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Bengtsson, Gunnar

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LECTURE NOTES ON ALGEBRAIC METHODS IN CONTROL THEORY - REALIZATION THEORY

G. BENGTSSON

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Lund Institute of Technology

LECTURE NOTES ON ALGEBRAIC METHODS IN CONTROL THEORY

- REALIZATION THEORY

Gunnar Bengtsson

Abstract

This report contains lecture notes on algebraic methods in control theory, delivered at Department of Automatic Control in Lund. A parallell treatment of the basic realization theorem for state, fraction and general operator representation of a linear time invariant system is made. Relationships between them are also established.

1. INTRODUCTION

A linear time invariant dynamic system can be described in several different ways, leading to different frameworks for algebraic manipulations in e.g. a regulator synthesis procedure. For control design purposes, there is clearly a need to know (a) relationships between different types of representations of a linear system (b) an algebraic framework to perform the formal manipulations.

In this report, we intend to give a brief and selfcontained survey of some different representations of a linear time invariant dynamic system and the relationships between them. This will be done using realization theory, i.e. by describing the relationships between external and internal descriptions of linear systems, and the conclusions that can be made from it.

Realization theory is basically an outgrowth of the work on controllability and observability by Kalman [1]. A rigorous (axiomatic) theory is given in Kalman [2] and Kalman et al [3]. In this report we show how to arrive to the basic realization theorem for different types of linear systems. The results are taken from [3] (state equations), [4] (fraction representations) and [5] (general operator representations). The proofs are more or less standard for state and fraction representations, but we offer a new (and somewhat more direct) proof of the basic realization theorem for operator representations.

2. SOME ELEMENTARY ALGEBRA

We will assume that the reader is familiar with elementary algebra such as fields, rings etc. The following notations are used for some common fields and rings.

$\mathbb{R} \triangleq$ the field of real numbers

$\mathbb{C} \triangleq$ the field of complex numbers

$\mathbb{Z} \triangleq$ the ring of integers

Polynomial Matrices

Let K be an arbitrary field. A polynomial form with coefficients in K is an expression of the form

$$p_0 + p_1 s + \dots + p_n s^n$$

where $p_i \in K$, $n \in \mathbb{Z}$ and s is an indeterminate. With the usual definitions of summation and multiplication of polynomials, they form a ring, denoted $K[s]$. In the sequel we will only use the ring $\mathbb{R}[s]$.

It is assumed that the reader is familiar with the elementary results on polynomials. We will concentrate our effort on some useful results on matrices of polynomials. A polynomial matrix $P(s)$ is a matrix whose elements belong to $\mathbb{R}[s]$. An elementary row (column) operation on $P(s)$ is one of the following operations.

(a) multiply any row (column) with a real nonzero number

(b) interchange any two rows (columns)

(c) add a polynomial times one row (column) to any other row (column)

A polynomial matrix $P(s)$ is said to be <u>unimodular</u> if $P(s)^{-1}$ exists and is also a polynomial matrix.
Since

$$\det\left(P(s)^{-1}\right) = \frac{1}{\det P(s)}$$

and $P(s)^{-1}$ is a polynomial matrix, it follows by the Kramer's rule for the inverse that $P(s)$ is unimodular if and only if $\det P(s)$ is a nonzero real number. Any sequence of elementary row operations performed on $P(s)$ gives as result a polynomial matrix of the form $M(s)P(s)$, where $M(s)$ is unimodular. Conversely, any unimodular matrix $M(s)$ can be written as a product of matrices defining elementary row (column) operations.

Two polynomial matrices $P_1(s)$ and $P_2(s)$ are <u>equivalent</u> if there are two unimodular matrices $M(s)$ and $N(s)$ such that $P_1(s) = M(s)P_2(s)N(s)$. They are <u>row equivalent</u>, if there is a unimodular matrix $M(s)$ such that $P_1(s) = M(s)P_2(s)$, and they are <u>column equivalent</u> if there is a unimodular matrix $N(s)$ such that $P_1(s) = P_2(s)N(s)$. It is trivial to show that all the conditions for an equivalence relation are satisfied.

The following theorem is basic:

<u>Theorem 2.1</u> (Smith's form)

Any polynomial matrix $P(s)$ is equivalent to a unique polynomial matrix of the form

$$\begin{bmatrix} \varepsilon_1(s) & & & \\ & \ddots & & \bigcirc \\ & & \varepsilon_r(s) & \\ \bigcirc & & & \bigcirc \end{bmatrix} \qquad (2.1)$$

where $\varepsilon_i(s)$, $i \in [1,2,\ldots,r]$ are nonzero monic polynomials such that $\varepsilon_i(s)$ divides $\varepsilon_{i+1}(s)$ (written $\varepsilon_i(s) | \varepsilon_{i+1}(s)$).

Moreover, if $D_i(s)$ is the greatest common divisor of all minors of order i, then

$$\varepsilon_i(s) = \frac{D_i(s)}{D_{i-1}(s)} .$$

□

The polynomials $\varepsilon_i(s)$ in the Smith's form of P(s) are called <u>the invariant factors</u> of P(s). From the theorem we immediately have

<u>Corollary 2.1</u>

Two polynomial matrices are equivalent if and only if they have the same invariant factors, i.e. the same Smith's form.

□

Let R(s), $P_1(s)$ and $P_2(s)$ be polynomial matrices. Then, R(s) is a <u>left divisor</u> of $P_1(s)$ if there is another polynomial matrix $\hat{P}_1(s)$ such that $P(s) = R(s)\hat{P}_1(s)$. Also, $P_1(s)$ and $P_2(s)$ are said to be <u>relatively left prime</u> if the only common left divisors are unimodular matrices. The analogous definitions are made for right divisors.

Now

<u>Theorem 2.2</u>

Let $P_1(s)$ and $P_2(s)$ be polynomial matrices of dimensions $r \times m$ and $r \times s$ respectively. The following statements are equivalent.

