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A DESIGN SCHEME FOR INCOMPLETE STATE OR OUTPUT FEEDBACK WITH APPLICATIONS TO BOILER AND POWER SYSTEM CONTROL.

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A rational method of designing a controller with prescibed structure, using the best fit on the dominant eigenspace, has been developed and applied to the control of a boiler and a power system.

SUMMARY

The problem of designing a linear feedback when all state variables are not available is discussed. The design scheme is based on computation of a complete state feedback and a reduction to a specified structure. The reduction is made by approximation on the eigenspace corresponding to a set of dominant eigenvalues. The method consists of successive choices of weightings on this space. The method is applied to the control of a boiler and a three-machine power system. In the power system case the complete state feedback can be replaced by local output feedback without any significant decrease in performance.

The examples indicate that the proposed method is a realistic design method for multivariable systems.

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1. INTRODUCTION.

The concept of state feedback plays an important role in existing control theory for linear systems. Linear quadratic control theory [1] and pole assignment theory [3,4] are two well-known examples. Unfortunately the whole state vector is, however, rarely available for measurement. Even if it was available a state feedback control would sometimes result in far too complex control systems. The standard way to bypass these difficulties is to measure only a small set of outputs and reconstruct the full state vector using a Kalman filter [2] or an observer [7]. The result is, however, still somewhat unsatisfactory since the reconstruction by itself might produce high order dynamics in the control function.

These facts justify the demands for simpler or suboptimal control policies. Practical constraints on the feedback system must be considered. A limited number of measurements is one obvious constraint. In large systems consisting of several coupled subprocesses, such as power systems, there may be a desire to control the system with local feedbacks on the different processes, eventually with the addition of a small number of interconnections. There are, however, no rational ways to design such hierarchical control schemes. Another example is diagonally controlled systems where the design philosophy is the classical one with each input variable controlling a single output variable.

A few methods exist to treat problems of these types. The use of dynamic feedback [15,16,17] have the same disadvantages as the observer approach above, i.e. the control may be unnecessarily complex. The problem can also be tackled by direct optimazation methods [8,10,19]. However, this technique does not seem to be practical when applied to large systems. A special version of modal control [4,5] has also been used in this class of problems. Quite recently frequency domain methods have been developed which extend the classical Nyquist criteria to multivariable systems. A survey of these results can be found in [20]. These criteria seems, however, to be difficult to use for large systems with several inputs and outputs. There is one considerable difference between the approach of this paper and the frequency domain techniques. In this paper we start with an "optimal" solution which is made suboptimal by imposing constraints in the control structure. In the frequency domain approach one attempts to successively improve the solution from an initial guess by varying the gains in the control.

In this paper a state feedback control is used as the starting point. This is quite a realistic assumption, since there are straightforward methods to find such controllers even for fairly large systems. See for instance [1] and [4]. The step taken is then to fit this control into another "similar" control with a predefined structure. The idea behind this fit is to make it as accurate as possible on the eigenspace corresponding to a dominant set of eigenvalues to the closed loop system. It is illustrated by examples that satisfactory controllers may be obtained in this way after a few iterations. It should be noticed that the method does not depend on how the state feedback controller is obtained. The reduction technique is thus applicable to any method that results in a linear feedback from the state.

Notice that this reduction procedure is a rational way of designing hierarchical control systems. Sometimes it is not possible to control the system satisfactorily by output feedback only. In such cases the reduction scheme can be used to find controllers of PD-types, where the derivative term will give additional information about the state of the system and thus make the system easier to stabilize.

In the boiler case it is shown that the feedback from all five states can be replaced by the feedback from two outputs. In this case it is possible to avoid the Kalman filter, proposed for the reconstruction of the state, without any significant decrease in performance.

The power system is an example of a system, consisting of geographically distributed subsystems. State estimation and feedback can be organized in a centralized or a decentralized manner. In both cases large amount of data has to be transmitted. Although data transmission systems are under construction it is desirable to have control schemes, which do not require high speed data transmission. The whole state vector could be reconstructed locally if the system is observable via locally available outputs. The dimension of the Kalman filter, however, becomes very high.

In this paper we consider a three-machine power system with 15 states. The complete state feedback can be replaced with local output feedback without any significant decrease in damping. Also in this case it is possible to avoid high order Kalman filters. The results also indicate that very little can be gained from complete centralized control schemes and that properly designed local controllers are sufficient for dynamic control.

In large systems, such as power systems, the computational effort is of importance. The major computational burden in this case lies on an initial eigenvalue-eigenvector calculation, which corresponds to approximately $8n^3$ operations. An additional eigenvalue calculation may have to be done to check if the reduced control law has an acceptable degree of stability.

This method could be an effective tool for the design of multivariable controllers in an interactive mode.

2. STATEMENT OF THE PROBLEM.

Consider a linear time invariant system in state space form

$$\dot{x} = Ax + Bu \tag{2.1}$$

where x is the n-vector of states and u is the m-vector of control inputs. A and B are real-valued matrices of compatible dimensions. Moreover, assume that a state feedback controller

$$u = Lx + v \tag{2.2}$$

where v is some external input and L an m×n matrix, is found such that the system (2.1) with the controller (2.2) has the desired properties.

In controlling the system (2.1) we will set certain constraints on the feedback system. The intention is then to "reduce" the control law (2.2) such that these constraints are satisfied. Two specific types of constraints will be considered corresponding to different degrees of complexity in the control function. These definitions should cover a large variety of practical constraints that might be imposed on the structure of a feedback system.

In order to simplify the notations we will use stars (*) to indicate properties associated with the reduced control laws.

The simplest kind of constraint is to permit output feedback. Let $y = Cx$ denote the output of (2.1) where C is a real $r \times n$ matrix. A control of the form

$$u = K^{*}Cx + v \tag{2.3}$$

will be referred as a control with a <u>single constrained feedback structure</u>.

A more complex structure is obtained if the i:th input component vector is restricted to be a function of certain specified outputs. Let $y_i = C_i x$, $i = 1, 2, \ldots, q$, denote q sets of output variables to (2.1) where C_i is an $r_i \times n$ matrix. Moreover, let $u^T = [u_1^T \; u_2^T \; \ldots \; u_q^T]$ be a partition of the control vector into an appropriate set of q subvectors. A control of the form

$$u_i = K_i^{*}C_i x + v_i \qquad i = 1, 2, \ldots, q \tag{2.4}$$

will be referred as a control with a <u>multiple constrained feedback structure</u>. It is easily verified that local as well as hierarchical types of control systems are included in this formulation. Notice that the control (2.3) is a special case of (2.4) with $q = 1$. An illustration of the two concepts is given in Fig. 2.1 and Fig. 2.2.

A common way to do the kind of reductions considered here is to simply neglect those entries of the state feedback matrix that are "small" in comparison with the others.

There are, however, several difficulties involved in such a procedure, and it requires frequently a fairly deep understanding of the process dynamics. Moreover, there is no rational way to "compensate" the remaining entries for the approximations made. The approach of this paper will instead be to construct a certain subspace of the state space where the reduction is made. In this way the "compensating" problem is avoided and converted to the problem of finding the appropriate subspace.

3. CONTROL REDUCTION.

Assume that a state feedback control is given. This control is then replaced with a "similar" control with a predefined feedback structure. It is shown that this can be done in such a way that a certain number of eigenvalues remain invariant (mode preservation). Since there is an upper bound on the number of invariant eigenvalues a different reduction is also given which minimizes a weighted shift of the eigenvalues (mode weighting). Controls of derivative types will be considered at the end of the section.

Mode Preservation.

Consider the system (2.1) with the control (2.2). The closed loop system becomes

$$\dot{x} = (A + BL)x + Bv \tag{3.1}$$

We will attempt to replace the control (2.2) with a similar control of the multiple constrained form (2.4). For this control the closed loop system becomes

$$\dot{x} = (A + \sum_{i=1}^{q} B_i K_i^* C_i)x + Bv \tag{3.2}$$

where $B = [B_1 \; B_2 \; \dots \; B_q]$ is a partition of the input matrix compatible with the partition of the control vector in (2.4). Moreover, the reduced control law shall be selected so that some dominant properties of (3.1) are preserved in (3.2).

Partition the state feedback matrix as

$$L = \begin{pmatrix} L_1 \\ L_2 \\ \vdots \\ L_q \end{pmatrix}$$

where L_i is $m_i \times n$. Then if $K_i C_i = L_i$ have solutions K_i^* for $i = 1, 2, \ldots, q$, the exact and the reduced control laws would be identical. However, such solutions rarely exist, and therefore approximations must be made. The following theorem describes one rational way to do such approximations.

Theorem 1: Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_p\}$ be a symmetric set of eigenvalues to $A + BL$ and let Q be a real basis matrix for the corresponding eigenspace. Then if

$$K_i C_i Q = L_i Q \tag{3.3}$$

have solutions K_i^* for $i = 1, 2, \ldots, q$, then Λ is also a set of eigenvalues to

$$A + \sum_{i=1}^{q} B_i K_i^* C_i$$

Moreover, if $T = [Q \quad \hat{Q}]$ where the columns of $\hat{Q}$ are any set of vectors that extend the columns of Q to a basis in R^n then

$$T^{-1}(A + BL)T = \begin{pmatrix} A_{11}^o & A_{12}^o \\ 0 & A_{22}^o \end{pmatrix} \tag{3.4}$$

and

$$T^{-1}(A + \sum_{i=1}^{q} B_i K_i^* C_i)T = \begin{bmatrix} A_{11}^o & A_{12}^o + \sum_{i=1}^{q} B_i^1 \Delta L_i^* \hat{Q} \\ 0 & A_{22}^o + \sum_{i=1}^{q} B_i^2 \Delta L_i^* \hat{Q} \end{bmatrix} \qquad (3.5)$$

where

$$\Delta L_i^* = K_i^* C_i - L_i$$

and

$$T^{-1} B_i = \begin{bmatrix} B_i^1 \\ B_i^2 \end{bmatrix}$$

<u>Proof</u>: Introduce $A_o = A + BL$ and

$$A_o^* = A + \sum_{i=1}^{q} B_i K_i^* C_i$$

From (3.3) we have

$$(A + \sum_{i=1}^{q} B_i K_i^* C_i)w = (A + \sum_{i=1}^{q} B_i L_i)w = (A + BL)w \qquad (3.6)$$

for any $w \in \{Q\}$. Since $\{Q\}$ is A_o invariant by construction, it follows from (3.6) that $\{Q\}$ is also A_o^* invariant and $A_o Q = A_o^* Q$. Let the columns of $\hat{Q}$ be any set of vectors that extend the columns of Q to a basis in R^n. Choose $T = [Q\ \hat{Q}]$ and write

$$T^{-1} = \begin{pmatrix} V \\ \hat{V} \end{pmatrix}$$

We then have

$$T^{-1}A_oT = \begin{pmatrix} VA_oQ & VA_o\hat{Q} \\ 0 & \hat{V}A_o\hat{Q} \end{pmatrix} = \begin{pmatrix} A^o_{11} & A^o_{12} \\ 0 & A^o_{22} \end{pmatrix} \tag{3.7}$$

$$T^{-1}A_o^*T = \begin{pmatrix} VA_o^*Q & VA_o^*\hat{Q} \\ 0 & \hat{V}A_o^*\hat{Q} \end{pmatrix} = \begin{pmatrix} A^o_{11} & VA_o^*\hat{Q} \\ 0 & \hat{V}A_o^*\hat{Q} \end{pmatrix} \tag{3.8}$$

The set of eigenvalues of A^o_{11} equals Λ. From (3.8) it then follows that Λ is also a set of eigenvalues to A_o^*. Moreover, we have

$$VA_o^*\hat{Q} = VA_o\hat{Q} + V(\sum_{i=1}^{q} B_iK_i^*C_i - L)\hat{Q} = A^o_{12} + V\sum_{i=1}^{q} B_i(K_i^*C_i - L_i)\hat{Q} =$$

$$= A^o_{12} + \sum_{i=1}^{q} B_i^1 \Delta L_i^* \hat{Q}$$

and in the same way

$$\hat{V}A_o^*\hat{Q} = A^o_{22} + \sum_{i=1}^{q} B_i^2 \Delta L_i^* \hat{Q}$$

□

Comments:

1. A real basis for the eigenspace can be constructed from the eigenvectors as was described in ~~Section~~ Appendix 1 2.
2. A comparison between the matrices (3.4) and (3.5) clearly illustrates the kind of approximations that

are made. The upper left block corresponding to eigenvalues Λ are identical in both systems. The remaining blocks are changed by an amount depending on ΔL_i^*, i.e. the difference between the exact and the reduced control laws.

3. The remaining eigenvalues of $A + BL$ are different from those of $A + BK^*C$. Observe, however, that the effect of the approximations are only localized to the part of $A + BL$ that contains the less dominant modes. The case when the approximations still cause an unacceptable change in the system is covered below.

4. Theorem 1 also yields an algorithm for pole assignment via output feedback. It has been shown in [5] that if rank $C = r$, then a symmetric set of r eigenvalues may be "almost" freely assigned. If a state feedback matrix L has been found so that r eigenvalues to the closed loop system takes some prescribed values Theorem 1 may be used to find a corresponding output feedback matrix (assuming (3.3) is solvable).

Mode Weighting.

The condition that (3.3) shall be solvable for K_i gives an upper bound on the number of eigenvalues that can be held fixed. This bound mostly equals r_i, i.e. the number of measured variables. One trivial exception is $C = L$, where $K = I$ preserves all the eigenvalues. It may, however, still happen that some of the remaining eigenvalues move to undesired locations in the complex plane. The solution to this problem is to include a larger number of eigenvalues looking for least square solutions of (3.3).

Introduce the matrix norm

$$||M|| = \left(\mathrm{tr}\{MM^T\}\right)^{1/2}$$

valid for an arbitrary real matrix M.

Consider first the case when there is more than one solution to (3.3). Let R_i, $i = 1, 2, \ldots, q$, be nonsingular $r_i \times r_i$ matrices. Then one solution is given by

$$K_i^* = L_i Q(R_i^{-1} C_i Q)^{\dagger} R_i^{-1} \tag{3.9}$$

Moreover, this solution is the one that minimizes the norm $||K_i R_i||$, i.e. a solution with small feedback gains is selected. The matrices R_i are used to scale the output variables.

Consider now the opposite case when there is no solution of (3.3). We may then attempt to minimize the norm

$$||(K_i C_i Q - L_i Q)W|| \tag{3.10}$$

where W is a nonsingular p×p matrix. In fact the minimum is obtained by taking

$$K_i^* = L_i QW(C_i QW)^{\dagger}$$

Now remember the special choice of basis that ~~was~~ is made in ~~(2.6)~~ (A1.2), i.e.

$$Q = [a_1\ a_2\ \ldots\ a_s;\ \mathrm{Re}\{a_{s+1}\}\ \mathrm{Im}\{a_{s+1}\}\ \mathrm{Re}\{a_{s+2}\}\ldots] \tag{3.11}$$

where a_k is the eigenvector corresponding to λ_k. If we choose $W = \mathrm{diag}(w_1, w_2, \ldots, w_p)$ where $w_k \neq 0$, (3.10) may be rewritten as

$$||(K_iC_iQ - L_iQ)W||^2 = \sum_{k=1}^{p} w_k^2|K_iC_ia_k - L_ia_k|^2 \tag{3.12}$$

From the last expression we see that a successive increase in w_k causes a successive better fit of the eigenvalue λ_k in the closed loop system, cf. Th.1. In this way W may be interpreted as a weighting matrix for the eigenvalues we desire to hold fixed. This point is further clearified by examples later.

Finally we observe that (3.9) and (3.11) may be combined to

$$K_i^* = L_iQW(R_i^{-1}C_iQW)^+R_i^{-1} \tag{3.13}$$

Proportional and Derivative Control.

In some cases acceptable degree of stability cannot be achieved by output feedback only. The classical way to bypass this difficulty is to include derivatives of the outputs in the feedback loop.

We will now permit a control of the form

$$u = K_1^*y + K_2^*P\dot{y} \tag{3.14}$$

where P is a given m×r matrix and y = Cx. Only the single constrained case will be considered. The extension to the multiple constrained case is straightforward. In classical control terms the control (3.14) is of PD-type. The derivative term will set some constraints on the quality of the measured signals, especially the presence of high frequency noise. This kind of control has, however, turned out to be successful in many applications.

By some simple manipulations the control (3.14) is transformed to the standard form (2.3). Using (2.1) we have

$$u = K_1^* y + K_2^* P C \dot{x} = K_1^* C x + K_2^* P C (Ax + Bu)$$

Assuming $I - K_2^* PCB$ is invertible the last expression may be solved for u

$$u = (I - K_2^* PCB)^{-1} K_1^* C x + (I - K_2^* PCB)^{-1} K_2^* PCAx =$$

$$= \hat{K}_1 C x + \hat{K}_2 PCAx \qquad (3.15)$$

Now defining a new output vector $\hat{y}$ as

$$\hat{y} = \hat{C} x = \begin{pmatrix} C \\ PCA \end{pmatrix} x \qquad (3.16)$$

The equation (3.15) can then be rewritten as

$$u = (\hat{K}_1 \hat{K}_2) \hat{C} x = \hat{K} \hat{C} x \qquad (3.17)$$

The previous results can now be used to find an appropriate $\hat{K}$. The feedback gains in (3.14) are then calculated as

$$K_1^* = (I - K_2^* PCB) \hat{K}_1 \qquad (3.18)$$

$$K_2^* = \hat{K}_2 (I + PCB \hat{K}_2)^{-1} \qquad (3.19)$$

The benefit of this kind of control is apparent from (3.16) and (3.17). By having a larger portion of the state available we are also, in view of the reduction technique above, able to keep a larger number of eigenvalues fixed. Moreover, if rank $\{\hat{C}\} = n$ then the reduced and the exact control laws become identical.

Examples.

Finally we will give some examples to illustrate the ideas of the section. Two more farreaching examples are considered in the next sections. A common feature in these examples is that the neglected modes become more dominant in the reduced control system. This could be expected since restrictions are imposed on the feedback structure, which naturally results in some decrease in performance.

Example 1:

$$\dot{x} = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} u$$

$$u = \begin{pmatrix} -5 & -1 \\ 2 & -5 \end{pmatrix} x + v$$

The closed loop system becomes

$$\dot{x} = \begin{pmatrix} -4 & 1 \\ 1 & -4 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} v$$

and the closed loop eigenvalues equal $\lambda_1 = -3$ and $\lambda_2 = -5$. Assume we shall hold $\lambda_1 = -3$ fixed. The eigenvector corresponding to λ_1 is

$$a_1 = 1/\sqrt{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Assume we permit a feedback of the form

$$u = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} x + v$$

This control is then of the multiple constrained type (2.4). The feedback structure becomes

$$u_1 = k_1 \begin{pmatrix} 1 & 0 \end{pmatrix} x + v_1$$

$$u_2 = k_2 \begin{pmatrix} 0 & 1 \end{pmatrix} x + v_2$$

Solving (3.3) for i = 1,2 we have

$$u = \begin{pmatrix} -6 & 0 \\ 0 & -3 \end{pmatrix} x + v$$

The eigenvalues of

$$A + \sum_{i=1}^{q} B_i k_i^* C_i$$

becomes $\nu_1 = -3$ and $\nu_2 = -4$.

Example 2 :

$$\dot{x} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} x$$

$$u = \begin{pmatrix} -2 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix} x + v$$

The closed loop system is

$$\dot{x} = \begin{pmatrix} -2 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & -2 & -2 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix} v$$

The eigenvalues of A + BL are $\lambda_{1,2} = -1 \pm j$ and $\lambda_3 = -2$.

