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Abstract

Several stability conditions for hybrid systems are formulated as LMI problems. This includes the computations of continuous piecewise quadratic Lyapunov functions and multiple Lyapunov functions. The proposed methods are related to classical frequency domain methods, such as the Circle and Popov criteria. Finally, it is pointed out how controller design for hybrid systems can be formulated as an LMI problem. Several examples that highlight the proposed methods are included.

Key words
Lyapunov stability, nonlinear systems, hybrid systems, LMIs

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Computation of Piecewise Quadratic Lyapunov Functions for Hybrid Systems

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Abstract The search for a piecewise quadratic Lyapunov functions for nonlinear and hybrid systems is stated as a convex optimization problem in terms of linear matrix inequalities. Several examples are included to demonstrate the flexibility and power of the approach.

Keywords Lyapunov stability, nonlinear systems, hybrid systems, LMIs

1. Introduction

Construction of Lyapunov functions is one of the most fundamental problems of systems theory. The most direct application is of course stability analysis, but analog problems appear more or less implicitly also in performance analysis, controller synthesis and system identification. Closely related to Lyapunov functions are for example the notions of storage functions, energy functions and loss functions [Willems, 1972].

The objective of this paper is develop a general method for computation of Lyapunov functions for nonlinear and hybrid systems. A need for such methods have been recognized for several decades, but only recently an appropriate computational tool has appeared, namely efficient algorithms for convex optimization in terms of linear matrix inequalities (LMI's) [Nesterov and Nemirovski, 1993, Boyd *et al.*, 1994, Elghaoui, 1995, Gahinet *et al.*, 1995]. These algorithms have quickly become a powerful tool in design and analysis, but so far mainly for linear systems.

Working exclusively with piecewise linear system dynamics, our main idea is to state the search for a piecewise quadratic Lyapunov function as an LMI

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optimization problem. The Lyapunov function will sometimes be forced to be continuous, and will always be decreasing with time. For hybrid systems, that have both continuous and discrete states, the Lyapunov function will be quadratic in the continuous state. The dependence on the discrete state will be arbitrary except that the function must be non-increasing at the jumps of the discrete state.

Looking back, a rich theory has been developed based on linear systems and quadratic Lyapunov/loss functions. A beautiful early representative of the theory is the circle criterion [Yakubovich, 1964], which considers a linear system interconnected with a nonlinear element represented by a quadratic inequality. The theorem gives a frequency domain condition on the transfer function of the linear system, that is sufficient for existence of a quadratic Lyapunov function for the interconnection. Such frequency domain criteria give valuable insight and were particularly important before the computer era, since they allowed for simple geometrical verification, rather than solving difficult matrix inequalities in the time domain.

It was soon recognized that the circle criterion was unnecessarily conservative and that it could easily be strengthened under some additional assumptions on the nonlinearity. The tool for such improvements was the so called "multipliers", and the results can often be interpreted as a generalization of the set of allowable Lyapunov functions [Desoer and Vidyasagar, 1975]. However, the multiplier conditions had an abstract form and only in the most simple cases, like the Popov criterion [Popov, 1962], it was possible to modify the graphical methods of verification.

It took until the 80-s before the situation changed thanks to new computational tools. Multi-loop generalizations of the circle criterion were then introduced for problems involving parametric uncertainty. The appropriate multipliers were computed frequency by frequency and the results were presented in terms of so called structured singular values [Doyle, 1982, Safonov and Athans, 1981].

A general approach to computation of multipliers using LMI optimization was introduced in [Megretski and Rantzer, 1995]. Some of the stability conditions used there can also be as criteria for existence of quadratic Lyapunov functions on an extended state space.

Several types of methods have been proposed for analysis of systems with piecewise linear dynamics. These range from qualitative methods such as phase plane analysis via computational geometry methods [Pettit and Wellstead, 1995] to algebraic approaches [Sontag, 1996].

The work on Lyapunov stability of hybrid systems dates back to the sixties [Pavlidis, 1967]. Some recent and rather general results for hybrid systems have been developed in [Branicky, 1994] and [Peleties and DeCarlo, 1991] using so-called 'multiple Lyapunov functions'. When it comes to construction of the Lyapunov functions, however, only simple examples have been considered and no general methods have been proposed.

The most general approach for construction of Lyapunov functions that has been suggested for piecewise linear systems, is known as quadratic stability. This means to search for a globally quadratic Lyapunov function

$$V(x) = x^T P x$$

that is valid globally, and for all linear systems $\dot{x} = A_i x$, $i \in I$. Although conservative, this method has several attractive features. The search for a common P matrix requires simultaneous solution of the matrix inequalities

$$\begin{aligned} P &= P^T > 0 \\ A_i^T P + P A_i &< 0 \end{aligned} \tag{1}$$

for $i \in I$. This is a convex optimization problem in P, which can be solved efficiently using LMI software. This allows for automated stability analysis of systems with a large number of A_i-matrices.