(i) $P_1(s)$ and $P_2(s)$ are relatively left prime

(ii) $\text{rank}\,[P_1(s); P_2(s)] = r$ for all $s \in \mathbb{C}$

(iii) $[P_1(s); P_2(s)]$ has Smith's form $[I_r \;\, 0]$

(iv) There are polynomial matrices $X_1(s)$ and $X_2(s)$ such that

$$P_1(s)X_1(s) + P_2(s)X_2(s) = I_r$$

□

The analoguous results hold for relatively right prime polynomial matrices (take transposes above).

Rational Matrices

Let K be an infinite field. A quotient of the form

$$\frac{p(s)}{q(s)} \tag{2.2}$$

where $p(s),\ q(s) \in K[s],\quad q(s) \neq 0,$ is called a rational form. The equivalence classes $\{\frac{p(s)}{q(s)} \mid p(s)q_1(s) = q(s)p_1(s)\}$ form a field denoted K(s). As a representative of the equivalence class, we can take (2.2) where p(s) and q(s) have no common factors and use the ordinary definitions of summation and multiplications for rational forms. In the sequel we only treat $\mathbb{R}(s)$. As in the polynomial case, we assume that the reader is familiar with how to manipulate rational forms and give only a result for matrices of rational forms, i.e. rational matrices.

Let us first derive a form for rational matrices which corresponds to the Smith's form in the polynomial case. Let T(s) be a rational matrix and write

$$T(s) = \frac{1}{d(s)} \cdot P(s) \tag{2.3}$$

where d(s) is the least common denominator of all the entries of T(s) and P(s) is a polynomial matrix. By multiplying P(s) from left and right by unimodular matrices M(s) and N(s), we can transform P(s) to Smith's form according to Theorem 2.1. Dividing each diagonal element in the Smith's form by d(s), we then obtain the following

<u>Theorem 2.3</u> (Smith-McMillan form)

Let T(s) be a rational matrix. There are unimodular matrices M(s) and N(s) such that M(s)T(s)N(s) is of the unique form

$$\begin{bmatrix} \frac{\varepsilon_1(s)}{\Psi_1(s)} & & & \bigcirc \\ & \ddots & & \\ & & \frac{\varepsilon_r(s)}{\Psi_r(s)} & \\ \bigcirc & & & \bigcirc \end{bmatrix} \tag{2.4}$$

where $\varepsilon_i(s)$, $\Psi_i(s)$ are monic nonzero relatively prime polynomials such that $\varepsilon_i(s) | \varepsilon_{i+1}(s)$ and $\Psi_{i+1}(s) | \Psi_i(s)$.

□

In the sequel a rational matrix T(s) represents the transfer matrix for a linear dynamical system. Let us define the characteristic polynomial and the order for such a matrix in the following way. The characteristic polynomial of a transfer matrix is

$$d\big(T(s)\big) \triangleq \prod_{i=1}^{r} \Psi_i(s) \tag{2.5}$$

where $\Psi_i(s)$ are the denominator polynomials in the Smith--McMillan form for T(s). The order of a transfer matrix is defined as

$$n\big(T(s)\big) \triangleq \deg d\big(T(s)\big) \tag{2.6}$$

i.e. the degree of the characteristic polynomial. Also, a rational matrix T(s) is said to be proper if the degree of the denominator polynomial is greater or equal to the degree of the numerator polynomial for each element of T(s).

3. DIFFERENTIAL EQUATIONS

Before going into the algebraic machinery, let us just briefly describe some different ways to represent linear timeinvariant differential equations. A differential equation, written as

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t) \qquad x(t_o) = x_o \\ y(t) &= C\,x(t) + D\,u(t) \end{aligned} \tag{3.1}$$

is said to be a state representation. Here, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ and $x(t) \in \mathbb{R}^p$ are called the input, output and the state vectors respectively; A, B, C and D are real matrices of compatible dimensions.

A differential equation of the form

$$\begin{aligned} &P(s)\,y(t) = A(s)\,u(t) \\ &s = \frac{d}{dt} \end{aligned} \tag{3.2}$$

where P(s) and Q(s) are polynomial matrices with $\det P(s) \neq 0$ (the zero polynomial), is called a fraction representation. Finally, a differential equation of the form

$$\begin{aligned} &P(s)\,z(t) = Q(s)\,u(t) \\ &y(t) = R(s)\,z(t) + D(s)\,u(t) \\ &s = \frac{d}{dt} \end{aligned} \tag{3.3}$$

is called a general operator representation. In (3.3) P(s), Q(s), R(s) and D(s) are polynomial matrices with elements in $\mathbb{R}[s]$ such that $\det P(s) \neq 0$. The vector z(t) is an internal variable, called the partial state. Seemingly, the operator representation (3.3) contains the state representation as a special case. However, from this we can not draw the conclusion that (3.3) is "more general" than (3.1). As will be seen, this conjecture is not true.

Algebraically, we represent the dynamical systems (3.1-3) as

$$
\begin{array}{lll}
\Sigma: (A,B,C,D) & \text{state representations} & \\
\Sigma: \big(P(s),Q(s)\big) & \text{fraction representations} & (3.4) \\
\Sigma: \big(P(s),Q(s),R(s),D(s)\big) & \text{operator representations} &
\end{array}
$$

For each of these representations we will define the following

- a notion of equivalence, i.e. a transformation which characterizes systems that are "the same"
- a notion of characteristic polynomial, i.e. a polynomial whose zeros determine the stability properties of the system
- a notion of dynamical order.

Now, from standard theory for differential equations, it follows that given an input u(t) whose laplace transform exists, there exists a unique solution y(t) to (3.1-3) for each choice of initial conditions x_0 in (3.1), $y^i(0)$ in (3.2) and $z^i(0)$ in (3.3). Hence, if we set these initial conditions equal to zero, the differential equations (3.1-3) also define a mapping from the input function u to the output function y, called the input/output map. A representative of this mapping is obtained by taking a laplace transform which yields

$$y(s) = T(s)\, u(s) \tag{3.5}$$

where

$$T(s) = C(s-A)^{-1} B + D$$

$$T(s) = P(s)^{-1} Q(s) \tag{3.6}$$

$$T(s) = R(s)\ P(s)^{-1}\ Q(s) + D(s)$$

We call the matrix T(s) obtained in this way the transfer matrix of the system. We see that a transfer matrix is a rational matrix.