Assume we permit a feedback of the form

$$u = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} y + v$$

First we attempt to keep the eigenvalues $\lambda_{1,2} = -1 \pm j$ fixed. The corresponding eigenvectors are

$$a_1 = \begin{bmatrix} 0 \\ -0.5 \\ 1 \end{bmatrix} + j \begin{bmatrix} 0.5 \\ -0.5 \\ 0 \end{bmatrix} \qquad a_2 = \begin{bmatrix} 0 \\ -0.5 \\ 1 \end{bmatrix} - j \begin{bmatrix} 0.5 \\ -0.5 \\ 0 \end{bmatrix}$$

The basis matrix Q for the eigenspace is then selected according to (2.6) as
(A1.2)

$$a = \begin{bmatrix} 0 & 0.5 \\ -0.5 & -0.5 \\ 1 & 0 \end{bmatrix}$$

Solving (3.3) for K we have

$$K_I^* = \begin{bmatrix} 0.67 & 0.33 \\ -0.33 & -1.67 \end{bmatrix}$$

and the eigenvalues of $A + BK_I^*C$ are $\nu_{1,2} = -1 \pm j$ as desired and $\nu_3 = 1.33$. The third mode has become unstable and therefore we include also this mode looking for least square solutions according to (3.12) and (3.13).

The eigenvector corresponding to $\lambda_3 = -2$ is

$$a_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The basis matrix for the eigenspace then becomes

$$Q = \begin{pmatrix} 0 & 0.5 & 1 \\ -0.5 & -0.5 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

We choose the weighting matrix W as W = diag(1,1,0.1) where a relatively small weight has been laid on λ_3. Using (3.13) we now have

$$K^{*}_{II} = \begin{pmatrix} 0.64 & 0.32 \\ -0.32 & -1.66 \end{pmatrix}$$

and the closed loop eigenvalues are $\nu_{1,2} = -0.995 \pm 0.998j$ and 1.29. The weight on the third mode was obviously too small. Take instead W = diag(1,1,0.8). We then obtain

$$K^{*}_{III} = \begin{pmatrix} -0.30 & -0.15 \\ 0.15 & -1.42 \end{pmatrix}.$$

The closed loop eigenvalues are now $\nu_{1,2} = -0.57 \pm 1.14j$ and $\nu_3 = -0.73$, which is considered to be satisfactory in this case. The solution was obtained after a few iterations by successively altering the weighting factors. In the general case there is no guarantee that a satisfactory solution can be obtained. However, if a satisfactory solution is difficult to obtain by altering the weighting factors, this indicates that the system is difficult to control with the prescribed feedback structure.

4. APPLICATION TO BOILER CONTROL.

A computer program for control reduction has been written based upon Theorem 1 and the least square solution (3.13). This program has been used to find simple control strategies for a boiler. The starting point is here a linear quadratic control law, which is used to fit a certain feedback structure. By simulations we show that a reduction can be made without any significant decrease in performance. In fact, the responses of the reduced control system are very similar to the responses of the system controlled by complete state feedback.

Control of a Boiler.

Different types of models for a drum boiler are thoroughly described in [9]. Here we will use a fifth order model from [10].

The linearized equations for a boiler around a certain operating point can be written as

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

where the state variables are

x_1 = drum pressure (bar)

x_2 = drum liquid level (m)

x_3 = drum liquid temperature ($^{\circ}$C)

x_4 = riser wall temperature ($^{\circ}$C)

x_5 = steam quality (%)

The control variables are

u_1 = heat flow to the risers (kJ/s)

u_2 = feedwater flow (kg/s)

Numerical values for A, B and C for a power station boiler with a maximum steam flow of about 350 t/h are calculated in [10]. The drum pressure is 140 bar and the operating point is 90% full load. From [10] we have

$$A = \begin{pmatrix} -0.129 & 0.000 & 0.396\times10^{-1} & 0.250\times10^{-1} & 0.191\times10^{-1} \\ 0.329\times10^{-2} & 0.000 & -0.779\times10^{-4} & 0.122\times10^{-3} & -0.621 \\ 0.718\times10^{-1} & 0.000 & -0.100 & 0.887\times10^{-3} & -3.851 \\ 0.411\times10^{-1} & 0.000 & 0.000 & -0.822\times10^{-1} & 0.000 \\ 0.361\times10^{-3} & 0.000 & 0.350\times10^{-4} & 0.426\times10^{-4} & -0.743\times10^{-1} \end{pmatrix}$$

$$B = \begin{pmatrix} 0.000 & 0.139\times10^{-2} \\ 0.000 & 0.359\times10^{-4} \\ 0.000 & -0.989\times10^{-2} \\ 0.249\times10^{-4} & 0.000 \\ 0.000 & -0.543\times10^{-5} \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

A state feedback matrix can be calculated using linear quadratic theory. In [10] it is shown that the following feedback matrix gives satisfactory responses.

$$L = \begin{pmatrix} -0.668\times10^{4} & -0.418\times10^{6} & -0.136\times10^{4} & -0.137\times10^{4} & 0.175\times10^{7} \\ -0.803\times10^{1} & -0.908\times10^{3} & -0.486 & -0.815 & 0.431\times10^{4} \end{pmatrix} \qquad (4.1)$$

The intention is now to replace the control $u = Lx$ with a simpler control using only output feedback, i.e.

$$u = K^{*}y = K^{*}Cx$$

where K^{*} shall be properly chosen. The eigenvalues of $A + BL$ are

$$\lambda_1 = -0.490\times10^{-1}$$

$$\lambda_{2,3} = -0.755\times10^{-1}\pm j\cdot0.511\times10^{-1}$$

$$\lambda_{4,5} = -0.141\pm j\cdot0.170\times10^{-1}$$

and they are shown in Fig. 4.1a.

First we attempt to include only the three eigenvalues $\lambda_{1,2,3}$ of $A + BL$ having the least real part. Somewhat arbitrarily we choose the corresponding factors as $W =$ $= \mathrm{diag}(1,1,1)$. Using (3.13) we have

$$K_I^{*} = \begin{pmatrix} 0.924\times10^{4} & -0.347\times10^{6} \\ 0.403\times10^{2} & -0.827\times10^{3} \end{pmatrix} \qquad (4.2)$$

The eigenvalues of $A + BK_I^{*}C$ are shown in Fig. 4.1b. We observe that the relative damping of the neglected pair $\lambda_{4,5} = -0.141 \pm 0.017$ has decreased in the reduced control system. In order to increase the damping we include also $\lambda_{4,5}$ in the solution and choose the weighting factors as $W = \mathrm{diag}(1,1,1,0.2,0.2)$, where the smaller weight has been laid on $\lambda_{4,5}$. The least square solution (3.13) becomes now

$$K_{II}^{*} = \begin{pmatrix} 0.569\times10^{3} & -0.286\times10^{6} \\ 0.870\times10^{1} & -0.601\times10^{3} \end{pmatrix} \qquad (4.3)$$

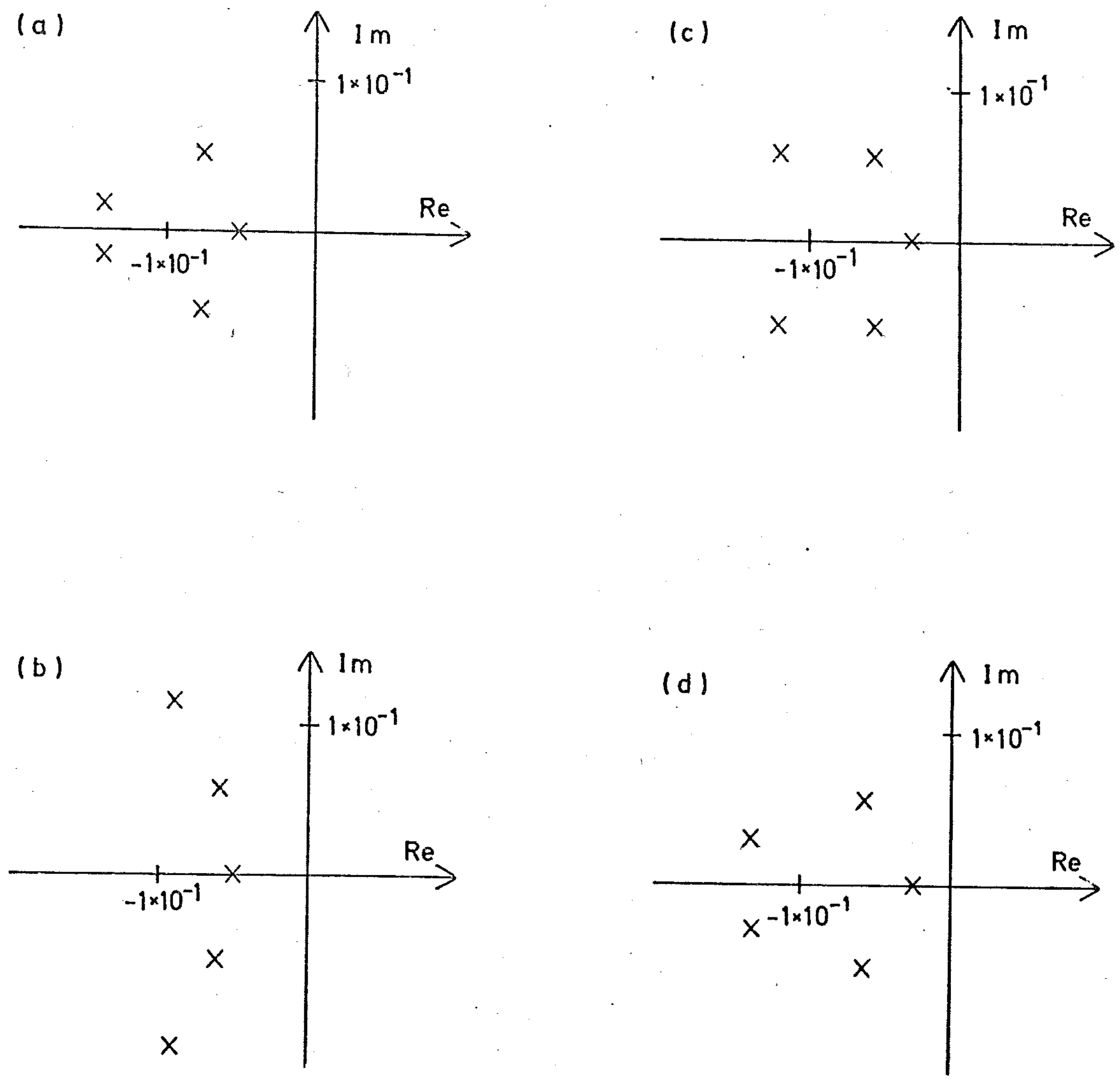
and the eigenvalues of $A + BK^{*}_{II}C$ are as shown in Fig. 4.1c. As can be seen the damping of the second complex pair has increased, but at the expense that the rightmost eigenvalue has moved somewhat nearer the imaginary axis. A further iteration with W = diag(1,1,1,0.5,0.5) gives

$$K^{*}_{III} = \begin{bmatrix} -0.265\times10^{4} & -0.263\times10^{6} \\ -0.167\times10^{1} & -0.528\times10^{3} \end{bmatrix} \tag{4.4}$$

The corresponding eigenvalue configuration is shown in Fig. 4.1d.

Simulations show that K^{*}_{II} is the most satisfactory choice in this case. The output feedback matrix can be compared with the corresponding elements in the state feedback matrix (4.1) (the two leftmost columns). As can be seen the feedback gains are slightly less in K^{*}_{II}, but of the same magnitude. However, the relations between the individual feedback gains differ considerably. This is due to the fact that compensations have been made in K^{*}_{II} for the remaining columns in L.

In Fig. 4.2 - 3 the system is simulated with control laws (4.1) and (4.3). Fig. 4.2 shows the responses for an initial condition in drum level of 0.02 m and Fig. 4.3 the same responses for an initial condition in drum pressure of 1 bar. As can be seen the difference between the exact and the reduced control laws is astonishingly small, indicating that a control only using feedback from the measured variables will be sufficient in this case.



<u>Fig. 4.1</u> - The pole configurations for the exact and the reduced control laws.

(a) exact control law (4.1)

(b) reduced control law (4.2)

(c) reduced control law (4.3)

(d) reduced control law (4.4)

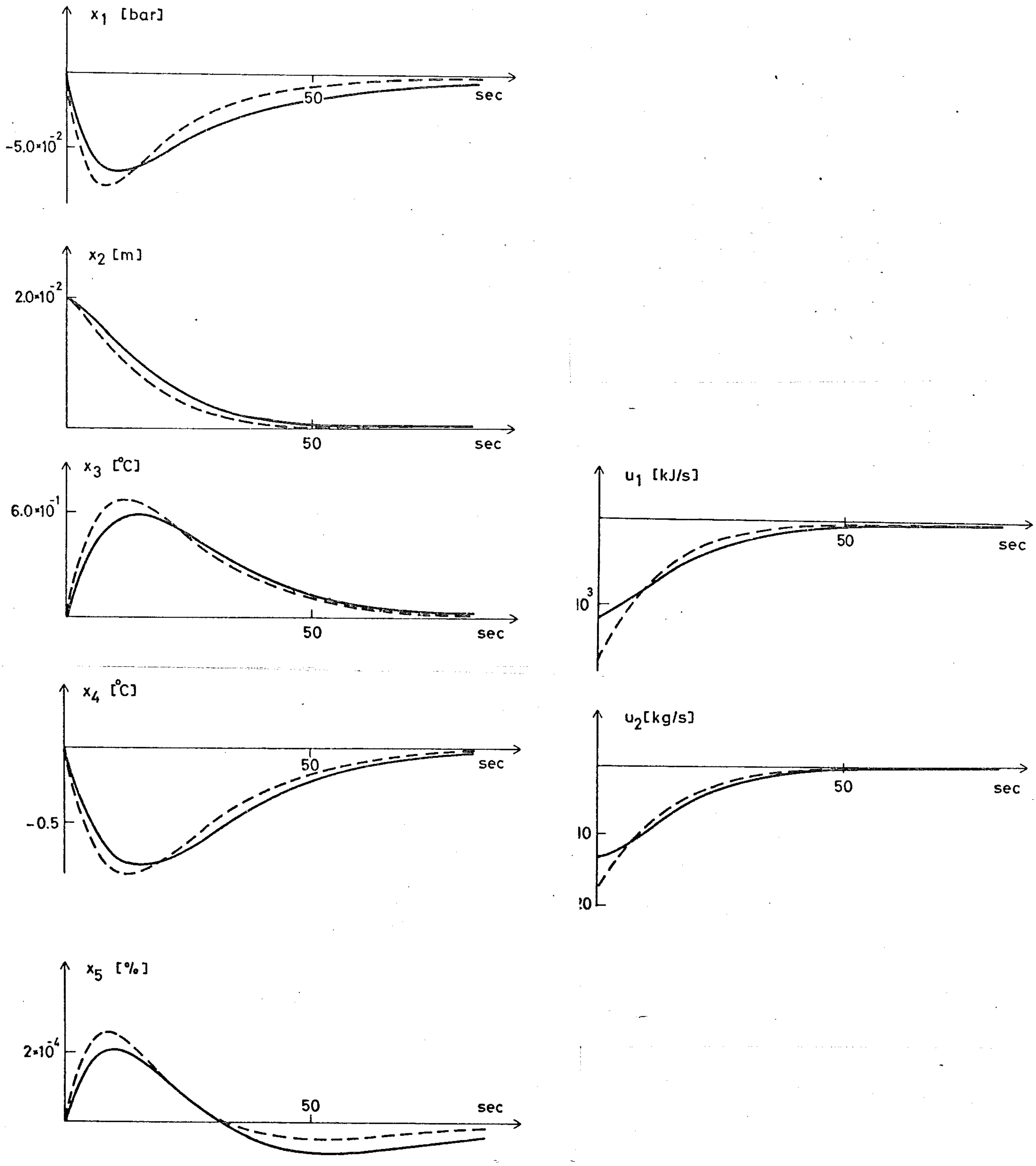


Fig. 4.2 - Responses for an initial condition in drum level of 0.02 m.

-------- exact control law (4.1)

——————— reduced control law (4.3)

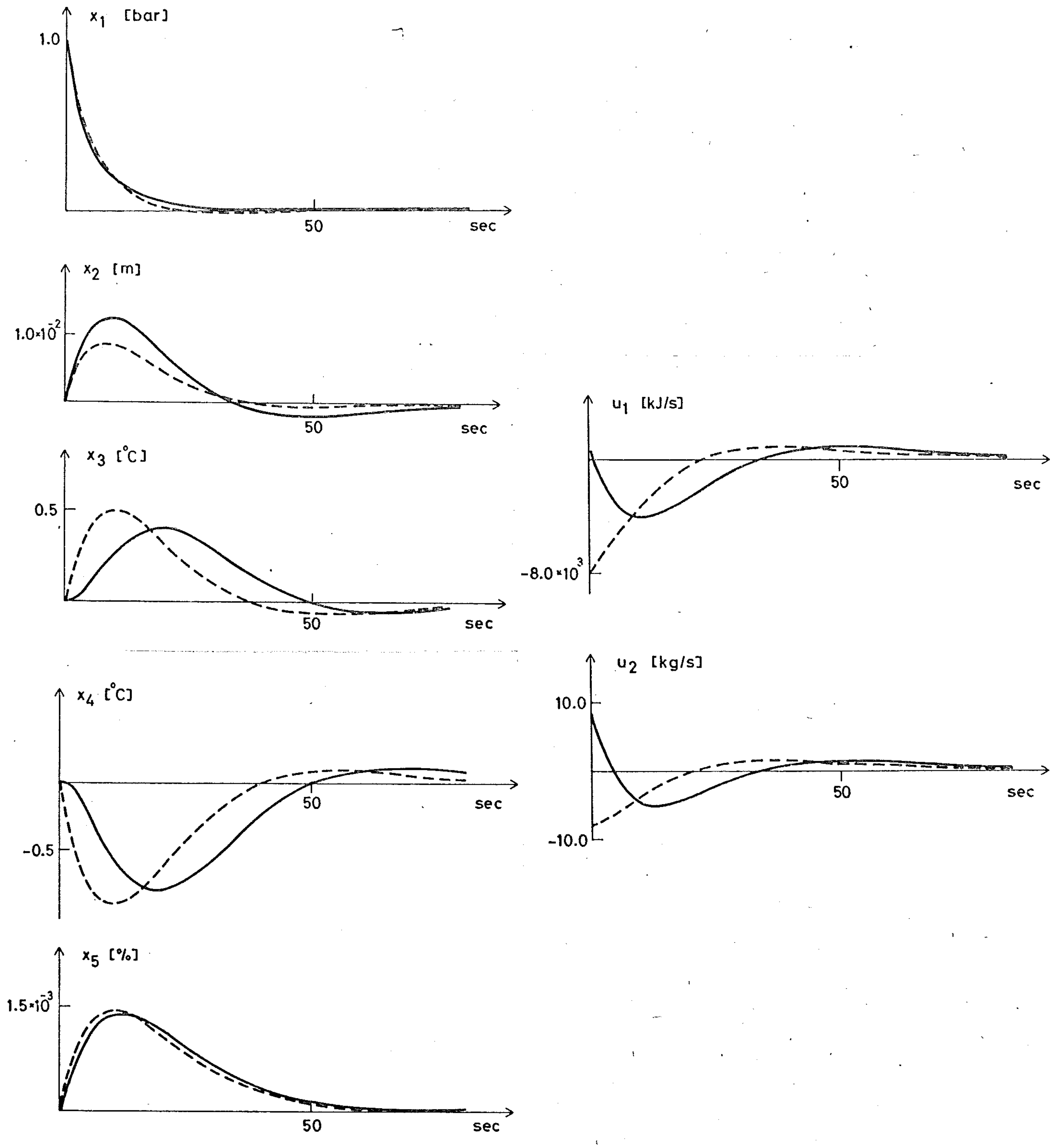


Fig. 4.3 - Responses for an initial condition in drum pressure of 1.0 bar.

-------- exact control law (4.1)

———— reduced control law (4.3)

5. APPLICATION TO POWER SYSTEM CONTROL.

We consider a reduced model of the Scandinavian network, which consists of approximately 150 nodes and 250 lines. The reduction of the model has been performed in two steps at the Swedish State Power Board. The original model has been verified by experiments. The accuracy of the reduced model can, of course, be questioned but in any case it is a typically power system model.

The model has three generators, one in North Sweden (GNOSVE), one in South Sweden (GSYSVE), and one in Norway (GNGE). The generators in North Sweden and in Norway have hydro turbines and the generator in South Sweden has a steam turbine.

The modeling of a multimachine power system has been treated in [13] and will not be discussed here.

