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ANALYTICAL SOLUTION OF A SIMPLE DUAL CONTROL PROBLEM

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Lund Institute of Technology
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ANALYTICAL SOLUTION OF A SIMPLE DUAL CONTROL PROBLEM.

Jan Sternby

ABSTRACT

A stochastic control problem for which the optimal dual control law can be calculated analytically is given. The system is a four state Markov chain with transition probabilities that depend on the control variable. The performance of the optimal dual control law and of some suboptimal control laws are calculated and compared.

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TABLE OF CONTENTS

1. INTRODUCTION.

It is in general very difficult to solve control problems leading to dual control laws in the sense of Feldbaum (1965). A few examples of this type have been solved numerically by extensive computer calculations, see e.g. Bohlin (1969), Jacobs and Langdon (1969) or Åström and Wittenmark (1971). The motivation for solving these necessarily rather simple problems has been to obtain some insight into how dual control laws actually work and thus to understand how to make good suboptimal dual controllers for more difficult problems.

But numerical solutions do not, however, give as much and as detailed information as analytical solutions. For one thing, with just a numerical solution it is not known what happens between data points. For this reason an example is given in this report which is completely solvable over any time interval. With the analytical solution given, one can study in detail how the dual controller works. An analytical comparison is also made between the performance of different control strategies. However, this is a very special problem, and consequently nothing can be said in general about other and more realistic problems.

The example is based on governed Markov chains as in Åström (1965), but may also be looked upon as a simplification of the example in Jacobs and Langdon (1969). In Chapter 2 two problems are formulated corresponding to open loop control and closed loop control. Functional equations for the two cases are derived in Chapter 3, and are solved in Chapter 4. From this we derive different control laws, including an open loop feedback control, see e.g. Bar-Shalom and Sivan (1969), Tse and Athans (1972) or Ku and Athans (1973). The performance of these controls are analyzed in Chapter 5. The last chapter is a discussion of the results.

2. TWO PROBLEMS.

Consider a Markov chain with four states called x_1 to x_4. The transition probabilities depend on a control variable u ($0 \leq u \leq 1$), and are shown in Table 1.

next state \ current state	x_1	x_2	x_3	x_4
x_1	$p_1(u)$	$p_1(u)$	$p_2(u)$	$p_2(u)$
x_2	$p_3(u)$	$p_3(u)$	0	0
x_3	0	0	$p_4(u)$	$p_4(u)$
x_4	$p_5(u)$	$p_5(u)$	$p_6(u)$	$p_6(u)$

Table 1.

The functions p_i are given in Fig. 1.

For every u we have

$$p_1 + p_3 + p_5 = p_2 + p_4 + p_6 = 1$$

The p_i:s are chosen piecewisely linear in u to make the calculations simple and, for the same reason, some of the transition probabilities are identical and others are zero. The desire to achieve a dual effect in the resulting regulator also restricts the possible choices of p_i:s.

A loss function h(x) is now introduced that assigns a loss to each of the states as follows

$$h(x_1) = h(x_4) = 1; \quad h(x_2) = h(x_3) = 0$$

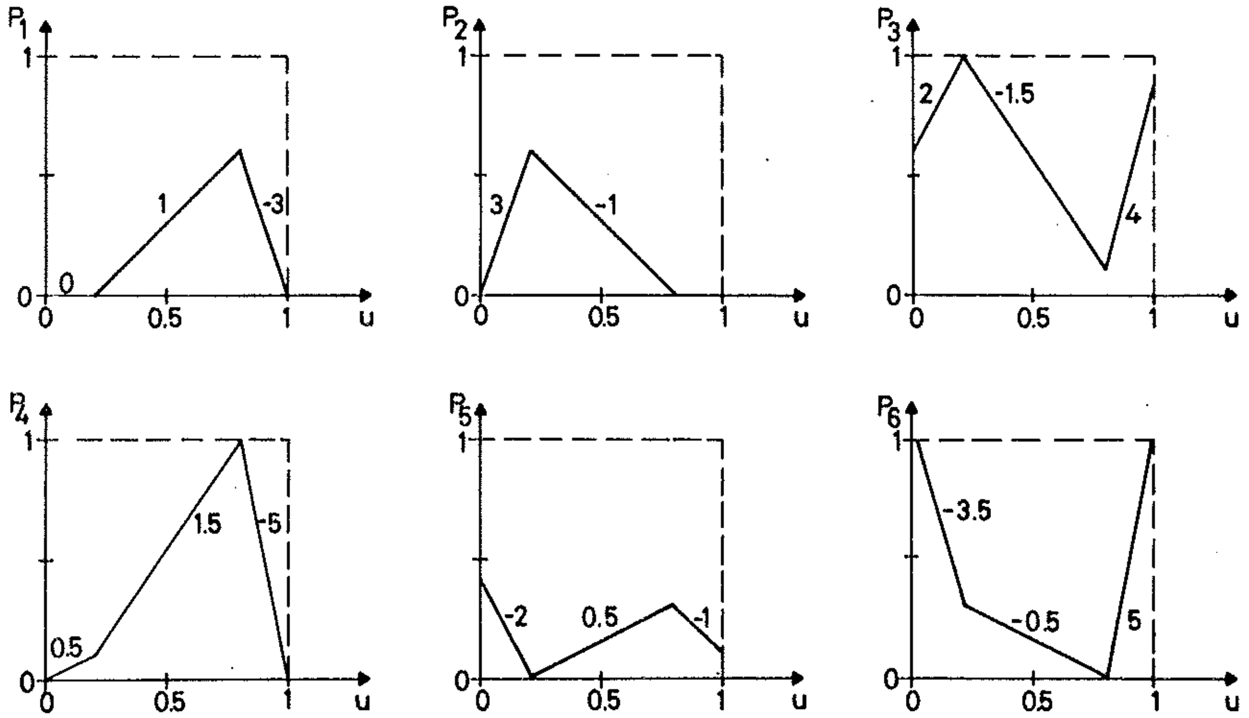

Fig. 1 - The transition probabilities as functions of u. Each line is marked by its slope.

The loss function thus puts the states together into two groups: x_1 and x_4 in one and x_2 and x_3 in another. For obvious reasons these groups will be called the one-states and the zero-states respectively.

Now we are ready to formulate the first problem, which is one of an open loop. No measurements are made and thus the only available information about the state is its initial probability distribution which is known in advance. The value of the state at time t is denoted by x^t.

Problem 1: Determine at $t = t_0$ a sequence $\bar{v}$ of values for the control variable to minimize the criterion

$$\bar{W}_n = E \sum_{t=t_0+1}^{t_0+n} h(x^t)$$

where n is chosen in advance.

The second problem is the corresponding closed loop situation. At every time t, $t_0+1 \leq t \leq t_0+n$, the value of the loss function is measured. This means that we are able to know whether the state is a zero-state or a one-state but we cannot separate at all the two states within the group.

Admissible control laws at time t may use all information available at that time, which is now the initial probability distribution for the state and the outcome of all measurements up to and including that at time t. Due to the Markov property this information is contained in the conditional probability distribution for x^{t+1}.

Now let us state

Problem 2: Find a sequence v of admissible control laws to minimize the criterion

$$W_n = E \sum_{t=t_0+1}^{t_0+n} h(x^t)$$

where n is chosen in advance.

This is the problem which leads to a dual control law.

3. DERIVATION OF FUNCTIONAL EQUATIONS.

In this chapter we will derive a functional equation for the minimal loss for each of the two problems in the previous section. In order to do this, dynamic programming in the number of steps is used. We then also find the resulting optimal control strategies. The system is completely time-invariant, and so we can always let the minimization in the dynamic programming equation be a minimization with respect to the initial control variable at $t = t_0$.

Since all the transition probabilities from states x_1 and x_2 are equal the future development of the process will be the same if the current state is x_1 or x_2. The same property is also true for states x_3 and x_4. Therefore we introduce q as the probability for the initial state to be x_1 or x_2. Then the corresponding probability for x_3 or x_4 will be 1-q.

