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On a First-order Stochastic Differential Equation†

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ABSTRACT

A first-order system with random parameters and random forcing is studied. The analysis is concentrated on the probability distributions. It is shown that considerable qualitative information can be obtained from Feller's classification of the singular points of the forward and backward Kolmogorov equations. It is found that there is a drastic difference between the cases of uncorrelated and strongly correlated disturbances.

The existence of stationary distributions is shown and their structure is analysed; it is found that the steady-state distributions are of the Pearson type. Some examples exhibit in detail the differences between uncorrelated and strongly correlated disturbances, giving the rather surprising effect that by making the fluctuations of the parameters sufficiently large, the probability of finding the state of the system in an interval around the origin can be made arbitrarily close to one 'peaking'. The results of some numerical computations are presented.

A case where the energy of the fluctuation in parameters is limited within a certain frequency band shows that this situation is different from the case of ' white noise '. For example, ' peaking ' of the distributions does not occur. It is found that, for the purpose of analysing probability distributions the system obtained can be approximated by a different system with white noise coefficients. These results are also illustrated by numerical computations.

§ 1. INTRODUCTION

THE problem discussed in this paper arose from the study of control systems subject to random disturbances. Linear systems with additive disturbances have been extensively studied and are well understood; but linear systems with random disturbances in the parameters are not so well understood. Adaptive control systems are typical examples.

† Communicated by the Author.

It is of interest to analyse the statistical properties of the output when those of the disturbances are known. Among the problems which naturally arise we mention the following: stability, influence of correlation between fluctuations in parameters and driving functions, ergodicity, character of the steady-state distribution, etc.

Some aspects of linear systems with random fluctuation in the parameters have been studied previously. The case of weakly stationary gaussian disturbances is treated by Tikhonov (1958) and Samuels and Eringen (1959). 'White noise' fluctuation in parameters is discussed in Samuels (1960), Åström (1962), Bogdanoff and Kozin (1962) and Caughey and Dienes (1962), analyses of the stability of systems with random fluctuations in the parameter have been given by Samuels (1960), Khas'minskii (1962) and Kozin (1963).

There are also some studies of a more general scope which contain material on the subject, e.g. Adomian (1963), Bharucha-Reid (1964) and Merklinger (1963). The problem discussed in this paper is similar to that treated by Caughey and Dienes (1962). Our analysis does, however, differ in two respects. Our study is limited to a first-order system with random parameters and random forcing. The analysis is concentrated on the probability distributions. Caughey and Dienes (1962) studied an nth-order system with emphasis on the moments of the distribution. Also, we do not agree with some of the results of Caughey and Dienes (1962). In such cases our results confirm those of Bogdanoff and Kozin (1962).

The problem is stated in § 2. The moments of the first probability distribution are given in § 3, and § 4 is devoted to a detailed study of the first probability distribution. In particular it is shown that considerable qualitative information can be obtained from Feller's classification of the singular points of the forward and backward Kolmogorov equations. It is found that there is a drastic difference between the cases of uncorrelated and strongly correlated disturbances.

In § 5 the existence of stationary distributions is shown and their structure is analysed. It is found that the steady-state distributions are of the Pearson type. Some examples are considered in § 6, which exhibit in detail the differences between uncorrelated and strongly correlated disturbances. In these examples we also find the rather surprising effect that by making the fluctuations of the parameters sufficiently large, the probability of finding the state of the system in an interval around the origin can be made arbitrarily close to one. In § 7 we present the results of some numerical computations.

In § 8 we consider a case where the energy of the fluctuation in parameters is limited within a certain frequency band. It is found that this situation is different from the case of 'white noise'. For example, peaking of the distributions does not occur. It is also found that, for the purpose of analysing probability distributions, the system obtained can be approximated by a different system with white noise coefficients. These results are also illustrated by numerical computations.

§ 2. Statement of the Problem

Consider the scalar stochastic differential equation:

$$dx = x\,dw_1 + dw_2, \tag{2.1}$$

where w_1 and w_2 are Wiener processes, i.e. random functions having independent gaussian increments such that

$$E\Delta w_1 = -m_1 h, \tag{2.2}$$

$$E\Delta w_2 = m_2 h, \tag{2.3}$$

$$\text{var}\,(\Delta w_1) = 2A_1 h, \tag{2.4}$$

$$\text{cov}\,(\Delta w_1, \Delta w_2) = 2A_{12} h, \tag{2.5}$$

$$\text{var}\,(\Delta w_2) = 2A_2 h, \tag{2.6}$$

where

$$\Delta w_i = w_i(t+h) - w_i(t), \quad i = 1, 2 \tag{2.7}$$

and

$$m_1 > 0, m_2 \geq 0, A_1 \geq 0, A_2 \geq 0 \quad \text{and} \quad A_{12}{}^2 \leq A_1 A_2. \tag{2.8}$$

There are basically two ways of interpreting eqn. (2.1). In one approach (Ito 1951, Doob 1953), the solution is defined by means of Picard iteration of the corresponding integral equation with the integral interpreted as a stochastic integral. In the other approach (Gihman 1955), the solution of (2.1) is defined as the limit of a stochastic difference equation. Both interpretations show that the solution of (2.1) $x = x(t, \omega)$ is a Markov process with continuous trajectories whose infinitesimal incremental moments are given by:

$$b(x) = \lim_{h \to 0} \frac{1}{h} E[\Delta x | x] = -m_1 x + m_2, \tag{2.9}$$

$$a(x) = \frac{1}{2} \lim_{h \to 0} \frac{1}{h} E[(\Delta x)^2 | x] = A_1 x^2 + 2x A_{12} + A_2. \tag{2.10}$$

See Appendix. Equation (2.8) does not agree with eqn. (2.5) of Caughey and Dienes (1962) but agrees with eqn. (9) Bogdanoff and Kozin (1962). The functions $a(x)$ and $b(x)$ clearly satisfy the conditions given by Doob (1953, p. 277) and Gihman (1955, Theorem 2). The solution of (2.1) will thus exist for all t. The problem we will consider is that of analysing the stochastic properties of the solution.

§ 3. Moments of the Solution

The moments of the solution of (2.1) are well known. See, e.g., Bogdanoff and Kozin (1962). For the purpose of easy reference we will give the expressions here.

Introducing

$$\alpha_n(t) = E x^n(t), \tag{3.1}$$

where E denotes mathematical expectation, we arrive easily at the following equations for $\alpha_n(t)$:

$$\left.\begin{aligned}
&\alpha_0 = 1,\\
&\frac{d\alpha_1}{dt} = -m_1\alpha_1 + m_2,\\
&\cdot\\
&\cdot\\
&\cdot\\
&\frac{d\alpha_n}{dt} = [-nm_1 + n(n-1)A_1]\alpha_n + [nm_2 + 2n(n-1)A_{12}]\alpha_{n-1}\\
&\qquad + n(n-1)A_2\alpha_{n-2}, \quad n \geq 2.
\end{aligned}\right\} \tag{3.2}$$

This system of equations is linear and has constant coefficients. Solutions will thus exist for all finite t. As t approaches infinity solutions may, however, cease to exist. The condition

$$(n-1)A_1 < m_1 \tag{3.3}$$

is sufficient for $\alpha_n(t)$ to be finite, as t tends to infinity.

§ 4. First Probability Density

Let $P(t, x, \Gamma)$ be the transition probability of the Markov process defined by (2.1), i.e. the probability that a trajector of (2.1), starting at x, after time t will be Γ. Further, let $P(t, x, z)$ be the corresponding density. The transition probability $P(t, x, \Gamma)$ satisfies the Chapman–Kolmogorov equation:

$$P(t+s, x, \Gamma) = \int P(t, z, \Gamma)P(s, x, dz), \tag{4.1}$$

and defines a one-parameter family of transformations:

$$T_t f = \int P(t, ., dy) f(y). \tag{4.2}$$

This family of transformations form a semi-group $\{T_t\}$ which preserves positivity and has the infinitesimal generator:

$$\Omega = a(x)\frac{\partial^2}{\partial x^2} + b(x)\frac{\partial}{\partial x}, \tag{4.3}$$

where $a(x)$ and $b(x)$ are given by (2.9) and (2.10). The transition probability satisfies the Kolmogorov equation:

$$-\frac{\partial P}{\partial t} = \Omega P = a(x)\frac{\partial^2 P}{\partial x^2} + b(x)\frac{\partial P}{\partial x}. \tag{4.4}$$

The character of the Markov process defined by (2.1) depends on the singularities of the operator Ω given by (4.3):

We have to separate two cases:

$$\left.\begin{aligned}
\mathrm{I}:&\quad A_{12}^2 < A_1A_2;\\
\mathrm{II}:&\quad A_{12}^2 = A_1A_2.
\end{aligned}\right\} \tag{4.5}$$

In case I eqn. (4.4) has singular points only at $x = \pm\infty$. In case II there is also a singularity at $x = -\sqrt{(A_2/A_1)}$ in the sense that the coefficient of the highest derivative of the right member vanishes.