In some cases, it is of interest to verify that no common solution P exists. This verification can be made by solving the following dual problem. If there exists positive definite matrices R_i with $i \in I$ satisfying

$$\begin{aligned} R_i &= R_i^T > 0 \\ \sum_{i \in I} A_i^T R_i + R_i A_i &< 0 \end{aligned} \tag{2}$$

then (1) does not admit a positive definite solution $P = P^T$.

Another attractive feature of having a globally quadratic Lyapunov function is that stability can be guaranteed independently of the cell partition, and for a large class of switching schemes. For examples, see the work on Lyapunov stability for Fuzzy Systems [Zhao, 1995], [Tanaka *et al.*, 1996]. However, it is not hard to come up with systems which are stable, but for which no common globally quadratic Lyapunov function on the form $V(x) = x^T P x$ exists. Indeed, this is the case for all examples given in this paper.

2. System Description

Systems with piecewise linear or affine dynamics can be represented in several ways. In this report, we shall use the notion of a cell partition (cf [Caines and Ortega, 1995]). We shall assume that the state space $\mathbb{R}^n$ is partitioned into a finite number of closed sets X_i, $i \in I$ called cells. A typical cell partition is shown in Figure 1.

Each cell constructed as the intersection of a p half planes, each given by an affine inequality

$$(c_{i,j})^T x + d_{i,j} \geq 0$$

Thus, each cell can be characterized by the vector inequality

$$C_i x + D_i \geq 0 \qquad x \in X_i$$

with $C_i \in \mathbb{R}^{p \times n}$ and $D_i \in \mathbb{R}^p$. The vector inequality $x \geq 0$ means that each entry of x should be non-negative. We shall further assume that the cells are constructed so that $C_i x + D_i = C_j x + D_j$ for $x \in X_i \cap X_j$, $i, j \in I$. Within each cell, the dynamics is piecewise affine

$$\dot{x}(t) = A_i x(t) + B_i$$

If $B_i = 0\, \forall i \in I$, we shall call the dynamics piecewise linear.

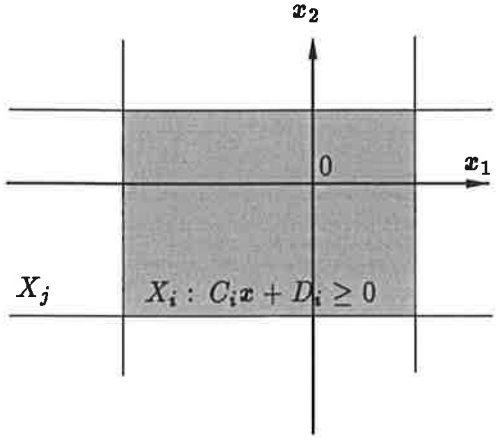

Figure 1 A typical cell partition.

In a more general setting, we also consider hybrid systems with autonomous switching between a set of piecewise affine dynamics [Branicky, 1995]. We model these systems as

$$\begin{aligned} \dot{x}(t) &= A_i x(t) + B_i \\ i(t) &= \nu(x(t), i(t-)) \end{aligned}$$

The discrete state $i(t) \in I$ is piecewise constant. The notation $t-$ indicates that the discrete state is piecewise continuous from the right. In our setting, changes in the discrete state $i(t)$ occur when the continuous state $x(t)$ hits a cell boundary, i.e.

$$(c_{i,j})^T x(t) + d_{i,j} = 0$$

for some $i, j \in I$. For a nice review of hybrid phenomena and models, the reader is referred to the thesis [Branicky, 1995].

3. A Simple Example

As a simple and illustrative example, consider the following nonlinear differential equation

$$\dot{x} = \begin{cases} A_1 x, & \text{if } x_1 < 0 \\ A_2 x, & \text{if } x_1 \geq 0 \end{cases}$$

where

$$A_1 = \begin{bmatrix} -5 & -4 \\ -1 & -2 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -2 & -4 \\ 20 & -2 \end{bmatrix}$$

By solving the dual problem stated in (2), one can verify that there is no globally quadratic Lyapunov function $V(x) = x^T P x$ that assures stability of the system. Still, the simulations shown in Figure 2 indicates that the system is stable.

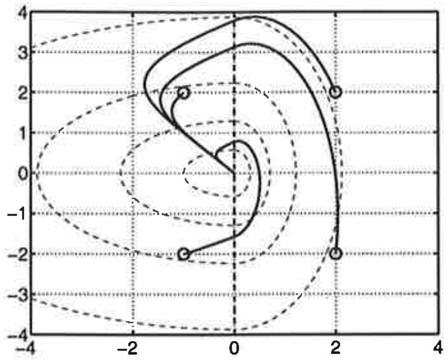

Figure 2 Trajectories of the simple switched system.

As an alternative to a globally quadratic Lyapunov function, it is natural to consider the following Lyapunov function candidate

$$V(x) = \begin{cases} x^T P x, & \text{if } x_1 < 0 \\ x^T P x + \eta x_1^2, & \text{if } x_1 \geq 0 \end{cases} \tag{3}$$

where P and $\eta \in \mathbb{R}$ are chosen so that both quadratic forms are positive definite. Note that the Lyapunov function candidate is constructed to be continuous and piecewise quadratic. The search for appropriate values of η and P can be done by numerical solution of the following linear matrix inequalities

$$\begin{aligned} P = P^T &> 0 \\ A_1^T P + P A_1 &< 0 \\ P + \eta C^T C &> 0 \\ A_2^T (P + \eta C^T C) + (P + \eta C^T C) A_2 &< 0 \end{aligned}$$

with $C = [1 \quad 0]$. One feasible solution is given by $P = \text{diag}\{1, 3\}$ and $\eta = 7$. The level surfaces of the computed Lyapunov function is plotted with dashed lines in Figure 2.