Conversely, given a rational matrix T(s), the problem of finding a system Σ which corresponds to it is the realization problem. We may also regard the systems (2.4) as different ways of rewriting a rational matrix T(s) according to (3.6).

Remark

The argument s denotes some different things above. In the representation (3.2-3), $s = \frac{d}{dt}$. In the corresponding algebraic description (3.4), we let s be an indeterminate. Taking the laplace transform in (3.5), $s \in \mathbb{C}$. However, in the realization problem, the equality (3.6) shall hold with s being an indeterminate. The different interpretations of the argument s do not pose any problem, especially since we in the realization problem (which is the problem to be solved) always regard s as an indeterminate.

4. STATE REPRESENTATIONS

A state representation of a linear time invariant system is thus

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{4.1}$$

where the state x(t), the input u(t) and the output y(t) take their values in $\mathbb{R}^n$, $\mathbb{R}^m$ and $\mathbb{R}^p$ respectively and A, B, C and D are real matrices (linear maps) with appropriate dimensions. Algebraically, we represent the system by the quadruple of linear maps (matrices)

$$\Sigma: \ (A,B,C,D) \tag{4.2}$$

The characteristic polynomial for Σ is

$$d(\Sigma) \triangleq \det\ (s-A) \tag{4.3}$$

and the order of Σ, written $n(\Sigma)$, is the integer n. Note that $n(\Sigma) = \deg\big(d(\Sigma)\big)$ as for transfer matrices. Furthermore, we say that two systems $\Sigma(A,B,C,D)$ and $\Sigma_1(A_1,B_1,C_1,D_1)$ are equivalent if they are related via a nonsingular state transformation in (4.1), i.e. if there is a nonsingular matrix T such that

$$TAT^{-1} = A_1; \quad TB = B_1; \quad CT^{-1} = C_1; \quad D = D_1 \tag{4.4}$$

It is easily seen that this is an equivalence relation.

We regard $\Sigma(A,B,C,D)$ as the internal description of the system. The corresponding external description is the transfer matrix T(s) defined by

$$T(s) = C(s-A)^{-1}\ B + D \tag{4.5}$$

Conversely, any system $\Sigma(A,B,C,D)$ satisfying (4.5) is said to be realization of $T(s)$. A realization is said to be minimal if its order is the least possible.

First, we have the following

Proposition 4.1

Equivalent systems have the same transfer matrix, order and characteristic polynomial.

□

The trivial proof is omitted.

In order to establish the realization theorem we need two further concepts. The system $\Sigma(A,B,C,D)$ is said to be controllable if

$$\text{rank}\ [B \quad AB \ \dots \ A^{n-1} \ B] = n$$

and observable if

$$\text{rank} \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} = n$$

where n is the order of Σ. We note that the rank of these matrices are uneffected by a nonsingular state transformation of the type (4.4).

Proposition 4.2

A system is controllable if and only if there is no equivalent system of the form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \qquad B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix} \qquad C = \begin{pmatrix} C_1 & C_2 \end{pmatrix} \qquad (4.8)$$

Proof

Let R be the column space of the matrix (4.6). By the Cayleigh-Hamilton theorem $AR \subset R$. If the system is not controllable $r = \dim R < n$. Moreover, R contains the column space of B. Therefore, if we make a basis change in the system Σ such that the r first basis vectors span R, the transformed system must have the form (4.8). Conversely, since all equivalent systems yield the same rank of the matrix (4.6), it follows directly by evaluating the rank in (4.6) with (4.8) that the system is not controllable.

□

First, we must show that realizations exist:

Proposition 4.3

Any proper T(s) has a realization Σ: (A,B,C,D).

Proof

First write $T(s) = T_1(s) + D_1$ where D_1 is a real matrix and $T_1(s)$ is a strictly proper rational matrix. Also, write $T_1(s) = \frac{1}{d(s)} P(s)$ where $P(s) = P_1 s^{n-1} + P_2 s^{n-2} + \ldots + \ldots + P_n$ and $d(s) = s^n + a_1 s^{n-1} + \ldots + a_n$ is the least common denominator of $T_1(s)$.
Take

$$A = \begin{pmatrix} -a_1 I_m & -a_2 I_m & \cdots & -a_n I_m \\ I_m & 0 & \cdots & 0 \\ 0 & \ddots & I_m & 0 \end{pmatrix} \qquad B = \begin{pmatrix} I_m \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$C = \begin{pmatrix} P_1 & P_2 & \cdots & P_n \end{pmatrix} \qquad D = D_1$$

It is easily verified that Σ(A,B,C,D) is a realization of T(s).

□

We are now able to state the basic realization theorem for state representations:

Theorem 4.1

Let $\Sigma(A,B,C,D)$ be a realization of $T(s)$. Then

(i) Σ is minimal if and only if it is controllable and observable

(ii) All minimal realizations of $T(s)$ are equivalent

(iii) If Σ is minimal, then $d(\Sigma) = d\big(T(s)\big)$ and therefore $n(\Sigma) = n\big(T(s)\big)$.

Proof

(i) If $\Sigma(A,B,C,D)$ is minimal but not controllable, there is an equivalent system of the form (4.8). Then, $\Sigma_1(A_{11},B_1,C_1,D)$ is a realization of $T(s)$ of less order, which is a contradiction. Hence, Σ is controllable. By symmetry Σ is also observable.

Conversely, let $\Sigma_i(A_i,B_i,C_i,D_i)$, $i = 1,2$, be two realizations of $T(s)$ with Σ_1 controllable and observable and Σ_2 minimal. As has been shown above, Σ_2 is observable and controllable. Since Σ_1 and Σ_2 are realizations of the same transfer matrix:

$$C_1(s-A_1)^{-1} B_1 + D_1 = C_2(s-A_2)^{-1} B_2 + D_2$$

A power expansion $(s-A_i)^{-1} = \sum_{k=1}^{\infty} \frac{1}{s^k} A_i^{k-1}$ of both sides, yields $D_1 = D_2$ and

$$C_1 A_1^k B_1 = C_2 A_2^k B_2 \qquad k \geq 0 \qquad (4.9$$

Now, introduce

$$Q_i = \begin{pmatrix} C_i \\ C_iA_i \\ \vdots \\ C_iA_i^{n-1} \end{pmatrix} \qquad R_i = \begin{pmatrix} B_i & A_iB_i & \ldots & A_i^{n-1} & B_i \end{pmatrix} \tag{4.10}$$

where $n \geq \max\ (n_1, n_2)$ and $n_i = n(\Sigma_i)$.