To find the character of the boundaries, we introduce Feller's canonical scale and Feller's canonical measure (Ito 1957, Bharucha-Reid 1964):

$$s(x) = \int_{x_0}^{x} \exp\left\{-\int_{x_0}^{x} b(s)a^{-1}(s)\,ds\right\} dx, \tag{4.6}$$

$$m(x) = \int_{x_0}^{x} a^{-1}(x) \exp\left\{\int_{x_0}^{x} b(s)a^{-1}(s)\,ds\right\} dx. \tag{4.7}$$

We get:

$$\sigma_2 = \int\int dm(x)\,ds(y) = \infty, \quad a < x < y < \infty, \tag{4.8}$$

$$\mu_2 = \int\int ds(x)\,dm(y) = \infty, \quad a < x < y < \infty. \tag{4.9}$$

The singular point $x = +\infty$ is thus natural (Ito 1957, pp. 5–57). In the same way we conclude that $x = -\infty$ also is a natural boundary for eqn. (4.4).

As the boundaries are natural it follows that no boundary conditions are required for the solution of (4.4) or for its formal adjoint:

$$\frac{\partial p}{\partial t} = \Omega^* p = \frac{\partial^2}{\partial x^2}(a(x)p) - \frac{\partial}{\partial x}(b(x)p).$$

See Feller (1952, p. 517).

In case II we find that the point $x = -\sqrt{(A_2/A_1)}$ is also singular. This point is a right translational point if

$$m_1\sqrt{(A_2/A_1)} + m_2 > 0. \tag{4.10}$$

See Feller (1954a, Definition 7). The condition (2.8) thus implies that $x = a = -\sqrt{(A_2/A_1)}$ is a right translational point in case II. In this case we can separately consider the processes in $[-\infty, -\sqrt{(A_2/A_1)}]$ and $[-\sqrt{A_2/A_1}), \infty]$. We get for the process in $[-\infty, -\sqrt{(A_2/A_1)}]$:

$$\sigma_2 = \int\int dm(x)\,ds(y) < \infty, \quad a' < x < y < a,$$

$$\mu_2 = \int\int dm(y)\,ds(x) = \infty, \quad a' < x < y < a,$$

and for the process in $[-\sqrt{(A_2/A_1)}, \infty]$:

$$\sigma_1 = \int\int dm(x)\,ds(y) = \infty, \quad a < y < x < a',$$

$$\mu_1 = \int\int dm(y)\,ds(x) < \infty, \quad a < y < x < a'.$$

The point $x = -\sqrt{(A_2/A_1)}$ is thus an exit boundary for the process in $[-\infty, -\sqrt{(A_2/A_1)}]$ and an entrance boundary for the process in

$[-\sqrt{(A_2/A_1)}, \infty]$. The forward equation (4.8) then has a unique solution in $[-\infty, -\sqrt{(A_2/A_1)}]$ without any boundary conditions Feller (1952, p. 517). To solve the forward eqn. (4.8) for the process in $[-\sqrt{(A_2/A_1)}, \infty]$ one boundary condition is required. This is obtained from the following condition:

$$\lim_{x \downarrow -\sqrt{(A_2/A_1)}} [(af)_x - bf] = \lim_{x \uparrow -\sqrt{(A_2/A_1)}} [(af)_x - bf], \tag{4.11}$$

where the right-hand member is obtained from the unique solution of (4.8) in $[-\infty, -\sqrt{(A_2/A_1)}]$. The condition (4.11) implies that there is no accumulation of probability mass at $x = -\sqrt{(A_2/A_1)}$.

Using another classification (Feller 1954 a, p. 11) we find that in case II the boundary $x = -\infty$ is *inaccessible* and the boundary $x = -\sqrt{(A_2/A_1)}$ is *accessible*. Consider the probability that the process in $[-\infty, \sqrt{(A_2/A_1)}]$ will ultimately leave this interval. It follows directly from Theorem 5 of Feller (1954 a) and the fact that $x = -\sqrt{(A_2/A_1)}$ is an exit boundary for the process that this probability equals one. In the same way we find that the boundaries $x = -\sqrt{(A_2/A_1)}$ and $x = \infty$ are inaccessible boundaries for the process in $[-\sqrt{(A_2/A_1)}, \infty]$. In case II a solution of (2.1) starting in the interval $[-\infty, -\sqrt{(A_2/A_1)}]$ will thus, with probability one, leave this interval while a solution starting in $[-\sqrt{(A_2/A_1)}, \infty]$ will stay in that interval with probability one.

Thus, in itself the analysis of the singular points of eqn. (4.4) will give a good qualitative picture of the behaviour of the solutions of the stochastic differential eqn. (2.1).

§ 5. Stationary Probability Distribution

We will now turn to a more detailed analysis of the stochastic properties of the solution of (2.1). The first probability distribution of the solution of (2.1) is calculated as:

$$p_t = T_t^* P_0, \tag{5.1}$$

where T_t^* is the adjoint of the operator T_t and p_0 is the density of the initial distribution. The infinitesimal generator of T_t^* is Ω^*.

As the transformations T_t^* are norm-preserving, we can immediately conclude that the limiting distributions of p_t exist in the abelian sense (Hille and Philips 1957, p. 502). The density of the limit distribution is the eigenfunction of Ω^* corresponding to $\lambda = 0$ and it is given by:

$$\Omega^* f_0 = \frac{d^2}{dx^2}(af_0) - \frac{d}{dx}(bf_0) = 0. \tag{5.2}$$

Integrating once we get:

$$\frac{d}{dx}(af_0) - bf_0 = \text{constant}. \tag{5.3}$$

As the boundaries $x = \pm\infty$ are inaccessible, the constant has to vanish and the density of the stationary distribution is then given by the solution of

$$\frac{d}{dx}(af_0) - bf_0 = 0, \tag{5.4}$$

which has L_1 norm equal to one.

It is natural to consider the operator Ω^* on the space of measures and Ω on the space of continuous functions. For the purpose of discussing the steady-state solution we will, however, consider Ω^* on the Hilbert space whose scalar product is defined by:

$$(f, g) = \int f(x)g(x)w(x)\,dx, \tag{5.5}$$

where the integration is performed over the interval $(-\infty, \infty)$.

The operator Ω^* can be written as:

$$\Omega^* = \frac{d}{dx}\left(w^{-2}p\frac{d}{dx}(w\,\cdot)\right), \tag{5.6}$$

where

$$p(x) = a^2(x)\exp\int^x -b(s)a^{-1}(s)\,ds \tag{5.7}$$

and

$$w(x) = a(x)\exp\int^x -b(s)a^{-1}(s)\,ds. \tag{5.8}$$

We have

$$\begin{aligned}(g, \Omega^*f) &= \int gw\frac{d}{dx}\left(w^{-2}p\frac{d}{dx}(wf)\right)dx\\ &= (gw)\left(w^{-2}p\frac{d}{dx}(wf)\right) - \int w^{-2}p\frac{d}{dx}(wf)\frac{d}{dx}(wg)\,dx\\ &= (gw)\left(w^{-2}p\frac{d}{dx}(wf)\right) - (wf)w^{-2}p\frac{d}{dx}(wg)\\ &\quad + \int wf\frac{d}{dx}\left[w^{-2}p\frac{d}{dx}(wg)\right].\end{aligned} \tag{5.9}$$

As the boundaries are natural, the out integrated parts vanish and Ω^* is thus self-adjoint. Further, it follows from (5.9) that

$$(f, \Omega^*f) = -\int w^{-2}p\left[\frac{d}{dx}(wf)\right]^2 dx.$$

Hence, the operator Ω^* is also non-positive. We can thus conclude that the non-positive axis contains the spectrum of Ω^*. One eigenvalue of Ω^* is equal to zero. The solution of (5.1) $p_t(\cdot)$ is thus positive and L_1

integrable. Its Laplace transform has the pole $s=0$ and no other singularities for $\operatorname{Re} s \geq 0$. It then follows from Ikeharas Tauberian Theorem (Widder 1946, p. 233) that the limit $\lim\limits_{t \to \infty} p(t,x)$ exists and equals $f_0(x)$.