3.1. The Open Loop Case.

If we put

$$\bar{F}_n = \min_{\bar{v}} \bar{W}_n$$

where n is the number of steps considered, $\bar{F}_n$ will be a function of q and

$$\bar{F}_1(q) = \min_{\bar{v}_0}\left\{q[p_1(\bar{v}_0) + p_5(\bar{v}_0)] + (1-q)[p_2(\bar{v}_0) + p_6(\bar{v}_0)]\right\} \quad (1)$$

The expression to be minimized with respect to $\bar{v}_0$ is simply the probability for the state x^{t_0+1} at time

$t = t_0 + 1$ to be a one-state, given that the probability for the initial state to be x_1 or x_2 is q.

Introduce $\bar{q}(\bar{v}_0)$ as

$\bar{q}(\bar{v}_0)$ = the probability for x^{t_0+1} (the state at time $t = t_0 + 1$) to be x_1 or x_2, if the control variable at $t = t_0$ is $\bar{v}_0$.

Then from Table 1

$$\bar{q}(\bar{v}_0) = q[p_1(\bar{v}_0) + p_3(\bar{v}_0)] + (1-q) \cdot p_2(\bar{v}_0) \tag{2}$$

Now, using dynamic programming the expected n-step loss will be computed as the sum of the immediate (one-step) loss and the minimal value of the expected n-1-step loss.

All transition probabilities and hence all state probability distributions are time-invariant. Therefore, the expected n-step loss $\bar{F}_n$ will be the same function of q for any value of the initial time t_0. Then we have for $n \geq 2$:

$$\bar{F}_n(q) = \min_{\bar{v}_0}\Big\{q[p_1(\bar{v}_0) + p_5(\bar{v}_0)] + $$

$$+ (1-q)[p_2(\bar{v}_0) + p_6(\bar{v}_0)] + \bar{F}_{n-1}\Big(\bar{q}(\bar{v}_0)\Big)\Big\} \tag{3}$$

The expression within brackets will be denoted by $\bar{J}_n(q,\bar{v}_0)$. In $\bar{J}_n(q,\bar{v}_0)$ the first terms are the immediate loss, i.e. $\bar{J}_1(q,\bar{v}_0)$, the probability for the next state to be a one-state. The last term is the expected loss for the last n-1 steps, since we know in advance that the probability for the state to be x_1 or x_2 at time $t = t_0 + 1$ will be $\bar{q}(\bar{v}_0)$.

3.2. The Closed Loop Case.

Let us now derive a similar equation to (3) for Problem 2. As in the previous section we put (without bars)

$$F_n = \min_{v} W_n$$

and obtain

$$F_1(q) = \min_{v_0}\Big\{ q[p_1(v_0) + p_5(v_0)] + $$

$$+ (1-q)[p_2(v_0) + p_6(v_0)]\Big\} \qquad (4)$$

Now, instead of $\bar{q}(\bar{v}_0)$, $q^0(v_0)$ and $q^1(v_0)$ are introduced as

$q^0(v_0)$ = the probability for x^{t_0+1} (the state at time $t = t_0 + 1$) to be x_2, given that it is a zero-state;

$q^1(v_0)$ = the probability for x^{t_0+1} to be x_1, given that it is a one-state.

By Bayes' rule this means that

$$q^0(v_0) = \frac{qp_3(v_0)}{qp_3(v_0) + (1-q)p_4(v_0)} \qquad (5)$$

and

$$q^1(v_0) = \frac{qp_1(v_0) + (1-q)p_2(v_0)}{q[p_1(v_0) + p_5(v_0)] + (1-q)[p_2(v_0) + p_6(v_0)]} \qquad (6)$$

Next introduce $P^0(v_0)$ and $P^1(v_0)$ as

$P^0(v_0)$ = the probability for x^{t_0+1} to be a zero-state if the control variable at $t = t_0$ was v_0;

$P^1(v_0)$ = the corresponding probability for a one-state.

In this notation we can rewrite (4) as

$$F_1(q) = \min_{v_0} P^1(v_0)$$

As for Problem 1 dynamic programming is used, but since we are going to measure $h(x^{t_0+1})$ and will learn if the state x^{t_0+1} is a zero-state or a one-state, the term corresponding to the last one in (3) can be split in two, one for each possible outcome of the measurement. Then for $n \geq 2$

$$F_n(q) = \min_{v_0}\Big\{P^1(v_0) + F_{n-1}\Big(q^1(v_0)\Big) \cdot P^1(v_0) + \\ + F_{n-1}\Big(q^0(v_0)\Big) \cdot P^0(v_0)\Big\} \quad (7)$$

Similarly to Problem 1 the expression within brackets will be denoted by $J_n(q,v_0)$. In it the first term is the immediate loss, while the last two terms add up to the expected loss for the next n-1 steps knowing that measurements are going to be made.

4. SOLUTIONS TO THE TWO PROBLEMS.

In both cases the one-step loss, $\bar{F}_1(q)$ and $F_1(q)$ respectively, is calculated by considering the slope of $\bar{J}(q,\bar{v}_0)$ $(J(q,v_0))$ as a function of $\bar{v}_0$ (v_0) for different values of q, and it is easily verified that

$$\bar{F}_1(q) = F_1(q) = \begin{cases} 0.9q & q \leq 1/2 \\ 0.9(1-q) & q \geq 1/2 \end{cases}$$

and

$$\bar{v}_0^{opt}(q) = v_0^{opt}(q) = \begin{cases} 0.8 & q < 1/2 \\ 0.2 & q > 1/2 \end{cases} \qquad \text{(either for } q = 1/2\text{)}$$

4.1. The Open Loop Case.

To obtain $\bar{F}_n(q)$ for $n \geq 2$ we need the following lemma.

Lemma 1: Let $\bar{F}_{n-1}(q) = \min[g_1(q), g_2(q)]$ with g_1 and g_2 linear in q, $p^1(\bar{v}_0)$ and $\bar{q}(\bar{v}_0)$ linear in $\bar{v}_0$. Then $\bar{H}_{n-1}(\bar{v}_0) = p^1(\bar{v}_0) + \bar{F}_{n-1}(\bar{q}(\bar{v}_0))$ is concave.

A proof of the lemma is found in Appendix 1.

Applying this lemma to (3) for n = 2 we find that for each value of q $\bar{J}_2(q,\bar{v}_0)$ is a concave function of $\bar{v}_0$ within each of the intervals where the transition probabilities are linear. Thus $\bar{J}_2$ is minimized for $\bar{v}_0 = 0$, 0.2, 0.8 or 1. Considering these four $\bar{v}_0$-values it can be shown that

$$\bar{F}_2(q) = \min[1.53q,\ 1.26(1-q)]$$

and

$$\bar{v}_0^{opt} = \begin{cases} 0.8 & q < 14/31 \\ 0.2 & q > 14/31 \end{cases} \quad \text{(either for } q = 14/31\text{)}$$

Now comparing $\bar{F}_1(q)$ and $\bar{F}_2(q)$ it seems reasonable to assume that for $n \geq 1$

$$\bar{F}_n(q) = \min\left[\bar{K}_1^n q,\ \bar{K}_2^n (1-q)\right]$$

with

$$\bar{v}_0^{opt} = \begin{cases} 0.8 & \text{small } q\text{'s} \\ 0.2 & \text{big } q\text{'s} \end{cases}$$

In Appendix 1 this is shown to be true with

$$\bar{K}_1^n = 3(1-0.7^n) \rightarrow 3 \qquad n \rightarrow \infty$$

$$\bar{K}_2^n = 1.5(1-0.4^n) \rightarrow 1.5 \qquad n \rightarrow \infty$$

The limiting optimal control is then

$$\bar{v}_0^{opt} = \begin{cases} 0.8 & q < 1/3 \\ 0.2 & q > 1/3 \end{cases} \quad \text{(either for } q = 1/3\text{)}$$

This control scheme can be used either in an open loop mode (if no measurements are made) or as the suboptimal closed loop control law called "open loop feedback control", Bar-Shalom and Sivan (1969), or "open loop feedback optimal control", Tse and Athans (1972) and Ku and Athans (1973). In Chapter 5 the performance is computed analytically for these two and for some other cases. Note that all control laws in this report will be discontinuous as functions of q. In the following they will be chosen to be right-continuous.