Using (4.9):

$$Q_1R_1 = Q_2R_2 \tag{4.11}$$

Since Σ_1 and Σ_2 are controllable and observable, Q_1 and Q_2 has rank n_1 and n_2 respectively. Also, R_1 and R_2 has rank n_1 and n_2 respectively. This implies that $n_1 = n_2$, and Σ_1 is minimal.

(ii) Let $\Sigma_i(A_i, B_i, C_i, D_i)$ be two minimal realizations. It has been shown above (i) that $D_1 = D_2$. Also, define Q_i and R_i as in (4.10). Since both systems are controllable and observable, we can take $\hat{Q}_1 = (Q_1^T\ Q_1)^{-1}\ Q_1^T$ and $\hat{R}_1 = R_1^T(R_1\ R_1^T)^{-1}$.

By (4.11)

$$I = \hat{Q}_1\ Q_2\ R_2\ \hat{R}_1$$

Set

$$T = \hat{Q}_1\ Q_2$$

Then

$$T^{-1} = R_2\ \hat{R}_1$$

Again, using (4.11) we have

$$Q_1\ A_1\ R_1 = Q_2\ A_2\ R_2$$

$$Q_1\ B_1 = Q_2\ B_2$$

$$C_1\ R_1 = C_2\ R_2$$

i.e.

$$A_1 = \hat{Q}_1 Q_2 A_2 R_2 \hat{R}_1$$

$$B_1 = \hat{Q}_1 Q_2 B_2$$

$$C_1 = C_2 R_2 \hat{R}_1$$

which shows that the systems are equivalent.

(iii) This part will be proven later.

□

From this theorem we can draw the following conclusions. From (i) we see that the order reduction that is possible in a nonminimal state representation is completely described by the notions of controllability and observability (via Prop. 4.2). From (ii), it follows that our notion of equivalence is rich enough to characterize all minimal realizations of any given proper transfer matrix. Finally, (iii) relates the order and the characteristic polynomial of a minimal system to the corresponding concepts for the transfer matrix.

5. OPERATOR REPRESENTATIONS

Below, we treat two types of operator representations for a linear system, the fraction representation and the general operator representation. The reason why we treat them separately is that the algebra becomes much simpler for the first one. The fraction representation is also a sufficiently rich representation for many applications.

Fraction Representations

The system is described by the differential relation

$$P(s)y = Q(s)u \qquad s = \frac{d}{dt} \tag{5.1}$$

where $P(s)$ and $Q(s)$ are $p \times p$ and $p \times m$ polynomial matrices such that $\det P(s) \neq 0$ and $s = \frac{d}{dt}$ is the differentiation operator. The output $y(t)$ and the input $u(t)$ take their values in $\mathbb{R}^p$ and $\mathbb{R}^m$ respectively.

The system is thus represented by a pair of polynomial matrices $\Sigma\big(P(s),Q(s)\big)$ subject to the conditions above.

The characteristic polynomial d_Σ for Σ is defined by

$$d(\Sigma) \triangleq r \cdot \det P(s) \tag{5.2}$$

where r is a nonzero real number such that $d(\Sigma)$ is monic. The order of Σ is the integer

$$n(\Sigma) = \deg \big(d(\Sigma)\big) \tag{5.3}$$

Furthermore, we say that two fraction representations

$\Sigma(P(s),Q(s))$ and $\Sigma_1(P_1(s),Q_1(s))$ are <u>equivalent</u> if there exists a unimodular matrix $M(s)$ such that

$$M(s)P_1(s) = P(s) \; ; \qquad M(s)Q_1(s) = Q(s)$$

The <u>external</u> description of the system $\Sigma(P(s),Q(s))$ is its transfer matrix

$$T(s) = P(s)^{-1}Q(s) \tag{5.4}$$

Conversely, any system $\Sigma(P(s),Q(s))$ satisfying (5.4) is said to be a (fraction) <u>realization</u> of $T(s)$. The realization is minimal if the order $n(\Sigma)$ is the least possible.

First we have the analogy of Prop. 4.1 for fraction representations

<u>Proposition 5.1</u>

Equivalent systems $\Sigma(P(s),Q(s))$ have the same transfer matrix, the same ch.p. and the same order. □

(The trivial proof is omitted.)

Also:

<u>Proposition 5.2</u>

Any rational matrix $T(s)$ has a realization $\Sigma(P(s),Q(s))$.

<u>Proof</u>

Write $T(s) = \frac{1}{d(s)} R(s)$ where $d(s)$ is the least common denominator and $R(s)$ a polynomial matrix. Obviously, $\Sigma(d(s)I,R(s))$ is a realization.

□

The basic realization theorem for fraction representations is now as follows.

Theorem 5.1

Let $\Sigma\left(P(s),Q(s)\right)$ be a realization of $T(s)$. Then

(i) Σ is minimal if and only if $P(s)$ and $Q(s)$ are relatively left prime

(ii) All minimal realizations are equivalent

(iii) If Σ is minimal, $d\left(T(s)\right) = d(\Sigma)$ and therefore also $n\left(T(s)\right) = n(\Sigma)$.

Proof

For convenience, we omit the argument s in polynomial and rational matrices.

(i) (Only if) Let $\Sigma(P,Q)$ be minimal. Assume that P and Q have a common left divisor D which is not unimodular, i.e. $P = DP_1$ and $Q = DQ_1$, where det D has degree at least one. Then $\Sigma_1(P_1,Q_1)$ is also a realization with order det det P_1 < deg det P, i.e. $\Sigma(P,Q)$ is not minimal which is a contradiction. Hence, P and Q are relatively left prime.