For the Markov process defined by (2.1), where $a(x)$ and $b(x)$ are respectively quadratic and linear functions of x, the steady state-distribution will be of the Pearson type. An account of these distributions is found in Elderton (1938).

The character of the distributions will depend on the character of the zeros of the quadratic $a(x)$ and we will thus have to consider cases I and II of § 4 separately.

In case I the polynomial $a(x)$ has two imaginary roots and the distribution is of Pearson type IV. Integrating (5.4) we get the following expression for the density of the steady-state distribution:

$$f(x)=C\cdot(A_1x^2+2A_{12}x+A_2)-1-\frac{m_1}{2A_1}$$

$$\exp\left\{\frac{m_2A_1+m_1A_{12}}{A_1\sqrt{(A_1A_2-A_{12}{}^2)}}\operatorname{arc\,tg}\frac{A_1x+A_{12}}{\sqrt{(A_1A_2-A_{12}{}^2)}}\right\}. \tag{5.10}$$

If this is to be a density function, we must require that

$$\frac{m_1}{A_1}>-1. \tag{5.11}$$

If this condition is fulfilled the function $f(x)$ is also in $L_2(w)$.

In case II the quadratic $a(x)$, a double real zero, the distribution is of Pearson type V. In this case the density function of the stationary distribution is:

$$f(x)=\begin{cases}\dfrac{A_1{}^{3/2}}{\Gamma(1+m_1/A_1)(m_2\sqrt{A_1}+m_1\sqrt{A_2})}G2+\dfrac{m_1}{A_1}(x)\exp\text{—}G(x) & x>-\sqrt{\dfrac{A_2}{A_1}}, \\ 0 & x\leq-\sqrt{\dfrac{A_2}{A_1}}, \end{cases} \tag{5.12}$$

where

$$G(x)=\frac{1}{A_1}\,\frac{m_2\sqrt{(A_1)}+m_1\sqrt{(A_2)}}{x\sqrt{(A_1)}+\sqrt{(A_2)}}. \tag{5.13}$$

If this is to be a density function eqn. (5.11) must be satisfied. In both cases distributions have the mode:

$$x_0=\frac{m_2-2A_{12}}{m_1+2A_1}. \tag{5.14}$$

The moments are obtained from the stationary solutions of the equations given in § 3. The first two moments of the stationary distribution are:

$$Ex = \frac{m_2}{m_1}, \tag{5.15}$$

$$Ex^2 = \left\{ \begin{array}{ll} \dfrac{m_2(2m_2 + 4A_{12}) + 2m_1A_2}{2m_1(m_1 - A_1)} & m_1 < A_1, \\ & \\ +\infty & m_1 \geq A_1. \end{array} \right\} \tag{5.16}$$

If A_1 tends to zero, both steady-state distributions tend toward

$$\lim_{A_1 \to 0} f(x) = \left(\frac{m_1}{2\pi A_2}\right)^{1/2} \exp \frac{m_1}{2A_2}\left(x - \frac{m_2}{m_1}\right)^2. \tag{5.17}$$

When the function w_1 is deterministic and linear in t the first probability density of the solution of (2.1) is thus normal $N[m_2/m_1, \sqrt{(A_2/m_2)}]$.

§ 6. Examples

We will now consider two examples.

Example 1

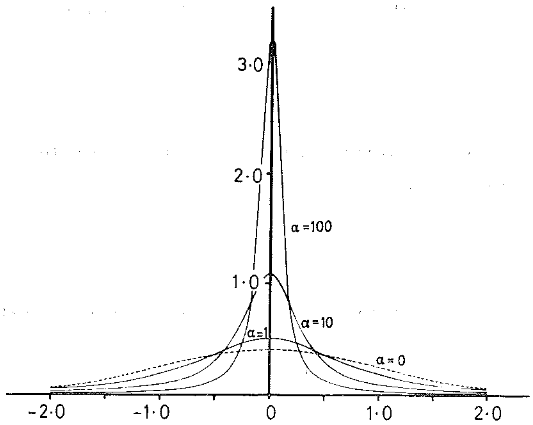
Assume that the random functions w_1 and w_2 are uncorrelated and that w_2 has a zero average i.e.

$$A_{12} = m_2 = 0. \tag{6.1}$$

The steady-state distribution is then symmetrical and of Pearson type VII. i.e. a student distribution. The density of the steady-state distribution is:

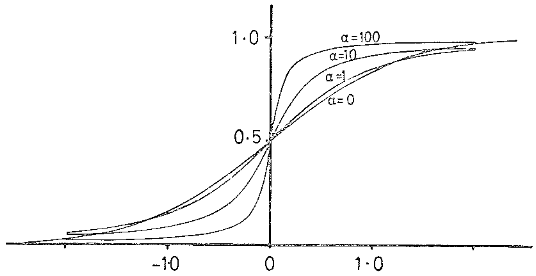
$$f(x) = \left(\frac{A_1}{\pi A_2}\right)^{1/2} \frac{\Gamma[1 + (m_1/2A_1)]}{\Gamma[\frac{1}{2} + (m_1/2A_1)]} \left(1 + x^2\frac{A_1}{A_2}\right)^{-1-\frac{m_1}{2A_1}}. \tag{6.2}$$

Fig. 1



Density function of the steady-state distribution for the case of uncorrelated disturbances.

Fig. 2



Steady-state distribution function for the case of uncorrelated disturbances.

The moments of the stationary distribution are:

$$Ex^{2n+1}=0, \tag{6.3}$$

$$Ex^{2n}\begin{cases}\left(\dfrac{A_2}{A_1}\right)^n \dfrac{(n-\frac{1}{2})(n-\frac{3}{2})\dots\frac{1}{2}}{[(m_1/2A_1)-\frac{1}{2}][(m_1/2A_1)-\frac{3}{2}]\dots[(m_1/2A_1)-n+1/2]} & \dfrac{A_1}{m_1}<\dfrac{2}{n+1}, \\ +\infty & \dfrac{A_1}{m_1}\geq\dfrac{2}{n+1}. \end{cases} \tag{6.4}$$

As the stationary distribution (6.2) for $A_1=0$ becomes normal $N[0, \sqrt{(A_2/m_1)}]$, it is natural to normalize by choosing

$$A_2=m_1. \tag{6.5}$$

The function $f(x)$ then depends only on the parameter

$$\alpha=\frac{A_1}{A_2}=\frac{A_1}{m_1}. \tag{6.6}$$

Graphs of the density functions and the corresponding distributions are shown in fig. 1 and fig. 2.

