the terms F_v^* and T_v^* although these factors are important to include in the identification algorithm. When a control law described by (4.28) is applied then the influence of v on the output is given by

$$T_F^* T_1^* A_M^* e = F_v^* v \tag{4.31}$$

It remains to show that the application of the feedforward term $T_v^* v$ really is such that the influence of v on the output is in some sense small. The approach taken here is to show that no feedforward terms influence the terms of F_v^* of powers less than that of the corresponding element of B_S^*. It is on the other hand possible to eliminate all terms of F_v^* of powers greater than that of the corresponding element of

B^*_S. This result is similar to that of §3.4. Then F^*_v represents a necessary correlation matrix for the input disturbance v with respect to the control input channel. It is possible - since B^*_S is diagonal - to make the decomposition of C^*_v in (4.24) into F^*_v and T^*_v row by row such that the highest power of the elements in each row of F^*_v is lower than the highest power of the corresponding diagonal element of B^*_S. This is done by standard use of the division algorithm for polynomials row by row. Let this be formulated in the following theorem.

<u>Theorem 4.4</u>

There exists a polynomial matrix solution

$$\begin{bmatrix} F^*_v & T^*_v \end{bmatrix} \tag{4.33}$$

satisfying the relation - where B^*_S is diagonal

$$C^*_v = F^*_v - B^*_S T^*_v \tag{4.34}$$

and such that the highest power of each row of F^*_v is lower than the power of the corresponding entry in the diagonal matrix B^*_S.

The proof is given in appendix 4:1.

It is now straightforward to see that the proposed control law (4.28) or (4.29) realizes the choice of T^*_{ff} in the control law (3.31) with least degrees of the corresponding F^*_v. The disturbance influence on the output is given by

$$G^*_{vy_1} = B^*_{1u}\left[R^*_u A^*_{uu} + S^*_y B^*_{\cdot u}\right]^{-1} T^*_{ff} + \left[U^* T^*_2 T^*_1 A^*_M\right]^{-1} H^*_v \qquad (4.35)$$

which is the transfer function from v (cf.(3.37)). If the polynomial identities (3.50) are used to simplify the first term above, it follows that

$$G^*_{vy_1} = \left[U^* T^*_2 T^*_1 A^*_M\right]^{-1}\left[B^*_S U^* T^*_{ff} + F^*_v - B^*_S T^*_v\right] \qquad (4.36)$$

The 'best' possible choice of T^*_{ff} is such that

$$U^* T^*_{ff} = T^*_v \qquad (4.37)$$

since F^*_v is impossible to manipulate by choices of T^*_{ff}. Finally - since (3.31) and (4.28)-(4.29) differ by the common factor U^* that appears in (4.37) it is clear that the feedforward of (4.28)-(4.29) satisfy the optimality condition of a minimal degree input polynomial with respect to v.

Example 4.2

A common choice of a feedforward is such that the static influence from v on the output is eliminated i.e.

$$G^*_{vy_1}(1) = 0$$

This may be achieved if the component T^*_v of (4.28) or (4.29) is replaced by the feedforward component

$$T^*_v - F^*_v(1)$$

which is verified by direct substitution into (4.35).

4.10 Control Input LIP-Models

Consider a parametric model of the type (4.19). Denote the elements of the matrices T_f^*, R_O, R^*, S^*, T_v^* and F_v^* by their lower case equivalents such that

$$T_f^* = \left[(t_f^*)_{ij}\right]$$

$$R_O = \left[(r_O)_{ij}\right] \quad \text{etc.} \tag{4.38}$$

Let the notation

$$(t_f^*)_{i\cdot} \quad \text{and} \quad (s^*)_{\cdot j} \tag{4.39}$$

mean the i:th row and j:th column of T_f^* and S^* respectively.

Each row i of (4.19) may then be expressed as

$$(t_f^*)_{i\cdot} y_f =$$

$$= (b_i^*)\left[u + (r_O)_{i\cdot} u + (r^*)_{i\cdot} u + (s^*)_{i\cdot} y - (t_v^*)_{i\cdot} v\right] +$$

$$+ (f_v^*)_{i\cdot} v \tag{4.40}$$

It is possible to commute any two multiplied polynomials since all the polynomials are time-independent by assumption. Let the following notation be introduced for any quantity x_i

$$(\bar{x}_i)_j = b_j^* x_i \tag{4.41}$$

or when i=j

$$\bar{x}_i = b_i^* x_i$$

Overbar '¯' denotes that the quantity x has been filtered by b_i^*.