4.2. The Closed Loop Case.

As was shown in the beginning of this chapter we have

$$F_1(q) = \min[0.9q, 0.9(1-q)]$$

and

$$v_0^{opt} = \begin{cases} 0.8 & q < 1/2 \\ 0.2 & q \geq 1/2 \end{cases} \qquad \text{(for } q = 1/2 \quad 0.2 \text{ by choice)}$$

These choices of v_0 can be compared to the ones for a completely known state, i.e. $q = 0$ (x_3 or x_4) or $q = 1$ (x_1 or x_2). For $q = 0$ we find $v_0 = 0.8$ by minimizing $p_2 + p_6$ of Fig. 1 and similarly $v_0 = 0.2$ for $q = 1$. The p-functions are chosen so that for these v_0-values (and $q = 0$ or 1) we are sure to arrive at a zero-state (i.e. no immediate loss), and, moreover, $q^0(v_0)$ and $q^1(v_0)$ will be zero or one, so that we will continue to know exactly where we are (i.e. no future loss either).

Thus the best one-step regulator chooses v_0 as if the most probable state was the true one.

To calculate $F_n(q)$ for $n \geq 2$ we need

Lemma 2: Let $F_{n-1}(q) = \min[g_1(q), \ldots, g_m(q)]$ with $g_1, \ldots, g_m$ linear in q. Also let $P^0(v_0)$, $P^1(v_0)$, $q^0(v_0)P^0(v_0)$ and $q^1(v_0)P^1(v_0)$ be linear in v_0. Then

$$H_{n-1}(v_0) = P^1(v_0) + F_{n-1}\Big(q^1(v_0)\Big)P^1(v_0) + $$
$$+ F_{n-1}\Big(q^0(v_0)\Big)P^0(v_0)$$

is concave.

This lemma is proved in Appendix 2.

Now Lemma 2 is applied to (7) for n = 2. Then for each value of q we find that $J_2(q,v_0)$ is a concave function of v_0 in each of the intervals where the transition probabilities are linear. Thus only $v_0 = 0$, 0.2, 0.8 or 1 could minimize $J_2(q,v_0)$. Considering these four v_0-values we have

$$F_2(q) = \begin{cases} 1.26q & 0 \leq q < 25/54 \\ 1-0.9q & 25/54 \leq q < 39/54 \\ 1.26(1-q) & 39/54 \leq q \leq 1 \end{cases}$$

and

$$v_0^{opt}(q) = \begin{cases} 0.8 & 0 \leq q < 25/54 \\ 1 & 25/54 \leq q < 39/54 \\ 0.2 & 39/54 \leq q \leq 1 \end{cases}$$

For q-values close to 1/2 $v_0 = 1$ is chosen. This v_0-value will never be used by the one-step regulator (or if the state is completely known), and so the two-step regulator is essentially different.

The value $v_0 = 1$ gives an identification step since $q^0(1) = 1$

and $q^1(1) = 0$, i.e. the exact state becomes known. Then the future loss will be zero.

Thus the two-step regulator works as follows: For q:s close to zero or one, i.e. good knowledge about the current state, v_0 will be chosen as by the one-step regulator, whereas for q:s close to 1/2, i.e. poor knowledge about the current state, an identification step will be taken.

Since the cost of choosing $v_0 = 1$ is $1 - 0.9q$ and the future loss will then be zero, one might guess that for $n \geqslant 2$

$$F_n(q) = \begin{cases} K^n q \\ 1 - 0.9q \\ K^n(1-q) \end{cases} \quad \text{with} \quad v_0^{opt} = \begin{cases} 0.8 & \text{small } q\text{'s} \\ 1 & q \sim 1/2 \\ 0.2 & \text{big } q\text{'s} \end{cases}$$

As a matter of fact the transition probabilities p were chosen to give $F_n(q) = 1 - 0.9q$ when $q \sim 1/2$ for all n by assuring that the total knowledge about the state gained in the identification step is preserved in the future (to give no loss).

In Appendix 2 it is shown by induction that the guess is correct and that

$$K^n = 1.4(1-0.1^{n-1}) \rightarrow 1.4 \qquad n \rightarrow \infty$$

The function F_n is shown in Fig. 2 for some values of n. Note that the "identification loss line" 1-0.9q is the same for all $n \geqslant 2$.

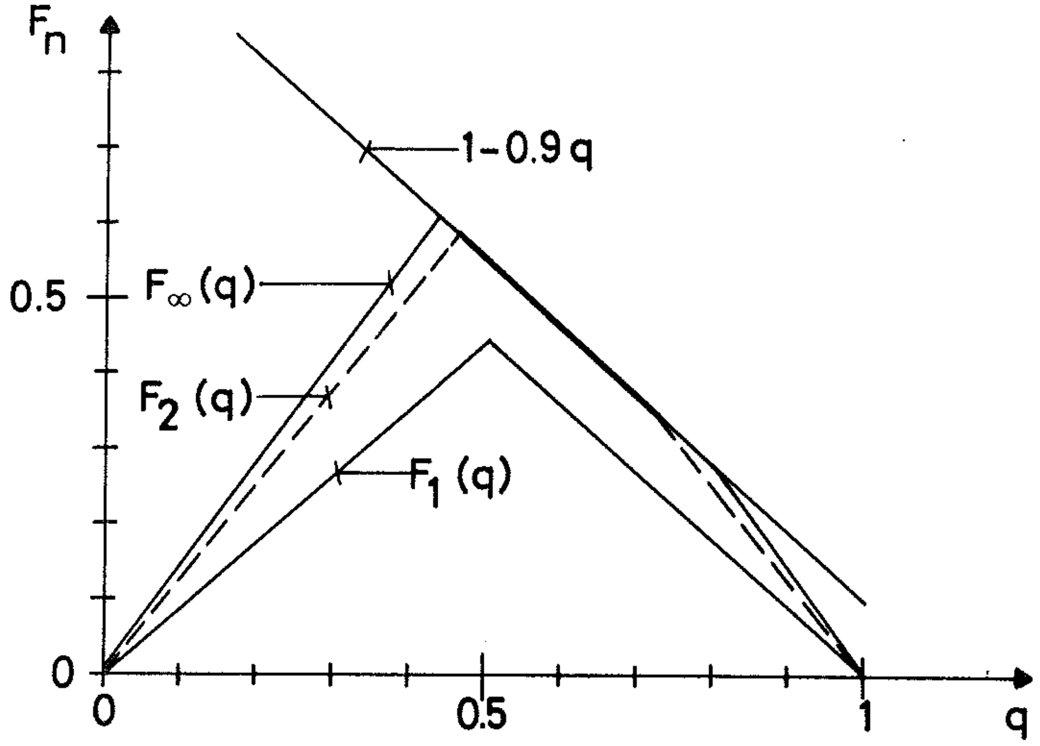

Fig. 2 - Graphs of the function F_n for some values of n.

It is instructive to study J_∞ as a function of v_0 for different values of q

$$\begin{aligned}J_\infty(q,v_0) = {} & q[p_1(v_0)+p_5(v_0)] + (1-q)[p_2(v_0)+p_6(v_0)] + \\ & + F_\infty\Big(q^0(v_0)\Big)[qp_3(v_0)+(1-q)p_4(v_0)] + \\ & + F_\infty\Big(q^1(v_0)\Big)\Big\{q[p_1(v_0)+p_5(v_0)] + \\ & + (1-q)[p_2(v_0)+p_6(v_0)]\Big\}\end{aligned}$$

This function is shown in Fig. 3 for q = 0.4, 0.5, 0.7, 0.8 and 0.9.

In Bohlin (1969) it is shown that a dual control law may be discontinuous as a function of the current hyperstate due to several local minima of the loss function.