The linearized equations for the power system can be written as

$$\frac{dx}{dt} = Ax + Bu$$

$$y = Cx$$

where the state variables are:

x_1, x_6, x_{11} = rotor angle; GNOSVE, GSYSVE, GNGE

x_2, x_7, x_{12} = rotor angular velocity; GNOSVE, GSYSVE, GNGE

x_3, x_8, x_{13} = flux linkage of field winding; GNOSVE, GSYSVE, GNGE

x_4, x_9, x_{14} = excitation voltage; GNOSVE, GSYSVE, GNGE

x_5, x_{15} = velocity of water; GNOSVE, GSYSVE, GNGE

x_{10} = steam pressure; GSYSVE

The input variables are:

u_1, u_3, u_6 = excitation input; GNOSVE, GSYSVE, GNGE

u_2, u_7 = gate opening, GNOSVE,GNGE

u_4 = steam valve setting; GSYSVE

u_5 = fuel flow, GSYSVE

The output variables are:

y_1, y_4, y_8 = rotor angular velocity; GNOSVE,GSYSVE,GNGE

y_2, y_5, y_9 = te-minal voltage; GNOSVE, GSYSVE, GNGE

y_3, y_6, y_{10} = excitation voltage; GNOSVE, GSYSVE, GNGE

y_7 = steam pressure, GSYSVE

Numerical values for A, B and C for the power system are given in Appendix 3. The operating point corresponds to the expected peak load 1975 with high transmission from North Sweden to South Sweden.

In [14] linear quadratic control theory was used to find a suitable state feedback matrix L. Numerical values of -L is given in Appendix 4. The intention is now to replace the state feedback u=Lx with local output feedback $u_i = K_i y_i$, where u_i are the inputs and y_i are the outputs at generator i. The eigenvalues of A+BL are:

$\lambda_1 = -7.33 \cdot 10^{-3}$

$\lambda_2 = -2.09 \cdot 10^{-1}$

$\lambda_{3,4} = -2.77 \cdot 10^{-1} \pm i3.55 \cdot 10^{-1}$

$\lambda_5 = -3.17 \cdot 10^{-1}$

$\lambda_{6,7} = -3.83 \cdot 10^{-1} \pm i2.53 \cdot 10^{-1}$

$\lambda_8 = -5.14 \cdot 10^{-1}$

$\lambda_{9,10} = -1.36 \pm i3.12$

$\lambda_{11,12} = -1.37 \pm i4.18$

$\lambda_{13,14} = -1.49 \pm i3.79 \cdot 10^{-2}$

$\lambda_{15} = -2.4613$

and they are shown in Fig. 5.1.

The control reduction was performed in two steps: In the first step we allowed local state feedback and attempted to preserve all eigenvalues. By adjusting the weighting factors (w) in five iterations we found a subset of seven critical eigenvalues. These were λ_3, λ_4, λ_6, λ_7, λ_8, λ_{13}, and λ_{14}. In the second step we attempted to preserve only the critical eigenvalues and used the same weighting factors as in the first step. The least square solution (3.13) is given in Appendix 5 and the eigenvalues of $A + BK^*C$ are:

$\lambda_1 = -5.56 \cdot 10^{-8}$

$\lambda_{2,3} = -3.33 \cdot 10^{-1} \pm i2.70 \cdot 10^{-1}$

$\lambda_{4,5} = -3.44 \cdot 10^{-1} \pm i2.52 \cdot 10^{-1}$

$\lambda_{6,7} = -3.84 \cdot 10^{-1} \pm i2.10 \cdot 10^{-1}$

$\lambda_8 = -4.65 \cdot 10^{-1}$

$\lambda_{9,10} = -7.95 \cdot 10^{-1} \pm i3.00$

$\lambda_{11,12} = -1.19 \cdot i4.05$

$\lambda_{13} = -1.47$

$\lambda_{14} = -1.66$

$\lambda_{15} = -2.44$

The eigenvalues of $A + BK^{*}C$ are shown in Fig. 5.2.

The power system is simulated with the complete state feedback given in Appendix 2 and the responses are shown in Fig. 5.3. The power system is also simulated with the local output feedback given in Appendix 3 and the responses are shown in Fig. 5.4. In both cases the rotor angle of the generator in North Sweden (x_1) is given an initial value of 0.5 rad.

It is surprising how well the local output feedback behaves. The linear quadratic control law is designed to keep the angle differences small, the frequency and the terminal voltages small. We observe that x_1, x_6 and x_{11} are close together when state feedback and local output feedback are employed. We also observe that the rotor angles are not reduced to zero with local output feedback. This explains the zero eigenvalue λ_1. The angles could be reduced to zero if one of them is included in the output vector. In practice the frequency error is integrated and fed to the controllers.

The responses of x_2, x_7 and x_{12} (angular velocity) are less but still very satisfactory damped in the second case. The responses of y_2, y_5 and y_9 (terminal voltage) are also less damped but the damping is still acceptable.

The proposed local output feedback does not require larger control effort than linerar quadratic control does. It is interesting to note that u_1 is initially negative in the linear quadratic control simulation but positive in the local output control simulation. From fig. 5.4b we observe that the terminal voltage is negative at t=0 and using only this information it is an understandable reaction to increase the excitation input in order to increase the terminal voltage. The linear quadratic control law decreases the excitation input in a coordinated action to reduce power swings and restore the state to zero. We also observe that u_2 and u_7 (gate opening) are almost zero in the second case. This is an advantage of the proposed local output feedback and is consistent with the current trend to use the excitation control to improve ~~damping~~ power system stability.

The state variables associated with the prime movers (x_5,x_{10}, and x_{15}) are almost zero in the second case. This is a direct consequence of the fact that the prime mover inputs are almost zero.

It is concluded that very little can be gained from complete centralized control schemes for dynamic control and that properly designed local controllers would be sufficient. The proposed method of approximating complete state feedback with output feedback could be an effective tool for the design of such local controllers.

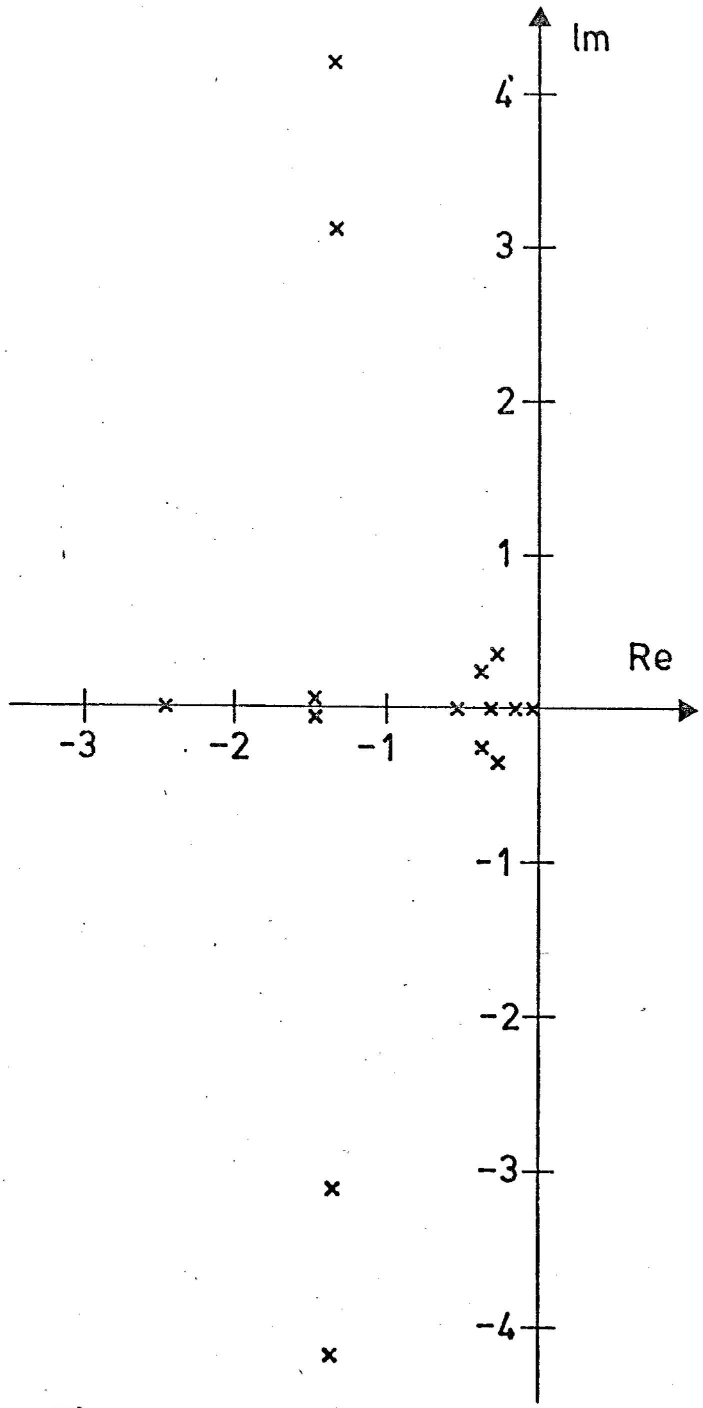


Fig. 5.1 - Eigenvalues for the power system with complete state feedback.

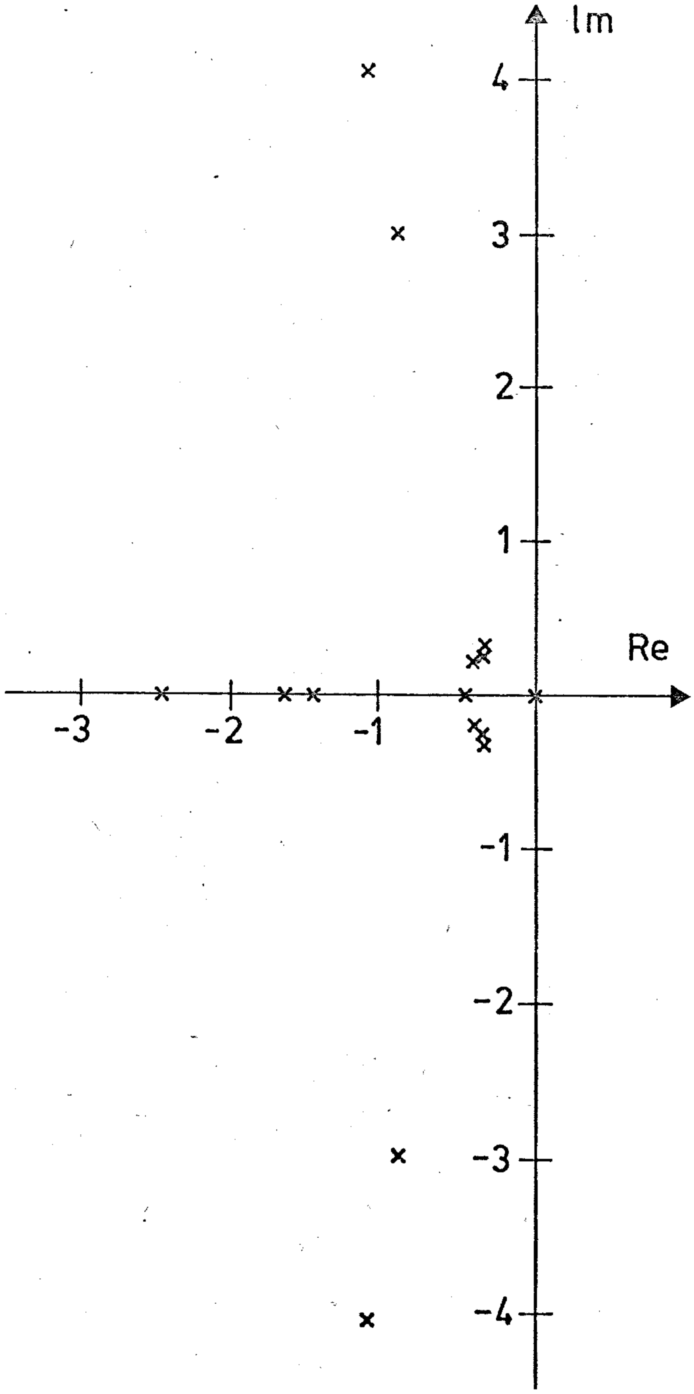
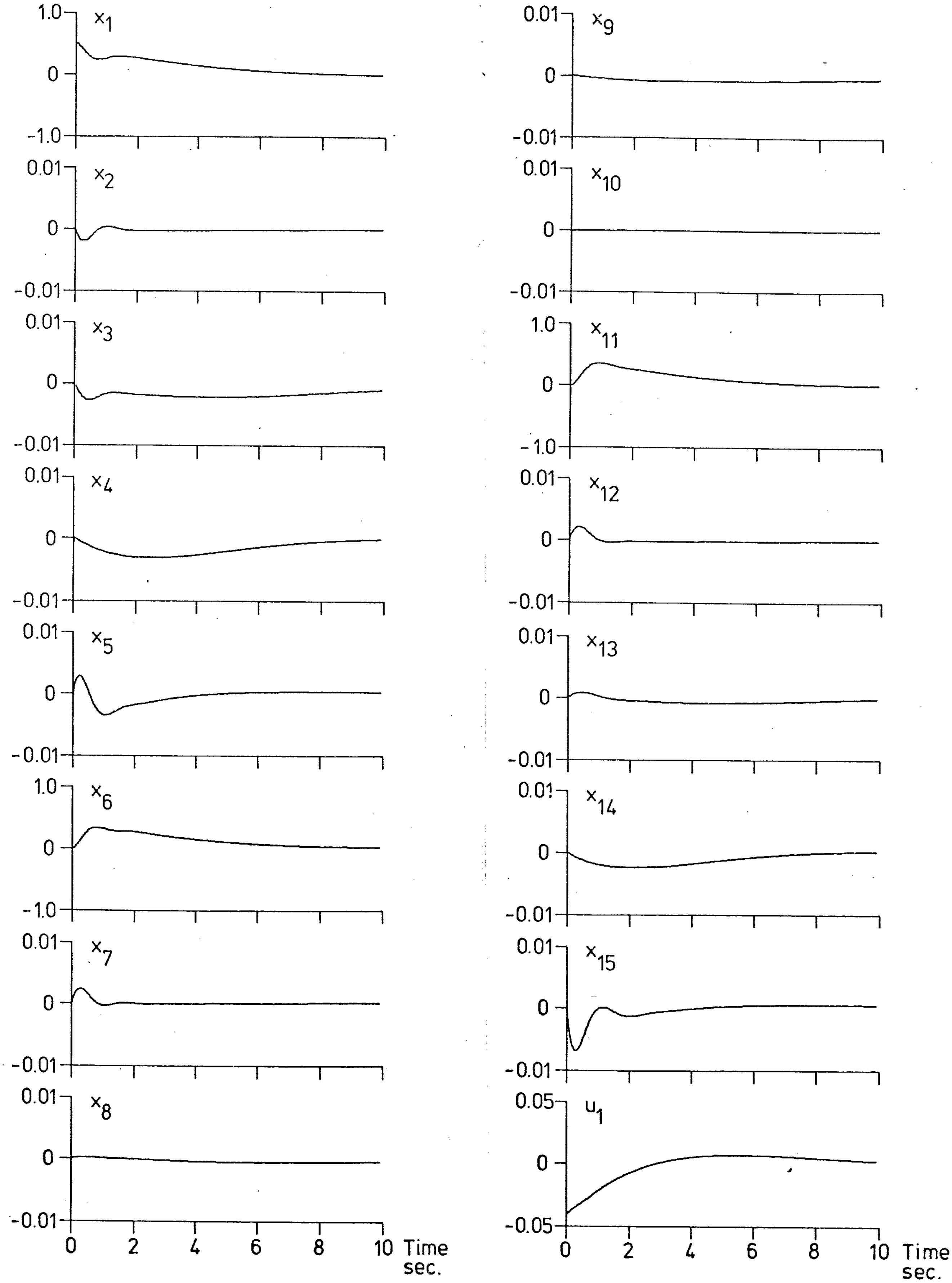


Fig. 5.2 - Eigenvalues for the power system with local output feedback.



<u>Fig. 5.3a</u> - Responses of the power system with complete state feedback.

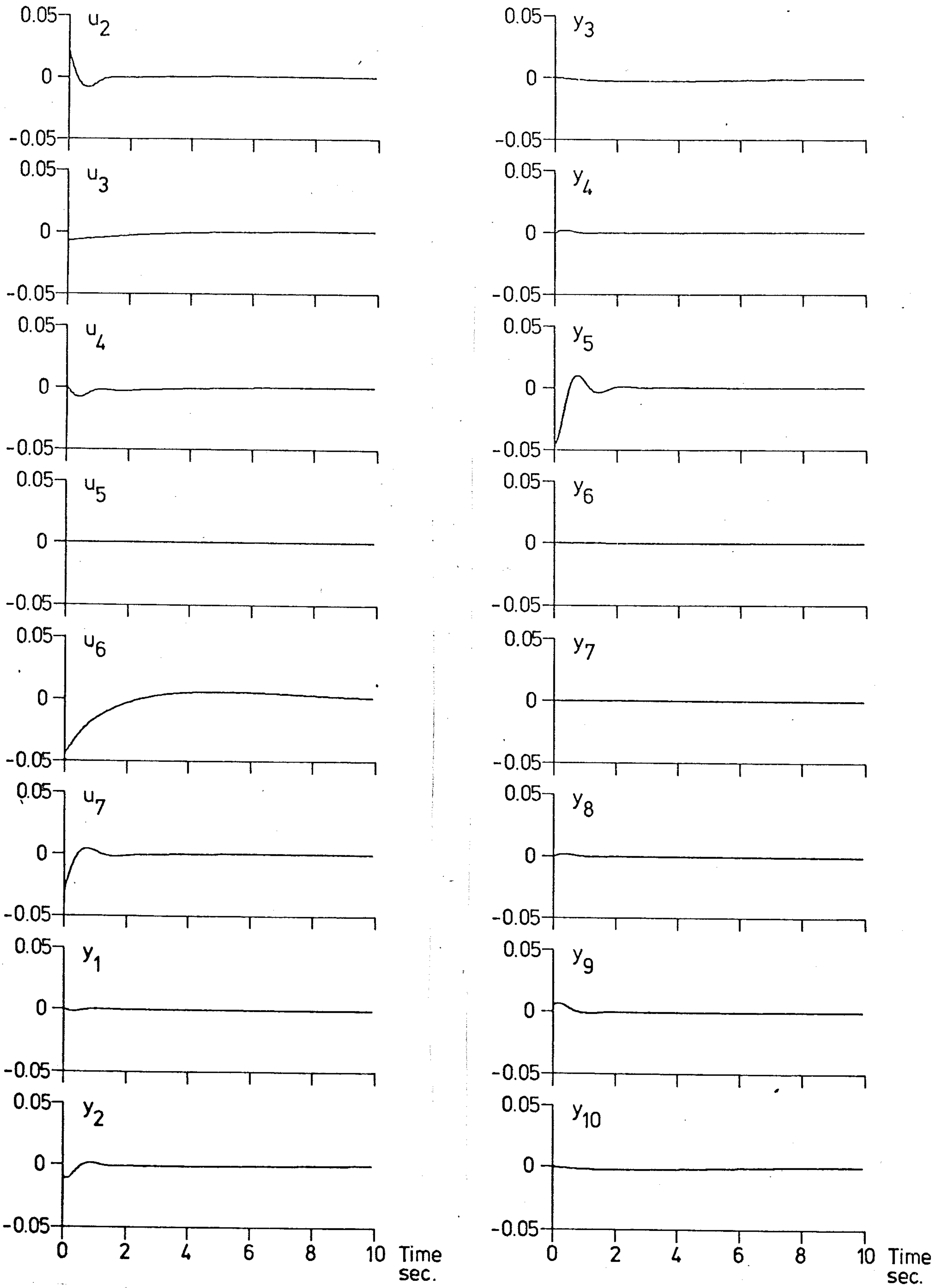


Fig. 5.3b - Responses of the power system with complete state feedback.