It is instructive to compare this solution to an application of the classical circle and the Popov criteria, to the simple switched system. Noting that

$$A_2 = A_1 + BC$$

with

$$B = \begin{bmatrix} 3 \\ 21 \end{bmatrix}, \qquad C = [1 \quad 0]$$

we can re-write the system equation as

$$\begin{aligned} \dot{x} &= A_1 x - B\phi(Cx) \\ \phi(x) &= \begin{cases} 0, & \text{if } y < 0 \\ y, & \text{if } y \geq 0 \end{cases} \end{aligned}$$

This situation is illustrated in Figure 3.

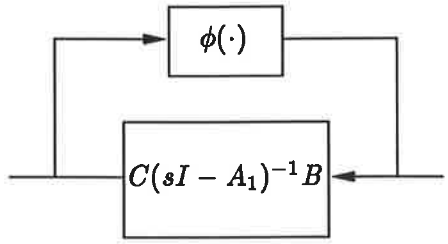

Figure 3 Switched system as feedback connection.

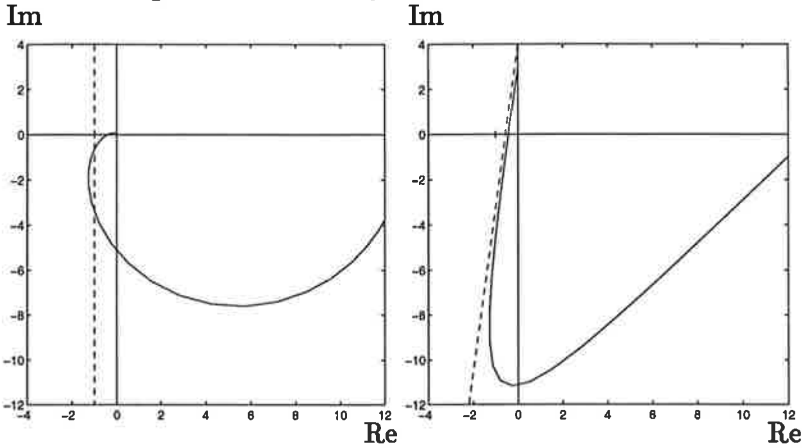

Figure 4 The circle criterion (left) fails to prove stability. The Popov plot (right) is separated from -1 by a straight line of slope $1/\eta$. Hence stability follows.

Defining $G(s) = C(sI - A_1)^{-1}B$, we obtain the frequency condition

$$\text{Re } G(i\omega) > -1 \qquad \forall\omega \in [0, \infty]$$

for the circle criterion and

$$\text{Re } [(1 + i\omega\eta)G(i\omega)] > -1 \qquad \forall\omega \in [0, \infty]$$

for the Popov criterion. Inspection of the Nyquist and Popov plots of Figure 4, reveals that stability follows from the Popov criterion but not from the circle criterion.

The failure of the circle criterion comes as no surprise, as the circle criterion relies on the existence of a common Lypuanov function on the form $V(x) = x^T Px$ [Khalil, 1992], which we know does not exist. The standard proof of the Popov criteria, on the other hand, is based on a Lyapunov function on the form

$$V(x) = x^T Px + 2\eta \int_0^{Cx} \phi(\sigma)d\sigma$$

For the simple switched system, this Lyapunov function reads

$$V(x) = \begin{cases} x^T Px, & \text{if } x_1 < 0 \\ x^T(P + \eta C^T C)x, & \text{if } x_1 \geq 0 \end{cases}$$

which is identical to (3), used successfully in the numerical optimization above.

4. Piecewise Linear Systems

Although classical frequency domain methods can be applied to assess stability of simple switching systems, it is not clear how to generalize these methods to more realistic cases, with several switches and more varying dynamics. Using the LMI formulation, on the other hand, it is possible to generalize the stability conditions to the case when we have a cell partition with an arbitrary number of cells with boundaries $(c_{i,j})^T x = 0$ through the origin.