Example 2

Consider now the case when the random functions w_1 and w_2 are strongly correlated.

$$A_{12}{}^2=A_1A_2. \tag{6.7}$$

Further assuming

$$m_2=0 \tag{6.8}$$

and normalizing in the same way as in the previous example, compare eqn. (6.5), the density of the steady-state distribution then becomes:

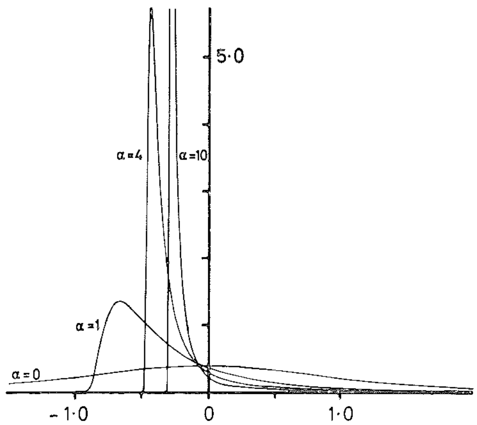
$$f(x)=\begin{cases}\left(\dfrac{A_1}{A_2}\right)^{3/2}\dfrac{1}{\Gamma[1+(A_2/A_1)]}\left(\dfrac{A_2}{A_1}\,\dfrac{1}{1+x\surd(A_1/A_2)}\right)^{+2+\frac{A_2}{A_1}} \\ \qquad\qquad \exp-\dfrac{A_2}{A_1}.\,\dfrac{1}{1+x\surd(A_1/A_2)}, \\ \qquad\qquad x>-\sqrt{\dfrac{A_2}{A_1}}, \\ \qquad\qquad x\geq-\sqrt{\dfrac{A_2}{A_1}}.\end{cases} \qquad (6.9)$$

Graphs of this density function and the corresponding distribution are shown in figs. 3 and 4. The stochastic differential eqn. (2.1) can be regarded as a perturbation of the following stochastic differential equation:

$$dx=-m_1x\,dt+dv_2, \qquad (6.10)$$

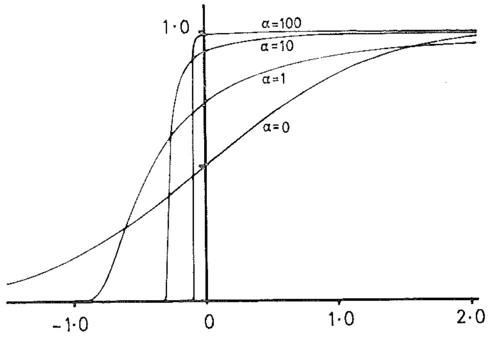
whose solution is a normal process whose steady-state distribution is normal $N[0,\surd(A_2/m_1)]$. The analysis just performed will show the effect of a stochastic perturbation of the coefficient m_1. The intensity of this perturbation is proportional to A_1. In example 1 the perturbation of the coefficient is independent of the driving function v_2, while in example 2 the perturbation is strongly correlated to the driving function v_2. For perturbation with low intensities the steady-state distribution is close to the gaussian distribution at the origin. For large values of A_1 the steady-state distribution does, however, differ from the gaussian; this is strongly reflected in the moments of the distribution.

Fig. 3



Density function of the steady-state distribution for the case of strongly correlated disturbances.

Fig. 4



Steady-state distribution function for the case of strongly correlated disturbances.

As the intensity of the perturbation increases, the character of the steady-state distribution changes. The qualitative features of the changes are seen in fig. 1 and fig. 3. Notice in particular the different behaviour of case I and II. Also notice that in both cases the probability mass in an interval around the origin increases with increasing intensity of the perturbation. In case I the steady-state probability of finding the variable x in the interval $(-\epsilon, \epsilon)$ is thus:

$$P\{|x|\leqq\epsilon\}=1-2\left(\frac{\alpha}{\pi}\right)^{1/2}\frac{\Gamma[1+(1/2\alpha)]}{\Gamma[\frac{1}{2}+(1/2\alpha)]}\int_{\epsilon}^{\infty}(1+x^2\alpha)^{-1-(2/\alpha)}\,dx$$

$$\geqq 1-\frac{2}{\sqrt{\pi}}\cdot\frac{\Gamma[1+(1/2\alpha)]}{\Gamma[\frac{1}{2}+(1/2\alpha)]}\cdot\frac{\alpha}{4+\alpha}\cdot(\epsilon\sqrt{\alpha})^{-1-(4/\alpha)}. \tag{6.11}$$

Similarly we have in case II:

$$P\{|x|\leq\epsilon\}\geq 1-\frac{1}{\Gamma[2+(1/\alpha)]}(\epsilon\alpha\sqrt{\alpha})^{-1-(1/\alpha)}. \tag{6.12}$$

In both cases we thus find that

$$P\{|x|\leq\epsilon\}\Rightarrow 1 \quad \text{as} \quad A_1\to\infty. \tag{6.13}$$

On the other hand, all the moments of the distributions will diverge as $A_1\Rightarrow\infty$. Compare with eqn. (3.3).

§ 7. Numerical Calculations

As an illustration, some numerical calculations were performed. To do these we utilize the fact that the stochastic differential eqn. (2.1) can be interpreted as the limit of the following stochastic difference equation:

$$\left.\begin{aligned} x_h(t_{n+1}) &= (1-hm_1)x_h(t_n)+m_2h+x_h(t_n)e_1(t_n)\sqrt{h}+e_2(t_n)\sqrt{h}, \\ t_{n+1} &= t_n+h, \end{aligned}\right\} \tag{7.1}$$

where $\{e_1(t_i)\}$ and $\{e_2(t_j)\}$ are sequences at normal random variables with zero means and the covariances:

$$\operatorname{cov}[e_1(t_i), e_1(t_j)] = 2A_1\delta_{ij}, \tag{7.2}$$

$$\operatorname{cov}[e_1(t_i), e_2(t_j)] = 2A_{12}\delta_{ij}, \tag{7.3}$$

$$\operatorname{cov}[e_2(t_i), e_2(t_j)] = 2A_2\delta_{ij}, \tag{7.4}$$

where δ_{ij} is the Kronecker delta. The difference eqn. (7.1) converges formally to the stochastic differential eqn. (2.1), and the mean and variance of the incremental moments satisfy Lipschitz conditions in x and their expectations are of the order h. According to Gihman (1955, Theorem 2) the solution of (7.1) will then converge (in the sense of mean square) to the solution of (2.1). The distribution law of $x_h(t)$ defined by (7.1) will also converge to the distribution law of the solution of (2.1). The purpose of the numerical calculations is to demonstrate that, for sufficiently small values of h the solution of (7.1) and its distribution law will be arbitrarily close to the solution of (2.1) and its distribution law. In the calculations we will evaluate the steady-state values of the first probability distribution. The solution of (7.1) was computed on the IBM 7090. The sequences of normal random numbers $\{e_1(t_i)\}$ and $\{e_2(t_i)\}$ were obtained from standard routines for the computer installation. The standard routines operate in the following way. Pseudo-random numbers with a rectangular distribution are generated using the power residue method. The following recursive equation is used:

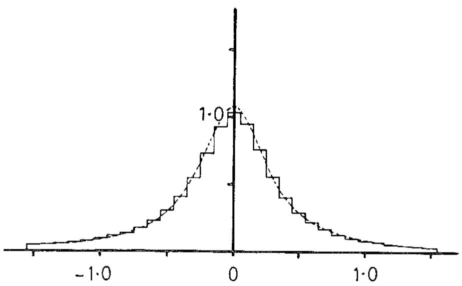
$$R_{n+1} = R_n(2^{18}+3) \quad \bmod 2^{35}.$$

This calculation is done in integer arithmetic and the result is converted to a floating point number in the interval $(0, 1)$. To obtain normally distributed random numbers, twelve rectangularly distributed pseudo-random numbers R_i are generated and the number

$$G = \sum_{i=1}^{12} R_i - 6$$

is formed. The numbers G will be approximately normal $N(0, 1)$. In the numerical calculations the solution of (7.1) for $(0, T)$ with the initial

Fig. 5



Histogram of the steady-state solution for an experiment with uncorrelated disturbances.