It is straightforward to obtain

$$(t_f^*)_{i.} y_f =$$

$$= \bar{u}_i + (r_0)_{i.}(\bar{u})_i + (r^*)_{i.}(\bar{u})_i + (s^*)_{i.}(y)_i -$$

$$- (t_v^*)_{i.}(\bar{v})_i + (f_v^*)_{i.} v \tag{4.42}$$

which may be rearranged into

$$\bar{u}_i = - \left[(r_0)_{i.}(\bar{u})_i + (r^*)_{i.}(\bar{u})_{i.} + (s^*)_{i.}(\bar{y})_i - (t_v^*)_{i.}(\bar{v})_i \right]$$

$$+ (t_f^*)_{i.} y_f - (f_v^*)_{i.} v \tag{4.43}$$

Let $(\theta_1)_i$ be a vector containing the coefficients of the polynomials in $(r_0)_{i.}$, $(r^*)_{i.}$, $(s^*)_{i.}$ and $(t_v^*)_{i.}$. Let analogously $(\theta_2)_i$ and $(\theta_v)_i$ denote the two vectors of coefficients of the polynomials in $(t_f^*)_{i.}$ and $(f_v^*)_{i.}$ respectively. Then define the vector $(\varphi_1)_i$, of the same dimensions as $(\theta_1)_i$, and containing outputs and inputs y, u and v corresponding to the polynomials represented by $(\theta_1)_i$. Let similarly $(\varphi_2)_i$ and $(\varphi_v)_i$ contain y_f and v up to sufficient powers of z^* - corresponding to the dimensions of $(\theta_2)_i$ and $(\theta_v)_i$ respectively.

Now the following LIP-model for each control input u_i in the square case (4.19) may be formulated.

$$\bar{u}_i = -(\theta_1)_i^T(\bar{\varphi}_1)_i + (\theta_2)_i^T(\varphi_2)_i - (\theta_v)_i^T(\varphi_v)_i \tag{4.44}$$

These models give possibilities for identification since $\bar{u}_i$ and $(\bar{\varphi}_1)_i$ may be computed and $(\varphi_2)_i$ and $(\varphi_v)_i$ are known.

Define $(\varphi_2^m)_i$ as a vector of the same kind as $(\varphi_2)_i$ where the entries contain components of

$$y_f^m = T_1^* A_M^* y^m = T_1^* B_M^* u_c \tag{4.45}$$

instead of the y_f. Now the control input (4.28) associated with the pole placement could be written as

$$u_i = -(\theta_1)_i^T(\varphi_1) + (\theta_2)_i^T(\varphi_2^m) \tag{4.46}$$

Obviously the θ_v-dependent component is omitted since this corresponds to unknown future disturbances v.

The adaptive control law in which the θ:s are substituted by their estimated counterparts $\hat{\theta}$ becomes

$$u_i = -(\hat{\theta}_1)_i^T(\varphi_1) + (\hat{\theta}_2)_i^T(\varphi_2^m) \tag{4.47}$$

In the servo case $\bar{\varphi}_2$ has to be replaced by a vector $\bar{\varphi}_2^e$ where y_f in the elements has been replaced by

$$e_f = y_f - y_f^M \tag{4.48}$$

It is also necessary to introduce a parameter vector θ_3 corresponding to the parameters of T_m^* in (4.29) which is essentially T_3^* (cf.(4.21)). The corresponding reference values are given by the data vector φ_3.

The estimation model is

$$\bar{u}_i = -(\theta_1)_i^T(\bar{\varphi}_1) + (\theta_2)_i^T(\varphi_2^e) - (\theta_v)_i^T(\varphi_v) - (\theta_3)_i^T(\bar{\varphi}_3) \tag{4.49}$$

The associated adaptive control law is

$$u_i = -(\hat{\theta}_1)_i^T(\varphi_1) - (\hat{\theta}_3)_i^T(\varphi_3) \tag{4.50}$$

5 EXAMPLES

5.1 Example 1

Consider a SISO-system of the usual configuration

$$y(t) = b_0 q^{-(k+1)} \frac{B^*(q^{-1})}{A^*(q^{-1})} u(t) =$$

$$= b_0 q^{-(k+1)} \frac{1 + b_1 q^{-1} + \ldots\ldots + b_m q^{-m}}{1 + a_1 q^{-1} + \ldots\ldots\ldots + a_p q^{-n}} u(t) \tag{5.1}$$

where B^* has all zeros outside the unit circle.

The model is

$$y^M(t) = q^{-(k+1)} y^m(t) = q^{-(k+1)} \frac{B_M^*(q^{-1})}{A_M^*(q^{-1})} u_c(t) \tag{5.2}$$

The equation

$$R_u^* A^* + b_0 q^{-(k+1)} S^* B^* = T_1^* A_M^* \tag{5.3}$$

where it e.g. is required that $T_1^*(0) A_M^*(0) = 1$, gives the parametric model

$$\frac{1}{b_0}\left[T_1^*(q^{-1}) A_M^*(q^{-1})\right] y(t) =$$

$$= q^{-(k+1)}\left[u + R^*(q^{-1})u + S^*(q^{-1})y(t) \right] \tag{5.4}$$

Rename the quantities above

$$T_2^*(q^{-1}) = \frac{1}{b_0} = \theta_2$$

$$T_1^*(q^{-1})A_M^*(q^{-1})y(t) = \varphi_2(t)$$

$$R^*(q^{-1})u(t) + S^*(q^{-1})y(t) = \theta_1^T\varphi_1(t) \tag{5.5}$$

The identification is done with the expression

$$u(t-k-1) = -\theta_1^T\varphi_1(t-k-1) + \theta_2^T\varphi_2(t) \tag{5.6}$$

The control law is

$$u(t) = -\theta_1^T\varphi_1(t) + \theta_2^T\varphi_2^m(t) \tag{5.7}$$

where

$$\varphi_2^m(t) = T_1^*(q^{-1})A_M^*(q^{-1})y^m(t) \tag{5.8}$$

where θ_i are substituted by estimates $\hat{\theta}_i$ in the adaptive control law.