(If) Let $\Sigma_1(P_1,Q_1)$ be a realization of T with P_1 and Q_1 relatively left prime and let $\Sigma_2(P_2,Q_2)$ be a minimal realization of T. As has been shown above, P_2 and Q_2 are relatively left prime. Using Theorem 2.2, there are polynomial matrices X and Y such that $P_2X + Q_2Y = I$, i.e.

$$X + TY = P_2^{-1}$$

$$X + P_1^{-1}Q_1Y = P_2^{-1}$$

$$P_1X + Q_1Y = P_1P_2^{-1}$$

which shows that $P_1P_2^{-1}$ is a polynomial matrix. In the same way one shows that $P_2P_1^{-1}$ is a polynomial matrix, i.e. $M = P_1P_2^{-1}$ has a polynomial inverse and is therefore unimodular. Hence, det P_1 = det $P_2 \cdot r$, where r = det M is a nonzero real number. There follows $n(\Sigma_1) = n(\Sigma_2)$ and hence Σ_1 is minimal.

(ii) Let $\Sigma_1(P_1,Q_1)$ and $\Sigma_2(P_2,Q_2)$ be two minimal realizations of T. According to the if-art of the proof above, there is a unimodular matrix M such that $P_1 = MP_2$. Then $T = P_2^{-1}M^{-1}Q_1 = P_2^{-1}Q_2$. A premultiplication with P_2 yields $Q_1 = MP_2$, i.e. Σ_1 and Σ_2 are equivalent.

(iii) Using Theorem 2.3, find unimodular matrices M and N such that T = NDM and D is in Smith-McMillan form, i.e.

$$D = \begin{bmatrix} \frac{\varepsilon_1}{\psi_1} & & & 0 \\ & \ddots & & \\ & & \frac{\varepsilon_r}{\psi_r} & \\ 0 & & & 0 \end{bmatrix}$$

Take

$$Q_1 = \begin{bmatrix} \varepsilon_1 & & & 0 \\ & \ddots & & \\ & & \varepsilon_r & \\ 0 & & & 0 \end{bmatrix} \quad p \times m$$

$$P_1 = \begin{bmatrix} \psi_1 & & & & 0 \\ & \ddots & & & \\ & & \psi_{r_1} & & \\ & & & \ddots & \\ 0 & & & & 1 \end{bmatrix} \quad p \times p$$

It follows directly that $D = P_1^{-1}Q_1$.
Take $\Sigma(P_1N^{-1},Q_1M)$, noting that N and M are unimodular.
Then Σ is a realization of T since

$$(P_1N^{-1})^{-1} Q_1M = NP_1^{-1}Q_1M = NDM = T$$

Since, ε_i and Ψ_i are relatively prime polynomials, there are polynomials x_i and y_i such that $x_i\varepsilon_i + y_i\Psi_i = 1$, $i = 1,2,..,r$, cf Theorem 2.2. Take

$$X_1 = \begin{bmatrix} x_1 & & & \\ & \ddots & & 0 \\ & & x_r & \\ & 0 & & 0 \end{bmatrix} \quad m \times p$$

$$Y_1 = \begin{bmatrix} y_1 & & & & 0 \\ & \ddots & & & \\ & & y_{r_1} & & \\ & 0 & & \ddots & \\ & & & & 1 \end{bmatrix} \quad p \times p$$

It follows immediately that $Q_1X_1 + P_1Y_1 = I$ and therefore also $(Q_1M)(M^{-1}X_1) + (P_1N^{-1})NY_1 = I$. Hence Q_1M and P_1N^{-1} are relatively left prime by Theorem 2.2 and therefore Σ is minimal by (i) in the theorem. Now

$$\begin{aligned} d(\Sigma) &= r_0 \det (P_1N^{-1}) \\ &= r_0r \det P_1 = \prod_{i=1}^{r} \Psi_i(s) \\ &= d\big(T(s)\big). \end{aligned}$$

□

Theorem 5.1 gives the same verifications of the validity of the definitions of equivalence, order etc. as Theorem 4.1 for state representation. From (i) it follows that the <u>order reduction</u> that is possible in a fraction representation Σ $\big(P(s),Q(s)\big)$ can be performed by removing left divisors (which are not unimodular) from the pair $\big(P(s),Q(s)\big)$. Condition (ii) implies that our definition of equivalence is rich enough to generate <u>all</u> minimal fraction realizations of the same transfer matrix T(s), and condition (iii) implies that the definitions of order and ch.p. is compatible with the definition for a rational matrix (and also for state representations by Theorem 4.1(iii)).

To see the relationship between the "redundance" concepts in Theorem 4.1(i) and Theorem 5.1(i), let us consider the following special fraction representation

$$(s-A)y = Bu \tag{5.5}$$

where A and B are real matrices. This representation corresponds to (4.1) with C = I.

Proposition 5.3

The polynomial matrices (s-A) and B are relatively left prime if and only if (A,B) is controllable.

Proof

First, (S-A) and B are relatively left prime if and only if rank $[s-A \quad B] = n \quad \forall s \in \mathbb{C}$ by Theorem 2.2.

(If) Assume (A,B) controllable but (s-A) and B are not relatively left prime. Then there exists $s_0 \in \mathbb{C}$ such that rank $[s_0-A \quad B] < n$, implying that there exists a nonzero vector $v \in \mathbb{C}^n$ such that $v^T[s_0-A \quad B] = 0$, i.e. $v^T(s_0-A) = 0$ and $v^TB = 0$. Then $v^T[B \quad AB \ldots A^{n-1}\, B] =$ $= [v^TB \quad s_0v^TB \ \ldots \ s_0{}^{n-1}v^TB] = 0$, which shows that rank $[B \quad AB \ \ldots \ A^{n-1}\, B] < n$, which is a contradiction. Hence (s-A) and B are relatively left prime.