Table 1

Centre of class-interval	1	2	3	4	Average	Density function
−2·0	0·0258	0·0200	0·0224	0·0256	0·0234	0·0218
−1·9	0·0292	0·0230	0·0256	0·0284	0·0266	0·0242
−1·8	0·0310	0·0280	0·0272	0·0314	0·0294	0·0270
−1·7	0·0348	0·0280	0·0332	0·0388	0·0337	0·0304
−1·6	0·0404	0·0330	0·0378	0·0432	0·0386	0·0343
−1·5	0·0478	0·0322	0·0420	0·0520	0·0435	0·0391
−1·4	0·0496	0·0402	0·0450	0·0580	0·0482	0·0449
−1·3	0·0650	0·0494	0·0502	0·0662	0·0577	0·0520
−1·2	0·0634	0·0564	0·0608	0·0712	0·0630	0·0610
−1·1	0·0836	0·0676	0·0696	0·0910	0·0780	0·0723
−1·0	0·1002	0·0788	0·0822	0·1076	0·0922	0·0868
−0·9	0·1110	0·1066	0·0996	0·1318	0·1122	0·1060
−0·8	0·1422	0·1272	0·1168	0·1536	0·1350	0·1318
−0·7	0·1706	0·1758	0·1518	0·1936	0·1730	0·1673
−0·6	0·2132	0·2280	0·2006	0·2708	0·2282	0·2174
−0·5	0·2910	0·3116	0·2754	0·3420	0·3050	0·2898
−0·4	0·3852	0·4154	0·3720	0·4454	0·4045	0·3961
−0·3	0·5304	0·5572	0·5088	0·5814	0·5444	0·5502
−0·2	0·7112	0·7624	0·6896	0·7362	0·7248	0·7556
−0·1	0·9144	0·9586	0·8832	0·9428	0·9248	0·9674
0·0	1·0594	1·0502	1·0056	1·0128	1·0320	1·0654
0·1	1·0130	0·9492	0·9288	0·9024	0·9484	0·9674
0·2	0·8358	0·7382	0·7528	0·6938	0·7552	0·7556
0·3	0·5726	0·5552	0·5598	0·5236	0·5528	0·5502
0·4	0·4186	0·4128	0·3808	0·3858	0·3995	0·3961
0·5	0·2828	0·2906	0·2980	0·2804	0·2880	0·2898
0·6	0·2156	0·2164	0·2252	0·2104	0·2169	0·2174
0·7	0·1594	0·1696	0·1818	0·1642	0·1688	0·1673
0·8	0·1128	0·1260	0·1422	0·1278	0·1272	0·1318
0·9	0·0826	0·1034	0·1144	0·1070	0·1018	0·1060
1·0	0·0700	0·0916	0·0886	0·0844	0·0836	0·0868
1·1	0·0588	0·0750	0·0770	0·0722	0·0708	0·0723
1·2	0·0458	0·0510	0·0580	0·0618	0·0542	0·0610
1·3	0·0380	0·0510	0·0538	0·0580	0·0502	0·0520
1·4	0·0366	0·0412	0·0494	0·0438	0·0428	0·0449
1·5	0·0330	0·0340	0·0378	0·0462	0·0378	0·0391
1·6	0·0304	0·0346	0·0378	0·0390	0·0354	0·0343
1·7	0·0252	0·0302	0·0372	0·0318	0·0311	0·0304
1·8	0·0298	0·0294	0·0272	0·0260	0·0281	0·0270
1·9	0·0248	0·0216	0·0240	0·0246	0·0238	0·0242
2·0	0·0218	0·0220	0·0274	0·0176	0·0222	0·0218

value $x_h(0)=0$, was calculated with the random numbers $\{e_1(t_i)\}$ and $\{e_2(t_i)\}$ generated as described above. For a particular solution the quantities

$$\frac{\Delta t(\Gamma)}{T}=\left\{\frac{t}{T};\, x(t)\in\Gamma\right\} \tag{7.5}$$

were formed. These quantities will approximate the steady-state probability $P\{x\in\Gamma\}$.

The sets Γ_i were chosen as adjacent equidistant intervals of length B. In table 1 we give the results obtained for four solutions with the data:

$$A_1=10,\quad A_2=1,\quad m_1=0,\quad m_2=1,\quad h=0{\cdot}002,\quad T=0{\cdot}1,\quad B=0{\cdot}1.$$

The quantities $\Delta t(\Gamma_i)/T$, which represent the ordinates of the histogram of the steady-state distribution, are tabulated as a function of the centres of the class intervals. The steady-state distribution of the solution of (2.1) for $A_1=10$, $A_2=1$, $m_1=0$ and $m_2=1$ has the density function:

$$f(x)=\left(\frac{10}{\pi}\right)^{1/2}\cdot\frac{\Gamma(1{\cdot}05)}{\Gamma(0{\cdot}55)}\,(1+10x^2)^{-1{\cdot}05}. \tag{7.6}$$

See example 1 of § 6. Figure 5 shows the average histogram obtained from the four experiments and the density function (7.6).

Table 2 shows the results obtained for a case with strongly correlated disturbances. Five solutions of (7.1) with the data:

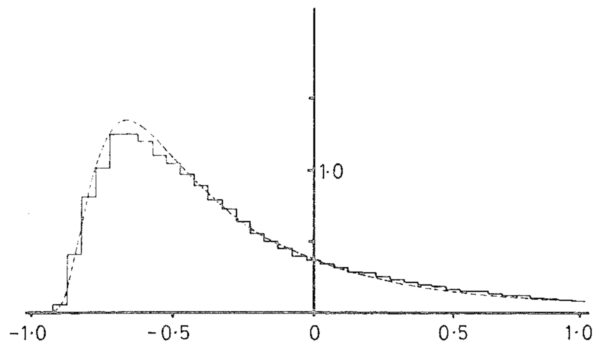
$$A_1=1,\quad A_2=1,\quad m_1=0,\quad m_2=1,\quad h=0{\cdot}002,\quad T=100,\quad B=0{\cdot}05,$$

were generated and the quantities $\Delta t(\Gamma)/T$ are shown. The steady-state distribution of (2.1) for $A_1=A_2=m_2=1$ and $m_1=0$ has the density function:

$$f(x)=(1+x)^{-3}\exp-(1+x). \tag{7.7}$$

This density function and the average histogram of the five experiments are shown in fig. 6.

Fig. 6



Histogram of the steady-state solution for an experiment with strongly correlated disturbances $k=1$.

Table 2

Centre of class-interval	1	2	3	4	5	Average	Density function
−1·00	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000
−0·95	0·0000	0·0000	0·0016	0·0000	0·0000	0·0003	0·0000
−0·90	0·0532	0·0228	0·0852	0·0548	0·0600	0·0552	0·0454
−0·85	0·3612	0·3632	0·5428	0·3480	0·4144	0·4059	0·3771
−0·80	0·7156	0·9000	0·9252	0·7248	0·9316	0·8394	0·8422
−0·75	1·0012	1·2228	1·0496	0·9404	1·2408	1·0910	1·1722
−0·70	1·2864	1·4108	1·0824	1·0780	1·3640	1·2443	1·3213
−0·65	1·1624	1·4200	1·1900	1·1396	1·3208	1·2466	1·3395
−0·60	1·0468	1·3496	1·1552	1·1128	1·3076	1·1944	1·2826
−0·55	1·0216	1·1164	1·0588	1·0288	1·2616	1·0974	1·1892
−0·50	1·0504	1·0820	1·0272	0·9164	1·1304	1·0413	1·0827
−0·45	0·9508	0·9728	0·9728	0·8936	1·0476	0·9675	0·9756
−0·40	0·8648	0·8972	0·9084	0·8320	0·9324	0·8870	0·8744
−0·35	0·8012	0·8104	0·8092	0·7064	0·8220	0·7898	0·7818
−0·30	0·6860	0·7592	0·7484	0·6764	0·7708	0·7282	0·6987
−0·25	0·6372	0·6328	0·6704	0·5944	0·6572	0·6384	0·6248
−0·20	0·5244	0·5552	0·5672	0·5440	0·5724	0·5526	0·5596
−0·15	0·4468	0·4920	0·5212	0·5400	0·4948	0·4990	0·5021
−0·10	0·3976	0·4508	0·4940	0·4500	0·4532	0·4491	0·4516
−0·05	0·3764	0·3808	0·4544	0·3896	0·3808	0·3964	0·4071
0·0	0·3820	0·3540	0·3872	0·3620	0·3452	0·3661	0·3679
0·05	0·3652	0·3276	0·3396	0·3476	0·3256	0·3411	0·3333
0·10	0·3508	0·2880	0·2908	0·3400	0·3260	0·3191	0·3027
0·15	0·3432	0·2708	0·2632	0·3084	0·2584	0·2888	0·2756
0·20	0·3344	0·2572	0·2380	0·2924	0·2916	0·2827	0·2515
0·25	0·2984	0·2296	0·2048	0·2940	0·2580	0·2570	0·2301
0·30	0·2768	0·2204	0·1836	0·2448	0·2640	0·2379	0·2109
0·35	0·2464	0·2096	0·1824	0·2352	0·2212	0·2190	0·1938
0·40	0·2348	0·1636	0·1792	0·2180	0·1976	0·1986	0·1784
0·45	0·2148	0·1648	0·1624	0·1956	0·1808	0·1837	0·1646
0·50	0·2048	0·1440	0·1532	0·1844	0·1544	0·1682	0·1521
0·55	0·1840	0·1348	0·1492	0·1724	0·1244	0·1530	0·1409
0·60	0·2004	0·1396	0·1272	0·1756	0·1412	0·1568	0·1307
0·65	0·1704	0·1228	0·1208	0·1352	0·1296	0·1358	0·1214
0·70	0·1612	0·1184	0·1072	0·1448	0·1024	0·1268	0·1130
0·75	0·1548	0·1096	0·1096	0·1376	0·1020	0·1227	0·1054
0·80	0·1436	0·0968	0·0848	0·1268	0·0828	0·1070	0·0984
0·85	0·1352	0·0944	0·0816	0·1356	0·0708	0·1035	0·0920
0·90	0·1192	0·0716	0·0712	0·1132	0·0768	0·0904	0·0861
0·95	0·1004	0·0752	0·0700	0·1160	0·0744	0·0872	0·0808
1·00	0·0956	0·0584	0·0732	0·0968	0·0668	0·0782	0·0758