Requiring only continuity of the Lyapunov function across the cell boundaries, we can use more general quadratic forms than $\eta C^T C$ as 'local patches' in the Lyapunov function. Let

$$V_i(x) = x^T P_i x \tag{4}$$

be a Lyapunov function for cell X_i and let the boundary between X_i and a neighbouring cell X_j be given by $(c_{i,j})^T x = 0$. Then, the composite Lyapunov function is continuous if and only if the Lyapunov function in cell X_j has the form

$$V_j(x) = x^T \left(P_i + c_{i,j}(e_{i,j})^T + e_{i,j}(c_{i,j})^T\right) x$$

The LMI conditions in the previous section are stated globally. This is unnecessarily restrictive, since the Lyapunov function V_i only has to decrease for $x(t) \in X_i$. One way to decrease conservatism in the stability conditions is to modify the condition

$$A_i^T P_i + P_i A_i < 0 \tag{5}$$

to

$$A_i^T P_i + P_i A_i + C_i^T R_i C_i < 0 \tag{6}$$

where the matrix R_i has non-negative coefficients. Since $C_i x > 0$, (6) implies that the Lyapunov function decreases along the trajectories of the system within X_i. For x outside of X_i, $C_i^T R_i C_i < 0$, which may make the inequality simpler to satisfy. This method is known as the S-procedure [Aiserman and Gantmacher, 1965]. We summarize the discussion in the following stability theorem.

THEOREM 1
For every index $i \in I$, let X_i be a closed subset of $\mathbb{R}^n$ and let $C_i \subset \mathbb{R}^{p\times n}$. Suppose that $C_i x \geq 0$ for $x \in X_i$ and $C_i x = C_j x$ for $x \in X_i \cap X_j$, $i, j \in I$. If there exist matrices Q and R_i, where R_i has non-negative coefficients, such that the matrices

$$P_i = \begin{bmatrix} C_i \\ I \end{bmatrix}^T Q \begin{bmatrix} C_i \\ I \end{bmatrix} \tag{7}$$

satisfy

$$\begin{aligned} 0 &< P_i \\ 0 &< P_i A_i + A_i^T P_i + C_i^T R_i C_i \end{aligned}$$

for $i \in I$, then every continuous piecewise C^1 trajectory in $\cup_{i \in I} X_i$ with

$$\dot{x}(t) = A_i x(t) \text{ for } x(t) \in X_i$$

tends to zero exponentially. □

The proof of this theorem is omitted, since it is a special case of Theorem 2, which will be proved in the next section. To show the applicability of Theorem 1, we consider the following piecewise linear system.

EXAMPLE 1—FLOWER SYSTEM
Consider the following piecewise linear system

$$\dot{x}(t) = A_i x(t) \tag{8}$$

$$i(t) = \begin{cases} 1, & x_1^2(t) - x_2^2(t) < 0 \\ 2, & x_1^2(t) - x_2^2(t) \geq 0 \end{cases} \tag{9}$$

where the continuous dynamics are given by

$$A_1 = \begin{bmatrix} -\epsilon & \omega \\ -\alpha\omega & -\epsilon \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -\epsilon & \alpha\omega \\ -\omega & -\epsilon \end{bmatrix} \tag{10}$$

Letting $\alpha = 5$, $\omega = 1$ and $\epsilon = 0.1$, the trajectory of an initial value simulation from $x_0 = (\,-2.5, \quad 0\,)^T$ moves towards the origin in a flower-like trajectory, as shown in Figure 5. Using the enumeration of the cells indicated in Figure 5,

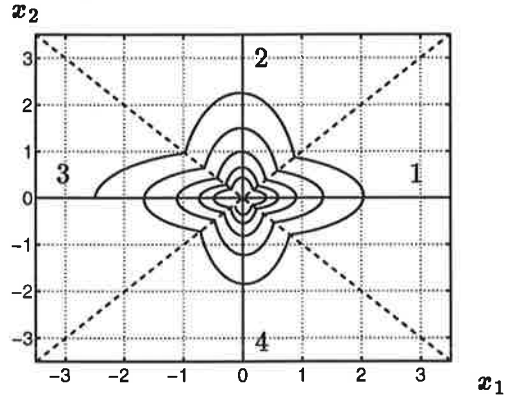

Figure 5 Initial value simulation (full) and cell boundaries (dashed).

the linear inequalities characterizing the cells are given by the matrices

$$C_1 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \qquad C_2 = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \tag{11}$$

$$C_3 = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}, \qquad C_4 = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} \tag{12}$$

From the conditions for a continuous piecewise quadratic Lyapunov function, as stated in Theorem 1 we find

$$Q = \begin{bmatrix} Q_{11} & 0 \\ 0 & \tilde{P} \end{bmatrix} \tag{13}$$

with

$$Q_{11} = \frac{1}{4}\begin{bmatrix} 0 & \alpha-1 \\ \alpha-1 & 0 \end{bmatrix}, \qquad \tilde{P} = \frac{1}{2}\begin{bmatrix} \alpha+1 & 0 \\ 0 & \alpha+1 \end{bmatrix} \tag{14}$$

The level surfaces of the computed Lyapunov function along with a typical system trajectory is shown in Figure 6. □

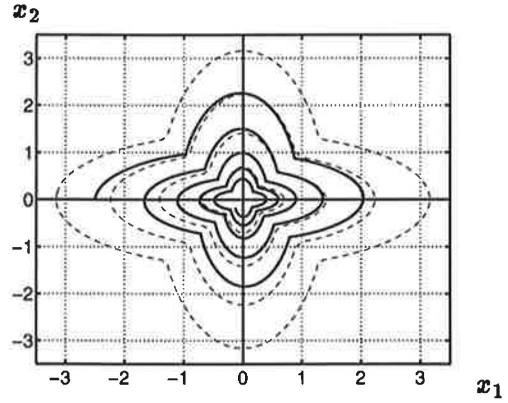

Figure 6 Initial value simulation (full) and level surfaces of the computed Lyapunov function (dashed).