(Only if) Assume that (s-A) and B are relatively left prime but (A,B) is not controllable. Using Proposition 4.2, there is a nonsingular matrix T such that

$$TAT^{-1} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \qquad TB = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}$$

Let s_0 be any eigenvalue to A_{22} and let v_0^T be any (left) eigenvector corresponding to s_0, i.e. $v_0^T A_{22} = s_0 v_0^T$. Take $v^T = [0 \quad v_0^T]T$. Then $v^T A = s_0 v^T$ and $v^T B = 0$, i.e. $v^T[s_0 - A \quad B] = 0$. This shows that rank $[s_0 - A \quad B] < n$, i.e. that (s-A) and B are not relatively prime. This is a contradiction and therefore (A,B) is controllable.

□

The relationship between the controllability of (A,B) and the left primeness of the polynomial matrices (s-A) and B goes still further. Consider (5.5) and assume that (A,B) is not controllable. Using Proposition 4.2 we have

$$T(s-A)T^{-1}T\dot{y} = TBu$$

$$\begin{pmatrix} s-A_{11} & -A_{12} \\ 0 & s-A_{22} \end{pmatrix} Ty = \begin{pmatrix} B_1 \\ 0 \end{pmatrix} u$$

The greatest common left divisor of the polynomial matrices on both sides is

$$\pi(s) = \begin{pmatrix} I & 0 \\ 0 & s-A_{22} \end{pmatrix}$$

Removing this divisor, we get a minimal fraction representation:

$$\begin{pmatrix} s-A_{11} & -A_{12} \\ 0 & I \end{pmatrix} Ty = \begin{pmatrix} B_1 \\ 0 \end{pmatrix} u$$

The greatest common divisor of (s-A) and B is then

$$\begin{pmatrix} I & 0 \\ 0 & s-A_{22} \end{pmatrix} T^{-1} \qquad (5.5)$$

from this we see that the greatest common divisor of s-A and B corresponds directly to the uncontrollable part as expressed by (4.8).

The results above are valid for a representation of a rational matrix T(s) in the form

$$T(s) = P(s)^{-1}Q(s)$$

By symmetry, we immediately can state the corresponding results for representations of the form

$$T(s) = Q_1(s)P_1(s)^{-1}$$

which corresponds to an operator representation as

$$P_1(s)z = u$$

$$y = Q_1(s)z$$

This is left as an exercise for the reader.

General Operator Representations

The system is described by the differential relation

$$P(s)z(t) = Q(s)u(t)$$

$$y(t) = R(s)z(t) + D(s)u(t) \qquad (5.6)$$

$$s = \frac{d}{dt}$$

where P(s), Q(s), R(s) and D(s) are polynomial matrices with det P(s) $\neq$ 0. The input u(t), the output y(t) and the internal variable z(t) take their values in $\mathbb{R}^m$, $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively. The internal variable z(t) is sometimes called the partial state.

We represent the system (5.6) as

$$\Sigma: \left(P(s), Q(s), R(s), D(s)\right)$$

The <u>characteristic polynomial</u> for Σ is

$$d(\Sigma) \triangleq r.\det P(s) \tag{5.7}$$

where r is a nonzero real number such that $d(\Sigma)$ is monic. The <u>order</u> of Σ is the integer

$$n(\Sigma) \triangleq \deg\left(d(\Sigma)\right) \tag{5.8}$$

Two systems $\Sigma_1\left(P_1(s),Q_1(s),R_1(s),D_1(s)\right)$ and $\Sigma_2\left(P_2(s),Q_2(s),R_2(s),D_2(s)\right)$ are said to be <u>equivalent</u> if there exist unit matrices I_1, I_2 of suitable sizes, unimodular matrices $N_1(s)$, $N_2(s)$ and polynomial matrices $X_1(s)$ and $X_2(s)$ such that (omitting the arguments)

$$\begin{pmatrix} N_1 & 0 \\ X_1 & I \end{pmatrix} \left(\begin{array}{cc|c} I_1 & 0 & 0 \\ 0 & P_1 & Q_1 \\ \hline 0 & -R_1 & D_1 \end{array}\right) \begin{pmatrix} N_2 & X_2 \\ 0 & I \end{pmatrix} = \left(\begin{array}{cc|c} I_2 & 0 & 0 \\ 0 & P_2 & Q_2 \\ \hline 0 & -R_2 & D_2 \end{array}\right)$$

This notion of equivalence needs some further comments. The expansion with the unit matrix I_1 just means that we add some auxiliary internal variables in Σ_1 according to

$$\tilde{z}_1(t) = 0$$

Yielding e.g. for Σ_1:

$$\begin{pmatrix} I_1 & 0 \\ 0 & P_1(s) \end{pmatrix} \begin{pmatrix} \tilde{z}_1(t) \\ z_1(t) \end{pmatrix} = \begin{pmatrix} 0 \\ Q_1(s) \end{pmatrix} u(t)$$

$$y(t) = \begin{pmatrix} 0 & R_1(s) \end{pmatrix} \begin{pmatrix} \tilde{z}_1(t) \\ z_1(t) \end{pmatrix} + D(s)u(t)$$

The expansion I_2 means the same thing for the system Σ_2. Assuming these expansions are made, let the systems be

$$\Sigma_1: \quad \hat{P}_1(s)\hat{z}_1(t) = \hat{Q}_1(s)u(t) \tag{5.10a}$$

$$y = \hat{R}_1(s)\hat{z}_1(t) + D_1(s)u(t) \tag{5.10b}$$

and

$$\Sigma_2: \begin{array}{ll} \hat{P}_2(s)\hat{z}_2(t) = \hat{Q}_2(s)u(t) & (5.11a) \\ y = \hat{R}_2(s)\hat{z}_2(t) + \hat{D}_2(s)u(t) & (5.11b) \end{array}$$

Our notion of equivalence means that we obtain the system Σ_2 from Σ_1 by the following sequence of transformations

(1) Premultiply (5.10a) by the unimodular matrix $N_1(s)$

(2) Add $X_1(s)\left(-P_1(s)\hat{z}_1(t) + Q_1(s)u(t)\right)$ to (5.10b). Note that the expression within paranthesis is zero by (5.10a)

(3) Select a new internal variable $\hat{z}_2(t) = N_2(s)^{-1} \cdot \left(\hat{z}_1(t) + X_2(s)u(t)\right)$ where $N_2(s)$ is unimodular.