5.2 Example 2

Consider the system

$$\begin{cases} \dot{x}(t) = k_1 x(t) + u(t) \\ y(t) = k_2 x(t) \end{cases} \tag{5.9}$$

where k_1 and k_2 are unknown, real constants such that $k_2 \neq 0$. Assume that y is measurable.

An output feedback should be designed such that the closed loop transfer function becomes

$$\frac{3}{p+3} \tag{5.10}$$

The system may be rewritten

$$\begin{cases} \frac{p-k_1}{p+1} x = \frac{1}{p+1} u \\ y = k_2 x \end{cases} \tag{5.11}$$

Introduce the transformed variable z^*

$$z^* = \frac{1}{p+1} \tag{5.12}$$

which gives the system description

$$\begin{cases} \left[1 - (k_1 + 1) z^* \right] x = z^* u \\ y = k_2 x \end{cases} \tag{5.13}$$

Define

$$A^*(z^*) = 1 - (1+k_1)z^* \overset{\Delta}{=} \left[1 - \alpha_1 z^* \right]$$

$$B^*(z^*) = k_2 z^* \overset{\Delta}{=} \beta_1 z^* \qquad (5.14)$$

Also make the decomposition

$$B_L^* = \beta_1 \qquad B_S^* = z^* \qquad \text{and} \qquad B_R^* = 1 \qquad (5.15)$$

Introduce the new internal variable ξ such that

$$\begin{cases} A^*(z^*)\ \xi(t) = u \\ y(t) = B^*(z^*)\ \xi(t) \end{cases} \qquad (5.16)$$

that is

$$\begin{cases} \left[1 - \alpha_1 z^* \right] \xi = u \\ y = \beta_1 z^* \xi \end{cases} \qquad (5.17)$$

The reference model is

$$\frac{B_S^*\ B_M^*}{A_M^*} = \frac{3}{p+3} = \frac{3z^*}{1 + 2z^*} \qquad (5.18)$$

This equality holds for

$$A_M^* = 1 + 2z^* \qquad \text{and} \qquad B_M^* = 3 \qquad (5.19)$$

The reference model transfer function is realizable since the model has the same structure polynomial as the control object. Let T_1^* arbitrarily be defined as

$$T_1^* = 1 \qquad (5.20)$$

The polynomial equation

$$R^*A^* + S^*B^* = T_1^*A_M^*B_R^* \qquad (5.21)$$

becomes

$$\left[1 - \alpha_1 z^*\right]R^* + \beta_1 z^* S^* = 1 + 2z^* \qquad (5.22)$$

which is satisfied for

$$R^* = 1 \quad \text{and} \quad S^* = \frac{2 + \alpha_1}{\beta_1} \overset{\Delta}{=} s_0 \qquad (5.23)$$

The control object is

$$y = \beta_1 z^* \xi = \beta_1 z^* \frac{1}{1+2z^*}\left[R^* u + S^* y\right] =$$

$$= \frac{\beta_1 z^*}{1+2z^*}\left[u + s_0 y\right] \qquad (5.24)$$

Define

$$y_f = T_1^* A_M^* y = \left[1+2z^*\right]y \qquad (5.25)$$

The quantity T_2^* is found as

$$T_2^* = \frac{1}{\beta_1} \qquad (5.26)$$

and

$$T_2^* T_1^* A_M^* y = \frac{1}{\beta_1} y_f = z^*\left[u + s_0 y\right] \qquad (5.27)$$

Define

$$\bar{u} = z^* u \quad \text{etc.} \qquad (5.28)$$

and

$$\theta_1 = s_0 \qquad \text{and} \qquad \theta_2 = \frac{1}{\beta_1} \tag{5.29}$$

The model for parameter estimation is then

$$\bar{u} = -\theta_1 \bar{y} + \theta_2 y_f \tag{5.30}$$

and the correct control law is

$$u = -\theta_1 y + \theta_2 y_f^m \tag{5.31}$$

where

$$y_f^m = T_1^* A_M^* y^m = T_1^* B_M^* u_c = 3u_c \tag{5.32}$$

The correct parameters are

$$\theta_1 = s_0 = \frac{3+k_1}{k_1} \qquad \text{and} \qquad \theta_2 = \frac{1}{k_1} \tag{5.33}$$

which give the closed loop system

$$\dot{x} = k_1 x + \left[-\frac{3+k_1}{k_2}(k_2 x) + \frac{1}{k_2} 3u_c \right] = -3x + \frac{3}{k_2} u_c \tag{5.34}$$

The transfer function to y becomes

$$y = \frac{3}{p+3} u_c \tag{5.35}$$

as required.

5.3 Example 3

Consider the system

$$y = \begin{bmatrix} b_1 q^{-3} & b_2 q^{-1} \\ 0 & b_3 q^{-2} \end{bmatrix} u \overset{\Delta}{=} B^*(q^{-1})u \qquad (5.36)$$

where it is assumed that b_1, b_2 and b_3 are unknown non-zero constants.

The objective is to find an adaptive scheme which gives a diagonal transfer function from the reference value to the output.