5. Piecewise Affine Systems

In many situations, the requirements that the dynamics should be linear and that the cell boundaries should cross through origin are too restrictive. For example, approximating a nonlinear function by linearizations around a set of operating conditions results in a piecewise affine system. In hybrid systems, switches often occur when state variables exceed certain limits, which is typically stated as affine conditions in the continuous state. Thus, it is important to consider stability of piecewise affine systems. Conditions for continuous piecewise quadratic Lyapunov functions for such systems are formulated next.

THEOREM 2

For every index $i \in I$, let $C_i \subset \mathbb{R}^{p\times n}$, $D_i \in \mathbb{R}^p$ and let X_i be a closed subset of $\mathbb{R}^n$. Suppose that $C_i x + D_i \geq 0$ for $x \in X_i$ and $C_i x + D_i = C_j x + D_j$ for $x \in X_i \cap X_j$, $i, j \in I$. If there exist $\epsilon > 0$, vectors r_i and matrices Q and R_i, where r_i and R_i have non-negative coefficients, such that

$$P_i = \begin{bmatrix} C_i & D_i \\ I & 0 \\ 0 & 1 \end{bmatrix}^T Q \begin{bmatrix} C_i & D_i \\ I & 0 \\ 0 & 1 \end{bmatrix}, \qquad i \in I$$

satisfy

$$\begin{bmatrix} \epsilon & 0 \\ 0 & 0 \end{bmatrix} \leq P_i \leq \begin{bmatrix} \epsilon^{-1}I & 0 \\ 0 & -(r_i)^T D_i \end{bmatrix} \tag{15}$$

$$\begin{bmatrix} -\epsilon & 0 \\ 0 & 0 \end{bmatrix} \geq P_i \begin{bmatrix} A_i & B_i \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} A_i & B_i \\ 0 & 0 \end{bmatrix}^T P_i + \begin{bmatrix} C_i & D_i \\ 0 & 1 \end{bmatrix}^T R_i \begin{bmatrix} C_i & D_i \\ 0 & 1 \end{bmatrix} \tag{16}$$

for $i \in I$, then every continuous piecewise C^1 trajectory in $\cup_{i \in I} X_i$ with

$$\dot{x}(t) = A_i x(t) + B_i \text{ for } x(t) \in X_i$$

tends to zero exponentially. □

REMARK 1
The non-strict inequalities in the formulation of this theorem can usually be replaced by strict inequalities in the numerical treatment. However, for all regions that contain $x = 0$, the last row and column of the corresponding P_i are zero, so one has to restrict to the upper left corner, before imposing srict inequalities. □

The proof of the theorem relies on the following standard lemma.

LEMMA 1
Let $V(t)$ be decreasing and piecewise C^1. If there exist $\alpha, \beta, \gamma > 0$ such that

$$\alpha |x(t)|^2 < V(t) < \beta |x(t)|^2$$
$$\frac{dV}{dt}(t) \leq -\gamma |x(t)|^2$$

then $|x(t)|^2 \leq \beta \alpha^{-1} e^{-\gamma t/\beta} |x(0)|^2$. □

Proof. The inequalities

$$\frac{dV}{dt} \leq -\gamma |x|^2 \leq -\frac{\gamma}{\beta} V$$

and the fact that V is positive imply that

$$\frac{d}{dt} \ln V \leq -\frac{\gamma}{\beta}$$

Hence

$$|x(t)|^2 \leq \frac{1}{\alpha} V(t) \leq \frac{V(0)}{\alpha} \exp\left(-\frac{\gamma}{\beta} t\right)$$

□

Proof of Theorem 2. Define $i(t)$ such that $x(t) \in X_{i(t)}$ and let

$$V(t) = \begin{bmatrix} x(t) \\ 1 \end{bmatrix}^T P_{i(t)} \begin{bmatrix} x(t) \\ 1 \end{bmatrix}$$

The upper bound for P_i implies that the last row and column are zero if $D_i \geq 0$, i.e. if $x = 0$ is admissible. Hence there exists a β for which the upper bound on V in Lemma 1 holds. The inequalities $\epsilon |x(t)|^2 \leq V(t)$ and $\dot{V}(t) \leq -\epsilon |x(t)|^2$ follow directly from (15) and (16) after multiplication by $(x, 1)$ from the left and right and noting that

$$\begin{bmatrix} C_i x + D_i \\ 1 \end{bmatrix}^T R_i \begin{bmatrix} C_i x + D_i \\ 1 \end{bmatrix} \geq 0, \qquad x \in X_i$$

because of the non-negative coefficients in R_i. Hence Lemma 1 completes the proof. □

An example of the application of Theorem 2 is given next.