§ 8. An Application

Consider the differential equation:

$$\frac{dz}{dt} = -az + \dot{e}(t), \tag{8.1}$$

where $\dot{e}(t)$ is a stationary random process whose power spectrum is essentially flat up to frequencies ω_B. For frequencies higher than ω_B the power spectrum decreases with ω. It is also assumed that ω_B is considerably higher than a. It is well known that the probability distributions of $z(t)$ are independent of the detailed shape of the power spectrum of $e(t)$ and of the actual value of ω_B, if ω_B is large enough. It is also well known that in such a case the probability distributions of the solutions of (8.1) are well approximated by those of the stochastic differential equation:

$$dx = -ax\,dt + dv, \tag{8.2}$$

where v is a Wiener process whose increments have zero means and the variances

$$\operatorname{var}[v(t+h) - v(t)] = 2\pi N_0 h, \tag{8.3}$$

where N_0 is the power density of $e(t)$ at the origin. The joint distribution of $z(t_1)$, $z(t_2), \ldots, z(t_n)$ is thus well approximated by $x(t_1), x(t_2), \ldots, x(t_n)$ if

$$\omega_B \gg \frac{1}{\min\limits_{i \neq j} |t_i - t_j|}.$$

This fact is widely used in the analysis and design of linear systems with additive disturbances.

We will now investigate if a similar result will hold for the stochastic differential equation:

$$\frac{dz}{dt} = \dot{e}_1 z + \dot{e}_2, \tag{8.4}$$

where

$$\dot{e}_1 = \frac{de_1}{dt} \quad \text{and} \quad \dot{e}_2 = \frac{de_2}{dt}$$

are stationary random processes whose power spectra are essentially flat up to the frequency ω_B and decreasing for higher values of ω.

Let $-n_1$ and n_2, $n_1 \geq 0$, $n_2 \geq 0$, denote the mean values and $r_1(t)$, $r_{12}(t)$ and $r_2(t)$ denote the convariance functions of $\dot{e}_1$ and $\dot{e}_2$. Further, let $\dot{e}_1 = \dot{e}_1 + n_1$ and $\dot{e}_2 = \dot{e}_2 - n_2$. It is assumed that the covariance functions are integrable over $(0, \infty)$. To analyse eqn. (8.4) we will now use the technique outlined in the Appendix. We find that (8.4) can be written as:

$$z = Kz + g, \tag{8.5}$$

where

$$g_0(t) = \exp\left[-n_1(t-t_0)\right] x(t_0) + \frac{n_2}{n_1}\{1-\exp\left[-n_1(t-t_0)\right]\}, \tag{8.6}$$

$$g_1(t) = \int_{t_0}^{t} \exp\left[-n_1(t-s)\right] de_1(s), \tag{8.7}$$

$$g(t) = g_0(t) + g_1(t), \tag{8.8}$$

$$Kf(t) = \int_{t_0}^{t} \exp\left[-n_1(t-s)\right] f(s)\, de_2(s). \tag{8.9}$$

The integrals appearing on the right-hand side of (8.7) and (8.9) are stochastic integrals (Doob 1953, p. 426). They exist if the covariance functions $r_i(t)$ are integrable. Under these conditions the operations of integration with respect to time and the taking of mathematical expectation can be interchanged (Loeve 1963, p. 472).

Now let $r_i(t)$, $i=1, 12, 2$ depend on a parameter n, i.e.

$$r_i(t) = r_i^{(n)}(t), \quad i = 1, 12, 2.$$

Assume that $r_i^{(n)}(t)$ converges weakly as $n \to \infty$ in such a way that

$$\int_{-a}^{a} r_i^{(n)}(t) f(t)\, dt \to 2B_i f(0) \tag{8.10}$$

for every continuous $f(t)$. This implies that the corresponding power spectra converges to:

$$\phi_i^{(n)}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left(-{}_i\omega_t\right) r_i^{(n)}(t)\, dt \to \frac{B_i}{\pi} \tag{8.11}$$

for all finite ω.

Now choose h fixed but arbitrarily small and consider the increment of the random process defined by (8.4) over the interval $(t, t+h)$.

Solving eqn. (8.5) by successive approximations we find that

$$E[z(t+n) - z(t) | z(t)] \to -[n_2 - n_1 z(t) + B_{12} + B_1 z(t)]h + 0(h^{3/2}), \tag{8.12}$$

$$E[(z(t+n) - z(t))^2 | z(t)] \to [B_1 z^2(t) + 2B_{12} z(t) + B_2]h + 0(h^{3/2}), \tag{8.13}$$

as $n \to \infty$.

We notice that due to (8.10) the increments of $z(t)$ over intervals h converges to zero as $n \to \infty$. Hence, if the process $z(t)$ is sampled at intervals of fixed but arbitrarily small lengths h we find that (a) the correlations of successive increments of $z(t)$ are arbitrarily small and (b) the first two moments of the increment of $z(t)$ are arbitrarily close to those of the process x defined by (2.1) if the parameters of (2.1) are chosen as:

$$A_i = B_i, \quad i = 1, 12, 2, \tag{8.14}$$

$$m_1 = n_1 - B_1 - B_{12}, \tag{8.15}$$

$$m_2 = n_2, \tag{8.16}$$

Table 3

Centre of class interval	1	2	3	4	Average	Density function
−2·0	0·0276	0·0090	0·0352	0·0174	0·0223	0·0192
−1·9	0·0290	0·0108	0·0302	0·0102	0·0200	0·0203
−1·8	0·0294	0·0060	0·0326	0·0132	0·0202	0·0215
−1·7	0·0314	0·0082	0·0320	0·0188	0·0226	0·0228
−1·6	0·0294	0·0112	0·0310	0·0274	0·0248	0·0244
−1·5	0·0400	0·0082	0·0290	0·0310	0·0270	0·0261
−1·4	0·0500	0·0112	0·0242	0·0348	0·0300	0·0280
−1·3	0·0432	0·0134	0·0260	0·0324	0·0288	0·0303
−1·2	0·0454	0·0118	0·0362	0·0336	0·0318	0·0329
−1·1	0·0498	0·0140	0·0476	0·0320	0·0358	0·0360
−1·0	0·0514	0·0168	0·0446	0·0332	0·0365	0·0396
−0·9	0·0570	0·0200	0·0342	0·0280	0·0348	0·0440
−0·8	0·0570	0·0282	0·0400	0·0422	0·0418	0·0493
−0·7	0·0618	0·0342	0·0434	0·0584	0·0494	0·0558
−0·6	0·0802	0·0498	0·0538	0·0750	0·0647	0·0640
−0·5	0·0982	0·0624	0·0682	0·0922	0·0802	0·0744
−0·4	0·1302	0·0822	0·0700	0·1144	0·0992	0·0876
−0·3	0·1602	0·0980	0·0918	0·1282	0·1196	0·1040
−0·2	0·1924	0·1348	0·1146	0·1382	0·1450	0·1231
−0·1	0·1986	0·1678	0·1308	0·1392	0·1591	0·1405
0·0	0·2240	0·1852	0·1492	0·1710	0·1824	0·1481
0·1	0·1986	0·2030	0·1224	0·1720	0·1740	0·1405
0·2	0·1802	0·1848	0·1218	0·1352	0·1555	0·1231
0·3	0·1798	0·1772	0·1200	0·1126	0·1474	0·1040
0·4	0·1534	0·1374	0·0986	0·0912	0·1202	0·0876
0·5	0·1246	0·1026	0·0850	0·0720	0·0960	0·0744
0·6	0·1132	0·0878	0·0716	0·0566	0·0823	0·0640
0·7	0·0988	0·0674	0·0628	0·0384	0·0668	0·0558
0·8	0·0916	0·0604	0·0722	0·0384	0·0656	0·0493
0·9	0·0706	0·0614	0·0624	0·0334	0·0570	0·0440
1·0	0·0566	0·0560	0·0486	0·0330	0·0486	0·0396
1·1	0·0416	0·0504	0·0344	0·0284	0·0387	0·0360
1·2	0·0396	0·0402	0·0270	0·0294	0·0340	0·0329
1·3	0·0364	0·0392	0·0208	0·0370	0·0334	0·0303
1·4	0·0362	0·0352	0·0166	0·0248	0·0282	0·0280
1·5	0·0356	0·0390	0·0124	0·0314	0·0296	0·0261
1·6	0·0328	0·0270	0·0124	0·0270	0·0248	0·0244
1·7	0·0248	0·0210	0·0188	0·0226	0·0218	0·0228
1·8	0·0228	0·0230	0·0160	0·0190	0·0202	0·0215
1·9	0·0214	0·0182	0·0160	0·0228	0·0196	0·0203
2·0	0·0222	0·0212	0·0178	0·0224	0·0209	0·0192

if n is sufficiently large. Thus, the joint distribution of $z(t_1), z(t_2), \ldots, z(t_n)$ for $\min_{i \neq j} |t_i - t_j| > h$ will be arbitrarily close to the distribution of $x(t_1), x(t_2), \ldots, x(t_n)$ if n is large enough.

Notice in particular eqn. (8.15). Back-tracking we will find that the last two terms of the right-hand member are due to the increments of $e_1(t)$ being correlated. Compare eqns. (A 15) and (A 16) of the Appendix. It is thus clear that there will be a drastic change in the character of the solution of (8.4) depending on whether the disturbance $e_1(t)$ has uncorrelated increments or not. This can be demonstrated numerically by introducing correlation in the sequence $\{e(t_i)\}$ of the difference approximation (8.1) of (2.1). Consider for example, the following system:

$$\left.\begin{aligned} x_h(t_{n+1}) &= (1 - hm_1')x_h(t_n) + m_2'h + x_h(t_n)u(t_n)\sqrt{h} + e_2(t_n)\sqrt{h}, \\ u(t_{n+1}) &= (1 - hb_1)u(t_n) + b_1 e_3(t_n)\sqrt{h}, \\ t_{n+1} &= t_n + h. \end{aligned}\right\} \tag{8.17}$$

where $\{e_2(t_i)\}$ and $\{e_3(t_i)\}$ are sequences of normal random variables with zero means and the covariances:

$$\operatorname{cov}[e_2(t_i), e_2(t_j)] = 2A_2\delta_{ij}, \tag{8.18}$$

$$\operatorname{cov}[e_3(t_i), e_3(t_j)] = 2A_3\delta_{ij}. \tag{8.19}$$

These equations correspond to $A_{12} = 0$ and

$$\operatorname{cov}[e_1(t_i), e_1(t_j)] = 2(1 - hb_1)^{i-j}A_1. \tag{8.20}$$

In table 3 we present the histograms of four experimentally obtained steady-state distributions of (8.17) for:

$$A_1 = 10, \quad A_3 = 1, \quad m_1' = 1, \quad m_2' = 0, \quad h = 0{\cdot}002, \quad T = 100, \quad B = 0{\cdot}1,$$
$$b_1 = 10, \quad x(0) = 0{\cdot}0.$$

The solutions were obtained by the difference eqns. (8.17) using pseudo-random numbers for $\{e_2(t_i)\}$ and $\{e_3(t_i)\}$ as described in § 7. In table 3 we have tabulated $\Delta t(\Gamma_i)/T$ versus the middle of the class intervals Γ_i.

The stochastic differential equation which approximates (8.17) has

$$A_1 = 10, \quad A_{12} = 0, \quad A_2 = 1, \quad m_1 = m_1' - A_1 - A_{12} = -9, \quad m_2 = m_2' = 0,$$

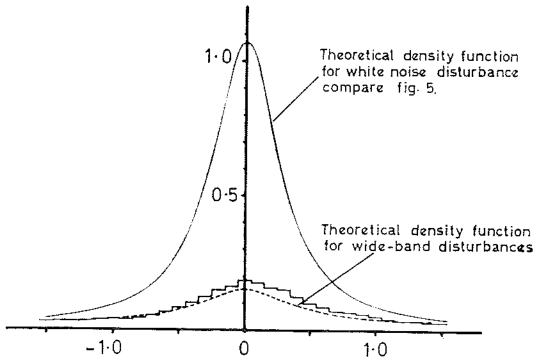
and its solution has a steady-state distribution with the density function

$$f(x) = \left(\frac{10}{\pi}\right)^{1/2} \frac{\Gamma(0{\cdot}55)}{\Gamma(0{\cdot}05)} (1 + 10x^2)^{-0{\cdot}55}. \tag{8.21}$$

This density function is tabulated in the last column of table 3. It is also graphed in fig. 7 together with the average histogram. In fig. 7 we have also shown the density function (8.21).

Notice the difference between the case of no correlation between the increments of $e_1(t_i)$. Compare figs. 5 and 7, and eqns. (7.6) and (8.21).

Fig. 7

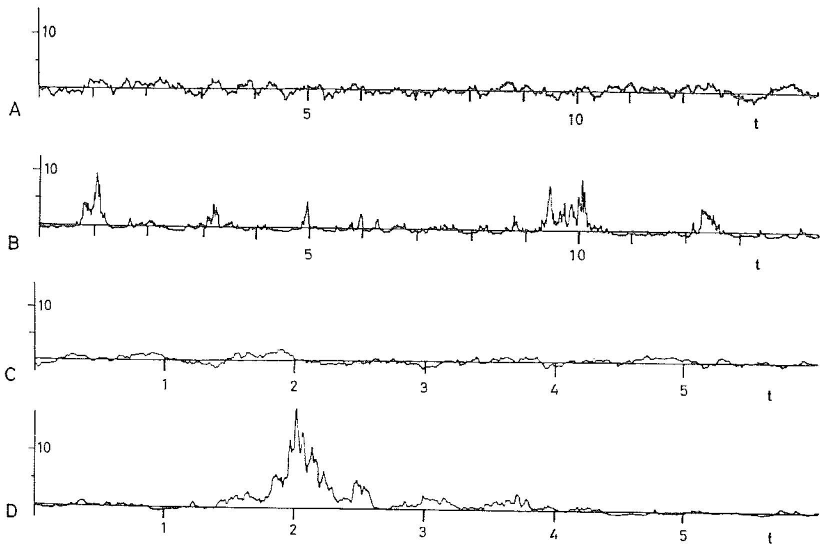


Histogram of the steady-state solution for an experiment with uncorrelated wide-band disturbances.

The stochastic differential eqn. (8.4) has also been studied on an analogue computer. The qualitative features of the solution have been verified. In fig. 8 we show some typical results for the equation $\dot{z}+6z=(1+kz)\dot{e}$, where $\dot{e}$ is a stationary process whose spectrum is essentially flat up to 200 rad $\sec^{-1}$ with a power density

$$N_0=\frac{6}{\pi}V^2\,\text{sec}.$$

Fig. 8



Sample function of the solution for the case of strongly correlated wide-band, $k=0$ in A and C and $k=1$ in B and D.