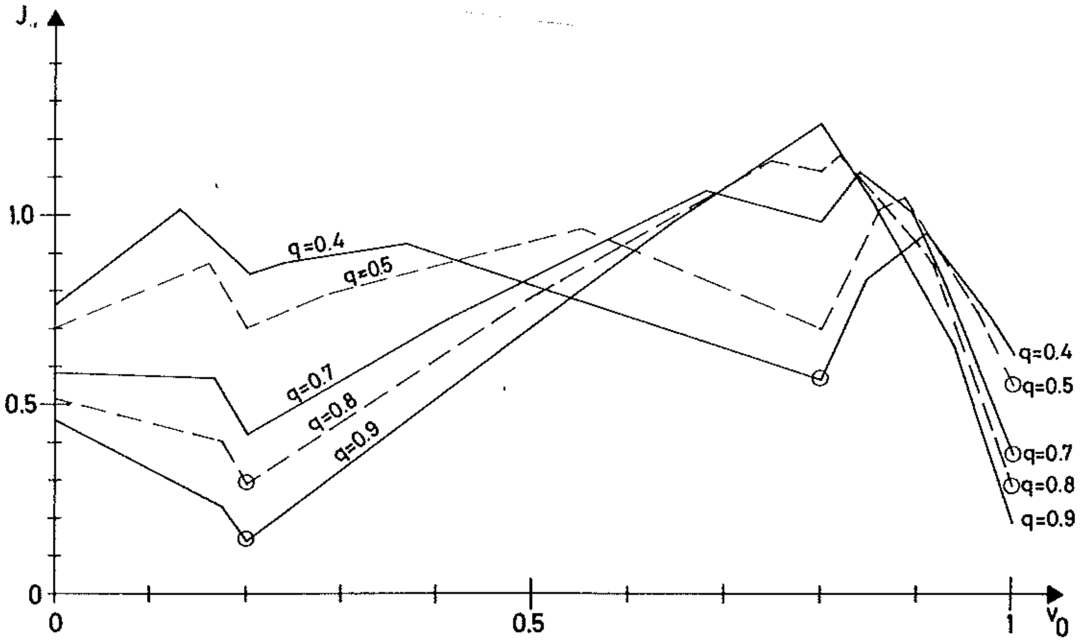

Fig. 3 - The expected loss for infinitely many steps as a function of v_0 plotted for some values of q. The rings mark out the global minimum for each q.

For the example discussed here, the discontinuity in $v_0(q)$ between $v_0 = 0.2$ and $v_0 = 0.8$ is not a dual effect, but is due to the choice of piecewisely linear transition probabilities. One can, however, imagine that nonlinear p:s might make one local minimum go continuously from $v_0 = 0.8$ to $v_0 = 0.2$, but for certain q:s the local minimum at $v_0 = 1$ would be smaller, thus giving a discontinuous control law in any case.

In the next chapter the performance of the system is computed analytically when using the one-step regulator, the two-step regulator and the optimal regulator.

5. COMPARISON OF DIFFERENT STRATEGIES.

In this chapter the expected n-step loss is calculated using six different control strategies. The loss will be denoted by $V_n^j(q)$, j = a, b, c, d, e or f according to

V^a - open loop control (no measurements)

V^b - open loop feedback control

V^c - one-step regulator

V^d - two-step regulator

V^e - approximate multistep regulator

V^f - optimal dual regulator

The approximate multistep regulator is calculated in the following way. When equation (7) is minimized to give v_0 the optimal open loop loss for an infinite number of steps, $\bar{F}_\infty$, is used instead of the corresponding closed loop loss. This corresponds closely to what is done for more general systems in Tse, Bar-Shalom and Meier (1973). A similar method giving a "neutral" control is proposed in Jacobs (1974).

To calculate the V:s an equation similar to (3) (Case a) or (7) is used, where the minimization w.r.t. v_0 is removed. Instead the v_0-value inserted should be some function of q, depending on which regulator is used.

In this chapter v_k will denote the control used at time $t = t_0 + k$, a function of q_k, the a posteriori probability for the state at time $t = t_0 + k$ to be x_1 or x_2.

5.1. Open Loop Control.

The control law is now

$$v_k = \begin{cases} 0.8 & q_k < 1/3 \\ 0.2 & q_k \geq 1/3 \end{cases}$$

No measurements are made, and so q_k must be computed from a formula similar to (2)

$$q_{k+1} = q_k[p_1(v_k) + p_3(v_k)] + (1-q_k)p_2(v_k)$$

Thus we can, of course, calculate all q_k:s and v_k:s in advance.

As is shown in Appendix 3, $\bar{q}$ will be less (greater) than 1/3 if q is, and so <u>all</u> the v_k:s will be equal.
From Appendix 3

$$V_n^a = \begin{cases} 3(1-0.7^n)q & q < 1/3 \\ 1.5(1-0.4^n)(1-q) & q \geq 1/3 \end{cases} \rightarrow \begin{cases} 3q \\ 1.5(1-q) \end{cases} \quad \text{as } n \rightarrow \infty$$

As expected $V_\infty^a = \bar{F}_\infty$ of Chapter 4.

5.2. Open Loop Feedback Control.

Again the regulator is

$$v_k = \begin{cases} 0.8 & q_k < 1/3 \\ 0.2 & q_k \geq 1/3 \end{cases}$$

but now the measurements are used to up-date q_k so that, depending on the measurements, an equation similar to (5) or (6) will be used where $q^0(v_0)$ and $q^1(v_0)$ are replaced by q_{k+1}, v_0 by v_k and q by q_k. Then from Appendix 4

$$v_n^b = \begin{cases} 1.5(1-0.4^n)q & q < 1/3 \\ 1.5(1-0.4^n)(1-q) & q \geq 1/3 \end{cases}$$

5.3. One-Step Regulator.

Now we use

$$v_k = \begin{cases} 0.8 & q_k < 1/2 \\ 0.2 & q_k \geq 1/2 \end{cases}$$

where q_k is calculated as in 5.2. From Appendix 4

$$v_n^c = \begin{cases} 1.5(1-0.4^n)q & q < 1/2 \\ 1.5(1-0.4^n)(1-q) & q \geq 1/2 \end{cases}$$

5.4. Two-Step Regulator.

The control law is

$$v_k = \begin{cases} 0.8 & 0 \leq q_k < 25/54 & (\sim 0 \leq q_k < 0.46) \\ 1 & 25/54 \leq q_k < 39/54 & (\sim 0.46 \leq q_k < 0.72) \\ 0.2 & 39/54 \leq q_k \leq 1 & (\sim 0.72 \leq q_k \leq 1) \end{cases}$$

and q_k is again computed as in 5.2. From Appendix 4 we get

$$V_n^d = \begin{cases} (1.4-5\cdot 0.1^n)q & 0 \leq q < 25/54 \\ 1 - 0.9q & 25/54 \leq q < 39/54 \\ (1.4-5\cdot 0.1^n)(1-q) & 39/54 \leq q \leq 1 \end{cases}$$

5.5. Approximate Multistep Regulator.

To obtain the control law we have to minimize

$$J^*(q_k,v_k) = P^1(v_k) + \bar{F}_\infty\Big(q^1(v_k)\Big)P^1(v_k) + \bar{F}_\infty\Big(q^0(v_k)\Big)P^0(v_k)$$

with respect to v_k.

According to Lemma 2 the minimizing v_k must be 0, 0.2, 0.8 or 1. Considering these four v_k-values we find

$$v_k = \begin{cases} 0.8 & 0 \leq q_k < 120/306 & (\sim 0 \leq q_k < 0.39) \\ 1 & 120/306 \leq q_k < 255/306 & (\sim 0.39 \leq q_k < 0.83) \\ 0.2 & 255/306 \leq q_k \leq 1 & (\sim 0.83 \leq q_k \leq 1) \end{cases}$$

As previously q_k should be computed as in 5.2. From Appendix 4 we have

$$V_n^e = \begin{cases} (1.4-5\cdot 0.1^n)q & 0 \leq q < 120/306 \\ 1 - 0.9q & 120/306 \leq q < 255/306 \\ (1.4-5\cdot 0.1^n)(1-q) & 255/306 \leq q \leq 1 \end{cases}$$

5.6. Optimal Dual Regulator.

This regulator is

$$v_k = \begin{cases} 0.8 & 0 \leq q_k < 50/115 \quad (\sim 0 \leq q_k < 0.43) \\ 1 & 50/115 \leq q_k < 92/115 \quad (\sim 0.43 \leq q_k < 0.80) \\ 0.2 & 92/115 \leq q_k \leq 1 \quad (\sim 0.80 \leq q_k \leq 1) \end{cases}$$

with q_k as in 5.2 - 5.4. Appendix 4 gives

$$V_n^f = \begin{cases} (1.4-5\cdot 0.1^n)q & 0 \leq q < 50/115 \\ 1 - 0.9q & 50/115 \leq q < 92/115 \\ (1.4-5\cdot 0.1^n)(1-q) & 92/115 \leq q \leq 1 \end{cases}$$

Now let us compare the expected loss for infinitely many steps when using these six regulators. This is done in Fig. 4.