Example 2—A Piecewise Affine System

Consider the piecewise affine system

$$\dot{x} = A_i x + B_i \tag{17}$$

where the dynamics are given by

$$A_1 = \begin{bmatrix} -10 & -10.5 \\ 10.5 & 9 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} -11 \\ 7.5 \end{bmatrix} \tag{18}$$

$$A_2 = \begin{bmatrix} -1 & -2.5 \\ 1 & -1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{19}$$

$$A_3 = \begin{bmatrix} -10 & -10.5 \\ 10.5 & -20 \end{bmatrix}, \qquad B_3 = \begin{bmatrix} 11.0 \\ 50.5 \end{bmatrix} \tag{20}$$

and the switching conditions are given by

$$i = \begin{cases} 1, & x_1 < -1 \\ 2, & -1 \le x_1 < 1 \\ 3, & 1 \le x_1 \end{cases} \tag{21}$$

Letting $z = [\,x \quad 1\,]$, the convex optimization routines return the following

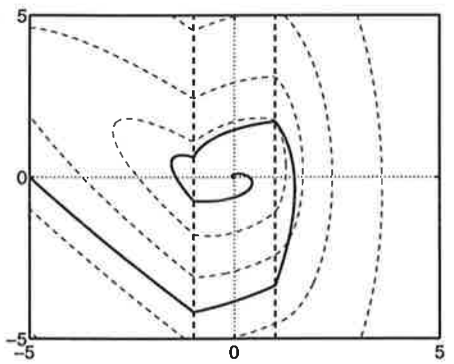

Figure 7 Initial value simulation and Lyapunov function level surfaces.

Lyapunov function.

$$V_1 = z^T \begin{bmatrix} 0.3434 & 0.3066 & 0.3379 \\ 0.3066 & 0.3408 & 0.4224 \\ 0.3379 & 0.4224 & 0.6796 \end{bmatrix} z \tag{22}$$

$$V_2 = z^T \begin{bmatrix} 0.3471 & -0.1159 & 0 \\ -0.1159 & 0.3408 & 0 \\ 0 & 0 & 0 \end{bmatrix} z \tag{23}$$

$$V_3 = z^T \begin{bmatrix} 2.8553 & 0.0850 & -1.9526 \\ 0.0850 & 0.3408 & -0.2008 \\ -1.9526 & -0.2008 & 1.3969 \end{bmatrix} z \tag{24}$$

The level surfaces of the computed Lyapunov function are indicated by the dashed lines in Figure 7. □

5.1 A Remark on Sliding Modes

Up to this point, we have disregarded the fact that piecewise affine systems may be of variable structure type, and as such can exhibit sliding modes along the cell boundaries [Utkin, 1977]. It can be interesting to investigate the implications of the derived stability criteria on the behaviour of the system along possible sliding modes.
According to [Utkin, 1977], the equations of the sliding modes on a cell boundary ∂X_i of r adjacent cells can be calculated by the equivalent dynamics

$$\dot{x}(t) = \sum_{j=1}^{r} \mu_j f(x(t), j) \qquad x \in \partial X_i \tag{25}$$

where the coefficients $\mu_j \geq 0$ and $\sum_{j=1}^{r} \mu_j = 1$ holds. It is simple to verify that the value of the continuous piecewise quadratic Lyapunov functions derived so far strictly decreases along the possible sliding modes of the system.

6. Lyapunov Functions for Hybrid Systems

It is trivial to extend the application of Theorem 1 to a class of hybrid systems.

THEOREM 3
For every $i \in I$ let $C_i \in \mathbb{R}^{p \times n}$. Let $x(t)$ be a continuous piecewise C^1 trajectory in $\mathbb{R}^n$ and let $i(t) \in I$ be piecewise constant and such that

$$\begin{aligned} \dot{x}(t) &= A_{i(t)} x(t) && \text{a.e.} \\ C_{i(t)} x(t) &\geq 0 && \forall t \\ C_{i(t-)} x(t) &= C_{i(t)} x(t) && \forall t \end{aligned}$$

If there exist matrices P, E and R_i, where R_i has non-negative coefficients, such that the matrices $P_i = P + C_i^T E + E^T C_i$ satisfy

$$\begin{aligned} 0 &< P_i \\ 0 &> P_i A_i + A_i^T P_i + C_i^T R_i C_i \end{aligned}$$

for $i \in I$, then $x(t)$ tends to zero exponentially. □

Proof. The result follows from Lemma 1 with $V(t) = x(t)^T P_{i(t)} x(t)$. □

For hybrid systems, there are many situations where continuous Lyapunov functions are too restrictive. Inspired by [Pettersson and Lennartson, 1996], we give the following motivating example.

EXAMPLE 3—HYBRID SYSTEM WITH SWITCHING STATE
Figure 8 shows an initial value simulation of the piecewise linear hybrid system given by

$$\dot{x}(t) = A_{i(t)} x(t) \tag{26}$$

$$i(t) = \begin{cases} 2, & \text{if } i(t-) = 1 \text{ and } (c_{1,2})^T x = 0 \\ 1, & \text{if } i(t-) = 2 \text{ and } (c_{2,1})^T x = 0 \end{cases} \tag{27}$$

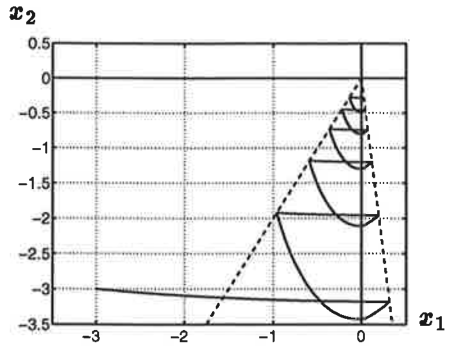

Figure 8 Trajectories of the hybrid system.