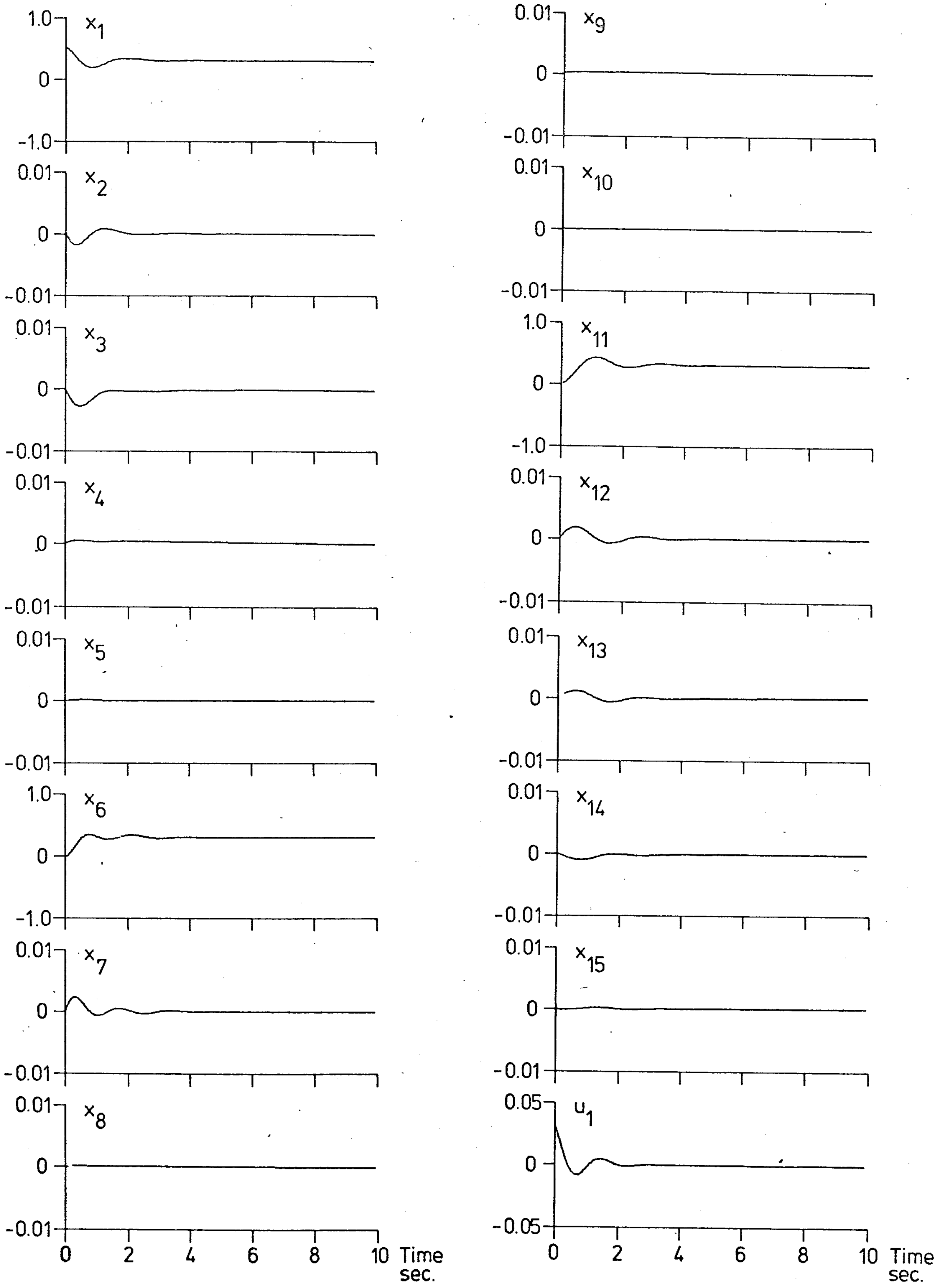


Fig. 5.4a - Responses of the power system with local output feedback.

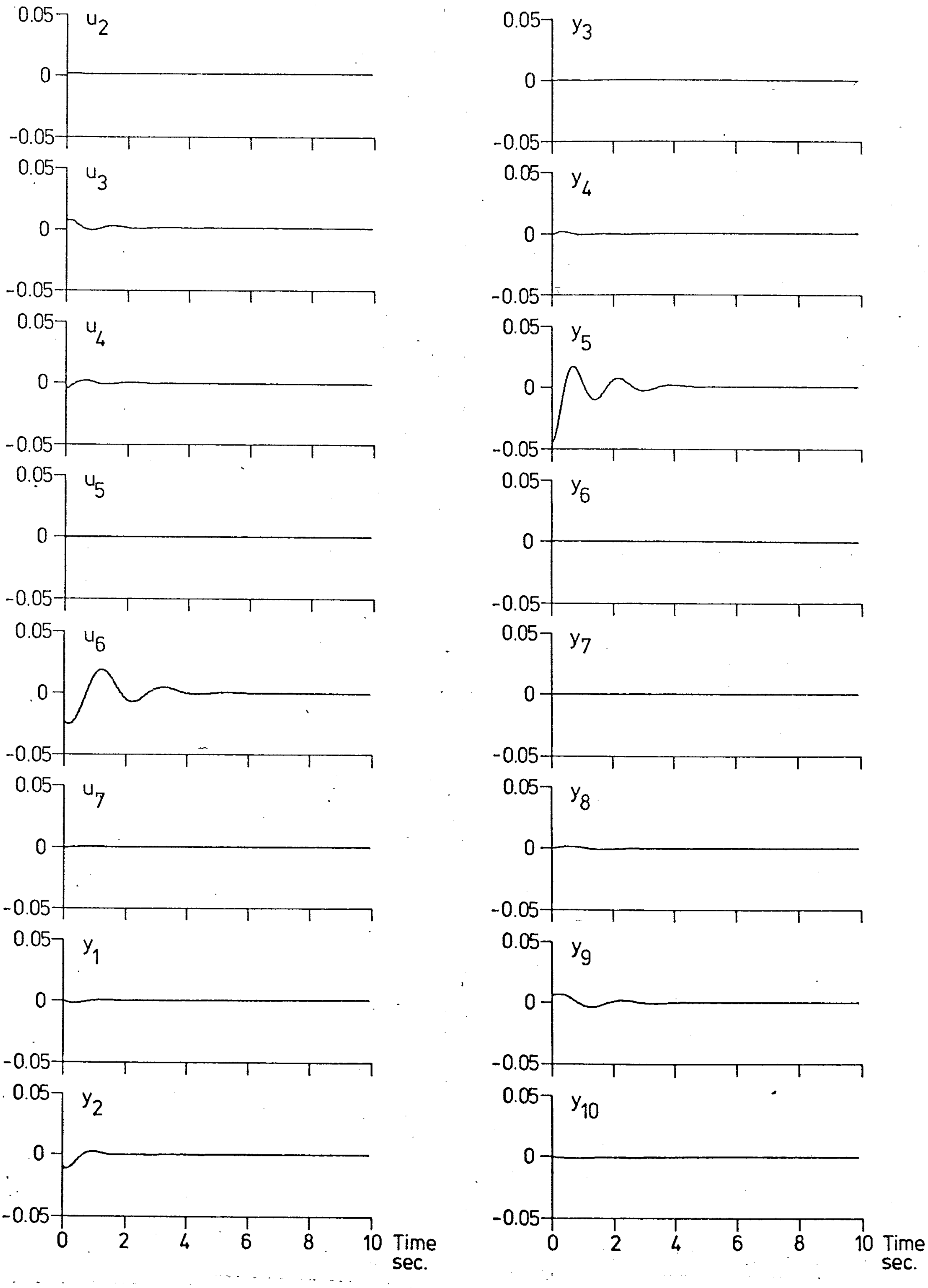


Fig. 5.4b - Responses of the power system with local output feedback.

6. CONCLUSIONS

In this paper we have attempted to devolope a rational approach to fit a known structural controller, based on output measurement by considering the best fit in the eigenspace corresponding to the dominant eigenvalues of the complete state feedback controller. This sate feedback controller can be computed in many ways e.g. by linear quadratic contro theory. An iterative procedure is developed by allowing the designer to change weighting factors for the dominant modes. Computatinally the design method requires an eigenvalue-eigenvector calculation and one least-squares solution as well as one eigenvalue calculation per iteration. The feasibility of the method has been d-monstrated on a number of examples. It has been demonstrated that an output feedback controller, using only two outputs, behaves almost as well as feeding back the full state. It has also been demonstrated that very little can be gained from complete centralized control schemes for a power system and that properly designed local controllers are sufficient. In both cases state reconstruction has been avoided and the second the power system example shows the feasibility of the method for large decentralised systems.

7. ACKNOWLEDGEMENT

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APPENDIX 1

Invariant Eigenspaces.

Let A be an arbitrary n×n matrix and let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_p\}$ be a given subset of eigenvalues to A.

Assume that Λ is a symmetric set, i.e. if $\lambda \in \Lambda$ then also $\bar{\lambda} \in \Lambda$, where the bar indicates complex conjugation. First consider the case when A is cyclic, i.e. there are n linearly independent eigenvectors to A. Then an invariant subspace is simply obtained from the eigenvectors $a_1, a_2, \ldots, a_p$ corresponding to Λ, i.e.

$$Q = [a_1 \;\; a_2 \; \ldots \; a_p] \tag{A1.1}$$

~~(2.5)~~

is a basis matrix for the eigenspace.

If A is non-cyclic the concept of generalized eigenvectors is introduced. Le λ_i be an eigenvalue of multiplicity $\sigma_i > 1$. The generalized eigenvectors $a_i^1, a_i^2, \ldots, a_i^{\sigma_i}$ corresponding to λ_i are then defined as the nontrivial solutions of

$$(A - \lambda_i I)a_i^1 = 0$$

$$(A - \lambda_i I)a_i^k = a_i^{k-1}; \qquad k = 1, 2, \ldots, \sigma_i$$

The basis for the eigenspace is then constructed according to the following rule. If a_i^ℓ is selected, then a_i^k, $k = 1, 2, \ldots, \ell-1$, must also be selected as members of the basis if an invariant subspace shall be obtained. In this way an invariant eigenspace may be constructed corresponding to any set of eigenvalues to A.

Finally, observe that since Λ is assumed to be a symmetric set and A is assumed to be real, a real basis for the eigenspace is obtained by taking

$$Q = [a_1 \; a_2 \; \ldots \; a_s; \; \mathrm{Re}\{a_{s+1}\} \; \mathrm{Im}\{a_{s+1}\} \; \mathrm{Re}\{a_{s+2}\} \; \ldots] \qquad (A1.2)$$

where a_1, a_2, ..., a_s are assumed to be real and a_{s+1}, a_{s+2}, ..., a_p are assumed to be complex. For any pair λ, $\bar{\lambda}$ belonging to Λ then choose Re{a}, Im{a} as members of the basis where a is the eigenvector corresponding to λ. In this way complex arithmetic is avoided in the sequel, and is only needed in the eigenvector calculation.

APPENDIX 2

Pseudo Inverses.

Let M be an arbitrary real matrix. The pseudo inverse $M^{\dagger}$ of M is then defined by the following four conditions:

1° $M^{\dagger}M\,M^{\dagger} = M^{\dagger}$

2° $M\,M^{\dagger}M = M$

3° $M^{\dagger}M$ is symmetric

4° $M\,M^{\dagger}$ is symmetric

It is shown in [11] that $M^{\dagger}$ is uniquely defined by these conditions. Numerical algorithms exist to find such inverses, see for instance [12].

The pseudo inverse has some nice properties in minimization on inner product spaces. Consider the equation

$$Mx = y$$

which shall be solved for x. Then $x_o = M^{\dagger}y$ has the following properties:

1° x_o minimizes $||Mx - y||$ where $||\cdot||$ denotes the ordinary euclidian quadratic norm.

2° amongst the possible candidates for the minimum of $||Mx - y||$, x_o is the one that minimizes $||x||$.

A-MATRIX, PAGE 1

A(1, 2)= .314159+03	A(2, 1)=-.242249-01	A(2, 2)=-.322929+01
A(2, 3)= .162980+00	A(2, 5)= .340985+00	A(2, 6)= .113810-01
A(2, 7)=-.664848-02	A(2, 8)=-.684552-02	A(2,11)= .128439-01
A(2,12)=-.998426-02	A(2,13)=-.712887-02	A(3, 1)=-.213677-01
A(3, 2)=-.676561-01	A(3, 3)=-.304433+00	A(3, 4)= .250453+00
A(3, 6)= .147243-01	A(3, 7)=-.884903-02	A(3, 8)= .554234-03
A(3,11)= .664346-02	A(3,12)= .672120-02	A(3,13)= .827741-02
A(4, 4)=-.769231-01	A(5, 5)=-.140858+01	A(6, 7)= .314159+03
A(7, 1)= .310481-01	A(7, 2)=-.207229-01	A(7, 3)=-.138751-01
A(7, 6)=-.499094-01	A(7, 7)=-.242749+01	A(7, 8)= .465573-01
A(7,10)= .159024+00	A(7,11)= .188614-01	A(7,12)=-.141278-01
A(7,13)=-.993110-02	A(8, 1)= .180088-02	A(8, 2)= .602946-02
A(8, 3)= .695069-02	A(8, 6)=-.227243-02	A(8, 7)=-.282597-01
A(8, 8)=-.360985+00	A(8, 9)= .336492+00	A(8,11)= .471545-03
A(8,12)= .390225-02	A(8,13)= .403555-02	A(9, 9)=-.100000+00
A(10,10)=-.732244-02	A(11,12)= .314159+03	A(12, 1)= .130472-01
A(12, 2)=-.106559-01	A(12, 3)=-.791938-02	A(12, 6)= .649000-02
A(12, 7)=-.511231-02	A(12, 8)=-.462964-02	A(12,11)=-.195372-01
A(12,12)=-.887218+00	A(12,13)= .165403+00	A(12,15)= .441134+00
A(13, 1)= .728335-02	A(13, 2)= .410596-02	A(13, 3)= .636214-02
A(13, 6)= .723718-02	A(13, 7)=-.444029-02	A(13, 8)=-.930530-04
A(13,11)=-.145205-01	A(13,12)=-.536106-01	A(13,13)=-.288581+00
A(13,14)= .247081+00	A(14,14)=-.769231-01	A(15,15)=-.183560+01

69 NONZERO ELEMENTS
156 ZERO ELEMENTS

B-MATRIX, PAGE 1

B(2, 2)=-.227323+00	B(4, 1)= .769231-01	B(5, 2)= .140858+01
B(7, 4)= .162229+00	B(9, 3)= .100000+00	B(10, 4)=-.783733-02
B(10, 5)= .730000-02	B(12, 7)=-.294089+00	B(14, 6)= .769231-01
B(15, 7)= .183560+01		

10 NONZERO ELEMENTS
95 ZERO ELEMENTS

C-MATRIX, PAGE 1

C(1, 2)= .100000+01	C(2, 1)=-.212729-01	C(2, 2)= .932214+00
C(2, 3)= .889955+00	C(2, 6)= .194903-01	C(2, 7)=-.100683-01
C(2, 8)= .734920-02	C(2,11)= .178260-02	C(2,12)= .227022-01
C(2,13)= .232592-01	C(3, 4)= .100000+01	C(4, 7)= .100000+01
C(5, 1)=-.913649-01	C(5, 2)= .240934+00	C(5, 3)= .233823+00
C(5, 6)= .162358+00	C(5, 7)= .277790+00	C(5, 8)= .286660+00
C(5,11)=-.709951-01	C(5,12)= .159072+00	C(5,13)= .143982+00
C(6, 9)= .100000+01	C(7,10)= .100000+01	C(8,12)= .100000+01
C(9, 1)= .108115-01	C(9, 2)= .114114-01	C(9, 3)= .151457-01
C(9, 6)= .126540-01	C(9, 7)=-.743007-02	C(9, 8)= .117922-02
C(9,11)=-.234655-01	C(9,12)= .102604+01	C(9,13)= .912437+00
C(10,14)= .100000+01		

34 NONZERO ELEMENTS
116 ZERO ELEMENTS

```
L-MATRIX, PAGE 1

L( 1, 1)= .145461+00   L( 1, 2)= .146007+02   L( 1, 3)= .653267+01
L( 1, 4)= .514367+01   L( 1, 5)= .387752+01   L( 1, 6)= .718818-01
L( 1, 7)= .752995+01   L( 1, 8)= .311316+00   L( 1, 9)= .322513+00
L( 1,10)= .519761+01   L( 1,11)= .620178-01   L( 1,12)= .181910+02
L( 1,13)= .438563+01   L( 1,14)= .188907+01   L( 1,15)= .365105+01
L( 2, 1)=-.130166-01   L( 2, 2)=-.550412+00   L( 2, 3)= .135702+00
L( 2, 4)= .445665-01   L( 2, 5)= .100274+00   L( 2, 6)= .576800-02
L( 2, 7)= .322695+00   L( 2, 8)= .448535-03   L( 2, 9)= .278974-02
L( 2,10)= .433868-01   L( 2,11)= .112001-01   L( 2,12)= .581754+00
L( 2,13)=-.537885-02   L( 2,14)= .188049-01   L( 2,15)=-.374450-01
L( 3, 1)= .279223-01   L( 3, 2)= .262411+01   L( 3, 3)= .681950+00
L( 3, 4)= .419267+00   L( 3, 5)= .547274+00   L( 3, 6)= .596462-02
L( 3, 7)= .966605+00   L( 3, 8)= .261260+00   L( 3, 9)= .639725+00
L( 3,10)= .490135+01   L( 3,11)= .125576-01   L( 3,12)= .306954+01
L( 3,13)= .512731+00   L( 3,14)= .181930+00   L( 3,15)= .548821+00
L( 4, 1)= .117973-02   L( 4, 2)=-.292205-01   L( 4, 3)= .508446-01
L( 4, 4)= .245616-01   L( 4, 5)= .322084-01   L( 4, 6)= .817428-02
L( 4, 7)= .978703+00   L( 4, 8)= .975280-02   L( 4, 9)= .189437-02
L( 4,10)= .557202-01   L( 4,11)=-.434347-02   L( 4,12)=-.283600+00
L( 4,13)= .699094-01   L( 4,14)= .235587-01   L( 4,15)= .459637-01
L( 5, 1)= .589446-04   L( 5, 2)= .495338-02   L( 5, 3)= .135419-02
L( 5, 4)= .789205-03   L( 5, 5)= .102425-02   L( 5, 6)= .738593-05
L( 5, 7)= .308770-02   L( 5, 8)= .321558-03   L( 5, 9)= .572478-03
L( 5,10)= .120141-01   L( 5,11)= .247140-04   L( 5,12)= .450175-02
L( 5,13)= .122347-02   L( 5,14)= .570772-03   L( 5,15)= .936447-03
L( 6, 1)= .173255+00   L( 6, 2)= .153267+02   L( 6, 3)= .391431+01
L( 6, 4)= .188907+01   L( 6, 5)= .311533+01   L( 6, 6)= .728300-01
L( 6, 7)= .716327+01   L( 6, 8)= .168458+00   L( 6, 9)= .139947+00
L( 6,10)= .375904+01   L( 6,11)=-.747451-02   L( 6,12)= .150397+02
L( 6,13)= .830344+01   L( 6,14)= .577019+01   L( 6,15)= .460714+01
L( 7, 1)= .233525-01   L( 7, 2)= .150466+01   L( 7, 3)= .496477-01
L( 7, 4)= .281233-01   L( 7, 5)= .725717-01   L( 7, 6)= .126299-01
L( 7, 7)= .103660+01   L( 7, 8)=-.486669-02   L( 7, 9)= .167516-02
L( 7,10)= .541131-01   L( 7,11)=-.296693-01   L( 7,12)=-.105230+01
L( 7,13)= .384120+00   L( 7,14)= .839044-01   L( 7,15)= .338981+00

      105 NONZERO ELEMENTS
        0 ZERO ELEMENTS
```

```
K-MATRIX, PAGE 1

K( 1, 1)=-.867660+01  K( 1, 2)= .303300+01  K( 1, 3)= .396280+01
K( 2, 1)= .106700+00  K( 2, 2)= .389030-01  K( 2, 3)= .234150-01
K( 3, 4)= .500470+00  K( 3, 5)= .116950+00  K( 3, 6)=-.294870+00
K( 3, 7)=-.561590+03  K( 4, 4)=-.524640+00  K( 4, 5)=-.102910+00
K( 4, 6)=-.354950+00  K( 4, 7)=-.841009+02  K( 5, 4)=-.240000-06
K( 5, 5)=-.860000-07  K( 5, 6)=-.430000-05  K( 5, 7)=-.912980+00
K( 6, 8)=-.133535+01  K( 6, 9)= .436700+01  K( 6,10)= .319935+01
K( 7, 8)=-.377860+00  K( 7, 9)= .102900+00  K( 7,10)=-.423256-02

     24 NONZERO ELEMENTS
     46 ZERO ELEMENTS
```

A DESIGN SCHEME FOR INCOMPLETE STATE OR OUTPUT FEEDBACK WITH APPLICATIONS TO BOILER AND POWER SYSTEM CONTROL

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A DESIGN SCHEME FOR INCOMPLETE STATE OR OUTPUT FEEDBACK WITH APPLICATIONS TO BOILER AND POWER SYSTEM CONTROL.