From Fig. 4 we can classify the six regulators into three groups. The first group is just the open loop control, which, of course, gives the biggest loss.
The second group consists of the open loop feedback control and the one-step regulator. For these two the slope of V_∞ is ±1.5 depending on the q. These strategies do not use any identification steps and so this type of regulator is called passively adaptive by Bar-Shalom and Tse (1974). Note that the one-step regulator is better than open loop feedback control for this example.
The third group consists of the two-step regulator, the approximate multistep regulator and the optimal dual control. Here the slope of V_∞ is ±1.4 for small and big q's, but the loss is decreased by taking an identification step when q_k is close to 1/2. These controls are called actively adaptive in the terms of Bar-Shalom and Tse (1974).

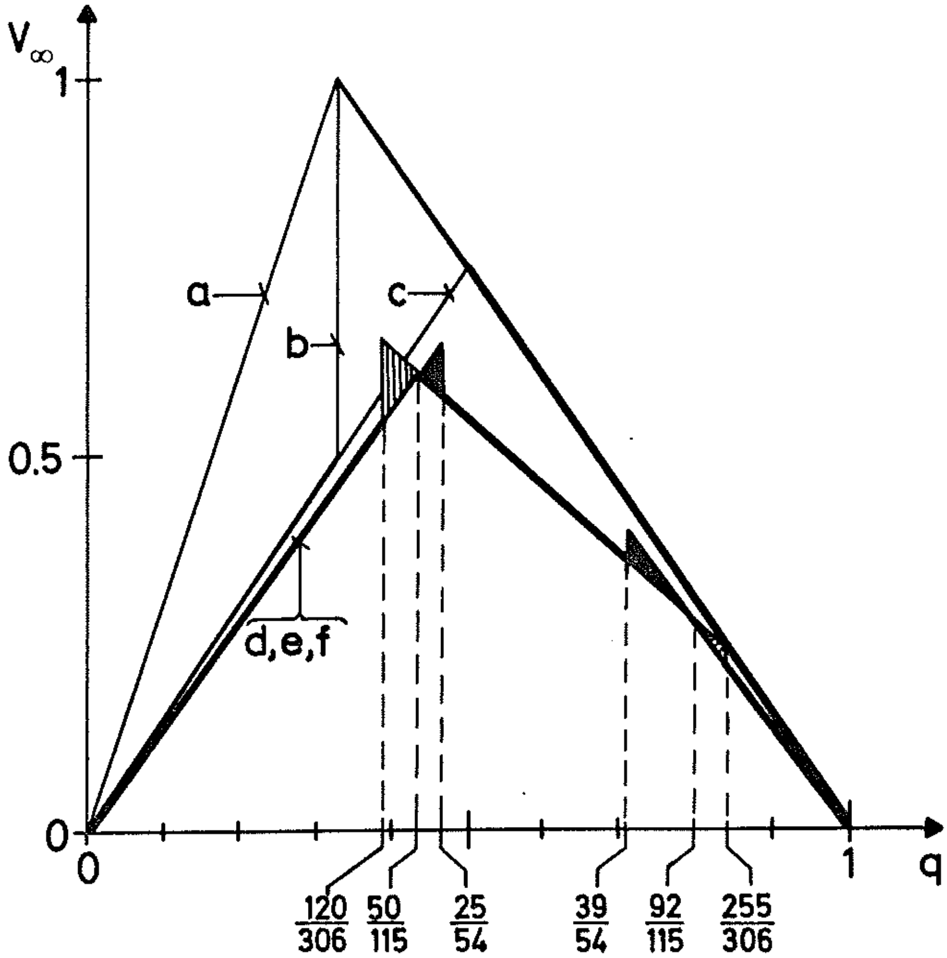

Fig. 4 - The expected loss for infinitely many steps, V_∞, as a function of the probability for the initial state to be x_1 or x_2 when using different control strategies.

a - open loop control
b - open loop feedback control
c - one-step regulator
d - two-step regulator
e - approximate multistep regulator
f - optimal dual regulator

The bottom curve is f, from which d and e differ only by the two black and dashed areas respectively.

Note that most of the dual effect is present for the two suboptimal dual controllers. The difference between the three can be explained as follows:

The two-step regulator is designed to be used for only two steps, and the need for identification is therefore less than with the optimal dual controller. This means that q must be closer to 1/2 before an identification step is taken.

With the approximate multistep regulator, however, we must try to find a good estimate of the state immediately, since the controller is designed as if no measurements are made after the first one.

6. CONCLUDING REMARKS.

The most interesting feature of the example in this report is that the problem is solvable analytically. This means that it has been possible to examine in detail how different strategies work.

It turns out that in this case the best one-step regulator is equivalent to certainty equivalence control. The best two-step regulator, however, is essentially different in that it gives the possibility for making identification steps. Having this feature built in it performs nearly as good as the optimal dual regulator. The same thing is also true for the approximate multistep regulator.

An interesting detail is that for this example the one-step regulator performs better than open loop feedback control, but, of course, the problem is a very special one, and again nothing can be said in general.

Nevertheless, as the analytical expressions are given it is also possible to see how the dual control law is created. The expected loss for the next n steps has three local minima as a function of the control variable. Two of these correspond to control actions taken when knowledge about the state is good, while the third one corresponds to a control giving an identification step. For the one-step regulator and the open loop feedback control this last minimum is never the lowest one, and so this value for the control variable is never used. Multiple minima in the expected loss as a function of the control variable are also reported by Bohlin (1969) for a different example.

Since the two-step regulator and the approximate multistep regulator both give nearly minimal loss, it seems as if a good way to derive suboptimal dual regulators is to include some approximation of the future loss when taking expectation and minimizing in order to find the current value of the control variable. However, the example in this report is a special one and for a more general case we can only say that it may be interesting to examine the effect of such approximations.

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APPENDIX 1

Calculation of the Optimal Open Loop Control Program.

As is said in Chapter 4 it is easily found that

$$\bar{F}_1(q) = \min[0.9q,\ 0.9(1-q)]$$

and

$$\bar{v}_0^{opt} = \begin{cases} 0.8 & q < 1/2 \\ 0.2 & q > 1/2 \end{cases} \qquad \text{(either for } q = 1/2)$$

To continue we need Lemma 1 (see page 9).

Proof of Lemma 1:

$$\bar{F}_{n-1}(\bar{q}(\bar{v}_0)) = \min\left[g_1(\bar{q}(\bar{v}_0)),\ g_2(\bar{q}(\bar{v}_0))\right]$$

Since g_1, g_2 and $\bar{q}$ are all linear $\bar{F}_{n-1}(\bar{q}(\bar{v}_0))$ is the minimum of two straight lines and is thus concave. $P^1(\bar{v}_0)$ is concave (linear) and so $\bar{H}_{n-1}$ is concave being the sum of two concave functions.

Now we want to prove by induction that

$$\bar{F}_n(q) = \min\left[\bar{K}_1^n q,\ \bar{K}_2^n(1-q)\right]$$

where

$\bar{K}_1^n \in [0.9,3]$, $\bar{K}_2^n \in [0.9,1.5]$, $\bar{K}_1^n \geq \bar{K}_2^n$ and $\bar{K}_1^n \geq 1$ for $n \geq 2$.