The control object has a transfer function which gives a relatively r.c.f. with $A^*=I$.

A decomposition of $G^* = B^*$ is given by elementary row and column operations on B^*.

$$B^*(q^{-1}) = \begin{bmatrix} 0 & b_2 \\ -\frac{b_1 b_3}{b_2} & b_3 q^{-1} \end{bmatrix} \begin{bmatrix} q^{-4} & 0 \\ 0 & q^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{b_1}{b_2} q^{-2} & 1 \end{bmatrix} \qquad (5.37)$$

Let

$$T_1^* A_M^* = I \qquad (5.38)$$

A solution to

$$R^* A^* + S^* B^* = T_1^* A_M^* B_R^* \qquad (5.39)$$

is given by

$$R^{*} = \begin{bmatrix} 1 & 0 \\ \frac{b_1}{b_2} q^{-2} & 1 \end{bmatrix} \quad \text{and} \quad S^{*} = 0 \tag{5.40}$$

$$T_2^{*} = \begin{bmatrix} \frac{1}{b_1} q^{-1} & -\frac{b_2}{b_1 b_3} \\ \frac{1}{b_2} & 0 \end{bmatrix} \tag{5.41}$$

A feasible B_D^{*} is given by

$$B_D^{*} = \begin{bmatrix} q^{-3} & 0 \\ 0 & q^{-4} \end{bmatrix} \tag{5.42}$$

The corresponding T_3^{*} is

$$T_3^{*} = \begin{bmatrix} \frac{1}{b_1} & -\frac{b_2}{b_1 b_3} \\ \frac{1}{b_2} q^{-2} & 0 \end{bmatrix} \tag{5.43}$$

The 'closed loop' system becomes

$$y = B^{*} R^{*-1} T_3^{*} y^{m} = B_D^{*} y^{m} \tag{5.44}$$

The parameter estimation is obtained from

$$T_2^{*}(q^{-1}) e = B_S^{*}\left[u + R^{*}(q^{-1})u - T_3^{*}(q^{-1})y^{m}\right] \tag{5.45}$$

This gives a possibilities for parameter estimation from the following relations

$$u_1(t-4) = \frac{1}{b_1} e_1(t-1) - \frac{b_2}{b_1 b_3} e_2(t) + \frac{1}{b_1} y_1^m(t-4) - \frac{b_2}{b_1 b_3} y_2^m(t-4)$$

$$u_2(t-1) = \frac{1}{b_2} e_1(t) - \frac{b_1}{b_2} u_1(t-3) + \frac{1}{b_2} y_1^m(t-3) \qquad (5.46)$$

for the control inputs

$$u_1(t) = \frac{1}{b_1} y_1^m(t) - \frac{b_2}{b_1 b_3} y_2^m(t)$$

$$u_2(t) = \frac{1}{b_2} y_1^m(t-2) - \frac{b_1}{b_2} u_1(t-2) \qquad (5.47)$$

5.4 Example 4

Consider the transfer function

$$\begin{bmatrix} \frac{1}{p-2} & \frac{1}{p^2-4} \\ \frac{p-3}{(p+1)(p-2)} & \frac{p^2-2p-1}{(p+1)(p^2-4)} \end{bmatrix} \qquad (5.48)$$

A $r.Z^*.c.f.$ is given by

$$A(p)\ \xi(t) = \begin{bmatrix} 1 & \frac{p-2}{p+1} \\ -3\frac{p+2}{p+1} & 0 \end{bmatrix} \xi(t) = u(t) \qquad (5.49)$$

$$y(t) = B(p)\,\xi(t) = \begin{bmatrix} \dfrac{1}{p+1} & \dfrac{1}{p+1} \\ -\,2\dfrac{p}{(p+1)^2} & \dfrac{p-3}{(p+1)^2} \end{bmatrix} \xi(t)$$

Choose $z^* = 1/(p+1)$. Then

$$A^*(z^*) = \begin{bmatrix} (1-3z^*) & -z^* \\ 0 & (1-z^*) \end{bmatrix}$$

$$B^*(z^*) = \begin{bmatrix} z^* & 0 \\ z^*(1-4z^*) & z^*(1-2z^*) \end{bmatrix} \tag{5.50}$$

The B^*-matrix is easily decomposed into

$$B_L^*(z^*) = \begin{bmatrix} 1 & 0 \\ 1-4z^* & -1 \end{bmatrix} ; \quad B_S^*(z^*) = \begin{bmatrix} z^* & 0 \\ 0 & -z^*(1-2z^*) \end{bmatrix} \tag{5.51}$$

and $B_R^* = I$.

If $T_1^* A_M^* = I$ we obtain

$$R^*(z^*) = \begin{bmatrix} (1-2z^*) & 0 \\ 2z^* & 1 \end{bmatrix} ; \quad S^*(z^*) = \begin{bmatrix} (4-2z^*) & 1 \\ (-1+2z^*) & -1 \end{bmatrix} \tag{5.52}$$

and

$$T_2^*(z^*) = \begin{bmatrix} 1 & 0 \\ (1-4z^*) & -1 \end{bmatrix} \tag{5.53}$$

and the closed loop system is

$$B_L^* B_S^* T_2^* = \begin{bmatrix} z^* & 0 \\ z^*(2-10z^* + 8z^{*2}) & -z^*(1-2z^*) \end{bmatrix} \tag{5.54}$$

which is triangular but statically diagonal($z^* = 1$).