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ABSTRACT.

The problem of designing a linear feedback when all state variables are not available is discussed. The design scheme is based on computation of a complete state feedback and a reduction to a specified structure. The reduction is made by approximation on the eigenspace corresponding to a set of dominant eigenvalues. The method consists of successive choices of weightings on this space. The method is applied to the control of a boiler and a three-machine power system. In the power system case the complete state feedback can be replaced by local output feedback without any significant decrease in performance.

The examples indicate that the proposed method is a realistic design method for multivariable systems.

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1. INTRODUCTION.

The concept of state feedback plays an important role in existing control theory for linear systems. Linear quadratic control theory [1] and pole assignment theory [3,4] are two well-known examples. Unfortunately the whole state vector is, however, rarely available for measurement. Even if it was available a state feedback control would sometimes result in far too complex control systems. The standard way to bypass these difficulties is to measure only a small set of outputs and reconstruct the full state vector using a Kalman filter [2] or an observer [7]. The result is, however, still somewhat unsatisfactory since the reconstruction by itself might produce high order dynamics in the control function.

These facts justify the demands for simpler or suboptimal control policies. Practical constraints on the feedback system must be considered. A limited number of measurements is one obvious constraint. In large systems consisting of several coupled subprocesses, such as power systems, there may be a desire to control the system with local feedbacks on the different processes, eventually with the addition of a small number of interconnections. There are, however, no rational ways to design such hierarchical control schemes. Another example is diagonally controlled systems where the design philosophy is the classical one with each input variable controlling a single output variable.

A few methods exist to solve some problems of these types. Here we will only mention modal control [4,5] and the suboptimal linear quadratic regulator [8]. In [8] the usual quadratic performance criterion is minimized with respect to a selected number of feedback gains, using a function minimization algorithm. Aside from the fact that the convergence toward a unique global minimum has not been shown, this technique does not seem to be practical when applied to large systems.

In this paper a state feedback control is used as the starting point. This is quite a realistic assumption, since there are straightforward methods to find such controllers even for fairly large systems. See for instance [1] and [4]. The step taken is then to fit this control into another "similar" control with a predefined structure. The idea behind this fit is to make it as accurate as possible on the eigenspace corresponding to a dominant set of eigenvalues to the closed loop system. It is illustrated by examples that satisfactory controllers may be obtained in this way after a few iterations. It should be noticed that the method does not depend on how the state feedback controller is obtained. The reduction technique is thus applicable to any method that results in a linear feedback from the state.

Notice that this reduction procedure is a rational way of designing hierarchical control systems. Sometimes it is not possible to control the system satisfactorily by output feedback only. In such cases the reduction scheme can be used to find controllers of PD-types, where the derivative term will give additional information about the state of the system and thus making the system easier to stabilize.

The paper is organized as follows. Some mathematical preliminaries are given in Section 2. The control reduction scheme is presented in Section 3. Applications to boiler control and power system control are finally discussed in Sections 4 and 5.

In the boiler case it is shown that the feedback from all five states can be replaced by the feedback from two outputs. In this case it is possible to avoid the Kalman filter, proposed for the reconstruction of the state, without any significant decrease in performance.

The power system is an example of a system, consisting of geographically distributed subsystems. State estimation and feedback can be organized in a centralized or a decentralized manner. In both cases large amount of data has to be transmitted. Although data transmission systems are under construction it is desirable to have control schemes, which do not require high speed data transmission. The whole state vector could be reconstructed locally if the system is observable via locally available outputs. The dimension of the Kalman filter, however, becomes very high.

In this paper we consider a three-machine power system with 15 states. The complete state feedback can be replaced with local output feedback without any significant decrease in damping. Also in this case it is possible to avoid high order Kalman filters. The results also indicate that very little can be gained from complete centralized control schemes and that properly designed local controllers are sufficient.

In large systems, such as power systems, the computational effort is of importance. The major computational burden in this case lies on an initial eigenvalue-eigenvector calculation, which corresponds to approximately $8n^3$ operations. An additional eigenvalue calculation may have to be done to check if the reduced control law has an acceptable degree of stability.

This method could be an effective tool for the design of multivariable controllers in an interactive mode.

2. PRELIMINARIES.

In this section we will give a formal statement of the problem. The concept of constrained feedback structures will be concisely defined. For completeness some well-known properties of invariant eigenspaces and generalized inverses are also given, since these two concepts will be frequently used in the sequel.

Statement of the Problem.

Consider a linear time invariant system in state space form

$$\dot{x} = Ax + Bu \tag{2.1}$$

where x is the n-vector of states and u is the m-vector of control inputs. A and B are real-valued matrices of compatible dimensions. Moreover, assume that a state feedback controller

$$u = Lx + v \tag{2.2}$$

where L is an $m \times n$ real-valued matrix is found such that the system (2.1) with the controller (2.2) has the desired properties.

In controlling the system (2.1) we will set certain constraints on the feedback system. The intention is then to "reduce" the control law (2.2) such that these constraints are satisfied. In specific two types of constraints will be considered corresponding to different degrees of complexity in the control function. These definitions should cover a large variety of practical constraints that might be imposed on the structure of a feedback system.

In order to simplify the notations we will use stars (*) to indicate properties associated with the reduced control laws.

The simplest kind of constraint is to permit output feedback. Let $y = Cx$ denote the output of (2.1) where C is a real $r \times n$ matrix. A control of the form

$$u = K^{*}Cx + v \tag{2.3}$$

will be referred as a control with a <u>single constrained feedback structure</u>.

A more complex structure is obtained if the i:th input variable is restricted to be a function of certain specified outputs. Let $y_i = C_i x$, $i = 1, 2, \ldots, q$, denote q sets of output variables to (2.1) where C_i is an $r_i \times n$ matrix. Moreover, let $u^T = [u_1^T \; u_2^T \; \ldots \; u_q^T]$ be a partition of the control vector into an appropriate set of q subvectors. A control of the form

$$u_i = K_i^{*} C_i x + v_i \qquad i = 1, 2, \ldots, q \tag{2.4}$$

will be referred as a control with a <u>multiple constrained feedback structure</u>. It is easily verified that as well local as hierarchical types of control systems are included in this formulation. Notice that the control (2.3) is a special case of (2.4) with $q = 1$. An illustration of the two concepts is given in Fig. 2.1 and Fig. 2.2.

A common way to do the kind of reductions considered here is to simply neglect those entries of the state feedback matrix that are "small" in comparison with the others.

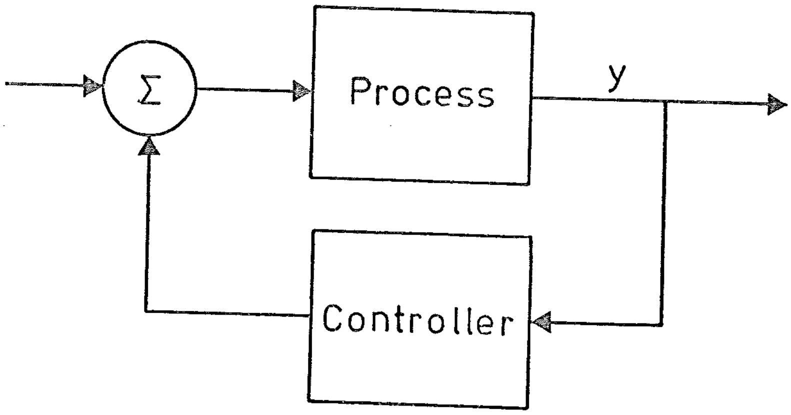


Fig. 2.1 - A system controlled via single constrained feedback structure.

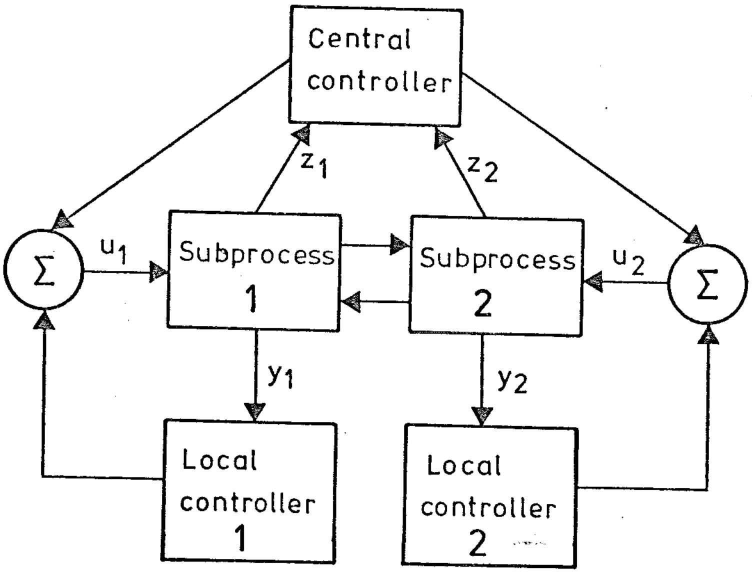


Fig. 2.2 - Two coupled systems controlled via multiple constrained feedback structure.

There are, however, several difficulties involved in such a procedure, and it requires frequently a fairly deep understanding of the process dynamics. Moreover, there is no rational way to "compensate" the remaining entries for the approximations made. The approach of this paper will instead be to construct a certain subspace of the state space where the reduction is made. In this way the "compensating" problem is avoided and converted to the problem of finding the appropriate subspace. However, there are rational ways to construct such subspaces and some of them will be described below.

Invariant Eigenspaces.

Let A be an arbitrary n×n matrix and let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_p\}$ be a given subset of eigenvalues to A.

Assume that Λ is a symmetric set, i.e. if $\lambda \in \Lambda$ then also $\bar{\lambda} \in \Lambda$, where the bar indicates complex conjugation. First consider the case when A is cyclic, i.e. there are n linearly independent eigenvectors to A. Then an invariant subspace is simply obtained from the eigenvectors $a_1, a_2, \ldots, a_p$ corresponding to Λ, i.e.

$$Q = [a_1 \quad a_2 \ \ldots \ a_p] \tag{2.5}$$

is a basis matrix for the eigenspace.

If A is non-cyclic the concept of generalized eigenvectors is introduced. Le λ_i be an eigenvalue of multiplicity $\sigma_i > 1$. The generalized eigenvectors $a_i^1, a_i^2, \ldots, a_i^{\sigma_i}$ corresponding to λ_i are then defined as the nontrivial solutions of

$$(A - \lambda_i I)a_i^1 = 0$$

$$(A - \lambda_i I)a_i^k = a_i^{k-1}; \qquad k = 1, 2, \ldots, \sigma_i$$

The basis for the eigenspace is then constructed according to the following rule. If a_i^ℓ is selected, then a_i^k, $k = 1, 2, \ldots, \ell-1$, must also be selected as members of the basis if an invariant subspace shall be obtained. In this way an invariant eigenspace may be constructed corresponding to any set of eigenvalues to A.

Finally, observe that since Λ is assumed to be a symmetric set and A is assumed to be real, a real basis for the eigenspace is obtained by taking

$$Q = [a_1\ a_2\ \ldots\ a_s;\ \mathrm{Re}\{a_{s+1}\}\ \mathrm{Im}\{a_{s+1}\}\ \mathrm{Re}\{a_{s+2}\}\ \ldots] \qquad (2.6)$$

where $a_1, a_2, \ldots, a_s$ are assumed to be real and a_{s+1}, $a_{s+2}, \ldots, a_p$ are assumed to be complex. For any pair λ, $\bar{\lambda}$ belonging to Λ then choose Re{a}, Im{a} as members of the basis where a is the eigenvector corresponding to λ. In this way complex arithmetic is avoided in the sequel, and is only needed in the eigenvector calculation.

Pseudo Inverses.

Let M be an arbitrary real matrix. The pseudo inverse $M^\dagger$ of M is then defined by the following four conditions:

1° $M^\dagger M M^\dagger = M^\dagger$

2° $M M^\dagger M = M$

3° $M^\dagger M$ is symmetric

4° $M M^\dagger$ is symmetric

It is shown in [11] that $M^{\dagger}$ is uniquely defined by these conditions. Numerical algorithms exist to find such inverses, see for instance [12].

The pseudo inverse has some nice properties in minimization on inner product spaces. Consider the equation

$$Mx = y$$

which shall be solved for x. Then $x_o = M^{\dagger}y$ has the following properties:

1^o x_o minimizes $||Mx - y||$ where $||\cdot||$ denotes the ordinary euclidian quadratic norm.

2^o amongst the possible candidates for the minimum of $||Mx - y||$, x_o is the one that minimizes $||x||$.

3. CONTROL REDUCTION.

Assume that a state feedback control is given. This control is then replaced with a "similar" control with a predefined feedback structure. It is shown that this can be done in such a way that a certain number of eigenvalues remain invariant (mode preservation). Since there is an upper bound on the number of invariant eigenvalues a different reduction is also given which minimizes a weighted shift of the eigenvalues (mode weighting). Controls of derivative types will be considered at the end of the section.

Mode Preservation.

Consider the system (2.1) with the control (2.2). The closed loop system becomes

$$\dot{x} = (A + BL)x + Bv \tag{3.1}$$

We will attempt to replace the control (2.2) with a similar control of the multiple constrained form (2.4). For this control the closed loop system becomes

$$\dot{x} = (A + \sum_{i=1}^{q} B_i K_i^* C_i)x + Bv \tag{3.2}$$

where $B = [B_1\ B_2\ \dots\ B_q]$ is a partition of the input matrix compatible with the partition of the control vector in (2.4). Moreover, the reduced control law shall be selected so that some dominant properties of (3.1) are preserved in (3.2).

Partition the state feedback matrix as

$$L = \begin{pmatrix} L_1 \\ L_2 \\ \vdots \\ L_q \end{pmatrix}$$

where L_i is $m_i \times n$. Then if $K_i C_i = L_i$ have solutions K_i^* for $i = 1, 2, \ldots, q$, the exact and the reduced control laws would be identical. However, such solutions rarely exist, and therefore approximations must be made. The following theorem describes one rational way to do such approximations.

Theorem 1: Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_p\}$ be a symmetric set of eigenvalues to $A + BL$ and let Q be a real basis matrix for the corresponding eigenspace. Then if

$$K_i C_i Q = L_i Q \tag{3.3}$$

have solutions K_i^* for $i = 1, 2, \ldots, q$, then Λ is also a set of eigenvalues to

$$A + \sum_{i=1}^{q} B_i K_i^* C_i$$

Moreover, if $T = [Q \quad \hat{Q}]$ where the columns of $\hat{Q}$ are any set of vectors that extend the columns of Q to a basis in R^n then

$$T^{-1}(A + BL)T = \begin{pmatrix} A_{11}^o & A_{12}^o \\ 0 & A_{22}^o \end{pmatrix} \tag{3.4}$$

and

$$T^{-1}(A + \sum_{i=1}^{q} B_i K_i^* C_i)T = \begin{pmatrix} A_{11}^o & A_{12}^o + \sum_{i=1}^{q} B_i^1 \Delta L_i^* \hat{Q} \\ 0 & A_{22}^o + \sum_{i=1}^{q} B_i^2 \Delta L_i^* \hat{Q} \end{pmatrix} \qquad (3.5)$$

where

$$\Delta L_i^* = K_i^* C_i - L_i$$

and

$$T^{-1} B_i = \begin{pmatrix} B_i^1 \\ B_i^2 \end{pmatrix}$$

Proof: Introduce $A_o = A + BL$ and

$$A_o^* = A + \sum_{i=1}^{q} B_i K_i^* C_i$$

From (3.3) we have

$$(A + \sum_{i=1}^{q} B_i K_i^* C_i)w = (A + \sum_{i=1}^{q} B_i L_i)w = (A + BL)w \qquad (3.6)$$

for any $w \in \{Q\}$. Since $\{Q\}$ is A_o invariant by construction, it follows from (3.6) that $\{Q\}$ is also A_o^* invariant and $A_oQ = A_o^*Q$. Let the columns of $\hat{Q}$ be any set of vectors that extend the columns of Q to a basis in R^n. Choose $T = [Q\ \hat{Q}]$ and write

$$T^{-1} = \begin{pmatrix} V \\ \hat{V} \end{pmatrix}$$

We then have

$$T^{-1}A_oT = \begin{pmatrix} VA_oQ & VA_o\hat{Q} \\ 0 & VA_o\hat{Q} \end{pmatrix} = \begin{pmatrix} A^o_{11} & A^o_{12} \\ 0 & A^o_{22} \end{pmatrix} \tag{3.7}$$

$$T^{-1}A^*_oT = \begin{pmatrix} VA^*_oQ & VA^*_o\hat{Q} \\ 0 & \hat{V}A^*_o\hat{Q} \end{pmatrix} = \begin{pmatrix} A^o_{11} & VA^*_o\hat{Q} \\ 0 & \hat{V}A^*_o\hat{Q} \end{pmatrix} \tag{3.8}$$

The set of eigenvalues of A^o_{11} equals Λ. From (3.8) it then follows that Λ is also a set of eigenvalues to A^*_o. Moreover, we have

$$VA^*_o\hat{Q} = VA_o\hat{Q} + V(\sum_{i=1}^{q} B_iK^*_iC_i - L)\hat{Q} = A^o_{12} + V\sum_{i=1}^{q} B_i(K^*_iC_i - L_i)\hat{Q} =$$

$$= A^o_{12} + \sum_{i=1}^{q} B^1_i\Delta L^*_i\hat{Q}$$

and in the same way

$$\hat{V}A^*_o\hat{Q} = A^o_{22} + \sum_{i=1}^{q} B^2_i\Delta L^*_i\hat{Q}$$

Comments:

1. A real basis for the eigenspace can be constructed from the eigenvectors as was described in Section 2.
2. A comparison between the matrices (3.4) and (3.5) clearly illustrates the kind of approximations that

are made. The upper left block corresponding to eigenvalues Λ are identical in both systems. The remaining blocks are changed by an amount depending on ΔL_i^*, i.e. the difference between the exact and the reduced control laws.

3. The remaining eigenvalues of A + BL are different from those of $A + BK^*C$. Observe, however, that the effect of the approximations are only localized to the part of A + BL that contains the less dominant modes. The case when the approximations still cause an unacceptable change in the system is covered below.

4. Theorem 1 also yields an algorithm for pole assignment via output feedback. It has been shown in [5] that if rank C = r, then a symmetric set of r eigenvalues may be "almost" freely assigned. If a state feedback matrix L has been found so that r eigenvalues to the closed loop system takes some prescribed values Theorem 1 may be used to find a corresponding output feedback matrix (assuming (3.3) is solvable).

Mode Weighting.

The condition that (3.3) shall be solvable for K_i gives an upper bound on the number of eigenvalues that can be held fixed. This bound mostly equals r_i, i.e. the number of measured variables. One trivial exception is C = L, where K = I preserves all the eigenvalues. It may, however, still happen that some of the remaining eigenvalues move to undesired locations in the complex plane. The solution to this problem is to include a larger number of eigenvalues looking for least square solutions of (3.3).

Introduce the matrix norm

$$||M|| = \left(\mathrm{tr}\{MM^T\}\right)^{1/2}$$

valid for an arbitrary real matrix M.

Consider first the case when there is more than one solution to (3.3). Let R_i, $i = 1, 2, \ldots, q$, be nonsingular $r_i \times r_i$ matrices. Then one solution is given by

$$K_i^* = L_i Q(R_i^{-1} C_i Q)^{\dagger} R_i^{-1} \tag{3.9}$$

Moreover, this solution is the one that minimizes the norm $||K_i \; R_i||$, i.e. a solution with small feedback gains is selected.