It is true for n = 1. Suppose it is true for n. Then

$$\bar{J}_{n+1}(q,\bar{v}_0) = q[p_1(\bar{v}_0)+p_5(\bar{v}_0)] + (1-q)[p_2(\bar{v}_0)+p_6(\bar{v}_0)] + $$
$$+ \bar{F}_n\left(\bar{q}(\bar{v}_0)\right)$$

from (3). By Lemma 1 with all terms except the last one included in $P^1(\bar{v}_0)$, $\bar{J}_{n+1}(q,\bar{v}_0)$ is a concave function of $\bar{v}_0$ in each of the intervals [0,0.2], [0.2,0.8] and [0.8,1]. This means that $\bar{J}_{n+1}(q,\bar{v}_0)$ is minimized for $\bar{v}_0$ = 0, 0.2, 0.8 or 1.

We have

$$\bar{q} = \begin{cases} 0.6q \\ 0.6 + 0.4q \\ 0.7q \\ 0.9q \end{cases} \text{ and } P^1 = \begin{cases} 1 - 0.6q \\ 0.9(1-q) \\ 0.9q \\ 1 - 0.9q \end{cases} \text{ for } \bar{v}_0 = \begin{cases} 0 \\ 0.2 \\ 0.8 \\ 1 \end{cases}$$

and so

$$\bar{J}_{n+1}(q,\bar{v}_0) = \begin{cases} \min[1 - 0.6q + \bar{K}_1^n \cdot 0.6q \quad , 1 - 0.6q + \bar{K}_2^n(1-0.6q)] \\ \min[0.9(1-q) + \bar{K}_1^n(0.6+0.4q), 0.9(1-q) + 0.4\bar{K}_2^n(1-q)] \\ \min[0.9q + \bar{K}_1^n \cdot 0.7q \quad , 0.9q + \bar{K}_2^n(1-0.7q)] \\ \min[1 - 0.9q + \bar{K}_1^n \cdot 0.9q \quad , 1 - 0.9q + \bar{K}_2^n(1-0.9q)] \end{cases}$$

for these $\bar{v}_0$-values. This matrix of possible expressions for $\bar{J}_{n+1}(q,\bar{v}_0)$ is denoted by $\bar{S}$.

Then

$\bar{S}_{21} > \bar{S}_{22}$ since $\bar{K}_1^n \geq \bar{K}_2^n$

$\bar{S}_{12} \geq \bar{S}_{42}$

$$\begin{cases} \bar{S}_{41} \geq \bar{S}_{11} \text{ if } \bar{K}_1^n \geq 1 \text{ (i.e. } n \geq 2) \\ \bar{S}_{41} > \bar{S}_{31} \text{ if } n = 1 \; (\bar{K}_1^1 = 0.9) \text{ and } q \leq 1/2 \\ \bar{S}_{41} > \bar{S}_{42} \text{ if } n = 1 \; (\bar{K}_2^1 = 0.9) \text{ and } q > 1/2 \end{cases}$$

$$\bar{S}_{42} > \bar{S}_{22} \text{ since } \bar{S}_{42} = (1+\bar{K}_2^n)(1-0.9q)$$
$$\text{and } \quad \bar{S}_{22} = (0.9+0.4\bar{K}_2^n)(1-q)$$

$$\begin{cases} \bar{S}_{11} > \bar{S}_{31} \text{ for } q < 1/3 \text{ since it is true for } q = 0, \\ \qquad \text{and } \bar{S}_{11} \neq \bar{S}_{31} \text{ for } q \in [0,1/3] \\ \bar{S}_{11} > \bar{S}_{22} \text{ for } q \geq 1/3 \text{ and } \bar{K}_1^n \geq 1, \text{ i.e. } n \geq 2 \text{ since} \\ \qquad \bar{S}_{11} = 1 + 0.6(\bar{K}_1^n - 1) \geq 1 \text{ and} \\ \qquad \bar{S}_{22} = (0.9+0.4\bar{K}_2^n)(1-q) \leq 1.5(1-q) \leq 1; \; q \geq 1/3 \\ \bar{S}_{11} > \bar{S}_{22} \text{ for } q \geq 1/3 \text{ and } \bar{K}_1^n = \bar{K}_2^n = 0.9, \text{ i.e. } n = 1 \text{ since} \\ \qquad \bar{S}_{11} = 1 - 0.06q \text{ and } \bar{S}_{22} = 1.26(1-q) \end{cases}$$

$$\begin{cases} \bar{S}_{32} > \bar{S}_{22} \text{ for } q > \frac{4}{17} \text{ since } \bar{K}_2^n \in [0.9,\ 1.5] \\ \bar{S}_{32} > 0.9 \cdot q + 0.5 > \bar{S}_{31} \text{ for } q \leq \frac{4}{17} \end{cases}$$

Thus $\bar{S}_{21}$, $\bar{S}_{12}$, $\bar{S}_{41}$, $\bar{S}_{42}$, $\bar{S}_{11}$ and $\bar{S}_{32}$ cannot be the minimum of $\bar{J}_{n+1}(q,\bar{v}_0)$, so that

$$\bar{F}_{n+1}(q) = \min[\bar{S}_{31},\bar{S}_{22}] =$$
$$= \min\left[(0.9+0.7\bar{K}_1^n)q,\ (0.9+0.4\bar{K}_2^n)(1-q)\right] =$$
$$= \min\left[\bar{K}_1^{n+1}q,\ \bar{K}_2^{n+1}(1-q)\right]$$

where

$$\bar{K}_1^{n+1} = 0.9 + 0.7\bar{K}_1^n \quad \text{and} \quad \bar{K}_2^{n+1} = 0.9 + 0.4\bar{K}_2^n$$

Now $\bar{K}_1^n \in [0.9,3] \Rightarrow \bar{K}_1^{n+1} \in [0.9,3]$

$\bar{K}_2^n \in [0.9,1.5] \Rightarrow \bar{K}_2^{n+1} \in [0.9,1.5]$

$$\bar{K}_1^n \geq \bar{K}_2^n \Rightarrow \bar{K}_1^{n+1} \geq \bar{K}_2^{n+1}$$
$$\bar{K}_1^n \geq 0.9 \Rightarrow \bar{K}_1^{n+1} \geq 1 \text{ so that } \bar{K}_1^n \geq 1 \text{ for } n \geq 2$$

This completes the induction.

$\bar{S}_{31}$ and $\bar{S}_{22}$ correspond to $\bar{v}_0 = 0.8$ and 0.2 respectively so that

$$\bar{v}_0^{opt} = \begin{cases} 0.8 \\ 0.2 \end{cases} \quad \text{when} \quad \bar{F}_n(q) = \begin{cases} \bar{K}_1^n q \\ \bar{K}_2^n (1-q) \end{cases}$$

Finally

$$\bar{K}_1^{n+1} = 0.9 + 0.7\bar{K}_1^n \text{ with } \bar{K}_1^1 = 0.9 \Rightarrow \bar{K}_1^n = 3(1-0.7^n)$$

and

$$\bar{K}_2^{n+1} = 0.9 + 0.4\bar{K}_2^n \text{ with } \bar{K}_2^1 = 0.9 \Rightarrow \bar{K}_2^n = 1.5(1-0.4^n)$$

APPENDIX 2

Calculation of the Optimal Dual Control Law.

Proof of Lemma 2 (see page 12 for the statement):

$$F_{n-1}(q) = \min[g_1(q), g_2(q), \ldots, g_m(q)]$$

Then for $j = 0, 1$ and for every v_0

$$F_{n-1}\left(q^j(v_0)\right)P^j(v_0) = \min\left[g_1\left(q^j(v_0)\right), \ldots, g_m\left(q^j(v_0)\right)\right]P^j(v_0) =$$
$$= \min\left[g_1\left(q^j(v_0)\right)P^j(v_0), \ldots, g_m\left(q^j(v_0)\right)P^j(v_0)\right]$$

Now, since all the g_i:s are linear and $q^0(v_0)P^0(v_0)$ and $q^1(v_0)P^1(v_0)$ are also linear according to assumptions $F_{n-1}(q^j(v_0))P^j(v_0)$ will be the minimum of m straight lines and therefore concave. Since $P^1(v_0)$ is also concave (linear) $H_{n-1}(v_0)$ will be concave being the sum of three concave functions. This completes the proof.