In this example, the switching boundaries are given by

$$c_{1,2} = [\,-10 \quad -1\,]^T \tag{28}$$
$$c_{2,1} = [\,2 \quad -1\,]^T \tag{29}$$

and the system matrices used are

$$A_1 = \begin{bmatrix} -1 & -100 \\ 10 & -1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & 10 \\ -100 & 1 \end{bmatrix} \tag{30}$$

The simulations indicate that the system is asymptotically stable. From the simulated trajectory of the system, however, it is also clear that it is not possible to find a single quadratic Lyapunov function that guarantees stability of the hybrid system. □

Despite the fact that there is no single Lyapunov function that assures stability of the hybrid system of Example 3, it is still possible to use Lyapunov theory to prove stability of the switching scheme. One way of doing this is to use multiple Lyapunov functions and require that the value of the Lyapunov functions should be non-increasing at the switching instants [Branicky, 1994]. The following discussion shows that these 'compatibility conditions' on the Lyapunov functions along the switching boundaries can be formulated as linear matrix inequalities.

Consider the situation depicted in Figure 9. Initially, the discrete state has the value i. The system trajectory evolves within the cell X_i, for which the Lyapunov function $V_i = x^T P_i x$ is valid. As the continuous state hits the cell boundary $(c_{i,j})^T x = 0$, the discrete state changes its value to j, and the Lyapunov function $V_j = x^T P_j x$ should be used. The condition that the value of the multiple Lyapunov function should be non-increasing at the switching instants can be written as

$$x^T P_j x \le x^T P_i x \qquad\qquad x : (c_{i,j})^T x = 0 \tag{31}$$

This condition can be expressed as the linear matrix inequality

$$P_i - P_j + c_{i,j}(e_{i,j})^T + (e_{i,j})(c_{i,j})^T \ge 0 \tag{32}$$

for $e_{i,j} \in \mathbb{R}^{1\times n}$. We formalize the discussion in the following stability theorem.

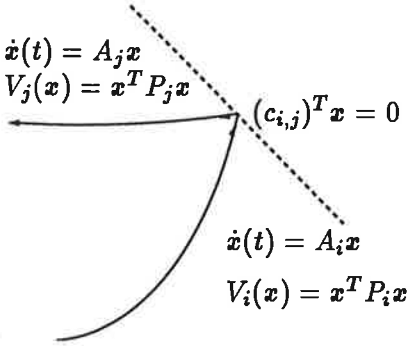

Figure 9 The switchings of Lyapunov functions occur as the continuous state hits a cell boundary.

THEOREM 4
For every $i, j \in I$ let $C_i \in \mathbb{R}^p$ and $c_{i,j} \in \mathbb{R}^n$ with $c_{i,i} = 0$ for all i. Let $x(t)$ be a continuous piecewise C^1 trajectory in $\mathbb{R}^n$ and let $i(t) \in I$ be piecewise constant and such that

$$\begin{aligned} \dot{x}(t) &= A_{i(t)} x(t) && \text{a.e.} \\ C_{i(t)} x(t) &\geq 0 && \forall t \\ c_{i(t-),i(t)} x(t) &= 0 && \forall t \end{aligned}$$

If there exist vectors $s_{i,j}$ and matrices P_i and R_i, where R_i has non-negative coefficients, such that

$$\begin{aligned} &0 < P_i \\ &0 \leq P_i - P_j + c_{i,j}(s_{i,j})^T + s_{i,j}(c_{i,j})^T \\ &0 > P_i A_i + A_i^T P_i + C_i^T R_i C_i \end{aligned}$$

for $i, j \in I$, then $x(t)$ tends to zero exponentially. □

Proof. The result follows from Lemma 1 with $V(t) = x(t)^T P_{i(t)} x(t)$. □

As an application of Theorem 4, we return to the hybrid system with a switching state.

EXAMPLE 4—STABILITY OF SWITCHING STATE SYSTEM
Letting

$$C = \begin{bmatrix} -10 & -1 \\ 2 & -1 \end{bmatrix} \tag{33}$$

The linear matrix inequalities of Theorem 4 for the system of Example 3 are readily found as

$$P_1, P_2 > 0 \tag{34}$$
$$P_1 - P_2 + c_{1,2}(e_{1,2})^T + e_{1,2}(c_{1,2})^T > 0 \tag{35}$$
$$P_2 - P_1 + c_{2,1}(e_{2,1})^T + e_{2,1}(c_{2,1})^T > 0 \tag{36}$$
$$A_1^T P_1 + P_1 A_1 + C^T \begin{bmatrix} 0 & r_1 \\ r_1 & 0 \end{bmatrix} C < 0 \tag{37}$$
$$A_2^T P_2 + P_2 A_2 + C^T \begin{bmatrix} 0 & r_2 \\ r_2 & 0 \end{bmatrix} C < 0 \tag{38}$$