Consider now the opposite case when there is no solution of (3.3). We may then attempt to minimize the norm

$$||(K_i C_i Q - L_i Q)W|| \tag{3.10}$$

where W is a nonsingular $p \times p$ matrix. In fact the minimum is obtained by taking

$$K_i^* = L_i QW(C_i QW)^{\dagger}$$

Now remember the special choice of basis that was made in (2.6), i.e.

$$Q = [a_1 \; a_2 \; \ldots \; a_s; \; \mathrm{Re}\{a_{s+1}\} \; \mathrm{Im}\{a_{s+1}\} \; \mathrm{Re}\{a_{s+2}\} \ldots]$$

where a_k is the eigenvector corresponding to λ_k. If we choose $W = \mathrm{diag}(w_1, w_2, \ldots, w_p)$ where $w_k \neq 0$, (3.10) may be rewritten as

$$||(K_iC_iQ - L_iQ)W||^2 = \sum_{k=1}^{p} w_k^2|K_iC_ia_k - L_ia_k|^2 \tag{3.12}$$

From the last expression we see that a successive increase in w_k causes a successive better fit of the eigenvalue λ_k in the closed loop system. In this way W may be interpreted as a weighting matrix for the eigenvalues we desire to hold fixed. This point is further clearified by examples later.

Finally we observe that (3.9) and (3.11) may be combined to

$$K_i^* = L_iQW(R_i^{-1}C_iQW)^{\dagger}R_i^{-1} \tag{3.13}$$

Proportional and Derivative Control.

In some cases acceptable degree of stability cannot be achieved by output feedback only. The classical way to bypass this difficulty is to include derivatives of the outputs in the feedback loop.

We will now permit a control of the form

$$u = K_1^*y + K_2^*P\dot{y} \tag{3.14}$$

where P is a given m×r matrix and y = Cx. Only the single constrained case will be considered. The extension to the multiple constrained case is straightforward. In classical control terms the control (3.14) is of PD-type. The drivative term will set some constraints on the qualitative of the measured signals, especially the presence of high frequency noise. This kind of control has, however, turned out to be successful in many applications.

By some simple manipulations the control (3.14) is transformed to the standard form (2.3). Using (2.1) we have

$$u = K_1^* y + K_2^* PC\dot{x} = K_1^* Cx + K_2^* PC(Ax + Bu)$$

Assuming $I - K_2^* PCB$ is invertible the last expression may be solved for u

$$u = (I - K_2^* PCB)^{-1} K_1^* Cx + (I - K_2^* PCB)^{-1} K_2^* PCAx =$$

$$= \hat{K}_1 Cx + \hat{K}_2 PCAx \tag{3.15}$$

Now defining a new output vector $\hat{y}$ as

$$\hat{y} = \hat{C}x = \begin{pmatrix} C \\ PCA \end{pmatrix} x \tag{3.16}$$

The equation (3.15) can then be rewritten as

$$u = (\hat{K}_1 \hat{K}_2)\hat{C}x = \hat{K}\hat{C}x \tag{3.17}$$

The previous results can now be used to find an appropriate $\hat{K}$. The feedback gains in (3.14) are then calculated as

$$K_1^* = (I - K_2^* PCB)\hat{K}_1 \tag{3.18}$$

$$K_2^* = \hat{K}_2 (I + PCB\hat{K}_2)^{-1} \tag{3.19}$$

The benefit of this kind of control is apparent from (3.16) and (3.17). By having a larger portion of the state available we are also, in view of the reduction technique above, able to keep a larger number of eigenvalues fixed. Moreover, if rank $\{\hat{C}\} = n$ then the reduced and the exact control laws become identical.

Examples.

Finally we will give some examples to illustrate the ideas of the section. Two more farreaching examples are considered in the next sections.

Example 1:

$$\dot{x} = \begin{pmatrix} -2 & 0 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} x$$

$$u = \begin{pmatrix} -1 & -2 \end{pmatrix} x + v$$

The closed loop system becomes

$$\dot{x} = \begin{pmatrix} -3 & -2 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$

and the eigenvalues of A + BL equal $\lambda_1 = -1$ and $\lambda_2 = -2$. Assume we permit a feedback of the form

$$u = ky + v$$

and that $\lambda_1 = -1$ shall remain fixed. The eigenvector corresponding to λ_1 is

$$a_1 = 1/\sqrt{2} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

In this simple case we have $Q = a_1$. Solving (3.3) we obtain

$$k^{*} = -1$$

and the eigenvalues of A + Bk*C become $\nu_1 = -1$ and $\nu_2 = -1$, i.e. one eigenvalue equals -1 as desired.

Example 2:

$$\dot{x} = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} u$$

$$u = \begin{pmatrix} -5 & -1 \\ 2 & -5 \end{pmatrix} x + v$$

The closed loop system becomes

$$\dot{x} = \begin{pmatrix} -4 & 1 \\ 1 & -4 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} v$$

and the closed loop eigenvalues equal $\lambda_1 = -3$ and $\lambda_2 = -5$. Assume we shall hold $\lambda_1 = -3$ fixed. The eigenvector corresponding to λ_1 is

$$a_1 = 1/\sqrt{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Assume we permit a feedback of the form

$$u = \begin{pmatrix} k_{11} & 0 \\ 0 & k_{22} \end{pmatrix} x + v$$

This control is then of the multiple constrained type (2.4). The feedback structure becomes

$$u_1 = k_1(1 \quad 0)x + v_1$$

$$u_2 = k_2 \begin{pmatrix} 0 & 1 \end{pmatrix} x + v_2$$

Solving (3.3) for i = 1,2 we have

$$u = \begin{pmatrix} -6 & 0 \\ 0 & -3 \end{pmatrix} x + v$$

The eigenvalues of

$$A + \sum_{i=1}^{q} B_i k_i^* C_i$$

becomes $\nu_1 = -3$ and $\nu_2 = -4$.

Example 3:

$$\dot{x} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} x$$

$$u = \begin{pmatrix} -2 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix} x + v$$

The closed loop system is

$$\dot{x} = \begin{pmatrix} -2 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & -2 & -2 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix} v$$

The eigenvalues of A + BL are $\lambda_{1,2} = -1 \pm j$ and $\lambda_3 = -2$.

Assume we permit a feedback of the form

$$u = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} y + v$$

First we attempt to keep the eigenvalues $\lambda_{1,2} = -1 \pm j$ fixed. The corresponding eigenvectors are

$$a_1 = \begin{pmatrix} 0 \\ -0.5 \\ 1 \end{pmatrix} + j \begin{pmatrix} 0.5 \\ -0.5 \\ 0 \end{pmatrix} \qquad a_2 = \begin{pmatrix} 0 \\ -0.5 \\ 1 \end{pmatrix} - j \begin{pmatrix} 0.5 \\ -0.5 \\ 0 \end{pmatrix}$$

The basis matrix Q for the eigenspace is then selected according to (2.6) as

$$a = \begin{pmatrix} 0 & 0.5 \\ -0.5 & -0.5 \\ 1 & 0 \end{pmatrix}$$

Solving (3.3) for K we have

$$K_I^* = \begin{pmatrix} 0.67 & 0.33 \\ -0.33 & -1.67 \end{pmatrix}$$

and the eigenvalues of $A + BK_I^*C$ are $\nu_{1,2} = -1 \pm j$ as desired and $\nu_3 = 1.33$. The third mode has become unstable and therefore we include also this mode looking for least square solutions according to (3.12) and (3.13).

The eigenvector corresponding to $\lambda_3 = -2$ is

$$a_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

The basis matrix for the eigenspace then becomes

$$Q = \begin{pmatrix} 0 & 0.5 & 1 \\ -0.5 & -0.5 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

We choose the weighting matrix W as W = diag(1,1,0.1) where a relatively small weight has been laid on λ_3. Using (3.13) we now have

$$K^*_{II} = \begin{pmatrix} 0.64 & 0.32 \\ -0.32 & -1.66 \end{pmatrix}$$

and the closed loop eigenvalues are $\nu_{1,2} = -0.995 \pm 0.998j$ and 1.29. The weight on the third mode was obviously too small. Take instead W = diag(1,1,0.8). We then obtain

$$K^*_{III} = \begin{pmatrix} -0.30 & -0.15 \\ 0.15 & -1.42 \end{pmatrix}$$

The closed loop eigenvalues are now $\nu_{1,2} = -0.57 \pm 1.14j$ and $\nu_3 = -0.73$. In this way, by successively altering the weighting factors, a satisfactory solution can be obtained after a few iterations.

4. APPLICATION TO BOILER CONTROL.

A computer program for control reduction has been written based upon Theorem 1 and the least square solution (3.13). This program has been used to find simple control strategies for a boiler. The starting point is here a linear quadratic control law, which is used to fit a certain feedback structure. By simulations we show that a reduction can be made without any significant decrease in performance. In fact, the responses of the reduced control system are very similar to the responses of the system controlled by complete state feedback.

Control of a Boiler.

Different types of models for a drum boiler are thoroughly described in [9]. Here we will use a fifth order model from [10].

The linearized equations for a boiler around a certain operating point can be written as

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

where the state variables are

x_1 = drum pressure (bar)

x_2 = drum liquid level (m)

x_3 = drum liquid temperature ($^{\circ}$C)

x_4 = riser wall temperature ($^{\circ}$C)

x_5 = steam quality (%)

The control variables are

u_1 = heat flow to the risers (kJ/s)

u_2 = feedwater flow (kg/s)

Numerical values for A, B and C for a power station boiler with a maximum steam flow of about 350 t/h are calculated in [10]. The drum pressure is 140 bar and the operating point is 90% full load. From [10] we have

$$A = \begin{pmatrix} -0.129 & 0.000 & 0.396\times10^{-1} & 0.250\times10^{-1} & 0.191\times10^{-1} \\ 0.329\times10^{-2} & 0.000 & -0.779\times10^{-4} & 0.122\times10^{-3} & -0.621 \\ 0.718\times10^{-1} & 0.000 & -0.100 & 0.887\times10^{-3} & -3.851 \\ 0.411\times10^{-1} & 0.000 & 0.000 & -0.822\times10^{-1} & 0.000 \\ 0.361\times10^{-3} & 0.000 & 0.350\times10^{-4} & 0.426\times10^{-4} & -0.743\times10^{-1} \end{pmatrix}$$

$$B = \begin{pmatrix} 0.000 & 0.139\times10^{-2} \\ 0.000 & 0.359\times10^{-4} \\ 0.000 & -0.989\times10^{-2} \\ 0.249\times10^{-4} & 0.000 \\ 0.000 & -0.543\times10^{-5} \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

A state feedback matrix can be calculated using linear quadratic theory. In [10] it is shown that the following feedback matrix gives satisfactory responses.

$$L = \begin{pmatrix} -0.668\times10^{4} & -0.418\times10^{6} & -0.136\times10^{4} & -0.137\times10^{4} & 0.175\times10^{7} \\ -0.803\times10^{1} & -0.908\times10^{3} & -0.486 & -0.815 & 0.431\times10^{4} \end{pmatrix} \quad (4.1)$$

The intention is now to replace the control $u = Lx$ with a simpler control using only output feedback, i.e.

$$u = K^{*}y = K^{*}Cx$$

where K^{*} shall be properly chosen. The eigenvalues of $A + BL$ are

$$\lambda_1 = -0.490\times10^{-1}$$

$$\lambda_{2,3} = -0.755\times10^{-1}\pm j\cdot0.511\times10^{-1}$$

$$\lambda_{4,5} = -0.141\pm j\cdot0.170\times10^{-1}$$

and they are shown in Fig. 4.1a.

First we attempt to include only the three eigenvalues $\lambda_{1,2,3}$ of $A + BL$ having the least real part. Somewhat arbitrarily we choose the corresponding factors as $W =$ $= \mathrm{diag}(1,1,1)$. Using (3.13) we have

$$K_I^{*} = \begin{pmatrix} 0.924\times10^{4} & -0.347\times10^{6} \\ 0.403\times10^{2} & -0.827\times10^{3} \end{pmatrix} \qquad (4.2)$$

The eigenvalues of $A + BK_I^{*}C$ are shown in Fig. 4.1b. We see that the neglected pair $\lambda_{4,5} = -0.141\pm0.016j$ of $A + BL$ has become a much less damping in the reduced control system. In order to increase the damping we include also $\lambda_{4,5}$ in the solution and choose the weighting factors as $W = \mathrm{diag}(1,1,1,0.2,0.2)$, where the smaller weight has been laid on $\lambda_{4,5}$. The least square solution (3.13) becomes now

$$K_{II}^{*} = \begin{pmatrix} 0.569\times10^{3} & -0.286\times10^{6} \\ 0.870\times10^{1} & -0.601\times10^{3} \end{pmatrix} \qquad (4.3)$$

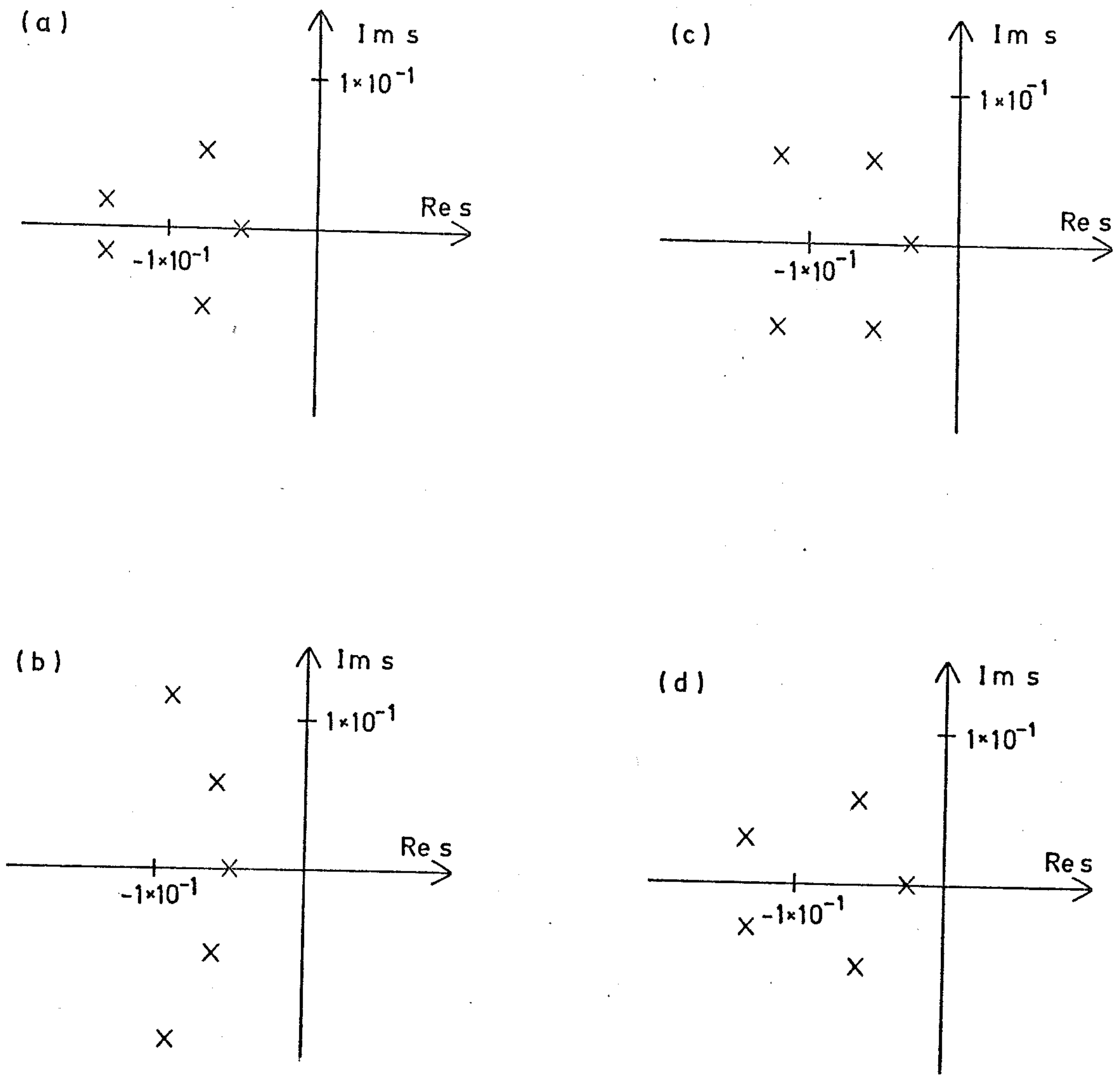
and the eigenvalues of $A + BK^{*}_{II}C$ are as shown in Fig. 4.1c. As can be seen the damping of the second complex pair has increased, but at the expense that the rightmost eigenvalue has moved somewhat nearer the imaginary axis. A further iteration with W = diag(1,1,1,0.5,0.5) gives

$$K^{*}_{III} = \begin{bmatrix} -0.265\times 10^{4} & -0.263\times 10^{6} \\ -0.167\times 10^{1} & -0.528\times 10^{3} \end{bmatrix} \tag{4.4}$$

The corresponding eigenvalue configuration is shown in Fig. 4.1d.

Simulations show that K^{*}_{II} is the most satisfactory choice in this case. The output feedback matrix can be compared with the corresponding elements in the state feedback matrix (4.1) (the two leftmost columns). As can be seen the feedback gains are slightly less in K^{*}_{II}, but of the same magnitude. However, the relations between the individual feedback gains differ considerably. This is due to the fact that compensations have been made in K^{*}_{II} for the remaining columns in L.

In Fig. 4.2 - 3 the system is simulated with control laws (4.1) and (4.3). Fig. 4.2 shows the responses for an initial condition in drum level of 0.02 m and Fig. 4.3 the same responses for an initial condition in drum pressure of 1 bar. As can be seen the difference between the exact and the reduced control laws is astonishingly small, indicating that a control only using feedback from the measured variables will be sufficient in this case.



<u>Fig. 4.1</u> - The pole configurations for the exact and the reduced control laws.

(a) exact control law (4.1)
(b) reduced control law (4.2)
(c) reduced control law (4.3)
(d) reduced control law (4.4)

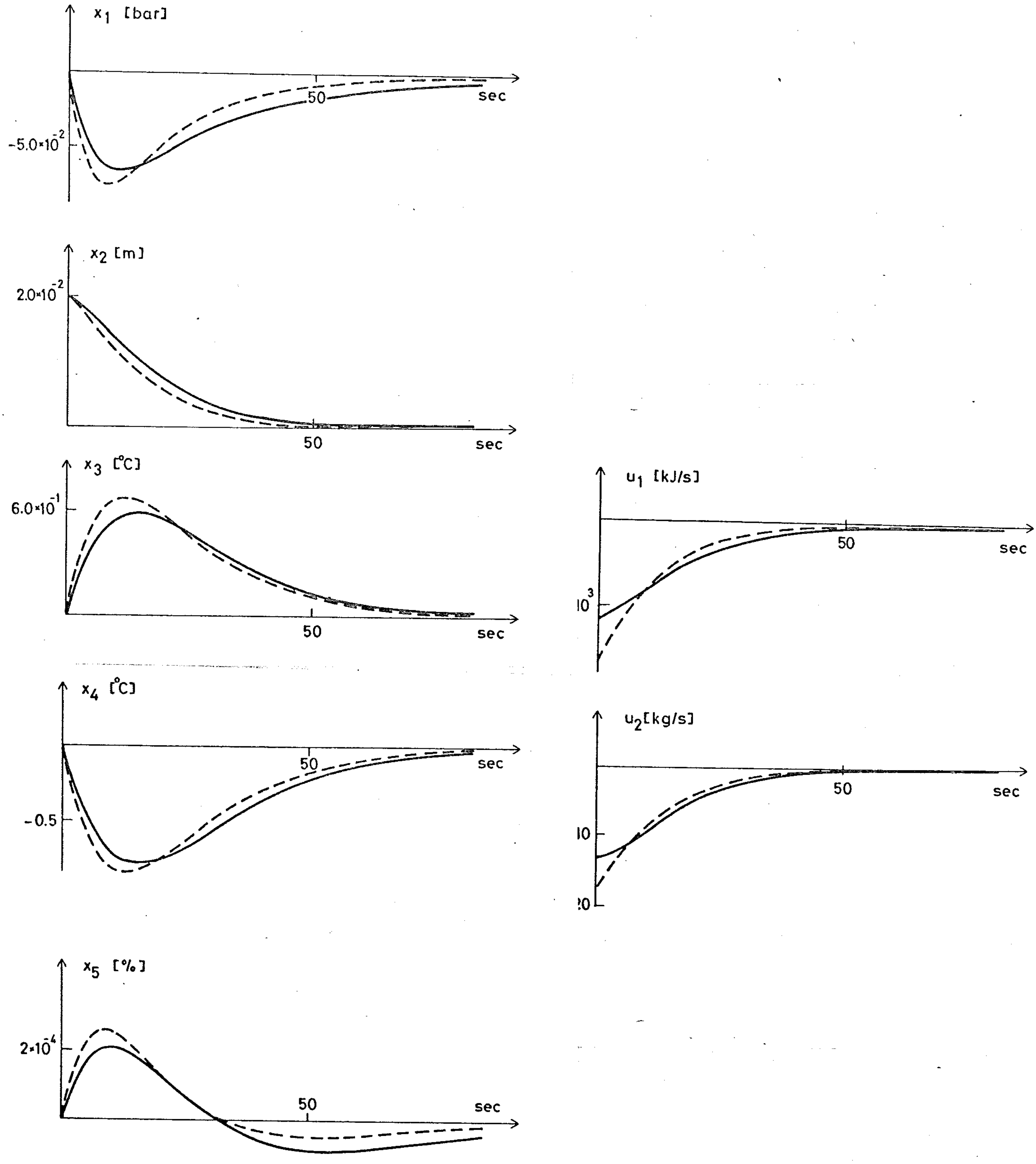


Fig. 4.2 - Responses for an initial condition in drum level of 0.02 m.