A parametric model is given by R^*, S^* and T_2^* above and the correct parameters are given by

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -4 & 2 & -1 & 2 \\ 1 & -2 & 1 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ z^* y_1 \\ y_2 \\ z^* u_1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 1 & -4 & -1 \end{bmatrix} \begin{bmatrix} y_1^m \\ z^* y_1^m \\ y_2^m \end{bmatrix} \tag{5.55}$$

The identification algorithm will use LS-identification of the model

$$T_2^* y = B_S^* \left[u + R^* u + S^* y \right] \tag{5.56}$$

The parameters of T_2^* and (R^*, S^*) are called θ_2 and θ_1 respectively. The corresponding datavectors are called φ_2 and φ_1. Overbar '‾' denotes a datavector filtered by B_S^*.

$$\theta_2^T \varphi_2 = B_S^* \left[u + \theta_1^T \varphi_1 \right] \tag{5.57}$$

Since B_S^* is diagonal we obtain

$$\bar{u} = - \theta_1^T \bar{\varphi}_1 + \theta_2^T \varphi_2 \tag{5.58}$$

without parameter cross couplings.
The control law is

$$u = - \theta_1^T \varphi_1 + \theta_2^T \varphi_2^m \tag{5.59}$$

where

$$\bar{\varphi}_2^m = \varphi_2^M \tag{5.60}$$

and φ_2^m is the vector corresponding to φ_2 containing y^m.

In the adaptive control law θ_i are substituted by estimates $\hat{\theta}_i$.

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APPENDIX 3:1

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<u>Lemma 3.1</u>

A quadratic (nxn) transfer function G^* of full rank corresponding to a strictly proper transfer function G may be decomposed into a relatively r.Z^*.c.f (A^*, B^*). The factorization is such that A^* contains all the Z^*-poles of G^*, B^* contains all Z^*-zeros of G^* and such that

$$G^* = B^* A^{*-1} = B^*_L B^*_S B^*_R A^{*-1} \qquad (3.24)$$

where

A^* is a quadratic polynomial matrix

B^*_L and B^*_R are polynomial Z^*-unimodular matrices

satisfying

$$A^*(0) = I$$

$$B^*_S(1) = I \qquad (3.25)$$

$$B^*_R(0) = \left[b_{ij}\right]_R \text{ is upper right triangular and invertible}$$

$$(b_{ii})_R = 1 \quad \text{for} \quad 1 \leq i \leq n$$

where B^*_S contain all the zeros of G^* in Z^*_- but has no zeros in Z^*_+. The stability region is assumed to be according to (2.7).

Proof

Step 1

The quadratic transfer function G^* of full rank may be factorized into a relatively r.Z^*.c.f. (A_1^*, B_1^*) where A_1^* and B_1^* are polynomial matrices

$$G^* = B_1^* A_1^{*-1} \qquad (A3.1.1)$$

as known from §2.4.

Step 2

Since the transfer function is strictly proper, there are no poles at $z=\infty$ i.e. $z^*=0$. Via full rank- and r.Z^*.c.f.-conditions it follows that A^* does not loose rank at $z^*=0$ and it is possible to form

$$A_2^*(z^*) = A_1^*(z^*)A_1^{*-1}(0)$$

$$B_2^*(z^*) = B_1^*(z^*)A_1^{*-1}(0) \qquad (A3.1.2)$$

Then

$$A_2^*(0) = I \qquad (A3.1.3)$$

Step 3

The polynomial matrix B_2^* has a unique Z^*-Smith form S^* such that

$$B_2^* = L_1^* \; S^* \; R_1^* \tag{A3.1.4}$$

where L_1^* and R_1^* are Z^*-unimodular matrices. Then $R_1^*(0)$ is invertible. Define a lower left triangular constant matrix L and an upper right triangular constant matrix R such that

$$R_1^*(0) = L \; R \tag{A3.1.5}$$

where both L and R are invertible, constant matrices. The decomposition becomes unique by choosing the diagonal elements of R equal to one i.e.

$$r_{ii} = 1 \qquad \text{for} \quad 1 \leq i \leq n \tag{A3.1.6}$$

A decomposition of the type (A3.1.5) does not always have a straightforward solution since the important l_{ii}-elements may become zero. A reordering of the rows of $R_1^*(0)$ solves the problem. It is however necessary to rearrange the matrices L_1^* and S^* correspondingly. Introduce the permutation matrices P_1 and P_2 and define

$$L_2R_2 = P_1R_1^*(0)$$

$$S_2^* = P_2S^*P_1^{-1}$$

$$L_3^* = L_1^*P_2^{-1} \tag{A3.1.7}$$

where P_1 is chosen such that $P_1R_1^*(0)$ may be decomposed on the form (A3.1.5) and P_2 is chosen such that S_2^* becomes diagonal. If S^*=diag(s_1^*,...,s_n^*) then S_2^* may be written

$$S_2^* = \mathrm{diag}\left[s_{k_1}^*, \ldots, s_{k_n}^*\right] \qquad \text{(A3.1.8)}$$

for some permutation $\{k_1, \ldots, k_n\}$ of the numbers $\{1, \ldots, n\}$.