The <u>external</u> description of $\Sigma\left(P(s),Q(s),R(s),D(s)\right)$ is its transfer matrix, i.e.

$$T(s) = R(s)P(s)^{-1}Q(s) + D(s) \qquad (5.12)$$

Conversely, any system $\Sigma\left(P(s),Q(s),R(s),D(s)\right)$ satisfying (5.12) is said to be a <u>realization</u>. The realization is minimal if its order $n(\Sigma)$ is the least possible.

First the analogy of Proposition 4.1:

<u>Proposition 5.4</u>

Equivalent systems have the same ch.p., order and transfer matrix.

The trivial proof is omitted.

The existence of realizations is established in

<u>Proposition 5.5</u>

Any rational matrix has a realization $\Sigma\left(P(s),Q(s),R(s),D(s)\right)$.

Proof

By proposition 5.2, there exists a fraction realization $\Sigma(P(s),Q(s))$. Then $\Sigma(P(s),Q(s),I,0)$ is an operator realization.

□

The basic realization theorem for general operator representations is now as follows.

Theorem 5.1

Let $\Sigma(P(s),Q(s),R(s),D(s))$ be a realization of $T(s)$. Then

(i) Σ is minimal if and only if $P(s),Q(s)$ are relatively left prime and $P(s),R(s)$ are relatively right prime

(ii) All minimal realizations of the same $T(s)$ are equivalent

(iii) If Σ is minimal, then $d(\Sigma) = d(T(s))$ and therefore also $n(\Sigma) = n(T(s))$.

Proof

(i) (Only if) Proven in the same way as the corresponding part of Theorem 5.1.

(If) Assume that $\Sigma:(P,Q,R,D)$ satisfies the primeness conditions and let $\Sigma_1(P_1,Q_1,R_1,D_1)$ be minimal. We intend to show that $n(\Sigma) = n(\Sigma_1)$. First, Σ_1 satisfies the primeness conditions since otherwise its order can be reduced according to above. Now, since Σ satisfies the primeness conditions, we have

$$T = RP^{-1}Q + D = \tilde{P}^{-1}(\tilde{R}Q + \tilde{P}D) \quad (5.13)$$

where

$$\tilde{P}^{-1}\tilde{R} = RP^{-1} \quad (5.14a)$$

and both fraction representations in (5.14a) are minimal, RP^{-1} by assumption and $\tilde{P}^{-1}\tilde{R}$ by construction. Then by Theorem 5.1(iii)

$$\det \tilde{P} = r \cdot \det P \tag{5.14b}$$

for some nonzero real number r. Since both pairs $(\tilde{P},\tilde{R})$ and (P,Q) are relatively left prime, there are polynomial matrices X, Y, $\tilde{X}$ and $\tilde{Y}$ such that (cf Theorem 2.2)

$$\begin{aligned} PX + QY &= I \\ \tilde{P}\tilde{X} + \tilde{R}\tilde{Y} &= I \end{aligned} \tag{5.15}$$

Multiply the first expression from left by $\tilde{R}$ and use (5.14a). This yields

$$\tilde{P}RX + \tilde{R}QY = \tilde{R}$$

Then multiply from right by $\tilde{Y}$ and substitute $\tilde{R}\tilde{Y}$ from (5.15), i.e.

$$\tilde{P}(RX\tilde{Y} + \tilde{X}) + (\tilde{R}Q)Y\tilde{Y} = I$$

and by a straightforward manipulation

$$\tilde{P}(RX\tilde{Y} + \tilde{X} - DY\tilde{Y}) + (\tilde{R}Q + \tilde{P}D)(Y\tilde{Y}) = I$$

By Theorem 2.2 and 5.1 it follows that (5.13) is a minimal fraction representation. Since Σ_1 also satisfies the primeness conditions it can be represented as in (5.13) and since all minimal fraction representations have the same order this implies that Σ and Σ_1 have the same order by (5.14b) and therefore Σ is minimal.

(ii) Let $T = P_1^{-1}Q_1$ be a minimal fraction representation. According to above $\Sigma_1:(P_1,Q_1,I_p,0)$ is a minimal (operator) realization. Also let $\Sigma:(P,Q,R,D)$ be an arbitrary minimal operator realization. We intend to show that $\Sigma \sim \Sigma_1$ which in turn implies that all minimal (operator) realizations are equivalent since Σ is arbitrary. First,

$$T = RP^{-1}Q + D = \tilde{P}^{-1}(\tilde{R}Q + \tilde{P}D) \tag{5.16}$$

where

$$RP^{-1} = \tilde{P}^{-1}\tilde{R} \tag{5.17}$$

and both are minimal fraction representations. Now, by (i) above $\tilde{P}$, $\tilde{R}Q + \tilde{P}D$ are relatively left prime and therefore (5.16) is a minimal fraction representation. Hence, by Theorem 5.1 (ii), there exists a unimodular matrix N such that

$$\begin{aligned} N\tilde{P} &= P_1 \\ N(\tilde{R}Q + \tilde{P}D) &= Q_1 \end{aligned} \tag{5.18}$$

Moreover, introduce polynomial matrices X, $\tilde{X}$, Y and $\tilde{Y}$ as

$$\begin{aligned} XR + YP &= I \\ \tilde{R}\tilde{X} + \tilde{P}\tilde{Y} &= I \end{aligned} \tag{5.19}$$

which is possible according to Theorem 2.2.
Using (5.17)-(5.19) it is trivial to show that the following matrix identity holds

$$\left(\begin{array}{cc|c} X & Y & 0 \\ -P_1 & NR & 0 \\ \hline I & 0 & I \end{array}\right)\left(\begin{array}{cc|c} I & 0 & 0 \\ 0 & P & Q \\ \hline 0 & -R & D \end{array}\right) = \left(\begin{array}{cc|c} I & 0 & 0 \\ 0 & P_1 & Q_1 \\ \hline 0 & -I & 0 \end{array}\right)\left(\begin{array}{cc|c} X & YP & YQ \\ -I & R & -D \\ \hline 0 & 0 & I \end{array}\right) \tag{5.20}$$