□

Now we can compute $F_2(q)$. From (7) we get

$$J_2(q,v_0) = P^1(v_0) + F_1\left(q^{1'}(v_0)\right)P^1(v_0) + F_1\left(q^0(v_0)\right)P^0(v_0)$$

where $F_1(q) = \min[0.9q, 0.9(1-q)]$. Then for each q by Lemma 2 $J_2(q,v_0)$ is a concave function of v_0 in each of the intervals [0,0.2], [0.2,0.8] and [0.8,1] so that $J_2(q,v_0)$ is minimized for $v_0 = 0, 0.2, 0.8$ or 1.

Thus we need P^0, P^1, q^0 and q^1 for these v_0-values.

$$p^0 = \begin{cases} 0.6q \\ q + 0.1(1-q) \\ 1 - 0.9q \\ 0.9q \end{cases} \quad p^1 = \begin{cases} 1 - 0.6q \\ 0.9(1-q) \\ 0.9q \\ 1 - 0.9q \end{cases} \quad \text{for } v_0 = \begin{cases} 0 \\ 0.2 \\ 0.8 \\ 1 \end{cases} \qquad (2.1)$$

and

$$q^0 = \begin{cases} 1 \\ \dfrac{q}{q + 0.1(1-q)} \\ \dfrac{0.1q}{1 - 0.9q} \\ 1 \end{cases} \quad q^1 = \begin{cases} 0 \\ \dfrac{2}{3} \\ \dfrac{2}{3} \\ 0 \end{cases} \quad \text{for } v_0 = \begin{cases} 0 \\ 0.2 \\ 0.8 \\ 1 \end{cases} \qquad (2.2)$$

$F_1(1) = F_1(0) = 0$ and $F_1(2/3) = 0.9\left(1 - (2/3)\right) = 0.3$ so that

$$J_2(q,v_0) = \begin{cases} 1 - 0.6q \\ 0.9(1-q) + 0.3\cdot 0.9(1-q) + 0.9\cdot\min[q, 0.1(1-q)] \\ 0.9q + 0.3\cdot 0.9q + 0.9\cdot\min[0.1q, 1-q] \\ 1 - 0.9q \end{cases}$$

for these four v_0-values. This means that

$$F_2(q) = \min[1 - 0.6q,\ 1.3\cdot 0.9(1-q) + 0.9q,$$

$$1.3\cdot 0.9(1-q) + 0.9\cdot 0.1(1-q),$$

$$1.3\cdot 0.9q + 0.9\cdot 0.1q,$$

$$1.3\cdot 0.9q + 0.9(1-q),\ 1 - 0.9q] =$$

$$= \min[S_1, \ldots, S_6]$$

Now $S_1 \geq S_6$

For $q \leq 1/2$:

$S_2 \geq S_4$ and $S_5 \geq S_4$

and for $q \geq 1/2$:

$S_2 \geq S_3$ and $S_5 \geq S_3$

and so S_1, S_2 and S_5 can be removed such that

$$F_2(q) = \min[1.26q,\ 1-0.9q,\ 1.26(1-q)]$$

with

$$v_0^{opt} = \begin{cases} 0.8 \\ 1 \\ 0.2 \end{cases} \quad \text{when} \quad F_2(q) = \begin{cases} 1.26q \\ 1-0.9q \\ 1.26(1-q) \end{cases}$$

□

By induction we will now show that

$$F_n(q) = \min[K^n q,\ 1-0.9q,\ K^n(1-q)]$$

for $n \geq 2$, where $K^n \in [1.26, 1.4]$. This is true for $n = 2$. Suppose that it is also true for n. Then again from (7)

$$J_{n+1}(q,v_0) = P^1(v_0) + F_n\left(q^1(v_0)\right)P^1(v_0) + F_n\left(q^0(v_0)\right)P^0(v_0)$$

so that $J_{n+1}(q,v_0)$ is a concave function of v_0 in each of the intervals [0,0.2], [0.2,0.8] and [0.8,1]. J_{n+1} is thus minimized for $v_0 = 0$, 0.2, 0.8 or 1 and again we need P^0, P^1, q^0 and q^1 for these v_0-values (see 2.1 and 2.2).

$F_n(0) = F_n(1) = 0$ but $F_n(2/3) = 1 - 0.9 \cdot 2/3 = 0.4$ since $K^n \in [1.26, 1.4]$.

Then

$$J_{n+1}(q,v_0) = \begin{cases} 1 - 0.6q \\ 1.4 \cdot 0.9(1-q) + \min[K^n q, 0.1, K^n \cdot 0.1(1-q)] \\ 1.4 \cdot 0.9q + \min[K^n \cdot 0.1q, 1-0.99q, K^n(1-q)] \\ 1 - 0.9q \end{cases}$$

so that

$$F_{n+1}(q) = \min[1 - 0.6q,\ 1.26(1-q) + K^n q,\ 1.26(1-q) + 0.1,$$
$$1.26(1-q) + 0.1K^n(1-q),\ 1.26q + 0.1K^n q,$$
$$1.26q + 1 - 0.99q,\ 1.26q + K^n(1-q),\ 1 - 0.9q] =$$
$$= \min[T_1, \ldots, T_8]$$

Some of the T:s can now be removed

$T_1 \geq T_8$ and $T_3 \geq T_8$.

$$\begin{cases} T_2 \geq T_5 \\ T_6 \geq T_5 \\ T_7 \geq T_5 \end{cases} \text{for } q \leq 1/2 \text{ and } \begin{cases} T_2 \geq T_4 \\ T_6 \geq T_4 \\ T_7 \geq T_4 \end{cases} \text{for } q \geq 1/2$$

Thus T_1, T_3, T_2, T_6 and T_7 can be removed and so

$$F_{n+1}(q) = \min[(1.26+0.1K^n)q,\ 1 - 0.9q,\ (1.26+0.1K^n)(1-q)] =$$
$$= \min[K^{n+1}q,\ 1 - 0.9q,\ K^{n+1}(1-q)]$$

with

$$K^{n+1} = 1.26 + 0.1K^n \in [1.26,1.4] \text{ if } K^n \in [1.26,1.4]$$

This completes the induction.

T_4, T_5 and T_8 correspond to $v_0 = 0.2,\ 0.8$ and 1 respectively, so that

$$v_0^{opt} = \begin{cases} 0.8 \\ 1 \\ 0.2 \end{cases} \quad \text{when} \quad F_n(q) = \begin{cases} K^n q \\ 1 - 0.9q \\ K^n(1-q) \end{cases}$$

Finally $K^{n+1} = 1.26 + 0.1K^n$ with $K^2 = 1.26$ gives

$$K^n = 1.4(1-10\cdot 0.1^n) \rightarrow 1.4 \quad n \rightarrow \infty.$$

APPENDIX 3

The Expected Loss When No Measurements Are Made.

For this case the control law is

$$v_k = \begin{cases} 0.8 & q_k < 1/3 \\ 0.2 & q_k \geq 1/3 \end{cases}$$

The expected one-step loss is then

$$V_1^a(q) = \begin{cases} P^1(0.8) = 0.9q & q < 1/3 \\ P^1(0.2) = 0.9(1-q) & q \geq 1/3 \end{cases}$$

To calculate $V_n^a(q)$ we need an equation similar to (3)

$$V_n^a(q) = \begin{cases} P^1(0.8) + V_{n-1}^a\left(\bar{q}(0.8)\right) & q < 1/3 \\ P^1(0.2) + V_{n-1}^a\left(\bar{q}(0.2)\right) & q \geq 1/3 \end{cases}$$

Now

$$\begin{cases} \bar{q}(0.2) = 0.6 + 0.4q > 1/3 \\ \bar{q}(0.8) = 0.7q < 1/3 \text{ if } q \leq 1/3 \end{cases}$$

so that it is easy to show by induction that

$$V_n^a(q) = \begin{cases} K_{a1}^n q & q < 1/3 \\ K_{a2}^n (1-q) & q \geq 1/3 \end{cases}$$