Step 4

Define

$$R_3^*(z^*) = R_1^{*-1}(0) R_1^*(z^*) \qquad \text{(A3.1.9)}$$

The decomposition of B_2^* may now be rewritten

$$B_2^* = L_3^* \, S_2^* \, L_2 R_2 R_3^* \qquad \text{(A3.1.10)}$$

Step 5

Consider the constant matrix

$$L_2 = \begin{bmatrix} 1_{11} & & 0 \\ \vdots & \ddots & \\ 1_{n1} & \cdots & 1_{nn} \end{bmatrix} \qquad \text{(A3.1.11)}$$

which multiplied from the left by S_2^* gives

$$M^* \triangleq S_2^* L_2 = \begin{bmatrix} 1_{11} s_{k_1}^* & & 0 \\ \vdots & \ddots & \\ 1_{n1} s_{k_n}^* & \cdots & 1_{nn} s_{k_n}^* \end{bmatrix} \qquad \text{(A3.1.12)}$$

It is desirable to factor out the invariant polynomials to the right. This is not possible to do directly since it is not known whether

$$s^*_{k_i} | s^*_{k_j} \quad \text{for } i<j \qquad (A3.1.13)$$

A construction proof is given in the following steps.

Step 6

Consider the first column of M^*. Factor out the g.c.Z^*.d of the column elements and assign

$$p^*_1 = g.c.Z^*.d(m^*_{i1}) \quad ; \quad 1 \leq i \leq n \qquad (A3.1.14)$$

The factor p^*_1 must correspond to the least k_i for which $l_{i1} \neq 0$ since

$$s^*_{k_i} | s^*_{k_j} \quad \text{for } k_i < k_j \qquad (A3.1.15)$$

This gives

$$p^*_1 = s^*_{k_i} \qquad (A3.1.16)$$

It is now possible to eliminate the elements in row i of all the columns 2,...,n corresponding to the common factor p^*_1 since row i of M^* has this common factor.

Step 7

This procedure may be repeated for each column. Factor out the g.c.Z^*.d. of column j and assign this to p^*_j. Eliminate the elements of M^* of the row where p^*_j was found, from each of the remaining columns j+1,..,n of M^*. Notice that all elements of a row has a common factor originating from S_2^*.

This may be written

$$M^*R_4 = L_4^*S_3^* \qquad (A3.1.17)$$

where R_4 is an upper right triangular, constant matrix with '1':s on the principal diagonal. The matrix S_3^* is

$$S_3^* = \mathrm{diag}(p_1^*,\ldots,p_n^*) \qquad (A3.1.18)$$

Through this procedure of elimination it is obvious that the factors $\{p_i^*\}$ represent some new permutation of

$$s_{k_1}^*,\ldots,s_{k_n}^* \qquad (A3.1.19)$$

These matrix manipulations give the intermediate result

$$B_2^* = L_3^*\ S_2^*L_2\ R_2^*R_3^* = L_3^*L_4^*\ S_3^*\ R_4^{-1}R_2R_3^* \qquad (A3.1.20)$$

where the S_3^* matrix is diagonal and $R_4^{-1}R_2$ is upper triangular.

Step_8

In order to obtain the requirements stated in the lemma it is necessary to scale the monic polynomials in the diagonal entries of S_3^*. Define

$$D = S_3^*(1)$$

$$L_5^* = L_4^*\ D$$

$$S_4^* = D^{-1} S_3^* \qquad (A3.1.21)$$

The matrix D is invertible since B^* is of full rank and since S_3^* is a polynomial matrix that is assumed to have no zeros at $z^*=1$ (cf.(2.7)).

Step 9

The matrix decomposition in the statement of the lemma is now obtained by assigning

$$B_L^* = L_3^* L_5^*$$

$$B_S^* = S_4^*$$

$$B_R^* = R_4^{-1} R_2 R_3^* \qquad (A3.1.22)$$

since R_2 and R_4 are upper right triangular matrices and $R_3^*(0) = I$. The diagonal elements of R_2 and R_4^{-1} are all '1'. This guarantees that $B_R^*(0)$ is upper triangular.

APPENDIX 3:2

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Lemma 3.2

There are polynomial matrix solutions R_u^*, S^*, T_2^*, T_3^* and a diagonal B_D^* matrix solution to the equations

$$T_2^*T_1^*A_M^* B_L^*B_S^* = B_S^* T_1^*A_M^*$$

$$R_u^*A_{uu}^* + S^*B_{.u}^* = T_1^*A_M^*B_R^* ; \qquad R_u^*(0) = B_R^*(0) \triangleq B_0$$

$$T_2^*T_1^*A_M^*B_D^* = B_S^*T_3^*$$

(3.50a-c)

under the conditions on A_M^*, B_{LS}^*, B_S^*, B_R^*, $B_{.u}^*$ and T_1^* given by (3.27-30) and (3.48-49).

Proof:

Step 1 Existence of T_2^*

Since T_1^* and A_M^* are both lower triangular, polynomial, Z^*-unimodular matrices (c.f. (3.48-49)), the product of these matrices becomes triangular. The matrix T_2^* is possible to express as

$$T_2^* = B_S^* T_1^*A_M^*B_S^{*-1} B_L^{*-1}\left[T_1^*A_M^*\right]^{-1} \qquad \text{(A3.2.1)}$$

In order to guarantee a polynomial matrix solution it is necessary to investigate the expressions closer.