In fig. 8 we show in (*a*) a sample of z with $k=0$, in (*b*) sample of z with $k=1$, (*c*) and (*d*) respectively the same things with an expanded time scale. The theoretical distribution, functions are:

$$P\{z \leq a\} = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{a} \exp - \tfrac{1}{2} t^2 \, dt$$

and

$$P\{z \leq a\} = \exp - \frac{1}{1+a}$$

respectively.

§ 9. Conclusions

(1) The analysis clearly shows that the moments do not conveniently characterize the probability distributions for the solutions of the stochastic differential eqn. (2.1). We have found that the order of moments which exists depends on the intensity of the disturbance on the parameter. Given a specified level of disturbance there will always exist an integer n such that the nth order moment of the first probability distribution does not exist (see eqn. (3.3)). In the examples illustrated in § 7, moments of an order higher than the first do not exist, and in § 8 we have an example where not even the first moment exists.

(2) We have also found that the character of the solutions depends strongly upon the correlation between the forcing disturbance and the disturbance on the parameter. Compare examples 1 and 2 of § 6 which represent the extremes of uncorrelated and linearly dependent disturbances.

(3) It is also found that an analysis of the singular points of Kolmogorov's equation using Feller's classification will give considerable insight into the qualitative features of the solution.

(4) The analysis also shows that there is a great difference between the cases of 'white noise' and 'band-limited' disturbances. Thus, it is not possible to express the case for 'white noise' disturbance by solving the problem for a disturbance with finite bandwidth and afterwards letting the bandwidth go to infinity. The case of band-limited disturbance on the parameter can, however, be approximated with 'white noise' disturbance on the parameter, the 'white noise' disturbance having a different mean than the band-limited disturbances (see eqns. (8.16), (8.17), (8.19)).

(5) In § 6 we have also found the rather surprising effect that the probability distributions tend to concentrate at $x=0$ with an increasing intensity in the disturbance on the parameter.

Although the analysis in this paper is limited to a first-order system we are led to ask if the conclusions arrived at will also hold for higher-order systems. Conclusion (1) obviously generalizes. Equations for the moments are easily obtained in the higher-order cases, and it is easily seen that they are given by linear equations similar to those of first-order cases. Higher-order moments do not exist for sufficiently large intensities of disturbances on the parameters.

The peculiar effect that the probability distributions tend to concentrate around the origin with increasing intensity of the disturbance on the parameters does not seem to extend to higher-order systems. The phenomenon seems to be related to the character of the singularity of the Green-function of the operator Ω, different in one, two and three dimensions. A related effect is mentioned by Khas'minskii (1962, p. 1562) where it is found that an unstable first-order linear system can be stabilized by ' white noise ' disturbance on the parameter, but not a higher-order system. The same effect is seen in the different character of the absorption probabilities for random walks (Feller 1958, p. 327) in one, two and three dimensions.

APPENDIX

Solution of the Stochastic Differential Equation by Successive Approximations and Calculation of the Incremental Moments

Equation (2.1) can be written as:

$$dx = (-xm_1 + m_2)\,dt + x\,dv_1 + dv_2, \tag{A 1}$$

where v_1 and v_2 are Wiener processes whose increments have zero means and the same covariances as $w_1(t)$ and $w_2(t)$. See eqns. (2.4)–(2.6). Re-writing (A 1) as an integral equation, we get:

$$x(t) = \exp[-m_1(t-t_0)]x(t_0) + \frac{m_2}{m_1}\{1-\exp[-m_1(t-t_0)]\}$$
$$+\int_{t_0}^{t_1} \exp[-m_1(t-s)]x(s)\,dv_1(s) + \int_{t_0}^{t_1} \exp[-m_1(t-s)]\,dv_2(s), \tag{A 2}$$

where the last two integrals are stochastic integrals (Doob 1953, pp. 426–451).

Introduce the functions

$$g_0(t) = \exp[-m_1(t-t_0)]x(t_0) + \frac{m_2}{m_1}\{1-\exp[-m_1(t-t_0)]\}, \tag{A 3}$$

$$g_1(t) = \int_{t_0}^{t} \exp[-m_1(t-s)]\,dv_2(s), \tag{A 4}$$

$$g(t) = g_0(t) + g_1(t), \tag{A 5}$$

and the operator K defined as:

$$Kf(t) = \int_{t_0}^{t} \exp[-m_1(t-s)]f(s)\,dv_1(s). \tag{A 6}$$

The eqn. (A 2) now becomes:

$$x = Kx + g. \tag{A 7}$$

Let $f(t,\omega)$ be a random function whose second moment exists. The norm of such a function is defined by:

$$||f(t,\omega)||=\left[\max_{t_0\leqslant t\leqslant t_1} Ef^2(t,\omega)\right]^{1/2}. \tag{A 8}$$

Hence

$$||g_0||=|x_0|+\left|\frac{m_2}{m_1}\right|.\{1-\exp[-m_1(t_1-t_0)]\}, \tag{A 9}$$

$$||g_1||=\left\{\frac{A_2}{m_1}[\exp[2m_1(t_1-t_0)]-1]\right\}^{1/2}. \tag{A 10}$$

To find the operator norm subordinate to (A 8) we proceed as follows:

$$\begin{aligned}||Kf||^2&=\max_{t_0\leqslant t\leqslant t_1} E\int_{t_0}^{t}\int_{t_0}^{t}\exp[-m_1(2t-s'-s'')]f(s)f(s'')\,dv_1(s')\,dv_1(s'')\\ &\leq ||f||^2\max_{t_0\leqslant t\leqslant t_1} E\int_{t_0}^{t}\int_{t_0}^{t}\exp[-m_1(2t-s'-s'')]\,dv_1(s')\,dv_1(s'')\\ &\leq ||f||^2\frac{A_1}{m_1}\{1-\exp[-2m_1(t_1-t_0)]\},\end{aligned} \tag{A 11}$$

where the first inequality is equivalent to the definition (A 8), the second inequality follows from the fact that v_1 and v_2 have uncorrelated increments, and the third inequality is obtained by interchanging expectation and integration. Equality is obtained for $f(t,\omega)=1$ with probability one. Hence

$$||K||=\left\{\frac{A_1}{m_1}(1-\exp[-2m_1(t_1-t_0)])\right\}^{1/2}. \tag{A 12}$$

If $||K||<1$, which obviously can be achieved by choosing h sufficiently small, the solution of (A 7) is given by the Neuman series:

$$x=g+Kg+K^2g+\dots .$$

In order to obtain the incremental moments we retain only terms of order h. We notice that

$$\begin{aligned}||g_0||&=0(1),\\ ||g_1||&=0(h^{1/2}),\\ ||K||&=0(h^{1/2}),\end{aligned}$$

and we get

$$\begin{aligned}\Delta x&=x(t_1)-x(t_0)\\ &=g_0-x(t_0)+g_1+Kg_0+Kg_1+K^2g_0+0(h^{3/2}).\end{aligned} \tag{A 13}$$

The third term is fully written in (A 4). We write out the last three completely and get:

$$Kg_0 = \int_{t_0}^{t_1} \exp\left[-m_1(t_1 - t_0)\right] g_0(s)\, dv_1(s), \tag{A 14}$$

$$Kg_1 = \int_{t_0}^{t_1} \exp\left[-m_1(t_1 - s)\right] dv_1(s) \int_{t_0}^{s} \exp\left[-m_1(t - s)\right] dv_2(s'), \tag{A 15}$$

$$K^2 g_0 = \int_{t_0}^{t_1} \exp\left[-m_1(t_1 - s)\right] dv_1(s) \int_{t_0}^{s} \exp\left[-m_1(s - s'_0)\right] g_0(s')\, dv_1(s'). \tag{A 16}$$

As the increments of v_1 and v_2 have zero means, the expectation of the third and the fourth terms of (A 13) will thus vanish. Also the expectation of the fourth and fifth terms vanish because the increments of v_1 and v_2 are uncorrelated. By letting h tend to zero in (A 13), we obtain (2.8) and (2.9) in the same way.

In the same way, we find that

$$(\Delta x)^2 = g_1{}^2 + (Kg_0)^2 + Kg_0 g_1 + g_1 K g_0 + 0(h).$$

By letting h tend to zero we get (2.9).

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