Moreover, using (5.18) and (5.19)

$$\begin{pmatrix} X & Y \\ -P_1 & N\tilde{R} \end{pmatrix}\begin{pmatrix} R & -\tilde{Y} \\ P & \tilde{X} \end{pmatrix} = \begin{pmatrix} I & +X\tilde{Y}-Y\tilde{X} \\ 0 & N \end{pmatrix}.$$

The righthand side is unimodular and therefore the matrix

$$\begin{pmatrix} X & Y \\ -P_1 & NR \end{pmatrix}$$

is also unimodular. Moreover, using (5.19)

$$\begin{pmatrix} X & YP \\ -I & R \end{pmatrix} = \begin{pmatrix} X & I-XR \\ -I & R \end{pmatrix} = \begin{pmatrix} I & -X \\ 0 & I \end{pmatrix}\begin{pmatrix} 0 & I \\ -I & R \end{pmatrix}$$

which shows that

$$\begin{pmatrix} X & YP \\ -I & R \end{pmatrix}$$

is unimodular. Hence (5.20) implies that $\Sigma \sim \Sigma_1$ which shows the result.

(iii) Follows trivially from the fact that a minimal operator realization can always be taken as a minimal fraction realization and the result holds for the latter, cf Theorem 5.1. This also proves Theorem 4.1(iii).

□

As before, Theorem 5.2 (i) shows that the order reduction of a nonminimal system $\Sigma(P,Q,R,D)$ is performed by removing left divisors from the pair $P(s),Q(s)$, yielding $\hat{P}(s),\hat{Q}(s)$, and thereafter removing right factors from the pair $\hat{P}(s),R(s)$, or the other way around. Condition (ii) implies that our notion of equivalence is rich enough to be able to generate all possible minimal realizations of any transfer matrix $T(s)$, and condition (iii) implies that the definition of ch.p. and order is compatible with that of a rational matrix.

Furthermore, an operator representation of the form

$$(s-A)\ z(t) = Bu(t)$$

$$y = Cx(t) + Du(t)$$

corresponds to the state representation (4.1). In this case, minimality is equivalent to $s-A$, B being relatively left prime and $s-A$, C being relatively right prime. Using Proposition 5.3 we see that these conditions are equivalent to (A,B) being controllable and (A,C) being observable. Hence, there is a direct relationship between the redundancy concepts in Theorem 4.1(i) and Theorem 5.2(i). As was shown in the preceding section, this relationship between controllability and observability and the primeness conditions goes still further. The greatest common left divisor of (s-A) and B represents the "uncontrollable part" of Σ, cf. (5.5) and (1.8). Quite analogously, the greatest common right divisor of s-A and C represents the "unobservable part" of Σ.

6. COMMON FEATURES

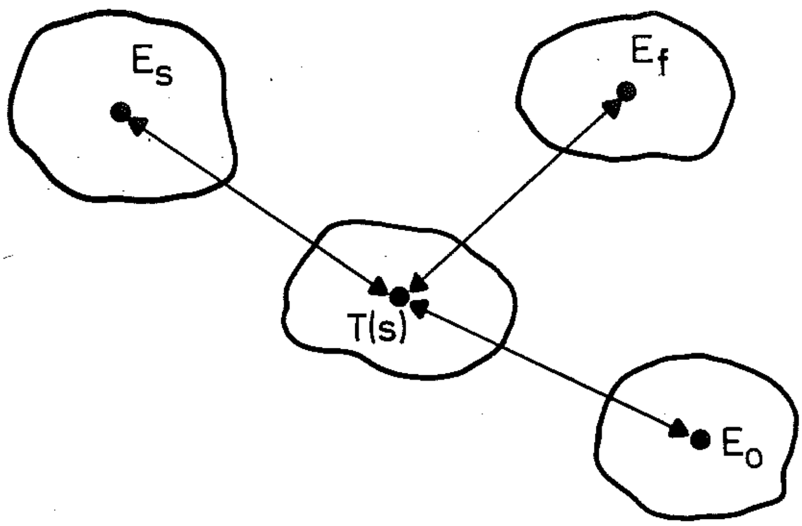
The realization theorems above imply that we established bijections between the equivalence classes under (state) equivalence, (fraction) equivalence and (operator) equivalence for minimal systems by the transfer matrix. Denote these equivalence classes

- $\{E_s\}$ set of equivalence classes for state representations
- $\{E_f\}$ set of equivalence classes for fraction representations
- $\{E_o\}$ set of equivalence classes for operator representations

For either type of representation we have shown that

- o all equivalent systems have the same transfer matrix T(s)
- o all minimal realizations of T(s) are equivalent

which establish the bijections for <u>minimal systems</u> and <u>proper transfer matrices</u> (T(s) was assumed to be proper in the state representation). The bijections are illustrated below.



Moreover, for either type of representation we have shown that

$$d(\Sigma) = d\big(T(s)\big)$$
$$n(\Sigma) = n\big(T(s)\big)$$

which also means that the notions of characteristic polynomial and order are <u>invariant</u> under the bijections above (note that equivalent systems always have the same characteristic polynomial and the same order).

This feature thus holds for minimal systems with proper transfer matrices. The extension to minimal systems and nonproper transfer matrices is easily made by first writing

$$T(s) = T_1(s) + D(s)$$

where $T_1(s)$ is strictly proper and $D(s)$ a polynomial matrix, and then letting $\Sigma\big(A,B,C,D(s)\big)$ be the state realization where $\Sigma(A,B,C,0)$ is a minimal realization of $T_1(s)$. If the equivalence between representations of the form $\Sigma\big(A,B,C,D(s)\big)$ are expressed in the same way as in (4.4) with D replaced by D(s), all what have been said above still holds.

For <u>nonminimal</u> system, Rosenbrock [5] has established the bijection between $\{E_s\}$ and $\{E_o\}$ by proving that

- within each equivalence class under operator equivalence there is a representation of the form $\Sigma\big(s-A,B,C,D(s)\big)$.

- All representations of the form $\Sigma\big(s-A,B,C,D(s)\big)$ within the same equivalence class under operator equivalence are related via a transformation of the type (4.4) (allowing D to be a polynomial matrix).

7. REFERENCES

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