This is true for n = 1. Supposing it is true for n we get

$$V^a_{n+1}(q) = \begin{cases} P^1(0.8) + V^a_n\left(\bar{q}(0.8)\right) \\ P^1(0.2) + V^a_n\left(\bar{q}(0.2)\right) \end{cases} =$$

$$= \begin{cases} 0.9q + K^n_{a1}\bar{q}(0.8) \\ 0.9(1-q) + K^n_{a2}\left(1 - \bar{q}(0.2)\right) \end{cases} =$$

$$= \begin{cases} (0.9+0.7K^n_{a1})q \\ (0.9+0.4K^n_{a2})(1-q) \end{cases} = \begin{cases} K^{n+1}_{a1}q & q < 1/3 \\ K^{n+1}_{a2}(1-q) & q \geq 1/3 \end{cases}$$

and the induction is completed.

$$\begin{cases} K^{n+1}_{a1} = 0.9 + 0.7K^n_{a1} \text{ with } K^1_{a1} = 0.9 \Rightarrow K^n_{a1} = 3(1-0.7^n) \\ K^{n+1}_{a2} = 0.9 + 0.4K^n_{a2} \text{ with } K^1_{a2} = 0.9 \Rightarrow K^n_{a2} = 1.5(1-0.4^n) \end{cases}$$

Thus $K^n_{a1} \to 3$ and $K^n_{a2} \to 1.5$ as $n \to \infty$ so that

$$V^a_\infty(q) = \begin{cases} 3q & q < 1/3 \\ 1.5(1-q) & q \geq 1/3 \end{cases}$$

APPENDIX 4

The Expected Loss When Measurements Are Made.

Five different control laws will be tested in this case.

b. Open Loop Feedback Control

$$v_k^b = \begin{cases} 0.8 & q_k < 1/3 \\ 0.2 & q_k \geq 1/3 \end{cases}$$

c. One-Step Regulator

$$v_k^c = \begin{cases} 0.8 & q_k < 1/2 \\ 0.2 & q_k \geq 1/2 \end{cases}$$

d. Two-Step Regulator

$$v_k^d = \begin{cases} 0.8 & 0 \leq q_k < 25/54 \\ 1 & 25/54 \leq q_k < 39/54 \\ 0.2 & 39/54 \leq q_k \leq 1 \end{cases}$$

e. Approximate Multistep Regulator

$$v_k^e = \begin{cases} 0.8 & 0 \leq q_k < 120/306 \\ 1 & 120/306 \leq q_k < 255/306 \\ 0.2 & 255/306 \leq q_k \leq 1 \end{cases}$$

f. Optimal Dual Regulator

$$v_k^f = \begin{cases} 0.8 & 0 \leq q_k < 50/115 \\ 1 & 50/115 \leq q_k < 92/115 \\ 0.2 & 92/115 \leq q_k \leq 1 \end{cases}$$

These five cases will be considered simultaneously and so the superscripts on the V are dropped.

The expected one-step loss is

$$V_1(q) = \begin{cases} P^1(0.8) = 0.9q & q < 1/3,\ 1/2,\ 25/54,\ 120/306 \text{ and } 50/115 \text{ respectively} \\ P^1(1) = 1 - 0.9q & \text{cases d, e and f: } q \text{ close to } 1/2 \\ P^1(0.2) = 0.9(1-q) & q \geq 1/3,\ 1/2,\ 39/54,\ 255/306 \text{ and } 92/115 \text{ respectively} \end{cases}$$

For $V_{n+1}(q)$ we have the following equation

$$V_{n+1}(q) = \begin{cases} P^1(0.8) + V_n\left(q^1(0.8)\right)P^1(0.8) + \\ \qquad + V_n\left(q^0(0.8)\right)P^0(0.8) \\ P^1(1) + V_n\left(q^1(1)\right)P^1(1) + V_n\left(q^0(1)\right)P^0(1) \\ P^1(0.2) + V_n\left(q^1(0.2)\right)P^1(0.2) + \\ \qquad + V_n\left(q^0(0.2)\right)P^0(0.2) \end{cases}$$

The choice of equation depends on the q according to the different regulators.

Now

$$\begin{cases} q^0(0.8) = \dfrac{0.1q}{1 - 0.9q} \\ q^0(1) = 1 \\ q^0(0.2) = \dfrac{q}{q + 0.1(1-q)} \end{cases} \quad \text{and} \quad \begin{cases} q^1(0.8) = 2/3 \\ q^1(1) = 0 \\ q^1(0.2) = 2/3 \end{cases}$$

so that $q^0(0.8) \leq q$ and $q^0(0.2) \geq q$.

For cases b and c suppose that

$$V_n(q) = \begin{cases} K^n q & q < 1/3 \text{ and } 1/2 \text{ respectively} \\ K^n(1-q) & q \geq 1/3 \text{ and } 1/2 \text{ respectively} \end{cases}$$

Then

$$V_{n+1}(q) = \begin{cases} 0.9q + K^n\left(1 - q^1(0.8)\right)P^1(0.8) + K^n \cdot q^0(0.8)P^0(0.8) \\ 0.9(1-q) + K^n\left(1 - q^1(0.2)\right)P^1(0.2) + K^n\left(1 - q^0(0.2)\right)P^0(0.2) \end{cases}$$

$$= \begin{cases} (0.9 + K^n \cdot 0.3 + K^n \cdot 0.1)q \\ (0.9 + K^n \cdot 0.3 + K^n \cdot 0.1)(1-q) \end{cases} = \begin{cases} K^{n+1} q \\ K^{n+1}(1-q) \end{cases}$$

This proves that

$$V_n^b(q) = \begin{cases} K_b^n q & q < 1/3 \\ K_b^n(1-q) & q \geq 1/3 \end{cases}$$

and

$$V_n^c(q) = \begin{cases} K_c^n q & q < 1/2 \\ K_c^n(1-q) & q \geq 1/2 \end{cases}$$

with $K_b^n = K_c^n = 1.5(1-0.4^n) \rightarrow 1.5 \quad n \rightarrow \infty$.

For cases d, e and f suppose that

$$V_n(q) = \begin{cases} K^n q & q < 25/54,\ 120/306 \text{ and } 50/115 \text{ resp.} \\ 1 - 0.9q & q \text{ close to } 1/2 \\ K^n(1-q) & q \geq 39/54,\ 255/306 \text{ and } 92/115 \text{ resp.} \end{cases}$$

Then

$$V_{n+1}(q) = \begin{cases} 0.9q + \left(1 - 0.9q^1(0.8)\right)P^1(0.8) + K^n \cdot q^0(0.8)P^0(0.8) \\ 1 - 0.9q + 0 \cdot P^1(1) + 0 \cdot P^0(1) \\ 0.9(1-q) + \left(1 - 0.9q^1(0.2)\right)P^1(0.2) + K^n\left(1 - q^0(0.2)\right)P^0(0.2) \end{cases}$$

$$= \begin{cases} (0.9 + 0.36 + K^n \cdot 0.1)q \\ 1 - 0.9q \\ (0.9 + 0.36 + K^n \cdot 0.1)(1-q) \end{cases} = \begin{cases} K^{n+1}q \\ 1 - 0.9q \\ K^{n+1}(1-q) \end{cases}$$

Thus we have proved that

$$V_n^d(q) = \begin{cases} K_d^n q & 0 \leq q < 25/54 \\ 1 - 0.9q & 25/54 \leq q < 39/54 \\ K_d^n(1-q) & 39/54 \leq q \leq 1 \end{cases}$$

$$V_n^e(q) = \begin{cases} K_e^n q & 0 \leq q < 120/306 \\ 1 - 0.9q & 120/306 \leq q < 255/306 \\ K_e^n(1-q) & 255/306 \leq q \leq 1 \end{cases}$$

and

$$V_n^f(q) = \begin{cases} K_f^n q & 0 \leq q < 50/115 \\ 1 - 0.9q & 50/115 \leq q < 92/115 \\ K_f^n(1-q) & 92/115 \leq q \leq 1 \end{cases}$$

with $K_d^n = K_e^n = K_f^n = 1.4 - 5 \cdot 0.1^n \to 1.4 \quad n \to \infty.$