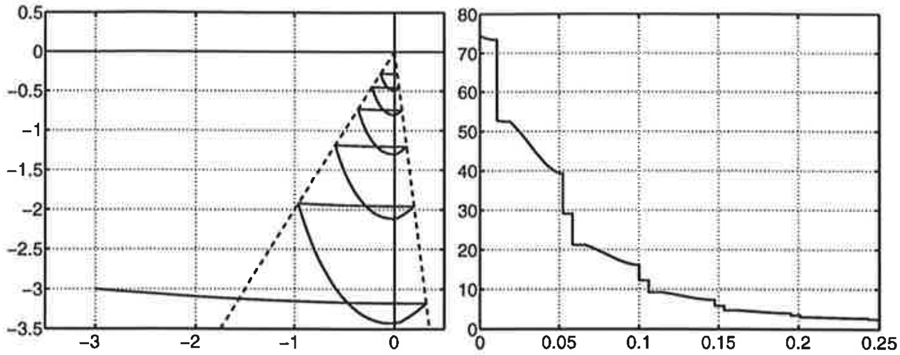

Figure 10 Initial value simulation and the corresponding values of the multiple Lyapunov function.

The convex optimization routines of LMI lab returns the feasible solution

$$P_1 = \begin{pmatrix} 0.8444 & 0.0665 \\ 0.0665 & 7.0790 \end{pmatrix} \tag{39}$$

$$P_2 = \begin{pmatrix} 32.2811 & -2.2415 \\ -2.2415 & 4.2587 \end{pmatrix} \tag{40}$$

A trajectory of the system is simulated and shown in Figure 10 along with the values of the active Lyapunov function as function of time. Notice the discontinuities in the value of the multiple Lyapunov function at the switching instants. □

The stability conditions for piecewise linear dynamics can be extended to the case of hybrid systems with piecewise affine dynamics.

THEOREM 5
For every $i, j \in I$ let $C_i \in \mathbb{R}^{p \times n}$, $D_i \in \mathbb{R}^p$, $c_{i,j} \in \mathbb{R}^n$ and $d_{i,j} \in \mathbb{R}$ with $c_{i,i} = 0$ and $d_{i,i} = 0$ for all i. Let $x(t)$ be a continuous piecewise C^1 trajectory in $\mathbb{R}^n$ and let $i(t) \in I$ be piecewise constant and such that

$$\begin{aligned} \dot{x}(t) &= A_{i(t)}x(t) + B_{i(t)} && \text{a.e.} \\ C_{i(t)}x(t) + D_{i(t)} &\geq 0 && \forall t \\ (c_{i(t-),i(t)})^T x(t) + d_{i(t-),i(t)} &= 0 && \forall t \end{aligned}$$

If there exist $\epsilon > 0$ vectors $s_{i,j}, r_i$ and matrices P_i and R_i, where r_i and R_i have non-negative coefficients, such that

$$\begin{bmatrix} \epsilon & 0 \\ 0 & 0 \end{bmatrix} \leq P_i \leq \begin{bmatrix} \epsilon^{-1}I & 0 \\ 0 & -(r_i)^T D_i \end{bmatrix} \tag{41}$$

$$0 \leq P_i - P_j + \begin{bmatrix} c_{i,j} \\ d_{i,j} \end{bmatrix} (s_{i,j})^T + s_{i,j} \begin{bmatrix} c_{i,j} \\ d_{i,j} \end{bmatrix}^T \tag{42}$$

$$\begin{bmatrix} -\epsilon & 0 \\ 0 & 0 \end{bmatrix} \geq P_i \begin{bmatrix} A_i & B_i \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} A_i & B_i \\ 0 & 0 \end{bmatrix}^T P_i + \begin{bmatrix} C_i & D_i \\ 0 & 1 \end{bmatrix}^T R_i \begin{bmatrix} C_i & D_i \\ 0 & 1 \end{bmatrix} \tag{43}$$

for $i, j \in I$, then $x(t)$ tends to zero exponentially. □

Proof. Define

$$V(t) = \begin{bmatrix} x(t) \\ 1 \end{bmatrix}^T P_{i(t)} \begin{bmatrix} x(t) \\ 1 \end{bmatrix}$$

The upper bound for P_i implies that the last row and column are zero if $D_i \geq 0$, i.e. if $x = 0$ is admissible. Hence there exists a β for which the upper bound on V in Lemma 1 holds. The derivative bound is obtained by multiplication of (43) from the left and right by $(x, 1)$. Hence the theorem follows from Lemma 1. □

7. Conclusions

The search for a piecewise quadratic Lyapunov functions for nonlinear and hybrid systems has been stated stated as a convex optimization problem in terms of linear matrix inequalities. The power of this approach appears to be very strong and we believe that the ideas can be generalized in a large number of directions including

- performance analysis
- global linearization
- controller optimization
- model approximation

and we hope to return to such issues in later publications.

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