The matrix

$$B^*_S T^*_1 A^*_M B^{*-1}_S \tag{A3.2.2}$$

is a polynomial, Z^*-unimodular, lower triangular matrix since the matrices B^*_S and B^{*-1}_S cancel each other perfectly on the diagonal and since the specification on the off-diagonal entries assure that these remain polynomials. Each element below the diagonal of the above matrix may be written

$$b^*_i p^*_{ij} \frac{1}{b^*_j} = b^*_i p'^*_{ij} \tag{A3.2.3}$$

which is a polynomial.

The resulting T^*_2 is a polynomial, Z^*-unimodular matrix since the matrix of (A3.2.2) is a polynomial matrix, since B^*_L is Z^*-unimodular (lemma 3.1), and since $T^*_1 A^*_M$ is a polynomial, Z^*-unimodular matrix by assumption.

In order to guarantee a polynomial matrix solution T^*_2 it is possible to use the non-uniqueness of the factorization of the polynomial matrix B^* for a given B^*_S.

Consider a tentative Z^*-generalized matrix T'^*_2 obtained as shown above. Find the diagonal polynomial matrix U^*, where the elements consist of the least common denominators of the columns of T'^*_2. Then $T'^*_2 U^*$ is a polynomial matrix. Since T'^*_2 is Z^*-unimodular it is clear that U^* is Z^*-unimodular.

Since U^* is Z^*-unimodular and commutes with B_S^*, it is possible to write

$$B^* = B_L^* B_S^* B_R^* = \left[B_L^* U^{*-1}\right] B_S^* \left[U^* B_R^*\right] \qquad \text{(A3.2.4)}$$

whereby a new factorization - with the desirable property of assuring a polynomial Z^*-unimodular matrix T_2^* - is obtained.

$$T_2^* = T_2'^* U^* \qquad \text{(A3.2.5)}$$

The matrix $U^* B_R^*$ is then still guaranteed to be a polynomial Z^*-unimodular matrix.

<u>Step 2</u> Existence of R_u^* and S^*

It is well known that the polynomial matrix equation (3.50b) has polynomial matrix solutions R_u^* and S^* (cf. e.g. [Per 1]:II:2.5) when there are no common left factors of A_{uu}^* and $B_{\cdot u}^*$. All solutions may be written as

$$\begin{aligned} R_u^* &= R_P^* + K^* R_H^* \\ S^* &= S_P^* + K^* S_H^* \end{aligned} \qquad \text{(A3.2.6)}$$

where $\left[R_P^* \;\; S_P^*\right]$ is a particular solution satisfying

$$R_P^* A_{uu}^* + S_P^* B_{\cdot u}^* = T_1^* A_M^* B_R^* \quad , \qquad \text{(A3.2.7)}$$

and $\left[R_H^* \;\; S_H^*\right]$ satisfies the homogenous equation

$$R_H^* A_{uu}^* + S_H^* B_{\cdot u}^* = 0 \quad , \qquad (A3.2.8)$$

and - finally - K^* is any polynomial matrix of appropriate dimensions.

<u>Step 3</u> Existence of T_3^* and B_D^*

The equation

$$T_2^* T_1^* A_M^* B_D^* = B_S^* T_3^* \qquad (A3.2.9)$$

has a solution T_3^* for a diagonal B_D^* and given polynomial matrices A_M^*, B_S^*, T_1^* and T_2^*. The solution is

$$T_3^* = B_S^{*-1} T_2^* T_1^* A_M^* B_D^* \qquad (A3.2.10)$$

A polynomial matrix solution T_3^* for a diagonal matrix B_D^* with a minimal number of Z^*-zeros is obtained by inspecting the least common denominators of the columns of the matrix

$$B_S^{*-1} T_2^* T_1^* A_M^* \qquad (A3.2.11)$$

and assigning these l.c.d. as the corresponding elements in the diagonal entries of the diagonal matrix B_D^*. Then a suitable B_D^* is obtained. A polynomial matrix solution for T_3^* is also immediately obtained from (A3.2.10), when B_D^* has been determined, since T_2^*, T_1^* and A_M^* are polynomial matrices, and since all denominators of (A3.2.11) derive from B_S^{*-1}.

Step 4

The matrix

$$R_u^*(0) \stackrel{\Delta}{=} B_0 \qquad (3.50b)$$

should satisfy (3.50b)

$$R_u^*(0)A_{uu}^*(0) + S_y^*(0)B_{\cdot u}^*(0) = T_1^*(0)A_M^*(0)B_R^*(0) \qquad (A3.2.12)$$

Via (3.25), (3.29), (3.46) and the properness assumptions in (3.2) it follows that

$$R_u^*(0) = B_R^*(0) \qquad (A3.2.13)$$

The proof of the lemma is now finished. The existence of the desired matrix solutions has been shown.