-------- exact control law (4.1)

———— reduced control law (4.3)

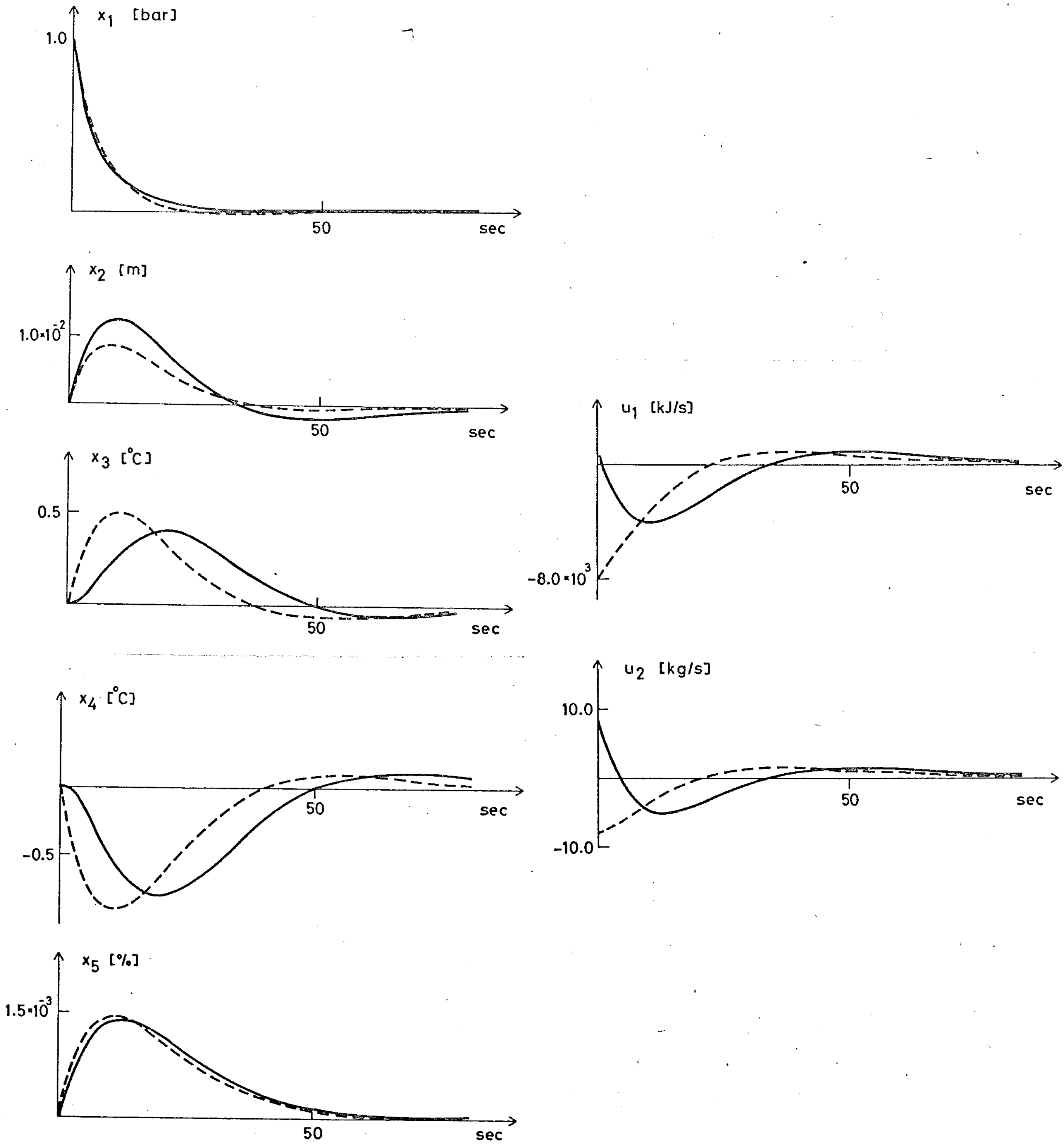


Fig. 4.3 - Responses for an initial condition in drum pressure of 1.0 bar.

-------- exact control law (4.1)

———— reduced control law (4.3)

5. APPLICATION TO POWER SYSTEM CONTROL.

We consider a reduced model of the Scandinavian network, which consists of approximately 150 nodes and 250 lines. The reduction of the model has been performed in two steps at the Swedish State Power Board. The original model has been verified by experiments. The accuracy of the reduced model can, of course, be questioned but in any case it is a typically power system model.

The model has three generators, one in North Sweden (GNOSVE), one in South Sweden (GSYSVE), and one in Norway (GNGE). The generators in North Sweden and in Norway have hydro turbines and the generator in South Sweden has a steam turbine.

The modeling of a multimachine power system has been treated in [13] and will not be discussed here.

The linearized equations for the power system can be written as

$$\frac{dx}{dt} = Ax + Bu$$

$$y = Cx$$

where the state variables are:

x_1 = rotor angle, GNOSVE

x_2 = rotor angular velocity, GNOSVE

x_3 = flux linkage of field winding, GNOSVE

x_4 = excitation voltage, GNOSVE

x_5 = velocity of water, GNOSVE

x_6 = rotor angle, GSYSVE

x_7 = rotor angular velocity, GSYSVE

x_8 = flux linkage of field winding, GSYSVE

x_9 = excitation voltage, GSYSVE

x_{10} = steam pressure

x_{11} = rotor angle, GNGE

x_{12} = rotor angular velocity, GNGE

x_{13} = flux linkage of field winding, GNGE

x_{14} = excitation voltage, GNGE

x_{15} = velocity of water, GNGE

The input variables are:

u_1 = excitation input, GNOSVE

u_2 = gate opening, GNOSVE

u_3 = excitation input, GSYSVE

u_4 = steam valve setting, GSYSVE

u_5 = fuel flow, GSYSVE

u_6 = excitation, GNGE

u_7 = gate opening, GNGE

The output variables are:

y_1 = rotor angular velocity, GNOSVE

y_2 = terminal voltage, GNOSVE

y_3 = excitation voltage, GNOSVE

y_4 = rotor angular velocity, GSYSVE

y_5 = terminal voltage, GSYSVE

y_6 = excitation voltage, GSYSVE

y_7 = steam pressure, GSYSVE

y_8 = rotor angular velocity, GNGE

y_9 = terminal voltage, GNGE

y_{10} = excitation voltage, GNGE

where GNOSVE, GSYSVE, and GNGE denote the generators in North Sweden, South Sweden, and Norway respectively.

Numerical values for A, B and C for the power system are given in Appendix 1. The operating point corresponds to the expected peak load 1975 with high transmission from North Sweden to South Sweden.

The initial state feedback matrix is calculated using linear quadratic control theory. In [14] it is shown that the state feedback matrix given in Appendix 2 gives satisfactory responses.

The intention is now to replace the control $u = -Lx$ with a simpler control using only local output feedback

$$u_i = - K_i^* y_i$$

where u_i are the inputs at station i and y_i are the outputs measured at station i. The eigenvalues of $A - BL$ are:

$$\lambda_1 = -7.33 \cdot 10^{-3}$$
$$\lambda_2 = -2.09 \cdot 10^{-1}$$
$$\lambda_{3,4} = -2.77 \cdot 10^{-1} \pm i3.55 \cdot 10^{-1}$$
$$\lambda_5 = -3.17 \cdot 10^{-1}$$
$$\lambda_{6,7} = -3.83 \cdot 10^{-1} \pm i2.53 \cdot 10^{-1}$$
$$\lambda_8 = -5.14 \cdot 10^{-1}$$
$$\lambda_{9,10} = -1.36 \pm i3.12$$

$$\lambda_{11,12} = -1.37 \pm i4.18$$

$$\lambda_{13,14} = -1.49 \pm i3.79 \cdot 10^{-2}$$

$$\lambda_{15} = -2.4613$$

and they are shown in Fig. 5.1.

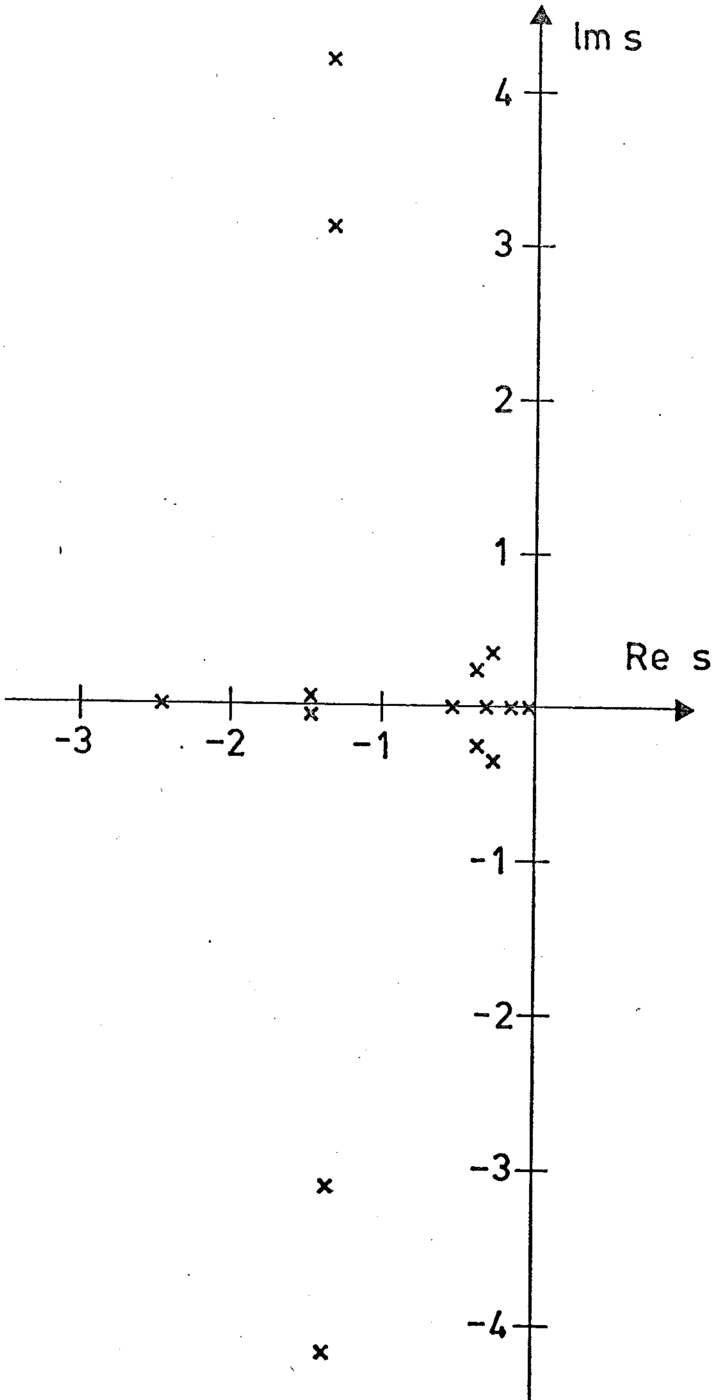


Fig. 5.1 - Eigenvalues for the power system with complete state feedback.

The control reduction was performed in two steps: In the first step we allowed local state feedback and attempted to preserve all eigenvalues. By adjusting the weighting factors (w) in five iterations we found a subset of seven critical eigenvalues. These were λ_3, λ_4, λ_6, λ_7, λ_8, λ_{13}, and λ_{14}. In the second step we attempted to preserve only the critical eigenvalues and used the same weighting factors as in the first step. The least square solution (3.13) is given in Appendix 3 and the eigenvalues of $A - K^{*}C$ are:

$$\lambda_1 = -5.56\cdot 10^{-8}$$
$$\lambda_{2,3} = -3.33\cdot 10^{-1} \pm i2.70\cdot 10^{-1}$$
$$\lambda_{4,5} = -3.44\cdot 10^{-1} \pm i2.52\cdot 10^{-1}$$
$$\lambda_{6,7} = -3.84\cdot 10^{-1} \pm i2.10\cdot 10^{-1}$$
$$\lambda_8 = -4.65\cdot 10^{-1}$$
$$\lambda_{9,10} = -7.95\cdot 10^{-1} \pm i3.00$$
$$\lambda_{11,12} = -1.19\cdot i4.05$$
$$\lambda_{13} = -1.47$$
$$\lambda_{14} = -1.66$$
$$\lambda_{15} = -2.44$$

The eigenvalues of $A - K^{*}C$ are shown in Fig. 5.2.

The power system is simulated with the complete state feedback given in Appendix 2 and the responses are shown in Fig. 5.3. The power system is also simulated with the local output feedback given in Appendix 3 and the responses are shown in Fig. 5.4. In both cases the rotor angle of the generator in North Sweden (x_1) is given an initial value of 0.5 rad.

It is surprising how well the local output feedback behaves. We observe that the rotor angles are not reduced to zero. This is an explanation to the zero eigenvalue λ_1. The angles can be reduced to zero if one of them is included in the output vector. In practice the frequency error is integrated and fed to the controllers.

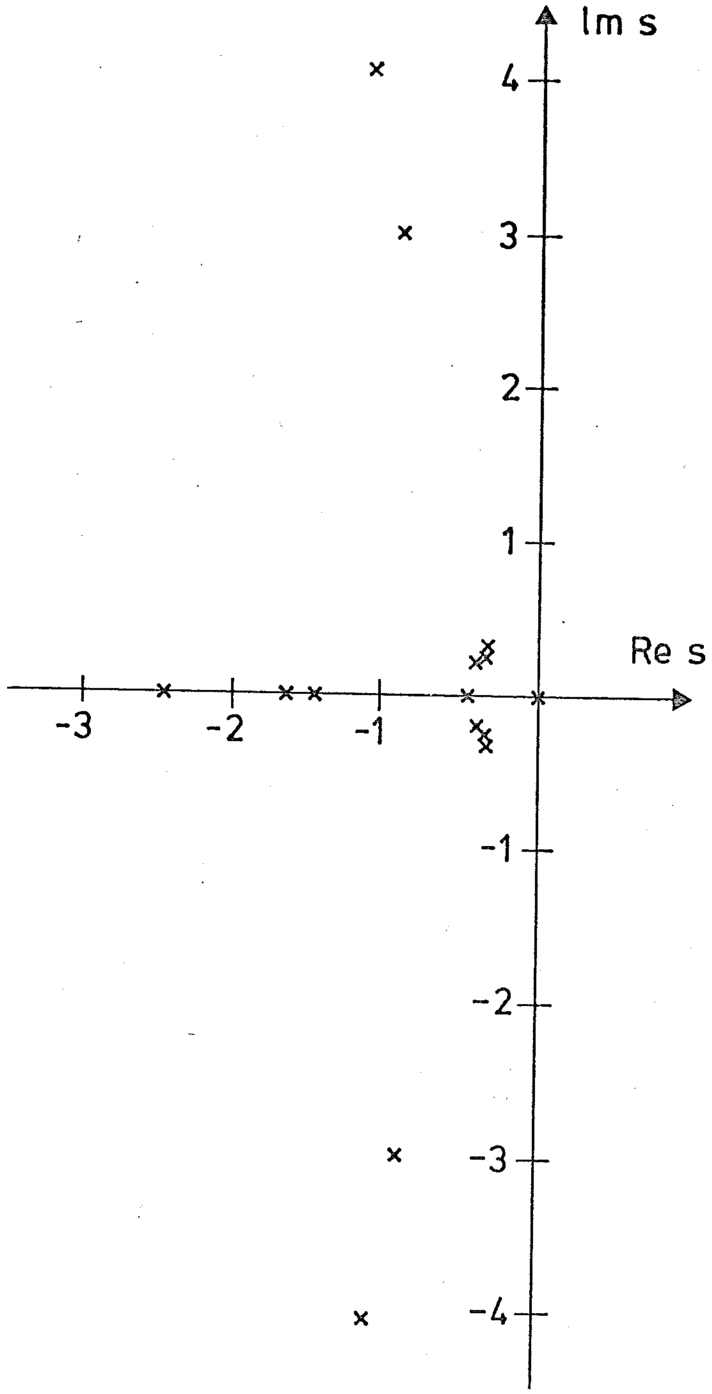


Fig. 5.2 - Eigenvalues for the power system with local output feedback.

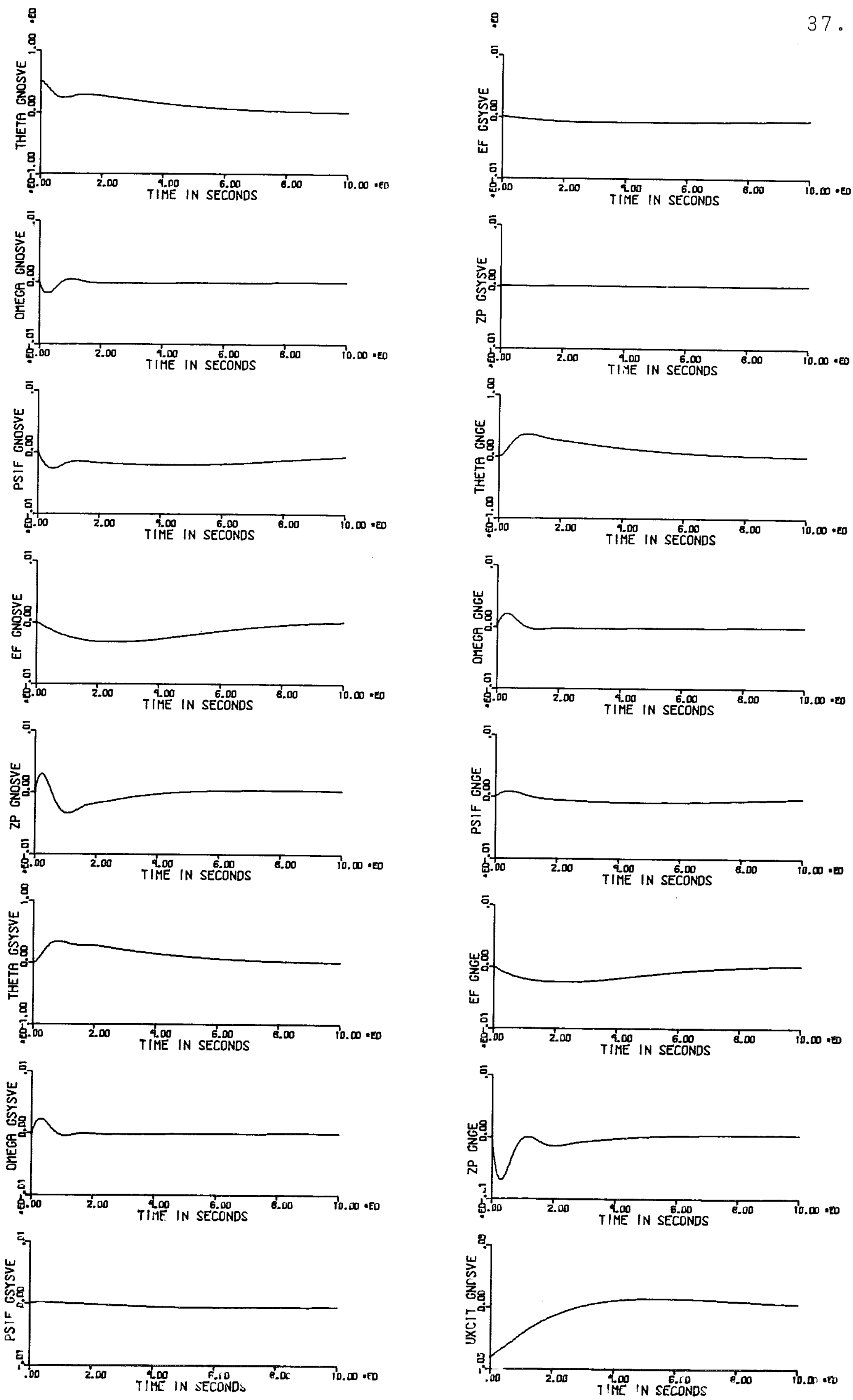


Fig. 5.3a - Responses of the power system with complete state feedback.

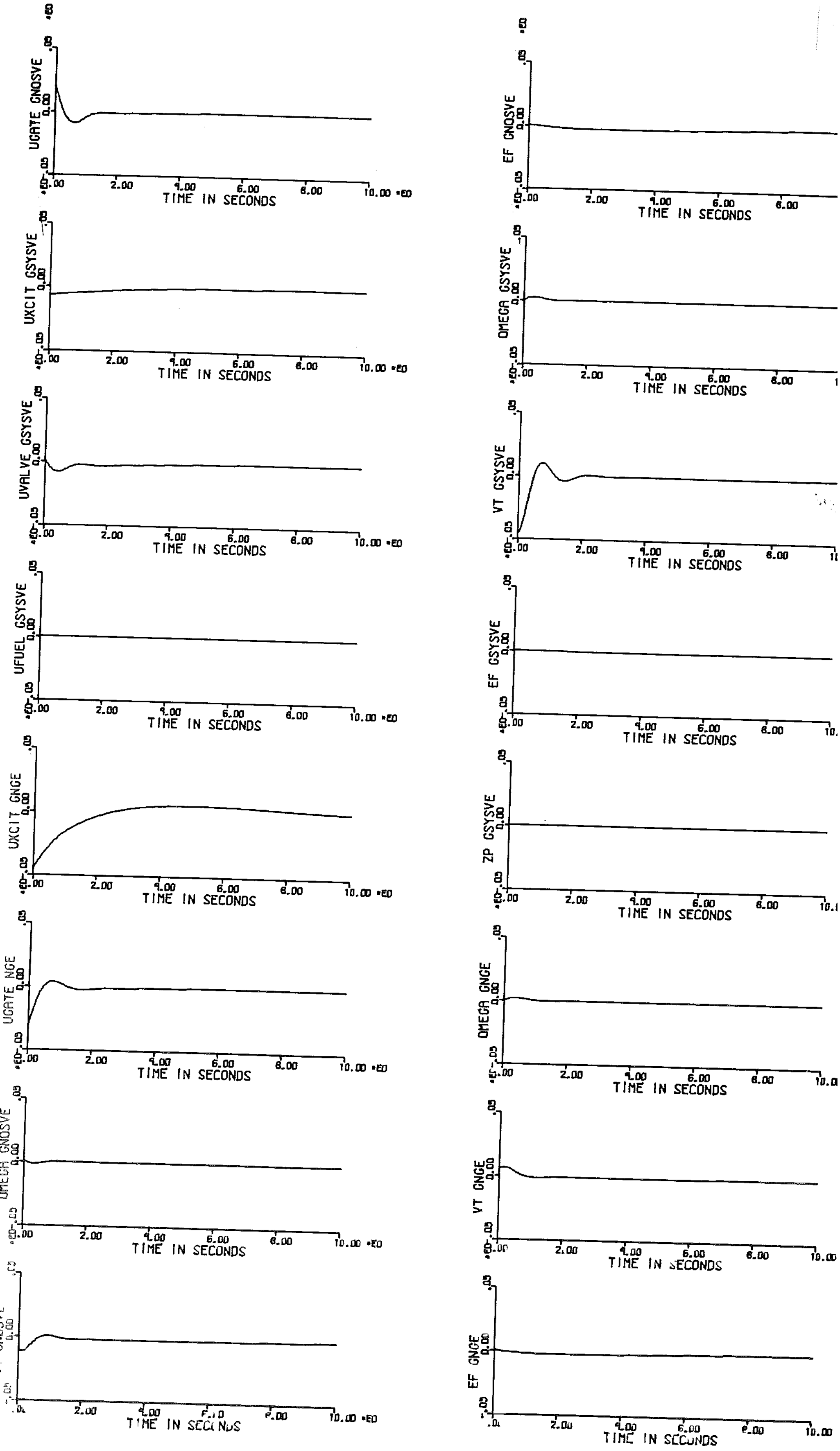

UGATE GNOSVE
TIME IN SECONDS
EF GNOSVE
TIME IN SECONDS
UKCIT GSYSVE
TIME IN SECONDS
OMEGA GSYSVE
TIME IN SECONDS
UVALVE GSYSVE
TIME IN SECONDS
VT GSYSVE
TIME IN SECONDS
UFUEL GSYSVE
TIME IN SECONDS
EF GSYSVE
TIME IN SECONDS
UKCIT GNGE
TIME IN SECONDS
ZP GSYSVE
TIME IN SECONDS
UGATE NGE
TIME IN SECONDS
OMEGA GNGE
TIME IN SECONDS
OMEGA GNOSVE
TIME IN SECONDS
VT GNGE
TIME IN SECONDS
TIME IN SECONDS
EF GNGE
TIME IN SECONDS

Fig. 5.3b - Responses of the power system with complete state feedback.

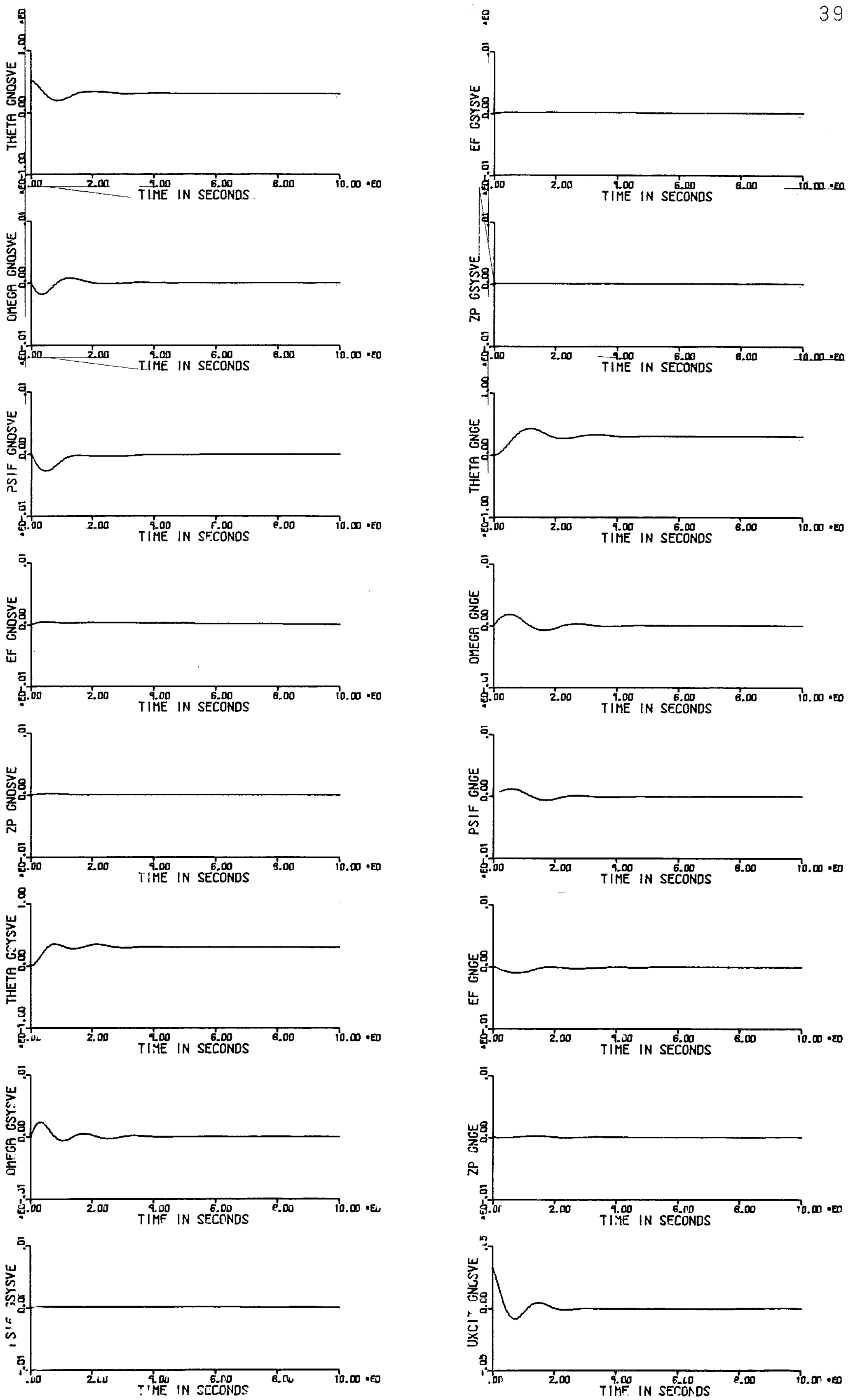


Fig. 5.4a - Responses of the power system with local output feedback.

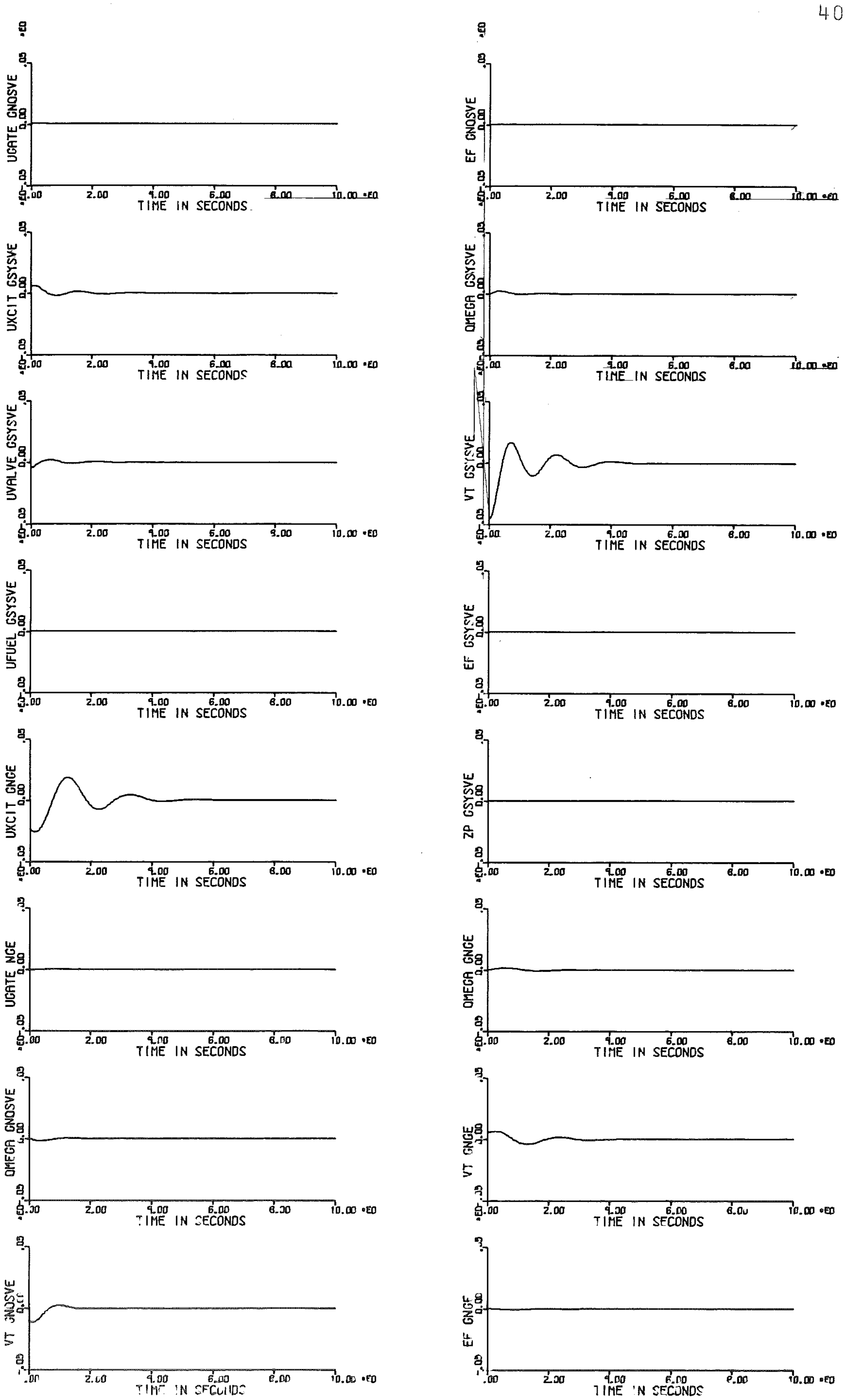


Fig. 5.4b - Responses of the power system with local output feedback.

6. ACKNOWLEDGEMENT

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A-MATRIX, PAGE 1

A( 1, 2)= .314159+03  A( 2, 1)=-.242249-01  A( 2, 2)=-.322929+01
A( 2, 3)= .162980+00  A( 2, 5)= .340985+00  A( 2, 6)= .113810-01
A( 2, 7)=-.664848-02  A( 2, 8)=-.684552-02  A( 2,11)= .128439-01
A( 2,12)=-.998426-02  A( 2,13)=-.712887-02  A( 3, 1)=-.213677-01
A( 3, 2)=-.676561-01  A( 3, 3)=-.304433+00  A( 3, 4)= .250453+00
A( 3, 6)= .147243-01  A( 3, 7)=-.884903-02  A( 3, 8)= .554234-03
A( 3,11)= .664346-02  A( 3,12)= .672120-02  A( 3,13)= .827741-02
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A( 8, 8)=-.360985+00  A( 8, 9)= .336492+00  A( 8,11)= .471545-03
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A(12, 2)=-.106559-01  A(12, 3)=-.791938-02  A(12, 6)= .649000-02
A(12, 7)=-.511231-02  A(12, 8)=-.462964-02  A(12,11)=-.195372-01
A(12,12)=-.887218+00  A(12,13)= .165403+00  A(12,15)= .441134+00
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   156 ZERO ELEMENTS


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B(15, 7)= .183560+01

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C( 2,13)= .232592-01  C( 3, 4)= .100000+01  C( 4, 7)= .100000+01
C( 5, 1)=-.913649-01  C( 5, 2)= .240934+00  C( 5, 3)= .233823+00
C( 5, 6)= .162358+00  C( 5, 7)= .277790+00  C( 5, 8)= .286660+00
C( 5,11)=-.709931-01  C( 5,12)= .159072+00  C( 5,13)= .143982+00
C( 6, 9)= .100000+01  C( 7,10)= .100000+01  C( 8,12)= .100000+01
C( 9, 1)= .108115-01  C( 9, 2)= .114114-01  C( 9, 3)= .151457-01
C( 9, 6)= .126540-01  C( 9, 7)=-.743007-02  C( 9, 8)= .117922-02
C( 9,11)=-.234655-01  C( 9,12)= .102604+01  C( 9,13)= .912437+00
C(10,14)= .100000+01

    34 NONZERO ELEMENTS
   116 ZERO ELEMENTS
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L-MATRIX, PAGE 1

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L( 1, 7)= .752995+01   L( 1, 8)= .311316+00   L( 1, 9)= .322513+00
L( 1,10)= .519761+01   L( 1,11)= .620178-01   L( 1,12)= .181910+02
L( 1,13)= .438583+01   L( 1,14)= .188907+01   L( 1,15)= .365105+01
L( 2, 1)=-.130186-01   L( 2, 2)=-.550412+00   L( 2, 3)= .135702+00
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L( 2, 7)= .322695+00   L( 2, 8)= .448535-03   L( 2, 9)= .278974-02
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L( 3, 7)= .966605+00   L( 3, 8)= .261260+00   L( 3, 9)= .639725+00
L( 3,10)= .490135+01   L( 3,11)= .125576-01   L( 3,12)= .306954+01
L( 3,13)= .512731+00   L( 3,14)= .181930+00   L( 3,15)= .548821+00
L( 4, 1)= .117973-02   L( 4, 2)=-.292205-01   L( 4, 3)= .508446-01
L( 4, 4)= .245616-01   L( 4, 5)= .322084-01   L( 4, 6)= .817428-02
L( 4, 7)= .978703+00   L( 4, 8)= .975280-02   L( 4, 9)= .189437-02
L( 4,10)= .557202-01   L( 4,11)=-.434347-02   L( 4,12)=-.283600+00
L( 4,13)= .699094-01   L( 4,14)= .235587-01   L( 4,15)= .459637-01
L( 5, 1)= .589446-04   L( 5, 2)= .495338-02   L( 5, 3)= .135419-02
L( 5, 4)= .789205-03   L( 5, 5)= .102425-02   L( 5, 6)= .738593-05
L( 5, 7)= .308770-02   L( 5, 8)= .321558-03   L( 5, 9)= .572478-03
L( 5,10)= .120141-01   L( 5,11)= .247140-04   L( 5,12)= .450175-02
L( 5,13)= .122347-02   L( 5,14)= .570772-03   L( 5,15)= .936447-03
L( 6, 1)= .173255+00   L( 6, 2)= .153267+02   L( 6, 3)= .391431+01
L( 6, 4)= .188907+01   L( 6, 5)= .311533+01   L( 6, 6)= .728300-01
L( 6, 7)= .716327+01   L( 6, 8)= .168458+00   L( 6, 9)= .139947+00
L( 6,10)= .375904+01   L( 6,11)=-.747451-02   L( 6,12)= .150397+02
L( 6,13)= .830344+01   L( 6,14)= .577019+01   L( 6,15)= .460714+01
L( 7, 1)= .233525-01   L( 7, 2)= .150466+01   L( 7, 3)= .496477-01
L( 7, 4)= .281233-01   L( 7, 5)= .725717-01   L( 7, 6)= .126299-01
L( 7, 7)= .103680+01   L( 7, 8)=-.486669-02   L( 7, 9)= .167516-02
L( 7,10)= .541131-01   L( 7,11)=-.296693-01   L( 7,12)=-.105230+01
L( 7,13)= .384120+00   L( 7,14)= .839044-01   L( 7,15)= .338981+00
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105 NONZERO ELEMENTS
0 ZERO ELEMENTS

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K-MATRIX, PAGE 1

K( 1, 1)=-.867660+01   K( 1, 2)= .303300+01   K( 1, 3)= .396280+01
K( 2, 1)= .106700+00   K( 2, 2)= .389030-01   K( 2, 3)= .234150-01
K( 3, 4)= .500470+00   K( 3, 5)= .116950+00   K( 3, 6)=-.294870+00
K( 3, 7)=-.561590+03   K( 4, 4)=-.524640+00   K( 4, 5)=-.102910+00
K( 4, 6)=-.354950+00   K( 4, 7)=-.841009+02   K( 5, 4)=-.240000-06
K( 5, 5)=-.860000-07   K( 5, 6)=-.430000-05   K( 5, 7)=-.912980+00
K( 6, 8)=-.133535+01   K( 6, 9)= .436700+01   K( 6,10)= .319935+01
K( 7, 8)=-.377860+00   K( 7, 9)= .102900+00   K( 7,10)=-.423256-02

     24 NONZERO ELEMENTS
     46 ZERO ELEMENTS
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