APPENDIX 4:1

APPENDIX 4:1

Theorem 4.3

There exists a polynomial matrix solution

$$\begin{bmatrix} F_v^* & T_v^* \end{bmatrix} \tag{4.33}$$

satisfying the relation - where B_S^* is diagonal

$$C_v^* = F_v^* - B_S^* T_v^* \tag{4.34}$$

and such that the highest power of each row of F_v^* is lower than the greatest power of the corresponding entry in the diagonal matrix B_S^*.

Proof

Consider eq. (3.23) with a given choice of B_{1v}^* and A_{vv}^*. Any pair of matrices

$$\begin{cases} B'^*_{1v} = B_{1v}^* + B_{1u}^* K^* \\ A'^*_{uv} = A_{uv}^* + A_{uu}^* K^* \end{cases} \tag{A4.1.1}$$

where K^* is any stable Z^*-generalized polynomial matrix, may substitute A_{uv}^* and B_{1v}^*. The matrices A_{uu}^*, A_{vv}^* and $B_{\cdot u}^*$ are not changed by any such substitution. This means that column operations are performed on A^* and B^* of (3.23) such that A_{uu}^*, A_{vv}^* and $B_{\cdot u}^*$ are left unaltered.

This fact may be exploited in the expressions for v of §4.4.

All the v-dependent terms of (4.19) and (4.22) may be expressed as

$$H_v^* v \overset{\Delta}{=} \left[F_v^* - B_S^* T_v^*\right] v \tag{A4.1.2}$$

The polynomial matrix H_v^* may be written (cf.§4.4)

$$H_v^* = U^* \left[T_2^* T_1^* A_M^* B_{1v}^* - B_S^* \left[R_u^* A_{uu}^* + S_y^* B_{\cdot v}^*\right]\right] A_{vv}^{*-1} \tag{A4.1.3}$$

Recall the relations (3.50a-b) to obtain

$$M^* \overset{\Delta}{=} T_2^* T_1^* A_M^* B_{1u}^* = B_S^* T_1^* A_M^* B_R^* = B_S^* \left[R_u^* A_{uu}^* + S_y^* B_{\cdot u}^*\right] \tag{A4.1.4}$$

Use this relation on

$$H_v^* = H_v^* + U^* M^* K^* A_{vv}^{*-1} - U^* M^* K^* A_{vv}^{*-1} \tag{A4.1.5}$$

where K^* is a Z^*-generalized polynomial matrix to be specified. Through substitution of M^* given by (A4.1.4) into (A4.1.5) it is confirmed that

$$H_v^* = U^* \left[T_2^* T_1^* A_M^* B_{1v}' - B_S^* \left[R_u^* A_{uv}' + S_y^* B_{\cdot v}'\right]\right] A_{vv}^{*-1} \tag{A4.1.6}$$

is independent of the choice of K^*. Consider now all the Z^*-generalized matrices F_v^* that can be achieved for given matrices A_{uu}^*, A_{vv}^* and $B_{\cdot u}^*$. These may be written

$$F_v^* = U^* T_2^* T_1^* A_M^* B_{1v}' A_{vv}^{*-1} = U^* T_2^* T_1^* A_M^* \left[B_{1v}^* + B_{1u}^* K^*\right] A_{vv}^{*-1} =$$

$$= U^* T_2^* T_1^* A_M^* B_{1v}^* A_{vv}^{*-1} + B_S^* U^* T_1^* A_M^* B_R^* K^* A_{vv}^{*-1} \tag{A4.1.7}$$

The choice of U^* in §4.4 guarantees

$$N^* \stackrel{\Delta}{=} U^* T_2^* T_1^* A_M^* B_{1v}^* A_{vv}^{*-1} \tag{A4.1.8}$$

to be a polynomial matrix. Now find polynomial matrices X^* and Y^* such that

$$N^* = X^* + B_S^* Y^* \tag{A4.1.9}$$

where the polynomial elements of X^* should have the lowest possible degree with respect to z^*. This is achieved by an elementwise solution of the polynomial equations

$$n_{ij}^* = x_{ij}^* + b_i^* y_{ij}^* \tag{A4.1.10}$$

The division algorithm for polynomials assures existence of a solution where each x_{ij}^* has lower degree than b_i^*.
Now choose

$$K^* = - B_R^{*-1} A_M^{*-1} T_1^{*-1} U^{*-1} Y^* A_{vv}^* \tag{A4.1.11}$$

The final choice is

$$F_v^* = X^* \tag{A4.1.12}$$

and

$$T_v^* = U^* \left[R_u^* A'^*_{uv} + S_y^* B'^*_{\cdot v} \right] A_{vv}^{*-1} =$$

$$= U^* \left[R_u^* A_{uv}^* + S_y^* B_{\cdot v}^* \right] A_{vv}^{*-1} + U^* \left[R_u^* A_{uu}^* + S_y^* B_{\cdot u}^* \right] K^* A_{vv}^{*-1} =$$

$$= U^* \left[R_u^* A_{uv}^* + S_y^* B_{\cdot v} \right] A_{vv}^{*-1} - Y^* \tag{A4.1.13}$$

The choice of U^* in §4.4 guarantees the first term to be a polynomial matrix. The matrix Y^* is a polynomial matrix according to (A4.1.9) above.

The proof is now finished since polynomial matrices F_v^* and T_v^* with the prescribed properties and compatible with (3.23) and the control law (3.31) have been shown to exist.